EC402: Simultaneous Equations

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- We'll discuss systems of regression equations
- This structure is appealing since we often think of economic variables as being interrelated
- This is why simultaneous equation models used to be very popular.
  For example, Central Banks used this a lot before:
  1. the Lucas' critique;
  2. the introduction of Vector Autoregression Models (VAR) that are more appropriate in modeling macro time series.

Note: several results of the simultaneous equation models will extend to VAR

Outline

1. Multivariate Regressions
   - Estimation
2. The General Simultaneous Equations Model
   - Identification

Outline

Multivariate Regressions

- This is characterized by a set of N reduced form regressions.
- The first building block is to extend single to multivariate regression. So y_t becomes an N vector,

$$y_t = X_t\beta + \varepsilon_t, \qquad t = 1, ..., T$$

where

$$y_t = \begin{bmatrix} y_{1t} \\ y_{2t} \\ \vdots \\ y_{Nt} \end{bmatrix}, X_t = \begin{bmatrix} x'_{1t} & 0 & \cdots & 0 \\ 0 & x'_{2t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x'_{Nt} \end{bmatrix}, \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_N \end{bmatrix}, \varepsilon_t = \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \vdots \\ \varepsilon_{Nt} \end{bmatrix}.$$

Note: each of the i equations does not need have the same regressors.

- Assume that the $i-th$ equation has K_i regressors. Hence X_t is $N \times K$ where $K = \sum_i K_i$.
- The errors are assumed correlated across the i,

$$E\left(\varepsilon_t \varepsilon_t'\right) = \Omega$$

an $N \times N$ matrix, *but uncorrelated across* t.

Note: This is a better model where we have multiple observations on each unit in a random sample (e.g. multiple demands by a household or firm) – VAR models are generally better for macro time series (in a VAR all the variables are endogenous, as normally is the case in a General Equilibrium model)

- This is known as SUR (*seemingly unrelated regressions*): equations appear to be unrelated but are in fact related through the errors.

- A possible estimator for β is least squares on each of the i equations individually,

$$\hat{\beta} = \left(\sum_{t=1}^{T} X_t' X_t \right)^{-1} \sum_{t=1}^{T} X_t' y_t.$$

- If the regressors are *process independent*, the estimates are unbiased with standard small sample properties. BUT this estimator is not in general efficient.
- An alternative if Ω is known is GLS:

$$\hat{\beta}_{gls} = \left(\sum_{t=1}^{T} X_t' \Omega^{-1} X_t \right)^{-1} \sum_{t=1}^{T} X_t' \Omega^{-1} y_t.$$

- As usual, if regressors are process independent the GLS estimator is optimal.

- In general Ω is not known and we need to fall back on feasible GLS. The approach is absolutely standard:
  1. Estimate the N equations separately using least squares
  2. Calculate the least squares residuals and estimate Ω as

$$\hat{\Omega} = \frac{1}{T} \sum_t \hat{\varepsilon}_t \hat{\varepsilon}_t'.$$

  3. Then use the GLS formula, with $\hat{\Omega}$ for Ω to obtain the F-GLS estimator.

Note under two circumstances the GLS estimator simplifies to least squares:

1. (Obviously) if Ω is known to be diagonal.
2. (Less obviously) if the regressors in each of the N equations are the same (this result will also extend to the VAR models we'll discuss later on)

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General Simultaneous Equations Model

- Suppose y_t is an $N \times 1$ vector of endogenous variables. Each of these is a linear function of the other endogenous variables and of a set of K exogenous variables, x_t.
- Usually x_t are contemporaneously independent of the errors or may be process independent.

Note: x_t may include lagged endogenous variables.

- The set of N equations is written

$$\Gamma y_t = Bx_t + \varepsilon_t \qquad t = 1, ..., T, \tag{1}$$

where Γ is $N \times N$, B is $N \times K$, ε_t is $N \times 1$, x_t is $K \times 1$, y_t is $N \times 1$.

- (1) is called the Structural Form.

