Unpublished Appendices for "Technological Revolutions"

Appendix D discusses what happens if learning costs are time - as opposed to consumption - costs. Appendices E, F, and G prove the results of Section 3. Appendix H presents a version of the model that incorporates a learning externality. Appendix I presents an extension of the model where there are two industries.

Appendix D: Proportional Learning Costs

The learning cost σ has been modeled as additive throughout the paper. Hence, it is best interpreted as capturing the payment of a fee, e.g. tuition. It is reasonable to suppose, however, that learning costs will also include a time component. In this case part of the learning cost is proportional to the wage, so as to capture the opportunity cost of learning. To see how the inclusion of time costs would change the results, consider the extreme case in which the learning cost is purely proportional (i.e. no fee component). The impact effect of the revolution would be identical to the impact in the additive-cost case. However, the post-revolutionary dynamics would be different. The fraction of the labor force working in technology 1 would depend on the wage *ratio* and not, as in the additive case, on the wage differential. But, as we have seen, the no-arbitrage condition dictates that the wage ratio is constant during the transition. Hence, there would be no change in L^1 after the revolutionary period, and the economy would always converge to a steady state with a two-tiered labor market. In terms of Figures 1 and 2, L^1_{t+1} becomes a flat function of K_{t+1}. Hence, the model would still explain why a technological revolution leads to an increase in inequality and a fall in the unskilled wage, but would not explain slow diffusion of the new technology. In fact, there would be no diffusion at all. Observe, however, that the "true" model is probably a combination of the purely additive and purely proportional, i.e. learning costs have both a fee and a time component. Based on the results in the paper, it is reasonable to conjecture that the additive component in this "true" model will still work to generate both revolutionary and post- revolutionary dynamics as in the purely additive case.

Appendix E: Proof of No-Hole Condition, and Derivation of (12)

Arriving at these results is conceptually straightforward but quite tedious. It helps to break down the reasoning in smaller pieces.

Step 1: *If $a, b, c \in I_t$, and $a > b > c$, then:*

$$\frac{W_t^a - W_t^b}{\overline{\sigma}_a - \overline{\sigma}_b} < \frac{W_t^a - W_t^c}{\overline{\sigma}_a - \overline{\sigma}_c} < \frac{W_t^b - W_t^c}{\overline{\sigma}_b - \overline{\sigma}_c} \tag{A.3}$$

For, assume that, on the contrary, the first inequality was reversed. This implies that the second inequality is also reversed. To see this use equation (11) to show that reversal of the first of the inequalities in (A.3) implies

$$A^{\frac{a-c}{1-\alpha}}(\overline{\sigma}_b - \overline{\sigma}_c) - \sigma_b > A^{\frac{b-c}{1-\alpha}}(\overline{\sigma}_a - \overline{\sigma}_c) - \sigma_a$$

Now add $\overline{\sigma}_c$ to both sides and note that the resulting expression implies that also the second of the inequalities above is reversed. With the resulting configuration of cost-adjusted wage differentials all workers x such that $x < (W_t^a - W_t^c)/(\overline{\sigma}_a - \overline{\sigma}_c)$ will strictly prefer to work in a than either in b or c, while workers x such that $x > (W_t^a - W_t^c)/(\overline{\sigma}_a - \overline{\sigma}_c)$ will strictly prefer c to either a and b. No workers, therefore, are willing to operate b machines. This contradicts the assumption that $b \in I_t$. Hence, the first inequality in (A.3) must hold. Repeating the identical thought process, it can readily be seen that (i) the first inequality implies the second, and that, as a corollary, all workers x such that $x < (W_t^a - W_t^b)/(\overline{\sigma}_a - \overline{\sigma}_b)$ will strictly prefer to work in

a than either in b or c, all workers x such that $x > (W_t^b - W_t^c)/(\overline{\sigma}_b - \overline{\sigma}_c)$ will strictly prefer c to either a and b, and all workers x such that $(W_t^a - W_t^b)/(\overline{\sigma}_a - \overline{\sigma}_b) < x < (W_t^b - W_t^c)/(\overline{\sigma}_b - \overline{\sigma}_c)$ will strictly prefer b to either a and c.

