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Optimal investment with inside information and parameter uncertainty

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Abstract An optimal investment problem is solved for an insider who has access to noisy information related to a future stock price, but who does not know the stock price drift. The drift is filtered from a combination of price observations and the privileged information, fusing a partial information scenario with enlargement of filtration techniques. We apply a variant of the Kalman–Bucy filter to infer a signal, given a combination of an observation process and some additional information. This converts the combined partial and inside information model to a full information model, and the associated investment problem for HARA utility is explicitly solved via duality methods. We consider the cases in which the agent has information on the terminal value of the Brownian motion driving the stock, and on the terminal stock price itself. Comparisons are drawn with the classical partial information case without insider knowledge. The parameter uncertainty results in stock price inside information being more valuable than Brownian information, and perfect knowledge of the future stock price leads to infinite additional utility. This is in contrast to the conventional case in which the stock drift is assumed known, in which perfect information of any kind leads to unbounded additional utility, since stock price information is then indistinguishable from Brownian information.

Keywords Insider trading · Enlargement of filtration · Filtering · Optimal investment · Partial information

1 Introduction

The goal of this paper is to examine the combined influence of inside information and drift parameter uncertainty on optimal investment rules. To this end, we explicitly solve a

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one-dimensional Merton-style investment problem for an insider who possesses, at time zero, additional information beyond that of regular traders, but who is not assumed to know the value of the stock's appreciation rate. Although the assumption of a one-dimensional model with unknown constant drift is restrictive, it has the benefit of allowing us to derive fully explicit solutions, so that the effect of the inside information on the parameter estimation and the optimal investment rule can be fully gauged.

This work thus combines elements of partial information models such as those of Rogers [28] or Björk, Davis and Landén [7], with enlargement of filtration techniques to incorporate the insider's additional information, as pioneered by Pikovsky and Karatzas [27].

The insider has knowledge of the value of a random variable L, corresponding to (usually, noisy) knowledge of the terminal stock price or of the terminal value of the Brownian motion driving the stock. But the insider does not have access to the Brownian filtration. Her trading strategies are required to be adapted to the stock price filtration, but enlarged by the additional information. The stock's risk premium is then an unobservable signal process which is estimated via a Kalman–Bucy filter. The filtering algorithm, Proposition 3, computes the best estimate of the stock's risk premium given both stock price observations and the additional information. In this case the usual Kalman–Bucy equations hold, but with modified initial conditions reflecting the additional information.

We begin with lognormal stock price dynamics written under some background filtration $\mathbb{F}$, under which the stock drift is known. We enlarge $\mathbb{F}$ by $\sigma(L)$, using classical techniques of initial enlargement of filtration [17,24,30]. Denoting the enlarged filtration by $\mathbb{F}^L$, the effect of the enlargement is to decompose an $\mathbb{F}$-Brownian motion into an $\mathbb{F}^L$-Brownian motion plus an *information drift*, ν^L, an $\mathbb{F}^L$-adapted process. The risk premium then becomes an $\mathbb{F}^L$-adapted process λ^L. With the dynamics of the stock price and its risk premium written under the enlarged filtration $\mathbb{F}^L$, the filtering algorithm infers the insider's unknown risk premium based on stock price observations as well as the additional information $\sigma(L)$.

If we denote by $\widehat{\mathbb{F}}^L$ the stock price filtration enlarged by $\sigma(L)$, the effect of the filtering is to convert the partial and inside information model to a standard full information model with random drift that is adapted to $\widehat{\mathbb{F}}^L$. Having restored a full information scenario, we solve the insider's utility maximisation problem using duality methods, giving closed form expressions for the maximum utility and optimal trading strategy.

We compare the results with the corresponding quantities for a regular agent who does not have inside information, but who must still filter the stock price drift. We find that stock price inside information is more valuable than information on the Brownian motion driving the stock. This can be traced directly to the parameter uncertainty, which requires stock price observations in order to be resolved. Hence, we find that exact terminal stock price information can lead to unbounded additional utility for the insider, but exact knowledge of the terminal Brownian motion does not. This is to be contrasted with the seminal insider trading model of Pikovsky and Karatzas [27]. In [27] there was no parameter uncertainty, so exact information of any kind led to unbounded logarithmic utility, as advance Brownian knowledge is equivalent to stock price knowledge when the stock's drift is known.

There are many papers on partial information investment models [7,8,22,26,28,29], in which trading strategies are required to be adapted to the stock price filtration. There is also a rich literature on insider trading models, dating back to the classical equilibrium models of Kyle [21] and Back [5], built upon by Cho [11], Campi and Çetin [10], Danilova [13], and Aase, Bjuland and Øksendal [1]. In these models, the insider can use his additional information to influence the stock price, a feature that also appears in Kohatsu-Higa and Sulem [19].

Insider trading models in which techniques of enlargement of filtration play a direct role stem from Pikovsky and Karatzas [27]. Amendinger et al. [3] gave an entropic characterisation of the additional utility achievable by an insider, extended to a semimartingale setting by Ankirchner et al. [4]. Amendinger et al. [2] used indifference arguments to give a monetary value to inside information in portfolio optimisation. Imkeller [15,16] used the notion of progressive enlargement of filtration to model inside information on a random time that is not a stopping time for regular traders, and used Malliavin calculus to characterise the information drift. Corcuera et al [12] considered a dynamic flow of inside information, Baudoin and Nguyen-Ngoc [6] considered so-called weak information, involving knowledge of the law of some random variable, Hillairet [14] compared optimal strategies of insiders with different forms of side-information, and Campi [9] treated a quadratic hedging problem. To the best of our knowledge this paper is the first to combine partial and inside information scenarios.

The rest of the paper is as follows. In Sect. 2 we solve an investment problem involving a stock with a Gaussian drift process, a result we need as subsequent problems can be rendered into this form. Section 3 details the model and the investment problems, and sets up generic notation and methodology that is common to all the problems. Section 4 gives the solution to the partial information investment problem with no inside information, used for comparison with later results. The main results, the solutions of the investment problems with inside information and parameter uncertainty, are given in Sect. 5 and proven in Sect. 6. Section 7 concludes. An Appendix contains enlargement of filtration and filtering results that are used in Sect. 6.

2 Optimal investment with Gaussian drift process

In this section we solve a generic optimal investment problem for a stock with a Gaussian risk premium. We shall see that all the utility maximisation problems of the paper are of this form.

We are given a probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$. We do not assume that $\mathcal{F}_0$ is trivial, in contrast to more standard scenarios, and this will be a pertinent ingredient in subsequent optimal investment problems.

A stock price $S = (S_t)_{0\leq t\leq T}$ follows

$$\mathrm{d}S_t = \sigma S_t(\delta_t \mathrm{d}t + \mathrm{d}B_t), \tag{1}$$

with $\sigma > 0$ constant, B an $\mathbb{F}$-Brownian motion and $\delta = (\delta_t)_{0\leq t\leq T}$ an $\mathbb{F}$-adapted process. Assume that δ is given by

$$\delta_t = \delta_0 + \int_0^t \mathrm{w}_s \mathrm{d}B_s, \quad 0 \leq t \leq T, \tag{2}$$

for some $\mathcal{F}_0$-measurable Gaussian random variable δ_0 independent of B, and where w is a deterministic function of time given by

$$\mathrm{w}_t = \frac{\mathrm{w}_0}{1+\mathrm{w}_0 t}, \quad 0 \leq t \leq T,$$

for some constant $\mathrm{w}_0 > -1/T$. We do not assume $\mathrm{w}_0 > 0$, and this will be manifested later in applications of the result in this section. Note that w satisfies the ordinary differential equation (ODE)

$$\frac{\mathrm{dw}_t}{\mathrm{d}t} = -\mathrm{w}_t^2. \tag{3}$$

Note that δ satisfies the integrability condition

$$E\left[\int_0^T \delta_t^2 \mathrm{d}t \middle| \mathcal{F}_0\right] < \infty, \quad \text{a.s.} \tag{4}$$

Consider optimal investment in S to maximise expected utility of terminal wealth, over $\mathbb{F}$-adapted self-financing portfolios. Let $\theta = (\theta_t)_{0 \le t \le T}$ denote an $\mathbb{F}$-adapted trading strategy representing the proportion of wealth invested in the stock. Assume the interest rate is zero, for simplicity. Then the wealth process is $X = (X_t)_{0 \le t \le T}$, following

$$\mathrm{d}X_t = \sigma\theta_t X_t(\delta_t \mathrm{d}t + \mathrm{d}B_t), \quad X_0 = x > 0. \tag{5}$$

An $\mathbb{F}$-adapted portfolio process θ is *admissible* if we have

$$\int_0^T \theta_t^2 \mathrm{d}t < \infty, \quad \text{and } X_t \ge 0 \text{ almost surely, for all } t \in [0, T], \tag{6}$$

and denote the set of admissible strategies over $[0, T]$ by $\mathcal{A}(T; \mathbb{F})$.

The utility function $U_p : \mathbb{R}^+ \to \mathbb{R}$ is of the HARA class, defined by

$$U_p(x) := \begin{cases} x^p/p, & p < 1, \quad p \ne 0, \\ \log x, & p = 0. \end{cases} \tag{7}$$

The value function of an agent who uses $\mathbb{F}$-adapted strategies will be defined as $u : \mathbb{R}^+ \to \mathbb{R}$, given by

$$u(x; T, \mathbb{F}) := \sup_{\theta \in \mathcal{A}(T;\mathbb{F})} E[U_p(X_T)|\mathcal{F}_0], \tag{8}$$

and we write $u(x) \equiv u(x; T, \mathbb{F})$ when we do not need to emphasise the investment horizon and filtration governing the investment strategies.

Theorem 1 *Assume that* $1 + \mathrm{w}_0 T > 0$. *For* $p \ne 0$, *define* q *by* $p^{-1} + q^{-1} = 1$. *For* $0 < p < 1$ *assume also that*

$$1 + q\mathrm{w}_0 T > 0, \quad 0 < p < 1. \tag{9}$$

Then the value function in (8) is given by

$$u(x, T; \mathbb{F}) = \begin{cases} (x^p/p)C, & p < 1, \quad p \ne 0, \\ \log x + K/2, & p = 0, \end{cases} \tag{10}$$

where C, K *are* $\mathcal{F}_0$*-measurable random variables given by*

$$\begin{aligned} C &= \left[(1 + \mathrm{w}_0 T)^p (1 + q\mathrm{w}_0 T)^{1-p}\right]^{-1/2} \exp\left(-\frac{q\delta_0^2 T}{2(1 + q\mathrm{w}_0 T)}\right), \\ K &= (\delta_0^2 + \mathrm{w}_0)T - \log(1 + \mathrm{w}_0 T), \end{aligned} \tag{11}$$

The optimal $\mathbb{F}$*-adapted trading strategy* θ^* *achieving the supremum in (8) is given by*

$$\theta_t^* = \begin{Bmatrix} \delta_t[\sigma(1-p)(1 + q\mathrm{w}_t(T-t))]^{-1}, & p < 1, \quad p \ne 0, \\ \delta_t/\sigma, & p = 0, \end{Bmatrix} \quad 0 \le t \le T. \tag{12}$$

Remark 1 The condition (9) will be necessary and sufficient to guarantee a finite dual value function when $p \in (0, 1)$. If (9) is violated, the dual value function is unbounded for all values of the dual Lagrange multiplier and hence, by Theorem 2.0 in Kramkov and Schachermayer [20] the primal value function will be infinite for any initial capital x.

