EC402: MEI. Spring Term: The Analysis of Time Series*

Lecture Notes #4: Cointegration and Error Correction Representation

1. Integrated and Cointegrated Processes

1.1. Integrated Processes

If a process x_t is stationary, then the process

$$y_t = x_t + y_{t-1} = \sum_{s=0}^{\infty} x_{t-s}$$

is called *integrated of order one*, $I(1)$. y_t has the obvious property by construction that its first difference is x_t and is, hence, stationary ($I(0)$)

$$\Delta y_t = y_t - y_{t-1} = x_t.$$

If y_t is $I(1)$ and

$$z_t = y_t + z_{t-1},$$

then z_t is said to be integrated of order two, $I(2)$, and the second difference of z_t is $I(0)$.

More generally, if y_t is integrated of order p, $I(p)$, and

$$z_t = y_t + z_{t-1},$$

then z_t is said to be integrated of order $p+1$, $I(p+1)$.

Note that the difference of a stationary process is stationary.

1.2. Cointegrated Process

Suppose the processes x_t and y_t are both $I(1)$ (that is, non-stationary with a unit root). If it exist a stationary linear combination of these processes, the processes are x and y are said to be cointegrated.

More generally, an $(n \times 1)$ vector time series $Y_t = [y_{1t}, ..., y_{2t}]'$ is said to be cointegrated if each of the series is individually $I(1)$, while some linear combination of the series $a'Y_t$ is stationary $(I(0))$ for some non-zero $(n \times 1)$ vector a.

Example 1. *Consider the following bivariate system*

$$\begin{aligned} y_{1t} &= \gamma y_{2t} + u_{1t} \\ y_{2t} &= y_{2,t-1} + u_{2t} \end{aligned}$$

with u_1 and u_2 uncorrelated with noise.
Clearly, both processes are non-stationary and contain a unit root i.e. are $I(1)$. Nevertheless, the linear combination $(y_1 - \gamma y_{2t})$ is stationary. Hence we would say that $Y_t = (y_{1t}, y_{2t})'$ is cointegrated with cointegrating vector $a' = (1, -\gamma)$.

Cointegration means that although many developments can cause permanent changes in the individual elements of Y_t, there is some long-run equilibrium relation tying the individual components together, and this is represented by the linear combination $a'Y_t$.

Example 2. *Suppose the optimal consumption rule for an agent is always to consume a share α_t of her current wealth W i.e.*

$$C_t = \alpha_t W_t,$$

where $\log \alpha_t \sim iid(\bar{\alpha}, \sigma_\alpha^2)$. Moreover, assume that log *wealth follows a random walk process with drift*

$$\log W_t = \mu + \log W_{t-1} + \varepsilon_t, \qquad \varepsilon_t \sim iid\left(0, \sigma_\varepsilon^2\right)$$

Clearly, neither wealth nor consumption are stationary (they both grow exponentially), but i) the log *of each of these variables is* $I(1)$ *and ii)* $\log C_t$ *and* $\log W_t$ *are cointegrated with cointegration vector* $a\prime = (1, -1)$ *since*

$$\log C_t - \log W_t = \log \alpha_t$$

Moreover, we have that, on average, C *will grow at the same rate* μ *as* X *since*

$$\begin{aligned}
\log C_t &= \log \alpha_t + \log W_t \\
&\rightarrow \log C_t - \log C_{t-1} = \log \alpha_t - \log \alpha_{t-1} + \log W_t - \log W_{t-1} \\
&\rightarrow E\left[\log C_t - \log C_{t-1}\right] = E\left[\log W_t - \log W_{t-1}\right] = \mu \\
&\rightarrow E\left[\log C_{t+T} - \log C_t\right] = E\left[\log W_{t+T} - \log W_t\right] = \mu \times T
\end{aligned}$$

that is, they share the same long-run trend, and the expected log *consumption-wealth ratio is constant*

$$E\left[\log \frac{C_t}{W_t}\right] = \bar{\alpha}.$$

This means that if we observe $\log C_t/W_t > \bar{\alpha}$ *(*$< \bar{\alpha}$*) we should expect the consumption-wealth ratio to reduce (increase) in the future.*

2. The Basic Time Series Model

The basic model is the stochastic difference equation,

$$A(L)y_t = D(L)x_t + \varepsilon_t. \tag{2.1}$$

$$x_t \text{ contemporaneously independent, } \varepsilon_t \text{ iid } (0, \sigma^2).$$

where L is the “lag operator” defined as $L^s z_t = z_{t-s}$, and $A(L)$ and $D(L)$ are polynomials in the lag operator.

