EC402: MEI. Spring Term: The Analysis of Time Series*

Problem Set 1

1. Regression with $MA(1)$ errors

Consider a regression model with $MA(1)$ distrurbance term

$$\begin{aligned} y_t &= x_t'\beta + u_t \\ u_t &= \varepsilon_t + \theta\varepsilon_{t-1}, \quad t = 1, ..., T \end{aligned} \tag{1.1}$$

where $\varepsilon_t \sim iid\left(0, \sigma^2\right)$ with $\varepsilon_0 = 0$ and x_t non-stochastic.

1. Derive an expression for the covariance matrix, $\sigma^2\Omega$, of the vector of disturbances $u = (u_1, ..., u_T)'$ in terms of θ.

   Answer. *Since*

$$\begin{aligned} E(u_t) &= 0,\, var(u_t) = E(u_t^2) = \sigma^2(1+\theta^2) \qquad t \geq 2 \\ E(u_t u_{t-1}) &= \theta\sigma^2,\, E(u_t u_{t-s}) = 0 \qquad s > 1 \end{aligned}$$

   we have that

$$\sigma^2\Omega = \sigma^2 \begin{bmatrix} 1 & \theta & 0 & \cdots & 0 \\ \theta & 1+\theta^2 & \ddots & & \vdots \\ 0 & \theta & \ddots & \ddots & 0 \\ \vdots & \ddots & & \ddots & \theta \\ 0 & \cdots & 0 & \theta & 1+\theta^2 \end{bmatrix}$$

2. By using $\varepsilon_t = u_t - \theta\varepsilon_{t-1}$ recursively, starting from $\varepsilon_0 = 0$, $\varepsilon_1 = u_1$, find the lower triangular matrix L such that $\varepsilon = Lu$ where $\varepsilon = (\varepsilon_1, ..., \varepsilon_T)'$.

   Answer. *Since* $\varepsilon_t = u_t - \theta\varepsilon_{t-1}$ *and* $\varepsilon_0 = 0$,*we have that*

$$
\begin{aligned}
\varepsilon_1 &= u_1 \\
\varepsilon_2 &= u_2 - \theta u_1 \\
\varepsilon_3 &= u_3 - \theta u_2 + \theta^2 u_1 \\
&\dots
\end{aligned}
$$

   Hence

$$
\begin{aligned}
\varepsilon &= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -\theta & 1 & 0 & \cdots & 0 \\ \theta^2 & -\theta & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ (-\theta)^{T-1} & (-\theta)^{T-2} & \cdots & -\theta & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ \vdots \\ \vdots \\ u_T \end{bmatrix} \\
\varepsilon &= Lu.
\end{aligned}
$$

3. Assume θ is known. Noting that $Var\,(Lu) = \sigma^2 I$, determine a method for computing the best linear unbiased estimator of β which does not require the construction and inversion of Ω.

   Answer. *To estimate the model with known* θ, *rewrite (1.1) in matrix form and multiply through by* L *obtaining*

$$
\begin{aligned}
Ly &= LX\beta + Lu \\
&= LX\beta + \varepsilon \qquad \varepsilon \sim (0, \sigma^2 I).
\end{aligned}
$$

   Hence OLS on this transformed model is BLUE.

2. $F - GLS$ estimation of the regression model with $AR\,(1)$ errors

Consider the model

$$
\begin{aligned}
y_t &= x_t'\beta + u_t \qquad (2.1) \\
u_t &= \phi u_{t-1} + \varepsilon_t \quad t = 1, ..., T
\end{aligned}
$$

where $|\phi| < 1$, $\varepsilon_t \sim iid\,(0, \sigma^2)$, x_t' and ε_t are process independent and

$$p\lim \frac{1}{T}\sum_{t=1}^{T} x_t x_t' = \Sigma_{xx};\; p\lim \frac{1}{T}\sum_{t=1}^{T} x_t x_{t-1}' = \Sigma_{xx_-};$$

and Σ_{xx} is non-singular.