Proof For brevity, write $u(x, T; \mathbb{F}) \equiv u(x)$ in this proof. With the integrability condition (4) and the stock dynamics (1), we are within the classical framework for portfolio optimisation via convex duality, as surveyed in Karatzas [18], for example.

Let Q denote the unique martingale measure for this market. The change of measure martingale $Z := (Z_t)_{0\le t\le T}$ is given by

$$Z_t := \left.\frac{\mathrm{d}Q}{\mathrm{d}P}\right|_{\mathcal{F}_t} = \mathcal{E}(-\delta \cdot B)_t = \left(\frac{\mathrm{w}_0}{\mathrm{w}_t}\right)^{1/2} \exp\left[-\frac{1}{2}\left(\frac{\delta_t^2}{\mathrm{w}_t} - \frac{\delta_0^2}{\mathrm{w}_0}\right)\right], \quad 0 \le t \le T. \quad (13)$$

One can verify this formula by utilising the SDE for Z and making the *ansatz* $Z_t = f(t, \delta_t)$ for some smooth f. Note also that (13) is indeed well-defined even for $\mathrm{w}_0 = 0$, since as $\mathrm{w}_0 \to 0$ we have

$$\lim_{\mathrm{w}_0\to 0} Z_t = \mathcal{E}(-\delta_0 B)_t = \exp\left(-\delta_0 B_t - \frac{1}{2}\delta_0^2 t\right). \quad (14)$$

Consider the utility maximisation problem (8) when $p < 1$, $p \neq 0$. The proof for $p = 0$ follows identical arguments (with more straightforward computations).

For $p < 1$, $p \neq 0$ the convex conjugate $\widetilde{U}_p$ of the utility function is given by

$$\widetilde{U}_p(y) = -\frac{y^q}{q}, \quad y > 0, \quad \frac{1}{p} + \frac{1}{q} = 1. \quad (15)$$

Therefore, the dual value function is given by

$$\tilde{u}(y) := E[\widetilde{U}_p(yZ_T)|\mathcal{F}_0] = -\frac{y^q}{q} E[Z_T^q|\mathcal{F}_0]. \quad y > 0, \quad (16)$$

By classical results on portfolio optimisation via convex duality ([18], for instance) the primal and dual value functions are conjugate. Hence, using (16), the primal value function u is given by (10), with C given by

$$C = \left(E[Z_T^q|\mathcal{F}_0]\right)^{1-p}. \quad (17)$$

It remains to show that C is equal to the expression in (11), and that the optimal strategy is given by (12).

By classical duality results [18], the optimal terminal wealth X_T^* is given by

$$X_T^* = -\widetilde{U}_p'(u'(x)Z_T).$$

Hence, using the form (10) for u, we obtain

$$X_T^* = \frac{x}{E[Z_T^q|\mathcal{F}_0]} Z_T^{-(1-q)}.$$

The optimal wealth process X^* is a $(Q, \mathbb{F})$-martingale (see Theorem 2.3.2 in [18]), so that

$$X_t^* = E^Q[X_T^*|\mathcal{F}_t] = \frac{1}{Z_t} E[Z_T X_T^*|\mathcal{F}_t] = \frac{x}{Z_t E[Z_T^q|\mathcal{F}_0]} E[Z_T^q|\mathcal{F}_t], \quad 0 \le t \le T, \quad (18)$$

where E^Q denotes expectation under Q. So, to compute explicit formulae for C in (17) and the optimal wealth process, we need to evaluate the last conditional expectation in (18).

From (2) and (3), for $t \le T$, and conditional on $\mathcal{F}_t$, δ_T is Gaussian according to

$$\mathrm{Law}(\delta_T|\mathcal{F}_t) = \mathrm{N}(\delta_t, \mathrm{w}_t - \mathrm{w}_T), \quad 0 \le t \le T.$$

So direct calculation yields that C is indeed given by (11), which is well-defined since $1 + \mathrm{w}_0 T > 0$ and, for $p \in (0,1)$, $1 + q\mathrm{w}_0 T > 0$. If the condition (9) is violated then $E[Z_T^q|\mathcal{F}_0] = +\infty$ so the dual value function (16) is infinite for all $y > 0$ as claimed in Remark 1.

For the optimal wealth process we obtain the formula

$$X_t^* = x\left(\frac{\Psi_t}{\Psi_0}\right)^{1/2} \exp\left(\frac{1}{2}(1-q)(\Phi_t - \Phi_0)\right), \quad 0 \le t \le T, \tag{19}$$

where

$$\Psi_t := \frac{\mathrm{w}_t}{1 + q\mathrm{w}_t(T-t)}, \quad \Phi_t := \frac{\delta_t^2}{\mathrm{w}_t(1 + q\mathrm{w}_t(T-t))}, \quad 0 \le t \le T.$$

To compute the optimal trading strategy θ^*, we apply the Itô formula to (19), use the SDE (2) for δ and the ODE (3) for w, and compare the coefficient of $\mathrm{d}B_t$ in dX_t^* with that in (5) for the case of the optimal wealth process. This gives (12). $\square$

3 The model

On a complete probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $\mathbb{F} := (\mathcal{F}_t)_{0\le t\le T}$, our model comprises a stock with price process $S = (S_t)_{0\le t\le T}$ following

$$\mathrm{d}S_t = \sigma S_t(\lambda \mathrm{d}t + \mathrm{d}B_t), \tag{20}$$

where $B = (B_t)_{0\le t\le T}$ is a one-dimensional $\mathbb{F}$-Brownian motion. We shall assume the risk premium λ is an unknown constant, so we take it to be an $\mathcal{F}_0$-measurable Gaussian random variable, independent of B. The volatility $\sigma > 0$ is assumed to be known. With continuous price monitoring it could be inferred exactly from the quadratic variation of S, and we make this approximation to focus on the more severe problem of drift uncertainty. See Rogers [28] for an account of the relative severity of drift versus volatility uncertainty. For simplicity, we take the interest rate to be zero.

Define the process $\xi = (\xi_t)_{0\le t\le T}$ by

$$\xi_t := \frac{1}{\sigma}\int_0^t \frac{\mathrm{d}S_s}{S_s} = \lambda t + B_t = \frac{1}{\sigma}\log\left(\frac{S_t}{S_0}\right) + \frac{1}{2}\sigma t, \quad 0 \le t \le T, \tag{21}$$

and denote by $\widehat{\mathbb{F}} = (\widehat{\mathcal{F}}_t)_{0\le t\le T}$ the filtration generated by ξ:

$$\widehat{\mathcal{F}}_t := \sigma(\xi_s; 0 \le s \le t), \quad 0 \le t \le T.$$

Then $\widehat{\mathbb{F}}$ coincides with the stock price filtration, and we have $\widehat{\mathcal{F}}_t \subseteq \mathcal{F}_t$, for all $t \in [0,T]$. We shall sometimes refer to an agent whose information set is $\widehat{\mathbb{F}}$ as a regular agent (or regular trader), to distinguish such an agent from an insider, who will have additional information as well as that provided by the data from ξ.

Let L denote an $\mathcal{F}$-measurable random variable. We model an *insider* as an agent who has knowledge at time zero of the value of L, where L will represent (typically, noisy) knowledge of an $\mathcal{F}_T$-measurable random variable, either ξ_T (equivalently, S_T) or the terminal Brownian motion B_T. In addition, the insider will not know the value of the risk premium λ, and her trading strategies will be adapted to the regular observation filtration $\widehat{\mathbb{F}}$ augmented by the inside information, represented by $\sigma(L)$, the sigma-field generated by L.

The uncertainty in the random variable λ will be modelled by assuming that its prior distribution conditional on $\widehat{\mathcal{F}}_0$ is Gaussian, according to the following standing assumption.

Assumption 1 The distribution of λ conditional on $\widehat{\mathcal{F}}_0$ is Gaussian, independent of B, with

$$E[\lambda|\widehat{\mathcal{F}}_0] = E[\lambda] = \lambda_0, \quad \mathrm{var}[\lambda|\widehat{\mathcal{F}}_0] = \mathrm{var}[\lambda] = \mathrm{v}_0,$$

for given constants λ_0 and $\mathrm{v}_0 \geq 0$.[1]

Of course, $\widehat{\mathcal{F}}_0$ is trivial, but we highlight its implicit presence in the prior for λ to contrast this case with the prior when additional information is available at time zero.

Given the filtration $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ and the random variable L, denote the enlarged filtration by $\mathbb{F}^L = (\mathcal{F}^L_t)_{0\leq t\leq T}$, given by

$$\mathcal{F}^L_t := \mathcal{F}_t \vee \sigma(L), \quad 0 \leq t \leq T. \tag{22}$$

Similarly, the insider's observation filtration will be denoted by $\widehat{\mathbb{F}}^L := (\widehat{\mathcal{F}}^L_t)_{0\leq t\leq T}$, given by

$$\widehat{\mathcal{F}}^L_t := \widehat{\mathcal{F}}_t \vee \sigma(L), \quad 0 \leq t \leq T, \tag{23}$$

that is the regular agent's observation filtration enlarged by $\sigma(L)$. We shall sometimes write $\widehat{\mathbb{F}}^0 \equiv \widehat{\mathbb{F}}$ to signify that the case without inside information may be considered as that with $L \equiv 0$.

3.1 The utility maximisation problems

The agents in this article will be restricted to using $\widehat{\mathbb{F}}^L$-adapted strategies (if there is inside information) or $\widehat{\mathbb{F}}$-adapted strategies if there is no inside information.

With the convention that $L \equiv 0$ corresponds to the situation with no inside information, denote by $\theta^L = (\theta^L_t)_{0\leq t\leq T}$ the agent's $\widehat{\mathbb{F}}^L$-adapted trading strategy, representing the proportion of wealth invested in the stock. Denote the corresponding wealth process by $X^L = (X^L_t)_{0\leq t\leq T}$. The set $\mathcal{A}(T;\widehat{\mathbb{F}}^L)$ of admissible strategies over $[0, T]$ is defined by the analogue of (6), as

$$\mathcal{A}(T;\widehat{\mathbb{F}}^L) := \left\{\theta^L : \theta^L \text{is } \widehat{\mathbb{F}}^L\text{-adapted}, \int_0^T (\theta^L_t)^2 \mathrm{d}t < \infty \text{ a.s., with } X^L_t \geq 0 \text{ a.s. } \forall t \in [0, T]\right\}.$$

Given initial capital $x > 0$, the insider's value function is u_L, defined by

$$u_L(x; T, \widehat{\mathbb{F}}^L) \equiv u_L(x) := \sup_{\theta^L\in\mathcal{A}(T;\widehat{\mathbb{F}}^L)} E[U_p(X^L_T)|\widehat{\mathcal{F}}^L_0], \tag{24}$$

with U_p given by (7), and we write $u_L(x) \equiv u_L(x; T; \widehat{\mathbb{F}}^L)$ when we do not need to emphasise the investment horizon and filtration under consideration.

