CONDORCET MEETS ELLSBERG

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Abstract. The Condorcet Jury Theorem states that given subjective expected utility maximization and common values, the equilibrium probability that the correct candidate wins goes to one as the size of the electorate goes to infinity. This paper studies strategic voting when voters have pure common values but may be ambiguity averse – exhibit Ellsberg-type behavior – as modeled by maxmin expected utility preferences. It provides sufficient conditions so that the equilibrium probability of the correct candidate winning the election is bounded above by one half in at least one state. As a consequence, there is no equilibrium in which information aggregates.

1. Introduction

When deciding how to vote, each individual may have private information about which of the two candidates will be better. Both the information itself and how others react to it affect how a rational voter casts her ballot. If each voter maximizes subjective expected utility (henceforth, SEU) and voters have common values, then there exists an equilibrium to the voting game in which all private information is revealed for a large enough electorate.[1] This result, known as the Condorcet Jury Theorem, provides an important efficiency justification for democracy as a political system. It describes conditions under which democracy is superior to even a benevolent dictatorship, since the probability of selecting the better policy is higher when an election rather than a privately informed dictator picks the policy.

This paper shows that when voters are ambiguity averse and their private information is ambiguous, there may not exist an equilibrium in which information aggregates, regardless of the size of the electorate. Theorems 2 and 4 show that for many pure common-values voting games, *no equilibrium of the game aggregates information.* A rational voter conditions her action on the probability that her vote changes the outcome of the election. Consequently, each voter's equilibrium strategy may differ from the action that her private information

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[1]For instance, see Austen-Smith and Banks [1996], Feddersen and Pesendorfer [1997, 1999], Myerson [1998] or Wit [1998] .

would suggest is best if she disregarded others' strategies. Given SEU, each vote noisily reveals private information, and with enough voters, information aggregates. When voters are ambiguity averse, each picks a voting strategy as if to insure herself against altering the outcome in favor of the worse candidate. Theorem 4 relates this behavior to an extreme "swing voter's curse": if others play a strategy profile that would aggregate information, each voter best responds by minimizing the probability that she casts a pivotal vote. She either plays a mixed strategy (Theorem 2) or abstains (Theorem 4). In any equilibrium, no vote reveals information, precluding aggregation.

A large literature, initiated by Ellsberg [1961], criticizes SEU on both normative and descriptive grounds. When payoffs depend on ambiguous events – that is, events about which the decision maker has only vague information – SEU does not accurately describe preferences. Agents typically prefer betting on unambiguous events to ambiguous ones. For example, a bet on an event E, which is known to occur with probability 0.5, may be preferred both to a bet on the event F and a bet on its complement F^c when no information is provided about F. Ambiguity aversion explains evidence from asset markets that contradicts SEU (for instance, see Epstein and Schneider [2010]).

Many important policy decisions are made under ambiguity.[2] A policy to cap carbon emissions deals with poorly understood costs, base case emissions, and tails of the probability distribution of temperature changes. The recession of 2008-2009 resulted at least in part from an unprecedented event (systematic default in AAA rated bonds) in the credit market. The Federal Reserve decided whether or not to bail out banks and hedge funds based on their beliefs about the poorly understood connection between this default, these companies and the financial system as a whole. Many foreign policy decisions must be made despite possessing only poor quality information, such as that leading to the 2003 invasion of Iraq.

To accommodate ambiguity averse voters, this paper assumes that voter preference conforms to maxmin expected utility (henceforth, MEU; introduced and axiomatized in Gilboa and Schmeidler [1989]). Voters consider a set of probability measures and evaluate an act by taking its minimum expected utility with respect to every measure in that set. Formally, for some set of probability measures Π and a von Neumann-Morgenstern index $u(\cdot)$, the utility of an act f can be written as

$$\min_{p\in\Pi} \mathbb{E}_p[u \circ f].$$

SEU is the special case when Π is singleton. When Π is not singleton, the behavior in the Ellsberg paradox can be rationalized.

Section 2 gives an example that illustrates how ambiguity averse voters behave differently from their SEU counterparts. Section 3 introduces ambiguous Poisson games and proves

[2]Papers that address political economy questions with ambiguity averse voters or candidates include Berliant and Konishi [2005], Ashworth [2005], Ghirardato and Katz [2006] and Bade [2011], though none consider strategic interaction between voters.

existence of an equilibrium. Section 4 describes a common values voting game when voter preferences are MEU and presents the paper's main results. Theorem 2 shows that ambiguity aversion can preclude the existence of any equilibrium that aggregates information. Theorem 3 provides sufficient conditions for existence of an equilibrium that aggregates information. Section 5 modifies the setup by allowing voters to abstain strategically. Theorem 4 shows that information may fail to aggregate in this setting as well. Section 6 concludes by relating the main results to other works that show failure of information aggregation in voting games. Proofs are collected in appendices.

2. Sincere Voting and Ambiguity

This section offers an example showing how ambiguity aversion alters the set of equilibria to voting games. Formal definitions of the game and equilibrium are deferred to Appendix A.

Consider an election with 101 voters who vote for one of two candidates, A and B. The candidate with the most votes wins. Suppose there are two states of the world, a and b, and all voters agree that A's policy is better in state a but B's policy is better in state b. Before voting, all voters observe a signal from the set $\{1, 2\}$. They believe that signal 1 occurs with probability 0.6 in state a, that signal 2 occurs with probability 0.6 in state b, and that signals are independently distributed conditional on the state of the world. After observing signal t, each voter considers the set of posteriors Π_t consisting of the Bayesian updates of the probability measures in some set Π. Because the state space is one dimensional, it is convenient to represent Π_t and Π as intervals, $[\underline{p}_t, \bar{p}_t]$ and $[\underline{p}, \bar{p}]$ respectively, corresponding to the probability each of the measures in the set assigns to a. For simplicity, suppose that the interval $[\underline{p}, \bar{p}]$ is symmetric about $\frac{1}{2}$. Voters get utility equal to 1 if the correct candidate is elected but 0 otherwise. After observing signal t, voter preference is represented by

$$\min_{p \in [\underline{p}_t, \bar{p}_t]} [p(Pr(A \text{ wins}|a)) + (1-p)Pr(B \text{ wins}|b)]. \tag{1}$$

Because of the noted symmetry, a voter (strictly) prefers to bet on a over b if she observes signal 1 and vice versa if she observes signal 2. If all voters who observe 1 vote for A and all those who observe 2 vote for B, information aggregates. If voters were SEU ($\underline{p} = \bar{p} = \frac{1}{2}$), then McLennan [1998, Thm. 1] would show that this sincere voting strategy is an equilibrium. In that equilibrium, information aggregates and each voter receives the same expected utility in each state, about 0.979. However, when $\underline{p} < 0.4$ and $0.6 < \overline{p}$ sincere voting is not an equilibrium because all players best respond by voting for both A and B with equal probability.

For instance, assume that $\underline{p} = .39$ and $\overline{p} = .61$. After updating, players who observe signal 1 use $\Pi_1 = [0.49, 0.7]$ and players who observe signal 2 use $\Pi_2 = [0.3, 0.51]$. Consider the problem of an arbitrary voter when all others vote sincerely. If this voter observes signal 1, then she picks her vote to maximize

$$\min_{p\in[0.49,0.7]}[pPr(A \text{ wins}|a)+(1-p)Pr(B \text{ wins}|b)]. \tag{2}$$

She affects the outcome only when she is pivotal, or when exactly 50 of the others vote for A. Since all others vote sincerely,

$$Pr(A \text{ has 50 votes}|a)=Pr(B \text{ has 50 votes}|b)=\begin{pmatrix}100\\50\end{pmatrix}.6^{50}.4^{50}=\rho,$$

which is approximately 0.01, and

$$Pr(\text{ 51+ votes for } A|a)=Pr(\text{ 51+ votes for } B|b)=\sum_{j=51}^{100}\begin{pmatrix}100\\j\end{pmatrix}.6^{j}.4^{100-j}=\theta,$$

which is approximately 0.973. If she votes for A with probability α, then

$$Pr(A \text{ wins}|a)=\theta+\rho\alpha$$

and

$$Pr(B \text{ wins}|b)=\theta+\rho(1-\alpha).$$

Therefore, this voter's utility from voting for A with probability α is

$$\min_{p\in[0.49,0.7]} p[\alpha\rho+\theta]+(1-p)[(1-\alpha)\rho+\theta]. \tag{3}$$

If she voted sincerely, then she would always vote for A ($\alpha=1$) and her utility would be

$$\min_{p\in[0.49,0.7]} p[\rho+\theta]+(1-p)\theta=\theta+.49\rho,$$

about 0.9779. If she played her other pure strategy, voting for B ($\alpha=0$), then she would get utility

$$\min_{p\in[0.49,0.7]} p\theta+(1-p)[\theta+\rho]=\theta+.3\rho,$$

about 0.976 which is less than if she voted for A.

When the voter picks her strategy, the state of the world is determined but unknown. By randomizing, she replaces subjective uncertainty with objective risk. While she prefers to follow her signal rather than vote against it, voting for A and B with equal probability insures her against ambiguity. By doing so, she receives utility equal to

$$\min_{p\in[0.49,0.7]} p[\theta+.5\rho]+(1-p)[\theta+.5\rho]=\theta+.5\rho,$$

about 0.978, so she prefers this mixture to sincere voting. A symmetric argument shows that the voter also prefers to mix in this way after observing signal 2. Hence, her best response is to randomize between voting for A and B regardless of the signal she observes.

As in the SEU case, each voter picks her strategy based on her "beliefs" about the state of the world if her vote decides the election. If all voters were SEU, then each vote would reveal something about the voter's private information, and as the number of voters approached infinity, the outcome of the election would reflect all private information. In contrast, in the example the voter minimizes the probability that she makes a mistake (conditional on her being pivotal) by randomizing between voting for A and B. She thinks that if she is pivotal, she will make a mistake with probability as high as 0.51 by voting for A or as high as 0.7 by voting for B. By mixing, she makes a mistake with precisely probability 0.5. Because the voter is ambiguity averse, she strictly prefers the latter strategy. Should the whole electorate play this strategy, information could not aggregate because no individual's vote reveals the underlying signal. Indeed, all voters randomizing as above is an equilibrium to this game.

That sincere voting fails to be an equilibrium is not in itself surprising; in fact, Austen-Smith and Banks [1996] show this is typically the case even with SEU voters. However, Theorem 2 below shows that there is *no* equilibrium to the above game in which information aggregates: if σ is an equilibrium where the expected winner in state a is A, then the expected winner in state b is *not* B.[3] Theorem 2 extends the logic above to any strategy profile. If information would aggregate should voters play strategy profile σ, then some voter prefers to insure herself rather than follow her prescribed strategy. Consequently, σ cannot be an equilibrium.

3. Ambiguous Poisson Games

This section introduces ambiguous Poisson games, a generalization of Myerson [1998]'s notion of *extended Poisson games*. Extended Poisson games simplify the analysis of large population games with some underlying uncertainty. Myerson proves that if an extended Poisson voting game has a common prior, common values and informative signals, then there exists an equilibrium in which information aggregates. The notation and definition of equilibrium are adapted from Myerson [1998]. Theorem 1 proves existence of an equilibrium.

For any finite set E, denote by ΔE the set of probability measures on E.

[3]This paper's results are stated for games with Poisson population uncertainty, but only Theorem 4 relies on this assumption. Theorems 2 and 3 hold without population uncertainty. Details available upon request.

Definition. An *ambiguous Poisson game* Γ is a collection $(\Omega, C, T, U, (\Pi_t)_{t\in T}, r, n)$ where:

- Ω is a finite set of states.
- C is a finite set of actions. Define $Z(C) = \{x \in \mathbb{R}^C : x(c) \in \mathbb{N} \forall c \in C\}$, the set of all possible realized action profiles (the number of players taking each action).
- T is a finite set of types.
- $U : T \times C \times \Omega \times Z(C) \to \mathbb{R}$ is a bounded function that represents preference. $U(t, c, \omega, x)$ is the utility for a voter of type t who takes action c when the realized state is ω and the realized action profile is x.
- $\Pi_t \subset \Delta(\Omega)$ is a closed, non-empty and convex set, representing the set of posteriors for each type. If Π_t is a singleton for every t, then all players are SEU, though they may have different priors.
- $r : \Omega \to \Delta T$ maps each state to a probability measure over types. Types are drawn independently according to $r(\omega)$ in state ω.
- The number of players is a random variable distributed Poisson with mean $n \in \mathbb{R}_{++}$.

The timing of the game is as follows. Nature chooses the number of players according to the Poisson distribution with mean n and chooses the state of the world according to some unknown, unmodeled procedure. Each player learns her type and forms a set of posteriors.[4] Before learning the realized state, how many other players there are or what actions the other players have taken, she picks a strategy $s \in \Delta C$. When she picks this strategy, the state of the world is realized but unknown, so ambiguity aversion leads each player to act as if Nature picked the distribution over states with the goal of minimizing her utility. A mixed strategy may equalize her expected utility across states, limiting her exposure to Nature's choice. For this reason, she may find a mixed strategy to be the only best response. For a more in depth discussion of this issue see Lo [1996] or Klibanoff [1996].