1. You estimate (2.1) by OLS and obtain the residual $\hat{u}_t$. You estimate ϕ by

$$\hat{\phi} = \frac{\sum_{t=2}^{T} \hat{u}_t \hat{u}_{t-1}}{\sum_{t=2}^{T} \hat{u}_{t-1}^2}.$$

Assuming that $\hat{\beta}_{OLS}$ is consistent, show that $\hat{\phi}$ is a consistent estimator of ϕ.

Answer. $\widehat{u}_t$ *are the OLS residuals so:*

$$\begin{aligned} \widehat{u}_t &= y_t - \widehat{y}_t \\ &= x_t'\beta + u_t - x_t'\widehat{\beta} \\ &= -x_t'(\widehat{\beta} - \beta) + u_t \end{aligned}$$

$$\begin{aligned} \therefore\quad \frac{1}{T}\sum \widehat{u}_t\widehat{u}_{t-1} &= \frac{1}{T}\sum u_t u_{t-1} + (\widehat{\beta}-\beta)'\frac{1}{T}\sum x_t x_{t-1}'(\widehat{\beta}-\beta) \\ &-(\widehat{\beta}-\beta)'\frac{1}{T}\sum x_t u_{t-1} - (\widehat{\beta}-\beta)'\frac{1}{T}\sum x_{t-1}u_t. \end{aligned}$$

Since: i) $\widehat{\beta}$ *is consistent so* $p\lim(\widehat{\beta}-\beta) = 0$, *ii)* $p\lim \frac{1}{T}\sum x_{t-1}u_t = 0$, $p\lim \frac{1}{T}\sum x_t u_{t-1} = 0$ *(since x is process independent) and iii)* $p\lim \frac{1}{T}\sum_{t=1}^{T} x_t x_{t-1}' = \Sigma_{xx_-}$, *we have that*

$$p\lim \frac{1}{T}\sum \widehat{u}_t\widehat{u}_{t-1} = p\lim \frac{1}{T}\sum u_t u_{t-1}.$$

Similarly $p\lim \frac{1}{T}\sum \widehat{u}_{t-1}^2 = p\lim \frac{1}{T}\sum u_{t-1}^2$. *But u_t is an $AR(1)$ hence* $p\lim \frac{1}{T}\sum u_t^2 = \frac{\sigma^2}{(1-\phi^2)}$ *and* $p\lim \frac{1}{T}\sum u_t u_{t-1} = \frac{\phi\sigma^2}{(1-\phi^2)}$. *Hence*

$$\begin{aligned} p\lim \widehat{\phi} &= \frac{p\lim \frac{1}{T}\sum \widehat{u}_t\widehat{u}_{t-1}}{p\lim \frac{1}{T}\sum \widehat{u}_{t-1}^2} \\ &= \frac{\phi\sigma^2/(1-\phi^2)}{\sigma^2/(1-\phi^2)} = \phi. \end{aligned}$$

3. Maximum Likelihood of the ARMA(1,1)

Consider the $ARMA\,(1,1)$ process

$$y_t = \phi y_{t-1} + \varepsilon_t + \theta\varepsilon_{t-1} \quad t = 1, ..., T$$

where $|\phi| < 1$, $\varepsilon_t \sim iidN\,(0, \sigma^2)$

1. Assuming, $y_1 = \varepsilon_1 = 0$ write down the log likelihood.
2. Obtain the FOCs wrt ϕ and θ.
3. Obtain
$$\frac{1}{T} I\,(\psi)$$
where $\psi := [\phi, \theta, \sigma^2]'$.
4. In finite sample we can approximate the distribution of $\left(\hat{\psi} - \psi_0\right)$ as a mean zero normal with covariance depending on $I\left(\hat{\psi}\right)$. How would you estimate the covariance matrix?

