EC402: More on Simultaneous Equations Models

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Outline

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1. Estimation of Simultaneous Equations Models
2. Dynamic Simultaneous Equation Models
3. Example: a simple income-expenditure model

Estimation of Simultaneous Equations Models

$$\text{Structural Form} \quad : \qquad \Gamma y_t = Bx_t + \varepsilon_t \qquad (1)$$

$$\text{Reduced Form:} \qquad y_t = \Gamma^{-1}Bx_t + \Gamma^{-1}\varepsilon_t = \Pi x_t + v_t \qquad (2)$$

- Consider first equation in the structural from (1)

$$y_{1t} = Y'_{1t}\gamma_1 + X'_{1t}\beta_1 + \varepsilon_{1t} \qquad (3)$$

where:

  - Y'_{1t} = row vector of y variables included in first equation excluding y_{1t},
  - X'_{1t} = row vector of x variables included in first equation.

We can rewrite the first equation in matrix form as

$$y_1 = Y_1\gamma_1 + X_1\beta_1 + \varepsilon_1 \tag{4}$$

where

$$y_1 = \begin{bmatrix} y_{11} \\ y_{12} \\ \vdots \\ y_{1T} \end{bmatrix}, \qquad Y_1 = \begin{bmatrix} Y'_{11} \\ Y'_{12} \\ \vdots \\ Y'_{1T} \end{bmatrix} \text{ etc.}$$

$y_1 = T \times 1$, $Y_1 = T \times (N_1 - 1)$, $X_1 = T \times K_1$, where N_1 is number of endogenous variables in equation (3), K_1 is number of exogenous variables in equation (3).

Note:

- $N - N_1$ = number of excluded y variables,
- $K - K_1$ = number of excluded x variables.

Recall from (2), that each y is correlated with all errors. So, to estimate (4), we need to use IV (instrumental variables).

- We have $(N_1 - 1)$ right hand side endogenous variables correlated with the error.
- And we have $(K - K_1)$ spare (excluded) X variables to use as instruments.
- But we need at least as many instruments as endogenous variables, i.e. we need

$$(K - K_1) \geq (N_1 - 1) \Rightarrow (N - N_1) + (K - K_1) \geq N - 1$$

But the LHS is the number of zeros in first equation – so this is the *order condition* for identification.

So if the equation is identified – i.e. this order condition is satisfied – we can do IV

- Let $X_1^T = T \times (K - K_1)$ be the matrix of excluded exogenous variables.
- To proceed with estimation, form the matrices

$$Z_1 = [Y_1, X_1], \qquad X = [X_1, X_1^T], \qquad \delta_1 = \begin{pmatrix} \gamma_1 \\ \beta_1 \end{pmatrix}.$$

- So the first equation (4) can be written

$$y_1 = Z_1 \delta_1 + \varepsilon_1 \tag{5}$$

- Using X as the set of instruments, the IV estimator is

$$\hat{\delta}_1 = \left(Z_1' X (X'X)^{-1} X' Z_1 \right)^{-1} Z_1' X (X'X)^{-1} X' y_1 \tag{6}$$

$$\begin{aligned} var(\hat{\delta}_1) &= s_{iv}^2 \left((Z_1' X)(X'X)^{-1}(X' Z_1) \right)^{-1} \\ s_{iv}^2 &= \frac{1}{(T - N_1 - K_1)} \sum_t \left(y_t - Z_{1t}' \hat{\delta}_1 \right)^2 . \end{aligned}$$

Note: if $Z_1'X$ is square (i.e. $N_1 + K_1 - 1 = K$, so just identified), then $\hat{\delta}_1$ can be simplified, i.e.

$$\begin{aligned}\hat{\delta}_1 &= \left(Z_1'X(X'X)^{-1}X'Z_1\right)^{-1} Z_1'X(X'X)^{-1}X'y_1 \\ &= (X'Z_1)^{-1}(X'X)(Z_1'X)^{-1}(Z_1'X)(X'X)^{-1}X'y_1 \\ &= (X'Z_1)^{-1}X'y_1. \qquad (7)\end{aligned}$$

- In the case of over-identification, the IV estimator (6) is known as *Two Stage Least Squares* (2SLS).
- In the just-identified case, (7) is known as *Indirect Least Squares*.