[1] One way to choose λ_0, v_0 would be to use past data before time zero to obtain a point estimate of λ, and to use the distribution of the estimator as the prior, as in Monoyios [25].

Our goal is to compute the optimal strategy $\theta^{L,*}$ achieving the supremum in (24) as well as the value function u_L, for three particular cases of L, and hence three choices of filtration $\widehat{\mathbb{F}}^L$, as listed below.

Problem 0 (*Optimal investment with partial information)* Here, $L \equiv 0, \widehat{\mathbb{F}}^L = \widehat{\mathbb{F}}^0 \equiv \widehat{\mathbb{F}}$, so there is no inside information. This problem was considered by Rogers [28] among many others. We describe it here (for a wider range of risk aversion parameter than in [28]) to establish some notations and to facilitate subsequent comparisons between the maximal utilities and optimal trading strategies of the insider and the regular trader.

Problem 1 (*Investment with Brownian inside information and drift uncertainty)* Here, L is given by

$$L = L_B := aB_T + (1-a)\epsilon, \quad 0 < a \leq 1, \tag{25}$$

where B_T is the terminal value of the Brownian motion in (21) and ϵ is a random variable on $(\Omega, \mathcal{F})$ which is standard normal and independent of B, λ and ξ. The case $a = 1$ corresponds to exact information on B_T, while $a \in (0, 1)$ implies noisy information.

The relevant filtration is therefore $\widehat{\mathbb{F}}^{L_B}$. For brevity of notation we shall sometimes write $\widehat{\mathbb{F}}^{L_B} \equiv \widehat{\mathbb{F}}^B$, with a similar convention for other quantities, so the trading strategy will be denoted by θ^B, the wealth process by X^B, the value function by u_B, and so on.

Problem 2 (*Investment with stock price information and drift uncertainty)* Here, L is given by

$$L = L_S := a\xi_T + (1-a)\epsilon, \quad 0 < a \leq 1, \tag{26}$$

where ξ is defined in (21) and ϵ is again standard normal and independent of B, λ and ξ. The relevant filtration is therefore $\widehat{\mathbb{F}}^{L_S}$. For brevity of notation we shall sometimes write $\widehat{\mathbb{F}}^{L_S} \equiv \widehat{\mathbb{F}}^S$, with a similar convention for other quantities.

3.2 Outline of methodology and notation

In this section we establish generic notation and the procedure that pertains to the solution of Problems 0–2. We start with the model (20) written in the underlying filtration $\mathbb{F}$. If inside information is available, we form the enlarged filtration $\mathbb{F}^L = (\mathcal{F}_t^L)_{0\leq t\leq T}$ given by (22). We write the stock price dynamics with respect to $\mathbb{F}^L$, yielding an SDE of the form

$$\mathrm{d}S_t = \sigma S_t \left(\lambda_t^L \mathrm{d}t + \mathrm{d}B_t^L\right), \tag{27}$$

where B^L is an $\mathbb{F}^L$-Brownian motion and λ^L is an $\mathbb{F}^L$-adapted process, with B^L, λ^L given by

$$B_t^L = B_t - \int_0^t \nu_s^L \mathrm{d}s, \quad \lambda_t^L := \lambda + \nu_t^L, \quad 0 \leq t \leq T. \tag{28}$$

Here, $\nu^L = (\nu_t^L)_{0\leq t\leq T}$ is an $\mathbb{F}^L$-adapted process called the *information drift*. This is specified using classical enlargement of filtration results [17,24,30] and depends explicitly on the random variable L. When there is no inside information, $\nu^L \equiv 0$.

Since the value of λ is not known, the agent filters λ^L (and hence λ) given her observation filtration $\widehat{\mathbb{F}}^L$, and thus infers the conditional expectation

$$\widehat{\lambda}_t^L := E[\lambda_t^L | \widehat{\mathcal{F}}_t^L], \quad 0 \leq t \leq T.$$

The prior distribution of λ given $\widehat{\mathcal{F}}_0$ is given in Assumption 1. We shall show that this implies a Gaussian prior for λ^L given $\widehat{\mathcal{F}}^L_0$, denoted by

$$\mathrm{Law}(\lambda^L_0|\widehat{\mathcal{F}}^L_0) = \mathrm{N}(\widehat{\lambda}^L_0, V^L_0), \quad \text{independent of } B^L, \tag{29}$$

for some $\widehat{\mathcal{F}}^L_0$-measurable mean $\widehat{\lambda}^L_0$ and variance $V^L_0 \geq 0$, given in terms of λ_0, v_0, the parameters of the prior for λ given in Assumption 1.

The agent computes $\widehat{\lambda}^L = (\widehat{\lambda}^L_t)_{0\leq t\leq T}$ using a variant of the Kalman–Bucy filter, and this converts the partial information model (27) to a full information model written with respect to the stochastic basis $(\Omega, \widehat{\mathcal{F}}^L_T, \widehat{\mathbb{F}}^L, P)$, of the form

$$\mathrm{d}S_t = \sigma S_t(\widehat{\lambda}^L_t \mathrm{d}t + \mathrm{d}\widehat{B}^L_t), \tag{30}$$

where $\widehat{B}^L$ is an $\widehat{\mathbb{F}}^L$-Brownian motion and $\widehat{\lambda}^L$ turns out to be a Gaussian process. Once we have the completely observable model (30), we solve the utility maximisation problem (24) via duality.

4 Optimal investment with partial information

In this section we present the solution to Problem 0. Prior to filtering, the stock price follows (20) on the stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, P)$. The unknown risk premium λ is a signal process with prior distribution given in Assumption 1 and follows the trivial SDE

$$\mathrm{d}\lambda = 0.$$

The observation process is ξ, given in (21). The conditional mean and variance of λ given $\widehat{\mathbb{F}}$ are defined by

$$\widehat{\lambda}_t := E[\lambda|\widehat{\mathcal{F}}_t], \quad \mathrm{v}_t := E[(\lambda-\widehat{\lambda}_t)^2|\widehat{\mathcal{F}}_t] = E[(\lambda-\widehat{\lambda}_t)^2], \quad 0 \leq t \leq T.$$

Proposition 1 (Solution to Problem **0**) *For $0 < p < 1$, assume that $1 + q\mathrm{v}_0 T > 0$, where $p^{-1} + q^{-1} = 1$. The value function in (24) for $\widehat{\mathbb{F}}^L = \widehat{\mathbb{F}}$ is given by*

$$u_0(x) = \begin{cases} (x^p/p)C_0, & p < 1, \quad p \neq 0, \\ \log x + K_0/2, & \qquad\quad p = 0, \end{cases} \tag{31}$$

where C_0, K_0 are constants given by

$$C_0 = \left[(1+\mathrm{v}_0T)^p(1+q\mathrm{v}_0T)^{1-p}\right]^{-1/2} \exp\left[-\frac{q\lambda_0^2 T}{2(1+q\mathrm{v}_0T)}\right], \quad \frac{1}{p}+\frac{1}{q} = 1, \tag{32}$$

$$K_0 = (\lambda_0^2 + \mathrm{v}_0)T - \log(1+\mathrm{v}_0T).$$

The optimal $\widehat{\mathbb{F}}$-adapted trading strategy achieving the supremum in (24) is $\theta^{0,*}$, given by

$$\theta^{0,*}_t = \left\{\begin{array}{ll} \widehat{\lambda}_t[\sigma(1-p)(1+q\mathrm{v}_t(T-t))]^{-1}, \; p < 1, & p \neq 0, \\ \widehat{\lambda}_t/\sigma, & p = 0, \end{array}\right\} \quad 0 \leq t \leq T, \tag{33}$$

where $\widehat{\lambda}$, v are given by

$$\widehat{\lambda}_t = \frac{\lambda_0 + \mathrm{v}_0\xi_t}{1+\mathrm{v}_0 t}, \quad \mathrm{v}_t = \frac{\mathrm{v}_0}{1+\mathrm{v}_0 t}, \quad 0 \leq t \leq T, \tag{34}$$

and ξ is defined in (21).

Remark 2 The condition $1 + q\mathrm{v}_0 T > 0$ plays the same role as the condition (9) of Remark 1 and its violation leads to unbounded utility for any initial capital.

Proof By the Kalman–Bucy filter (Theorem 10.3 in Lipster and Shiryaev [23]) $\widehat{\lambda}$ satisfies

$$\widehat{\lambda}_t = \lambda_0 + \int_0^t \mathrm{v}_s \mathrm{d}\widehat{B}_s = \lambda_0 + \int_0^t \mathrm{v}_s(\mathrm{d}\xi_s - \widehat{\lambda}_s \mathrm{d}s), \quad 0 \le t \le T, \tag{35}$$

where $\widehat{B}$ is the innovations process, an $\widehat{\mathbb{F}}$-Brownian motion defined in terms of $\xi, \widehat{\lambda}$ by

$$\widehat{B}_t := \xi_t - \int_0^t \widehat{\lambda}_s \mathrm{d}s, \quad 0 \le t \le T, \tag{36}$$

and the conditional variance v satisfies the deterministic Riccati equation

$$\frac{\mathrm{d}\mathrm{v}_t}{\mathrm{d}t} = -\mathrm{v}_t^2, \tag{37}$$

with initial value v_0 and solution as given in (34), which together with (35) gives $\widehat{\lambda}$ as in (34).

From (36), the stock price SDE with respect to $\widehat{\mathbb{F}}$ is

$$\mathrm{d}S_t = \sigma S_t \mathrm{d}\xi_t = \sigma S_t(\widehat{\lambda}_t \mathrm{d}t + \mathrm{d}\widehat{B}_t). \tag{38}$$

Using these dynamics and noting that, due to (35), $E[\int_0^T \widehat{\lambda}_t^2 \mathrm{d}t | \widehat{\mathcal{F}}_0] < \infty$, a.s., we can apply Theorem 1 with $(\widehat{\mathbb{F}}, \widehat{B}, \widehat{\lambda}, \mathrm{v})$ in place of $(\mathbb{F}, B, \delta, \mathrm{w})$ and the theorem follows. $\square$

Naturally, in the limit $\mathrm{v}_0 \to 0$, the risk premium of the stock becomes the constant λ_0 and Proposition 1 gives the solution to the classical full information Merton optimal investment problem for a stock with constant risk premium λ_0 and volatility σ.

5 Optimal investment with inside information and drift uncertainty

In this section we give the solutions to Problems 1 and 2 and make comparisons between the maximal utilities and optimal trading strategies of the insider and the regular trader. The proofs are given in Sect. 6, using some enlargement of filtration and filtering results in the Appendix.

Define the modulated terminal time T_a by

$$T_a := T + \left(\frac{1-a}{a}\right)^2, \quad 0 < a \le 1, \tag{39}$$

which will appear frequently, and note that $\lim_{a\to 1} T_a = T$.

5.1 Brownian inside information

In Problem 1 the random variable L is L_B given in (25). For brevity of notation we shall often write $\widehat{\mathbb{F}}^{L_B} \equiv \widehat{\mathbb{F}}^B$, with a similar convention for the value function, $u_{L_B} \equiv u_B$, and other relevant quantities.

