EC402: MEI. Spring Term: The Analysis of Time Series*

Problem Set 3

1. Cointegration

Suppose that aggregate income, Y_t, follows a random walk with drift

$$Y_t = \mu_y + Y_{t-1} + \varepsilon_{yt}$$

and that the government always spends a fraction of the previous period output

$$G_t = \mu_g + gY_{t-1} + \varepsilon_{gt} \tag{1.1}$$

where $0 < g < 1$, ε_{yt} and ε_{gt} are mean zero serially independent *iid* variables.

Moreover, assume that the government runs a balanced budget ($T_t = G_t \ \forall t$) and that consumption follows

$$C_t = \alpha + \beta \left(Y_t - T_t \right) \tag{1.2}$$

with $\alpha, \beta > 0$

1. Are Y, C and G stationary? What are their orders of integration?

2. Are C_t and G_t cointegrated? Are Y_t and G_t cointegrated? Are C_t and Y_t cointegrated? Are C_t, Y_t and G_t cointegrated?

3. How many linearly independent cointegration vectors are there? Why?

4. What are the long-run trends of Y_t, G_t and C_t?

2. Dynamic Simultaneous Equations

Consider the following model

$$\begin{aligned} y_{1t} &= \gamma y_{2t} + \beta x_t + \varepsilon_{1t} \\ y_{2t} &= \alpha y_{1,t-1} + \varepsilon_{2t} \end{aligned}$$

where ε_{1t} and ε_{2t} are serially uncorrelated disturbances which may be contemporaneously correlated with each other.

1. Are the parameters identifiable?

2. Write down the final form and the autoregressive final form of the model.

3. Assume x_t is stationary. What is the necessary condition for the model to be stable?

3. VAR Estimation

Consider the VAR

$$y_t = c + \Phi_1 y_{t-1} + ... + \Phi_p y_{t-p} + \varepsilon_t$$

where y_t is a $n \times 1$ vector of time series, c is $n \times 1$ vector of constants, the Φ's are $n \times n$ matrixes of coefficients and ε_t is a vector of disturbances s.t. $\varepsilon_t \sim iidN(0, \Omega)$.

1. Define the $(np + 1) \times 1$ vector

$$x_t = \begin{bmatrix} 1 \\ y_{t-1} \\ \cdots \\ y_{t-p} \end{bmatrix}$$

and the $n \times (np+1)$ matrix

$$\Pi' = [c,\ \Phi_1, ..., \Phi_p].$$

Write down the sample log likelihood of this model.

2. Show that the term $\sum_{t=1}^{T} (y_t - \Pi' x_t)' \Omega^{-1} (y_t - \Pi' x_t)$ in the log likelihood you derived above can be rewritten as

$$\sum_{t=1}^{T} \left\{ \left[\hat{\varepsilon}_t + \left(\hat{\Pi} - \Pi \right)' x_t \right]' \Omega^{-1} \left[\hat{\varepsilon}_t + \left(\hat{\Pi} - \Pi \right) x_t \right] \right\} \tag{3.1}$$

where $\hat{\Pi}$ is the OLS equation-by-equation estimate of Π and $\hat{\varepsilon}_t$ is the vector of OLS residuals.

3. Show that (3.1) can be further simplified as

$$\sum_{t=1}^{T} \hat{\varepsilon}_t' \Omega^{-1} \hat{\varepsilon}_t + \sum_{t=1}^{T} \left[x_t' \left(\hat{\Pi} - \Pi \right) \Omega^{-1} \left(\hat{\Pi} - \Pi \right)' x_t \right] \tag{3.2}$$

[Hint: use the fact that (3.1) is a scalar and recall that the OLS residuals are, by construction, orthogonal to the regressors]

4. Use this last result to show that the OLS estimator of Π ($\hat{\Pi}$) is the MLE.