In the next lectures we'll generalize this setting to have multiple equations i.e. the case in which y_t and ε_t are vectors, x_t is a matrix and $A(L)$ and $D(L)$ are matrix of polynomials in the lag operator.

In the Appendix we present a number of examples of economic models that can be characterized as in equation (2.1).

2.1. The Error Correction Form

For expositional purposes, consider the case of only one x variable. The general dynamic model with one independent variable (with an explicit constant term), has the form

$$\begin{aligned} y_t &= \alpha_0 + \alpha_1 y_{t-1} + \alpha_2 y_{t-2} + ... + \alpha_m y_{t-m} + \beta_0 x_t + \beta_1 x_{t-1} \\ &\quad + \beta_2 x_{t-2} + ... + \beta_n x_{t-n} + u_t \qquad (u_t \text{ iid}) \end{aligned} \tag{2.2}$$

or $A(L)y_t = \alpha_0 + B(L)x_t + u_t$ where

$$\begin{aligned} A(L) &= 1 - \alpha_1 L - \alpha_2 L^2 - ... - \alpha_m L^m \\ B(L) &= \beta_0 + \beta_1 L + \beta_2 L^2 + ... + \beta_n L^n. \end{aligned}$$

In this model, the short-run multiplier is $\frac{\partial y_t}{\partial x_t} = \beta_0$. The long-run relationship between x and y is

$$A(1)y = \alpha_0 + B(1)x$$

or

$$y = \frac{\alpha_0}{A(1)} + \frac{B(1)}{A(1)}x. \tag{2.3}$$

The long-run multiplier is $\frac{\partial y}{\partial x} = \frac{B(1)}{A(1)}$. We can rewrite the model (2.2) to isolate this long-run relationship. Start by rewriting $A(L)y_t$ as a function of y_{t-1}, Δy_t, Δy_{t-1}, Δy_{t-2},..., Δy_{t-m+1} and $B(L)x_t$ as a function of x_{t-1}, $\Delta x_t, x_{t-1}, ..., \Delta x_{t-n+1}$. To do this, note

$$\begin{aligned} A(L) &= (1-L) + (1-\alpha_1)L - \alpha_2 L^2 - ... - \alpha_m L^m \\ &= (1-L) + (1-\alpha_1-\alpha_2-...-\alpha_m)L + \alpha_2(L-L^2) + \alpha_3(L-L^3) \\ &\quad +... + \alpha_m(L-L^m) \end{aligned}$$

$$\begin{aligned} &= (1-L) + A(1)L + \alpha_2(L-L^2) + \alpha_3\left((L-L^2)+(L^2-L^3)\right) \\ &\quad +... + \alpha_m\left((L-L^2)+(L^2-L^3)+...+(L^{m-1}-L^m)\right) \\ &= (1-L) + A(1)L + \alpha_2(1-L)L + \alpha_3(1-L)(L+L^2) \\ &\quad + \alpha_4(1-L)(L+L^2+L^3) + ... + \alpha_m(1-L)(L+L^2+...+L^{m-1}) \end{aligned}$$

$$
\begin{aligned}
&= (1-L) + A(1)L + (\alpha_2 + \alpha_3 + ... + \alpha_m)L(1-L) \\
&\quad +(\alpha_3 + \alpha_4 + ... + \alpha_m)L^2(1-L) + (\alpha_4 + \alpha_5 + ... + \alpha_m)L^3(1-L) \\
&\quad ... \\
&\quad +\alpha_m L^{m-1}(1-L).
\end{aligned}
$$

So

$$A(L)y_t = A(1)y_{t-1} + \Delta y_t + \alpha_1^*\Delta y_{t-1} + \alpha_2^*\Delta y_{t-2} + ... + \alpha_{m-1}^*\Delta y_{t-m+1}$$

where

$$\alpha_j^* = (\alpha_{j+1} + \alpha_{j+2} + ... + \alpha_m).$$

Similarly

$$
\begin{aligned}
B(L)x_t &= B(1)x_{t-1} + \beta_0^*\Delta x_t - \beta_1^*\Delta x_{t-1} - \beta_2^*\Delta x_{t-2} - ... - \beta_{n-1}^*\Delta x_{t-n+1} \\
\beta_0^* &= \beta_0, \qquad \beta_j^* = \left(\beta_{j+1} + \beta_{j+2} + ... + \beta_n\right).
\end{aligned}
$$

