Playing for the Winning Team: Selection, Performance, and the Longevity of Organizations

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Abstract. We model the interplay between the quality of a team, its longevity, and the talent pool of those who are willing to accept a position within it. The team's quality increases and it enjoys long-term survival if it can attract good recruits, but its fortunes decline if the talent pool is shallow. Higher-quality recruits require better career prospects (via a long-lived team) to be willing to serve. In a benchmark case there are multiple equilibria: a winning team is sustained by good recruits; in a bad equilibrium, pessimistic expectations mean that a declining team is unable to attract the talent needed to mount a recovery. However, if there is noise in the process of retirement, recruitment, and replenishment, then there is a unique equilibrium in which sufficiently talented recruits are available and so the team prospers if and only if the team's current quality exceeds a unique threshold. This threshold responds to the benefits of team membership, the attractiveness of outside opportunities, and the nature of the recruitment procedure. An application is a political setting in which a governing executive survives in office only if it retains a team of good politicians.

The performance of an organization often depends upon the talents of its members. In politics, a governing executive performs better if staffed by competent decision-makers; in sport, a soccer team wins trophies if it is able to sign skilled players; and in academia, a university maintains its international standing by attracting the best scholars. If an organization is unable to recruit from a sufficiently deep talent pool to replenish its membership, then its quality declines and it may fail: poorly performing governments lose elections, and failing sports teams are relegated to a lower league. The ongoing success and long-term survival of such organizations depend upon the recruitment of talent.

Whilst recruitment is influenced by individual incentives and rewards, the prospect of joining a successful and long-lived team is also important. A politician aspires to serve in the highest office; a soccer player dreams of lifting his team's trophies; a musician of performing in the best orchestras and at the grandest venues; and academics often wish to belong to a prestigious department. At a basic level, the prospect of joining a winning team is attractive simply because a recruit's tenure is likely to be long. A longer-lived (or even immortal) organization or one which is likely to maintain its high status (think of a soccer team's long-term membership of the highest league) offers an attractive career.

[1]Some elements of this paper are based upon results from the earlier (and now superseded) conference presentation "Selection, Performance, and Government Longevity." We thank colleagues and seminar audiences, in particular the discussants at the APSA Meetings (Ethan Bueno de Mesquita), MPSA Meetings (Georgy Egorov), and the LSE-NYU PSPE Conference (Cathy Hafer) for their comments on that earlier work.

We model the interplay between these two components. We consider a world in which the evolution of a team's quality depends on the talent pool available to it, and where that team (and so the careers of its members) dies if its quality falls too far. Meanwhile, talented recruits have better outside options and so join the talent pool only if they forecast a sufficiently long life for the team. These components are complementary: a winning team attracts the talent needed to sustain it, but the pessimistic expectations of a losing team can also be fulfilled in equilibrium. We confirm that in the absence of uncertainty (specifically, when there is a deterministic link between the talent pool and the evolution of a team's quality) there are multiple equilibria. Any Markov equilibrium is characterized by a critical quality threshold. If the organization's quality is above that threshold then it prospers and survives forever, but if its quality is below then the talent pool evaporates and the team withers and dies. However, that threshold can take any value within a wide range.

Our full model, however, uses a richer specification in which there is noise in the organization's recruitment process: the connection between a recruit's talent (and so his opportunities in the outside world) and his contribution to a team is not perfect. The presence of noise means that a winning team can (even if this is unlikely) experience a run of bad luck and so suffer a decline in quality. Similarly, a languishing team faces an outside chance of resurrection. Our central theoretical contribution is to show that the presence of such noise successfully pins down a unique equilibrium. When there are two types of recruit (good and bad, where good recruits are needed for the team to prosper) then there is a unique equilibrium threshold such that good recruits join the talent pool (and so the team's quality increases in expectation) if and only if the quality exceeds the threshold. This result also extends in a natural way to a world with many different types of recruit.

When the noise in the recruitment process is small we are able to offer clear comparative-static predictions. For example, in the two-type good-or-bad world, the quality threshold needed to sustain a winning team is (unsurprisingly) decreasing in the benefits of team membership and increasing in the attractiveness of outside opportunities. More interesting, however, it reacts to the properties of noise in the recruitment process. Specifically, the threshold is increasing in the noisiness of the recruitment process when the talent pool contains high-quality recruits, but decreasing in the noise when the pool contains only low-quality types. In the first situation, a winning team enjoys an expected increase in quality, and so any shocks to recruitment serve only to harm recruits' expectations of the team's longevity. In the second situation, however, noise in recruitment provides the chance that the team may benefit from a lucky streak and subsequent resurrection. There are implications for the choice of recruitment policy by an organization's leader: that leader is willing to trade lower expected talent when selecting a recruit in exchange for less risk if her team is winning; however, that trade is made for greater risk if her team is losing.

A further insight into the management of an organizational talent pool emerges from our general model with multiple types of recruit. We ask: does an organization benefit when the set of types is more heterogeneous? We establish that there is a gain from the presence of those with intermediate quality. Such types can act as stepping stones on a path toward a world where the organization can recruit the very best available talent. If such an intermediate type can arrest a team's decline, or, even better, nudge it onto the path to resurrection, then in so doing the value of team membership to higher type recruits is increased. Anticipating that the subsequent inflow of such higher types will accelerate the team's success, this reinforces the intermediate type's incentive to join. Overall, we show that broadening the set of feasible good types lowers the quality threshold needed for a winning team.

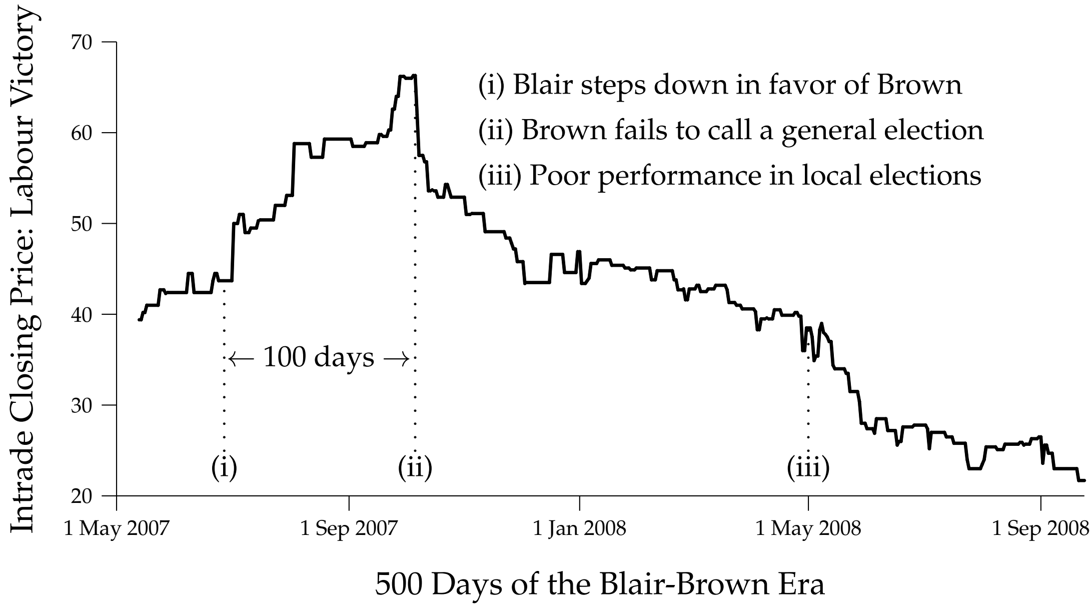
We proceed by describing our motivating applications, most notably the selection, performance, and longevity of a governing executive, before reviewing related literature. After describing our model, we analyze a deterministic benchmark before describing our main results. We extend to consider multiple types of recruit before offering concluding remarks.

1. Motivation: Winning Teams in Soccer, Politics, and Academia

Our insights apply whenever talented recruits wish to play for a winning and long-lived team. Such a desire is especially relevant when individual payments are relatively rigid. For example, explicit performance-related pay does not (to our knowledge) exist in political settings, where renumeration is typically set by legislature or committee. Similarly, academic and athletic salaries respond only imperfectly to individual characteristics. In the United Kingdom, for example, there are national pay awards in academia, and even the top soccer clubs have pay structures that they are reluctant to break. Under these circumstances, the decision to join an academic department, sign for a soccer club, or run for public office, can be determined by the desire to be part of a team that reaches the top and stays there.

The notion of a "winning team" is most salient when there are clear rankings that distinguish successful organizations from those that fail. In soccer, a team playing in the English Premier League, the Italian Seria A, the Spanish Primera Liga, or the German Bundesliga, succeeds when qualifying for the prestigious European Champions League that, in turn, requires a top three or four domestic finish. A less ambitious club succeeds when avoiding relegation to a lower league. In academia there are rankings of world universities such as those provided by Quacquarelli Symonds and Times Higher Education. In politics, ambitions are fulfilled when a politician joins a government that succeeds via re-election.

Soccer exemplifies the interplay between team success and the recruitment of talent. The composition of a team's squad evolves via a process of retirement and replenishment. Recruitment is, of course, influenced by the financial muscle that a club exerts. This, though, is only part of the story. Chelsea took several years to reach the Premiership summit and several more before winning the Champions League, despite the financial clout provided by billionaire owner Roman Abramovich. Manchester City and Paris St Germain have yet to make a Champions League impact, despite their oil wealth. To attract and retain top players requires more than money. A winning team needs the best players, but in turn the best players desire to play for a winning team. Ashley Cole stated the burning desire for trophies and medals as his motivation when he joined Chelsea in 2006. Sami Nasri and Robin van Persie left Arsenal in search of teams more likely, in their judgement, to win trophies. Manchester United striker Wayne Rooney stated concerns over the club's squad strength as



Notes. This figure is reproduced from Dewan and Myatt (2012). Data from the `intrade.com` prediction market show the daily closing prices for a Labour victory in a UK General Election. (A contract pays 100 if the event occurs.)

FIGURE 1. Prediction-Market Confidence in the UK Government

behind behind his decision to seek an exit from Old Trafford in 2011.[2] Other players have failed to play ball when dissatisfied with their team's prospects.

Academics are no different. When not devoting themselves to research and teaching they follow the job market to see where junior stars are placing and which senior faculty are on the move. They form assessments about future university and department rankings. These assessments influence their perceptions of their own prospects and feed into career choices: an academic who forms a pessimistic view of his institution's profile is more likely to go on the market, while a corresponding outside offer is more appealing when made by a department that is seen to be moving up.

In politics, we see similar dynamics at play. A public servant must sacrifice outside earning opportunities. Obtaining power can compensate, but only if a government is likely to stay in office. Such longevity is enhanced by talented ministers. A UK example demonstrating the dynamic evolution of talent and executive performance involves the Labour government of Gordon Brown that replaced that of Tony Blair in June 2007. Though initially buoyed by popular support, by 2009 it struggled in the opinion polls: Figure 1, from Dewan and Myatt (2012), illustrates the evolving confidence in the administration. Several high profile resignations in 2009 preceded a crushing defeat in elections to local councils and the European Parliament in May of that year. Within a few days of that defeat Brown had lost a further six ministers. The (appropriate) idiom of "rats leaving the sinking ship" was used and re-used

[2]A statement from Rooney, following a meeting with club director David Gill, read: "He did not give me any of the assurances I was seeking about the future squad," although Rooney subsequently stayed at the club.

by columnists and cartoonists in British media depicting this state of affairs.[3] The government's support did not recover. Indeed it was ousted from power in May 2010. The current UK government has also leaked talent. Africa Minister Mark Simmonds resigned in August 2014 while complaining that pay was insufficient, and several Conservative MPs have defected to the UK Independence Party as confidence in the Conservatives has waned.

Building on this example, a central application of our model is to a political setting in which a governing executive survives in office only if it performs well. This, in turn, requires it to attract talented individuals to serve in it. The assumptions we build into our model, as well as our main results, are related to a broad literature in political economy that studies the personal qualities of office holders. So, before providing the main details of our model, we reflect upon some ideas and findings in that literature and our contribution to it.

2. Related Literature

Our model relates government performance (or team performance, more generally) to the talents of office-holders. This relationship is central to the agency models of Fearon (1999) and Besley (2004, 2005, 2006) who, in contrast to studies of incentives (Barro, 1973; Ferejohn, 1986), emphasized the role of elections in selecting those most able to govern. Besley (2006) traced this view to Key, Jr. (1956) who argued that "the nature of the workings of government depends ultimately on the men who run it." Perhaps most notably, Fearon (1999) shared this view: "introduce any variation in politicians' attributes or propensities relevant to their performance in office, and it makes sense for the electorate to focus completely on choosing the best type when it comes time to vote." Empirical evidence supports the dependence of performance on talent. Brollo, Nannicini, Perotti, and Tabellini (2013), for example, showed that more competent politicians are able to provide services with smaller budgets.

A missing component is the supply side. If the skills required to govern are correlated with those valued by the private sector (Besley, Folke, Persson, and Rickne, 2013; Folke, Persson, and Rickne, 2014) then how can the talented be induced to serve? In politics, talent is scarce and yet rewards may not easily adjust to elicit an increase in supply.[4] The scarcity of talent has been the subject of comment. Dewan and Myatt (2010) documented the account by Tristan Garel-Jones (a whip in a United Kingdom government in the 1990s and a confidante of then Prime Minister John Major) of recruitment for a junior post. According to Paxman (2003, p. 209), Garel-Jones lamented "… you have to find maybe ninety people to form a government. You have perhaps 350 or so people to choose from. Once you've eliminated the bad, mad, drunk and over-the-hill, you've got rid of a hundred. You then have to pick ninety people out of a pool of 250. Is it any wonder that the calibre is so low?"

[3]Cartoonist Andy Haley depicted Brown standing at the helm of Her Majesty's Ship New Labour (The Sun, June 3rd, 2009). As it sinks, rats clutching ministerial briefcases are pictured scarpering from the deck.

[4]A theoretical literature, focusing on salaries, has reached different conclusions (Mattozzi and Merlo, 2008; Messner and Polborn, 2004). Ferraz and Finan (2008) presented evidence from Brazilian municipalities suggesting that salary increases have a positive impact on the quality of elected politicians. Gagliarducci and Nannicini (2013) disentangled the impact of salaries on selection from that on incentives, whereas Gagliarducci, Nannicini, and Naticchioni (2010) argued that allowing "moonlighting" by politicians attracts higher quality types but at the cost of lower effort. Besley (2004) has reviewed the (early) relevant literature.