As in Myerson [1998], assuming a Poisson population yields convenient properties. Because types are conditionally independent and the population is distributed Poisson, the number of players that take each action c in state ω is also distributed Poisson and is independent of the number of players taking action $c' \neq c$ in state ω. Moreover, each player's conditional expectation does not depend on her private information. If $\lambda(\omega)(c)$ is the expected number of players in state ω that take action c, the probability of any given action profile x in state ω is $p(x|\lambda(\omega))$ where

$$p(x|\lambda) = \prod_{c\in C} \frac{e^{-\lambda(c)}\lambda(c)^{x(c)}}{x(c)!}. \tag{4}$$

[4]Note that posterior beliefs rather than prior beliefs are taken as a primitive. One could specify a set of priors and an updating rule (in the example from Section 2, the updating rule is prior-by-prior Bayesian updating), which would constitute a special case of the above. However, there are no ex-ante actions so the set of priors only enters a voter's decision through her set of posteriors.

These properties imply that the best response correspondence is the same for any two players with the same type. A *strategy profile* σ is a map from types to strategies, $\sigma : T \to \Delta(C)$. A player of type t picks a strategy $\sigma_t \in \Delta C$ to maximize

$$V_t(\sigma_t, \sigma) = \min_{q \in \Pi_t} \int_{\Omega} \int_{Z(C)} \sum_{c \in C} \sigma_t(c) U(t, c, \omega, x) dp(x|\lambda(\sigma, \omega)) dq(\omega) \tag{5}$$

where

$$\lambda(\sigma, \omega)(a) = n \sum_{t \in T} \sigma(t)(a) r(t|\omega). \tag{6}$$

Definition. A strategy profile σ^* is an *equilibrium for* Γ if for each $t \in T$

$$\sigma^*(t) \in \arg\max_{\hat{\sigma} \in \Delta C} V_t(\hat{\sigma}, \sigma^*). \tag{7}$$

If σ^* is an equilibrium, then every player picks her strategy to maximize the minimum expected utility over all measures in her set of posteriors, given she knows that the other players follow the strategy profile σ^*. When Π_t is singleton for all $t \in T$, this definition is equivalent to the definition in Myerson [1998], though players may not have a common prior. Because each player maximizes her minimum expected utility given her beliefs and all player's beliefs agree, the behavior of each player is as in Lo [1996]'s "beliefs equilibrium with agreement." While he does not consider games with population uncertainty, this definition of equilibrium otherwise coincides with his.

In ambiguous Poisson games, a player perceives ambiguity about the distribution of the state of the world, not about the other players' strategies, so she best responds to her correct perception of the other players' strategies and her information. This paper focuses on failure of information aggregation in voting games, and this restriction stacks the deck in favor of the Condorcet Jury Theorem holding. This is because the Condorcet Jury Theorem shows that the strategy played by an SEU voter *given her correct perception of other voters' strategies* leads to information aggregation. If a player perceives ambiguity about other voters' strategies, then she no longer correctly perceives other voters' strategies, so one should not expect the result to hold.[5]

Theorem 1 shows that an equilibrium exists for every ambiguous Poisson game.

Theorem 1. *For any ambiguous Poisson game* Γ*, there exists a strategy profile* σ^* *that is an equilibrium for* Γ.

[5]If each player's perception of ambiguity about others' strategies is consistent with Lo [1996]'s equilibrium concept, then Theorem 2 continues to hold in large electorates if there are only two signals (details available upon request). However, analysis of most voting games where voters perceive ambiguity about others' strategies is intractable due to difficulties determining the minimizing strategy profile .

4. The Condorcet Jury Theorem

This section describes common values voting games with MEU players and discusses the limiting equilibria. Theorem 2 establishes the existence of voting games for which no equilibrium aggregates information. Theorem 3 shows that information aggregates in equilibrium for some voting games where no voter is SEU. Neither of these two results depends on population uncertainty – very similar arguments work when the population is fixed at n.

4.1. **Ambiguous voting games.** Candidates A and B each commit to a distinct policy. Voters cast a vote for one of them, and the candidate with the most votes wins; in a tie, each candidate is selected with equal probability. Voters have common values and are instrumentally rational: they care only about the policy outcome and they have the same preference over policies given the state. Depending on the state of the world, the policy is either good or bad. There are two states, a and b, representing which policy is the good one.

Formally, an *ambiguous voting game* is an ambiguous Poisson game where the action set is $C = \{A, B\}$, the set of states is $\Omega = \{a, b\}$ and the utility function of all types takes value 1 if the candidate elected matches the state and 0 otherwise. The action A is interpreted as a vote for candidate A, the action B is interpreted as a vote for B and the set of types T is interpreted as a set of signals. Given that others play strategy profile σ, the payoff to a voter of type t using strategy $\hat{\sigma} \in \Delta\{A, B\}$ is

$$\begin{aligned} V_t(\hat{\sigma}, \sigma) \quad = \quad & \min_{\pi \in \Pi_t} \{\pi(a)[\hat{\sigma}(A) Pr(A \text{ wins}|a, v_A, \sigma) + \hat{\sigma}(B) Pr(A \text{ wins}|a, v_B, \sigma)] + \\ & + \pi(b)[\hat{\sigma}(A) Pr(B \text{ wins}|b, v_A, \sigma) + \hat{\sigma}(B) Pr(B \text{ wins}|b, v_B, \sigma)]\}, \end{aligned}$$

where $Pr(c \text{ wins}|\omega, v_d, \sigma)$ is the probability candidate c wins in state ω if she votes for candidate d and others play strategy profile σ.

As in Section 2, represent Π_t by the interval of probabilities that the measures within it assign to a. That is, $\Pi_t \equiv [p_t, q_t]$ where $p_t = \min_{\rho \in \Pi_t} \rho(a)$ and $q_t = \max_{\rho \in \Pi_t} \rho(a)$.

4.2. **Main result.** This subsection describes a set of ambiguous voting games for which no equilibrium aggregates information. Theorem 2 below shows that if voters lack confidence, then no equilibrium aggregates information. Voters lack confidence when the following condition on posteriors holds.

Definition. An ambiguous voting game *has voters who lack confidence* if $p_t < \frac{1}{2} < q_t$ for all $t \in T$.

An outsider can detect when voters lack confidence through betting preferences. If voters lack confidence, then all voters strictly prefer betting on the outcome of a fair coin toss over betting on either a or b. Even if the voter thinks that a is a better bet than b, she lacks confidence in this judgment and strictly prefers to hedge her bet on a by mixing it with a bet on b. This is impossible with SEU: if a is at least as likely as b when a and b are the only two states, then a bet on a is at least as good as a fifty-fifty lottery.

This translates into the voting setting as follows. Suppose a random voter were made a dictator – whichever policy she chooses will be implemented. If, irrespective of the signal she receives, she strictly prefers to pick the policy implemented by flipping a fair coin rather than implementing either policy for sure, then, and only then, voters lack confidence.

To give a better sense of the meaning of lacking confidence, suppose that voters form posterior beliefs by updating a common set of priors Π using prior-by-prior Bayesian updating. Precision of signals and the set of priors both contribute to posterior beliefs. Voters lack confidence when the signals do not provide enough information to offset the prior ambiguity. With very precise signals (there is some t so that $\frac{r(t|b)}{r(t|a)}$ is very high or very low), Π must be very close to $[0, 1]$ for voters to lack confidence; however, if signals are not very precise ($\frac{r(t|b)}{r(t|a)}$ close to one for all t), then Π can be a much smaller interval. For instance, with the signal structure described in Section 2, voters lack confidence whenever $[.4, .6] \subset \Pi$ without equality, but if $r(1|a) = r(2|b) = .51$, then voters lack confidence whenever $[.49, .51] \subset \Pi$ without equality. If $\Pi = [.45, .55]$, then voters lack confidence given the second signal structure but not the first.

Theorem 2. *Suppose that Γ is an ambiguous voting game with voters who lack confidence. If σ is an equilibrium for Γ in which the expected vote share for A in state a is greater than $\frac{1}{2}$, then the expected vote share for B in state b is less than $\frac{1}{2}$.*

Theorem 2 implies that the equilibrium probability of the correct candidate winning the election is bounded above by $\frac{1}{2}$ in at least one state, prohibiting information aggregation. The following outlines the proof.

Suppose, for the sake of contradiction, that there is an equilibrium strategy profile σ where the expected winner is correct in both states of the world. The key is to show that the worst case scenario (the state in which the wrong candidate is more likely to be elected) is not independent of the voter's strategy when the others play σ. If the worst case scenario were independent, then the voter acts as if maximizing SEU according to the posterior that maximizes the probability of the worst case scenario and familiar arguments (for instance, Myerson [1998, Thm. 2]) would imply that there is an equilibrium in which information aggregates when n is large.

To see why the worst case scenario depends on the voter's strategy, suppose that it doesn't, that n is "large" and that σ calls for all voters to play a pure strategy. Since the ratio of pivot probabilities must not go to either 0 or infinity, Myerson [2000, Thm. 1] shows that the expected vote share for A in state a is the same as the expected vote share for B in state b. As a consequence, each voter thinks that if she abstains, then her conditional expected utility is equal across states. When a player votes for A for sure, her expected utility conditional on state a increases and her expected utility conditional on state b decreases (vice versa when voting for B). It follows that the worst case scenario depends on her vote. When voters lack confidence, this argument can be extended to any σ, regardless of n.

Because the worst case scenario depends on her vote, there is a mixed strategy that insures the voter against making a mistake and altering the election in favor of the wrong candidate when others play σ, as in Section 2. Since voters lack confidence, each voter weakly prefers to play this mixed strategy over any other strategy. If every voter insured herself, then information could not aggregate because this insurance strategy is independent of private information. Therefore, it must be that some voter is willing to play a different strategy. However, the only strategies that are at least as good as the insurance strategy assign higher probability to voting for the candidate that receives more votes from the insurance strategy. All voters expect to vote for the same candidate regardless of signal. This candidate is the expected winner in both states, a contradiction.

The following result characterizes one equilibrium to the game.

Proposition 1. *If an ambiguous voting game Γ has voters who lack confidence, then the strategy profile σ defined by $\sigma(t)(A) = \frac{1}{2}$ for all $t \in T$ is an equilibrium for Γ.*

In this equilibrium, both candidates are elected with equal probability regardless of the state. Therefore, knowing the winner of the election would not change the beliefs of a Bayesian agent. Neither Proposition 1 nor Theorem 2 show that this is the only equilibrium. However, Theorem 2 shows that if an equilibrium results in a higher probability of electing the correct candidate than this equilibrium in one state of the world, then it must result in a lower probability of electing the correct candidate in the other state of the world.

4.3. **Information aggregation.** This subsection provides a formal definition of information aggregation and proves that some ambiguous voting games have an equilibrium in which information aggregates. Because there is always some possibility of a mistake in a finite electorate, one cannot require full certainty that voters elect the proper candidate in a given game. Instead, the literature focuses on sequences of voting games where the probability of electing the wrong candidate vanishes along some sequence of equilibria. Below, a sequence of ambiguous voting games is indexed by the mean number of players, with all other primitives remaining the same.

Definition. A sequence of ambiguous voting games $(\Gamma_n)_{n=1}^{\infty}$ satisfies *full information equivalence (FIE)* if there exists a sequence of strategy profiles $(\sigma_n)_{n=1}^{\infty}$ so that σ_n is an equilibrium for Γ_n, and for any $\epsilon > 0$, there exists N so that $n > N$ implies that the correct candidate is elected in each state with probability higher than $1 - \epsilon$ when σ_n is played.[6]

[6]This definition is adapted from Feddersen and Pesendorfer [1997].

An implication of Theorem 2 is that FIE fails for many sequences of ambiguous voting games. In contrast, as long as the signal structure is informative (the conditional distribution of signals varies with the state), any sequence of SEU voting games satisfies FIE. Since SEU is a special case of MEU, some ambiguous voting games satisfy FIE. However, SEU is not necessary for information to aggregate. In fact, Theorem 3 proves the existence of a sequence of equilibria that aggregates information whenever the game has disjoint* posteriors.

Definition. An ambiguous Poisson game has *disjoint* posteriors* if for any distinct t and t' in T either $p_{t'} = q_t$, $p_t = q_{t'}$ or $[p_t, q_t] \cap [p_{t'}, q_{t'}]$ is empty.

If all voters are SEU, then each Π_t is a singleton and the ambiguous Poisson game has disjoint* posteriors. More generally, one can distinguish between SEU, disjoint* posteriors and voters who lack confidence using Lemma 1 (in the appendix). Consider an ambiguous voting game Γ. If Γ has singleton posteriors, then all voters act as SEU maximizers and none strictly prefer to randomize for any strategy profile. If Γ has disjoint* posteriors, then for any strategy profile at most one type of voter strictly prefers to randomize. If Γ has voters who lack confidence, then there exists a strategy profile such that all voters strictly prefer randomizing to playing a pure strategy.

Theorem 3. *Suppose that $(\Gamma_n)_{n=1}^{\infty}$ is a sequence of ambiguous voting games that have disjoint* posteriors, that $0 < p_t \leq q_t < 1$ for all t and that for each ω and t, $r(t|\omega) > 0$. If there is some $t \in T$ s.t. $r(t|a) \neq r(t|b)$, then $(\Gamma_n)_{n=1}^{\infty}$ satisfies FIE.*

The proof generalizes the construction from Myerson [1998, Thm. 2]. As in that paper, the equilibrium consists of a "step strategy": at most one type of voter randomizes, and all others play a pure strategy, determined by how likely they view a relative to the randomizing voter. Because of disjoint* posteriors, at most one type of voter has a strict preference for randomization. The proof shows that along this sequence, even if some voter strictly prefers to randomize for every element of the sequence, the strategy is the same as in the SEU game with the same signal structure at the limit. In fact, Myerson [1998, Thm. 2] is the special case where each Π_t is a singleton that results from Bayesian updating of a common prior.

