EC402: Cointegration and Error Correction

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- We'll formally define the concepts of Integrated process and Cointegrated processes.
- We'll introduce the error correction representation of a dynamic model and link it to the concept of cointegration.

Outline

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Integrated Processes

- If a process x_t is stationary, then the process
$$y_t = x_t + y_{t-1} = \sum_{s=0}^{\infty} x_{t-s}$$
is called *integrated of order one*, $I(1)$.
- y_t has the obvious property by construction that its first difference is x_t and is, hence, stationary, $I(0)$
- If y_t is $I(1)$ and
$$z_t = y_t + z_{t-1},$$
then z_t is said to be integrated of order two, $I(2)$, and the second difference of z_t is $I(0)$.
- More generally, if y_t is integrated of order p, $I(p)$, and
$$z_t = y_t + z_{t-1},$$
then z_t is said to be integrated of order $p+1$, $I(p+1)$.

Note: the difference of a stationary process is stationary.

Cointegrated Process

- Suppose the processes x_t and y_t are both $I(1)$ (that is, non-stationary with a unit root).
- If it exist a *stationary* linear combination of these processes, the processes are x and y are said to be cointegrated.
- More generally, an $(n \times 1)$ vector time series $Y_t = [y_{1\,t}, ..., y_{2\,t}]'$ is said to be cointegrated if:
  1. each of the series $y_{1\,t}, ..., y_{2\,t}$ is individually $I(1)$
  2. some linear combination of the series $a' Y_t$ is stationary ($I(0)$) for some non-zero $(n \times 1)$ vector a.

Example: Consider the following bivariate system

$$\begin{aligned} y_{1t} &= \gamma y_{2t} + u_{1t} \\ y_{2t} &= y_{2,t-1} + u_{2t} \end{aligned}$$

with u_1 and u_2 uncorrelated with noise.

- Clearly, both processes are are $I(1)$.

But: the linear combination $(y_1 - \gamma y_{2t})$ is stationary.

- Hence $Y_t = (y_{1t}, y_{2t})'$ is cointegrated with cointegrating vector $a' = (1, -\gamma)$.

Long-run interpretation

Cointegration means that although many developments can cause permanent changes in the individual elements of Y_t, there is some long-run equilibrium relation tying the individual components together, and this is represented by the linear combination $a'Y_t$.

Example: The Consumption-Wealth Ratio

- Suppose the optimal consumption choice an agent is always to consume a share α_t of her current wealth W

$$C_t = \alpha_t W_t,$$

where $\log \alpha_t \sim iid\left(\bar{\alpha}, \sigma_\alpha^2\right)$.

- Assume also that log wealth follows a random walk process with drift

$$\log W_t = \mu + \log W_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim iid\left(0, \sigma_\varepsilon^2\right)$$

Note: neither wealth nor consumption are stationary (they both grow exponentially)

But:

1. the log of each of these variables is $I(1)$
2. $\log C_t$ and $\log W_t$ are cointegrated with cointegration vector $a\prime = (1, -1)$ since

$$\log C_t - \log W_t = \log \alpha_t$$

Note: on average, C will grow at the same rate μ as X since

$$\log C_t = \log \alpha_t + \log W_t$$

$$\begin{aligned}
\rightarrow \quad & \log C_t - \log C_{t-1} = \log \alpha_t - \log \alpha_{t-1} + \log W_t - \log W_{t-1} \\
\rightarrow \quad & E\left[\log C_t - \log C_{t-1}\right] = E\left[\log W_t - \log W_{t-1}\right] = \mu \\
\rightarrow \quad & E\left[\log C_{t+T} - \log C_t\right] = E\left[\log W_{t+T} - \log W_t\right] = \mu \times T
\end{aligned}$$

- That is, C and W share the same long-run trend, and the expected long-run log consumption-wealth ratio is constant

$$E\left[\log \frac{C_t}{W_t}\right] = \bar{\alpha}.$$

- This means that if we observe $\log C_t/W_t > \bar{\alpha}$ ($< \bar{\alpha}$) we should expect the consumption-wealth ratio to reduce (increase) in the future.