Theorem 2 (Solution to Problem 1) *For* $0 < p < 1$, *assume that*

$$\mathrm{v}_0 T < \frac{1-p}{p} + \frac{T}{T_a}. \tag{40}$$

The value function in (24) when L is given by (25) and $a \neq 1$ is

$$u_B(x) = \begin{cases} (x^p/p)C_B, & p < 1, \quad p \neq 0, \\ \log x + K_B/2, & \qquad\quad p = 0, \end{cases}$$

where C_B, K_B are $\widehat{\mathcal{F}}^B_0$-measurable random variables given by

$$C_B = \left[(1 + \mathrm{v}^B_0 T)^p (1 + q\mathrm{v}^B_0 T)^{1-p}\right]^{-1/2} \exp\left(-\frac{q(\widehat{\lambda}^B_0)^2 T}{2(1 + q\mathrm{v}^B_0 T)}\right), \quad \frac{1}{p} + \frac{1}{q} = 1,$$
$$K_B = \left((\widehat{\lambda}^B_0)^2 + \mathrm{v}^B_0\right) T - \log(1 + \mathrm{v}^B_0 T),$$

and where $\widehat{\lambda}^B_0$, v^B_0 are given by

$$\widehat{\lambda}^B_0 = \lambda_0 + \frac{L_B}{aT_a}, \quad \mathrm{v}^B_0 = \mathrm{v}_0 - \frac{1}{T_a}.$$

The optimal $\widehat{\mathbb{F}}^B$-adapted trading strategy achieving the supremum in (24) is $\theta^{B,*}$, given by

$$\theta^{B,*}_t = \begin{Bmatrix} \widehat{\lambda}^B_t[\sigma(1-p)(1 + q\mathrm{v}^B_t(T-t))]^{-1}, & p < 1, \quad p \neq 0, \\ \widehat{\lambda}^B_t/\sigma, & \qquad\quad p = 0, \end{Bmatrix} \quad 0 \le t \le T,$$

where $\widehat{\lambda}^B$, v^B are given by

$$\widehat{\lambda}^B_t = \frac{\widehat{\lambda}^B_0 + \mathrm{v}^B_0 \xi_t}{1 + \mathrm{v}^B_0 t}, \quad \mathrm{v}^B_t = \frac{\mathrm{v}^B_0}{1 + \mathrm{v}^B_0 t}, \quad 0 \le t \le T. \tag{41}$$

For $a = 1$ and $\mathrm{v}_0 > 0$ the above results still hold, while for $a = 1$ and $\mathrm{v}_0 = 0$ the value function is unbounded for $p \in [0, 1)$ and equal to zero for $p < 0$.

Remark 3 The condition (40), which is equivalent to $1 + q\mathrm{v}^B_0 T > 0$, once again plays the same role as the condition (9) of Remark 1 and its violation leads to unbounded utility for any initial capital.

Remark 4 Notice that v^B_0 can potentially be negative. As we will see in the proof of the theorem, v^B plays the same role as w in Theorem 1, and this is the reason for not assuming $\mathrm{w}_0 > 0$ at the start of Sect. 2.

The proof will be given in Sect. 6. Naturally, the value function and optimal strategy depend on L. There are clear similarities in the structure of the solution to this problem with that of Problem 0, with $(\widehat{\lambda}, \mathrm{v})$ replaced by $(\widehat{\lambda}^B_0, \mathrm{v}^B)$. It turns out that v^B is related to (but not identical to) the conditional variance of the insider's unknown risk premium λ^B given $\widehat{\mathbb{F}}^B$, as we shall see later.

For $a = 1$ and $\mathrm{v}_0 > 0$ the value function and optimal strategy are well defined, even though the insider has *exact* knowledge of the value of terminal Brownian motion B_T. This is to be contrasted with the results of Pikovsky and Karatzas [27], in which there is no drift parameter uncertainty (corresponding to $\mathrm{v}_0 \to 0$ here). In [27], exact knowledge of any kind at time zero leads to unbounded logarithmic utility. Here, in contrast, provided $\mathrm{v}_0 > 0$, the parameter uncertainty means that exact Brownian inside information does *not* lead to an explosion in utility (for $0 \le p < 1$) or to zero value function (the maximum possible expected utility) for $p < 0$.

But for $\mathrm{v}_0 \to 0$ we should, and do, recover results consistent with [27], with unbounded utility for $p \in [0, 1)$ and zero for $p < 0$. In this case, since $\xi_T = B_T + \lambda T$, then when the value of λ is known with certainty, exact Brownian inside information is equivalent to exact terminal stock price information, and hence Pikovsky and Karatzas [27] find that exact information of any kind leads to unbounded logarithmic utility.

In contrast, we shall see shortly that, with $\mathrm{v}_0 > 0$, only exact terminal *stock price* knowledge leads to unbounded logarithmic utility. The intuition here is that, with the initial and terminal stock prices known at time zero, one immediately obtains the best possible estimate of λ at time zero, and the parameter uncertainty is eliminated as much as is possible from the outset, leading to a utility explosion, and to the filtering algorithm being rendered redundant, as we shall see in the proof of Problem 2.

5.1.1 Additional utility of the insider

To quantify the additional utility of the insider with Brownian information relative to the regular trader, let us define the value of the additional information as π_B, given implicitly by

$$E[u_B(x)|\widehat{\mathcal{F}}_0] = u_0(x + \pi_B(x)).$$

In other words, π_B is the additional wealth needed by the regular trader to achieve, on average, the same expected utility as the insider who knows the value of L_B. To compute π_B, we need the distribution of $\widehat{\lambda}_0^B$ given $\widehat{\mathcal{F}}_0$. But $\widehat{\mathcal{F}}_0$ is in fact the trivial sigma-field, so we have $\mathrm{Law}(\widehat{\lambda}_0^B|\widehat{\mathcal{F}}_0) = \mathrm{Law}(\widehat{\lambda}_0^B)$, given by

$$\widehat{\lambda}_0^B \sim \mathrm{N}(\lambda_0, 1/T_a) = \mathrm{N}(\lambda_0, \mathrm{v}_0 - \mathrm{v}_0^B).$$

When $0 < p < 1$, we assume that $1 + q\mathrm{v}_0 T > 0$, so that u_0 is well-defined. Under this condition we also have $1 + q\mathrm{v}_0^B T > 0$, so that u_B is also well-defined. Hence π_B is well-defined and given by

$$\pi_B(x)/x = [(1 + \mathrm{v}_0 T)(1 + q\mathrm{v}_0^B T)]^{1/2}[(1 + \mathrm{v}_0^B T)(1 + q\mathrm{v}_0 T)]^{-1/2} - 1, \quad p < 1.$$

It can be verified that $\pi_B(x) > 0$, reflecting the insider's additional expected utility, and that (formally) $\pi_B(x) \to \infty$ (uniformly in x) for $a = 1$, $\mathrm{v}_0 = 0$, for all values of $p < 1$.

5.2 Stock price inside information

When the insider has terminal stock price inside information at time zero, given by the random variable $L \equiv L_S$ in (26), the solution to the optimal investment problem is as follows.

Theorem 3 (Solution to Problem 2) *For $0 < p < 1$, assume that*

$$\left[1 - \left(1 + \frac{1}{p}\right)\frac{T}{T_a}\right]\mathrm{v}_0 T < \frac{1-p}{p} + \frac{T}{T_a}. \tag{42}$$

The value function in (24) when L is given by (26) and $a \neq 1$ is

$$u_S(x) = \begin{cases} (x^p/p)C_S, & p < 1, \quad p \neq 0, \\ \log x + K_S/2, & p = 0, \end{cases}$$

where C_S, K_S are $\widehat{\mathcal{F}}_0^S$-measurable random variables given by

$$C_S = \left[(1 + \mathrm{v}_0^S T)^p (1 + q\mathrm{v}_0^S T)^{1-p}\right]^{-1/2} \exp\left(-\frac{q(\widehat{\lambda}_0^S)^2 T}{2(1 + q\mathrm{v}_0^S T)}\right), \quad \frac{1}{p} + \frac{1}{q} = 1,$$

$$K_S = \left((\widehat{\lambda}_0^S)^2 + \mathrm{v}_0^S\right)T - \log(1 + \mathrm{v}_0^S T),$$

and where $\widehat{\lambda}_0^S$, v_0^S *are given by*

$$\widehat{\lambda}_0^S = \frac{\lambda_0(1 - T/T_a) + (1 + \mathrm{v}_0 T)(L_S/(aT_a))}{1 + \mathrm{v}_0 T(T/T_a)}, \quad \mathrm{v}_0^S = \frac{(1 - T/T_a)^2 \mathrm{v}_0}{1 + \mathrm{v}_0 T(T/T_a)} - \frac{1}{T_a}.$$

The optimal $\widehat{\mathbb{F}}^S$*-adapted trading strategy achieving the supremum in (24) is* $\theta^{S,*}$*, given by*

$$\theta_t^{S,*} = \left\{ \begin{array}{ll} \widehat{\lambda}_t^S[\sigma(1-p)(1 + q\mathrm{v}_t^S(T-t))]^{-1}, \ p < 1, & p \neq 0, \\ \widehat{\lambda}_t^S/\sigma, & p = 0, \end{array} \right\} \quad 0 \le t \le T,$$

where $\widehat{\lambda}^S$, v^S *are given by*

$$\widehat{\lambda}_t^S = \frac{\widehat{\lambda}_0^S + \mathrm{v}_0^S \xi_t}{1 + \mathrm{v}_0^S t}, \quad \mathrm{v}_t^S = \frac{\mathrm{v}_0^S}{1 + \mathrm{v}_0^S t}, \quad 0 \le t \le T. \tag{43}$$

For $a = 1$ *the value function is unbounded for* $p \in [0, 1)$ *and equal to zero for* $p < 0$.

Remark 5 The condition (42), which is equivalent to $1 + q\mathrm{v}_0^B T > 0$, again plays the same role as the condition (9) of Remark 1 and its violation leads to unbounded utility for any initial capital.

The proof will be given in Sect. 6. Again, we observe the similarity in the structure of the solution to this problem with that of Problems 0 and 1, with $(\widehat{\lambda}^S, \mathrm{v}^S)$ in place of $(\widehat{\lambda}, \mathrm{v})$ or $(\widehat{\lambda}^B, \mathrm{v}^B)$.

5.2.1 Additional utility of the insider

We can quantify the additional utility of the insider with terminal stock price information, as we did for the insider with Brownian information. The distribution of $\widehat{\lambda}_0^S$ in this case is given by

$$\widehat{\lambda}_0^S \sim \mathrm{N}\left(\lambda_0, \frac{(1 + \mathrm{v}_0 T)^2}{T_a(1 + \mathrm{v}_0 T(T/T_a))}\right) = \mathrm{N}(\lambda_0, \mathrm{v}_0 - \mathrm{v}_0^S).$$

When $0 < p < 1$, we assume that $1 + q\mathrm{v}_0 T > 0$, so that u_0 is well-defined. Under this condition we also have $1 + q\mathrm{v}_0^S T > 0$, so that u_S is also well-defined. Hence the value of the additional information, π_S, is well-defined and given by

$$\pi_S(x)/x = [(1 + \mathrm{v}_0 T)(1 + q\mathrm{v}_0^S T)]^{1/2}[(1 + \mathrm{v}_0^S T)(1 + q\mathrm{v}_0 T)]^{-1/2} - 1, \quad p < 1.$$

Once again, it can be verified that $\pi_S(x) > 0$, reflecting the insider's additional expected utility, and that (formally) $\pi_S(x) \to \infty$ (uniformly in x) for $a \to 1$, for all values of $p < 1$.