To see why the IV estimator (6) is a Two Stage Least Squares note that:

- regressing Z_1 on X we obtain the least square coefficients $(X'X)^{-1} X'Z_1$.
- we can therefore construct the "fitted" values $\hat{Z}_1 := X(X'X)^{-1} X'Z_1$.
- Regressing then y_1 on $\hat{Z}_1$ via least squares we get the coefficients

$$\begin{aligned}
\left(\hat{Z}_1'\hat{Z}_1\right)^{-1} \hat{Z}_1 y_1 &= \left[Z_1'X(X'X)^{-1} X'X(X'X)^{-1} X'Z_1\right]^{-1} X(X'X)^{-1} X'Z_1 y_1 \\
&= \left[Z_1'X(X'X)^{-1} X'Z_1\right]^{-1} X(X'X)^{-1} X'Z_1 y \\
&= : \hat{\delta}_1
\end{aligned}$$

Remarks:

(1) The above seems to imply that you can estimate an equation which passes the order condition but fails the rank condition.

But: an estimate of the variance of the limiting distribution of $\sqrt{T}(\hat{\delta}_1 - \delta_1)$ is

$$\sigma^2 \left[\frac{1}{T} Z_1' X \left(\frac{1}{T} X'X \right)^{-1} \frac{1}{T} X' Z_1 \right]^{-1}$$

- If an equation fails the rank condition, then in the limit $\frac{1}{T} Z_1' X$ tends to a matrix with rank less than the number of columns of Z_1. This in turn makes the matrix in square brackets singular and noninvertible. In finite samples you get very large estimated standard errors; the estimates are very poorly defined.

(1) The IV estimator is consistent but not efficient.

- There are more complex techniques which are asymptotically efficient, in particular Full Information Maximum Likelihood (FIML) and Three Stage Least Squares (3SLS).
- The underlying rationale of ML is exactly as you would expect. 3SLS is basically a feasible GLS technique. It uses 2SLS for the initial estimates, estimates Ω from the residuals and then does a final iteration of GLS based on the estimated Ω.

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Dynamic Simultaneous Equation Models

- The standard notation for a simultaneous system can be extended to simultaneous stochastic difference models,

$$A_0 y_t = A_1 y_{t-1} + \ldots + A_p y_{t-p} + B_0 x_t + \ldots + B_s x_{t-s} + \varepsilon_t. \quad (8)$$

Note: lagged y variables are contemporaneously independent so long as ε_t is iid.

- If the ε_t are serially independent the standard criteria for identification are valid.
- The reduced form is

$$y_t = A_0^{-1} A_1 y_{t-1} + \ldots + A_0^{-1} A_p y_{t-p} + A_0^{-1} B_0 x_t + \ldots + A_0^{-1} B_s x_{t-s} + A_0^{-1} \varepsilon_t. \quad (9)$$

Note: the lagged y variables are treated as exogenous variables in the previous model.

- Equation (8) can be re-expressed using lag polynomials,

$$A(L) y_t = B(L) x_t + \varepsilon_t. \quad (10)$$

where $A(L)$ and $B(L)$ are matrices of polynomials in L

Some linear algebra definitions:

- Given a $n \times n$ matrix A, consider the A_{ij} matrix obtained by deleting row i and column j. The $(i,j) - th$ minor of a A is defined as
$$M_{ij} = \det A_{ij}.$$
- The $(i,j) - th$ cofactor of a matrix A is defined as
$$C_{ij} = (-1)^{i+j} M_{ij}$$
where M_{ij} is the $(i,j) - th$ minor.
- Given a $n \times n$ matrix A, the adjoint matrix of A $(adj\,(A))$ is the $n \times n$ matrix whose $(i,j) - th$ entry is the $(j,i) - th$ cofactor of A.

Theorem

For any $n \times n$ matrix A, we have that
$$A \times adj\,(A) = \det A \times I$$

- This last result implies that, if $A(L)$ in invertible

$$A^{-1}(L) = \frac{A^*(L)}{|A(L)|} \tag{11}$$

where $A^*(L)$ is the adjoint of A

- For $A(L)$ to be invertible, $|A(L)| \neq 0$.
- If $A(L)$ is invertible, (8) can be written as

$$y_t = A^{-1}(L)B(L)x_t + A^{-1}(L)\varepsilon_t. \tag{12}$$

- This is the Final Form and it expresses y_t as a function of current and lagged exogenous variables and errors only (these lags are usually infinite).

- (12) can also be written as

$$|A(L)|\, y_t = A^*(L)B(L)x_t + A^*(L)\varepsilon_t. \qquad (13)$$

This is the Autoregressive Final Form.

Note:

- $|A(L)|$ is just a scalar polynomial of order $\underline{p \times N}$.
- Each equation in (13) is a s.d.e. in lagged values of the equation dependent variable, current and lagged values of the exogenous variables, and the vector of errors.
- In all N equations the coefficients on the lagged dependent variables are the same $\Rightarrow$ *common pattern of dynamic behavior for the endogenous variables in the system.*
- As usual the condition for stability is that $|A(L)|$ should have roots with absolute values less than one.