Step 2: The monotonicity of "cost-adjusted wage differentials," alluded to in the text, emerges as a further corollary. To see this, call $\underline{i}$ the least advanced technology being used. Relabel all other technologies in I_t so that their alphabetical order mirrors technological order: $I_t = \{\underline{i}, a, b, c, ...\}$, and $a > b > c > d > ... > \underline{i}$. Note that I have not proved the "no-hole" result yet, so I cannot presume that, say, $b = a - 1$. This is why I am forced to this cumbersome notational expedient. Notice, however, that we do know that $a = g(t)$. Now apply Step 1 to all triples formed by three successive elements, i.e. $\{a, b, c\}$, $\{b, c, d\}$, $\{c, d, e\}$,.... It readily emerges that

$$\frac{W_t^a - W_t^b}{\overline{\sigma}_a - \overline{\sigma}_b} < \frac{W_t^b - W_t^c}{\overline{\sigma}_b - \overline{\sigma}_c} < \frac{W_t^c - W_t^d}{\overline{\sigma}_c - \overline{\sigma}_d} < ...$$

Step 3: Finally, the desire of individuals to maximize their net incomes implies that: all workers x such that $x < (W_t^a - W_t^b)/(\overline{\sigma}_a - \overline{\sigma}_b)$ will strictly prefer to work in a than in any other technology; all worker x such that $(W_t^a - W_t^b)/(\overline{\sigma}_a - \overline{\sigma}_b) < x < (W_t^b - W_t^c)/(\overline{\sigma}_b - \overline{\sigma}_c)$ will strictly prefer b to any other technology; all worker x such that $(W_t^b - W_t^c)/(\overline{\sigma}_a - \overline{\sigma}_b) < x < (W_t^c - W_t^d)/(\overline{\sigma}_b - \overline{\sigma}_c)$ will strictly prefer c to any other technology; ... . Hence the pattern of employment is as follows:

$$L_t^a = \frac{W_t^a - W_t^b}{\overline{\sigma}_a - \overline{\sigma}_b}$$

$$L_t^b = \frac{W_t^b - W_t^c}{\overline{\sigma}_b - \overline{\sigma}_c} - \frac{W_t^a - W_t^b}{\overline{\sigma}_a - \overline{\sigma}_b}$$

...

$$L_t^{\underline{i}} = 1 - \sum_{i \in I_t, i \neq \underline{i}} L_t^i$$

Step 4: We can now prove the "no-hole" condition, and we do that by contradiction. Assume, then, that $i \notin I_t$, while $\exists h', z' \in I_t$ such that $h' > i > z'$. By Step 3 we then have that $\exists h, z \in I_t$, such that $h' \geq h > i > z \geq z'$, and that individual $x \equiv \frac{W_t^h - W_t^z}{\overline{\sigma}_h - \overline{\sigma}_z}$ (i) is indifferent between using technology h or z and (ii) prefers either h or z to any other technology in I_t. Consider a firm approaching this worker with a proposal to learn and use technology i, instead of z or h, for a wage of $W_t^z + x(\overline{\sigma}_i - \overline{\sigma}_z)$. Since this worker's best alternative is to make net income $W_t^z - x\overline{\sigma}_z$, this plan is feasible, and delivers profits

$$A^i k^\alpha - kR_t^z - W_t^z - x(\overline{\sigma}_i - \overline{\sigma}_z)$$

where R_t^z is the (common) interest rate at time t, and k is the amount of capital of type i with which the worker is endowed. Taking the first order condition with respect to k, and using (4), gives an optimal amount of capital of $A^{\frac{i-z}{1-\alpha}} K_t^z / L_t^z$. Using this, the definition of x, and equations (4) and (11) in the profit function gives maximum profits of:

$$\left[A^{i+\alpha\frac{i-z}{1-\alpha}} - \alpha A^{z+\frac{i-z}{1-\alpha}} - (1-\alpha)A^z - \frac{\overline{\sigma}_i - \overline{\sigma}_z}{\overline{\sigma}_h - \overline{\sigma}_z}(A^{\frac{h-z}{1-\alpha}} - 1)A^z(1-\alpha)\right]\left(\frac{K_t^z}{L_t^z}\right)^\alpha$$