5. Strategic Abstention

In SEU voting games, abstention typically improves the outcome of the election. This is due to the "swing voter's curse" (introduced in Feddersen and Pesendorfer [1996]): uninformed voters are more likely to abstain than informed voters. As a consequence, the expected percentage of votes for the correct candidate is larger than if voters could not abstain, so abstention improves the expected outcome of the election. The ambiguous voting games studied in Section 4 explicitly rule out the possibility of strategic abstention, leaving open the possibility that the conclusion of Theorem 2 fails when voters can choose to abstain.

This section shows that a version of Theorem 2 holds without mandatory voting. The analysis provides insight into the mechanism behind Theorem 2; namely, equilibrium behavior can be interpreted as an extreme swing voter's curse. Each voter prefers to minimize the chance that she casts a pivotal vote. If she abstained, then she would never be pivotal, which would be better than any available strategy. However, Theorem 2 assumes that she must vote. Among her available choices, her best option is to mimic abstention through a mixed strategy.

In order to allow for abstention, modify the ambiguous voting games from Section 4 by replacing the action set with $C = \{A, B, \emptyset\}$ and requiring that $T = \{1, 2\}$. The action $\emptyset$ corresponds to abstention. The payoffs for each voter are as in the previous section. The restriction to two types is for simplicity. Call such a game an *ambiguous voting game with abstention.*

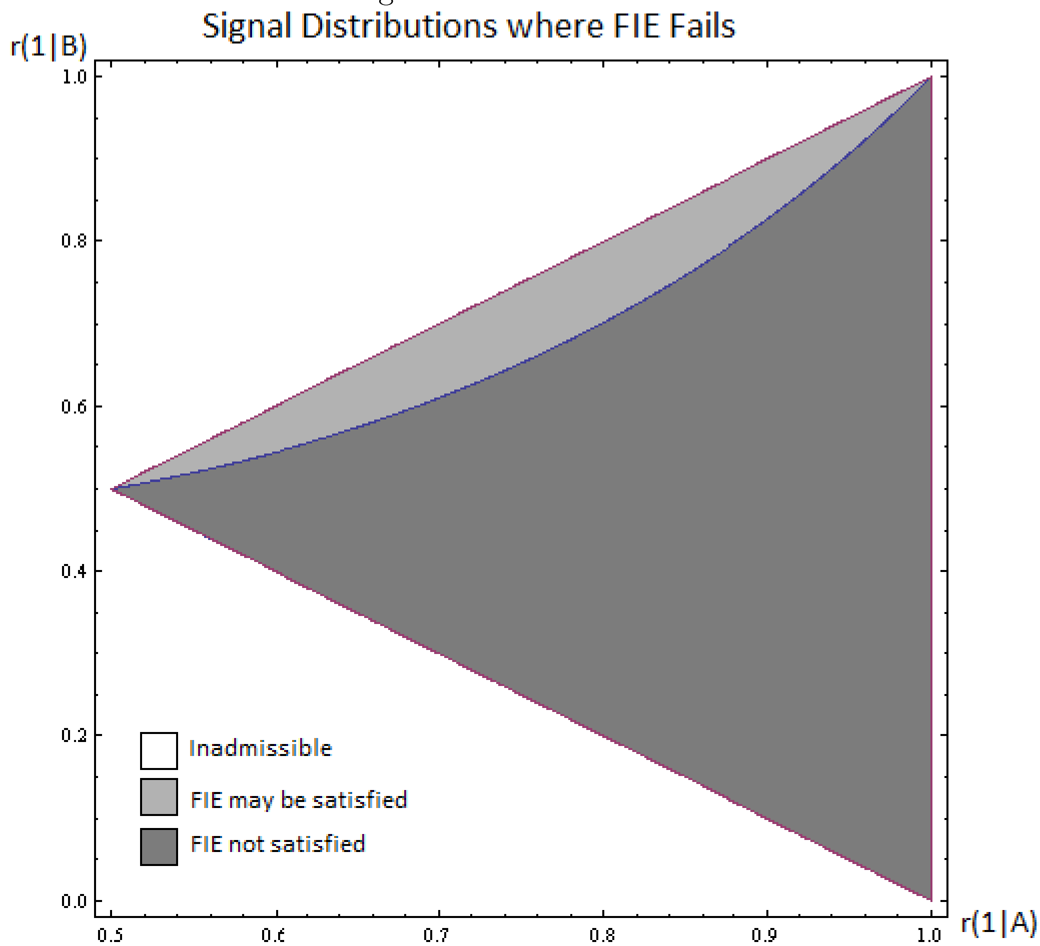
Say that an ambiguous Poisson game has *symmetric signals* if $r(1|a) = r(2|b)$ and that players *have posteriors that respect likelihood ratios* if $\frac{r(t|a)}{r(t|b)} > \frac{r(t'|a)}{r(t'|b)}$ implies that $\min_{\pi\in\Pi_t} \pi(a) \geq \min_{\pi\in\Pi_{t'}} \pi(a)$ *and* $\max_{\pi\in\Pi_t} \pi(a) \geq \max_{\pi\in\Pi_{t'}} \pi(a)$ for every $t, t' \in T$. In ambiguous voting games with abstention satisfying these two assumptions, information does not aggregate along any sequence of equilibria.

Theorem 4. *Suppose that $(\Gamma_n)_{n=1}^{\infty}$ is a sequence of ambiguous voting games with abstention and symmetric signals. If voters lack confidence and have posteriors that respect likelihood ratios, then $(\Gamma_n)_{n=1}^{\infty}$ does not satisfy FIE.*

For a sequence of ambiguous voting games with abstention to satisfy FIE, it is necessary that there exists a sequence of equilibrium strategy profiles where the winner is correct in both states and the expected number of votes in each state goes to infinity. The proof of Theorem 4 adapts and extends the arguments from Theorem 2 to show that either the number of votes is bounded above in some state or the expected winner is incorrect in at least one state. As a consequence, FIE must fail.

With SEU voters, Feddersen and Pesendorfer [1999] and Bouton and Castanheira [2009] show the swing voter's curse persists in voting games similar to those considered here. Given two SEU voters who observe different signals, the voter whose signal conveys less information about the state of the world is more likely to abstain in equilibrium. As a consequence, for a fixed signal structure, the percentage of votes cast by more informed voters is higher in an election with abstention compared to one with mandatory voting. In contrast, Theorem 4 demonstrates that ambiguity aversion strengthens the swing voter's curse. An ambiguity averse swing voter perceives the probability of making a mistake with her vote to be larger than her SEU counterpart. Allowing abstention leads to fewer votes in expectation but, unlike SEU, may not change the composition of the votes when voters lack confidence.

FIGURE 1. Signal Distributions Where FIE Fails



In the appendix, the symmetric signals assumption is relaxed substantially. Relabeling so that $r(1|a) + r(1|b) \geq r(2|a) + r(2|b)$ and $r(1|a) \geq r(1|b)$, the distributions of signals for which FIE does not hold are indicated by Figure 1.[7]

Proposition 2 constructs a set of equilibria for any ambiguous voting game with abstention in which voters lack confidence.

Proposition 2. *If* Γ *is an ambiguous voting game with abstention that has voters who lack confidence, then for any* $s \in [0,1]$*, the strategy profile* σ^* *defined by* $\sigma^*(t)(\emptyset) = s$ *and* $\sigma^*(t)(A) = \sigma^*(t)(B) = \frac{1-s}{2}$ *for every* $t \in T$ *is an equilibrium for* Γ.[8]

Proposition 2 shows that equilibrium expected turnout with MEU voters can be anywhere between zero and one hundred percent. Of particular interest are the equilibrium where $\sigma^*(t)(\emptyset) = 0$ for all $t \in T$ and the equilibrium where $\sigma^*(t)(\emptyset) = 1$ for all $t \in T$. In the

[7]While the proof only shows that FIE fails in the dark gray region, I conjecture that FIE fails in the light gray region as well. In fact, it's clear from the proof that FIE fails for at least part of this region.
[8]This result generalizes immediately to the case where T is any finite set.

former case, voters behave exactly as in the equilibrium constructed by Proposition 1: each voter plays a mixed strategy, voting for each candidate with equal probability. In the latter equilibrium, all voters play a pure strategy, abstain. Despite the different strategies, the expected outcome is the same for either equilibrium, as it is for *all* of the equilibria shown to exist by Proposition 2: each candidate is elected with equal probability, regardless of the state of the world that obtains. Consequently, the payoffs are the same for each voter, as is the information that observing the outcome would provide to an observer.

The equilibrium in which all voters abstain contrasts with Propositions 2 and 3 of Feddersen and Pesendorfer [1996] and Proposition 5 of Feddersen and Pesendorfer [1999]. In these papers, the fraction of voters who don't abstain remains bounded away from zero along any sequence of equilibria. This result is a consequence of SEU preferences: even a small difference in the expected benefits of voting for A instead of B induces a strict preference to vote for A.

6. Conclusion

Theorems 2 and 4 show that rational but ambiguity averse voters may find it optimal to insure themselves by minimizing the chance they cast a pivotal vote. This mechanism leads to a failure of information aggregation not documented by previous work. These papers show that the dimensionality of the uncertainty and the degree of commonality between voters are important in evaluating the efficiency of the election. In contrast, this paper suggests that how familiar the electorate is with the issues at stake also matters a good deal. By way of conclusion, this section reviews some of these results and contrasts them with Theorems 2 and 4.

Feddersen and Pesendorfer [1997] prove that if the distribution of preferences is unknown, then FIE fails generically. The problem is one of dimensionality; namely, each voter must infer both the distribution of signals and the distribution of preferences from these votes. Even if a voter knew which votes others cast and the electorate were large, she could not infer the state of the world. In contrast, this paper assumes common knowledge of the distribution of preferences. However, the distribution of votes may not vary with the state (see Proposition 1 or 2) because voters insure themselves against ambiguity by abstaining or randomizing.

Mandler [2011] shows that if the conditional distribution of signals is unknown, then FIE may fail. If all the signals were observed by each voter, then uncertainty would remain as to which state is correct even as the size of the electorate goes to infinity. In this paper, if all signals were observed, then the true state would be known with probability approaching 1 despite the prior ambiguity.

Bhattacharya [2008] drops the assumption of common values and characterizes the distributions of preferences for which FIE fails. For instance, FIE fails when any voter who receives information in favor of the Condorcet-winner with perfect information is very likely to strongly prefer the other candidate.[9] In contrast, this paper maintains pure common values.

Finally, the result in this paper relates to work that studies the effect of ambiguous information in other contexts. For instance, Condie and Ganguli [2011] demonstrates a failure of information transmission with ambiguity averse agents in general equilibrium. They show that a rational expectations equilibrium for an exchange economy may be partially revealing when agents are ambiguity averse; in contrast, fully revealing equilibria are generic with SEU agents. Two differences are worth pointing out. First, in their model agents do not act strategically – they are price takers. Second, they assume that only a subset of agents are ambiguity averse, while an ambiguous voting game has voters who lack confidence only if all voters are ambiguity averse.

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[9]Additionally, the non-aggregation result in this paper is stronger because of his more demanding definition of FIE, which requires the definition of FIE from this paper to hold for *every* sequence of symmetric, Bayesian Nash equilibria in undominated strategies. Unlike Theorem 2, his conditions do not rule out the existence of a different equilibrium in which information would aggregate. For example, the game depicted by his Figure 1 fails his definition of FIE but satisfies the definition in this paper.

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Appendix A. Details for Section 2

Consider $n = 101$, a set of players $I = \{1, ..., n\}$, a set of alternatives $\mathcal{A} = \{A, B\}$, set of types $T_i = \{1, 2\}$ for each $i \in I$ and $T_0 = \{a, b\}$. Set $T := T_0 \times T_1 \times ... \times T_n$. Each player has the same preference over state-alternative pairs given by a function $u : T \times \mathcal{A} \to \mathbb{R}$. Define $u(\cdot)$ by

$$u((t_0, ..., t_{101}), c) \equiv u(t_0, c) = \begin{cases} 1 & t_0 = c \\ 0 & t_0 \neq c \end{cases}$$

for all T. Player i's pure strategies are $S_i = \{A, B\}$; let $S = S_1 \times ... \times S_n$. An aggregation rule $f : S \to \mathcal{A}$ maps the profile of actions to an alternative. Set

$$f(s_0, ..., s_{101}) = \begin{cases} A & \text{if } \sum_{i=1}^{101} \chi_{s_i}(A) \geq 51 \\ B & otherwise \end{cases}$$

for all $(s_0, ..., s_{101}) \in S$, where $\chi_E(\cdot)$ is the indicator function of the set E. Fix a non-empty, closed and convex set of common priors $\Pi \in \Delta T$. Define Π by

$$\Pi = \{\pi \in \Delta T : \pi(\{a\} \times T_1 \times ... \times T_{101}) \in [\underline{p}, \bar{p}] \text{ and}$$
$$\frac{\pi(\{(a, t_1, ..., t_{101})\})}{\pi(\{a\} \times T_1 \times ... \times T_{101})} = \prod_{i=1}^{101} .6^{t_i}.4^{1-t_i} \text{ and}$$
$$\frac{\pi(\{(b, t_1, ..., t_{101})\})}{\pi(\{b\} \times T_1 \times ... \times T_{101})} = \prod_{i=1}^{101} .6^{1-t_i}.4^{t_i}\}$$

which gives the desired form of priors. The game is defined by the collection $(\mathcal{A}, I, T, u, S, f, \Pi)$.