- In view of the importance of stability, one of the first things that you want to know is the autoregressive final form and hence the roots of $|A(L)|$.
- How do we check this? Do the formal matrix algebra or, in small systems, solve by hand.

Outline

- Consider the simple income-expenditure model

$$\begin{aligned} c_t &= \beta_{11} + \gamma_{13} y_t + \gamma_{14} y_{t-1} + \varepsilon_{1t} & (14) \\ i_t &= \beta_{21} + \beta_{22} i_{t-1} + \varepsilon_{2t} & (15) \\ y_t &= c_t + i_t. & (16) \end{aligned}$$

Note: order and rank conditions are satisfied (check it!)

- Equation (15) is already in reduced form. To derive the rest of the reduced form:
  - Substitute (14) and (15) into (16)

$$y_t = \frac{\beta_{11} + \beta_{21}}{1 - \gamma_{13}} + \frac{\gamma_{14}}{1 - \gamma_{13}} y_{t-1} + \frac{\beta_{22}}{1 - \gamma_{13}} i_{t-1} + \frac{\varepsilon_{1t} + \varepsilon_{2t}}{1 - \gamma_{13}} \quad (17)$$

  - From the identity (16),and (15) and (17) we have

$$c_t = \frac{\beta_{11} + \beta_{21}\gamma_{13}}{1 - \gamma_{13}} + \frac{\gamma_{14}}{1 - \gamma_{13}} y_{t-1} + \frac{\beta_{22}\gamma_{13}}{1 - \gamma_{13}} i_{t-1} + \frac{\varepsilon_{1t} + \varepsilon_{2t}\gamma_{13}}{1 - \gamma_{13}}. \quad (18)$$

Note:

- To estimate (14), we can use i_{t-1} as instrument for y_t (which is correlated with ε_{1t}).
- (15) is already a reduced form, so it may be estimated by OLS.

- To assess the stability of the model, generate a condensed model consisting of only y and i equations, eliminating c_t (and, in general, all its lags, although there are no lags of c in this model).
- Substituting (14) into (16), we have a condensed model,

$$\begin{aligned} y_t &= \frac{\beta_{11}}{(1-\gamma_{13})} + \frac{1}{(1-\gamma_{13})} i_t + \frac{\gamma_{14}}{(1-\gamma_{13})} y_{t-1} + \frac{\varepsilon_{1t}}{(1-\gamma_{13})} \quad (19) \\ i_t &= \beta_{21} + \beta_{22} i_{t-1} + \varepsilon_{2t}. \end{aligned}$$

- Writing this condensed model in matrix form gives

$$\underbrace{\begin{bmatrix} 1 - \frac{\gamma_{14} L}{(1-\gamma_{13})} & \frac{-1}{1-\gamma_{13}} \\ 0 & (1-\beta_{22} L) \end{bmatrix}}_{A(L)} \begin{bmatrix} y_t \\ i_t \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\beta_{11}}{1-\gamma_{13}} \\ \beta_{21} \end{bmatrix}}_{B(L)} + \begin{bmatrix} \frac{\varepsilon_{1t}}{1-\gamma_{13}} \\ \varepsilon_{2t} \end{bmatrix}$$

Note: that the x variables include only the constant terms in this case.

We can then easily see that

$|A(L)|$:

$$|A(L)| = 1 - \left[\beta_{22} + \frac{\gamma_{14}}{1-\gamma_{13}}\right] L + \frac{\beta_{22}\gamma_{14}}{(1-\gamma_{13})} L^2.$$

$Adj(A)$:

$$A^*(L) = \begin{bmatrix} (1-\beta_{22}L) & \frac{1}{1-\gamma_{13}} \\ 0 & 1 - \frac{\gamma_{14}L}{(1-\gamma_{13})} \end{bmatrix}.$$

And we can then easily write the Autoregressive Final Form.

- The model is stable if the roots of $A(L)$ are less than 1 in absolute value.
- From $|A(L)|$, we see that the roots satisfy the quadratic equation

$$\lambda^2 - \left(\beta_{22} + \frac{\gamma_{14}}{1 - \gamma_{13}}\right)\lambda + \frac{\beta_{22}\gamma_{14}}{(1 - \gamma_{13})} = 0.$$

- The roots are $\frac{\gamma_{14}}{1-\gamma_{13}}$ are β_{22}. So the model is stable iff $\left|\frac{\gamma_{14}}{1-\gamma_{13}}\right| < 1$, $|\beta_{22}| < 1$.