Imposing the condition that this expression be non-positive leads, after a few simplifications and rearrangements, to the condition:

$$(\overline{\sigma}_h - \overline{\sigma}_z)(A^{\frac{i-z}{1-\alpha}} - 1) - (\overline{\sigma}_i - \overline{\sigma}_z)(A^{\frac{h-z}{1-\alpha}} - 1) \leq 0$$

Using (11) this no-profit condition can be rewritten as:

$$\frac{W_t^h - W_t^z}{\overline{\sigma}_h - \overline{\sigma}_z} \geq \frac{W_t^i - W_t^z}{\overline{\sigma}_i - \overline{\sigma}_z}$$

Since $h > i > z$, this contradicts (A.3).

Step 5: Armed with the just-obtained no-hole condition, we can go back to Step 4 and proceed to identify $b = a - 1 = c + 1 = d + 2 = \ldots.$ This gives equations (12).

Appendix F: Proof that $\mathrm{i} \in \mathrm{I}_{T_j}$, $\mathrm{i} \notin \mathrm{I}_{T_j+1} \Rightarrow \mathrm{I}_{T_j+1} = \{\mathrm{j}\}$

Using the no-hole condition it follows from the assumption that $\underline{i} \notin I_{T_j+1}$, where $\underline{i}$ is defined as $\underline{i} \in I_{T_j}$, $\underline{i} \leq h$, $\forall h \in I_{T_j}$. Now define $\underline{i}'$ as $\underline{i}' \in I_{T_j+1}$, $\underline{i}' \leq h$, $\forall h \in I_{T_j+1}$. Notice that, by the no-hole condition, $\underline{i}' \in I_{T_j}$ and $\underline{i}' > \underline{i}$. The statement of the claim now can be reformulated as $\underline{i}' = j$. The proof is by contradiction, and follows two steps.

Step 1: Suppose, by contradiction, that $\underline{i}' < j$. Then $W^{\underline{i}'}_{T_j+1} < W^{\underline{i}'}_{T_j}$.

To see this, suppose, again to the contrary, that $W^{\underline{i}'}_{T_j+1} \geq W^{\underline{i}'}_{T_j}$. From equation (11) this implies that

$$\sum_{i \in I_{T_j+1}} A^{\frac{i}{1-\alpha}} L^i_{T_j+1} \leq \sum_{i \in I_{T_j}} A^{\frac{i}{1-\alpha}} L^i_{T_j}. \tag{A.4}$$

In turn this implies, again from equation (11), that $W^i_{T_j+1} \geq W^i_{T_j}$, $\forall i \in I_{T_j}, I_{T_j+1}$, i.e. for all i such that $j > i \geq \underline{i}'$. Using this fact, always in combination with equation (11), in equations (12), we find that $L^j_{T_j+1} + L^{j-1}_{T_j+1} \geq L^{j-1}_{T_j}$, and $L^i_{T_j+1} \geq L^i_{T_j}$ for $j - 1 > i > \underline{i}'$. Now notice that the left side of (A.4) is (strictly) more than

$$A^{\frac{j-1}{1-\alpha}}(L^j_{T_j+1} + L^{j-1}_{T_j+1}) + A^{\frac{j-2}{1-\alpha}} L^{j-2}_{T_j+1} + \ldots + A^{\frac{\underline{i}'}{1-\alpha}} L^{\underline{i}'}_{T_j+1}$$

while the right side is (strictly) less than

$$A^{\frac{j-1}{1-\alpha}} L^{j-1}_{T_j} + A^{\frac{j-2}{1-\alpha}} L^{j-2}_{T_j} + \ldots + A^{\frac{\underline{i}'}{1-\alpha}}(L^{\underline{i}'}_{T_j} + L^{\underline{i}'-1}_{T_j} + \ldots + L^{\underline{i}}_{T_j}).$$