So our equation becomes

$$
\begin{aligned}
\Delta y_t &= \alpha_0 - A(1)y_{t-1} + B(1)x_{t-1} - \alpha_1^*\Delta y_{t-1} - \alpha_2^*\Delta y_{t-2} \\
&\quad -... - \alpha_{m-1}^*\Delta y_{t-m+1} + \beta_0^*\Delta x_t - \beta_1^*\Delta x_{t-1} - \beta_{12}^*\Delta x_{t-2} \\
&\quad -... - \beta_{n-1}^*\Delta x_{t-n+1} + u_t. \qquad (2.4)
\end{aligned}
$$

Note first, (2.2), (2.4) are exactly the same equation. Second, the long-run multiplier is $\left(\frac{-\text{ coefficient on } x_{t-1}}{\text{coefficient on } y_{t-1}}\right)$. So, the "long-run" solution can be read off immediately. Let y, x be constant, namely $\Delta y = \Delta x = 0$ all time periods (you might think of this as the steady state solution), then (2.4) reduces to

$$0 = \alpha_0 - A(1)y + B(1)x$$

or

$$y = \frac{\alpha_0}{A(1)} + \frac{B(1)}{A(1)}x,$$

which is the same as (2.3). If the equation (2.4) is rewritten as

$$
\begin{aligned}
\Delta y_t &= -A(1)\left(y_{t-1} - \frac{B(1)}{A(1)}x_{t-1} - \frac{\alpha_0}{A(1)}\right) \\
&\quad + \text{terms in } \Delta x_{t-i},\ \Delta y_{t-i},
\end{aligned}
$$

this is known as the *error correction form.* Why? $y_{t-1} - \frac{B(1)}{A(1)}x_{t-1} - \frac{\alpha_0}{A(1)}$ is the difference between y_{t-1} and the long-run equilibrium value of y corresponding to x_{t-1}, namely $\frac{B(1)}{A(1)}x_{t-1} + \frac{\alpha_0}{A(1)}$. So if y_{t-1} is above (below) this value, this tends to move y down (up). This difference is the "error" and the movement is the "correction". Note that any stochastic difference equation can be written in error correction form unless $A(1) = 0$. The remaining $\Delta x, \Delta y$ terms are often known as "short-run dynamics."

2.2. Error Correction Form and Cointegration

The error correction form has been extensively used to model the relationships among cointegrated variables. To fix idea consider the case of two $I(1)$ variable y_t and x_t. By definition, if y_t and x_t are cointegrated, there exist a vector $a' = (1, -\gamma)$ such that $a'[y_t, x_t]'$ is stationary. Moreover, the by assumption that y_t and x_t are $I(1)$, we know that their first differences are $I(0)$. This implies that, if we know the cointegrating vector, we can write the following error correction representation

$$\Delta y_t = c_1 + c_1 (y_t - \gamma x_t) + \text{ terms in } \Delta x_{t-i},\ \Delta y_{t-i}\ + u_t$$

where all the right hand side terms are stationary variables (and this allows us to invoke the standard MLE asymptotics). Moreover, the term $c_1 (y_t - \gamma x_t)$ will be the error correction component with a straightforward economic interpretation.

One of the problems with this is that often we don't know ex-ante whether two or more variables should be cointegrated (economic theory can help us along this dimension). So often applied researchers tend to test for cointegration among variables and then, if such a relationship is detected, they impose the cointegration structure to the data. This is problematic since cointegration tests tend to have very poor small sample properties. Moreover, if we impose a cointegration relationship when it is not there, we might make very misleading long run predictions.

In practices, researchers tend to proceed as follows:

1. In the first step they test whether each variable has a unit root. It will often be the case that you are unable to reject a unit root. This is a totally reasonable step, but

we shold be aware that the small sample proprtioes of unit root tests (like the DF test you have already seen) is extremly poor (never rely on only one unit root test!).