Several research papers have focused on the impact of the scarcity of talent in politics.

Galasso and Nannicini (2011) analyzed the interaction between elections and talent allocation. They explored a mechanism via which politicians are allocated to districts that differ according to how safe they are for the incumbent party. Their theoretical justification and empirical evidence support the claim that talented politicians are more likely to be allocated to marginal districts. Empirical evidence reveals that, ex post, these politicians perform better on a range of indicators. Based on the exploitation of exogenous changes in seat marginality from national coalition bargaining, this improved performance is related to inherent talent rather than to better incentives that arise in close races.

Dewan and Myatt (2010) modeled the dynamic relationship between a government's talent pool and its performance. Competent office-holders are required to sustain a government in office but they may fail to deliver—they may be hit by scandal or their policies may fail. The incentives to perform are provided by a leader who fires those deemed to have failed sufficiently. The severity of the (endogenous) firing rule depends upon the depth of the talent pool: it becomes more lenient as the pool dries up. This has policy consequences. Early in a government's tenure, with plentiful talent and strong incentives, it performs well. This performance declines and policy failures become more likely as the talent pool depletes.

Whereas Dewan and Myatt (2010) looked at politicians acting in isolation, here the focus is on strategic complementarities. Politics, like soccer, is a team sport. A minister's time in office depends ultimately on the joint performance of the government he serves. Such strategic complementarities were explored by Caselli and Morelli (2004). They modeled citizen-candidates who decide whether to enter politics. In equilibrium the share of good types running for office increases with the salary offered. Moreover, in their world the quality of politicians creates "ego rents" that are enjoyed by the entire class of politicians.[5] They described (amongst other equilibria) a "bad" equilibrium in which only incompetent politicians stand. Path dependencies sustain this equilibrium: if the existing politicians are bad, perhaps owing to historical accident, then when these outgoing politicians determine the ego rents enjoyed by an incoming group, a bad equilibrium will be sustained. This is also a theme of Svolik (2013). He argued that in newly established democracies, imperfect screening may result in the selection of crooked candidates who see election as a one-time chance to exploit the public. A normal type candidate, by contrast, would refrain from exploitation if provided with appropriate re-election incentives. This requires that citizens engage in costly monitoring of politicians. Svolik (2013) illustrated a "trap of pessimistic expectations" in which voters perceive all politicians as crooks, and so politicians of every type exploit their offices. Prophesies of this kind were also analyzed by Frisell (2009) who showed that the public trust or distrust in politicians' behavior may be self-fulfilling.

Path dependencies and coordination failure are also relevant in our model. A bad equilibrium is sustained when politicians form pessimistic expectations about the survival prospects

[5] Extensions include work by Fréchette, Maniquet, and Morelli (2008) and by Júlio, Paulo and Tavares (2014) which analyzed the effect of quota restrictions on quality.

of a government which, therefore, is unable to attract the talent that might reverse its fortunes. In Caselli and Morelli (2004) the bad equilibrium is one of many. Here, however, if noise is present in the recruitment process (a realistic feature) then there is a unique equilibrium and unambiguous comparative-static predictions.

A different take on the bad governance theme focuses on institutions. Elections may fail to deliver good politicians owing to the intermediary effects of parties who do not always choose the best candidates (Mattozzi and Merlo, 2011); voters may be unable to coordinate on the most competent candidate (Besley and Coate, 1997), emphasize other dimensions such as ethnicity over competence (Banerjee and Pande, 2007), or face trade-offs between politicians' competence and ideology (Beath, Christia, Egorov, and Enikolopov, 2014; Mattozzi and Snowberg, 2014); and, indeed, voters need not benefit from improvements in the human capital of elected candidates who face competing claims for their time from their party or faction (Buisseret and Prato, 2013). Relatedly, Folke, Persson, and Rickne (2014) analyzed data connecting intra-party competition with quality. Moreover, executive leaders may emphasize loyalty over ability when allocating positions (Egorov and Sonin, 2012).

For Acemoglu, Egorov, and Sonin (2010), performance is determined by an executive team of citizen candidates. Any voter-desired transition to another team is subject to approval by incumbents: incompetent executive members veto changes that foreclose their participation in future governments. Acemoglu, Egorov, and Sonin (2010) distinguished between perfect democracy, where the consent of the existing elite is not required, and non-democratic societies where incumbents have some veto power. Their model of dynamic political selection reveals that in all cases other than perfect democracy, a government that consists of the least competent ministers may persist indefinitely. Thus, as they show, a bad government may be sustained by institutions that make it difficult to replace.

We provide insights into the recruitment strategies deployed by an organization leader: she trades lower expected talent when selecting a recruit in exchange for less risk if her team is winning, while making the same trade in favor of greater risk if her team is losing. Similar trade-offs have been studied elsewhere. Downs and Rocke (1993) analyzed a leader who engages in risky foreign policy ventures. In their world a leader "gambles on resurrection" by continuing with an adventurous war because "cessation would, given the current state of the world, cause him to be removed from office." Such policy gambles were also studied by Majumdar and Mukand (2004) who show that an incumbent may make inefficient policy choices when a voter judges his ability from policy outcomes. The mean-variance trade off in our model also relates to work by Strömberg (2008). He studied campaigning across safe and less safe seats with two party competition under the electoral college.

In our model a governing team's quality evolves via a Brownian motion. This relates our work to recent dynamic models of political economy (Callander, 2011a,b; Callander and Hummel, 2014). Callander (2011a,b) studied policy choice and learning by politicians under uncertainty, where policy outcomes correspond to a Brownian motion that describes correlated multi-armed bandits. The modeling techniques we use here are most closely related to those in Dewan and Myatt (2012) who considered recruits of the same type (that is, there are no talent differences) but team members choose effort while in office. High effort reduces the risk of a bad outcome and the subsequent resignation of a team member. A familiar efficiency wage argument applies: a team member works hard only if his position is sufficiently valuable. That value is high when the team survives for a long time, which in turn is true whenever the overall effort level of the team is high. Just as in this paper, there are multiple equilibria when the team's performance evolves deterministically in response to its efforts; however, if noise is added to this process then there is a unique equilibrium in which team members work hard if and only if the current status of the team exceeds a unique threshold. The present paper differs from that earlier work in three ways. Firstly, we deal with variable types rather than variable effort choices. Secondly, Dewan and Myatt (2012) restricted to two actions choices whereas here we obtain substantial new results when there are many different types of recruit. Thirdly, here we allow the noise in the evolution of the team's quality to vary according to the nature of the marginal recruit. This gives insights into the situations in which greater noise is more or less desirable.

Moving beyond politics, we offer some insights into a general collective-action problem. A team's quality is a stock that helps it to survive; it is a common-pool resource that is replenished when talented recruits serve. Dynamic models of the common-pool-resource problem have been developed by Tornell and Velasco (1992) amongst others. Relatedly, Kremer and Morcom (2000) modeled the harvesting of storable open-access resources. There are multiple equilibria. If a resource is expected to survive then future harvests will be plentiful. Future prices and hence present prices (via the possibility of storage) of the harvest will be low. Low prices ensure that current exploitation is low, and so the resource survives. However, if the resource is expected to disappear, then high prices in both the future and present result in the high harvesting rates that are consistent with the resource depletion. There are multiple self-fulfilling prophesies. Similarly, Rowat and Dutta (2007) analyzed commons exploitation in the presence of capital markets and found multiple equilibria. Where these models fail, however, is in resolving the problem of multiple equilibria. In our model the addition of random events provides unique predictions.

3. A Model of a Team's Quality and Recruitment

A simple model incorporates two key components. Firstly, a team's evolving quality and so (ultimately) its long-term survival depend upon the talent pool of recruits that are available to replenish its membership. Secondly, the composition of that talent pool depends endogenously upon the team's quality: potential recruits use their observations of that quality to predict the longevity of the team and so the value of a career within it.

A Team's Evolving Quality. The $n+1$ types of recruit are indexed by $i \in \{0, 1, \ldots, n\}$, where a higher index indicates a better recruit. The quality of the team at time t is $x_t \in \mathcal{R}_+$. It evolves in response to the composition of the talent pool of those who are willing to join the team. If the best recruit available is type i, then

$$dx_t = \mu_i \, dt + \sigma_i \, dz_t, \tag{1}$$

where $\mu_i \neq 0$ and where dz_t is the increment of a Wiener process: the team's quality evolves noisily via a Brownian motion with a drift and volatility that are determined by those who are willing to join if asked. The parameters μ_i and σ_i reflect the properties of a (possibly imperfect) process of retirement, recruitment, and replenishment. There is some $i^\dagger \in \{1, \dots, n\}$ such that $\mu_i > 0$ if and only if $i \geq i^\dagger$, and so the team's quality drifts up in expectation if and only if the pool of talent available to it is sufficiently deep.

The team survives if and only if its quality is sufficiently high: if x_t hits an absorbing lower barrier at zero then the team (together with the careers of those within it) expires.[6]

The Talent Pool. A potential recruit (from a large outside population) decides whether to make himself available within the talent pool from which the team can replenish its membership. If he joins the team then he expects to enjoy a flow payoff $u > 0$, discounted in continuous time at rate ρ, for so long as the team survives. We write $U = (u/\rho)$ for the present value of a career within an immortal team. His outside option has present value W_i, where $0 = W_0 < W_1 < \cdots < W_n < U$; hence a better recruit has a better alternative opportunity. A recruit joins the talent pool if and only if the expected present value of his career within the team at least equals his outside option.[7] For example, if he believes that the organization will survive for a length of time T then he joins the talent pool if and only if $(1 - e^{-\rho T})U \geq W_i$. The lowest type ($i = 0$) is always willing to serve.

We assume that potential recruits are Markovian: they base their participation decisions entirely on the team's current quality. Thus, we write $V(x)$ for the expected present value of a career when that current quality is x, and so the talent pool is determined by the maximal i satisfying $W_i \leq V(x)$. Our main task will be to characterize $V(x)$.

The Two-Type Case. We have described here a specification with $n + 1$ different types of recruit. However, the key ideas are most clearly conveyed in the context of a two-type model, and so our early analysis (Sections 4 and 5) focuses on this simpler case. However, more general results (Section 6) and our formal proofs are for the full specification. When dealing with the two-type world we use the more natural subscripts H and L corresponding to good (H) and bad (L) recruits, where $\mu_H > 0 > \mu_L$ and $U > W_H > W_L = 0$.

Interpretation. We have (intentionally) used a stripped-down abstract form for our model. Before documenting our results, we pause to describe the connections between the model's components and the real-world organizations we have in mind.

We think of a potential recruit's type (his index i) as human capital that is valued in many sectors. For example, his educational qualifications may be appreciated by everyone; more generally, transferrable skills translate into better outside options. In politics, parliamentary

[6]There is no upper bound to the team's quality. However, our model can be extended to accommodate either a reflecting upper barrier or an absorbing upper barrier which rewards team members with some terminal prize. For expositional and analytical convenience we have removed the upper barrier.

[7]Implicitly we are assuming that any invitation to join the team is a now-or-never opportunity, and so a recruit cannot gain from delaying his participation to watch the progress of the team's quality. This is consistent with a situation in which there is a large outside population of potential recruits.

and committee service have, for example, been shown to enhance earning potential in the private sector (Diermeier, Keane, and Merlo, 2005; Keane and Merlo, 2010). In soccer, a high scoring record at one club increases a player's value to others. In academia, the publication record of an academic researcher can be directly relevant to the research funding of any host department. In short, we plausibly assume that an election-winning politician, an academic with a string of prestigious publications, or a soccer player with an impressive tally of goals, have better career options than those without such qualifications.

A recruit's type determines the evolution of the organization via equation (1). $\mu_i\,dt$ is the expected change in quality when a type i joins the team.[8] This quality may be the expertise of an executive board, the popularity of a government in the polls, or the league position of a sports team. A natural case is where $\mu_i < \mu_{i+1}$, so that those who have better outside options also (in expectation) add more to the team. In soccer, established scorers can be expected to improve a team; in academia, a recruit with top publications helps a department's ranking; and in politics, those with executive experience help a government to succeed while a novice might cause damage. In UK politics, the former types are often referred to as "big beasts" whose decisions to enter or exit political service are closely watched by commentators.[9]

The second term $\sigma_i\,dz_t$ in equation (1) is noise in the relationship between a recruit's talent in the outside market and his eventual contribution to the team. A basic source of this noise is the imperfect relationship between a recruit's general value and his personal match with the team. In soccer, a striker who scores with abundance at one club may sometimes fail to settle at another, yielding a lower than expected return to the team. Conversely, a player that may have failed at his old club could rediscover form at his new one.[10] In academia, a job market star may fail to deliver on new projects when joining a prestigious department, whereas a recruit who, on paper, looks less promising may flourish. Finally, the aforementioned big beast may alas no longer have what it takes to cut it at the highest level of politics.

An alternative interpretation of the noise term is that it represents imperfections in the recruitment technology. Put simply, the best available candidate may not get the job. Although an excellent soccer player may be willing to join a team, the deal may not materialize owing to bargaining failures or the limited scope of the club's scouting network. Similarly, a department may miss out on the recruitment of a talented academic owing to an incomplete search process. Similar distortions may arise in politics where, as in our quote from Paxman (2003), a Prime Minister hastily recruits from a shortlist drawn up by a confidante.

Our specification of payoffs also deserves discussion. The flow payoff u corresponds to the prestige of playing for a winning team, be it a trophy-laden soccer club, an academically

[8]Our focus on the best possible recruit is only one way of linking the dynamic evolution of the team's quality to the attractiveness of a position on it. Another interpretation is to think of i as the ability of the best player either on the team, or willing to join it, and μ_i as corresponding to the net talent inflow and outflow. A fuller model would, however, provide an explicit account of the inflow and outflow of talent.

[9]*Labour's Big Beasts Retire From The Jungle*, The Guardian, 27 May 2001.

[10]Recruiting from the youth teams has yielded unanticipated (by some) success at one of Europe's biggest clubs. In 1995 Alan Hanson, a UK pundit, quipped that "you can't win anything with kids." His comments followed the defeat by Aston Villa of a Manchester United team that contained several products (including David Beckham) from its youth team. This United team went on to win the championship at a canter.

thriving department, or an election-winning government. A key aspect of the current value of playing for a winning team is the expected duration of that winning status before falling from contention. This is the expected tenure of a soccer team in the premier division, the longevity of a department's status at the top, or the length of time before a government faces a successful electoral challenge. Our focus on the time a player spends at the top can be justified. Few remember the achievements of a player once he transfers to a team in the lower leagues. Similarly a political career is defined during the period in which a politician has decisive influence over policies, as when he forms part of the government.