Since employments must sum to one, $L^{\underline{i}'}_{T_j+1} = 1 - \sum_{\underline{i}' < i \leq j} L^i_{T_j+1}$, and $(L^{\underline{i}'}_{T_j} + L^{\underline{i}'-1}_{T_j} + \ldots + L^{\underline{i}}_{T_j}) = 1 - \sum_{\underline{i}' < i < j} L^i_{T_j}$. Hence, we get that condition (A.4) is satisfied only if

$$(A^{\frac{j-1}{1-\alpha}} - A^{\frac{\underline{i}'}{1-\alpha}})(L^j_{T_j+1} + L^{j-1}_{T_j+1} - L^{j-1}_{T_j}) + (A^{\frac{j-2}{1-\alpha}} - A^{\frac{\underline{i}'}{1-\alpha}})(L^{j-2}_{T_j+1} - L^{j-2}_{T_j}) + \ldots + (A^{\frac{\underline{i}'+1}{1-\alpha}} - A^{\frac{\underline{i}'}{1-\alpha}})(L^{\underline{i}'+1}_{T_j+1} - L^{\underline{i}'+1}_{T_j}) < 0,$$

which is a contradiction in view of the above-established fact that $L^{j}_{T_j+1} + L^{j-1}_{T_j+1} \geq L^{j-1}_{T_j}$, and $L^{i}_{T_j+1} \geq L^{i}_{T_j}$ for $j-1 > i > \underline{i}'$.

Step 2: $W^{\underline{i}'}_{T_j+1} < W^{\underline{i}'}_{T_j}$ implies that there are unexploited profit opportunities in period T_j+1.

To see this, consider a firm that, in period T_j+1, hires individual $x=1$ to work with technology $\underline{i}$. By the results in Appendix E this individual's best alternative is to work with technology $\underline{i}'$. Hence, this policy is feasible if the wage offered to the worker is $W^{\underline{i}'}_{T_j+1} - (\overline{\sigma}_{\underline{i}'} - \overline{\sigma}_{\underline{i}})$. Maximizing profits with respect to endowment of type $\underline{i}$ capital, taking interest rates as given, substituting the result back into the profit function, using equation (4) with $i=\underline{i}'$ to substitute for the interest rate and for the wage rate, we get that maximum profits from this strategy are:

$$\left[A^{\underline{i}+\alpha\frac{\underline{i}'-\underline{i}}{\alpha-1}} - \alpha A^{\underline{i}'+\frac{\underline{i}'-\underline{i}}{\alpha-1}} - (1-\alpha)A^{\underline{i}'}\right]\left(\frac{K^{\underline{i}'}_{T_j+1}}{L^{\underline{i}'}_{T_j+1}}\right)^{\alpha} + (\overline{\sigma}_{\underline{i}'} - \overline{\sigma}_{\underline{i}})$$

Now impose that this expression be non-positive, simplify, rearrange, and use again equation (4) to find that the non-profit condition is

$$\frac{1-A^{\frac{\underline{i}'-\underline{i}}{\alpha-1}}}{\overline{\sigma}_{\underline{i}'} - \overline{\sigma}_{\underline{i}}} W^{\underline{i}'}_{T_j+1} \geq 1. \tag{A.5}$$

Now note that by the results of Appendix E the corresponding individual $x=1$ of the generation born in T_j worked with technology $\underline{i}$. We then must have

$$\frac{W^{\underline{i}'}_{T_j} - W^{\underline{i}}_{T_j}}{\overline{\sigma}_{\underline{i}'} - \overline{\sigma}_{\underline{i}}} < 1,$$

otherwise $x=1$ would have preferred $\underline{i}'$ to $\underline{i}$. Using equation (11) the last condition can be rewritten as

$$\frac{1-A^{\frac{\underline{i}'-\underline{i}}{\alpha-1}}}{\overline{\sigma}_{\underline{i}'} - \overline{\sigma}_{\underline{i}}} W^{\underline{i}'}_{T_j} < 1$$

But Step 1 of this appendix establishes that $W^{i'}_{T_j+1} < W^{i'}_{T_j}$. Together with the last equation, this means that the no-profit condition (A.5) must necessarily be violated, and we have a contradiction.