Warning #1 Assuming that a variable has a unit root when it doesn't is not an innocuous mistake (even though many in practitioners think so!).

Example 3. *Suppose the true model is*

$$y_t = \beta x_t + \varepsilon_t \backsim iid\left(0, \sigma^2\right)$$

where x_t is only contemporaneously independent of ε_t.

Suppose the econometrician estimates instead the model in first differences

$$\Delta y_t = \beta \Delta x_t + u_t.$$

Since now $u_t = \varepsilon_t - \varepsilon_{t-1}$ this model has: $i)$ serial correlation in the errors – with the related problems outlined in the first lecture – and $ii)$ $E\left[\Delta x_t u_t\right] = E\left[\left(x_t - x_{t-1}\right)\left(\varepsilon_t - \varepsilon_{t-1}\right)\right] = -E\left[x_t \varepsilon_{t-1}\right] - E\left[x_{t-1}\varepsilon_t\right]$, and this does not need to be zero since x and ε are only contemporaneously independent, ie regressor and error term might be correlated.

2. Testing for Cointegration. If all variables are $I(1)$, write down the long-run relationship between y variable and the x variables, excluding the short-run dynamics e.g.

$$y_t = \beta_0 + \sum \beta_k x_{kt} + \varepsilon_t. \tag{2.5}$$

If there exists a long-run relationship cointegrating relationship we should be able to find a set of parameter estimates $\hat{\beta}_k$ so that

$$\left(y_t - \sum \hat{\beta}_k x_{ktfor}\right) \text{ is stationary}$$

If we can find no such parameter estimates, then y and all possible linear combinations of the x variables will tend to drift apart over time (no long-run relationship between y and the x variables).

How should one test for cointegration? A simple way is to:

do OLS on (2.5) to generate $\hat{\varepsilon}_t$. (we may also include short-run dynamics in (2.5).)

Then test the $\hat{\varepsilon}_t$ sequence for a unit root using the standard augmented DF procedure. Note: you cannot use the standard DF tables. The problem is that we have $\hat{\varepsilon}_t$ here and not ε_t. The appropriate statistics corresponding to $\hat{\tau}_\mu$ are given in the Davidson/MacKinnon text book (or Table 8.2 in Johnston/DiNardo)

Warning #2 Cointegration tests tend to have very poor small sample properties If we impose a cointegration relationship when it is not there, we might make very misleading long run predictions.

Example 4. *From a theoretical point of view, the log housing price index* ($\log P$) *and household aggregated log income* ($\log Y$) *do not need to be cointegrated. Nevertheless, this two variables in the past* seemed *to be cointegrated and this was judged to be consistent with some kind of common sense argument.*

If an econometrician were to believe that such a long run restriction holds, any time she observes P/Y *above its historical average she would predict that* P/Y *should go down some time in the future.*

Would this be reasonable? Let's consider as practical example the UK housing mortgage market. In the UK, and similarly in other countries, one of the standard banks' rules in determining the maximum loan value is given by the sum of 3.75 *times the gross yearly income of the primary borrower plus* 1 *time the income of the gross yearly income of the secondary borrower (typically the partner of the primary borrower). It is therefore obvious that a shift in the typical household composition – namely the increased share of single households – could cause a permanent upward shift in the mean of the* P/Y *ratio in the long run. This means that inference based on the assumption that* $\log P$ *and* $\log Y$ *are cointegrated might be very misleading.*

Fortunately, economic theory can help us deciding which long-run restriction are reasonable and which are not, and should be used to complement empirical tests.

3. If you can reject non-cointegration (i.e. reject a unit root in the above test), then estimate (by OLS) a general dynamic model in standard form

$$A(L)y_t = \alpha_0 + B(L)x_t + u_t$$

or in equivalent error correction form (to taste[1])

$$\begin{aligned}\Delta y_t &= \alpha_0 - A(1)y_{t-1} + B(1)x_{t-1} - \alpha_1^*\Delta y_{t-1} \\ &\quad -... - \alpha_{m-1}^*\Delta y_{t-m+1} + \beta_0^*\Delta x_t - \beta_1^*\Delta x_{t-1} - ... - \beta_{n-1}^*\Delta x_{t-n+1}.\end{aligned}$$

3. Appendix

What kind of economic models underlie dynamic equations like (2.1)? Here we present a number of examples where, for expositional purposes, we have only one x variable.