Of course, in some cases success may be more tangible. A soccer player wants trophies that will forever be associated with his name: a terminal prize. A reconfiguration of our model can capture this motivation. Note that in our model there is no upper bound to the team's quality. However, our model can be extended to accommodate either a reflecting or an absorbing upper barrier. This could be interpreted as the reward for a team that achieves a terminal prize. Such a terminal prize can be incorporated readily; however, for expositional convenience, we retain the flow payoff from membership of a successful team and simplify the statement of our results by removing any upper barrier to the team's quality.

4. A DETERMINISTIC BENCHMARK

Our benchmark is a world in which the team's quality evolves deterministically, so that $\sigma_i^2 = 0$ for all i. This corresponds to a perfect recruitment process, so the team always picks the best available type from the pool, where talent perfectly determines a player's team contribution. We focus here (and in Section 5) on the two-type (L and H) case.

Equilibria. An equilibrium is determined by the value $V(x)$ of a career in a team of quality x. This satisfies $V(x) \leq U$ (a recruit can do no better than in an immortal team) and it is (at least weakly) increasing in x (a better team survives for longer).

One possibility is that a winning team is expected to last forever. If so, then $V(x) = U$ and good recruits join the talent pool. Given that $V(x)$ is monotonic, $V(x) = U$ must also hold for larger values of x. Hence, there must be some $x^\star$ such that $V(x) = U$ for all $x > x^\star$.

The second possibility is that a losing team dies in finite time. This can happen only if good recruits are unavailable, and so $V(x) < W_H$. Given that the talent pool consists only of bad recruits, the evolution of the team's quality satisfies $dx/dt = \mu_L < 0$, and so the team expires after a period of time $x/|\mu_L|$. The value of a career in this finite-lived team is $V(x) = U(1 - e^{-\rho x/|\mu_L|})$. This can be an equilibrium only if good recruits are unwilling to join the talent pool, which holds if and only if $V(x) < W_H$. This requires

$$x < \bar{x} \quad \text{where} \quad \bar{x} \equiv \frac{|\mu_L|}{\rho} \log\left(\frac{U}{U - W_H}\right).$$

If this does not hold (that is, if the team's quality exceeds $\bar{x}$) then the team's lifetime is guaranteed to be long enough to attract the services of better recruits.

Proposition 1 (Equilibrium with Deterministically Evolving Quality). *For any $x^\star \in [0, \bar{x}]$ there is an equilibrium in which (i) the team is immortal (it is a winning team) beginning from any quality $x > x^\star$, but (ii) its quality decays and the team eventually fails if $x < x^\star$.*

There can be multiple self-fulfilling prophesies: winning teams that are expected to last (so that $x > x^\star$) attract the calibre required for this prophesy to be fulfilled; lower quality teams are expected to fail and so do so. However, it is hard to say anything more substantive: the threshold $x^\star$ that separates the optimistic and pessimistic worlds is arbitrary.

Equilibrium Selection. The problem of multiplicity (of equilibrium thresholds) is resolved fully once we return (in Section 5) to a world in which the team's quality evolves stochastically. Here, however, we discuss briefly a heuristic argument for equilibrium selection.

Consider an equilibrium with a threshold $x^\star$ between the "good" and "bad" worlds. Here,

$$V(x) = U \times \begin{cases} 1 - e^{-\rho x/|\mu_L|} & x < x^\star \\ 1 & x > x^\star \end{cases}.$$

Now imagine that a talented recruit (of type H) is contemplating his participation in the talent pool, and that the team's quality is just equal to the knife-edge threshold $x^\star$. What should he do? According to whether quality rises or falls in the next instant, his career value will be either U or $U(1 - e^{-\rho x/|\mu_L|})$. If he anticipates that positive and negative moves in the next instant are equally likely (as we show later, this is appropriate when $\mu_H = |\mu_L|$ and $\sigma_H = \sigma_L$) then his expected career value is $U(1 - \frac{1}{2}e^{-\rho x/|\mu_L|})$. Given that $x^\star$ is the critical threshold between the good and bad regimes, the talented recruit should be just indifferent between joining the talent pool and remaining outside it. That is,

$$U\left(1 - \frac{e^{-\rho x^\star/|\mu_L|}}{2}\right) = W_H \quad \Leftrightarrow \quad x^\star = \frac{|\mu_L|}{\rho}\log\left(\frac{U}{2(U - W_H)}\right).$$

This threshold has some natural properties: it is increasing in the relative attractiveness of a career outside the team, but decreasing in the impatience of the recruit. Note also that $x^\star$ is positive if and only if $W_H > \frac{U}{2}$. This suggests that the appropriate threshold between the good and bad worlds is zero (in essence, the team is always expected to win) if $W_H < \frac{U}{2}$. Hence, if this heuristic selection criterion is accepted then a team will successfully achieve a good outcome if the outside opportunities of good recruits are not too attractive.

Clearly, this is only suggestive. Nevertheless, as we show next, the solution given here emerges from a formal analysis of an environment with stochastically evolving quality.

5. Equilibrium with Noisily Evolving Quality

We now study our fuller model in which the team's quality evolves noisily via the stochastic differential equation $dx_t = \mu_i\, dt + \sigma_i\, dz_t$ whenever the best available recruit is i. As in Section 4, we focus here on the two-type good-or-bad specification.

Properties of Career Values. The noisy evolution of the team's quality means that a run of bad luck can induce failure. This implies that $V(x) < U$: a team member's payoff can never be as high as that obtained from a lifetime in post. Furthermore, $V(x)$ is a strictly increasing and twice continuously differentiable function of x. This continuity contrasts with the noiseless case of Section 4: recall that $V(x)$ had a sharp discontinuity at $x^\star$. This discontinuity separated the optimistic world of a long-lived team staffed by good appointees from that of a short-lived operation run by (relatively) incompetent staff. The presence of noise smooths out the gap between the optimistic and pessimistic worlds. The basic properties of the career value function $V(x)$, reported in Lemma 1, are that it increases smoothly from the absorbing lower barrier at $V(0) = 0$ and asymptotes to the upper bound U.

Lemma 1 (Career Values in a Two-Type World)**.** *The career value function $V(x)$ is a strictly increasing and twice differentiable function of the organization's quality x. It satisfies $V(0) = 0$ and $\lim_{x\to\infty} V(x) = U$. In a two-type world there is a threshold $x^\star \in (0, \bar{x})$ such that the good type recruit H is in the talent pool if and only the organization's quality exceeds $x^\star$.*

We now investigate further the characteristics of the value function $V(x)$. It must satisfy

$$\rho V(x)\, dt = \rho U\, dt + \mathrm{E}[dV(x)], \tag{2}$$

The left-hand side is the flow rental value of a career within the team. The first term on the right-hand side is the direct flow benefit from that position. The second term on the right-hand side is the expected flow change in the value of the career.

Now we write $V_i(x)$ for the segment of the career value function which applies in the range where the best available recruit is of type $i \in \{L, H\}$. That is,

$$V(x) = \begin{cases} V_L(x) & x < x^\star \\ V_H(x) & x > x^\star. \end{cases}$$

For each segment of the career value function we can use Itô's Lemma:

$$\mathrm{E}[dV_i(x)] = \mu_i V_i'(x)\, dt + \frac{\sigma_i^2 V_i''(x)}{2}\, dt. \tag{3}$$

The first term corresponds to the expected path of the quality process: the expected net flow of quality changes the career value of a team member via the derivative $V_i'(x)$. There is also a second-order effect owing to the presence of the random component. This is captured by the second term, which depends upon the curvature of the value function.

Bringing together the two equations (2) and (3), we obtain

$$\rho U - \rho V_i(x) + \mu_i V_i'(x) + \frac{\sigma_i^2 V_i''(x)}{2} = 0. \tag{4}$$

This is a constant-coefficient linear second-order differential equation, and so it is solved relatively straightforwardly. In fact, it has a general solution

$$V_i(x) = a_i^+ e^{-b_i^+ x} + a_i^- e^{-b_i^- x} + U \quad \text{where} \quad b_i^\pm \equiv \frac{\mu_i \pm \sqrt{\mu_i^2 + 2\rho\sigma_i^2}}{\sigma_i^2}, \tag{5}$$

where $b_i^+ > 0 > b_i^-$, and where the coefficients a_i^+ and a_i^- have yet to be determined.

The coefficients a_i^+ and a_i^- are determined by considering the boundary conditions at the top and bottom of each segment of the value function. These four boundary conditions are

$$V_L(0) = 0, \qquad V_L(x^\star) = V_H(x^\star) = W_H, \quad \text{and} \quad \lim_{x\to\infty} V_H(x) = U.$$

The first condition says that a career is worth nothing when the team expires. The second and third equations hold because $x^\star$ is the threshold where the good type is just willing to join the talent pool, and so the two segments of the value function join together at $x^\star$ to equal the outside-option W_H of the good type. The final condition says that the career value approaches the value U in an immortal team when the team's quality grows large.

Lemma 2 (Value Function)**.** *The lower and upper segments of the value function satisfy*

$$\begin{aligned} V_L(x) &= U - \frac{(U - W_H)(e^{-b_L^- x} - e^{-b_L^+ x}) + U(e^{-(b_L^- x^\star + b_L^+ x)} - e^{-(b_L^+ x^\star + b_L^- x)})}{e^{-b_L^- x^\star} - e^{-b_L^+ x^\star}} \\ V_H(x) &= U - e^{-b_H^+(x - x^\star)}(U - W_H). \end{aligned}$$

This lemma provides a full characterization of the value function so long as we know the the threshold $x^\star$ above which the organization attracts good recruits into the talent pool. To pin down an equilibrium, we need to characterize this threshold.

Equilibrium. The solutions for the two segments $V_L(x)$ and $V_H(x)$ of the career value function join together at the threshold $x^\star$, satisfying $V_L(x^\star) = V_H(x^\star) = W_H$. To find the threshold, we note that the value function is continuously differentiable across its entire range (Lemma 1) and so we impose the smooth pasting condition: $V_L'(x^\star) = V_H'(x^\star)$.

Differentiation of $V_L(x)$ and $V_H(x)$ with respect to x yields

$$\begin{aligned} V_L'(x^\star) &= \frac{(W_H - U)(b_L^+ e^{-b_L^+ x^\star} - b_L^- e^{-b_L^- x^\star}) + U(b_L^+ - b_L^-)e^{-(b_L^+ x^\star + b_L^- x^\star)}}{e^{-b_L^- x^\star} - e^{-b_L^+ x^\star}} \\ V_H'(x^\star) &= b_H^+(U - W_H). \end{aligned}$$

The expression for $V_L'(x^\star)$ is monotonic in $x^\star$, and so $V_L'(x^\star) = V_H'(x^\star)$ has a unique solution.

Proposition 2 (Equilibrium with Noisily Evolving Quality)**.** *In a two-type good-or-bad world there is a unique equilibrium with a quality threshold $x^\star$ such that good types are willing to join the talent pool if and only if the team's quality is at least $x^\star$. This threshold is the unique solution to*

$$b_H^+ = \frac{b_L^- e^{b_L^+ x^\star} - b_L^+ e^{b_L^- x^\star}}{e^{b_L^+ x^\star} - e^{b_L^- x^\star}} + \frac{U}{U - W_H} \frac{b_L^+ - b_L^-}{e^{b_L^+ x^\star} - e^{b_L^- x^\star}}, \tag{6}$$

where the coefficients b_H^+, b_L^-, and b_L^+ are

$$b_H^+ = \frac{\mu_H + \sqrt{\mu_H^2 + 2\rho\sigma_H^2}}{\sigma_H^2} \quad \text{and} \quad b_L^\pm = \frac{\mu_L \pm \sqrt{\mu_L^2 + 2\rho\sigma_L^2}}{\sigma_L^2}.$$

The equilibrium threshold $x^\star$ is increasing in the value W_H of the good recruit's outside opportunity, but decreasing in the value U of a permanent career in an immortal team.

This is the first major contribution of this paper. The complementarity between the recruitment of talent and the attractiveness of the team suggests the possibility of multiple equilibria. Our benchmark demonstrates (Proposition 1) that many such equilibria are possible when the team's quality evolves deterministically. Here, however, we see the presence of noise (even if small) eliminates many equilibria and pins down a unique threshold $x^\star$.

To see why, consider an equilibrium in which recruits use a threshold $x^\star = \varepsilon$ for some small $\varepsilon > 0$. Now suppose that there is some noise in the recruitment process. If $x = \varepsilon$, then a recruit recognizes that (for small ε) there is a significant risk that a shock in the next instant will push the team's quality into the lower barrier at zero and so cause the expiry of the team.[11] This limits the attractiveness of recruitment, and causes the equilibrium to unravel.

Equilibrium with Limited Noise. A cleaner expression for the equilibrium threshold emerges when the noise in the evolution of quality is small. To proceed here, we use a scaling parameter ξ to control this noise. Specifically, we now write equation (1) as

$$dx_t = \mu_i \, dt + \xi \sigma_i \, dz_t.$$

Allowing ξ to fall toward zero, the key coefficients $b_H^\pm$ and $b_L^\pm$ simplify in the limit.

Lemma 3 (Properties of the Coefficients $b_H^\pm$ and $b_L^\pm$). *Allowing noise to vanish ($\xi \to 0$),*

$$\lim_{\xi \to 0} \xi^2 b_H^+ = \frac{2\mu_H}{\sigma_H^2}, \quad \lim_{\xi \to 0} b_H^- = -\frac{\rho}{\mu_H}, \quad \lim_{\xi \to 0} b_L^+ = -\frac{\rho}{\mu_L}, \quad \textit{and} \quad \lim_{\xi \to 0} \xi^2 b_L^- = \frac{2\mu_L}{\sigma_L^2}.$$

Note that the first and last claims imply that $\lim_{\xi \to 0} b_H^+ = \infty$ and $\lim_{\xi \to 0} b_L^- = -\infty$.

