EC402: MEI. Spring Term: The Analysis of Time Series*

# Problem Set 2 – Solutions

## 1. Hypothesis test

1. Consider the $MA(1)$ model

$$y_t = \varepsilon_t + \theta\varepsilon_{t-1}, \quad \varepsilon_t \sim iidN\left(0, \sigma^2\right), \quad t = 1, ..., T, \quad \varepsilon_0 = 0$$

   1. Derive the $LM$ test of the null $\theta = 0$

   2. What is the advantages of the $LM$ test over the $LR$ test in this case?

      **Answer.** *ML estimation of $\theta$ is equivalent to non-linear least squares*

$$\min_{\theta} \sum_t \varepsilon_t(\theta)^2 \text{ where } \varepsilon_t(\theta) = y_t - \theta\varepsilon_{t-1}(\theta)$$

$$z_t = -\frac{\partial\varepsilon_t}{\partial\theta} = \varepsilon_{t-1} + \theta\frac{\partial\varepsilon_{t-1}}{\partial\theta}$$

$$FOC: \qquad \sum_t z_t\varepsilon_t = 0.$$

      *Quite complicated to compute.*

      <u>LM test</u>. *Under the null, $\theta = 0$ and $y_t = \varepsilon_t$. So $z_t = \varepsilon_{t-1}$ and $LM$ test is carried out by regressing $\varepsilon_t$ on $z_t$, i.e. in this case by regressing $y_t$ on $y_{t-1}$ and*

*computing $TR^2$. Since this is a one variable linear regression, $R^2$ is the square of the correlation coefficient, so*

$$TR^2 = T\frac{\left(\sum y_t y_{t-1}\right)^2}{\left(\sum y_t^2\right)\left(\sum y_{t-1}^2\right)} \sim \chi^2(1).$$

*Therefore, the LM test is straightforward to compute in this case.*

2. Consider the model

$$y_t = \beta_1 x_{1t} + \beta_2 \frac{(x_{2t}-\gamma)^{-2}}{2} + \varepsilon_t$$

where $\varepsilon_t \sim iidN\left(0, \sigma^2\right)$ and the regressors $x_1$ and $x_2$ are process indepemdent of the errors.

1. Obtain the log likelihood of the model and outline how you would obtain the NLE of the parameters $\beta_1,\ \beta_2,\ \gamma,\ \sigma^2$.
2. Construct a $LM$ test of the null $H_0 : \gamma = 0$.
3. Why is the $LM$ test easier to carry out than the Likelihood ratio test?

**Answer.**

$$\begin{aligned} \varepsilon_t &= y_t - \beta_1 x_{1t} - \beta_2 \frac{(x_{2t}-\gamma)^{-2}}{2} \\ \log L &= -\frac{T}{2}\log 2\pi - \frac{T}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum \varepsilon_t^2. \end{aligned}$$

*As this is nonlinear least squares maximize* $\log L$ *wrt* $(\beta_1, \beta_2, \gamma)$ *using derivations:*

$$\begin{bmatrix} \frac{\partial \log L}{\partial \beta_1} \\ \frac{\partial \log L}{\partial \beta_2} \\ \frac{\partial \log L}{\partial \gamma} \end{bmatrix} = \frac{1}{\sigma^2}\sum z_t \varepsilon_t \text{ where } z_t = -\begin{bmatrix} \frac{\partial \varepsilon_t}{\partial \beta_1} \\ \frac{\partial \varepsilon_t}{\partial \beta_2} \\ \frac{\partial \varepsilon_t}{\partial \gamma} \end{bmatrix} = \begin{bmatrix} x_{1t} \\ \frac{1}{2}(x_{2t}-\gamma)^{-2} \\ -\beta_2 (x_{2t}-\gamma)^{-3} \end{bmatrix}.$$

*Solving* $\sum z_t \varepsilon_t = 0$ *yields ML estimates of* $(\beta_1, \beta_2, \gamma)$. $\widehat{\sigma}^2 = \frac{1}{T}\sum \widehat{\varepsilon}_t^2$.
$H_0 : \gamma = 0$. *Under* $H_0$

$$\begin{aligned} y_t &= \beta_1 x_{1t} + \beta_2 \frac{1}{2x_{2t}^2} + \varepsilon_t \\ \varepsilon_t &= y_t - \beta_1 x_{1t} - \frac{\beta_2}{2x_{2t}^2}. \end{aligned}$$

*ML estimation of restricted model is least squares regression of $y_t$ on $x_{1t}$, $x_{2t}^{-2}$. Then calculate*

$$\widehat{\varepsilon}_t = y_t - \widehat{\beta}_1 x_{1t} - \widehat{\beta}_2 \frac{1}{2x_{2t}^2}.$$

*When* $\gamma = 0$, $z_t = \begin{bmatrix} x_{1t} \\ \frac{1}{2x_{2t}^2} \\ -\frac{\beta_2}{x_{2t}^3} \end{bmatrix}$. *LM test requires regressing $\widehat{\varepsilon}_t$ on $z_t$, calculate $TR^2$. Reject $H_0$ if $TR^2 > \chi^2_\alpha(1)$.*

*LM easier than LR as you do not have to do nonlinear least squares maximization.*

## 2. Distributed lags and transformations

1. Rewrite the distributed lag

$$B(L)\, x_t = \beta_0 x_t + \beta_1 x_{t-1} + ... + \beta_m x_{t-m}$$

in the form

$$\delta_0 x_t + \delta_1 \Delta x_t + \delta_2 \Delta x_{t-1} + ... + \delta_m \Delta x_{t-m+1}$$