More pertinently, if we compare the values of stock price and Brownian information, we find that for $a \neq 1$,

$$\pi_S(x) > \pi_B(x), \quad \text{for all } x,$$

since we have $\mathrm{v}_0^S < \mathrm{v}_0^B$. In other words, *stock price information is more valuable than Brownian information when the drift of the stock is unknown*.

Remark 6 Although stock price information is more valuable than Brownian information, the $\mathbb{F}^L$-information drifts in both cases are indistinguishable (see Eqs. 77 and 74 in the Appendix). The salient point is that under $\mathbb{F}^L$, since λ is a known parameter, advance knowledge of $\xi_T = B_T + \lambda T$ is indistinguishable from advance knowledge of B_T.

The two cases become distinct as soon as we move to the filtration $\widehat{\mathbb{F}}^L$, since the additional stock price information contributes to a better estimate of the unknown drift. This is why our results differ from Pikovsky and Karatzas [27] and why $\pi_S > \pi_B$.

This distinction becomes extreme when $a = 1$ and hence $L_S = \xi_T$. In this case the dynamics of S with respect to $\mathbb{F}^{S_T}$ become (see (77) in the Appendix)

$$\frac{\mathrm{d}S_t}{\sigma S_t} = \left(\mathrm{d}B_t^{S_T} + \frac{\xi_T - \xi_t}{T-t}\mathrm{d}t\right).$$

All terms except for B^{S_T} are manifestly adapted to $\widehat{\mathbb{F}}^{S_T}$, but not to $\widehat{\mathbb{F}}^{B_T}$. This implies that the $\mathbb{F}^{S_T}$-Brownian motion B^{S_T} is also a Brownian motion under the observation filtration $\widehat{\mathbb{F}}^{S_T}$, and the filtering procedure is *redundant*. Knowing the value of S_T at time zero immediately gives the insider the best estimate of λ, but this is not so given knowledge of B_T at time zero. This is because the best estimate of λ from observations of $\xi_t = \lambda t + B_t$ over $[0, T]$ is $\bar{\lambda}(T) := \xi_T/T$. This is the underlying reason for the well-known difficulty of estimating the mean return of a log-Brownian stock, as discussed in Rogers [28], for example.

This ultimately leads to an explosion in expected utility for the case with perfect stock price (but not Brownian) information. In contrast, in the absence of parameter uncertainty, Pikovsky and Karatzas [27] obtain unbounded utility with any form of perfect information.

5.3 Comparison of trading strategies of regular trader and insider

One can also carry out a comparison of how aggressive the insider is in taking a stock position relative to the regular trader. One way to make a meaningful comparison of $\widehat{\mathbb{F}}^L$-adapted strategies with $\widehat{\mathbb{F}}$-adapted strategies, is to condition the insider's optimal portfolio θ_t^L on $\widehat{\mathcal{F}}_t$.

For $0 < p < 1$ we always assume that $1+q\mathrm{v}_0T > 0$, and this guarantees that $1+q\mathrm{v}_0^LT > 0$, so that R^L, defined by

$$R_t^L := \frac{E[\theta_t^{L,*}|\widehat{\mathcal{F}}_t]}{\theta_t^{0,*}}, \quad L \in \{B, S\}, \quad 0 \le t \le T,$$

exists.

The effect of conditioning $\theta^{L,*}$ with respect to $\widehat{\mathbb{F}}$ isolates the multiplier $(1+q\mathrm{v}_t^L(T-t))^{-1}$. This measures the extent to which the insider's stock position is magnified or reduced in response to her estimate $\widehat{\lambda}^L$ of the risk premium.

Proposition 2 *For $p = 0$, $R_B = R_S = 1$, $\widehat{\mathbb{F}}$-a.s., while for $p \neq 0$, we have*

$$R_t^B = \frac{1+q\mathrm{v}_t(T-t)}{1+q\mathrm{v}_t^B(T-t)}, \quad R_t^S = \frac{1+q\mathrm{v}_t(T-t)}{1+q\mathrm{v}_t^S(T-t)}, \quad 0 \le t \le T.$$

Note that $\mathrm{v}_t^S < \mathrm{v}_t^B < \mathrm{v}_t$ for $t \in [0, T]$. So for $p \neq 0$ we have the ordering (recall $q < 0$ when $p \in (0, 1)$)

$$qR_t^S > qR_t^B > q, \quad 0 \le t \le T. \tag{44}$$

The insider takes a more aggressive holding in the stock than the regular trader when $p < 0$ (corresponding to relative risk aversion larger than 1). This effect is more marked when the inside information is on the stock price rather than the Brownian motion, because in this case the insider derives greater confidence in his estimate of the stock drift than with Brownian knowledge. This result seems intuitively plausible.

When $p = 0$ this effect is nullified. Interestingly, when $p \in (0, 1)$, this effect is reversed. In other words, when risk aversion becomes sufficiently small (and, as we have seen, rather complicated conditions on the prior variance are needed for a well-posed problem) an agent with progressively less uncertainty in her estimate of the stock's risk premium, nevertheless takes a *less* aggressive holding in the stock. This unexpected result has (to the best of our knowledge) not been previously observed.

At first glance this result is paradoxical, since it implies that agents prefer stocks with greater as opposed to less uncertainty in the drift. The reason for this apparent paradox lies in a combination of (i) drift uncertainty and (ii) the behaviour of $U_p(x)$ for $p \in (0, 1)$ as $x \to 0$ and $x \to \infty$. In this case, because $U_p(x)$ is *unbounded* as $x \to \infty$ (but is finite if $p < 0$) an agent may be tempted to try and achieve infinite utility by gambling in an attempt to drive wealth to infinity, and this gamble is only *finitely* penalised, because $U_p(x) \to 0$ as $x \to 0$ (but this limit is $-\infty$ when $p < 0$, corresponding to an infinite penalty if a gamble goes awry).

This possibility leads less informed agents towards aggressive investment in the face of drift uncertainty. In effect, the less informed agent takes a gamble, encouraged to do so by the chance of infinite utility, with only a finite penalty if this gamble fails, and takes a more aggresive holding in the stock than the insider. But this does not result in greater expected utility (and so does not contradict our earlier results on the ordering of the value of inside information). The uninformed agent's gamble will indeed fail on some occasions, and the insider is privy to this knowledge in cases when the uninformed agent is not. It is precisely this knowledge which leads the insider to hold back from being aggressive. Note that this effect did not appear in Rogers [28] as the case $p \in (0, 1)$ was not considered in that paper.

Proof (Proof of Proposition 2)

For any choice of L, due to the form of $\theta^{L,*}$, to prove the proposition it is sufficient to show that $E[\widehat{\lambda}_t^L|\widehat{\mathcal{F}}_t] = \widehat{\lambda}_t$, $0 \le t \le T$. Using the definition of $\widehat{\lambda}^L$ and the tower property we have

$$E[\widehat{\lambda}_t^L|\widehat{\mathcal{F}}_t] = E\left[E[\lambda_t^L|\widehat{\mathcal{F}}_t^L] \middle| \widehat{\mathcal{F}}_t \right] = E[\lambda_t^L|\widehat{\mathcal{F}}_t] = E[\lambda + \nu_t^L|\widehat{\mathcal{F}}_t] = E[E[\lambda + \nu_t^L|\mathcal{F}_t]|\widehat{\mathcal{F}}_t]$$

$$= E[\lambda + E[\nu_t^L|\mathcal{F}_t]|\widehat{\mathcal{F}}_t] = \widehat{\lambda}_t + E[E[\nu_t^L|\mathcal{F}_t]|\widehat{\mathcal{F}}_t].$$

So to finish the proof we need to establish that $E[\nu_t^L|\mathcal{F}_t] = 0$.

From (27) and (28), ξ satisfies, with respect to $\mathbb{F}^L$, the SDE

$$\mathrm{d}\xi_t = (\lambda + \nu_t^L)\mathrm{d}t + \mathrm{d}B_t^L. \tag{45}$$

Therefore, computing $E[\nu_t^L|\mathcal{F}_t]$ is tantamount to projecting the SDE (45) onto $\mathbb{F}$. From standard arguments in non-linear filtering this projection is given by

$$\xi_t = \lambda t + \int_0^t E[\nu_s^L|\mathcal{F}_s]\mathrm{d}s + \beta_t, \quad 0 \le t \le T,$$

for some $\mathbb{F}$-Brownian motion β. But we already have $\xi_t = \lambda t + B_t$, so $\beta = B$ and hence $E[\nu_t^L|\mathcal{F}_t] = 0$ for all $t \in [0, T]$. □

6 Proofs of the main theorems

Proof (Proof of Theorem 2)

Using Corollary 1 in the Appendix we write the stock price SDE (20) with respect to the enlarged filtration $\mathbb{F}^B$ to give

$$\mathrm{d}S_t = \sigma S_t(\lambda_t^B \mathrm{d}t + \mathrm{d}B_t^B), \tag{46}$$

where the $\mathbb{F}^B$-adapted risk premium $\lambda^B \equiv \lambda^{L_B}$ satisfies (see the Appendix)

$$\mathrm{d}\lambda_t^B = -\frac{1}{T_a - t}\mathrm{d}B_t^B, \quad \lambda_0^B = \lambda + \frac{L_B}{aT_a}. \tag{47}$$

With respect to $\mathbb{F}^B$ the returns process ξ in (21) satisfies

$$\mathrm{d}\xi_t = \lambda_t^B \mathrm{d}t + \mathrm{d}B_t^B. \tag{48}$$

Consider (47) and (48) as a linear signal and observation system in a filtering framework.

To apply our variant of the Kalman–Bucy filter, Proposition 3 in the Appendix, we need the prior distribution of λ_0^B given $\widehat{\mathcal{F}}_0^B$. Using Assumption 1 and that B_T and ϵ in (25) are independent of each other and of λ, the distribution of λ given $\widehat{\mathcal{F}}_0^B$ is the same as its distribution given $\widehat{\mathcal{F}}_0$. Using the initial condition in (47), the probability law for λ_0^B given $\widehat{\mathcal{F}}_0^B$ is therefore

$$\mathrm{Law}(\lambda_0^B|\widehat{\mathcal{F}}_0^B) = \mathrm{Law}\left(\lambda + \frac{L_B}{aT_a}\middle|\widehat{\mathcal{F}}_0^B\right) = \mathrm{N}\left(\lambda_0 + \frac{L_B}{aT_a}, \mathrm{v}_0\right), \quad \text{independent of} B^B.$$

Of course, since L_B is $\widehat{\mathcal{F}}_0^B$-measurable, it acts as a constant in the above computation, and since λ is independent of B and L_B, it is independent of B^B, and so λ_0^B is also independent of B^B.