The key equilibrium condition is equation (6) from Proposition 2. This can be written as

$$\xi^2 b_H^+ = \frac{1}{e^{b_L^+ x^\star} - e^{b_L^- x^\star}} \left(\xi^2 b_L^- e^{b_L^+ x^\star} - \xi^2 b_L^+ e^{b_L^- x^\star} + \frac{U \xi^2 (b_L^+ - b_L^-)}{U - W_H} \right). \tag{7}$$

We can use the properties reported in Lemma 3 to simplify this condition. For example: the left-hand side converges to $2\mu_H/\sigma_H^2$; $\xi^2(b_L^+ - b_L^-) \to 2\mu_L/\sigma_L^2$; and (at least for $x^\star > 0$) $e^{b_L^- x^\star} \to 0$. Taking limiting values for all of the components, equation (7) becomes

$$\frac{2\mu_H}{\sigma_H^2} = \frac{1}{e^{\rho x^\star/|\mu_L|}} \left(\frac{2\mu_L e^{\rho x^\star/|\mu_L|}}{\sigma_L^2} - \frac{2\mu_L U}{\sigma_L^2 (U - W_H)} \right). \tag{8}$$

This solves readily to yield a simple limiting solution for the threshold $x^\star$.

Proposition 3 (Equilibrium with Vanishing Noise). *Suppose that the team's quality evolves via $dx_t = \mu_i \, dt + \xi \sigma_i \, dz_t$. Allowing the noise to vanish, in the unique equilibrium $\lim_{\xi \to 0} x^\star = x^\diamond$ where*

$$x^\diamond = \frac{|\mu_L|}{\rho} \left[\log \left(\frac{|\mu_L|/\sigma_L^2}{(\mu_H/\sigma_H^2) + (|\mu_L|/\sigma_L^2)} \right) + \log \left(\frac{U}{U - W_H} \right) \right]. \tag{9}$$

The equilibrium career value in a team of quality x satisfies

$$\lim_{\xi \to 0} V(x) = U \times \begin{cases} 1 - e^{-\rho x/|\mu_L|} & x < x^\diamond \\ 1 & x > x^\diamond \end{cases}.$$

[11] The probability that the team's quality hits the zero barrier within the next instant approaches one as $\varepsilon \to 0$.

When noise is small the value of a career within the team approaches its immortal value if the team's quality exceeds $x^\diamond$. However, if the quality is lower then the career value approaches the value experienced in a deterministic finite-lived team, as described in Section 4.

A particular case of interest is when $\sigma_L^2 = \sigma_H^2$, so that the noise in the recruitment process is the same in both good (above $x^\star$) and bad (below $x^\star$) worlds, and when $\mu_H = |\mu_L|$. This second inequality means that the rate of increase in team quality when recruits are good matches the rate of decline when recruits are bad. In this situation,

$$x^\diamond = \frac{|\mu_L|}{\rho} \log\left(\frac{U}{2(U - W_H)}\right).$$

This is the threshold that we suggested heuristically at the end of Section 4.

Comparative-Static Properties. The key output from our analysis here is the limiting threshold $x^\diamond$. It sharply separates optimistic ($x > x^\diamond$) and pessimistic ($x < x^\diamond$) worlds. The simple solution for $x^\diamond$ allows further comparative-static predictions.

Proposition 4 (Comparative Statics)**.** *The threshold $x^\diamond$, for the case where the noise in evolution of the team's quality is small, is decreasing in the rate of quality increase μ_H when recruits are good, but increasing in the rate of decline $|\mu_L|$ when recruits are bad. It is increasing in σ_H^2/σ_L^2, which is the relative noisiness of recruitment in the good versus the bad world.*

The claims concerning μ_H and μ_L are unsurprising. The more interesting prediction (and one that forms a second key contribution) is the final one. Specifically, the key quality threshold responds in different directions to the relative noisiness of recruitment in the good (winning team) and bad (losing team) worlds.

When the team's quality exceeds the threshold, so that good recruits are present in the talent pool, the team's quality increases in expectation. In this situation, the team benefits from a complete absence of noise: its quality increases forever (or hits any upper barrier if one is imposed) and the team is immortal; any uncertainty in recruitment can only be harmful. Below the threshold, however, the team is doomed to eventual failure in the absence of any random events. Noise in the recruitment process provides the possibility that a run of good luck (where bad recruits turn out to work well in the team) allows team quality to rise above the critical threshold.

Discussion. Here we relate our results to the worlds of politics, soccer, and academia.

Proposition 1 confirms the intuition described in our introductory remarks concerning the interplay between recruitment, on the one hand, and team performance, on the other. In a deterministic world—that is, with a noiseless recruitment process—there are multiple self-fulfilling prophesies: organizations that recruit talented members can expect sustained success and such expected success, in turn, reinforces the ability to recruit the best; without such expectations an organization is unable to recruit at the highest level, its performance falters, and subsequently it is unable to alleviate the shallowness of its talent pool.

A key message is that shallows are dangerous. If an organization is unable to recruit from a sufficiently deep talent pool to replenish its membership, then its quality drops and it may fail: poorly performing governments lose elections and failing sports teams are relegated to a lower league. Related further to the world of politics, this result provides a novel perspective on performance. As in Caselli and Morelli (2004), it raises the prospect of a bad equilibrium in which only the lesser talents are willing to serve in government. Starting from a simple assumption—politicians are motivated by success—our model can account for the declining fortunes of governments such as that of Gordon Brown (Figure 1). As in Caselli and Morelli (2004), in this deterministic world, a bad equilibrium in which only those lacking talent are willing to serve is but one of multiple self-fulfilling prophesies. At the heart of that paper lies a coordination failure in which citizens end up in the bad state. Our argument and analysis, by contrast, does not rely on focusing upon a particular equilibrium. Instead, the addition of noise to the recruitment process adds a dose of realism while allowing us to pin down a unique equilibrium (Proposition 2) and to offer a set of comparative-static predictions (Proposition 4).

While some of these predictions are unsurprising, others provide novel insights into recruitment. Of particular interest is the insight that when the team's quality is above the critical threshold separating worlds of pessimism and optimism, it is helpful to restrict the noise in the recruitment process; the range of winning team qualities is widened by playing it safe when expectations are optimistic. By contrast, below the threshold, it pays the organization to take calculated risks in recruitment, for a lucky draw may push it above the threshold that allows it to enjoy sustained success. As noted earlier, in soccer, recruiting a top striker who has an established record of scoring at the highest level will likely enhance a team's quality and success. So it is that Chelsea, English Premier league runners-up and and Champions league semi-finalists, boosted their squad by signing Diego Costa; he had been the star striker in Atletico Madrid's Spanish La Liga winning side (and Champions League runners up). In recent seasons other clubs have looked to Spain for strikers. Few fancied Swansea City's chances of remaining in the top flight following their promotion in 2012. What they did was in large part due to their exploiting uncovered gems—the lucky draws—such as star striker Michu signed from Raya Vallecano, Atletico's less illustrious Madrid neighbors.

Such recruitment strategies are also relevant to a government that seeks to reverse its sliding fortunes. Returning to the example of the United Kingdom's Brown government it is noticeable that, in order to restore the government's prospects, Brown did not place his faith in Labour's big beasts. Instead he broadened the scope of his recruitment, building a "government of all the talents" or "goats" as the media subsequently dubbed them. These included Sir Ara Darzi (a consultant surgeon who became a health minister in the House of Lords), Sir Digby Jones (a former director general of the CBI who became minister of state for trade and investment), and Sir Alan West (the former head of the Royal Navy who became a security minister at the Home Office). On this occasion, the experiment was not a success and failed to reverse the government's slide. An academic report (Hazell and Yong, 2011) put it thus: "By the end of Brown's premiership … views about the success of these appointments were at best mixed. Some of the original 'goats' had left government after a relatively short time; commentators were scathing about their achievements." On this occasion, then, the "gamble for resurrection" did not succeed; however, without this gamble the Brown era of decline may also have been inevitable.

6. MULTIPLE TYPES OF RECRUIT

We now study our fuller model in which there are many types of recruit. Recall that in this general specification the expected path of the team's quality is upward (that is, $\mu_i > 0$) if and only if the best recruit in the talent pool satisfies $i \geq i^\dagger$. Thus the set of possible recruit type partitions into the set of bad types $\{0, 1, \ldots, i^\dagger - 1\}$ and the set of good types $\{i^\dagger, \ldots, n\}$.

The Deterministic Benchmark. We consider again a benchmark in which there is no noise in the evolution of the team's quality. Just as before, there is some $x^\star$ such that $V(x) = U$ for $x > x^\star$. In this range all good types join the talent pool. For $x < x^\star$, however, consistency requires the team's quality to fall and so only bad types are willing to join. Hence, $V(x) < W_{i^\dagger}$ for $x < x^\star$, and $V(x)$ is strictly and continuously increasing in this lower range.

Within that lower range, the team's quality and the value of a career within it both decline over time. Thus, for lower qualities, only the lower ranked types are willing to join the talent pool. Indeed, following the discussion in Section 4, only the very lowest type (that is, type $i = 0$) is willing to join the talent pool whenever

$$x < \bar{x}_1 \quad \text{where} \quad \bar{x}_1 = \frac{|\mu_0|}{\rho} \log\left(\frac{U}{U - W_1}\right).$$

If the quality of the team rises above $\bar{x}_1$ then type $i = 1$ is willing to serve the talent pool, and so the team's quality declines at rate $|\mu_1|$ rather than rate $|\mu_0|$. More generally, as the quality of the team grows then better recruits are attracted into the talent pool.

Proposition 5 (Equilibrium with Deterministic Quality and Many Types)**.** *For $i \leq i^\dagger$ define*

$$\bar{x}_i = \bar{x}_{i-1} + \frac{|\mu_{i-1}|}{\rho} \log\left(\frac{U - W_{i-1}}{U - W_i}\right), \tag{10}$$

where $\bar{x}_0 = 0$. For any $x^\star \in [0, \bar{x}_{i^\dagger}]$ there is an equilibrium in which (i) the team is immortal (it is a winning team) beginning from $x > x^\star$, but (ii) its quality decays to eventual failure if $x < x^\star$. Type $i < i^\dagger$ is the best available recruit if $\bar{x}_i < x < \min\{\bar{x}_{i+1}, x^\star\}$. The value of a career in the team is

$$V(x) = \begin{cases} U & x > x^\star \\ \bar{V}_i(x) & \bar{x}_i < x < \min\{\bar{x}_{i+1}, x^\star\}. \end{cases} \tag{11}$$

where $\bar{V}_i(x) = W_i + (U - W_i)(1 - e^{-\rho(x - \bar{x}_i)/|\mu_i|})$.

Just as in the two-type good-and-bad world, there are multiple equilibria: the threshold $x^\star$ that separates the optimistic and pessimistic worlds is (so long as $\sigma_i^2 = 0$ for all i) arbitrary.

Equilibrium with Stochastically Evolving Quality. In the presence of noisily evolving team quality, the properties of the career value functions are much as before: it increases smoothly from $V(0) = 0$ to $V(\bar{x}) = \bar{U}$, and there is an ascending sequence of quality thresholds such that a recruit of type i is willing to serve if and only if $x \geq x_i^\star$. For each segment $V_i(x)$, the differential equation (3) is satisfied. Moreover, this equation can be solved by using the boundary conditions $W_i = V_i(x_i^\star)$ and $W_{i+1} = V_i(x_{i+1}^\star)$, given knowledge of the set of thresholds. These observations are recorded in the next lemma.

Lemma 4 (Career Values in a Multiple-Type World). *The career value function is a strictly increasing and twice differentiable function of the team's quality x, and it satisfies $V(0) = 0$ and $\lim_{x\to\infty} V(x) = U$. There are n thresholds satisfying $0 < x_1^* < \cdots < x_n^*$ such that a recruit of type i is in the talent pool if and only if the team's quality exceeds x_i^*. Within the segment (x_i^*, x_{i+1}^*),*

$$V_i(x) = U + (W_{i+1} - U)\frac{e^{-b_i^-(x-x_i^*)} - e^{-b_i^+(x-x_i^*)}}{e^{-b_i^-(x_{i+1}^*-x_i^*)} - e^{-b_i^+(x_{i+1}^*-x_i^*)}} + (W_i - U)\frac{e^{b_i^+(x_{i+1}^*-x)} - e^{b_i^-(x_{i+1}^*-x)}}{e^{b_i^+(x_{i+1}^*-x_i^*)} - e^{b_i^-(x_{i+1}^*-x_i^*)}},$$

where we have defined $W_0 = 0$ and where this applies for $i \in \{0, 1, \ldots, n-1\}$. For the final segment:

$$V_n(x) = U + (W_n - U)e^{-b_n^+(x-x_n^*)}.$$

Lemma 4 characterizes the career value function contingent on the values taken by the n thresholds which separate the $n+1$ segments of that value function. To pin down those thresholds, we use smooth pasting conditions at each threshold: $V_{i-1}'(x_i^*) = V_i'(x_i^*)$ for each $i \in \{1, \ldots, n\}$. The properties of these derivatives are reported in Lemma 5.

Lemma 5 (Derivative of the Career Value Function). *(i) For $i \in \{0, 1, \ldots, n-1\}$ the career value function segments satisfy $V_i'(x_i^*) = D_{ii}(x_{i+1}^* - x_i^*)$ and $V_i'(x_{i+1}^*) = D_{i(i+1)}(x_{i+1}^* - x_i^*)$ where*

$$D_{ii}(y) \equiv \frac{(U - W_i)(b_i^+ e^{-b_i^- y} - b_i^- e^{-b_i^+ y}) - (U - W_{i+1})(b_i^+ - b_i^-)}{e^{-b_i^- y} - e^{-b_i^+ y}} \tag{12}$$

$$D_{i(i+1)}(y) \equiv \frac{(U - W_i)(b_i^+ - b_i^-) - (U - W_{i+1})(b_i^+ e^{b_i^- y} - b_i^- e^{b_i^+ y})}{e^{b_i^+ y} - e^{b_i^- y}}. \tag{13}$$

$D_{ii}(y)$ is strictly positive, continuous, $\lim_{y\to 0} D_{ii}(y) = \infty$ and $\lim_{y\to\infty} D_{ii}(y) = b_i^+(U - W_i)$. $D_{i(i+1)}(y)$ is strictly and continuously decreasing, $\lim_{y\to 0} D_{i(i+1)}(y) = \infty$, and there is a unique $\bar{y}_i > 0$ such that $D_{i(i+1)}(\bar{y}_i) = 0$. $D_{i(i+1)}^{-1}(\cdot) : \mathcal{R}_+ \mapsto [0, \bar{y}_i]$ is strictly and continuously decreasing.