For each i, let $\hat{S}_i : T_i \to \Delta S_i$ be a strategy for player i and let $\Sigma := \hat{S}_0 \times ... \times \hat{S}_n$ be the set of strategy profiles. This requires that the player's strategy be measurable with respect to her type. As is convention, let σ_i denote player i's strategy and let σ_{-i} represents the vector of strategies chosen by players other than i. A strategy profile $\sigma^* \in \Sigma$ is an *equilibrium* if

$$\sigma_i^*(t_i) \in \arg\max_{\sigma \in \Delta S_i} \min_{\pi \in \Pi} \mathbb{E}_\pi[\mathbb{E}_{\sigma^*_{-i}}[u((t_0, ..., t_n), f((s_0(t_0), ..., s_{i-1}(t_{i-1}), \sigma, s_{i+1}(t_{i+1}), ..., s_n(t_n))]|t_i]$$

for every $i \in I$.

Appendix B. Proof of Theorem 1

Proof. Define the set $\Lambda = \{l \in \mathbb{R}^{C \times \Omega} : \sum_{a \in C} l(a, \omega) = n\}$, noting that Λ is compact, and consider the correspondence $A_t : \Lambda \twoheadrightarrow \Delta C$ defined by

$$A_t(\lambda) = \arg\max_{\hat{\sigma} \in \Delta C} \min_{q \in \pi_t} \int_\omega \int_{Z(A)} \sum_{a \in C} \hat{\sigma}(a) U(x, t, a, \omega) dp(x|\lambda(\omega)) dq.$$

Define $A(\lambda) = \times_{t \in T} A_t(\lambda)$ and let $F : \Lambda \twoheadrightarrow \Lambda$ be defined by

$$F(\lambda) = \{n \sum_{t \in T} c_t(a) r(t|\omega) : c \in A(\lambda)\}.$$

If $\lambda \in F(\lambda)$ then equation (7) is satisfied for the strategy profile σ^* that generates λ. Hence, existence of an equilibrium is equivalent to showing that $F(\cdot)$ has a fixed point, which follows if F is convex and closed. Since $F(\lambda)$ is an affine transformation of $A(\lambda)$, need to show that $A(\cdot)$ is convex and closed. Show first that all A_t are convex, compact and UHC.

[Convex:] Define $\phi : C(\Omega) \to \mathbb{R}$ by $\phi(f) = \min_{q\in\pi_t} \int f dq$, where $C(\Omega)$ is the set of continuous functions from Ω to the real numbers). Then ϕ and $p(x|\lambda(\omega))U(x,t,\cdot,\omega)$ are both concave. So $g : \Delta C \to \mathbb{R}$ defined by $g(\hat{\sigma}) = \phi(p(x|\lambda(\omega))\sum_{a\in C}\hat{\sigma}(a)U(x,t,a,\omega))$ is also concave. Hence $g(x) = g(y) \implies g(\alpha x + (1-\alpha)y) \geq g(x) \forall \alpha \in [0,1]$ and $x, y \in A_t(\lambda) \implies \alpha x + (1-\alpha)y \in A_t(\lambda)$. Therefore $A_t(\lambda)$ is convex, from which it follows that $A(\cdot)$ is convex since a product of convex sets is convex. Since $A(\cdot)$ is convex, $F(\cdot)$ is convex.

[Closed:] ϕ is continuous by the Maximum Theorem (Theorem 17.31 of Aliprantis and Border [2006]; henceforth, AB). $p(x|\cdot)$ is continuous since it is a product of continuous functions. $U(x,t,\cdot,\omega)$ is continuous by assumption. So $\min_{q\in\pi_t}\int_\omega p(x|\lambda(\omega))[\sum_{a\in A}\hat{\sigma}(a)U(x,t,a,\omega)]dq$ is continuous. Hence $A_t(\lambda)$ is UHC and compact by the Maximum Theorem as the set of solutions to a maximization problem.

$A(\lambda)$ is compact for all λ by the Tychonoff product theorem (AB Theorem 2.61) because $A(\lambda)$ is a product of compact sets. By AB Theorem 17.20, it suffices to show that if $\lambda_n \to \lambda$, $x_n \in A(\lambda_n)$, and $x_n \to x$ then $x \in A(\lambda)$. Given such sequences, let $x_{n,t}$ be the t-th component of x_n and x_t the t-th component of x for any $t \in T$. By definition of the product topology, $x_n \to x \iff x_{n,t} \to x_t$ for all $t \in T$. By definition of $A(\cdot)$, $x_{n,t} \in A_t(\lambda_n)$ for each n. Because $A_t(\cdot)$ is UHC and compact, $x_t \in A_t(\lambda)$. Since t is arbitrary, $x_t \in A_t(\lambda)$ for all $t \in T$ and by definition of $A(\cdot)$, $x \in A(\lambda)$. Hence $A(\cdot)$ is UHC and compact. AB Theorem 17.10 establishes that $A(\cdot)$ is closed and thus $F(\cdot)$ is closed.

Since Λ is compact and $F(\cdot)$ is closed and convex, applying Kakutani's Fixed point theorem (AB Corollary 17.55) yields a λ^* such that $\lambda^* \in F(\lambda^*)$, establishing the existence of an equilibrium. □

Appendix C. Preliminaries for the Remaining Proofs

Lemma 1 relies on two functions of the strategy profile.

Formally, if other votes unfold so that the realized action profile is in the event

$$Piv_A = \{x \in Z(C) : x(A) = x(B) \text{ or } x(A) = x(B) - 1\}, \tag{8}$$

then the voter is pivotal for candidate A; let Piv_B be the corresponding event for B. Each voter's best response depends on the relationship between her set of posteriors and the function $b : \Omega \times (\Delta C)^T \to [0,1]$ given by $b(b,\sigma) =$

$$\frac{Pr(Piv_B|b,\sigma) + Pr(Piv_A|b,\sigma)}{Pr(Piv_B|b,\sigma) + Pr(Piv_B|a,\sigma) + Pr(Piv_A|b,\sigma) + Pr(Piv_A|a,\sigma)}. \tag{9}$$

and $b(a,\sigma) = 1 - b(b,\sigma)$. The probabilities in this function depend only on the strategy profile and not on an individual voter's type.

Another key equation is the *insurance strategy*, denoted $\hat{s}(\cdot, \sigma)$, is given by

$$\hat{s}(A,\sigma) = \frac{2(\mathbb{E}[U|b,\sigma] - \mathbb{E}[U|a,\sigma]) + Pr(Piv_B|b,\sigma) + Pr(Piv_B|a,\sigma)}{Pr(Piv_B|b,\sigma) + Pr(Piv_B|a,\sigma) + Pr(Piv_A|b,\sigma) + Pr(Piv_A|a,\sigma)}.$$

and $\hat{s}(B,\sigma) = 1 - \hat{s}(A,\sigma)$. This maps a strategy profile σ into the strategy a voter would play to ensure his expected utility is independent of the state if $\hat{s}(A,\sigma) \in [0,1]$. Otherwise, no strategy equalizes a voters expected utilities between states.

Notice that expected utility in state ω if the voter abstained is given by

$$\mathbb{E}[U|a,\sigma] = \sum_{n=0}^{\infty} \frac{e^{-\lambda(a)(A)}\lambda(a)(A)^n}{n!}[\sum_{j=0}^{n-1} \frac{e^{-\lambda(a)(B)}\lambda(a)(B)^j}{j!} + \frac{1}{2}\frac{e^{-\lambda(a)(A)}\lambda(a)(A)^n}{n!}]$$

where $\lambda(\omega)(c) = \mathbb{E}[x(c)|\omega,\sigma]$ as in equation (6). Define $\mathbb{E}[U|b,\sigma]$ analogously. This expression is precisely the probability that candidate ω wins in state ω. The expected utility of voting for candidate c in state ω when others play strategy profile σ is

$$\mathbb{E}[U|\omega, v_c, \sigma] = \mathbb{E}[U|\omega,\sigma] + [\chi_{\{\omega\}}(c) - \frac{1}{2}]Pr(Piv_c|\omega).$$

Additionally, let $\tau : C \times \Omega \times \Delta C^T \to [0,1]$ be the expected vote share for a candidate in a state given a strategy profile. Formally,

$$\tau(c|\omega,\sigma) = \sum_{t\in T} r(t|\omega)\sigma(t)(c).$$

Note that this does not depend on the number of voters.

Appendix D. Proofs from Section 4

Lemma 1 establishes the form of a voter's best response correspondence. This will be used to prove both Theorem 2 and Theorem 3.

Lemma 1. *For any σ^*, σ^* is an equilibrium if $\sigma_t^*(A) \in BR_t(\sigma^*)(A)$ where*

$$BR_t(\sigma)(A) = \begin{cases} \{0\} & \text{if } \mathbb{E}[U|a,v_B,\sigma] \geq \mathbb{E}[U|b,v_B,\sigma] \;\&\; b(b,\sigma) > p_t \\ & \quad \text{or } \mathbb{E}[U|b,v_A,\sigma] \geq \mathbb{E}[U|a,v_A,\sigma] \;\&\; b(b,\sigma) > q_t \\ [0,1] & \text{if } \mathbb{E}[U|a,v_B,\sigma] \geq \mathbb{E}[U|b,v_B,\sigma] \;\&\; b(b,\sigma) = p_t \\ & \quad \text{or } \mathbb{E}[U|b,v_A,\sigma] \geq \mathbb{E}[U|a,v_A,\sigma] \;\&\; b(b,\sigma) = q_t \\ \{1\} & \text{if } \mathbb{E}[U|a,v_B,\sigma] \geq \mathbb{E}[U|b,v_B,\sigma] \;\&\; b(b,\sigma) < p_t \\ & \quad \text{or } \mathbb{E}[U|b,v_A,\sigma] \geq \mathbb{E}[U|a,v_A,\sigma] \;\&\; b(b,\sigma) < q_t \\ \hat{BR}_t(\sigma)(A) & \text{otherwise} \end{cases}$$

and

$$\hat{BR}_t(\sigma)(A) = \begin{cases} \{0\} & \textit{if } b(b,\sigma) > q_t \\ [0, \hat{s}(A,\sigma)] & \textit{if } b(b,\sigma) = q_t \\ \{\hat{s}(A,\sigma)\} & \textit{if } q_t > b(b,\sigma) > p_t \\ [\hat{s}(A,\sigma), 1] & \textit{if } b(b,\sigma) = p_t \\ \{1\} & \textit{if } b(b,\sigma) < p_t \end{cases}$$

where $p_t = \min_{\rho\in\Pi_t} \rho(a)$ *and* $q_t = \max_{\rho\in\Pi_t} \rho(a)$. *If* $BR_t(\sigma) = \hat{BR}_t(\sigma)$ *then* $\hat{s}(A,\sigma) \in [0,1]$.

Proof. Throughout, a strategy is indexed solely by the probability of playing A. This is WLOG since ΔC is one dimensional. Let p_t and q_t be as in the statement of the Lemma.

A player of type t has a best response to σ of playing A with probability s if s maximizes

$$V_t(s,\sigma) = \min_{\rho\in\Pi_t} \mathbb{E}_\rho[\int [sU(t,A,\omega,x) + (1-s)U(t,B,\omega,x)]p(dx|\lambda(\omega))].$$

This function is not in general differentiable everywhere. Since $V_t(\cdot,\sigma)$ is concave as a minimum of a set of linear functions, the super-differential exists everywhere. By definition and adapted to this setting, the super-differential is given by

$$\partial V_t(s,\sigma) = \{x \in \mathbb{R}^\Omega : V_t(y,\sigma) \leq V_t(s,\sigma) + \sum_\omega [(y(\omega) - s(\omega))x(\omega)] \forall y \in \Delta A\}.$$

The best response correspondence is the set of all s s.t. $0 \in \partial V_t(s,\sigma)$ where $\partial V_t(s,\sigma)$ is the super-differential of $V_t(\cdot,\sigma)$ at s. This follows from the dual to Aliprantis and Border [2006, Lem 7.10], which states that s is a maximum of $V_t(\cdot,\sigma)$ if and only if $0 \in \partial V_t(s,\sigma)$.