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Basic Models

- The basic model is the stochastic difference equation,

$$A(L)y_t = D(L)x_t + \varepsilon_t.$$

  x_t variables contemporaneously independent, ε_t iid $(0, \sigma^2)$.

  where L is the "lag operator" defined as $L^s z_t = z_{t-s}$, $A(L)$ is a polynomials in the lag operator, and $D(L)$ is a row vector with elements that are polynomials in the lag operator.
- In future lectures we'll generalize this setting to have multiple equations i.e. the case in which y_t and ε_t are vectors, x_t is a matrix and $A(L)$ and $D(L)$ are matrix of polynomials in the lag operator.
- What models underlie these dynamic equations?

  (The handout presents a number of examples of economic models that can be represented in this form)

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The Error Correction Form

The general dynamic model with (for simplicity) one independent variable, has the form

$$y_t = \alpha_0 + \sum_{i=1}^{m} \alpha_i L^i y_{t-1} + \sum_{j=0}^{n} \beta_j L^j x_t + u_t \qquad (u_t \text{ iid}) \tag{1}$$

or

$$A(L) y_t = \alpha_0 + B(L) x_t + u_t \tag{2}$$

where

$$\begin{aligned} A(L) &= 1 - \alpha_1 L - \alpha_2 L^2 - \ldots - \alpha_m L^m \\ B(L) &= \beta_0 + \beta_1 L + \beta_2 L^2 + \ldots + \beta_n L^n. \end{aligned}$$

- In this model, the short-run multiplier is $\frac{\partial y_t}{\partial x_t} = \beta_0$.
- The long-run relationship between x and y is

$$A(1)y = \alpha_0 + B(1)x$$

$$\Rightarrow y = \frac{\alpha_0}{A(1)} + \frac{B(1)}{A(1)}x. \tag{3}$$

- The long-run multiplier is $\frac{\partial y}{\partial x} = \frac{B(1)}{A(1)}$ (this measures the impact of a permanent change in x on y)

- We can rewrite the model (1) to isolate the long-run relationship.To do this, note that (proof in lecture notes)

$$A(L)y_t = A(1)y_{t-1} + \Delta y_t + \alpha_1^* \Delta y_{t-1} + \ldots + \alpha_{m-1}^* \Delta y_{t-m+1}$$

$$\alpha_0^* = 1, \qquad \alpha_j^* = \left(\alpha_{j+1} + \alpha_{j+2} + \ldots + \alpha_m\right).$$

- Similarly

$$\begin{aligned} B(L)x_t &= B(1)x_{t-1} + \beta_0^* \Delta x_t - \beta_1^* \Delta x_{t-1} - \ldots - \beta_{n-1}^* \Delta x_{t-n+1} \\ \beta_0^* &= \beta_0, \qquad \beta_j^* = \left(\beta_{j+1} + \beta_{j+2} + \ldots + \beta_n\right). \end{aligned}$$

- So our model becomes

$$\begin{aligned} \Delta y_t &= \alpha_0 - A(1)y_{t-1} + B(1)x_{t-1} \\ &\quad - \sum_{i=1}^{m+1} \alpha_i^* L^i \Delta y_t + \sum_{j=0}^{n+1} \beta_j^* L^j \Delta x_t + u_t. \end{aligned} \tag{4}$$

Note:

1. (1), (4) are exactly the same equation.
2. the long-run multiplier $(B(1)/A(1))$ can be read off immediately in (4)
   Let y, x be constant, namely $\Delta y = \Delta x = 0$ all time periods (you might think of this as the steady state solution), then (4) reduces to

$$0 = \alpha_0 - A(1)y + B(1)x$$

$$\rightarrow y = \frac{\alpha_0}{A(1)} + \frac{B(1)}{A(1)}x,$$

- If the equation (4) is rewritten as

$$\Delta y_t = -A(1)\left(y_{t-1} - \frac{B(1)}{A(1)}x_{t-1} - \frac{\alpha_0}{A(1)}\right) + \text{terms in } \Delta x_{t-i},\ \Delta y_{t-i},$$

this is known as the *error correction form*.

Why? $y_{t-1} - \frac{B(1)}{A(1)}x_{t-1} - \frac{\alpha_0}{A(1)}$ is the difference between y_{t-1} and the long-run equilibrium value of y corresponding to x_{t-1}, namely $\frac{B(1)}{A(1)}x_{t-1} + \frac{\alpha_0}{A(1)}$.

- So if y_{t-1} is above (below) this value, this tends to move y down (up). The difference is the so called "error" and the movement is the "correction".

Note: any stochastic difference equation can be written in error correction form unless $A(1) = 0$.

- The remaining $\Delta x, \Delta y$ terms are often known as "short-run dynamics."