1. Show that $\delta_0 = B(1)$, the total effect. What is the relationship between the $\delta$'s and the $\beta$'s?

**Answer.**

$$\beta_0 x_t + \beta_1 x_{t-1} + ... + \beta_m x_{t-m} =$$

$$\delta_0 x_t + \delta_1 (x_t - x_{t-1}) + \delta_2 (x_{t-1} - x_{t-2}) + ... + \delta_m (x_{t-m+1} - x_{t-m})$$

$$\begin{array}{ll} \therefore \beta_m = -\delta_m & \delta_m = -\beta_m \\ \beta_{m-1} = \delta_m - \delta_{m-1} & \delta_{m-1} = -(\beta_{m-1} + \beta_m) \\ \beta_{m-2} = \delta_{m-1} - \delta_{m-2} & \delta_{m-2} = -(\beta_{m-2} + \beta_{m-1} + \beta_m) \\ \vdots & \\ \beta_1 = \delta_2 - \delta_1 & \delta_1 = -(\beta_1 + ... + \beta_m) \\ \beta_0 = \delta_0 + \delta_1 & \delta_0 = \beta_0 + \beta_1 + ... + \beta_m = B(1) \textit{ total effect.} \end{array}$$

2. If you write the second form as

$$\delta_0^+ x_{t-1} + \delta_1^+ \Delta x_t + \delta_2^+ \Delta x_{t-1} + ... + \delta_m^+ \Delta x_{t-m+1}$$

how are the $\delta^{+}$'s related to the $\beta$'s?

**Answer.**

$$\delta_0^{+} x_{t-1} + \delta_1^{+}(x_t - x_{t-1}) + \delta_2^{+}(x_{t-1} - x_{t-2}) + ... + \delta_m^{+}(x_{t-m+1} - x_{t-m})$$

$$\begin{array}{ll} \beta_m = -\delta_m^{+} & \therefore \delta_m^{+} = -\beta_m \\ \vdots & \\ \beta_2 = \delta_3^{+} - \delta_2^{+} & \therefore \delta_2^{+} = -(\beta_2 + ... + \beta_m) \\ \beta_0 = \delta_1^{+} & \therefore \delta_1^{+} = \beta_0 \\ \beta_1 = \delta_0^{+} - \delta_1^{+} + \delta_2^{+} & \therefore \delta_0^{+} = \beta_1 + \beta_0 + \beta_2 + ... + \beta_m = B(1) \end{array}$$

*Note: coeff. on levels variable unchanged (still total effect).*

2. Consider the Autoregressive Distributed Lag Model

$$A(L)\, y_t = \lambda + B(L)\, x_t + \varepsilon_t$$

where

$$\begin{aligned} A(L) &= 1 - \alpha_1 L - ... - \alpha_m L^m \\ B(L) &= \beta_0 + \beta_1 L + ... + \beta_n L^n. \end{aligned}$$

1. Show that you can rewrite this model in the error correction form

$$\Delta y_t = \lambda + \gamma_0 y_{t-1} + \sum_{i=1}^{m-1} \gamma_i \Delta y_{t-i} + \delta_0 x_{t-1} + \sum_{i=1}^{n} \delta_i \Delta x_{t-i+1} + \varepsilon_t.$$

What is the interpretation of $\gamma_0$ and $\delta_0$and hence of $\delta_0/\gamma_0$? [Hint: use the results you derived in the first question of this section]

**Answer.**

$$\begin{aligned} (y_t - y_{t-1}) &= \lambda + \gamma_0 y_{t-1} + \sum_{i=1}^{m-1} \gamma_i (y_{t-i} - y_{t-i-1}) \\ &\quad + \delta_0 x_{t-1} + \sum_{i=1}^{n} \delta_i (x_{t-i+1} - x_{t-i}) + \varepsilon_t \\ y_t &= \lambda + (\gamma_0 + 1) y_{t-1} + \sum_{i=1}^{m-1} \gamma_i (y_{t-i} - y_{t-i-1}) + B(L) x_t + \varepsilon_t \end{aligned}$$

*where* $\delta_0 = B(1)$ *and* $\delta_1 = \beta_0$, $\delta_i = -\sum_{j=i}^{n} \beta_j$ *from results above.*

$$y_t - (1+\gamma_0)y_{t-1} - \sum_{i=1}^{m-1} \gamma_i(y_{t-i} - y_{t-i-1}) = \lambda + B(L)x_t + \varepsilon_t.$$

$$LHS = y_t - (1+\gamma_0)y_{t-1} - \gamma_1(y_{t-1} - y_{t-2}) - \gamma_2(y_{t-2} - y_{t-3}) - ... - \gamma_{m-1}(y_{t-m+1} - y_{t-m})$$

$$\begin{array}{lll} \therefore & -\alpha_m = \gamma_{m-1} & \gamma_{m-1} = -\alpha_m \\ & -\alpha_{m-1} = \gamma_{m-2} - \gamma_{m-1} & \gamma_{m-2} = -(\alpha_{m-1} + \alpha_m) \\ & -\alpha_2 = \gamma_1 - \gamma_2 & \gamma_1 = -(\alpha_2 + ... + \alpha_m) \\ & -\alpha_1 = -(1+\gamma_0) - \gamma_1 & \gamma_0 = -(1 - \alpha_1 - \alpha_2 - ... - \alpha_m) \\ & & \quad = -A(1) \end{array}$$

$\frac{\delta_0}{\gamma_0} = -\frac{B(1)}{A(1)}$, *this is minus the steady state effect of a change in* $x_t$ *on* $y_t$.

2. If you estimate the original model and the model in error correction form, what is the relationship between the two sets of estimates and the two sets of residuals?

   **Answer.** *As these are non-singular linear transforms, the estimates have the same relationships as above and the residuals are the same.*