Consider the case where $\mathbb{E}[U|a, v_B, \sigma] \geq \mathbb{E}[U|b, v_B, \sigma]$. Note that

$$\begin{aligned} V_t(s,\sigma) &= \min_{p\in\Pi_t(A)} \{p[s\frac{1}{2}Pr(Piv_A|a) - (1-s)\frac{1}{2}Pr(Piv_B|a) + \mathbb{E}[U|a,\sigma]] + \\ &\quad +(1-p)[(1-s)\frac{1}{2}Pr(Piv_B|b) - s\frac{1}{2}Pr(Piv_A|b) + \mathbb{E}[U|b,\sigma]]\} \\ &= p_t[s\frac{1}{2}Pr(Piv_A|a) - (1-s)\frac{1}{2}Pr(Piv_B|a) + \mathbb{E}[U|a,\sigma]] + \\ &\quad +(1-p_t)[(1-s)\frac{1}{2}Pr(Piv_B|b) - s\frac{1}{2}Pr(Piv_A|b) + \mathbb{E}[U|b,\sigma]] \end{aligned}$$

because for every s

$$sPr(Piv_A|a) - (1-s)Pr(Piv_B|a) + 2\mathbb{E}[U|a,\sigma] \geq (1-s)Pr(Piv_B|b) - sPr(Piv_A|b) + 2\mathbb{E}[U|b,\sigma].$$

This occurs because the RHS reaches its minimum at $s = 0$ and the LHS reaches its maximum at $s = 0$. At $s = 0$ the RHS equals $\mathbb{E}[U|a, v_B, \sigma]$ and the LHS equals $\mathbb{E}[U|b, v_B, \sigma]$. By hypothesis, $\mathbb{E}[U|a, v_B, \sigma] \geq \mathbb{E}[U|b, v_B, \sigma]$ so for every s the RHS is larger than the LHS. Thus, $V_t(s,\sigma)$ is differentiable in $s \in (0,1)$. By Aliprantis and Border [2006, Cor 7.17]

$\partial V_t(s,\sigma)$ is singleton and coincides with the Gateaux derivative when it exists. Hence,

$$\partial V_t(s,\sigma) = \{p_t[\frac{1}{2}Pr(Piv_A|a) + \frac{1}{2}Pr(Piv_B|a)] - (1-p_t)[\frac{1}{2}Pr(Piv_A|b) + \frac{1}{2}Pr(Piv_B|b)]\}$$

and $0 \in \partial V_t(s,\sigma)$ only if $b(b,\sigma) = p_t$. If $b(b,\sigma) < p_t$ this is positive and if $b(b,\sigma) < p_t$ this is negative and hence no $s \in (0,1)$ is a maxima.

If $s = 1$ then the derivative is not defined since $V_t(1+\epsilon,\sigma)$ for any $\epsilon > 0$ does not exist. The super-differential does exist:

$$\partial V_t(1,\sigma) = \{x \in \mathbb{R} : V_t(y,\sigma) - V_t(1,\sigma) \leq (y-1)x \forall y \in [0,1]\}.$$

Since

$$V_t(y,\sigma) - V_t(1,\sigma) = (y-1)\frac{1}{2}(p_t[Pr(Piv_A|a) + Pr(Piv_B|a)] - (1-p_t)[Pr(Piv_A|b) + Pr(Piv_B|b)])$$

$0 \in \partial V_t(1,\sigma)$ if and only if

$$p_t[Pr(Piv_A|a) + Pr(Piv_B|a)] - (1-p_t)[Pr(Piv_A|b) + Pr(Piv_B|b) > 0.$$

As noted above, $b(b,\sigma) < p_t$ implies this is positive.

Additionally, if $s = 0$ the derivative is not defined since $V_t(0-\epsilon,\sigma)$ for any $\epsilon > 0$ does not exist. The super-differential does exist:

$$\partial V_t(0,\sigma) = \{x \in \mathbb{R} : V_t(y,\sigma) - V_t(0,\sigma) \leq y \cdot x \forall y \in [0,1]\}.$$

Since $V_t(y,\sigma) - V_t(0,\sigma) =$

$$y\frac{1}{2}(p_t[Pr(Piv_A|a) + Pr(Piv_B|a)] - (1-p_t)[Pr(Piv_A|n) + Pr(Piv_B|n)])$$

$0 \in \partial V_t(0,\sigma)$ if and only if

$$p_t[Pr(Piv_A|a) + Pr(Piv_B|a)] - (1-p_t)[Pr(Piv_A|b) + Pr(Piv_B|b)] < 0.$$

As noted above, $b(B,\sigma) > p_t$ implies this is negative.

The above observations show that if $\mathbb{E}[U|a,v_B,\sigma] \geq \mathbb{E}[U|b,v_B,\sigma]$, then the set of maximizers of $V_t(\cdot,\sigma)$ is

$$\arg\max_{s\in[0,1]} V_t(s,\sigma) = \begin{cases} \{1\} & \text{if } b(b,\sigma) > p_t \\ [0,1] & \text{if } b(b,\sigma) = p_t. \\ \{0\} & \text{if } b(b,\sigma) < p_t \end{cases}$$

If $\mathbb{E}[U|a,v_B,\sigma] \leq \mathbb{E}[U|b,v_B,\sigma]$, similar arguments show the same form of BR correspondence with the probability assigned to a equal to q_t instead of p_t.

Now, suppose that neither of the above inequalities hold. Then there exists an $\bar{s} \in (0,1)$ so that the conditional expected utilities in A and B are equal. Further, if $s > \bar{s}$ the conditional expected utility in state a is larger than that in state b and if $s < \bar{s}$ then the expected utility in state B is larger than that in state A. Algebra shows that

$$\bar{s} = \frac{2(\mathbb{E}[U|b,\sigma] - \mathbb{E}[U|a,\sigma]) + Pr(Piv_B|b) + Pr(Piv_B|a)}{Pr(Piv_B|a) + Pr(Piv_A|a) + Pr(Piv_B|b) + Pr(Piv_A|b)}$$

which is $\hat{s}(A,\sigma)$.

Since for all $s \in (0,\bar{s})$ and every $s \in (\bar{s},1)$ the minimizer is unique, the Gateaux derivative exists whenever $s \notin \{0,\bar{s},1\}$. If $s \in (\bar{s},1)$ then

$$\partial V_t(s,\sigma) = \{\frac{\partial}{\partial s}V_t(s,\sigma)\} = \{p_t\frac{1}{2}[Pr(Piv_A|a)+Pr(Piv_B|a)]-(1-p_t)\frac{1}{2}[Pr(Piv_A|b)+Pr(Piv_B|b)]\}.$$

If $s' \in (0,\bar{s})$ then

$$\partial V_t(s',\sigma) = \{\frac{\partial}{\partial s}V_t(s,\sigma)\} = \{q_t\frac{1}{2}[Pr(Piv_A|a)+Pr(Piv_B|a)]-(1-q_t)\frac{1}{2}[Pr(Piv_A|b)+Pr(Piv_B|b)]\}.$$

Thus any $s \in (\bar{s},1)$ is an optimum only if

$$p_t[Pr(Piv_A|a) + Pr(Piv_B|a)] - (1-p_t)[Pr(Piv_A|b) + Pr(Piv_B|b)] = 0,$$

which happens when $p_t = b(b,\sigma)$. Similarly, any $s \in (0,\bar{s})$ is an optimum when $q_t = b(b,\sigma)$. Otherwise there cannot be an optimum in $(0,1)\backslash\{\bar{s}\}$.

As above, when $s = 1$ then the derivative is not defined since $V_t(1+\epsilon,\sigma)$ for any $\epsilon > 0$ does not exist. The super-differential does exist:

$$\partial V_t(1,\sigma) = \{x \in \mathbb{R} : V_t(y,\sigma) - V_t(1,\sigma) \leq (y-1)x \forall y \leq 1\}.$$

Since $V_t(y,\sigma) - V_t(1,\sigma)$ is equal to

$$(y-1)(p_t\frac{1}{2}[Pr(Piv_A|a) + Pr(Piv_A|b)] - (1-p_t)\frac{1}{2}[Pr(Piv_B|a) + Pr(Piv_B|b)]),$$

$0 \in \partial V_t(1,\sigma)$ if and only if

$$V_t(y,\sigma) - V_t(1,\sigma) \leq 0 \iff b(b,\sigma) \leq p_t.$$

Hence $s = 1$ is optimal only if $b(b,\sigma) \geq p_t$. Similar arguments show then $0 \in \partial V_t(0,\sigma) \iff b(b,\sigma) \geq q_t$.

By the above, we have covered the cases where $b(b,\sigma) \geq q_t$ and $b(b,\sigma) \leq p_t$. Suppose $p_t < b(b,\sigma) < q_t$. In this case,

$$q_t[Pr(Piv_A|a) + Pr(Piv_B|a)] > (1-q_t)[Pr(Piv_A|b) + Pr(Piv_B|b)]$$

and

$$p_t[Pr(Piv_A|a) + Pr(Piv_B|a)] < (1-p_t)[Pr(Piv_A|b) + Pr(Piv_B|b)].$$

So for $s > \bar{s}$,

$$\partial V_t(s,\sigma) = \{p_t\frac{1}{2}[Pr(Piv_A|a) + Pr(Piv_B|a)] - (1-p_t)\frac{1}{2}[Pr(Piv_A|b) + Pr(Piv_B|b)]\}$$

is a singleton strictly smaller than zero. For $s' < \bar{s}$,

$$\partial V_t(s', \sigma) = \{q_t \frac{1}{2}[Pr(Piv_A|a) + Pr(Piv_B|a)] - (1 - q_t)\frac{1}{2}[Pr(Piv_A|b) + Pr(Piv_B|b)]\}$$

is a singleton strictly larger than zero. However, for $s = \bar{s}$

$$\partial V_t(\bar{s}, \sigma) = \{p(a)\frac{1}{2}[Pr(Piv_A|a) + Pr(Piv_B|a)] - p(b)\frac{1}{2}[Pr(Piv_A|b) + Pr(Piv_B|b)] : p \in \Pi_t\}$$

Since $q_t > \rho(A) > p_t$, $\exists \rho \in \Pi_t$ s.t.

$$\frac{\rho(a)}{1 - \rho(a)} = \frac{Pr(Piv_A|b) + Pr(Piv_B|b)}{Pr(Piv_A|a) + Pr(Piv_B|a)}$$

implying that $0 \in \partial V_t(\bar{s}, \sigma)$ and $\bar{s}$ is the only maximizer when $q_t > b(B, \sigma) > p_t$.

Combining the above results yields the desired form of the best response function. □

In order to prove Theorem 2, two more preliminary results are necessary. Lemma 2 and Lemma 3 allow characterization of the worst case scenario. The proof of Theorem 2 will use both these facts to show that no equilibrium exists where a voter thinks the worst case scenario is independent of her vote.

Lemma 2. *For any $n \geq 1$, $\mathbb{E}[U|a, \sigma_n] \geq \mathbb{E}[U|B, \sigma_n] \iff \tau(A|a, \sigma_n) \geq \tau(A|b, \sigma_n)$.*

Proof. Let $f(x, \lambda) = \frac{e^{-\lambda}\lambda^x}{x!}$, the probability mass function of the Poisson distribution with mean λ, and $F(x, \lambda)$ its CDF. The CDF of the Poisson distribution has the form $\frac{\Gamma([x+1], \lambda)}{[x]!}$ where $[z]$ is the greatest integer less than or equal to z and $\Gamma(z, y)$ is the generalized incomplete gamma function:

$$\Gamma(z, y) = \int_y^\infty e^{-t} t^{z-1} dt.$$

We can write

$$\mathbb{E}[U|a, \sigma_n] = Q(\tau(A|a, \sigma_n)n) + \frac{1}{2}\sum_{j=0}^{\infty} f(j, \tau(A|a, \sigma_n)n) f(j, \tau(B|a, \sigma_n)n)$$

where $Q(\cdot)$ is given by

$$Q(\lambda) = \sum_{j=0}^{\infty} f(j, \lambda) F(j - 1, n - \lambda).$$

Observe that

$$\frac{\partial Q}{\partial \lambda} = \sum_{x=1}^{\infty}[\frac{\partial f(j, \lambda)}{\partial \lambda} F(j - 1, n - \lambda) + f(j, \lambda)\frac{\partial F(j - 1, n - \lambda)}{\partial \lambda}].$$