3.1. Examples of Underlying Models

3.1.1. Adaptive Expectations Model[2]

$$\begin{aligned}y_t &= \beta x_{t+1}^e + \varepsilon_t \qquad \varepsilon_t \text{ iid } (0, \sigma^2) \\ x_{t+1}^e &= x_t^e + \gamma(x_t - x_t^e) \qquad 0 < \gamma < 1,\end{aligned}$$

where x_{t+1}^e is the expectation formed at t of x at $t+1$.

If

$$\begin{array}{ll}\gamma = 0, & \text{expectations do not adjust} \\ \gamma = 1, & \text{expectations adjust instantly.}\end{array}$$

The latter is often termed static expectations. The adaptive expectations model is rational iff x follows the process

$$x_{t+1} = x_t + \varepsilon_{t+1} - (1-\gamma)\varepsilon_t, \qquad \varepsilon_t \text{ iid } (0, \sigma^2).$$

[1] Frequentists have good reasons to prefer the second form, while Bayesians have good reasons to prefer the first form.

[2] This kind of model was often used in economis before the Lucas' rational expectation critique.

Note, rational means $x^e_{t+1} = E(x_{t+1} \,| I_t)$, where I_t is the information set at t, which includes $x_t, x_{t-1}, ...$ and ε_t.

<u>Proof:</u> Suppose

$$\begin{aligned}
x^e_{t+1} &= (1-\gamma)x^e_t + \gamma x_t \\
&= (1-\gamma)\left[(1-\gamma)x^e_{t-1} + \gamma x_{t-1}\right] + \gamma x_t \\
&= (1-\gamma)^2 x^e_{t-1} + (1-\gamma)\gamma x_{t-1} + \gamma x_t \\
&= (1-\gamma)^2\left[(1-\gamma)x^e_{t-2} + \gamma x_{t-2}\right] + ... \\
&= (1-\gamma)^3 x^e_{t-2} + (1-\gamma)^2\gamma x_{t-2} + (1-\gamma)\gamma x_{t-1} + \gamma x_t.
\end{aligned}$$

So,

$$x^e_{t+1} = \gamma\left[x_t + (1-\gamma)x_{t-1} + (1-\gamma)^2 x_{t-2} + ...\right].$$

This is rational if

$$\begin{aligned}
x_{t+1} &= \gamma\left[x_t + (1-\gamma)x_{t-1} + (1-\gamma)^2 x_{t-2} + ...\right] + \varepsilon_{t+1} \\
&= \gamma\left[1 + (1-\gamma)L + (1-\gamma)^2 L^2 + ...\right] x_t + \varepsilon_{t+1} \\
&= \frac{\gamma x_t}{1-(1-\gamma)L} + \varepsilon_{t+1}
\end{aligned}$$

or

$$x_{t+1} = x_t + \varepsilon_{t+1} - (1-\gamma)\varepsilon_t.$$

Going the other way round, suppose

$$x_{t+1} = x_t + \varepsilon_{t+1} - (1-\gamma)\varepsilon_t, \qquad \varepsilon_t \text{ iid } (0, \sigma^2).$$

Then

$$x^e_{t+1} = x_t - (1-\gamma)\varepsilon_t. \tag{3.1}$$

So

$$x^e_t = x_{t-1} - (1-\gamma)\varepsilon_{t-1}$$

or

$$x^e_t = x_t - \varepsilon_t$$

[because $x_t = x_{t-1} + \varepsilon_t - (1-\gamma)\varepsilon_{t-1}$]. Multiply this by $(1-\gamma)$ and subtract from (3.1)

$$x^e_{t+1} - (1-\gamma)x^e_t = \gamma x_t$$

or

$$x^e_{t+1} = x^e_t + \gamma(x_t - x^e_t). \tag{QED}$$

The adaptive expectations equation can be written

$$x^e_{t+1} = \frac{\gamma x_t}{[1-(1-\gamma)L]}.$$

Substitute this into $y_t = \beta x^e_{t+1} + \varepsilon_t$ to obtain

$$y_t = \frac{\beta\gamma x_t}{(1-(1-\gamma)L)} + \varepsilon_t$$

or

$$y_t = (1-\gamma)y_{t-1} + \beta\gamma x_t + \varepsilon_t - (1-\gamma)\varepsilon_{t-1}.$$

3.1.2. Partial adjustment

Desired or target outcome

$$y^*_t = \beta x_t.$$

Because it is "costly" to adjust, y_t reacts as

$$y_t = y_{t-1} + \gamma(y^*_t - y_{t-1}) + \varepsilon_t \qquad 0 < \gamma < 1.$$

$\gamma = 0$, no adjustment; $\gamma = 1$, complete adjustment. So γ is the "speed of adjustment".