(ii) The topmost segment of the value function satisfies $V_n'(x_n^*) = b_n^+(U - W_n)$.

Using the the properties of $D_{ii}(y)$ and $D_{i(i+1)}(y)$ from Lemma 5 it is straightforward to find the n quality thresholds that determine the presence of types $i \in \{1, \ldots, n\}$ in the talent pool.

To do this, we work with the gaps between those thresholds rather than the thresholds themselves: we define $y_i^* \equiv x_i^* - x_{i-1}^*$ for each $i \in \{1, \ldots, n\}$. For $i \in \{1, \ldots, n-1\}$,

$$V_{i-1}'(x_i^*) = V_i'(x_i^*) \quad \Leftrightarrow \quad y_i^* = D_{(i-1)i}^{-1}\left(D_{ii}(y_{i+1}^*)\right), \tag{14}$$

and so the gaps between the various thresholds can be found iteratively. For the top gap,

$$y_n^* = D_{(n-1)n}^{-1}\left(b_n^+(U - W_n)\right). \tag{15}$$

This system of n equations determines the equilibrium team-quality thresholds.

Proposition 6 (Equilibrium with Noisily Evolving Quality and Multiple Types). *There is a unique equilibrium in which a recruit of type i joins the talent pool if and only if the team's quality exceeds a threshold x_i^* where the set of n thresholds satisfy $0 < x_1^* < \cdots < x_n^*$. These thresholds satisfy $x_i^* = \sum_{j=1}^{i} y_j^*$, where the gaps between the thresholds are obtained from equations* (14)–(15).

As the team's quality improves, then so does the depth of the talent pool.

We have imposed only a single-crossing property on the relationship between recruit type and the expected change in the team's quality: our formal condition is that $\mu_i > 0$ if and only if $i \geq i^\dagger$. A natural tightening of this is a monotonicity condition, so that μ_i is increasing in i. If this is the case, then a winning team with higher quality enjoys a more rapid expected increase in that quality; similarly, a losing team declines more quickly as its quality falls.

Equilibrium with Limited Noise. Just as in the two-type world, our results are sharpened by taking the limit as the noise in the evolution of quality is eliminated. As before, we now write equation (1) as $dx_t = \mu_i\, dt + \xi\sigma_i\, dz_t$, where ξ is a scaling parameter that we take to zero.

The equilibrium in this multi-type world is characterized by equations (14)–(15), which iteratively determine gaps between the thresholds used by the different types of recruit:

$$D_{(n-1)n}(y_n^\star) = b_n^+(U - W_n) \quad \text{and} \quad D_{(i-1)i}(y_i^\star) = D_{ii}(y_{i+1}^\star) \quad \text{for} \quad i \in \{1, \ldots, n-1\}.$$

Of central importance are the properties of $D_{ii}(y)$ and $D_{i(i+1)}(y)$. Recall from Lemma 5 that there is a unique $\bar{y}_i > 0$ such that $D_{i(i+1)}(\bar{y}_i) = 0$ and that $D_{i(i+1)}(y) > 0$ if and only if $y < \bar{y}_i$.

Lemma 6 (Properties of $D_{ii}(y)$ and $D_{i(i+1)}(y)$). *For talent pools comprising good types (where $i \geq i^\dagger$), $\lim_{\xi\to 0} \bar{y}_i = 0$. For talent pools comprising bad types (where $i < i^\dagger$) and for $y > 0$,*

$$\lim_{\xi\to 0} D_{ii}(y) = \frac{\rho(U - W_i)}{|\mu_i|} \quad \textit{and} \quad \lim_{\xi\to 0} \bar{y}_i = \frac{|\mu_i|}{\rho} \log\left(\frac{U - W_i}{U - W_{i+1}}\right).$$

Furthermore, if $y < \lim_{\xi\to 0} \bar{y}_i$ then $\lim_{\xi\to 0} D_{i(i+1)}(y) = \infty$.

These properties lead directly to two conclusions regarding the gaps between the thresholds.

Firstly, note that the $D_{(i-1)i}(y_i^\star) > 0$ for each i, which implies that $y_i^\star < \bar{y}_{i-1}$. However, if $i > i^\star$ then $\bar{y}_{i-1} \to 0$ as $\xi \to 0$. This implies that $y_i^\star \to 0$ for all such i. This says that the gaps between thresholds used by all good recruits disappear as noise vanishes from the system. Equivalently, this means that there is a single critical threshold for the team's quality such that all good types are willing to serve once that threshold is met. The reason is that once one good type (the weakest of the good types, in fact) is willing to serve, then the expected movement of the team's quality is upward. As noise vanishes from the system, this means that the value of a career rapidly approaches U. Somewhat more formally,

$$x > \lim_{\xi^2\to 0} x_{i^\dagger}^\star \quad \Rightarrow \quad \lim_{\xi^2\to 0} V(x) = U.$$

This naturally means that the value of a career is now enough to entice all good types of recruit. In summary: once one good types is on board, then the others will follow.

Secondly, we can say something about the gaps between lower thresholds. For $i < i^\dagger$, suppose that $\lim_{\xi\to 0} y^\star_{i+1} > 0$. This implies (using Lemma 6) that $\lim_{\xi\to 0} D_{ii}(y^\star_{i+1}) > 0$. Noting that $\lim_{\xi\to 0} D_{i(i+1)}(y) = \infty$ if $y < \lim_{\xi\to 0} \bar{y}_i$, this implies that $\lim_{\xi\to 0} y^\star_i = \lim_{\xi\to 0} \bar{y}^\star_i$, and so

$$\lim_{\xi\to\infty} y^\star_{i+1} > 0 \quad \Rightarrow \quad \lim_{\xi\to\infty} y^\star_i = \frac{|\mu_i|}{\rho} \log\left(\frac{U - W_i}{U - W_{i+1}}\right).$$

Notice that this is the gap between thresholds used by bad recruits which apply in the deterministic equilibrium when the team is failing.

Proposition 7 (Equilibrium with Vanishing Noise and Multiple Types). *Suppose that the team's quality evolves via $dx_t = \mu_i\, dt + \xi\sigma_i\, dz_t$. There is a unique $x^\diamond < \bar{x}_{i^\dagger}$ such that*

$$\lim_{\xi\to 0} x^\star_i = \min\{x^\diamond, \bar{x}_i\} \quad \textit{and} \quad \lim_{\xi\to 0} V(x) = \begin{cases} U & x > x^\diamond \\ \bar{V}_i(x) & \bar{x}_{i-1} < x < \min\{\bar{x}_i, x^\diamond\}, \end{cases}$$

where both $\bar{x}_i$ and $\bar{V}_i(x)$ are taken from the statement of Proposition 5.

As in the two-type case, the elimination of noise selects a threshold $x^\diamond$ that divides winning and losing teams. However, Proposition 7 does not specify an explicit solution.

Nevertheless, such a close form is available. The next lemma identifies a critical type $i^\ddagger \in \{0, 1, \ldots, i^\dagger\}$ such that $y^\star_i \to 0$ for all $i > i^\ddagger$. The idea here is that once the team's quality is sufficiently strong to draw type $i^\ddagger$ into the talent pool, then all other higher types follow.

Lemma 7 (The Critical Recruit Type). *For notational convenience set $W_{n+1} \equiv U$ and $\bar{\mu}_j \equiv \mu_j/\sigma_j^2$. If $\sum_{j=0}^{i^\dagger-1} |\bar{\mu}_j|(W_{j+1} - W_j) \le \sum_{i^\dagger}^{n} \bar{\mu}_j(W_{j+1} - W_j)$ then set $i^\ddagger = 0$. Otherwise, set*

$$i^\ddagger = \max\left\{ i \in \{1, \ldots, i^\dagger\} \,:\, \sum_{j=i-1}^{i^\dagger-1} |\bar{\mu}_j|(W_{j+1} - W_j) \ge \sum_{i^\dagger}^{n} \bar{\mu}_j(W_{j+1} - W_j) \right\}.$$

If $i \le i^\ddagger$, then $\lim_{\xi\to 0} y^\star_i > 0$. However, if $i > i^\ddagger$ then

$$\lim_{\xi\to 0} \frac{y^\star_i}{\xi^2} = \frac{\log(1 + X_i)}{2\bar{\mu}_{i-1}} \quad \textit{where} \quad X_i = \frac{\bar{\mu}_{i-1}(W_i - W_{i-1})}{\sum_{j=i}^{n} \bar{\mu}_j(W_{j+1} - W_j)}.$$

Proposition 8 (Properties of Equilibrium as Noise Vanishes). *Consider the unique equilibrium in a world where the team's quality evolves via $dx_t = \mu_i\, dt + \xi\sigma_i\, dz_t$. If $i^\ddagger = 0$ then $x^\diamond = 0$, and so $\lim_{\xi\to 0} x^\star_i = 0$ for all i; the organization is always a winning team. Otherwise,*

$$x^\diamond = \bar{x}_{i^\ddagger-1} + \frac{|\mu_{i^\ddagger-1}|}{\rho} \log\left(\frac{(U - W_{i^\ddagger-1})|\bar{\mu}_{i^\ddagger-1}|}{\sum_{j=i^\ddagger}^{n} (\bar{\mu}_j + |\bar{\mu}_{i^\ddagger-1}|)(W_{j+1} - W_j)}\right).$$

A special case of this proposition is when all types $i \ge 1$ add value to the team, so that there is a single bad type. (Our two-type model studied in earlier sections is a special case of this.) In this case, $i^\ddagger \in \{0, 1\}$: either there is a winning team for all quality levels ($i^\ddagger = 0$, and hence $x^\diamond = 0$), or the critical type of recruit needed to sustain a winning team is simply the lowest good type ($i^\ddagger = 1$). For this case, the solution for $x^\diamond$ can be stated more cleanly.

Corollary (to Proposition 8). *Suppose that $i^\dagger = 1$. If $W_1|\bar{\mu}_0| < \sum_{i=1}^n \bar{\mu}_i(W_{i+1} - W_i)$ then $i^\ddagger = 0$ and $x^\diamond = 0$, and so (for $\xi \to 0$) there is always a winning team. Otherwise, $i^\ddagger = 1$ and*

$$x^\diamond = \frac{|\mu_0|}{\rho} \log\left(\frac{U|\bar{\mu}_0|}{\sum_{j=1}^n (\bar{\mu}_j + |\bar{\mu}_0|)(W_{j+1} - W_j)}\right) \quad \textit{where} \quad \bar{\mu}_j = \frac{\mu_j}{\sigma_j^2}.$$

For this $i^\dagger$ case, and where $i^\ddagger = 1$, the limiting threshold satisfies

$$(W_1 - \bar{V}_0(x^\diamond))|\mu_0| = (U - W_1)\sum_{j=1}^n \left(\frac{W_{j+1} - W_j}{U - W_1}\right)\bar{\mu}_j.$$

To interpret this, consider the lowest good type $i^\ddagger = 1$. His outside option is a career outside the organization worth W_1. If he joins the team, then there are two possibilities. Firstly, the team may decline into the losing team range ($x < x^\diamond$) resulting a reduction in his career value from W_1 to $\bar{V}_0(x^\diamond)$. (Recall that $\bar{V}_0(x^\diamond)$ is the value of a career in a deterministic world, when no good recruits are available and when the team's current quality is $x^\diamond$.) Secondly, the team may improve into the winning team range ($x > x^\diamond$) resulting in a gain in his career value from W_1 to U. These two possibilities are balanced by their relative likelihood. The (relative) chance of a step down is determined by $|\bar{\mu}_0| = \mu_0/\sigma_0^2$. In the two-type world, the (relative) chance of a step up is determined by $\bar{\mu}_1 = \mu_1/\sigma_1^2$. Here, however, that chance is determined by the weighted average of the rates $\bar{\mu}_i$ across all of the good types. Of course,

$$\sum_{j=1}^n \left(\frac{W_{j+1} - W_j}{U - W_1}\right)\bar{\mu}_j > \bar{\mu}_1,$$

so long as μ_1 is not too large; a sufficient condition is that $\bar{\mu}_1 < \bar{\mu}_j$ for $j > 1$. Hence the presence of higher good types raise the relative chance of the good outcome. A consequence is that $x^\diamond$ is lower than it would be if $i^\ddagger = 1$ were the only good type of recruit.

Why is this? Once the lowest type joins the team, then the quality moves upwards into the winning zone. This raises the value of a career within the team, which then attracts in the next good type. So long as that better type satisfies $\bar{\mu}_2 > \bar{\mu}_1$, this helps to cement the position of the winning team. This process continues as more types are brought into the team.

In fact, it is not just higher types that help the prospects of winning team by lowering the limiting threshold $x^\diamond$. Mathematically, this can be seen by contemplating the effect of removing some type i from the set of feasible good types. If $1 < i \leq n$ then this is equivalent to combining the ith and $(i-1)$th terms in the summation above. If $\bar{\mu}_i > \bar{\mu}_{i-1}$ then

$$\left(\frac{W_{i+1} - W_i}{U - W_1}\right)\bar{\mu}_i + \left(\frac{W_i - W_{i-1}}{U - W_1}\right)\bar{\mu}_{i-1} > \left(\frac{W_{i+1} - W_{i-1}}{U - W_1}\right)\bar{\mu}_{i-1}.$$

Putting this another way, adding an intermediate type—a stepping stone in the talent pool—to the set of possible recruits is helpful for team survival.

Proposition 9 (Stepping Stones). *Suppose that $\bar{\mu}_0 < 0 < \bar{\mu}_1 < \ldots < \bar{\mu}_n$, so that there is a single bad type and good types with better outside options generate a stronger (adjusting for σ_i^2) uplift in the team's quality. Removing a good type from the set of feasible recruit types increases $x^\diamond$. Hence, when noise is small, a greater variety of good types helps to sustain a winning team.*

Stepping Stones to a Winning Team. Our results here illustrate an important component in the complementary relationship between a team's success and its ability to attract talent. As in the two-state case, a recruit anticipates the effect of his own career choice on the value of joining a team. He also considers also how his decision affects the evolution of the talent pool via its impact on the choices of others. In particular, he considers the prospect that potentially higher quality recruits will join as a consequence of his own decision.

