EC402 and FM437: additional handout.

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This handout provides an outline of the proof of:

Theorem 0.1. *Under* $A1, A2, A3 \geq A3Rmi, A4, A5$*, and if* A *is* $1 \times K$ *(so that* $A\beta$ *is a scalar, linear combination of the elements of* β*)*

$$T = \frac{A\hat{\beta} - A\beta}{SE(A\hat{\beta})} \sim t(N-K)$$

where $SE(\hat{A\beta}) = \sqrt{\widehat{Var}(A\hat{\beta})} = \sqrt{s^2 A(X'X)^{-1}A'}$.

Proof. A random variable T is distributed with a Student's t-distribution with n degrees of freedom if and only if it can be written as $T = \frac{Z}{\sqrt{\frac{Q}{n}}}$, where $Z \sim \mathcal{N}(0,1)$, $Q \sim \chi^2(n)$, and Z and Q are independent.

In our case, denoting $Z = \frac{A\hat{\beta}-A\beta}{\sqrt{Var(A\hat{\beta})}} = \frac{A\hat{\beta}-A\beta}{\sqrt{\sigma_\epsilon^2 A(X'X)^{-1}A'}}$,

$$T = \frac{A\hat{\beta} - A\beta}{\sqrt{s^2 A(X'X)^{-1}A'}} = Z\sqrt{\frac{\sigma_\epsilon^2}{s^2}} = \frac{Z}{\sqrt{\frac{s^2}{\sigma_\epsilon^2}}} = \frac{Z}{\sqrt{\frac{\hat{\epsilon}'\hat{\epsilon}}{\sigma_\epsilon^2(N-K)}}} = \frac{Z}{\sqrt{\frac{Q}{N-K}}}$$

where $Q = \frac{\hat{\epsilon}'\hat{\epsilon}}{\sigma_\epsilon^2}$.

Z is standard normal, as a consequence of $\hat{\beta} - \beta | X \sim \mathcal{N}(0, \sigma_\epsilon^2(X'X)^{-1})$: so all we have left to do is show that $Q \sim \chi^2(N-K)$ and that Z and Q are independent.

$$Q = \frac{\hat{\epsilon}'\hat{\epsilon}}{\sigma_\epsilon^2} = \frac{\epsilon' M_X \epsilon}{\sigma_\epsilon^2} = \frac{\epsilon}{\sigma_\epsilon}' M_X \frac{\epsilon}{\sigma_\epsilon}$$

Which is, basically, the setup of PS2, question 7, where we showed that $Q \sim \chi^2(rank(M_X)) = \chi^2(N-K)$. See solution to that exercise.

Left to prove, then, is that Z and Q are independent. The only random thing in Z is $\hat{\beta}$; the only random thing in Q is s^2. The fact that $\hat{\beta}$ is independent from s^2 is a known property of the OLS estimator, e.g. in page 30 of your lecture notes. Hence Z and Q are also independent.

□