Substitute:

$$y_t = (1-\gamma)y_{t-1} + \gamma\beta x_t + \varepsilon_t.$$

Very similar to section (3.1.1) except error is no longer MA.

3.1.3. General model

y^* is "equilibrium" level of a variable (random variable). y is actual level of variable. Suppose

$$y_t^* = \beta_0 + \beta_1 x_t + u_t$$

where x_t is exogenous and follows some stochastic process. Agent (representative or otherwise) solves

$$\min \sum_{s=0}^{\infty} \alpha^s \left[\left(y_{t+s} - E_t(y_{t+s}^*) \right)^2 + \lambda \left(y_{t+s} - y_{t+s-1} \right)^2 \right]$$

where $0 < \alpha < 1$. The idea here is that the agent would like to keep y close to y^* but, because of adjustment costs, y must not move too rapidly. Many dynamic optimization models can be approximated by problems of this type.[3] For example, standard investment, consumption, labour demand and pricing models are often founded on this basic structure. Here I present the solution although you are not expected to know this for the examination.

$$\begin{aligned} \text{FOC}: \qquad & 2\alpha^s \left(y_{t+s} - E_t(y_{t+s}^*) \right) + 2\alpha^s \lambda \left(y_{t+s} - y_{t+s-1} \right) \\ & -2\alpha^{s+1} \lambda \left(y_{t+s+1} - y_{t+s} \right) = 0 \qquad s = 0, 1, ... \end{aligned}$$

or,

$$\alpha\lambda y_{t+s-1} - (1 + \alpha\lambda + \lambda) y_{t+s} + \lambda y_{t+s-1} = -E_t(y_{t+s}^*)$$

or,

$$\left(-\alpha\lambda L^{-1} + (1 + \alpha\lambda + \lambda) - \lambda L \right) y_{t+s} = E_t(y_{t+s}^*). \tag{3.2}$$

Now factorize this quadratic in the lag operator. Suppose the factors have the form

$$a_1 \left[1 - a_2 L^{-1} \right] \left[1 - a_3 L \right] \tag{3.3}$$

or

$$-a_1 a_2 L^{-1} + a_1(1 + a_2 a_3) - a_1 a_2 a_3 L.$$

[3] Such a representation often arises as the "Loss Function" of a central bank that wants to set monetary policy optimally.

Comparing coefficients

$$a_1 a_2 = \alpha\lambda, \qquad a_1(1 + a_2 a_3) = 1 + \alpha\lambda + \lambda, \qquad a_1 a_3 = \lambda.$$

So $a_1 = \frac{\lambda}{a_3}$, $a_2 = \alpha a_3$. Substituting into the second equation gives

$$\lambda\alpha a_3^2 - (1 + \alpha\lambda + \lambda)a_3 + \lambda = 0. \tag{3.4}$$

So a_3 is the solution to a quadratic equation. If we suppose $f(a_3) = \lambda\alpha a_3^2 - (1+\alpha\lambda+\lambda)a_3 + \lambda$, then $f(0) = \lambda > 0$, $f(1) = -1 < 0$, $f(\infty) > 0$. So this quadratic has one root in the interval (0,1) and one root in the interval $(1, \infty)$. In order to generate a solution to the difference equation which is stable, we select the "stable" root in the (0,1) interval. Call this root μ.