Answer. *Since*

$$y_t | y_{t-1}, \varepsilon_{t-1} \sim N\left(\phi y_{t-1} + \theta\varepsilon_{t-1}, \sigma^2\right)$$

if we condition on $y_1, \varepsilon_1 = 0$

$$\begin{aligned} LogL(\phi, \theta, \sigma^2) &= -\frac{(T-1)}{2}\log 2\pi - \frac{(T-1)}{2}\log\sigma^2 \\ &\quad -\frac{1}{2\sigma^2}\sum_{t=2}^{T}\left(y_t - \phi y_{t-1} - \theta\varepsilon_{t-1}\right)^2 \end{aligned}$$

This is nonlinear least squares where

$$\begin{aligned} \varepsilon_t &= y_t - \phi y_{t-1} - \theta\varepsilon_{t-1} \qquad t = 2, ..., T, \quad \varepsilon_1 = 0, \quad y_1 = 0 \\ z_t &= \begin{bmatrix} -\frac{\partial\varepsilon_t}{\partial\phi} \\ -\frac{\partial\varepsilon_t}{\partial\theta} \end{bmatrix} = \begin{bmatrix} y_{t-1} + \theta\frac{\partial\varepsilon_{t-1}}{\partial\phi} \\ \varepsilon_{t-1} + \theta\frac{\partial\varepsilon_{t-1}}{\partial\theta} \end{bmatrix}, \quad \frac{\partial\varepsilon_1}{\partial\phi} = \frac{\partial\varepsilon_1}{\partial\theta} = 0. \end{aligned}$$

Then

$$\begin{aligned}
\begin{bmatrix} \frac{\partial \log L}{\partial \phi} \\ \frac{\partial \log L}{\partial \theta} \end{bmatrix} &= \begin{bmatrix} \frac{1}{\sigma^2}\sum\left(y_{t-1}+\theta\frac{\partial \varepsilon_{t-1}}{\partial \phi}\right)\varepsilon_t \\ \frac{1}{\sigma^2}\sum\left(\varepsilon_{t-1}+\theta\frac{\partial \varepsilon_{t-1}}{\partial \theta}\right)\varepsilon_t \end{bmatrix} = \frac{1}{\sigma^2}\sum_{t=2}^{T} z_t\varepsilon_t \\
\frac{\partial \log L}{\partial \sigma^2} &= -\frac{(T-1)}{2\sigma^2}+\frac{1}{2(\sigma^2)^2}\sum_{t=2}^{T}\varepsilon_t^2.
\end{aligned}$$

Next, compute expected second derivatives, to compute $I(\psi)$. Recall

$$\begin{aligned}
I(\psi) &= -E\begin{bmatrix} \frac{\partial^2 \log L}{\partial \phi^2} & \frac{\partial^2 \log L}{\partial \phi\partial\theta} & \frac{\partial^2 \log L}{\partial \phi\partial\sigma^2} \\ \frac{\partial^2 \log L}{\partial \theta\partial\phi} & \frac{\partial^2 \log L}{\partial \theta^2} & \frac{\partial^2 \log L}{\partial \theta\partial\sigma^2} \\ \frac{\partial^2 \log L}{\partial \sigma^2\partial\phi} & \frac{\partial^2 \log L}{\partial \sigma^2\partial\theta} & \frac{\partial^2 \log L}{\partial (\sigma^2)^2} \end{bmatrix} \\
-E\left(\frac{\partial^2 \log L}{\partial \phi^2}\right) &= \frac{1}{\sigma^2}\left(\sum E\left(y_{t-1}+\frac{\theta\partial\varepsilon_{t-1}}{\partial\phi}\right)^2-\sum E\frac{\theta\partial^2\varepsilon_{t-1}}{\partial\phi^2}.\varepsilon_t\right).
\end{aligned}$$