Example: suppose

$$
\begin{aligned}
y_{1t} &= \gamma_{12} y_{2t} + \beta_{11} x_{1t} + \beta_{14} x_{4t} + \varepsilon_{1t} \\
y_{2t} &= \gamma_{23} y_{3t} + \beta_{22} x_{2t} + \beta_{23} x_{3t} + \varepsilon_{2t} \\
y_{3t} &= \gamma_{32} y_{2t} + \beta_{31} x_{1t} + \beta_{33} x_{3t} + \beta_{34} x_{4t} + \varepsilon_{3t}.
\end{aligned}
$$

- Then, in structural form, we have

$$
\underbrace{\begin{bmatrix} 1 & -\gamma_{12} & 0 \\ 0 & 1 & -\gamma_{23} \\ 0 & -\gamma_{32} & 1 \end{bmatrix}}_{\Gamma}
\begin{bmatrix} y_{1t} \\ y_{2t} \\ y_{3t} \end{bmatrix}
=
\underbrace{\begin{bmatrix} \beta_{11} & 0 & 0 & \beta_{14} \\ 0 & \beta_{22} & \beta_{23} & 0 \\ \beta_{31} & 0 & \beta_{33} & \beta_{34} \end{bmatrix}}_{B}
\begin{bmatrix} x_{1t} \\ x_{2t} \\ x_{3t} \\ x_{4t} \end{bmatrix}
+
\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{bmatrix}.
$$

- The solution for y_t in terms of x_t is called the Reduced Form, that is,

$$y_t = \Gamma^{-1} B x_t + \Gamma^{-1} \varepsilon_t \Rightarrow y_t = \Pi x_t + v_t \tag{2}$$

where $\Pi = \Gamma^{-1} B$, $N \times K$ and $v_t = \Gamma^{-1} \varepsilon_t$.

Note: each element of v_t is generally a function of all the elements of ε_t, so each endogenous variable depends on all the errors.

- The data may be thought of as being generated by the reduced form (2) (the SUR results apply to this form)

$\Rightarrow$ So, given the data, we can make estimates of the elements of Π and the variance matrix of v.

Identification

- If we know (estimate) the elements of Π, can we find out about the parameters Γ and B?
- This is a problem of identification and, in general, the answer is no.

Why?

- The basic equation is

$$B = \Gamma\Pi.$$

  Looking at each element in turn, we have $N \times K$ equations.
- Assuming that the diagonal elements of Γ are restricted to be equal to 1 (this is simply a normalization), there are $N \times K$ unknowns in B and $N \times (N - 1)$ unknowns in Γ.
- That is, we have many more unknowns than equations!

What are the conditions for identification?

1. Conditions apply to one equation at a time.
2. A necessary condition (order condition) is that the number of omitted x, y variables in the equation $\geq N - 1$.

Note: *a linear restriction counts as a omitted variable*.

3. A necessary and sufficient condition (rank condition) is that the matrix obtained from *all* the Γ, B coefficients in the other equations corresponding to the zero's in the equation concerned, is of rank $N - 1$ (note that this matrix has $(N - 1)$ rows).
4. If the nec./suff. conditions is passed and the number of zero's + linear restrictions $> N - 1$, the equation is over-identified; if equal to $N - 1$, it is just identified.
5. an identity (e.g. $y = c + i + g$) is always identified.

Example:

$$
\begin{aligned}
y_{1t} &= \gamma_{12} y_{2t} + \beta_{11} z_{1t} + \beta_{13} z_{3t} + \beta_{14} z_{4t} + u_{1t} \\
y_{2t} &= \gamma_{24} y_{4t} + \beta_{22} z_{2t} + \beta_{24} z_{4t} + u_{2t} \\
y_{3t} &= \gamma_{32} y_{2t} + \beta_{32} z_{2t} + \beta_{33} z_{3t} + u_{3t} \\
y_{4t} &= \gamma_{42} y_{2t} + \gamma_{43} y_{3t} + \beta_{41} z_{1t} + \beta_{42} z_{2t} + \beta_{44} z_{4t} + u_{4t}.
\end{aligned}
$$

- First, write in matrix form

$$
\underbrace{\begin{bmatrix}
1 & -\gamma_{12} & 0 & 0 & -\beta_{11} & 0 & -\beta_{13} & -\beta_{14} \\
0 & 1 & 0 & -\gamma_{24} & 0 & -\beta_{22} & 0 & -\beta_{24} \\
0 & -\gamma_{32} & 1 & 0 & 0 & -\beta_{32} & -\beta_{33} & 0 \\
0 & -\gamma_{42} & -\gamma_{43} & 1 & -\beta_{41} & -\beta_{42} & 0 & -\beta_{44}
\end{bmatrix}}_{[\Gamma, B]}
\begin{bmatrix} y_{1t} \\ y_{2t} \\ y_{3t} \\ y_{4t} \\ z_{1t} \\ z_{2t} \\ z_{3t} \\ z_{4t} \end{bmatrix}
=
\begin{bmatrix} u_{1t} \\ u_{2t} \\ u_{3t} \\ u_{4t} \end{bmatrix}
$$