By the fundamental theorem of calculus, $\frac{\partial F(x,\lambda)}{\partial \lambda} = -\frac{e^{-\lambda}\lambda^x}{x!}$ and $\frac{\partial f(x,\lambda)}{\partial \lambda} = \frac{e^{-\lambda}\lambda^{x-1}(x-\lambda)}{x!}$ whenever x is an integer. Given this, the above sum can be written as

$$\begin{aligned}
\frac{\partial Q}{\partial \lambda} &= \sum_{x=1}^{\infty}[\frac{\partial f(x,\lambda)}{\partial \lambda}F(x-1,n-\lambda)+f(x,\lambda)\frac{\partial F(x-1,n-\lambda)}{\partial \lambda}] \\
&= \sum_{x=1}^{\infty}[\frac{e^{-\lambda}\lambda^{x-1}(x-\lambda)}{x!}F(x-1,n-\lambda)+\frac{e^{-\lambda}\lambda^{x}}{x!}\frac{e^{-n+\lambda}(n-\lambda)^{x-1}}{x-1!}] \\
&= \sum_{x=1}^{\infty}[-\frac{\lambda^{x}e^{-\lambda}}{x!}F(x-1,n-\lambda)+\frac{x\lambda^{x-1}e^{-\lambda}}{x!}F(x-1,n-\lambda)+\frac{e^{-\lambda}\lambda^{x}}{x!}\frac{e^{-n+\lambda}(n-\lambda)^{x-1}}{x-1!}] \\
&= \sum_{x=1}^{\infty}\frac{\lambda^{x-1}e^{-\lambda}}{x-1!}F(x-1,n-\lambda)-\sum_{x=1}^{\infty}\frac{\lambda^{x}e^{-\lambda}}{x!}F(x-1,n-\lambda)+\sum_{x=1}^{\infty}\frac{e^{-\lambda}\lambda^{x}}{x!}\frac{e^{-n+\lambda}(n-\lambda)^{x-1}}{x-1!} \\
&= \sum_{x=1}^{\infty}\frac{\lambda^{x-1}e^{-\lambda}}{x-1!}[F(x-2,n-\lambda)+f(x-1,n-\lambda)]+\sum_{x=1}^{\infty}[\frac{e^{-\lambda}\lambda^{x}}{x!}\frac{e^{-n+\lambda}(n-\lambda)^{x-1}}{x-1!}-\frac{\lambda^{x}e^{-\lambda}}{x!}F(x-1,n-\lambda)] \\
&= \sum_{y=0}^{\infty}\frac{\lambda^{y}e^{-\lambda}}{y!}F(y-1,n-\lambda)+\sum_{x=1}^{\infty}[\frac{e^{-\lambda}\lambda^{x-1}}{x-1!}\frac{e^{-n+\lambda}(n-\lambda)^{x-1}}{x-1!}+\frac{e^{-\lambda}\lambda^{x}}{x!}\frac{e^{-n+\lambda}(n-\lambda)^{x-1}}{x-1!}-\frac{\lambda^{x}e^{-\lambda}}{x!}F(x-1,n-\lambda)] \\
&= \sum_{x=1}^{\infty}\frac{e^{-\lambda}\lambda^{x}}{x!}\frac{e^{-n+\lambda}(n-\lambda)^{x-1}}{x-1!}+\sum_{x=0}^{\infty}\frac{\lambda^{x}e^{-\lambda}}{x!}\frac{(n-\lambda)^{x}e^{-n+\lambda}}{x!} \\
&= e^{-n}[\sum_{x=1}^{\infty}\frac{\lambda^{x}}{x!}\frac{(n-\lambda)^{x-1}}{x-1!}+\sum_{x=0}^{\infty}\frac{\lambda^{x}}{x!}\frac{(n-\lambda)^{x}}{x!}
\end{aligned}$$

Now, we must deal with the second term.

$$\begin{aligned}
\frac{\partial}{\partial \lambda}\sum_{j=0}^{\infty}f(j,\lambda)f(j,n-\lambda) &= \sum_{x=0}^{\infty}\frac{\partial}{\partial \lambda}e^{-n}\frac{\lambda^{x}(n-\lambda)^{x}}{x!x!}] \\
&= \sum_{x=1}^{\infty}e^{-n}[\frac{x\lambda^{x-1}(n-\lambda)^{x}}{x!x!}-\frac{x\lambda^{x}(n-\lambda)^{x-1}}{x!x!}] \\
&= \sum_{x=1}^{\infty}e^{-n}[\frac{\lambda^{x-1}(n-\lambda)^{x}}{x!(x-1)!}-\frac{\lambda^{x}(n-\lambda)^{x-1}}{x!(x-1)!}]
\end{aligned}$$

Adding together shows that $\frac{\partial}{\partial \lambda}\mathbb{E}[U|a,\sigma_n,n]$ is equal to

$$e^{-n}[\sum_{x=0}^{\infty}\frac{\lambda^{x}}{x!}\frac{(n-\lambda)^{x}}{x!}+\frac{1}{2}\sum_{x=1}^{\infty}(\frac{\lambda^{x}}{x!}\frac{(n-\lambda)^{x-1}}{x-1!}+\frac{\lambda^{x-1}(n-\lambda)^{x}}{x!(x-1)!})].$$

Clearly, this term is positive. Recall that $\lambda=\tau(A|a,\sigma_n)n$ so that $\frac{\partial \mathbb{E}[U|a,\sigma_n]}{\partial \tau(A|a,\sigma_n)}=\frac{\partial \mathbb{E}[U|a,\sigma_n]}{\partial \lambda}\frac{\partial \lambda}{\partial \tau(A|A,\sigma_n)}=n\frac{\partial \mathbb{E}[U|a,\sigma_n]}{\partial \lambda}$. Since $\frac{\partial \mathbb{E}[U|a,\sigma_n]}{\partial \lambda}\geq 0$, so is $\frac{\partial \mathbb{E}[U|a,\sigma_n]}{\partial \tau(A|a,\sigma_n)}$.

Since $\frac{\partial \mathbb{E}[U|a,\sigma_n]}{\partial \tau(A|a,\sigma_n)}\geq 0$, as $\tau(A|a,\sigma_n)$ increases, $\mathbb{E}[U|a,\sigma_n]$ increases. Similarly for $\tau(A|b,\sigma_n)$ and $\mathbb{E}[U|b,\sigma_n]$. Since the expected number of voters in each state is equal, the terms $\mathbb{E}[U|a,\sigma_n]$ and $\mathbb{E}[U|b,\sigma_n]$ are equal whenever $\tau(A|a,\sigma_n)$ and $\tau(B|b,\sigma_n)$ are equal. This establishes the claim. □

Lemma 3. *If $\frac{1}{2}<\tau(B|b,\sigma_n)<\tau(A|a,\sigma_n)$, then $\hat{s}(A,\sigma_n)<\frac{1}{2}$. In particular, when the expected winner in each state is correct, $\hat{s}(A,\sigma_n)<\frac{1}{2}\iff b(b,\sigma_n)>\frac{1}{2}$.*

Proof. Suppose $\frac{1}{2}<\tau(B|b,\sigma_n)<\tau(A|a,\sigma_n)$. By Lemma 2, $\mathbb{E}[U|b,\sigma_n]<\mathbb{E}[U|a,\sigma_n]$. Consider the numerator of $\hat{s}(A,\sigma_n)$. Recall that it is

$$\phi(\sigma_n)=2(\mathbb{E}[u|b,\sigma_n]-\mathbb{E}[u|a,\sigma_n])+Pr(Piv_B|b,\sigma_n)+Pr(Piv_B|a,\sigma_n).$$

The fraction is less than $\frac{1}{2}$ if and only if

$$2\phi(\sigma_n)<[Pr(Piv_B|b,\sigma_n)+Pr(Piv_B|a,\sigma_n)+Pr(Piv_A|b,\sigma_n)+Pr(Piv_A|a,\sigma_n)].$$

Equivalently, this holds if and only if

$$4(\mathbb{E}[u|b,\sigma_n]-\mathbb{E}[u|a,\sigma_n])+Pr(Piv_B|b,\sigma_n)+Pr(Piv_B|a,\sigma_n)-Pr(Piv_A|b,\sigma_n)-Pr(Piv_A|a,\sigma_n)<0.$$

We can rewrite

$$\gamma=Pr(Piv_B|b,\sigma_n)+Pr(Piv_B|a,\sigma_n)-Pr(Piv_A|b,\sigma_n)-Pr(Piv_A|a,\sigma_n)$$

as a function only of $\tau(A|a,\sigma_n)$ and $\tau(B|b,\sigma_n)$. Set $t = \tau(B|b,\sigma_n)$ and $s = \tau(A|a,\sigma_n)$ for convenience. Expanding and writing in terms of t and s,

$$\begin{aligned}\gamma &= e^{-n}\sum_{j=0}^{\infty} n^{2j}[\frac{t^j(1-t)^j}{j!j!} + n\frac{t^j(1-t)^{j+1}}{j!j+1!} + \frac{s^j(1-s)^j}{j!j!} + n\frac{s^{j+1}(1-s)^j}{j!j+1!}] - \\ &\quad -e^{-n}\sum_{j=0}^{\infty} n^{2j}[\frac{t^j(1-t)^j}{j!j!} + n\frac{t^{j+1}(1-t)^j}{j!j+1!} + \frac{s^j(1-s)^j}{j!j!} + n\frac{s^j(1-s)^{j+1}}{j!j+1!}] \\ &= \sum_{j=0}^{\infty} e^{-n}n^{2j+1}[\frac{t^j(1-t)^{j+1}}{j!j+1!} + \frac{s^{j+1}(1-s)^j}{j!j+1!} - \frac{t^{j+1}(1-t)^j}{j!j+1!} - \frac{s^j(1-s)^{j+1}}{j!j+1!}].\end{aligned}$$

Recall that

$$\begin{aligned}\mathbb{E}[U|b,\sigma_n] &= \sum_{j=0}^{\infty} f(j;tn)F(j-1;(1-t)n) + \frac{1}{2}\sum_{j=0}^{\infty} f(j;tn)f(j;(1-t)n) \\ &:= \hat{\psi}_n(\tau(B|b,\sigma_n))\end{aligned}$$

and similarly $\mathbb{E}[U|a,\sigma_n] = \hat{\psi}_n(\tau(A|a,\sigma_n))$. Setting

$$\theta_n(t) = \sum_{j=0}^{\infty} e^{-n}n^{2j+1}[\frac{t^j(1-t)^{j+1}}{j!j+1!} - \frac{t^{j+1}(1-t)^j}{j!j+1!}]$$

and

$$\psi_n(x) = 4\hat{\psi}_n(x) + \theta_n(x).$$

gives that

$$\hat{s}(A,\sigma_n) < \frac{1}{2} \iff \psi_n(t) - \psi_n(s) < 0.$$

From Lemma 2 and writing $\lambda = nt$, we have that

$$\frac{\partial\hat{\psi}_n}{\partial\lambda} = e^{-n}[\sum_{x=0}^{\infty}\frac{\lambda^x}{x!}\frac{(n-\lambda)^x}{x!} + \frac{1}{2}\sum_{x=1}^{\infty}(\frac{\lambda^x}{x!}\frac{(n-\lambda)^{x-1}}{x-1!} + \frac{\lambda^{x-1}(n-\lambda)^x}{x!(x-1)!})].$$

which is positive. Now,

$$\theta_n(\lambda) = \sum_{j=0}^{\infty} e^{-n}\frac{\lambda^j(n-\lambda)^{j+1} - \lambda^{j+1}(n-\lambda)^j}{j!(j+1)!}$$

so that

$$\begin{aligned}
\frac{\partial \theta_n}{\partial \lambda} &= \sum_{j=0}^{\infty} e^{-n} \frac{\partial}{\partial \lambda} \frac{\lambda^j (n-\lambda)^{j+1} - \lambda^{j+1}(n-\lambda)^j}{j!(j+1)!} \\
&= \sum_{j=0}^{\infty} e^{-n} \frac{\lambda^{j-1}(n-\lambda)^{j+1}}{(j-1)!(j+1)!} - \frac{\lambda^j (n-\lambda)^j}{j!j!} - \frac{\lambda^j (n-\lambda)^j}{j!j!} + \frac{\lambda^{j+1}(n-\lambda)^{j-1}}{(j-1)!(j+1)!} \\
&= \sum_{j=1}^{\infty} e^{-n} \frac{\lambda^{j-1}(n-\lambda)^{j+1} + \lambda^{j+1}(n-\lambda)^{j-1}}{(j-1)!(j+1)!} - 2\sum_{j=0}^{\infty} e^{-n} \frac{\lambda^j (n-\lambda)^j}{j!j!}
\end{aligned}$$

Combining,

$$\begin{aligned}
\frac{\partial \psi_n}{\partial t} &= [4\frac{\partial \hat{\psi}_n}{\partial \lambda} + \frac{\partial \theta_n}{\partial \lambda}]\frac{\partial \lambda}{\partial t} \\
&= n[2\sum_{x=0}^{\infty} e^{-n} \frac{\lambda^x (n-\lambda)^x}{x!^2} + 3\sum_{j=1}^{\infty} e^{-n} \frac{\lambda^{j-1}(n-\lambda)^{j+1} + \lambda^{j+1}(n-\lambda)^{j-1}}{(j-1)!(j+1)!}]
\end{aligned}$$

which is clearly greater than 0.

To show that $\psi_n(\tau(B|b,\sigma_n)) - \psi_n(\tau(A|a,\sigma_n)) < 0$, recall that we can write this as $\int_s^t \frac{\partial \psi_n(x)}{\partial x} dx$ which is negative because the integrand is positive but $\tau(A|a,\sigma_n) > \tau(B|b,\sigma_n)$. Therefore, whenever $\tau(A|a,\sigma_n) > \tau(B|b,\sigma_n)$ it must be that $\hat{s}(A,\sigma_n) < \frac{1}{2}$.

To complete the second part of the Lemma, note the following.

Claim 1. $b(b,\sigma_n) > \frac{1}{2} \iff |\tau(A|a,\sigma_n) - \frac{1}{2}| > |\tau(B|b,\sigma_n) - \frac{1}{2}|$.