So $a_3 = \mu$. $a_2 = \alpha\mu$, $a_1 = \frac{\lambda}{\mu}$. Consequently the second order difference equation (3.2) may be factorized as (3.3) which has the form

$$\frac{\lambda}{\mu}\left(1 - \alpha\mu L^{-1}\right)\left(1 - \mu L\right) y_{t+2} = E_t(y^*_{t+s}), \qquad s = 0, 1, ...$$

This reduces to

$$(1 - \mu L) y_{t+s} = \frac{\mu}{\lambda}\frac{E_t(y^*_{t+s})}{(1 - \alpha\mu L^{-1})}.$$

Because $0 < \alpha\mu < 1$, $(1 - \alpha\mu L^{-1})^{-1}$ may be expanded as

$$\frac{1}{(1 - \alpha\mu L^{-1})} = 1 + \alpha\mu L^{-1} + (\alpha\mu)^2 L^{-2} + (\alpha\mu)^3 L^{-3} + ...$$

So, at time $s = 0$, the equation becomes

$$y_t - \mu y_{t-1} = \frac{\mu}{\lambda}\sum_{i=0}^{\infty}(\alpha\mu)^i E_t(y^*_{t+i}). \tag{3.5}$$

Because μ satisfies the quadratic (3.4), then

$$\lambda\alpha\mu^2 - (1 + \alpha\lambda + \lambda)\mu + \lambda = 0 \tag{3.6}$$

$$\Rightarrow \frac{\mu}{\lambda} = (1 - \mu)(1 - \alpha\mu).$$

Substituting this into (3.5) yields

$$y_t = \mu y_{t-1} + (1 - \mu)(1 - \alpha\mu)\sum_{i=0}^{\infty}(\alpha\mu)^i E_t(y^*_{t+i}). \tag{3.7}$$

This, along with the equation determining y_t^*, namely

$$y_t^* = \beta_0 + \beta_1 x_t + u_t, \tag{3.8}$$

is the solution to the agent's problem. Note that (3.7) is a partial adjustment model with $(1-\mu)$ as the speed of adjustment and $(1-\alpha\mu)\sum(\alpha\mu)^i E_t(y_{t+i}^*)$ as the target. Note that $(1-\alpha\mu)\sum_{i=0}^{\infty}(\alpha\mu)^i = 1$, so the target is a distributed lead on y^* with weights summing to 1. The higher is μ the lower the speed of adjustment and the more important the future. Furthermore, we can show that $\frac{\partial\mu}{\partial\lambda} > 0$. So the bigger are the costs of adjustment, the lower is the speed of adjustment.

In order to make the models (3.7), (3.8) operational, there are two basic methods, (A) and (B).

(A) Consider (3.7) written as

$$y_t = \mu y_{t-1} + (1-\mu)(1-\alpha\mu)\left[y_t^* + \alpha\mu E_t y_{t+1}^* + (\alpha\mu)^2 E_t y_{t+2}^* + ...\right].$$

Go one step forward

$$y_{t+1} = \mu y_t + (1-\mu)(1-\alpha\mu)\left[y_{t+1}^* + \alpha\mu E_{t+1} y_{t+2}^* + (\alpha\mu)^2 E_{t+1} y_{t+3}^* + ...\right].$$

Take expectations with respect to "t" dated information and multiply by $\alpha\mu$,

$$\alpha\mu E_t y_{t+1} = \alpha\mu^2 y_t + (1-\mu)(1-\alpha\mu)\left[\alpha\mu E_t y_{t+1}^* + (\alpha\mu)^2 E_t y_{t+2}^* + (\alpha\mu)^3 E_t y_{t+3}^* + ...\right].$$

Note that $E_t\left(E_{t+1}(y_{t+s}^*)\right) = E_t(y_{t+s}^*)$ by the law of iterated expectations (see Blanchard/Fisher p.218 or Romer p.263). Subtract this from (3.7) to obtain

$$y_t - \alpha\mu E_t y_{t+1} = \mu y_{t-1} - \alpha\mu^2 y_t + (1-\mu)(1-\alpha\mu) y_t^*$$

or

$$y_t = \frac{\alpha\mu}{(1+\alpha\mu^2)} E_t y_{t+1} + \frac{\mu}{(1+\alpha\mu^2)} y_{t-1} + \frac{(1-\mu)(1-\alpha\mu)}{(1+\alpha\mu^2)} y_t^*.$$