1. Assume $a \neq 1$. Proposition 3 with $L = L_B$ gives

$$\mathrm{d}\widehat{\lambda}_t^B = \left(V_t^B - \frac{1}{T_a - t}\right)\mathrm{d}\widehat{B}_t^B, \quad \widehat{\lambda}_0^B = \lambda_0 + \frac{L_B}{aT_a}, \tag{49}$$

   where we write $\widehat{\lambda}^{L_B} \equiv \widehat{\lambda}^B$, $V^{L_B} \equiv V^B$, $\widehat{B}^B \equiv \widehat{B}^{L_B}$ for brevity of notation, and the conditional variance satisfies (86) with $L = L_B$. Define $\mathrm{v}^B \equiv \mathrm{v}^{L_B}$ by

$$\mathrm{v}_t^B := V_t^B - \frac{1}{T_a - t}, \quad 0 \leq t \leq T.$$

   Due to (86) v^B satisfies a Riccati equation of the same form as (3):

$$\frac{\mathrm{d}\mathrm{v}_t^B}{\mathrm{d}t} = -\left(\mathrm{v}_t^B\right)^2, \quad \mathrm{v}_0^B = \mathrm{v}_0 - \frac{1}{T_a}, \tag{50}$$

   which has solution as given in (41). We also have $V^B \geq 0$ for all $t \in [0, T]$, which follows from direct calculation and $\mathrm{v}_0 > 0$.
   The solution to the SDE (49) for $\widehat{\lambda}^B \equiv \widehat{\lambda}^{L_B}$ is then

$$\widehat{\lambda}_t^B = \widehat{\lambda}_0^B + \int_0^t \mathrm{v}_s^B \mathrm{d}\widehat{B}_s^B, \quad 0 \leq t \leq T, \tag{51}$$

   or, in terms of ξ, as given in Eq. 41 of the theorem, where we have used (84).

The stock price SDE $\mathrm{d}S_t = \sigma S_t \mathrm{d}\xi_t$ with respect to $\widehat{\mathbb{F}}^B$ becomes

$$\mathrm{d}S_t = \sigma S_t(\widehat{\lambda}_t^B \mathrm{d}t + \mathrm{d}\widehat{B}_t^B), \tag{52}$$

and the insider is investing in this stock using an $\widehat{\mathbb{F}}^B$-adapted strategy θ^B.
From (50) and (51) we have $E[\int_0^T (\widehat{\lambda}_t^B)^2 \mathrm{d}t | \widehat{\mathcal{F}}_0^L] < \infty$, a.s. From (50–52) we see that we have recovered a model of the same form as that of Sect. 2, and the solution to the optimal investment problem for $a < 1$ then follows from Theorem 1 with $(\widehat{\mathbb{F}}^B, \widehat{B}^B, \widehat{\lambda}^B, \mathrm{v}^B)$ in place of $(\mathbb{F}, B, \delta, \mathrm{w})$, and the condition (40) on v_0 given in the theorem for $0 < p < 1$ ensures that the solution is well defined.

2. Suppose $a = 1$ and $\mathrm{v}_0 > 0$. Then $L = B_T$ and $T_a = T$, and the initial value of the effective variance v^{B_T} is given by

$$\mathrm{v}_0^{B_T} = \mathrm{v}_0 - \frac{1}{T}.$$

We can only apply the filtering algorithm up to some terminal time $T^* < T$, in accordance with Remark 7. Then the arguments leading to (50–52) are applicable over the time interval $[0, T)$.
To establish the results for $t = T$, define

$$\widehat{\lambda}_T^{B_T} := \lim_{t\to T} E[\lambda_t^{B_T} | \widehat{\mathcal{F}}_t^{B_T}] = \lim_{t\to T} \widehat{\lambda}_t^{B_T} = \lim_{t\to T} \frac{\widehat{\lambda}_0^{B_T} + \mathrm{v}_0^{B_T}\xi_t}{1 + \mathrm{v}_0^{B_T} t} = \frac{\widehat{\lambda}_0^{B_T} + \mathrm{v}_0^{B_T}\xi_T}{1 + \mathrm{v}_0^{B_T} T}, \tag{53}$$

which is well defined since ξ_t is continuous on $[0, T]$, and $1 + \mathrm{v}_0^{B_T} T = \mathrm{v}_0 T \neq 0$. Since $\widehat{B}_t^{B_T}$ is defined only for $t \in [0, T)$ we need to define $\widehat{B}_T^{B_T}$. To this end, define

$$\widehat{B}_T^{B_T} := \lim_{t\to T} \widehat{B}_t^{B_T} = \lim_{t\to T} \left[\xi_t - \int_0^t \widehat{\lambda}_s^{B_T} \mathrm{d}s \right],$$

which is well defined due to continuity of ξ_t and $\widehat{\lambda}_t^B$ on $[0, T]$ (the latter due to (53)). From this it follows that the relation

$$\xi_t = \int_0^t \widehat{\lambda}_s^{B_T} \mathrm{d}s + \widehat{B}_t^{B_T}, \quad 0 \le t \le T,$$

as well as (51), both hold over $[0, T]$. This, along with the Itô formula yields that $\widehat{\lambda}^{B_T}$ is given by

$$\widehat{\lambda}_t^{B_T} = \lambda_0 + \frac{B_T}{T} + \int_0^t \mathrm{v}_s^{B_T} \mathrm{d}\widehat{B}_s^{B_T} = \frac{\widehat{\lambda}^{B_T} + \mathrm{v}_0^{B_T}\xi_t}{1 + \mathrm{v}_0^{B_T} t}, \quad 0 \le t \le T,$$

with v^{B_T} given by

$$\mathrm{v}_t^{B_T} = \frac{\mathrm{v}_0^{B_T}}{1 + \mathrm{v}_0^{B_T} t}, \quad 0 \le t \le T,$$

for $t \in [0, T]$. This establishes that (50–52) do in fact hold over the entire time interval $[0, T]$ and the rest of the proof of the optimal investment result goes through unaltered.

3. For the case $a = 1$ and $v_0 = 0$, we use a similar argument to the one in Pikovsky and Karatzas [27], and consider an investment problem over a sub-interval $[0, T^*]$, for some $T^* < T$. Indeed, note that when $v_0 = 0$ there is no uncertainty in the risk premium λ and we are back in the scenario considered by [27]. The risk premium λ is equal to the constant λ_0 and the returns process ξ in (21) satisfies

$$\xi_t = \lambda_0 t + B_t, \quad 0 \le t \le T.$$

This implies that $\widehat{\mathbb{F}} = \mathbb{F}$ and also $\widehat{\mathbb{F}}^{B_T} = \mathbb{F}^{B_T}$. The decomposition of the Brownian motion B under $\mathbb{F}^{B_T}$ is given by the Brownian bridge relation (75). Then, with respect to $\mathbb{F}^{B_T}$ the stock price dynamics are

$$\mathrm{d}S_t = \sigma S_t(\lambda_t^{B_T}\mathrm{d}t + \mathrm{d}B_t^{B_T}),$$

where the $\mathbb{F}^{B_T}$-adapted risk premium process λ^{B_T} is given by

$$\lambda_t^{B_T} = \lambda_0 + \frac{B_T - B_t}{T - t}, \quad 0 \le t \le T,$$

and we have used the fact that $\lambda = \lambda_0$ for $v_0 = 0$. For any $T^* < T$ we have

$$E\left[\int_0^{T^*} (\lambda_t^{B_T})^2 \mathrm{d}t \,\middle|\, \mathcal{F}_0^{B_T}\right] < \infty, \quad \text{a.s.}$$

Note also that we have the dynamics (with respect to $\mathbb{F}^{B_T}$) for λ^{B_T}:

$$\mathrm{d}\lambda_t^{B_T} = -\frac{1}{T-t}\mathrm{d}B_t^{B_T}.$$

In this case define the effective variance v^{B_T} by

$$v_t^{B_T} := -\frac{1}{T-t}, \quad 0 \le t \le T^* < T,$$

and we then have

$$v_t^{B_T} = \frac{v_0^{B_T}}{1 + v_0^{B_T} t}, \quad \lambda_t^{B_T} = \lambda_0^{B_T} + \int_0^t v_s^{B_T}\mathrm{d}B_s^{B_T}, \quad 0 \le t \le T^* < T.$$

So we have an optimal investment model of the form in Sect. 2 over the sub-interval $[0, T^*]$, for any $T^* < T$.

We introduce a sequence of auxiliary optimal investment problems over a sub-interval $[0, T^n]$, for $T^n := T - 1/n$ and $n \in \mathbb{N}$. Define the value function

$$u_{B_T}(x; T^n, \mathbb{F}^{B_T}) := \sup_{\theta^{B_T} \in \mathcal{A}(T^n; \mathbb{F}^{B_T})} E[U_p(X_{T^n}^{B_T})|\mathcal{F}_0^{B_T}]. \tag{54}$$

which is the maximum expected utility over the class $\mathcal{A}(T^n; \mathbb{F}^{B_T})$ of $\mathbb{F}^{B_T}$-adapted portfolios on the subinterval $[0, T^n]$. The maximum utility for the problem (54) follows from Theorem 1 with $(\mathbb{F}^{B_T}, B^{B_T}, \lambda^{B_T}, v^{B_T})$ in place of $(\mathbb{F}, B, \delta, w)$, and with T^n in place of T, giving

$$u_{B_T}(x; T^n, \mathbb{F}^{B_T}) = \begin{cases} (x^p/p)C_{B_T}, & p < 1, \quad p \ne 0, \\ \log x + K_{B_T}/2, & p = 0, \end{cases} \tag{55}$$

where C_{B_T}, K_{B_T} are $\mathcal{F}_0^{B_T}$-measurable random variables given by

$$C_{B_T} = \left[(1 - T^n/T)^p (1 - qT^n/T)^{1-p}\right]^{-1/2} \exp\left(-\frac{q(\lambda_0^{B_T})^2 T^n}{2(1 - qT^n/T)}\right), \quad \frac{1}{p} + \frac{1}{q} = 1,$$
$$K_{B_T} = \left((\lambda_0^{B_T})^2 - 1/T\right) T^n - \log(1 - T^n/T),$$

and where we have used the explicit form for $\mathrm{v}_0^{B_T}$. The optimal $\mathbb{F}^{B_T}$-adapted trading strategy achieving the supremum in (54) is $\theta^{B_T,*}$, given by

$$\theta_t^{B_T,*} = \left\{ \begin{array}{ll} \lambda_t^{B_T}[\sigma(1-p)(1+q\mathrm{v}_t^{B_T}(T^n - t))]^{-1}, \; p < 1, & p \neq 0, \\ \lambda_t^{B_T}/\sigma, & p = 0, \end{array} \right\} \quad 0 \leq t \leq T^n.$$

To conclude that the value function of the insider is as claimed, consider the following strategy in $\mathcal{A}(T; \mathbb{F}^B)$. Define a portfolio $\theta^{B_T,n}$, for any $T^n < T$, by

$$\theta_t^{B_T,n} := \theta_t^{B_T,*} \mathbb{1}_{\{t \leq T^n\}}, \quad 0 \leq t \leq T, \tag{56}$$

which corresponds to the strategy of investing optimally up to T^n then moving all the investments to the risk-free asset and keeping them there until time T. This strategy is admissible for the original problem over the interval $[0, T]$, so $u_{B_T}(x; T^n, \mathbb{F}^{B_T}) \leq u_{B_T}(x; T, \mathbb{F}^{B_T})$, and taking limits in (55) yields

$$u_{B_T}(x; T, \mathbb{F}^{B_T}) \geq \lim_{n \to +\infty} u_{B_T}(x; T^n, \mathbb{F}^{B_T}) = \begin{cases} \infty, & p \in [0, 1), \\ 0, & p < 0, \end{cases}$$

Since $u_{B_T}(x) \leq +\infty$ for $p \in (0, 1)$ and $u_{B_T}(x) \leq 0$ for $p \in (-\infty, 0)$, the result follows. □

Proof (Proof of Theorem 3) Using Corollary 2 in the Appendix we write the stock price SDE (20) with respect to $\mathbb{F}^S$ to give

$$\mathrm{d}S_t = \sigma S_t(\lambda_t^S \mathrm{d}t + \mathrm{d}B_t^S), \tag{57}$$

where the $\mathbb{F}^S$-adapted risk premium $\lambda^S \equiv \lambda^{L_S}$ satisfies

$$\mathrm{d}\lambda_t^S = -\frac{1}{T_a - t}\mathrm{d}B_t^S, \quad \lambda_0^S = \lambda\left(1 - \frac{T}{T_a}\right) + \frac{L_S}{aT_a}. \tag{58}$$

With respect to $\mathbb{F}^S$ the returns process ξ in (21) satisfies

$$\mathrm{d}\xi_t = \lambda_t^S \mathrm{d}t + \mathrm{d}B_t^S. \tag{59}$$

Consider (58) and (59) as a linear signal and observation system in a filtering framework.