Appendix G: Proof that $I_{T_j} \subset I_{T_j+1} \Rightarrow W^i_{T_j+1} < W^i_{T_j}$, $\forall i \in I_{T_j}$

By contradiction, suppose that $\exists h \in I_{T_j}$, such that $W^h_{T_j+1} \geq W^h_{T_j}$. Repeating a line of argument already used in Appendix F this is readily seen to imply

$$\sum_{i \in I_{T_j+1}} A^{\frac{i}{1-\alpha}} L^i_{T_j+1} \leq \sum_{i \in I_{T_j}} A^{\frac{i}{1-\alpha}} L^i_{T_j}$$

and $L^i_{T_j+1} > L^i_{T_j}$ for all $i \in I_{T_j}$. Combined with $I_{T_j} \subset I_{T_j+1}$, these two facts are in contradiction.

Appendix H: Learning Externalities

In the text the distribution of learning costs was exogenous and time-invariant, and the post revolutionary dynamics were uniquely driven by capital accumulation. It seems likely, however, that an individual's cost of learning changes as the new technique diffuses throughout the economy. It is easier to pick up computer skills when there are many computer-literate individuals who can answer questions, and many machines to practice on. Also, it appears that, as a technology has been around for some time, younger generations have an easier time acquiring the necessary skills. This may reflect the fact that, once a technology becomes widespread, the mandatory schooling system incorporates the corresponding skills in the core curriculum. In addition, growing up in a household where the parents have already acquired certain skills will allow for a less costly transmission of the same skills to the children. When these endogenous changes in learning costs occur, the dynamics of the model are no longer uniquely driven by capital accumulation.

To model this learning "externality by osmosis," in this section I suppose that, at any date t, the learning cost σ is uniformly distributed in the interval $[0, \overline{\sigma}/K_t^1]$. In words, the distribution is uniform in every period, but the upper bound moves down as the stock of new capital increases. As the economy as a whole upgrades, everyone's cost of learning is pushed down. Hence, the distribution of learning costs is now endogenous and time-varying. Under this assumption the labor supply functions become:

$$L_t^1 = \max\left\{0,\ \min\left[1,\ (W_t^1 - W_t^0)\frac{K_t^1}{\overline{\sigma}}\right]\right\} \tag{A.6}$$

and $L_t^0 = 1 - L_t^1$, which have the same interpretation as (7), with $K_t^1/\overline{\sigma}$ the density of the new distribution.

The impact effects of a technological revolution in period T are identical to those with time-invariant cost distribution. Provided that the maximum learning cost, indexed by $\overline{\sigma}$, is large enough relative to the productivity gain A, some workers migrate to the new sector and enjoy wage gains, while some workers keep using the original technology and suffer wage losses. Appendix A can be easily adapted to prove this formally. There is only one slight complication. There is now a *minimum* level for the initial capital stock, $\overline{K}$, below which neither partial nor total adoption can occur.

The learning-externality variant delivers new insights, however, when post-revolutionary dynamics are considered. As in Appendices A and B, for a given level of total savings from the young in period $t-1$, K_t, the labor-1 supply equation, the no-arbitrage condition, the adding up constraint for labor and capital in the two sectors, and the four factor pricing equations, form a system of equations that determines factor employments and prices as functions of the state variable K_t. In this case, however, it turns out that this system can be solved explicitly. The solution includes

$$L_t^1 = \left[\frac{(1-\alpha)A}{\overline{\sigma}(1-A^{\frac{1}{\alpha-1}})^{\alpha}}\right]^{\frac{1}{1+\alpha}} K_t - \frac{1}{A^{\frac{1}{1-\alpha}}-1}. \tag{A.7}$$

and

$$K_t^1 = \frac{A^{\frac{1}{1-\alpha}}}{A^{\frac{1}{1-\alpha}}-1} K_t - \frac{\overline{\sigma}^{\frac{1}{1+\alpha}}}{\left[(1-\alpha)(A-A^{\frac{\alpha}{\alpha-1}})\right]^{\frac{1}{1+\alpha}}(A^{\frac{1}{1-\alpha}}-1)} \tag{A.8}$$

using which showing that K_t^1/L_t^1, W_t^1 and W_t^0 are constant is trivial.