Proof. $b(b,\sigma_n) > \frac{1}{2} \iff Pr(Piv_A|b) + Pr(Piv_B|b) > Pr(Piv_A|a) + Pr(Piv_B|a)$

Let $t = \tau(A|a,\sigma_n)$ so that $Pr(Piv_A|a) + Pr(Piv_B|a)$ equals

$$2\sum_{j=0}^{\infty} p(2j) \begin{pmatrix} 2j \\ j \end{pmatrix} t^j(1-t)^j + \sum_{j=0}^{\infty} p(2j+1) \begin{pmatrix} 2j+1 \\ j+1 \end{pmatrix} [t^j(1-t)^{j+1} + t^{j+1}(1-t)^j]$$

where $p(x) = \frac{e^{-n}n^x}{x!}$. Take the derivative with respect to t to get

$$(1-2t)[2\sum_{j=0}^{\infty} j\{p(2j) \begin{pmatrix} 2j \\ j \end{pmatrix} t^{j-1}(1-t)^{j-1} + \sum_{j=0}^{\infty} p(2j+1) \begin{pmatrix} 2j+1 \\ j+1 \end{pmatrix} t^{j-1}(1-t)^{j-1}\}]$$

which is positive whenever $t < .5$ and negative whenever $t > .5$. Similarly for $Pr(Piv_A|b) + Pr(Piv_B|b)$. Given the symmetry of $Pr(Piv_A|a) + Pr(Piv_B|b)$ with respect to $\tau(A|a,\sigma_n)$ and $Pr(Piv_A|b) + Pr(Piv_B|b)$ with respect to $\tau(B|b,\sigma_n)$, the claim follows immediately. □

From Claim 1, whenever $b(b,\sigma_n) > \frac{1}{2}$, $|\tau(A|a,\sigma_n) - \frac{1}{2}| > |\tau(B|a,\sigma_n) - \frac{1}{2}|$. Further, if the expected winners are correct, it must be that both $\tau(A|a,\sigma_n) > \frac{1}{2}$ and $\tau(B|b,\sigma_n) > \frac{1}{2}$. It follows that $\tau(A|a,\sigma_n) > \tau(B|b,\sigma_n)$, so $\hat{s}(A,\sigma_n) < \frac{1}{2}$. Similarly, suppose that $\hat{s}(A,\sigma_n) < \frac{1}{2}$

and the expected winners are correct. From the above, $\tau(A|a,\sigma_n) > \tau(B|b,\sigma_n) > \frac{1}{2}$, so by Claim 1 $b(b,\sigma_n) > \frac{1}{2}$. □

Proof of Theorem 2:

Proof. First, note that if there is no t so that $r(t|a) \neq r(t|b)$, vote shares must be equal across states, completing the proof. Therefore, assume that for some t, $r(t|a) \neq r(t|b)$.

Suppose, for the sake of contradiction, that σ_n is an equilibrium for Γ_n where $\tau(A|a,\sigma_n) > \frac{1}{2}$ and $\tau(B|b,\sigma_n) > \frac{1}{2}$.

Claim 2. $BR_t(\sigma_n) = \hat{BR}_t(\sigma_n)$ for all t.

Proof. If $BR_t(\sigma_n) \neq \hat{BR}_t(\sigma_n)$ then either

$$\mathbb{E}[U|a,\sigma_n] \geq \mathbb{E}[U|b,\sigma_n] + \frac{1}{2}(Pr(Piv_B|b) + Pr(Piv_B|a)) \tag{10}$$

or

$$\mathbb{E}[U|b,\sigma_n] \geq \mathbb{E}[U|a,\sigma_n] + \frac{1}{2}(Pr(Piv_A|b) + Pr(Piv_A|a)) \tag{11}$$

by Lemma 1.

In the first case, some type $\hat{t}$ plays a mixed strategy in σ_n. First, suppose equation (10) holds. In this case, because σ_n is an equilibrium, Lemma 1 implies that $b(b,\sigma_n) = p_{\hat{t}}$. Because voters lack confidence, $p_{\hat{t}} < \frac{1}{2}$ which implies that

$$|\tau(A|a,\sigma_n) - \frac{1}{2}| < |\tau(B|b,\sigma_n) - \frac{1}{2}|$$

by Claim 1. Since $\tau(A|a,\sigma_n), \tau(B|b,\sigma_n) > \frac{1}{2}$ it follows that $\tau(A|a,\sigma_n) < \tau(B|b,\sigma_n)$ and Lemma 2 implies that $\mathbb{E}[U|b,\sigma_n] > \mathbb{E}[U|a,\sigma_n]$, a contradiction.

Similarly, if instead equation (11), it must be that $b(n,\sigma_n) = q_{\hat{t}}$. Because voters lack confidence, $q_{\hat{t}} > \frac{1}{2}$ which implies that

$$|\tau(A|a,\sigma_n) - \frac{1}{2}| > |\tau(B|b,\sigma_n) - \frac{1}{2}$$

by Claim 1. Since $\tau(A|a,\sigma_n), \tau(B|b,\sigma_n) > \frac{1}{2}$ it follows that $\tau(A|a,\sigma_n) > \tau(B|b,\sigma_n)$, Lemma 2 implies that $\mathbb{E}[u|a,\sigma_n] > \mathbb{E}[U|b,\sigma_n]$, a contradiction.

In the second case, all types play pure strategies in σ_n. Further, at least one type (WLOG, 1) votes for A for sure and another type (WLOG, 2) votes for B for sure. By Lemma 1, if equation (10) holds then $p_2 \leq b(b,\sigma_n) \leq p_1 < \frac{1}{2}$. By Lemma 2, $\mathbb{E}[U|b,\sigma_n] > \mathbb{E}[U|a,\sigma_n]$, contradicting that equation (10) holds. Suppose instead that equation (11) holds. By Lemma 1, $\frac{1}{2} < q_2 \leq b(b,\sigma_n) \leq q_1$. By Lemma 2, $\mathbb{E}[U|a,\sigma_n] > \mathbb{E}[U|b,\sigma_n]$, contradicting that equation (11) holds.

Hence, $BR_t(\sigma_n) = \hat{BR}_t(\sigma_n)$ for all t. □

I now show that no type plays a pure strategy.

Claim 3. $\sigma_n(t) \in (0,1)$ for all t.

Proof. Suppose $\sigma_n(t)(A) \in \{0,1\}$ for some t. WLOG, assume that either $\sigma_n(2)(A) = 1$ or $\sigma_n(2)(B) = 1$.

Assume the former. Then it must be that $1 \in BR_2(\sigma_n)(A)$ so $b(b,\sigma_n) \leq p_2 < \frac{1}{2}$ by Lemma 1. By Lemma 3 it must be that $\hat{s}(A,\sigma_n) > \frac{1}{2}$. By assumption, some type of voter must vote for A with probability smaller than $\frac{1}{2}$. WLOG, assume this type is 1, so that $\sigma_n(1)(A) \leq \frac{1}{2} < \hat{s}(A,\sigma_n)$ for n high enough. Hence, it must be that $b(b,\sigma_n) \geq q_1$. Combining with $p_2 \geq b(b,\sigma_n)$ gives that $p_2 \geq q_1$, which is a contradiction of $p_2 < \frac{1}{2} < q_1$.

Now, assume the latter. It must be that $0 \in BR_2(\sigma_n)(A)$ so $b(B,\sigma_n) \geq q_2$ by Lemma 1. By Lemma 3 it must be that $\hat{s}(A,\sigma_n) < \frac{1}{2}$. By assumption, some type of voter must vote for A with probability larger than $\frac{1}{2}$. WLOG, assume this type is 1, so that $\sigma_n(1)(A) \geq \frac{1}{2} > \hat{s}(A,\sigma_n)$ for n high enough. Lemma 1 implies that $b(B,\sigma_n) \leq p_1$. Combining with $b(B,\sigma_n) \geq q_2$ gives that $p_1 \geq q_2$, which is a contradiction of $p_1 < \frac{1}{2} < q_2$. □

This claim shows that all types of voters must play a mixed strategy. Setting $[\underline{p},\bar{p}] = \cap_{t\in T}[p_t,q_t]$, $b(b,\sigma_n) \in [\underline{p},\bar{p}]$, since otherwise at least one type of voter plays a pure strategy by Lemma 1. Further, if $b(b,\sigma_n) \in (\underline{p},\bar{p})$, the best response of all voters is to play $\sigma_n(t)(A) = \hat{s}(A,\sigma_n)$. Because of this, vote shares in each state are the same, a contradiction.

Claim 4. Suppose that $b(b,\sigma_n) = \underline{p}$. Then the expected winner in state b is not B.

Proof. WLOG, assume that $\underline{p} = p_1$; in fact, $p_1 = \max_{t\in T} p_t$ so $q_t > b(b,\sigma_n) > p_t \forall t \neq 1$. By Lemma 1, $\sigma_n(1)(A) \geq \hat{s}(A,\sigma_n)$ and $\sigma_n(t)(A) = \hat{s}(A,\sigma_n)$ for all $t \neq 1$. Because $b(b,\sigma_n) = p_1 < \frac{1}{2}$, by Lemma 3 it must be that $\hat{s}(A,\sigma_n) > \frac{1}{2}$. Therefore $\sigma_n(t)(A) > \frac{1}{2}$ for all t. Therefore, $\tau(B|b,\sigma_n) < \frac{1}{2}$ and thus B is not the expected winner in state b. □

Claim 5. Suppose that $b(B,\sigma_n) = \bar{p}$. Then the expected winner in state a is not A.

Proof. WLOG, assume that $\bar{p} = q_1$. By Lemma 1, $\sigma_n(1)(A) \leq \hat{s}(A,\sigma_n)$ and $\sigma_n(t)(A) = \hat{s}(A,\sigma_n)$ for all $t \neq 1$. By Lemma 3 it must be that $\hat{s}(A,\sigma_n) < \frac{1}{2}$. Therefore $\sigma_n(t)(A) < \frac{1}{2}$ for all t, so $\tau(A|a,\sigma_n) < \frac{1}{2}$ and A is not the expected winner in state a. □

Therefore, there is no equilibrium where both $\tau(A|a,\sigma_n) > \frac{1}{2}$ and $\tau(B|b,\sigma_n) > \frac{1}{2}$. □

Proof of Proposition 1:

Proof. Suppose σ is played. Clearly, $\mathbb{E}[U|a,\sigma] = \mathbb{E}[U|b,\sigma] = \frac{1}{2}$. This implies that $BR_t(\sigma) = \hat{BR}_t(\sigma)$ for all t by Lemma 1. Further, note that $b(b,\sigma) = \frac{1}{2}$ since $Pr(Piv_c|a) = Pr(Piv_c|b)$ for $c \in \{A,B\}$ since the vote shares are equal in both states. Since $b(b,\sigma) \in [p_t,q_t]$, voters of type t are willing to play $\sigma(t)(A) = \hat{s}(A,\sigma) = \frac{1}{2}$. Therefore, σ is an equilibrium. □

Proof of Theorem 3:

Proof. This proof adapts the arguments of Myerson [1998] Theorem 2.

Relabel $T = \{1, 2, ..., T\}$ so that $\min_{p \in \Pi_i} p(a) < \min_{p \in \Pi_{i+1}} p(a)$ for every $i \in \{1, 2, ..., T-1\}$. Denote $[h] = \max_{z \in \mathbb{Z}} z \leq h$ and $\sigma(h)$ for some $h \in [1, T]$ the strategy profile such that if h is an integer then $\sigma(t)(A) = 0$ if $t \leq h$ and $\sigma(t)(A) = 1$ if $t > h$. If h is not an integer then $\sigma(h)$ is such that $\sigma(t)(A) = 0$ if $t < [h]$ and $\sigma(t)(A) = 1$ if $t > h$ and $\sigma([h])(A) = h - [h]$. The proof will show that for all n high enough, there is an $h(n)$ so that $\sigma(h(n))$ is an equilibrium and that the expected winner in a is A and the expected winner in b is B.

Define functions $z : [1, T] \times \mathbb{N} \to [0, 1]$ and $\beta : [1, T] \times \mathbb{N} \to [0, 1]$ by the formulas

$$z(h, n) := \begin{cases} \hat{s}(A, \sigma(h), n) & \hat{s}(A, \sigma(h), n) \in [0, 1] \\ 1 & \hat{s}(A, \sigma(h), n) > 1 \\ 0 & \hat{s}(A, \sigma(h), n) < 0 \end{cases}$$

where $\hat{s}(c, \sigma, n)$ is $\hat{s}(c, \sigma)$ when there are n expected players. Further, define $\beta(h, n)$ to be $b(b, \sigma(h))$ when there are n expected players.

Let $q_t = \max_{p \in \Pi_t} p(a)$ and $p_t = \min_{p \in \Pi_t} p(a)$. If $\hat{s}(A, \sigma, n) < 0$ then

$$\mathbb{E}[U|b, \sigma] - \mathbb{E}[U|a, \sigma] + \frac{1}{2}(Pr(Piv_B|b) + Pr(Piv_B|a)) < 0$$

so that $\mathbb{E}[U|a, v_B, \sigma] > \mathbb{E}[U|b, v_B, \sigma]$. Hence

$$BR_t(\sigma) = \begin{cases} 1 & b(b, \sigma) > p_t \\ [0, 1] & b(b, \sigma) = p_t \\ 0 & b(b, \sigma) < p_t \end{cases}$$

by Lemma 1. Similarly, if $\hat{s}(A, \sigma) > 1$ then $1 - \hat{s}(A, \sigma) < 0$ which implies

$$\mathbb{E}[U|a, \sigma] - \mathbb{E}[U|b, \sigma] + \frac{1}{2}((Pr(Piv_A|a) + Pr(Piv_A|b))) < 0$$

and thus $\mathbb{E}[U|a, v_B, \sigma] > \mathbb{E}[U|b, v_B, \sigma]$. Hence

$$BR_t(\sigma) = \begin{cases} 1 & b(b, \sigma) > q_t \\ [0, 1] & b(b, \sigma) = q_t \\ 0 & b(b, \sigma) < q_t \end{cases}$$

by Lemma 1. Otherwise, $BR_t(\sigma)(A) = \hat{BR}_t(\sigma)(A)$.