To apply the filtering algorithm, Proposition 3 in the Appendix, we need the prior distribution of λ_0^S given $\widehat{\mathcal{F}}_0^S$. The prior distribution of λ is given in Assumption 1. However, the distribution of λ conditional on $\widehat{\mathcal{F}}_0^S$ is altered from that in Assumption 1 because the inside information (related as it is to S_T) contributes to the estimation of λ. We have the following lemma.

Lemma 1 *Conditional on* $\widehat{\mathcal{F}}_0^S$, λ *is Gaussian, with*

$$E[\lambda|\widehat{\mathcal{F}}_0^S] = \frac{\lambda_0 + \mathrm{v}_0 T(L_S/aT_a)}{1 + \mathrm{v}_0 T(T/T_a)}, \quad \mathrm{var}[\lambda|\widehat{\mathcal{F}}_0^S] = \frac{\mathrm{v}_0}{1 + \mathrm{v}_0 T(T/T_a)}. \tag{60}$$

□

Proof For two independent Gaussian random variables X and Y distributed according to $\mathrm{N}(\mu_X, \sigma_X^2)$ and $\mathrm{N}(\mu_Y, \sigma_Y^2)$, respectively,

$$\mathrm{Law}[X|X+Y] = \mathrm{N}\left(\mu_X + \frac{\sigma_X^2}{\sigma_X^2 + \sigma_Y^2}(X + Y - \mu_X - \mu_Y), \frac{\sigma_X^2 \sigma_Y^2}{\sigma_X^2 + \sigma_Y^2}\right). \tag{61}$$

Using $\xi_T = B_T + \lambda T$ we have

$$\begin{aligned} E[\lambda|\widehat{\mathcal{F}}_0^S] &= E[\lambda|L_S] = E[\lambda|a(B_T + \lambda T) + (1-a)\epsilon] \\ &= \frac{1}{aT} E[a\lambda T | a\lambda T + aB_T + (1-a)\epsilon]. \end{aligned}$$

Applying (61) with $X = a\lambda T \sim \mathrm{N}(a\lambda_0 T, a^2 \mathrm{v}_0 T^2)$, $Y = aB_T + (1-a)\epsilon \sim \mathrm{N}(0, a^2 T_a)$, gives (60). □

Using this lemma and the formula for λ_0^S in (58), we find that, conditional on $\widehat{\mathcal{F}}_0^S$, λ_0^S is Gaussian according to $\mathrm{Law}(\lambda_0^S|\widehat{\mathcal{F}}_0^S) = \mathrm{N}(\widehat{\lambda}_0^S, V_0^S)$, with

$$\widehat{\lambda}_0^S = \frac{\lambda_0(1 - T/T_a) + (1 + \mathrm{v}_0 T)(L_S/(aT_a))}{1 + \mathrm{v}_0 T(T/T_a)}, \quad V_0^S = \frac{(1 - (T/T_a))^2 \mathrm{v}_0}{1 + \mathrm{v}_0 T(T/T_a)}, \tag{62}$$

which defines the prior distribution of the signal process λ^S.

1. Assume $a \neq 1$. Proposition 3 with $L = L_S$ gives

$$\mathrm{d}\widehat{\lambda}_t^S = \left(V_t^S - \frac{1}{T_a - t}\right)\mathrm{d}\widehat{B}_t^S, \quad \widehat{\lambda}_0^S = \frac{\lambda_0(1 - T/T_a) + (1 + \mathrm{v}_0 T)(L_S/(aT_a))}{1 + \mathrm{v}_0 T(T/T_a)}, \tag{63}$$

where we write $\widehat{\lambda}^{L_S} \equiv \widehat{\lambda}^S$, $V^{L_S} \equiv V^S$, $\widehat{B}^S \equiv \widehat{B}^{L_S}$ for brevity of notation, and the conditional variance satisfies (86) with $L = L_S$.
Define $\mathrm{v}^S \equiv \mathrm{v}^{L_S}$ by

$$\mathrm{v}_t^S := V_t^S - \frac{1}{T_a - t}, \quad 0 \le t \le T.$$

Due to (86) v^S satisfies a Riccati equation of the same form as (3):

$$\frac{\mathrm{d}\mathrm{v}_t^S}{\mathrm{d}t} = -\left(\mathrm{v}_t^S\right)^2, \quad \mathrm{v}_0^S = \frac{(1 - (T/T_a))^2 \mathrm{v}_0}{1 + \mathrm{v}_0 T(T/T_a)} - \frac{1}{T_a}, \tag{64}$$

which has solution as given in (43). It is easy to see that we have $V^S \ge 0$ for all $t \in [0, T]$. The solution to the SDE (63) for $\widehat{\lambda}^S \equiv \widehat{\lambda}^{L_S}$ is then

$$\widehat{\lambda}_t^S = \widehat{\lambda}_0^S + \int_0^t \mathrm{v}_s^S \mathrm{d}\widehat{B}_s^S, \quad 0 \le t \le T, \tag{65}$$

or, in terms of ξ, as given in Eq. 43 of the theorem, where we have used (84).

The stock price SDE $\mathrm{d}S_t = \sigma S_t \mathrm{d}\xi_t$ with respect to $\widehat{\mathbb{F}}^S$ becomes

$$\mathrm{d}S_t = \sigma S_t(\widehat{\lambda}_t^S \mathrm{d}t + \mathrm{d}\widehat{B}_t^S), \tag{66}$$

and the insider is investing in this stock using an $\widehat{\mathbb{F}}^S$-adapted strategy θ^S.
From (64) and (65) we have $E[\int_0^T(\widehat{\lambda}_t^S)^2 dt|\widehat{\mathcal{F}}_0^S] < \infty$ a.s. From (64–66) we see that we have recovered a model of the same form as that of Sect. 2, and the solution to the optimal investment problem for $a < 1$ then follows from Theorem 1 with $(\widehat{\mathbb{F}}^S, \widehat{B}^S, \widehat{\lambda}^S, \mathrm{v}^S)$ in place of $(\mathbb{F}, B, \delta, \mathrm{w})$, and the condition (42) on v_0 given in the theorem for $0 < p < 1$ ensures that the solution is well defined.

2. Suppose $a = 1$. Then from Corollary 2, the stock price SDE with respect to the enlarged filtration $\mathbb{F}^{S_T}$ is

$$\mathrm{d}S_t = \sigma S_t(\lambda_t^{S_T} \mathrm{d}t + \mathrm{d}B_t^{S_T}), \tag{67}$$

where λ^{S_T} is the $\mathbb{F}^{S_T}$-adapted process given by

$$\lambda_t^{S_T} = \frac{\xi_T - \xi_t}{T - t}, \quad 0 \le t \le T.$$

But this is also $\widehat{\mathbb{F}}^{S_T}$-adapted, so it immediately follows that

$$\widehat{\lambda}_t^{S_T} = \lambda_t^{S_T} = \frac{\xi_T - \xi_t}{T - t}, \quad 0 \le t \le T.$$

Since $\mathrm{d}\xi_t = \mathrm{d}S_t/(\sigma S_t)$, from (67) its dynamics with respect to $\mathbb{F}^{S_T}$ are

$$\mathrm{d}\xi_t = \lambda_t^{S_T} \mathrm{d}t + \mathrm{d}B_t^{S_T}, \tag{68}$$

which implies that B^{S_T} is $\widehat{\mathbb{F}}^{S_T}$-adapted, given that $\lambda^{S_T} = \widehat{\lambda}^{S_T}$ is $\widehat{\mathbb{F}}^{S_T}$-adapted. We now show that we have $B^{S_T} = \widehat{B}^{S_T}$, where $\widehat{B}^{S_T}$ is an $\widehat{\mathbb{F}}^{S_T}$-Brownian motion.
Setting $a = 1$ in (58) the signal process λ^{S_T} follows

$$\mathrm{d}\lambda_t^{S_T} = -\frac{1}{T - t} \mathrm{d}B_t^{S_T}. \tag{69}$$

By Remark 7 we apply Proposition 3 over $[0, T)$, giving the SDE for ξ with respect to $\widehat{\mathbb{F}}^{S_T}$:

$$\mathrm{d}\xi_t = \widehat{\lambda}_t^{S_T} \mathrm{d}t + \mathrm{d}\widehat{B}_t^{S_T}, \tag{70}$$

for all $t \in [0, T)$. Equating (68) and (70) gives $P[B_t^{S_T} = \widehat{B}_t^{S_T},\ t \in [0, T)] = 1$. Define $\widehat{B}_T^{S_T} := \limsup_{t \to T} \widehat{B}_t^{S_T}$. Then by continuity of Brownian motion we have $P[B_t^{S_T} = \widehat{B}_t^{S_T},\ t \in [0, T]] = 1$ and therefore the stock price follows (67) in the observation filtration, as claimed.
Hence filtering is redundant, and we are in a similar scenario to that of Theorem 2 for the case $a = 1, \mathrm{v}_0 = 0$. The remainder of the proof follows the same lines as in the proof of Theorem 2: we define a sequence of auxiliary optimal investment problems over a sub-interval $[0, T^n]$, for $T^n := T - 1/n$ and $n \in \mathbb{N}$ and show that there is a utility explosion. The arguments being the same, they are omitted. □

7 Conclusion

We have studied the effect of insider information on the estimation of an unknown stock price drift and on the optimal strategy to maximise expected utility of wealth, in a log-Brownian setting. We showed that filtering with additional information leaves the Kalman–Bucy equations intact, with the exception of modified initial conditions. Applying this filtering algorithm to the scenario where an insider estimates the unknown drift parameter based on his anticipative information (either Brownian or stock price-based) and stock return observations, we found that the insider has an advantage over the regular trader in all cases. Anticipative stock price information is superior to anticipative Brownian information in the sense that the variance of the estimate of the unknown drift is lower. This ultimately leads to an optimal trading strategy that is generally more aggressive in terms of stock holdings. In the case where the insider possesses precise knowledge of the future stock price, the maximal expected utility blows up, This is to be compared with the classical scenario considered in Pikovsky and Karatzas [27], in which the stock price drift is known with certainty, and in which precise information of any kind leads to unbounded utility.