The dynamics are described by period t's savings function:

$$\begin{aligned} K_{t+1} &= \frac{\beta}{1+\beta}\left[\int_0^{W_t^1-W_t^0}(W_t^1-\sigma)\frac{K_t^1}{\overline{\sigma}}d\sigma + \int_{W_t^1-W_t^0}^{\overline{\sigma}/K_t^1} W_t^0\frac{K_t^1}{\overline{\sigma}}d\sigma\right] \\ &= \frac{\beta}{1+\beta}\left[\frac{K_t^1}{2\overline{\sigma}}(W_t^1-W_t^0)^2 + W^0\right] \end{aligned}$$

Substituting from (4), (A.7) and (A.8) this yields:

$$K_{t+1} = \frac{\beta}{1+\beta}\frac{1}{2A^{\frac{\alpha}{1-\alpha}}}\left[\frac{(1-\alpha)\overline{\sigma}^{\alpha}}{(A-A^{\frac{\alpha}{\alpha-1}})^{\alpha}}\right]^{\frac{1}{1+\alpha}}\left\{1+A^{\frac{1}{1-\alpha}}\left[\frac{(1-\alpha)(A-A^{\frac{\alpha}{\alpha-1}})}{\overline{\sigma}}\right]^{\frac{1}{1+\alpha}} K_t\right\} \tag{A.9}$$

The notable feature of these relations is, of course, their linearity. Since the capital-labor ratio K_t^1/L_t^1 is independent of K_t, hence so are the wages W_t^1 and W_t^0. In (A.9) slope and intercept are positive. Using (A.7) and (A.9) one can easily draw the analogous of Figures 1 and 2. What will happen depends in part on the slope of (A.9). If the slope is less than 1, whether or not full absorption obtains depends on whether the point K^* is on the right or on the left of point $\tilde{K}$. In other words, if the slope of (A.9) is less than one we are in the same situation encountered in the case of the time-invariant distribution. Instead, if the slope of (A.9) is greater than 1, the point of full absorption of the labor force into the skilled technology is necessarily reached in finite time. After this, the economy returns to a standard-looking path of capital accumulation with decreasing returns and a unique technology.

So what do we learn from this variant to the model? As already noted, the special algebraic representation I have chosen for the learning externality implies that capital-labor ratios, and hence wages, are constant throughout the post-revolutionary dynamics. Now recall that increasing capital-labor ratios and wage differentials were the key driving forces for such dynamics in Section 2. Hence, all dynamic developments we observe in the present setting are entirely due to the mere presence of the learning externality. Specifically, as the revolution increases output and the capital stock, the reason why more and more workers move from sector 0 to sector 1 is not, as before, that wage differentials widen, but that the learning cost of an increasing number of workers fall below the (constant) wage differential. In turn, the increase in the relative amount of resources employed in the more productive sector increases output, and so on. The possibility of "steep" dynamics, such as when the slope of (A.9) is more than 1, implies that the learning externality by itself can have dramatic effects on the dynamics of the post-revolutionary economy.

More generally, this example points to the importance of the shape of the learning-cost distribution in determining wage inequality in the long-run. Even in the (general) case in which wage differentials increase over time, *the rate* at which workers move from sector 0 to sector 1 depends on the frequency distribution of the learning costs at the current wage differential. The same increase in $W^1 - W^0$ can generate widely different amounts of labor "migration" at different points of the cost distribution. In turn, such varying rates of migration can profoundly affect the rate of capital accumulation. If, for example, the rate at which workers move from sector 0 to sector 1 increases very sharply over time, the curves in the right sides of The analogous of Figures 1 and 2 may even become convex: rapidly increasing rates of reallocation of resources from the less to the more productive sector can more than compensate for the declining marginal products of capital generated by increasing capital-labor ratios.