Since ε_t is iid, $E\frac{\theta\partial^2\varepsilon_{t-1}}{\partial\phi^2}\varepsilon_t=\theta E\frac{\partial^2\varepsilon_{t-1}}{\partial\phi^2}E(\varepsilon_t)=0$. So

$$\begin{aligned}
-E\left(\frac{\partial^2 \log L}{\partial \phi^2}\right) &= \frac{1}{\sigma^2}\sum E\left(y_{t-1}+\frac{\theta\partial\varepsilon_{t-1}}{\partial\phi}\right)^2 \\
-E\left(\frac{\partial^2 \log L}{\partial \theta\partial\phi}\right) &= \frac{1}{\sigma^2}\left(\sum E\left(y_{t-1}+\frac{\theta\partial\varepsilon_{t-1}}{\partial\phi}\right)\left(\varepsilon_{t-1}+\frac{\theta\partial\varepsilon_{t-1}}{\partial\theta}\right)\right. \\
&\quad \left.-\sum E\frac{\theta\partial^2\varepsilon_{t-1}}{\partial\theta\partial\phi}.\varepsilon_t-\sum E\frac{\partial\varepsilon_{t-1}}{\partial\phi}\varepsilon_t\right) \\
&= \frac{1}{\sigma^2}\left(\sum E\left(y_{t-1}+\frac{\theta\partial\varepsilon_{t-1}}{\partial\phi}\right)\left(\varepsilon_{t-1}+\frac{\theta\partial\varepsilon_{t-1}}{\partial\theta}\right)\right).
\end{aligned}$$

The second and the third terms are zero by the same argument as above. Similarly

$$\begin{aligned}
-E\left(\frac{\partial^2 \log L}{\partial \theta^2}\right) &= \frac{1}{\sigma^2}\sum E\left(\varepsilon_{t-1}+\frac{\theta\partial\varepsilon_{t-1}}{\partial\theta}\right)^2 \\
-E\left(\frac{\partial^2 \log L}{\partial \sigma^2\partial\theta}\right) &= -E\left(\frac{\partial^2 \log L}{\partial \sigma^2\partial\phi}\right)=0 \\
-E\left(\frac{\partial^2 \log L}{\partial (\sigma^2)^2}\right) &= -E\left[\frac{(T-1)}{2(\sigma^2)^2}-\frac{1}{(\sigma^2)^3}\sum\varepsilon_t^2\right]=\frac{(T-1)}{2(\sigma^2)^2}
\end{aligned}$$

So

$$\begin{aligned}
I(\psi) &= \begin{bmatrix} \frac{\sum E\left(y_{t-1}+\theta\frac{\partial\varepsilon_{t-1}}{\partial\phi}\right)^2}{\sigma^2} & \frac{\sum E\left(y_{t-1}+\theta\frac{\partial\varepsilon_{t-1}}{\partial\phi}\right)\left(\varepsilon_{t-1}+\theta\frac{\partial\varepsilon_{t-1}}{\partial\theta}\right)}{\sigma^2} & 0 \\ \frac{\sum E\left(\varepsilon_{t-1}+\theta\frac{\partial\varepsilon_{t-1}}{\partial\theta}\right)\left(y_{t-1}+\theta\frac{\partial\varepsilon_{t-1}}{\partial\phi}\right)}{\sigma^2} & \frac{\sum E\left(\varepsilon_{t-1}+\theta\frac{\partial\varepsilon_{t-1}}{\partial\theta}\right)^2}{\sigma^2} & 0 \\ 0 & 0 & \frac{(T-1)}{2(\sigma^2)^2} \end{bmatrix} \\
&= \frac{1}{\sigma^2}\begin{bmatrix} \sum E(z_tz_t') & 0 \\ 0 & \frac{(T-1)}{2\sigma^2} \end{bmatrix}.
\end{aligned}$$

To approximate the variance-covariance matrix, replace θ, ϕ, σ^2 by $\widehat{\theta}$, $\widehat{\phi}$, $\widehat{\sigma}^2$ and replace expectations by sample moments, i.e.

$$I(\widehat{\psi})^{-1} \simeq \begin{bmatrix} \widehat{\sigma}^2 \left(\sum z_t z_t'\right)^{-1} & 0 \\ 0 & \frac{2\widehat{\sigma}^4}{(T-1)} \end{bmatrix}.$$