Note: $N = 4$, $N - 1 = 3$.

$$\rightarrow \underbrace{\begin{bmatrix} 1 & -\gamma_{12} & 0 & 0 & -\beta_{11} & 0 & -\beta_{13} & -\beta_{14} \\ 0 & 1 & 0 & -\gamma_{24} & 0 & -\beta_{22} & 0 & -\beta_{24} \\ 0 & -\gamma_{32} & 1 & 0 & 0 & -\beta_{32} & -\beta_{33} & 0 \\ 0 & -\gamma_{42} & -\gamma_{43} & 1 & -\beta_{41} & -\beta_{42} & 0 & -\beta_{44} \end{bmatrix}}_{[\Gamma, B]}$$

Equation 1

Order: No. of zeros = 3 = $N - 1$

Rank: $$Rank \begin{bmatrix} 0 & -\gamma_{24} & -\beta_{22} \\ 1 & 0 & -\beta_{32} \\ -\gamma_{43} & 1 & -\beta_{42} \end{bmatrix} = 3$$

($\det \neq 0$)

$\Rightarrow$ So just identified.

$$\rightarrow \underbrace{\begin{bmatrix} \boxed{1} & -\gamma_{12} & \boxed{0} & 0 & \boxed{-\beta_{11}} & 0 & \boxed{-\beta_{13}} & -\beta_{14} \\ 0 & 1 & 0 & -\gamma_{24} & 0 & -\beta_{22} & 0 & -\beta_{24} \\ 0 & -\gamma_{32} & 1 & 0 & 0 & -\beta_{32} & -\beta_{33} & 0 \\ 0 & -\gamma_{42} & -\gamma_{43} & 1 & -\beta_{41} & -\beta_{42} & 0 & -\beta_{44} \end{bmatrix}}_{[\Gamma, B]}$$

Equation 2

Order: No. of zeros = $4 > N - 1$

Rank: $Rank \begin{bmatrix} 1 & 0 & -\beta_{11} & -\beta_{13} \\ 0 & 1 & 0 & -\beta_{33} \\ 0 & -\gamma_{43} & -\beta_{41} & 0 \end{bmatrix} = 3$

(det. of 1st sub-matrix $\neq 0$)

$\Rightarrow$ So over-identified.

$$\rightarrow \underbrace{\begin{bmatrix} \boxed{1} & -\gamma_{12} & 0 & 0 & -\beta_{11} & 0 & -\beta_{13} & -\beta_{14} \\ 0 & 1 & 0 & -\gamma_{24} & 0 & -\beta_{22} & 0 & -\beta_{24} \\ 0 & -\gamma_{32} & 1 & 0 & 0 & -\beta_{32} & -\beta_{33} & 0 \\ 0 & -\gamma_{42} & -\gamma_{43} & 1 & -\beta_{41} & -\beta_{42} & 0 & -\beta_{44} \end{bmatrix}}_{[\Gamma, B]}$$

Equation 3

Order: No. of zeros = $4 > N - 1$

Rank: $Rank \begin{bmatrix} 1 & 0 & -\beta_{11} & -\beta_{14} \\ 0 & -\gamma_{24} & 0 & -\beta_{24} \\ 0 & 1 & -\beta_{41} & -\beta_{44} \end{bmatrix} = 3$

(det. of 1st sub-matrix $\neq 0$)

$\Rightarrow$ So over-identified.

$$\underbrace{\begin{bmatrix} 1 & -\gamma_{12} & 0 & 0 & -\beta_{11} & 0 & -\beta_{13} & -\beta_{14} \\ 0 & 1 & 0 & -\gamma_{24} & 0 & -\beta_{22} & 0 & -\beta_{24} \\ 0 & -\gamma_{32} & 1 & 0 & 0 & -\beta_{32} & -\beta_{33} & 0 \\ 0 & -\gamma_{42} & -\gamma_{43} & 1 & -\beta_{41} & -\beta_{42} & 0 & -\beta_{44} \end{bmatrix}}_{[\Gamma, B]}$$

Equation 4

Order: No. of zeros = $2 < N - 1$

$\Rightarrow$ So not identified.

- So, if order and rank conditions are satisfied we can estimate the model in reduced form and then use the restrictions to pin down the structural parameters

But

1. From (2), we know that each y is correlated with all errors. So, to estimate the model, we need to use IV (instrumental variables).
2. Moreover, if the system if over-identified (i.e. we have more equations than unknown), how do we recover the parameters of the structural form?

$\Rightarrow$ We'll discuss estimation in the next lecture