Appendix I: Two-Industry Extension of the Model(Sketch)

The model is a straightforward extension of the one in Section 2. For current purposes it is sufficient to focus on a static version, but adding dynamics is trivial. Consumers derive utility from the consumption of two goods, x and z. Tastes are identical among consumers and they are represented by the Cobb-Douglas function $X^{\zeta}Z^{1-\zeta}$ $(0 < \zeta < 1)$. Maximization of this objective under a budget constraint leads to the condition:

$$\frac{X}{Z} = p\frac{\zeta}{1-\zeta} \tag{A.10}$$

Where p is the relative price of good z. In the pre-revolutionary economy the two goods can only be produced by means of one technology:

$$\begin{aligned} X &= (K_x^0)^{\alpha}(L_x^0)^{1-\alpha} \\ Z &= (K_z^0)^{\beta}(L_z^0)^{1-\beta} \end{aligned} \tag{A.11}$$

where K_i^j (L_i^j) is employment of capital (labor) of type j in production of good i. To fix ideas assume that $\beta \geq \alpha$ so that z is the capital-intensive sector. Tools of type j require similar skills independently of the industry in which they are used. Hence labor of type 0 can be used in either industry, and the same full mobility applies to capital. Hence, assuming that all markets are perfectly competitive leads to the implication that there is a unique wage and a unique interest rate across the two sectors before the revolution. The model is closed by the adding-up constraints $L_x^0 + L_z^0 = 1$ and $K_x^0 + K_z^0 = 1$. Tedious but elementary algebra leads to the following solution for the pre-revolutionary economy.

$$
\begin{aligned}
L_z^0 = \overline{L}_z &= \frac{(1-\beta)(1-\zeta)}{(1-\beta)(1-\zeta)+(1-\alpha)\zeta} \\
K_z^0 = \overline{K}_z &= \frac{\beta(1-\zeta)}{\beta(1-\zeta)+\alpha\zeta} \\
p = \overline{p} &= \frac{\alpha^\alpha(1-\alpha)^{1-\alpha}}{\beta^\beta(1-\beta)^{1-\beta}}\left[\frac{(1-\beta)(1-\zeta)+(1-\alpha)\zeta}{\beta(1-\zeta)+\alpha\zeta}\right]^{\alpha-\beta}
\end{aligned}
\tag{A.12}
$$

Where as in the paper an overline denotes a pre-revolutionary equilibrium value. In the special case in which $\alpha = \beta$ we have: $\overline{X} = \overline{K}_x = \overline{L}_x = \zeta$, $\overline{Z} = \overline{K}_z = \overline{L}_z = 1-\zeta$, and $\overline{p} = \overline{k}_x = \overline{k}_z{=}1$

The technological revolution introduces capital of type 1, which, if operated by workers with the appropriate skills, is more productive than the original capital. I assume that a fraction L^1 of the population costlessly obtains the corresponding skills (endogenizing L^1 is simple but adds nothing for my present goals). The new production functions are:

$$
\begin{aligned}
X &= (K_x^0)^\alpha(L_x^0)^{1-\alpha} + A(K_x^1)^\alpha(L_x^1)^{1-\alpha} \\
Z &= (K_z^0)^\beta(L_z^0)^{1-\beta} + B(K_z^1)^\beta(L_z^1)^{1-\beta}
\end{aligned}
\tag{A.13}
$$

where I assume $A, B > 1$. The solution of the model changes slightly according to whether $A^{\frac{1}{1-\alpha}} \gtreqless B^{\frac{1}{1-\beta}}$, but the qualitative implications are independent of this. So let's focus on the special case in which $A^{\frac{1}{1-\alpha}} = B^{\frac{1}{1-\beta}}$. This generates an interior solution, i.e. a solution with $L_j^i > 0, \quad i = 0,1, \quad j = x,z$. (Other parameter configurations lead to $L_j^i = 0$ for at least one of the four i,j combinations). Define μ as the fraction of L^1 who work in sector x, i.e. $L_x^1 = \mu L^1$. μ is endogenous. The model is closed by the adding up constraints:

$$
\begin{aligned}
L_z^1 &= (1-\mu)L^1 \\
L_x^0 + L_z^0 &= (1-L^1) \\
K_x^0 + K_x^1 + K_z^0 + K_z^1 &= 1
\end{aligned}
\tag{A.14}
$$