A central lesson is that heterogeneity in an organizations talent pool is an ingredient of success. As an example, consider an academic department that is recruiting. How far should it cast its net in its talent search? Should it target only job market stars with the higher outside options? Or should it limit the search to those who are sure to accept a job if offered? Or should it entertain a mix of possible types? Our analysis suggests that the latter is the optimal course of action. Why? An intermediate recruit will realize that should he join then higher quality recruits will be tempted to join also, thus cementing the upward drift in the department's fortunes.

Our analysis can be related to recent work on the determinants of footballing success. In the popular book "The Numbers Game" Anderson and Sally (2013) have argued that football is a weakest-link game in which "success is determined by whichever team makes the fewest mistakes, whether they are individual or collective" and, so, "upgrading a weak link can help a club more than improving its best player." According to their empirical analysis lavishing millions on recruiting the latest superstar striker will yield a lower return (in terms of success) than upgrading the weakest link, perhaps an incompetent defender or mis-communicating midfielder. Our formal analysis complements those findings by highlighting dependencies between recruitment choices that can benefit a team whose talent pool is sufficiently diverse. A signing that provides a marginal improvement in one area of the pitch (an upgrade in the weakest link) can have a knock-on effect in making recruitment of even higher quality players possible. This is an illustration of the stepping stone effect.

The stepping stone effect demonstrates another lesson: the sequencing of recruitment is important to an organization's success. Following the logic of proposition 9 it is straightforward to see that an academic department or a football team that craves success should sequence offers in an ascending order: first μ_1, then μ_2, and so on. The first and subsequent recruits provide stepping stones along the path toward the creation of a winning team.[12]

[12] There are many anecdotal examples from football, academia, or politics that illustrate the "stepping stone" effect. However, a personal anecdote offers a more precise illustration. Around seven years ago, one of the authors of this study bought a house in the vicinity of a London primary school that had reversed its trajectory to become one of the better performing schools in the borough. The Report of the Office for Standards in Education (that provides five yearly reports on UK schools) stated the reasons for the success story: "the prime factor in the continuing success of the school is the outstanding leadership and management of the head teacher and senior leadership team. This provides the drive that is continually seeking to improve the school's performance." Wisdom on the street corroborated the view that the leadership team had been pivotal to the team's success. However, those closer to the school revealed a more nuanced take. The current head was indeed of the highest calibre but it was during the tenure of the previous less celebrated head teacher that the school had begun its reversal. The mechanism linking that earlier success to later form was the precisely the one indicated here: the calibre of the talent pool of applicants when the head teacher reached the end of her term was higher as a result of the upward trajectory established during her tenure. Subsequent improvements could then be directly linked to earlier ones via the endogenous formation of the talent pool.

7. Concluding Remarks

We have analyzed the interplay between the quality of a team and the talent pool of those who are willing to accept a position within it. A winning team sustains long-term success, where success is defined by a well understood ranking such as a league table. The team's quality increases and it enjoys long-term survival if it can attract good recruits, but its fortunes decline if its talent pool is shallow. Higher-quality recruits require better career prospects (via a long-lived team) to be willing to serve and share common knowledge of the process by which the teams talent evolves (a Brownian motion with a drift and volatility that are determined by those who are willing to join if asked). We have characterized a unique equilibrium in which sufficiently talented recruits are available and so the team prospers if and only if the team's quality exceeds a threshold. That threshold is decreasing in the benefits of team membership and increasing in the attractiveness of outside opportunities.

The properties of noise in the recruitment process have implications for recruitment policy. An organization's leader is willing to trade lower expected talent when selecting a recruit in exchange for less risk if her team is winning, but makes the trade for greater risk if her team is losing. In the first situation, a winning team enjoys an expected increase in quality, and so any shocks to recruitment serve only to harm recruits' expectations of the team's longevity. In the second situation, however, noise in recruitment provides the chance that the team may benefit from a lucky streak and so it pays to "gamble on resurrection." Our multiple-type model reveals additional lessons with respect to the recruitment process: heterogeneity in an organization's talent pool is good. Path dependencies between recruitment choices that benefit a team are realized only if the talent pool is sufficiently diverse.

We conclude this discussion by considering several possible extensions. In many of the examples we study, competition is present: a soccer team's transfer policy translates into success when other clubs underperform; an academic department rises in the rankings if recruiting successfully when others do not; a government benefits from a deep reservoir of available talent when the opposition's talent pool is shallow by contrast. In sum, whether a team is successful or not depends not only on its own recruitment, but on that of its competitors. So a natural extension is to consider how the threshold needed to sustain a winning team is affected by the success of its recruitment strategy relative to that of its rivals.

We have modeled the evolution of team performance as a function of the quality of its latest recruit. Strategic complementarities arise over time as the decision of one individual affects those made by subsequent (potential) recruits. A deeper specification would allow for different types of interactions within teams. For example, and straightforwardly, organizational performance is linked to the fit of a new recruit with the existing team. Here that aspect of the relationship is captured via the noise parameter rather than studied directly.

Moreover, and finally, we have analyzed recruitment in a world where (implicitly at least) the organization selects the best available from its talent pool. Future applications of our model involve exploring the implications of different recruitment technologies.

Omitted Proofs

Proof of Proposition 1. This follows from the preceding discussion in the main text. □

Proof of Lemma 1. The key claim is that $V(x)$ is sufficiently smooth; specifically, that it is twice continuously differentiable and satisfies equation (4). This follows most generally from Theorem 1 of Strulovici and Szydlowski (2014). They consider a optimal control problem with a one-dimensional, time-homogenous diffusion state. Here, there is no control problem but nevertheless our model satisfies their Assumptions 1–3. Their Theorem 1 shows that (Strulovici and Szydlowski, 2014, p. 2) "the value function of any optimal control problem is twice continuously differentiable and satisfies the HJB (Hamilton-Jacobi-Bellman) equation everywhere." Here, this implies the key smoothness property of $V(x)$, and the HJB equation corresponds here to our equation (4). □

Proof of Lemma 2. For the low segment, the two conditions $V_L(0) = 0$ and $V_L(x^\star) = W_H$ are explicitly $0 = a_L^+ + a_L^- + U$ and $W_H = a_L^+ e^{-b_L^+ x^\star} + a_L^- e^{-b_L^- x^\star} + U$. They solve to yield

$$a_L^+ = \frac{U(1 - e^{-b_L^- x^\star}) - W_H}{e^{-b_L^- x^\star} - e^{-b_L^+ x^\star}} \quad \text{and} \quad a_L^- = \frac{W_H - U(1 - e^{-b_L^+ x^\star})}{e^{-b_L^- x^\star} - e^{-b_L^+ x^\star}}.$$

Plugging these coefficients into equation (5) gives us $V_L(x)$. For the high segment, the conditions $V_H(x^\star) = W_H$ and $V_H'(\bar{x}) = 0$ are explicitly $W_H = a_H^+ e^{-b_H^+ x^\star} + a_H^- e^{-b_H^- x^\star} + U$ and $0 = a_H^+ b_H^+ e^{-b_H^+ \bar{x}} + a_H^- b_H^- e^{-b_H^- \bar{x}}$, and these two equations solve to yield

$$a_H^+ = -\frac{b_H^- e^{-b_H^- \bar{x}}(U - W_H)}{b_H^- e^{-b_H^- \bar{x}} e^{-b_H^+ x^\star} - b_H^+ e^{-b_H^- x^\star} e^{-b_H^+ \bar{x}}} \quad \text{and} \quad a_H^- = \frac{b_H^+ e^{-b_H^+ \bar{x}}(U - W_H)}{b_H^- e^{-b_H^- \bar{x}} e^{-b_H^+ x^\star} - b_H^+ e^{-b_H^- x^\star} e^{-b_H^+ \bar{x}}}.$$

Plugging these coefficients into equation (5) gives us the solution for $V_H(x)$. □

Proof of Proposition 2. This follows from the discussion in the text, which in turn builds upon Lemma 2. In particular, equation (6) in the proposition is a re-statement of the condition $V_L'(x^\star) = V_H'(x^\star)$. The concluding comparative-static claims follow by inspection. □

Proof of Lemma 3. The properties of b_H^+ and b_L^- are obtained straightforwardly, and the claims regarding b_H^- and b_L^+ are obtained by applying l'Hôpital's rule. □

Proof of Proposition 3. The expression for $x^\diamond$ is obtained by solving equation (8). The claims regarding the equilibrium career value are obtained by using the expressions for the segments of the value function from Lemma 2 and taking the limit as $\xi \to 0$. This limit uses the properties of the coefficients $b_H^\pm$ and $b_L^\pm$ reported in Lemma 3. □

Proof of Proposition 4. These claims follow from an inspection of $x^\diamond$ in equation (9). □

Proof of Proposition 5. If the team quality is $\bar{x}_{i-1}$, then type $i-1$ is indifferent to participation and so $V(\bar{x}_{i-1}) = W_{i-1}$. For $x \in (\bar{x}_{i-1}, \bar{x}_i)$ the team's quality declines at rate $|\mu_{i-1}|$ and so the team continues for a length of time $(x - \bar{x}_{i-1})/|\mu_{i-1}|$ until the quality hits $\bar{x}_{i-1}$. These observations yield equation (11). Setting $V(x)$ equal to W_i yields equation (10). □

Proof of Lemma 4. The smoothness of $V(x)$ follows from the discussion given in the proof of Lemma 1. As in Section 5, the general solution for each segment of the career value function is $V_i(x)$ is $V_i(x) = U + a_i^+ e^{-b_i^+ x} + a_i^- e^{-b_i^- x}$ where $b_i^\pm = [\mu_i \pm \sqrt{\mu_i^2 + 2\rho\sigma_i^2}]/\sigma_i^2$. The coefficients $a_i^\pm$ are determined by the boundary conditions $V_i(x_i^*) = W_i$ and $V_i(x_{i+1}^*) = W_{i+1}$, where for notational convenience we define $W_0 = 0$ and $W_{n+1} = U$. Explicitly, these conditions are $W_i = U + a_i^+ e^{-b_i^+ x_i^*} + a_i^- e^{-b_i^- x_i^*}$ and $W_{i+1} = U + a_i^+ e^{-b_i^+ x_{i+1}^*} + a_i^- e^{-b_i^- x_{i+1}^*}$, and they yield

$$a_i^\mp = \pm \frac{(U - W_{i+1})e^{-b_i^\pm x_i^*} - (U - W_i)e^{-b_i^\pm x_{i+1}^*}}{e^{-(b_i^+ x_{i+1}^* + b_i^- x_i^*)} - e^{-(b_i^+ x_i^* + b_i^- x_{i+1}^*)}}.$$

Plugging these coefficients in yields the explicit solution stated in the lemma. □

Proof of Lemma 5. Differentiation of $V_i(x)$ reported in Lemma 4 yields

$$V_i'(x) = (W_{i+1} - U)\frac{b_i^+ e^{-b_i^+(x - x_i^*)} - b_i^- e^{-b_i^-(x - x_i^*)}}{e^{-b_i^-(x_{i+1}^* - x_i^*)} - e^{-b_i^+(x_{i+1}^* - x_i^*)}} + (W_i - U)\frac{b_i^- e^{b_i^-(x_{i+1}^* - x)} - b_i^+ e^{b_i^+(x_{i+1}^* - x)}}{e^{b_i^+(x_{i+1}^* - x_i^*)} - e^{b_i^-(x_{i+1}^* - x_i^*)}}.$$

Evaluating at x_i^* and x_{i+1}^* yields equations (12) and (13).

We now verify the properties of $D_{ii}(y)$. From equation (12), and noting that $b_i^+ > 0 > b_i^-$, the denominator of $D_{ii}(y)$ is strictly positive for all $y > 0$. Hence $D_{ii}(y)$ is positive if and only if the numerator is positive. Noting that $W_{i+1} > W_i$ and hence $U - W_i > U - W_{i+1}$, a sufficient condition for the numerator to be positive is $b_i^+ e^{-b_i^- y} - b_i^- e^{-b_i^+ y} \geq b_i^+ - b_i^-$, or equivalently

$$B(y) \equiv b_i^+(e^{-b_i^- y} - 1) - b_i^-(e^{-b_i^+ y} - 1) \geq 0. \tag{16}$$

Observe that $B(0) = 0$, and $B'(y) = -b_i^- b_i^+(e^{-b_i^- y} - e^{-b_i^+ y}) \geq 0$. Hence $B(y)$ is positive, which implies that $D_{ii}(y) > 0$ as required. Next, as $y \to \infty$ the term $e^{-b_i^+ y}$ vanishes while $e^{-b_i^- y} \to \infty$, and so $D_{ii}(y) \to b_i^+(U - W_i)$, as claimed. Taking the limit as $y \to 0$, the denominator converges to zero while the numerator remains strictly positive, and hence $D_{ii}(y) \to \infty$.