Appendix

A Enlargement of filtration and filtering

In this section we give two results that we apply repeatedly in the proofs of the main theorems in the next section. The first result is a consequence of a classical enlargement of filtration theorem, giving the explicit semi-martingale decomposition of the $\mathbb{F}$-Brownian motion B with respect to $\mathbb{F}^L$ in our setting. The second result is a variant of the Kalman–Bucy filter which incorporates the inside information as well as stock price observations into the estimation of λ^L.

A.1 Initial enlargements of filtrations

Here we give the semimartingale decomposition of B with respect to $\mathbb{F}^L$.

We begin with the filtration $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ that contains the filtration of the Brownian motion B driving the stock price process in (20), and also $\sigma(\lambda) \subset \mathcal{F}_0$. Let L be an $\mathcal{F}$-measurable random variable.

Lemma 2 *Suppose L is such that it admits a conditional density $g : [0,T] \times \mathbb{R}^2 \to \mathbb{R}^+$ with respect to $\mathbb{F}$, so that for any test function $f : \mathbb{R} \to \mathbb{R}$,*

$$E[f(L)|\mathcal{F}_t] = \int_{\mathbb{R}} f(x) g(t, x, B_t) dx, \quad 0 \leq t \leq T.$$

Suppose that $g(t,x,y)$ is smooth enough in y for application of the Itô formula and satisfies

$$\int_0^t \left| \frac{g_y(s,x,y))}{g(s,x,y)} \right| ds < \infty, \quad \int_{\mathbb{R}} |g_y(t,x,y)| dx < \infty, \quad \text{for a.e. } y \in \mathbb{R} \text{ and } t \in [0,T]. \quad (71)$$

Then the semi-martingale decomposition of B with respect to $\mathbb{F}^L$ is

$$B_t = B_t^L + \int_0^t \frac{g_y(s, L, B_s)}{g(s, L, B_s)} ds, \quad 0 \le t \le T, \tag{72}$$

with B^L an $\mathbb{F}^L$-Brownian motion.

The process

$$\nu_t^L := \frac{g_y(t, L, B_t)}{g(t, L, B_t)}, \quad 0 \le t \le T,$$

is called the *information drift*.

Proof Apply Theorem 1.6 in Mansuy and Yor [24], and obtain the explicit representation (72) from calculations similar in spirit to Example 1.6 in [24], which are valid due to the conditions in (71). □

Corollary 1 *Take $L = L_B$ given in (25). Then* $\mathrm{Law}[L_B|\mathcal{F}_t] = \mathrm{N}(aB_t, a^2(T_a - t))$*, where T_a is defined in (39). Then Lemma 2 gives the semimartingale decomposition of B with respect to $\mathbb{F}^L = \mathbb{F}^{L_B} \equiv \mathbb{F}^B$ as*

$$B_t = B_t^B + \int_0^t \frac{L_B - aB_s}{a(T_a - s)} ds, \tag{73}$$

with L_B as in (25). Therefore the information drift in this case is

$$\nu_t^B = \frac{L_B - aB_t}{a(T_a - t)} = \frac{a(B_T - B_t) + (1-a)\epsilon}{a(T_a - t)}, \quad 0 \le t \le T. \tag{74}$$

For $a = 1$ we have $L_B = B_T$ and we obtain the Brownian bridge decomposition of B under $\mathbb{F}^{B_T}$:

$$B_t = B_t^{B_T} + \int_0^t \frac{B_T - B_s}{T - s} ds, \quad 0 \le t \le T. \tag{75}$$

Corollary 2 *Take $L = L_S$ given in (26). Then* $\mathrm{Law}[L_S|\mathcal{F}_t] = \mathrm{N}(a(B_t + \lambda T), a^2(T_a - t))$. *Lemma 2 yields that the semimartingale decomposition of B with respect to $\mathbb{F}^L = \mathbb{F}^{L_S} \equiv \mathbb{F}^S$ is*

$$B_t = B_t^S + \int_0^t \frac{L_S - a(B_s + \lambda T)}{a(T_a - s)} ds, \tag{76}$$

and where we have written $B^{L_S} \equiv B^S$ to ease notation. Therefore, the information drift ν^S is

$$\nu_t^S = \frac{L_S - a(B_t + \lambda T)}{a(T_a - t)} = \frac{a(B_T - B_t) + (1-a)\epsilon}{a(T_a - t)}, \quad 0 \le t \le T. \tag{77}$$

A.2 Linear filtering with initial enlargement

After writing the stock price dynamics with respect to the enlarged filtration $\mathbb{F}^L$, the resulting risk premium λ^L is an $\mathbb{F}^L$-adapted process which we treat as an unobservable signal process in a filtering framework. The filtering algorithm, presented below, estimates λ^L based on stock price observations augmented with the information provided at time zero by knowledge of the value of L.

It turns out that the usual form of the Kalman–Bucy filter equations hold in this case, with initial conditions reflecting the inside information. Technical integrability conditions, needed for the applicability of the Kalman filter, mean that this procedure is only valid over $[0, T]$ for $a < 1$, or if $a = 1$, the algorithm is only valid up to a time $T^* < T$, as will be seen shortly. The rather singular case corresponding to exact information on B_T or S_T requires some separate reasoning to obtain the optimal investment results.

In both cases of inside information, that is $L = L_B$ or $L = L_S$, the information drift is the same, given by (74) and (77), so we write $\nu^L \equiv \nu^B = \nu^S$ below.

The semimartingale decomposition of the $\mathbb{F}$-Brownian motion B with respect to $\mathbb{F}^L$ is

$$B_t = B_t^L + \int_0^t \nu_s^L ds, \quad 0 \le t \le T, \tag{78}$$

where B^L is an $\mathbb{F}^L$-Brownian motion and with ν^L given by

$$\nu_t^L = \frac{a(B_T - B_t) + (1-a)\epsilon}{a(T_a - t)}, \quad 0 \le t \le T.$$

With respect to $\mathbb{F}^L$, the process ξ in (21) follows

$$\mathrm{d}\xi_t = \lambda_t^L \mathrm{d}t + \mathrm{d}B_t^L, \quad \lambda_t^L := \lambda + \nu_t^L, \tag{79}$$

Applying the Itô formula under $\mathbb{F}^L$ gives

$$\mathrm{d}\lambda_t^L = -\frac{1}{T_a - t}\mathrm{d}B_t^L, \quad \lambda_0^L = \lambda + \frac{aB_T + (1-a)\epsilon}{aT_a}. \tag{80}$$

We consider λ^L is an unobservable signal process with $\mathbb{F}^L$-dynamics (80), and ξ is an observation process with $\mathbb{F}^L$-dynamics (79). We filter the conditional expectation and variance

$$\widehat{\lambda}_t^L := E[\lambda_t^L | \widehat{\mathcal{F}}_t^L], \quad V_t^L := E[(\lambda_t^L - \widehat{\lambda}_t^L)^2 | \widehat{\mathcal{F}}_t^L], \quad 0 \le t \le T. \tag{81}$$

To apply the filter we need the prior distribution of λ^L given $\widehat{\mathcal{F}}_0^L$. This can be deduced from the prior distribution of λ given $\widehat{\mathcal{F}}_0$ in Assumption 1, and turns out to be Gaussian, given by (29). The difference between the usual Kalman–Bucy filter and the situation here is that the initial filtration is not trivial. The proof follows the same lines as the conventional innovations-based proof of the usual Kalman–Bucy filter, so is omitted.

Proposition 3 *On a probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ and an $\mathcal{F}$-measurable random variable L, define the enlarged filtration $\mathbb{F}^L = (\mathcal{F}_t^L)_{0 \le t \le T}$ by*

$$\mathcal{F}_t^L := \mathcal{F}_t \vee \sigma(L), \quad 0 \le t \le T.$$

Let $\lambda^L = (\lambda_t^L)_{0 \le t \le T}$ be an $\mathbb{F}^L$-adapted signal process satisfying

$$d\lambda_t^L = -\frac{1}{T_a - t} dB_t^L,$$

where B^L is an $\mathbb{F}^L$-Brownian motion and $T_a > T$. Let $\xi = (\xi_t)_{0\le t\le T}$ be an $\mathbb{F}^L$-adapted observation process satisfying

$$d\xi_t = \lambda_t^L dt + dB_t^L, \quad \xi_0 = 0,$$

and let $\widehat{\mathbb{F}} = (\widehat{\mathcal{F}}_t)_{0\le t\le t}$ be the filtration generated by ξ. Define the filtration $\widehat{\mathbb{F}}^L = (\widehat{\mathcal{F}}_t^L)_{0\le t\le T}$ by

$$\widehat{\mathcal{F}}_t^L := \widehat{\mathcal{F}}_t \vee \sigma(L).$$

Suppose λ_0^L is an $\mathcal{F}_0^L$-measurable random variable with distribution given $\widehat{\mathcal{F}}_0^L$ that is Gaussian with mean m_L and variance Σ_L, independent of B^L. Then the conditional expectation

$$\widehat{\lambda}_t^L := E[\lambda_t^L | \widehat{\mathcal{F}}_t^L], \quad 0 \le t \le T, \tag{82}$$

satisfies

$$d\widehat{\lambda}_t^L = \left(V_t^L - \frac{1}{T_a - t}\right) d\widehat{B}_t^L, \quad \widehat{\lambda}_0^L = m_L, \tag{83}$$

where $\widehat{B}^L$ is an $\widehat{\mathbb{F}}^L$-Brownian motion, the innovations process, satisfying

$$d\widehat{B}_t^L = d\xi_t - \widehat{\lambda}_t^L dt, \tag{84}$$

and V^L is the conditional variance of λ^L, defined by

$$V_t^L := E[(\lambda_t^L - \widehat{\lambda}_t^L)^2 | \widehat{\mathcal{F}}_t], \quad 0 \le t \le T, \tag{85}$$

which is independent of $\widehat{\mathbb{F}}^L$ and satisfies the ordinary differential equation

$$\frac{dV^L}{dt} = \frac{2}{T_a - t} V_t^L - \left(V_t^L\right)^2, \quad V_0^L = \Sigma_L. \tag{86}$$

Remark 7 In the Proposition 3, the condition $T_a > T$ ensures that Theorem 10.3 of [23] is applicable.

If, instead, we have $T_a \ge T$, corresponding to $0 < a \le 1$, then we have

$$\int_0^{T^*} \left(\frac{1}{T_a - t}\right)^2 \mathrm{d}t < \infty,$$

for any $T^* < T$, and application of Theorem 10.3 in [23] yields that, with respect to $\widehat{\mathbb{F}}^L$, ξ satisfies the SDE (84) over $[0, T)$, and $\widehat{\lambda}^L$, V^L then satisfy (83) and (86) over $[0, T)$.

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