As in Section 2 of the paper, the solution features $W^0 < \overline{W} < W^1$, where $\overline{W}$ is the pre-revolution wage. Since both sectors have workers of both type 0 and type 1, this means that there is an *increase in within-industry wage inequality.* Some other properties of the post-revolutionary equilibrium are:

$$L_x^0 = \frac{(1-\alpha)\zeta[1 - L^1 + B^{\frac{1}{1-\beta}}(1-\mu)L^1] - (1-\beta)(1-\zeta)B^{\frac{1}{1-\beta}}\mu L^1}{(1-\alpha)\zeta + (1-\beta)(1-\zeta)} \tag{A.15}$$

$$L_z^0 = \frac{(1-\beta)(1-\zeta)[1 - L^1 + B^{\frac{1}{1-\beta}}\mu L^1] - (1-\alpha)\zeta[B^{\frac{1}{1-\beta}}(1-\mu)L^1]}{(1-\alpha)\zeta + (1-\beta)(1-\zeta)} \tag{A.16}$$

Defining $L_x = L_x^0 + L_x^1 = L_x^0 + \mu L^1$ we get:

$$L_x = \frac{(1-\alpha)\zeta[1 - (1 - B^{\frac{1}{1-\beta}})(1-\mu)L^1] - (1-\beta)(1-\zeta)(B^{\frac{1}{1-\beta}} - 1)\mu L^1}{(1-\alpha)\zeta + (1-\beta)(1-\zeta)} \tag{A.17}$$

and $L_z = 1 - L_x$. Also, we get:

$$K_x^0 + K_x^1 = K_x = \frac{\alpha\zeta}{\alpha\zeta + \beta(1-\zeta)} \tag{A.18}$$

and $K_z = 1 - K_x$. It turns out that the parameter μ itself is indeterminate. I.e. there is an infinity of equilibria, one for each value of $0 < \mu < 1$. Suppose then that we have an equilibrium where:

$$\mu < \frac{(1-\alpha)\zeta}{(1-\alpha)\zeta + (1-\beta)(1-\zeta)}$$

In other words, the capital intensive sector (z) has a relatively large proportion of type-1 workers. Then:

$$\frac{K_z/L_z}{K_x/L_x} > \frac{\overline{K}_z/\overline{L}_z}{\overline{K}_x/\overline{L}_x}$$

I.e. the *interindustry inequality of capital per worker increases,* relative to its pre-revolution level. Furthermore,

$$\frac{(L_z^0 W^0 + L_z^1 W^1)/L_z}{(L_x^0 W^0 + L_x^1 W^1)/L_x} > 1$$

implying that *interindustry average-wage inequality increases*, with the capital intensive industry experiencing the largest gains. Finally,

$$\frac{L_x}{L_z} > \frac{\overline{L}_x}{\overline{L}_z}$$

This means that *total employment falls in the capital-intensive industry and increases in the labor-intensive industry.* This last result is important to explain why the employment-weighted log-variance of the capital-labor ratio does not increase. Define $k_i = \log(K_i/L_i)$ The employment-weighted log-mean is $k = k_x L_x + k_z L_z$. The employment-weighted log-variance is $V = (k_x - k)^2 L_x + (k_z - k)^2 L_z$. For the pre-revolutionary economy, define $\overline{k}_i = \log(\overline{K}_i/\overline{L}_i)$. The employment-weighted log-mean is $\overline{k} = \overline{k}_x \overline{L}_x + \overline{k}_z \overline{L}_z$ and the employment-weighted log-variance is: $\overline{V} = (\overline{k}_x - \overline{k})^2 \overline{L}_x + (\overline{k}_z - \overline{k})^2 \overline{L}_z$. Now: while the unweighted variance *always increases*, the weighted variance may well decrease. This happens, for example, for the following set of values:

$$\alpha = 0.1; \zeta = 0.5; \beta = 0.6; \mu = 0.1; B^{\frac{1}{1-\beta}} = 2; L^1 = 0.5; \tag{A.19}$$

With these values, we have that the unweighted variance increases from 0.32 to 1.84, while the weighted variance falls from 0.27 to 0.08. The intuition for this is given above (last paragraph of Point 2) and in the paper.