We now verify the properties of $D_{i(i+1)}(y)$. This may be written as

$$\frac{D_{i(i+1)}(y)}{(U - W_{i+1})(b_i^+ - b_i^-)} = \frac{\eta_i - e^{b_i^- y}}{e^{b_i^+ y} - e^{b_i^- y}} + \frac{b_i^-}{b_i^+ - b_i^-} \quad \text{where} \quad \eta_i \equiv \frac{U - W_i}{U - W_{i+1}} > 1. \tag{17}$$

By inspection, $D_{i(i+1)}(y) \to \infty$ as $y \to 0$, and $D_{i(i+1)}(y) \to b_i^-(U - W_{i+1}) < 0$ as $y \to \infty$. Moreover, from straightforward differentiation

$$D_{i(i+1)}'(y) < 0 \quad \Leftrightarrow \quad -b_i^- e^{b_i^- y}(e^{b_i^+ y} - e^{b_i^- y}) < (\eta_i - e^{b_i^- y})(b_i^+ e^{b_i^+ y} - b_i^- e^{b_i^- y}). \tag{18}$$

A little more algebra yields the inequality $b_i^- e^{b_i^- y}(\xi_i - e^{b_i^+ y}) < b_i^+(\xi_i - e^{b_i^- y})e^{b_i^+ y}$. The left-hand side is decreasing in η_i while the right-hand side is increasing in η_i. Thus, it is sufficient to check this inequality holds for $\eta_i = 1$. That is, it holds if $B(y) \geq 0$, where $B(y)$ is defined in equation (16). We demonstrated earlier that $B(y)$ is positive. We conclude that $D_{i(i+1)}(y)$ is continuously decreasing in y, becoming negative for large y; hence there is a $\bar{y}_i$ where it crosses zero. Thus, its inverse is well defined and has the properties claimed. □

Proof of Proposition 6. This follows from the preceding discussion in the main text. □

Lemma (More General Statement of Lemma 6)**.** *Suppose that $dx_t = \mu_i dt + \xi\sigma_i dz_t$. For talent pools comprising bad types (that is, where $i < i^\dagger$) and for $y > 0$,*

$$\lim_{\xi\to 0} D_{ii}(y) = \frac{\rho(U-W_i)}{|\mu_i|} \quad \textit{and} \quad \lim_{\xi\to 0}\xi^2 D_{i(i+1)}(y) = \frac{2\mu_i\left((U-W_{i+1})-(U-W_i)e^{-\rho y/|\mu_i|}\right)}{\sigma_i^2}$$

For talent pools comprising good types (that is, where $i \geq i^\dagger$) and for $y > 0$,

$$\lim_{\xi\to 0}\xi^2 D_{ii}(y) = \frac{2\mu_i\left((U-W_i)-(U-W_{i+1})e^{-\rho y/\mu_i}\right)}{\sigma_i^2} \quad \textit{and} \quad \lim_{\xi\to 0} D_{i(i+1)}(y) = -\frac{\rho(U-W_{i+1})}{\mu_i}.$$

Proof. The properties of $b_H^\pm$ reported in Lemma 3 extend to $b_i^\pm$ for $i \geq i^\dagger$, and similarly the properties of $b_L^\pm$ from Lemma 3 extend to $b_i^\pm$ for $i < i^\dagger$. For the case $i < i^\dagger$,

$$\begin{aligned} D_{ii}(y) = \frac{e^{-b_i^- y}}{e^{-b_i^- y}-e^{-b_i^+ y}}\left((U-W_i)b_i^+ - (U-W_i)b_i^- e^{-(b_i^+-b_i^-)y} - (U-W_{i+1})(b_i^+-b_i^-)e^{b_i^- y}\right) \\ \to \quad (U-W_i)\lim_{\xi\to 0} b_i^+ = \frac{\rho(U-W_i)}{|\mu_i|}, \end{aligned}$$

where the limit is driven by $e^{b_i^- y}\to 0$. For the other case where $i \geq i^\dagger$:

$$\begin{aligned} \xi^2 D_{ii}(y) = \frac{e^{-b_i^- y}}{e^{-b_i^- y}-e^{-b_i^+ y}}\left((U-W_i)(\xi^2 b_i^+ - \xi^2 b_i^- e^{-(b_i^+-b_i^-)y)}) - \frac{(U-W_{i+1})\xi^2(b_i^+-b_i^-)}{e^{-b_i^- y}}\right) \\ \to \frac{2\mu_i\left(U-W_i-(U-W_{i+1})e^{-\rho y/\mu_i}\right)}{\sigma_i^2}. \end{aligned}$$

Similar limits can be taken to obtain the claims for $D_{i(i+1)}(y)$.

The statements of Lemma 6 follow as corollaries. □

Proof of Proposition 7. This follows from Lemma 6 and the arguments in the text. □

Lemma 8 (Properties of $D_{ii}(y_{i+1}^\star)$)**.** *If $\lim_{\xi\to 0} y_{i+1}^\star = 0$ and $\lim_{\xi\to 0}(y_{i+1}^\star/\xi^2) = Y_{i+1} \in (0,\infty)$ then*

$$\lim_{\xi\to 0}\xi^2 D_{ii}(y_{i+1}^\star) = \frac{2\mu_i}{\sigma_i^2}\frac{W_{i+1}-W_i}{1-e^{-2\mu_i Y_{i+1}/\sigma_i^2}}.$$

Proof of Lemma 8. Throughout "$\to$" indicates the limit as $\xi\to 0$. $\xi^2 D_{ii}(y_{i+1}^\star)$ satisfies

$$\begin{aligned} \xi^2 D_{ii}(y_{i+1}^\star) &= \frac{(U-W_i)(\xi^2 b_i^+ e^{-\xi^2 b_i^-(y_{i+1}^\star/\xi^2)} - \xi^2 b_i^- e^{-\xi^2 b_i^+(y_{i+1}^\star/\xi^2)}) - (U-W_{i+1})(\xi^2 b_i^+ - \xi^2 b_i^-)}{e^{-\xi^2 b_i^-(y_{i+1}^\star/\xi^2)} - e^{-\xi^2 b_i^+(y_{i+1}^\star/\xi^2)}} \\ &\to \quad \frac{(U-W_i)(\xi^2 b_i^+ e^{-\xi^2 b_i^- Y_{i+1}} - \xi^2 b_i^- e^{-\xi^2 b_i^+ Y_{i+1}}) - (U-W_{i+1})(\xi^2 b_i^+ - \xi^2 b_i^-)}{e^{-\xi^2 b_i^- Y_{i+1}} - e^{-\xi^2 b_i^+ Y_{i+1}}}. \end{aligned}$$

If $i \geq i^\dagger$ then $\mu_i > 0$ and $\xi^2 b_i^+$ has a positive finite limit, whereas $\xi^2 b_i^- \to 0$. Hence

$$\xi^2 D_{ii}(y_{i+1}^\star) \to \frac{(U-W_i)\xi^2 b_i^+ - (U-W_{i+1})\xi^2 b_i^+}{1-e^{-\xi^2 b_i^+ Y_{i+1}}} = \frac{(W_{i+1}-W_i)\xi^2 b_i^+}{1-e^{-\xi^2 b_i^+ Y_{i+1}}} \to \frac{2\mu_i}{\sigma_i^2}\frac{W_{i+1}-W_i}{1-e^{-2\mu_i Y_{i+1}/\sigma_i^2}},$$

where the final step is because $\xi^2 b_i^+ \to 2\mu_i/\sigma_i^2$. If $i < i^\dagger$ then $\mu_i < 0$ and $\xi^2 b_i^-$ has a negative finite limit, whereas $\xi^2 b_i^+ \to 0$. Hence in this case

$$\xi^2 D_{ii}(y_{i+1}^\star) \to \frac{(U-W_i)\xi^2 b_i^- - (U-W_{i+1})\xi^2 b_i^-}{1-e^{-\xi^2 b_i^- Y_{i+1}}} = \frac{(W_{i+1}-W_i)\xi^2 b_i^-}{1-e^{-\xi^2 b_i^- Y_{i+1}}} \to \frac{2\mu_i}{\sigma_i^2}\frac{W_{i+1}-W_i}{1-e^{-2\mu_i Y_{i+1}/\sigma_i^2}}.$$

$\square$

Lemma 9 (Properties of $D_{(i-1)i}(y_i^\star)$)**.** *If* $\lim_{\xi\to} \xi^2 D_{ii}(y_{i+1}^\star) = D_i \in (0,\infty)$ *then*

$$i-1 \geq i^\dagger \quad \Rightarrow \quad \lim_{\xi\to 0} \frac{y_i^\star}{\xi^2} = \frac{\sigma_{i-1}^2}{2\mu_{i-1}} \log\left(1 + \frac{2\mu_{i-1}}{\sigma_{i-1}^2}\frac{W_i - W_{i-1}}{D_i}\right).$$

If $i-1 < i^\dagger$ *then this claim also holds if and only if*

$$W_i - W_{i-1} < \frac{\sigma_{i-1}^2 D_i}{2|\mu_{i-1}|}.$$

The remaining case is when $i-1 < i^\dagger$ *and the above inequality fails. In this case*

$$\lim_{\xi\to 0} y_i^\star = -\frac{\mu_{i-1}}{\rho}\log\left(\frac{U - W_{i-1}}{U - W_i - \frac{\sigma_{i-1}^2 D_i}{2\mu_{i-1}}}\right)$$

Proof of Lemma 9. $\xi^2 D_{(i-1)i}(y_i^\star)$ satisfies

$$\xi^2 D_{(i-1)i}(y_i^\star) = \frac{(U-W_{i-1})(\xi^2 b_{i-1}^+ - \xi^2 b_{i-1}^-) - (U-W_i)(\xi^2 b_{i-1}^+ e^{b_{i-1}^- y_i^\star} - \xi^2 b_{i-1}^- e^{b_{i-1}^+ y_i^\star})}{e^{b_{i-1}^+ y_i^\star} - e^{b_{i-1}^- y_i^\star}}.$$

Suppose that $i-1 \geq i^\dagger$, so that $\mu_i > 0$, $\xi^2 b_{i-1}^+ \to 2\mu_{i-1}/\sigma_{i-1}^2$, and $\xi^2 b_{i-1}^- \to 0$. Hence:

$$D_i = \frac{2\mu_{i-1}}{\sigma_{i-1}^2}\lim_{\xi\to 0}\frac{(U-W_{i-1}) - (U-W_i)e^{b_{i-1}^- y_i^\star}}{e^{b_{i-1}^+ y_i^\star} - e^{b_{i-1}^- y_i^\star}} + (U-W_i)\lim_{\xi\to 0}\frac{\xi^2 b_{i-1}^- e^{b_{i-1}^+ y_i^\star}}{e^{b_{i-1}^+ y_i^\star} - e^{b_{i-1}^- y_i^\star}}.$$

We know that $y_i^\star \to 0$, and b_{i-1}^- converges. Hence $b_{i-1}^- y_i^\star \to 0$, which implies $e^{b_{i-1}^- y_i^\star} \to 1$. So:

$$D_i = \frac{2\mu_{i-1}}{\sigma_{i-1}^2}\lim_{\xi\to 0}\frac{W_i - W_{i-1}}{e^{b_{i-1}^+ y_i^\star} - 1} + (U-W_i)\lim_{\xi\to 0}\frac{\xi^2 b_{i-1}^- e^{b_{i-1}^+ y_i^\star}}{e^{b_{i-1}^+ y_i^\star} - 1}.$$

This equality is inconsistent with either $b_{i-1}^+ y_i^\star \to 0$ (the right-hand side diverges) or $b_{i-1}^+ y_i^\star \to \infty$ (the right-hand size converges to zero). $\lim_{\xi\to\infty} b_{i-1}^+ y_i^\star \in (0,\infty)$, which in turn (noting that $\xi^2 b_{i-1}^- \to 0$) implies that the second term above converges to zero. Hence

$$D_i = \frac{2\mu_{i-1}}{\sigma_{i-1}^2}\lim_{\xi\to 0}\frac{W_i - W_{i-1}}{e^{b_{i-1}^+ y_i^\star} - 1} \quad \Rightarrow \quad \lim_{\xi\to 0} b_{i-1}^+ y_i^\star = \log\left(1 + \frac{2\mu_{i-1}}{\sigma_{i-1}^2}\frac{W_i - W_{i-1}}{D_i}\right).$$

Of course, $b_{i-1}^+ y_i^\star = (\xi^2 b_{i-1}^+)(y_i^\star/\xi^2)$, and $\xi^2 b_{i-1}^+ \to 2\mu_{i-1}/\sigma_{i-1}^2$. This yields the first claim.

Suppose that $i-1 < i^\dagger$, so that $\mu_{i-1} < 0$, $\xi^2 b_{i-1}^- \to 2\mu_{i-1}/\sigma_{i-1}^2$, and $\xi^2 b_{i-1}^+ \to 0$. Hence:

$$D_i = -\frac{2\mu_{i-1}}{\sigma_{i-1}^2}\lim_{\xi\to 0}\frac{(U-W_{i-1}) - (U-W_i)e^{b_{i-1}^+ y_i^\star}}{e^{b_{i-1}^+ y_i^\star} - e^{b_{i-1}^- y_i^\star}} - (U-W_i)\lim_{\xi\to 0}\frac{\xi^2 b_{i-1}^+ e^{b_{i-1}^- y_i^\star}}{e^{b_{i-1}^+ y_i^\star} - e^{b_{i-1}^- y_i^\star}}.$$

One possibility is that $\lim_{\xi\to 0} y_i^\star > 0$. If this is the case, then $e^{b_{i-1}^- y_i^\star} \to 0$ whereas $e^{b_{i-1}^+ y_i^\star}$ remains finite. Given that $\xi^2 b_{i-1}^+ \to 0$, the second term above is zero. Hence:

$$D_i = -\frac{2\mu_{i-1}}{\sigma_{i-1}^2}\lim_{\xi\to 0}\frac{(U-W_{i-1}) - (U-W_i)e^{b_{i-1}^+ y_i^\star}}{e^{b_{i-1}^+ y_i^\star}}$$

$$\Rightarrow \quad \lim_{\xi\to 0} y_i^\star = -\frac{\mu_{i-1}}{\rho}\log\left(\frac{U-W_{i-1}}{U - W_i - \frac{\sigma_{i-1}^2 D_i}{2\mu_{i-1}}}\right),$$

where here we used the fact that $b_{i-1}^+ \to -\rho/\mu_{i-1}$. This works so long as the logarithm is positive, which requires its argument to exceed one. That is, we require

$$\frac{U - W_{i-1}}{U - W_i - \frac{\sigma_{i-1}^2 D_i}{2\mu_{i-1}}} > 1 \quad \Leftrightarrow \quad W_i - W_{i-1} > -\frac{\sigma_{i-1}^2 D_i}{2\mu_{i-1}}.$$

The other possibility is that $y_i^\star \to 0$. b_{i-1}^+ converges, and so $b_{i-1}^+ y_i^\star \to 0$ and $e^{b_{i-1}^+ y_i^\star} \to 1$. So:

$$D_i = -\frac{2\mu_{i-1}}{\sigma_{i-1}^2} \lim_{\xi \to 0} \frac{W_i - W_{i-1}}{1 - e^{b_{i-1}^- y_i^\star}} - (U - W_i) \lim_{\xi \to 0} \frac{\xi^2 b_{i-1}^+ e^{b_{i-1}^- y_i^\star}}{1 - e^{b_{i-1}^- y_i^\star}}.$$

Mimicking early arguments, $b_{i-1}^- y_i^\star$ cannot diverge or vanish to zero. Given that it converges, and that $\xi^2 b_{i-1}^+ \to 0$, the right-hand term vanishes to zero. Hence:

$$D_i = -\frac{2\mu_{i-1}}{\sigma_{i-1}^2} \lim_{\xi \to 0} \frac{W_i - W_{i-1}}{1 - e^{b_{i-1}^- y_i^\star}} \quad \Rightarrow \quad \lim_{\xi \to 0} b_{i-1}^- y_i^\star = \log\left(1 + \frac{2\mu_{i-1}}{\sigma_{i-1}^2} \frac{W_i - W_{i-1}}{D_i}\right).$$