Error Correction and Cointegration

The error correction form has been extensively used to model the relationships among cointegrated variables.

Example: consider the two $I(1)$ variable, y_t and x_t, that are cointegrated.

- By definition, there exist a vector $a' = (1, -\gamma)$ such that $a'[y_t, x_t]'$ is stationary.
- Therefore, if we know the cointegrating vector, we can write the following error correction representation

$$\Delta y_t = c_1 + c_1 \left(y_t - \gamma x_t\right) + \text{ terms in } \Delta x_{t-i} \text{ and } \Delta y_{t-i} \; + u_t$$

  where all the right hand side terms are stationary variables (and this allows us to invoke the standard MLE asymptotics).
- Moreover, the term $c_1(y_t - \gamma x_t)$ will be the error correction component with a straightforward economic interpretation.

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Error Correction and Cointegration in Practice

In practices, researchers tend to proceed as follows:

1. Taka each variable and investigate whether or not it has a unit root. It will often be the case that you are unable to reject a unit root.

Warning: Assuming that a variable has a unit root when it doesn't is not an innocuous mistake (many practitioners think so!).

Example: suppose the true model is

$$y_t = \beta x_t + \varepsilon_t \backsim iid\left(0, \sigma^2\right)$$

where x_t is *only* contemporaneously independent of ε_t.

- Suppose the econometrician estimates instead

$$\Delta y_t = \beta \Delta x_t + u_t.$$

- Since now $u_t = \varepsilon_t - \varepsilon_{t-1}$ this model has: *i*) serial correlation in the errors and *ii*) $E\left[\Delta x_t u_t\right] = E\left[\left(x_t - x_{t-1}\right)\left(\varepsilon_t - \varepsilon_{t-1}\right)\right] = -E\left[x_t \varepsilon_{t-1}\right] - E\left[x_{t-1}\varepsilon_t\right]$, and this does not need to be zero since x and ε are *only* contemporaneously.

2. **Testing for Cointegration.** Assuming these variables are all $I(1)$, write down the long-run relationship between the y and the $x's$ e.g.

$$y_t = \beta_0 + \sum \beta_k x_{kt} + \varepsilon_t. \tag{5}$$

- If they are cointegrated, we should be able to find estimates $\hat{\beta}_k$ so that

$$\left(y_t - \sum \hat{\beta}_k x_{kt}\right) \text{ is stationary}$$

- How to test for cointegration? A simple way is to:
  1. Do OLS on (5) to generate $\hat{\varepsilon}_t$. (we may include short-run dynamics in (5).)
  2. Then test the $\hat{\varepsilon}_t$ sequence for a unit root using the standard augmented DF procedure.

Note: you cannot use the standard DF tables (we have $\hat{\varepsilon}_t$ here and not ε_t, so there is extra sampling variation). The appropriate statistics are given in Davidson/MacKinnon text book and Johnston/DiNardo

Warning: Cointegration tests have very poor small sample properties! If we impose a cointegration relationship when it is not there, we might make very misleading long-run predictions.

Example: theoretically log housing prices ($\log P$) and log income ($\log P$) do not need to be cointegrated. Nevertheless, they *seemed* to be cointegrated in the past.

If an econometrician believes so, any time she observes P/Y above historical averages she would predict P/Y to go down in the future.

Is this be reasonable? Example: in the UK, a standard bank rules in determining the maximum loan value is the sum of 3.75 times the income of the primary borrower plus 1 time the income of the secondary borrower $\Rightarrow$ shift in household composition can cause a permanent shift in the mean of the P/Y ratio $\Rightarrow$ inference based on the assumption that $\log P$ and $\log Y$ are cointegrated might be very misleading!

But: economic theory can help us deciding which long-run restriction are reasonable and which are not.

3. If you can reject non-cointegration then estimate (by OLS) a general dynamic model in standard form

$$A(L)y_t = \alpha_0 + B(L)x_t + u_t$$

or in equivalent error correction form (to taste)

$$\begin{aligned}\Delta y_t &= \alpha_0 - A(1)y_{t-1} + B(1)x_{t-1} - \alpha_1^* \Delta y_{t-1} \\ &\quad ... - \alpha_{m-1}^* \Delta y_{t-m+1} + \beta_0^* \Delta x_t - ... - \beta_{n-1}^* \Delta x_{t-n+1} + u_t.\end{aligned}$$

These are the same (same estimated regressors and same residuals)