Defining $\nu_t = (E_t y_{t+1} - y_{t+1})$ and using (3.8), we have

$$\begin{aligned} y_t &= \frac{\alpha\mu}{(1+\alpha\mu^2)} y_{t+1} + \frac{\mu}{(1+\alpha\mu^2)} y_{t-1} + \frac{(1-\mu)(1-\alpha\mu)}{(1+\alpha\mu^2)} \beta_0 + \frac{(1-\mu)(1-\alpha\mu)}{(1+\alpha\mu^2)} \beta_1 x_t \\ &\quad + \underbrace{\left[\frac{(1-\mu)(1-\alpha\mu)}{(1+\alpha\mu^2)} u_t + \frac{\alpha\mu}{(1+\alpha\mu^2)} \nu_t \right]}_{\eta_t}. \end{aligned}$$

So we may estimate the model

$$y_t = \gamma_0 + \gamma_1 y_{t+1} + \gamma_2 y_{t-1} + \gamma_3 x_t + \eta_t$$

noting that y_{t+1} is correlated with η_t because η_t contains the term $\nu_t = (E_t y_{t+1} - y_{t+1})$. So we need *instruments* for y_{t+1} and $y_{t-1}, y_{t-2}, x_{t-1}, x_{t-2}$ are commonly used. Note that from our estimates $\hat{\gamma}_0, \hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3$, we can compute estimates of $\alpha, \mu, \beta_0, \beta_1$, using the equations

$$\frac{(1-\mu)(1-\alpha\mu)}{(1+\alpha\mu^2)}\beta_0 = \hat{\gamma}_0, \quad \frac{\alpha\mu}{(1+\alpha\mu^2)} = \hat{\gamma}_1, \quad \frac{\mu}{(1+\alpha\mu^2)} = \hat{\gamma}_2, \quad \frac{(1-\mu)(1-\alpha\mu)\beta_1}{(1+\alpha\mu^2)} = \hat{\gamma}_3.$$

(B) In this method, we specify the stochastic processes driving x_t and u_t. Suppose, for example, that u_t is iid $(0, \sigma_u^2)$ and that x_t satisfies

$$x_t = \rho x_{t-1} + \varepsilon_t, \qquad \varepsilon_t \text{ iid } (0, \sigma_\varepsilon^2). \tag{3.9}$$

Then from (3.8)

$$y^*_{t+i} = \beta_0 + \beta_1 x_{t+i} + u_{t+i}.$$

So $E_t y^*_{t+i} = \beta_0 + \beta_1 E_t x_{t+i}$.

From (3.9),

$$x_{t+i} = \rho x_{t+i-1} + \varepsilon_{t+i}.$$

Thus

$$\begin{aligned} E_t x_{t+i} &= \rho E_t x_{t+i-1} \text{ since } \varepsilon_t \text{ is iid} \\ &= \rho E_t (x_{t+i-2} + \varepsilon_{t+i-1}) = \rho^2 E_t x_{t+i-2} \text{ since } \varepsilon_t \text{ is iid} \\ &= \rho^i x_t \text{ by backward recursion.} \end{aligned}$$

So

$$\begin{aligned} E_t y^*_{t+i} &= \beta_0 + \beta_1 \rho^i x_t \qquad \text{(all } i > 0) \\ y^*_t &= \beta_1 + \beta_1 x_t + u_t. \end{aligned}$$

So

$$\begin{aligned} & \sum (\alpha\mu)^i E_t y^*_{t+i} \\ = & \sum (\alpha\mu)^i \left(\beta_0 + \beta_1 \rho^i x_t\right) + u_t \\ = & \frac{\beta_0}{1-\alpha\mu} + \frac{\beta_1 x_t}{1-\alpha\rho\mu} + u_t. \end{aligned}$$

Substituting into (3.7) yields

$$y_t = (1-\mu)\beta_0 + \mu y_{t-1} + \frac{\beta_1(1-\mu)(1-\alpha\mu)}{(1-\alpha\rho\mu)} x_t + (1-\mu)(1-\alpha\mu)u_t \tag{3.10}$$

and this, along with (3.9) gives the operational model.

The so-called deep parameters of the model are α, μ, β_0, β_1, ρ. If we have outside estimates of the parameter α, then estimating (3.9), (3.10) enables us to deduce the other deep parameters. Equation (3.10) is a simple stochastic difference equation model. Notice that the coefficients of (3.10)) depend on the parameter of the x process, ρ. So, if the parameters of the x process shift, then the coefficients of (3.10) will shift even though the "deep" parameters of the agents' model $(\alpha, \mu, \beta_0, \beta_1)$ are unchanged. This is the foundation of the "Lucas Critique".