Of course, $\xi^2 b_{i-1}^- \to 2\mu_{i-1}/\sigma_{i-1}^2$, which yields the claim of the lemma. This is valid so long as the argument of the logarithm is positive. This requires

$$1 + \frac{2\mu_{i-1}}{\sigma_{i-1}^2} \frac{W_i - W_{i-1}}{D_i} > 0 \quad \Leftrightarrow \quad W_i - W_{i-1} < -\frac{\sigma_{i-1}^2 D_i}{2\mu_{i-1}},$$

which is the inequality reported in the lemma. □

Proof of Lemma 7. Consider the topmost gap $y_n^\star$. This satisfies $\xi^2 D_{(n-1)n}(y_n^\star) = \xi^2 b_n^+ (U - W_n)$. Define $D_n = 2\mu_n (U - W_n)/\sigma_n^2 = 2\bar{\mu}_n (W_{n+1} - W_n)$ and note that $\xi^2 b_n^+ (U - W_n) \to D_n$. We use Lemma 9. One possibility is that $n > i^\dagger \geq i^\ddagger$. The first part of Lemma 9 applies. Hence:

$$\begin{aligned} \lim_{\xi \to 0} \frac{y_n^\star}{\xi^2} = \frac{\sigma_{n-1}^2}{2\mu_{n-1}} \log\left(1 + \frac{2\mu_{n-1}}{\sigma_{n-1}^2} \frac{W_n - W_{n-1}}{D_n}\right) \\ = \frac{1}{2\bar{\mu}_{n-1}} \log\left(1 + \frac{\bar{\mu}_{n-1}(W_n - W_{n-1})}{\bar{\mu}_n (W_{n+1} - W_n)}\right) = \frac{\log(1 + X_n)}{2\bar{\mu}_{n-1}}. \end{aligned}$$

A second possibility is that $n = i^\dagger > i^\ddagger$. If $n > i^\ddagger$ then (from the definition of $i^\ddagger$) it must be that $|\bar{\mu}_{n-1}|(W_n - W_{n-1}) < \bar{\mu}_n (W_{n+1} - W_n)$. The second part of Lemma 9 now applies, and hence $(y_n^\star/\xi^2) \to \log(1 + X_n)/(2\bar{\mu}_{n-1})$ as before. The final case to consider is when $n = i^\dagger = i^\ddagger$. The third part of Lemma 9 now applies, and so $\lim_{\xi \to 0} y_n^\star > 0$ as claimed.

We continue inductively. Suppose that the claimed properties apply for types $\{i+1, \ldots, n\}$. If $\lim_{\xi \to 0} y_{i+1}^\star > 0$ (which can only be true if $i < i^\dagger$) then (applying Lemma 6) $\xi^2 D_{ii}(y_{i+1}^\star) \to 0$, which implies (from the more general statement of Lemma 6) that $\xi^2 D_{i(i+1)}(y_i^\star) \to 0$. Using that general statement once more, this implies that $y_i^\star \to \bar{y}_i > 0$. Hence, the claimed properties apply for i. The other possibility is that $\lim_{\xi \to 0} y_{i+1}^\star = 0$. If this is so, then we apply Lemma 8. That is, $\xi^2 D_{ii}(y_{i+1}^\star) \to D_i$ where

$$D_i = \frac{2\mu_i}{\sigma_i^2} \frac{W_{i+1} - W_i}{1 - e^{-2\mu_i Y_{i+1}/\sigma_i^2}} = \frac{2\bar{\mu}_i (W_{i+1} - W_i)}{X_{i+1}/(1 + X_{i+1})}.$$

We then continue (as we did for the inductive basis of $i = n$) by applying Lemma 9. For example, if $i \geq i^\dagger + 1$, then the first part of Lemma 9 applies, and so

$$\lim_{\xi \to 0} \frac{y_i^\star}{\xi^2} = \frac{1}{2\bar{\mu}_{i-1}} \log\left(1 + \frac{2\bar{\mu}_{i-1}(W_i - W_{i-1})}{D_i}\right) = \frac{1}{2\bar{\mu}_{i-1}} \log\left(1 + \frac{\bar{\mu}_{i-1}(W_i - W_{i-1})X_{i+1}}{\bar{\mu}_i(W_{i+1} - W_i)(1 + X_{i+1})}\right)$$
$$= \frac{1}{2\bar{\mu}_{i-1}} \log\left(1 + \frac{\bar{\mu}_{i-1}(W_i - W_{i-1})}{\sum_{j=i}^n \bar{\mu}_j (W_{j+1} - W_j)}\right) = \frac{\log(1 + X_i)}{2\bar{\mu}_{i-1}}.$$

The other arguments used as part of the induction basis also apply for the cases where $i < i^\dagger + 1$. By the principle of induction, the claims of the lemma hold for all i. □

Proof of Proposition 8. Lemma 7 establishes that $y_i^\star \to 0$ for $i > i^\ddagger$ and that $(y_i^\star/\xi^2) \to \log(1 + X_i)/(2\bar{\mu}_{i-1})$. In particular, $y^*_{i^\ddagger+1} \to 0$. Applying Lemma 8, this means that

$$\lim_{\xi \to 0} \xi^2 D_{i^\ddagger i^\ddagger}(y^\star_{i^\ddagger+1}) = D_{i^\ddagger} \quad \text{where} \quad D_{i^\ddagger} = \frac{2\bar{\mu}_{i^\ddagger}(W_{i+1} - W_i)(1 + X_{i^\ddagger+1})}{X_{i^\ddagger+1}} = 2\sum_{j=i^\ddagger}^n \mu_j (W_{j+1} - W_j).$$

Turning to $y^\star_{i^\ddagger}$, the third part of Lemma 9 applies. Hence:

$$\lim_{\xi \to 0} y^\star_{i^\ddagger} = -\frac{\mu_{i^\ddagger-1}}{\rho} \log\left(\frac{U - W_{i^\ddagger-1}}{U - W_{i^\ddagger} - \frac{D_{i^\ddagger}}{2\mu_{i^\ddagger-1}}}\right)$$
$$= -\frac{\mu_{i^\ddagger-1}}{\rho} \log\left(\frac{(U - W_{i^\ddagger-1})\mu_{i^\ddagger-1}}{\mu_{i^\ddagger-1}(U - W_{i^\ddagger}) - \sum_{j=i^\ddagger}^n \mu_j (W_{j+1} - W_j)}\right)$$
$$= \frac{|\mu_{i^\ddagger-1}|}{\rho} \log\left(\frac{(U - W_{i^\ddagger-1})|\mu_{i^\ddagger-1}|}{\sum_{j=i^\ddagger}^n (\mu_j + |\mu_{i^\ddagger-1}|)(W_{j+1} - W_j)}\right). \quad \square$$

Proof of Proposition 9. The threshold $x^\diamond$ is determined by $\sum_{j=1}^n (\bar{\mu}_j + |\bar{\mu}_0|)(W_{j+1} - W_j)$. Removing the lowest type directly lowers this summation, by removing the first term. Removing any other type $i > 1$ is equivalently to combining two neighboring terms, and this again lowers the summation following an argument equivalent to that described in the main text. □

References

ACEMOGLU, D., G. EGOROV, AND K. SONIN (2010): "Political Selection and Persistence of Bad Governments," *Quarterly Journal of Economics*, 125(4), 1511–1575.

ANDERSON, C., AND D. SALLY (2013): *The Numbers Game: Why Everything You Know About Football is Wrong*. Penguin, New York.

BANERJEE, A. V., AND R. PANDE (2007): "Parochial Politics: Ethnic Preferences and Politician Corruption," .

BARRO, R. J. (1973): "The Control of Politicians: An Economic Model," *Public Choice*, 14(1), 19–42.

BEATH, A., F. CHRISTIA, G. EGOROV, AND R. ENIKOLOPOV (2014): "Electoral Rules and Political Selection: Theory and Evidence from a Field Experiment in Afghanistan," *Working Paper: World Bank, MIT, Northwestern, and UPF*.

BESLEY, T. (2004): "Paying Politicians: Theory and Evidence," *Journal of the European Economic Association*, 2(2-3), 193–215.

——— (2005): "Political Selection," *Journal of Economic Perspectives*, 19(3), 43–60.

——— (2006): *Principled Agents? The Political Economy of Good Government*. Oxford University Press, Oxford.

BESLEY, T., AND S. COATE (1997): "An Economic Model of Representative Democracy," *Quarterly Journal of Economics*, 112(1), 85–114.

BESLEY, T., O. FOLKE, T. PERSSON, AND J. RICKNE (2013): "Gender Quotas and the Crisis of the Mediocre Man: Theory and Evidence from Sweden," *Working Paper*.

BROLLO, F., T. NANNICINI, R. PEROTTI, AND G. TABELLINI (2013): "The Political Resource Curse," *American Economic Review*, 103(5), 1759–1796.

BUISSERET, P. E., AND C. PRATO (2013): "Electoral Control and the Human Capital of Politicians," *Working Paper: Warwick University and Georgetown University*.

CALLANDER, S. (2011a): "Search and Learning by Trial and Error," *American Economic Review*, 101(6), 2277–2308.

——— (2011b): "Searching for Good Policies," *American Political Science Review*, 105(4), 643–662.

CALLANDER, S., AND P. HUMMEL (2014): "Preemptive Policy Experimentation," *Econometrica*, 82(4), 1509–1528.

CASELLI, F., AND M. MORELLI (2004): "Bad Politicians," *Journal of Public Economics*, 88(3-4), 759–782.

DEWAN, T., AND D. P. MYATT (2010): "The Declining Talent Pool of Government," *American Journal of Political Science*, 54(2), 267–286.

——— (2012): "Dynamic Government Performance: Honeymoons and Crises of Confidence," *American Political Science Review*, 106(1), 123–145.

DIERMEIER, D., M. P. KEANE, AND A. MERLO (2005): "A Political Economy Model of Congressional Careers," *American Economic Review*, 95(1), 347–373.

DOWNS, G. W., AND D. M. ROCKE (1993): "Conflict, Agency, and Gambling for Resurrection: The Principal-Agent Problem Goes to War," *American Journal of Political Science*, 38(2), 362–380.

EGOROV, G., AND K. SONIN (2012): "Incumbency Advantages in Non-Democracies," *Working Paper: Northwestern University and New Economic School*.

FEARON, J. D. (1999): "Electoral Accountability and the Control of Politicians: Selecting Good Types versus Sanctioning Poor Performance," in *Democracy, Accountability, and Representation*, ed. by A. Przeworski, S. Stokes, and B. Manin, Cambridge. Cambridge University Press.

FEREJOHN, J. A. (1986): "Incumbent Performance and Electoral Control," *Public Choice*, 50(1-3), 5–25.

FERRAZ, C., AND F. FINAN (2008): "Motivating Politicians: The Impacts of Monetary Incentives on Quality and Performance," *Working Paper: IPEA and UCLA*.

FOLKE, O., T. PERSSON, AND J. RICKNE (2014): "Preferential Voting , Accountability and Promotions into Political Power: Evidence from Sweden," *Working Paper: Columbia University and IIES Stockholm*.

FRÉCHETTE, G. R., F. MANIQUET, AND M. MORELLI (2008): "Incuments' Interests and Genda Quotas," *American Journal of Political Science*, 52(4), 891–909.

FRISELL, L. (2009): "A Theory of Self-Fulfilling Political Expectations," *Journal of Public Economics*, 93(5-6), 715–720.

GAGLIARDUCCI, S., AND T. NANNICINI (2013): "Do Better Paid Politicians Perform Better? Disentangling Incentives From Selection," *Journal of the European Economic Association*, 11(2), 369–398.

GAGLIARDUCCI, S., T. NANNICINI, AND P. NATICCHIONI (2010): "Moonlighting Politicians," *Journal of Public Economics*, 94(9-10), 688–699.

GALASSO, V., AND T. NANNICINI (2011): "Competing on Good Politicians," *American Political Science Review*, 105(1), 79–99.

HAZELL, R., AND B. YONG (2011): "Putting Goats Amongst the Wolves: Appointing Ministers from Outside Parliament," *Discussion Paper: UCL Constitution Unit*.

JÚLIO, PAULO AND TAVARES, J. (2014): "The Good, the Bad, and the Different: Can Gender Quotas Raise the Quality of Politicians?," *Working Paper: Nova School of Business and Economics*, November.

KEANE, M. P., AND A. MERLO (2010): "Money , Political Ambition, and the Career Decisions of Politicians," *American Economic Journal: Microeconomics*, 2(3), 186–215.

KEY, JR., V. O. (1956): *American State Politics: An Introduction*. Alfred A. Knopf, New York.

KREMER, M., AND C. MORCOM (2000): "Elephants," *American Economic Review*, 90(1), 212–234.

MAJUMDAR, S., AND S. W. MUKAND (2004): "Policy Gambles," *American Economic Review*, 94(4), 1207–1222.

MATTOZZI, A., AND A. MERLO (2008): "Political Careers or Career Politicians?," *Journal of Public Economics*, 92(3-4), 597–608.

——— (2011): "Mediocracy," *PIER Working Paper 11-002, University of Pennsylvania*.

MATTOZZI, A., AND E. SNOWBERG (2014): "The Right Type of Legislator," *Working Paper: EUI and Caltech*.

MESSNER, M., AND M. K. POLBORN (2004): "Paying Politicians," *Journal of Public Economics*, 88(12), 2423–2445.

PAXMAN, J. (2003): *The Political Animal: An Anatomy*. Penguin, London.

ROWAT, C., AND J. DUTTA (2007): "The Commons with Capital Markets," *Economic Theory*, 31(2), 225–254.

STRÖMBERG, D. (2008): "How the Electoral College Influences Campaigns and Policy: The Probability of Being Florida," *American Economic Review*, 98(3), 769–807.

STRULOVICI, B., AND M. SZYDLOWSKI (2014): "On the Smoothness of Value Functions and the Existence of Optimal Strategies," *Working Paper: Northtwestern University and University of Minnesota*.

SVOLIK, M. W. (2013): "Learning to Love Democracy: Electoral Accountability and the Success of Democracy," *American Journal of Political Science*, 57(3), 685–702.

TORNELL, A., AND A. VELASCO (1992): "The Tragedy of the Commons and Economic Growth: Why Does Capital Flow from Poor to Rich Countries?," *Journal of Political Economy*, 100(6), 1208.

D23, D71, D72, D73, H11, J45