Given the above notes, Lemma 1 shows that σ_h is an equilibrium if $\beta(h, n) \in \eta(h, n)$

$$\eta(h,n) = \begin{cases} [q_h, p_{h+1}] & h \in \mathbb{Z} \\ q_{[h]} & h \in ([h]+z(h,n),[h]+1) \\ [p_{[h]}, q_{[h]}] & h = [h]+z(h,n) \\ p_{[h]} & h \in ([h],[h]+z(h,n)) \end{cases}.$$

It's clear that $\hat{s}(\cdot,\sigma_n)$ is continuous by construction. It follows that $z(\cdot,n)$ is continuous since it can be written as the minimum of two continuous functions. Therefore $\eta(\cdot,n)$ is UHC, compact and convex.

There exists numbers $I(a) \neq I(b)$ so that $\tau(A|\omega,\sigma_{I(\omega)}) = \tau(B|\omega,\sigma_{I(\omega)})$ for each $\omega \in \{a,b\}$ and for every $h \in (I(a),I(b))$ (or $(I(b),I(a))$ if $I(b) < I(a)$) $\tau(A|\omega,\sigma_h) \neq \tau(B|\omega,\sigma_h)$ for each ω because $r(\cdot|a) \neq r(\cdot|b)$ and $h \mapsto \tau(c|\omega,\sigma_h)$ is a continuous function with range equal to $[0,1]$. Assume WLOG that $I(a) < I(b)$. For n high enough, $\exists h(n)$ so that $\beta(h(n),n) \in \eta(h(n),n)$ and $h(n) \in (I(a),I(b))$. This follows from $\beta(I(a),n) \to 0$, $\beta(I(b),n) \to 1$, $\beta(\cdot,n)$ is continuous and $\eta(\cdot,n)$ is convex and UHC. Since $h(n) \in (I(a),I(b))$, $\tau(A|a,\sigma_{h(n)}) > \tau(B|a,\sigma_{h(n)})$ and $\tau(B|b,\sigma_{h(n)}) > \tau(A|b,\sigma_{h(n)})$. Define $\sigma_n^* = \sigma_{h(n)}$ when n is large enough; otherwise, σ_n^* let σ_n^* be an arbitrary equilibrium. Using the arguments of Myerson [1998] Theorem 2, the sequence of equilibrium vote shares from $(\sigma_n^*)_{n=1}^{\infty}$ must converge. Applying the law of large numbers gives that the correct candidate is elected with arbitrarily high probability in both states. Therefore, $(\Gamma_n)_{n=1}^{\infty}$ satisfies FIE. □

Appendix E. Proofs from Section 5

Theorem 4 follows from a special case of Theorems 5 and 6.

Theorem 5. *Suppose Γ_n is an ambiguous voting game with abstention that has voters who lack confidence and posteriors that respect likelihood ratios. If σ_n is an equilibrium for Γ_n where the worst case scenario for all voters is not independent of their vote and the expected vote share for A in state a is greater than $\frac{1}{2}$, then the expected vote share for B in state b is less than $\frac{1}{2}$.*

Proof. The proof will be by contradiction. Suppose σ_n is an equilibrium for Γ_n where the worst case scenario for all voters is not independent of their vote and the expected vote share for A in state a is greater than $\frac{1}{2}$ and the expected vote share for B in state b is also greater than $\frac{1}{2}$.

Begin by deriving the best response correspondence for voters when the worst case scenario varies with the strategy played. For any strategy $s \in \Delta C$, represent s by the ordered pair $(\frac{s(A)}{1-s(\emptyset)}, s(\emptyset))$ if $s(\emptyset) < 1$ and $(0,1)$ otherwise. Note that there is a bijection between these ordered pairs corresponds and each strategy profile. Now, define a function $\hat{s} : \Omega \times [0,1) \times$

$(\Delta C)^T \to \mathbb{R}$ by

$$\hat{s}(A;s,\sigma) = \frac{2(\mathbb{E}[U|b,\sigma]-\mathbb{E}[U|a,\sigma])+(1-s)[Pr(Piv_B|a,\sigma)+Pr(Piv_B|b,\sigma)]}{(1-s)[Pr(Piv_B|a,\sigma)+Pr(Piv_B|b,\sigma)+Pr(Piv_A|a,\sigma)+Pr(Piv_A|b,\sigma)]}$$

and $\bar{s}:\sigma \to [0,1]$ implicitly by

$$\hat{s}(A;\bar{s}(\sigma),\sigma) = \begin{cases} 1 & if\ \mathbb{E}[U|b,\sigma] > \mathbb{E}[U|a,\sigma] \\ 0 & if\ \mathbb{E}[U|b,\sigma] < \mathbb{E}[U|a,\sigma] \end{cases}$$

and $\bar{s}(\sigma) = 1$ if $\mathbb{E}[U|b,\sigma] = \mathbb{E}[U|a,\sigma]$. Note that if $\sigma(t)(\emptyset) < \bar{s}(\sigma)$, the voter's worst case scenario still changes with her vote. In this case, playing the strategy defined by $\sigma(t)(A) = \hat{s}(A;\sigma(t)(\emptyset),\sigma)$ equalizes the voter's expected utilities across states. On the other hand, if $\sigma(t)(\emptyset) \geq \bar{s}(\sigma)$, the voter is abstaining enough that her vote will no longer affect the worst case scenario.

Lemma 4. *Suppose that the worst case scenario is not independent of the strategy picked given σ and that the expected winner is correct in each state. If σ is an equilibrium and $b(b,\sigma) \in (p_t,q_t)$, then $\sigma(t) \in BR_t(\sigma)$ where*

$$BR_t(\sigma) = \begin{cases} \{(0,1)\} & if\ \mathbb{E}[U|b,\sigma] = \mathbb{E}[U|a,\sigma] \\ = \begin{cases} \{(0,1)\} & if\ \frac{q_t}{1-q_t} < \frac{Pr(Piv_A|b,\sigma)}{Pr(Piv_A|a,\sigma)} \\ \{1\}\times[\bar{s}(\sigma),1] & if\ \frac{q_t}{1-q_t} = \frac{Pr(Piv_A|b,\sigma)}{Pr(Piv_A|a,\sigma)} \\ \{(1,\bar{s}(\sigma))\} & if\ \frac{q_t}{1-q_t} > \frac{Pr(Piv_A|b,\sigma)}{Pr(Piv_A|a,\sigma)} \end{cases} & if\ \mathbb{E}[U|b,\sigma] > \mathbb{E}[U|a,\sigma] \\ = \begin{cases} \{(0,1)\} & if\ \frac{p_t}{1-p_t} > \frac{Pr(Piv_B|b,\sigma)}{Pr(Piv_B|a,\sigma)} \\ \{0\}\times[\bar{s}(\sigma),1] & if\ \frac{p_t}{1-p_t} = \frac{Pr(Piv_B|b,\sigma)}{Pr(Piv_B|a,\sigma)} \\ \{(0,\bar{s}(\sigma))\} & if\ \frac{p_t}{1-p_t} < \frac{Pr(Piv_B|b,\sigma)}{Pr(Piv_B|a,\sigma)} \end{cases} & if\ \mathbb{E}[U|b,\sigma] < \mathbb{E}[U|a,\sigma] \end{cases}.$$

Proof. (I drop the subscript t for convenience).

Suppose $p < b(b,\sigma) < q$. If the voter plays strategy (s,θ), she gets

$$\begin{aligned} V_t(s,\theta;\sigma) &= \min_{\pi\in[p,q]} \pi\{\mathbb{E}[U|a,\sigma]+(1-\theta)[sPr(Piv_A|a,\sigma)-(1-s)Pr(Piv_B|a,\sigma)]\} + \\ &\quad +(1-\pi)\{\mathbb{E}[U|b,\sigma]+(1-\theta)[(1-s)Pr(Piv_B|b,\sigma)-sPr(Piv_A|b,\sigma)]\}. \end{aligned}$$

Given a fixed $\theta < \bar{s}(\sigma)$, consider $v_{\theta\sigma}:[0,1]\to\mathbb{R}$ define by $v_{a\sigma}(s) = V_t(s,\theta;\sigma)$. Note that

$$\partial v_{a\sigma}(s) = \begin{cases} \{(1-\theta)[p[Pr(Piv_A|a,\sigma) + Pr(Piv_B|a,\sigma)] - & if\ s > \hat{s}(A;\theta,\sigma) \\ \quad -(1-p)[Pr(Piv_B|b,\sigma) + Pr(Piv_A|b,\sigma)]]\} & \\ \{(1-a\theta)[\pi[Pr(Piv_A|a,\sigma) + Pr(Piv_B|a,\sigma)] - & if\ s = \hat{s}(A,\theta,\sigma) \\ \quad -(1-\pi)[(Pr(Piv_B|b,\sigma) + Pr(Piv_A|b,\sigma)]] : \pi \in [p,q]\} & \\ \{(1-\theta)[q[Pr(Piv_A|a,\sigma) + Pr(Piv_B|a,\sigma)] + & if\ s < \hat{s}(A,\theta,\sigma) \\ \quad +(1-q)[Pr(Piv_B|b,\sigma) + Pr(Piv_A|b,\sigma)]]\} & \end{cases}$$

As in Lemma 1, given $p < b(b,\sigma) < q$, $0 \in \partial v_{\theta\sigma}(s)$ only if $s = \hat{s}(A,\theta,\sigma)$. Given this, consider $v_\sigma : [0,1] \to \mathbb{R}$ defined by $v_\sigma(\theta) = V_t(\hat{s}(A,\theta,\sigma),\theta,\sigma)$. Write $pc\omega = Pr(Piv_c|\omega,\sigma)$. By construction

$$\mathbb{E}[U|a,\sigma]+(1-\theta)[\hat{s}Pr(Piv_A|a,\sigma)-(1-\hat{s})Pr(Piv_B|a,\sigma)]] = [\mathbb{E}[U|b,\sigma]+(1-\theta)[(1-\hat{s})Pr(Piv_B|b,\sigma)-\hat{s}Pr(Piv$$

when $\hat{s} = \hat{s}(A,\theta,\sigma)$. So if $\theta < \bar{s}(\sigma)$

$$\begin{aligned} v_\sigma(\theta) &= \mathbb{E}[U|a,\sigma] + (1-\theta)[\hat{s}Pr(Piv_A|a,\sigma) - (1-\hat{s})Pr(Piv_B|a,\sigma)]] \\ \partial v_\sigma(\theta) &= \{\frac{\partial}{\partial\theta}[(1-\theta)\frac{2(\mathbb{E}[U|b,\sigma]-\mathbb{E}[U|a,\sigma]) + (1-\theta)[pBa+pBb]}{(1-\theta)[pAa+pBb+pBa+pAb]}pAa - \\ &\quad -(1-\theta)\frac{2(\mathbb{E}[U|a,\sigma]-\mathbb{E}[U|b,\sigma]) + (1-\theta)[pAa+pAb]}{(1-\theta)[pAa+pBb+pBa+pAb]}pBa]\} \\ &= \{\frac{\partial}{\partial\theta}[\frac{2(\mathbb{E}[U|b,\sigma]-\mathbb{E}[U|a,\sigma]) - 2(\mathbb{E}[U|a,\sigma]-\mathbb{E}[U|b,\sigma])}{pAa+pBb+pBa+pAb} + \\ &\quad + \frac{(1-\theta)[pBa+pBb]pAa - pBa[pAa+pAb]}{pAa+pBb+pBa+pAb}]\} \\ &= \{\frac{pBa[pAa+pAb] - [pBa+pBb]pAa}{pAa+pBb+pBa+pAb}\} \\ &= \{\frac{pBa(pAb) - pBb(pAa)}{pAa+pBb+pBa+pAb}\} \end{aligned}$$

Since FIE implies that $\frac{Pr(Piv_A|A,\sigma)}{Pr(Piv_B|A,\sigma)} < \frac{Pr(Piv_A|B,\sigma)}{Pr(Piv_B|B,\sigma)}$, no $\sigma(t)(\emptyset) < \bar{s}(\sigma)$ is optimal. Therefore, the voter abstains enough that worst case scenario is independent of whether she votes for A or B when she votes.

We can think of her as a SEU voter that assigns either probability p to a (if $\mathbb{E}[U|b,\sigma] < \mathbb{E}[U|a,\sigma]$) or q to a (if $\mathbb{E}[U|b,\sigma] > \mathbb{E}[U|a,\sigma]$). In this case, because $p < b(b,\sigma) < q$, the voter votes for B (in the first case) or A (in the second case) for sure conditional on voting. In the first case, she abstains for sure if $\frac{p}{1-p} > \frac{Pr(Piv_B|b,\sigma)}{Pr(Piv_B|a,\sigma)}$, and abstains with probability $\bar{s}(\sigma)$ if $\frac{p}{1-p} < \frac{Pr(Piv_B|b,\sigma)}{Pr(Piv_B|a,\sigma)}$. She is willing to abstain with any probability between $[\bar{s}(\sigma), 1]$ if $\frac{p}{1-p} = \frac{Pr(Piv_B|b,\sigma)}{Pr(Piv_B|a,\sigma)}$. In the second case, she abstains for sure if $\frac{q}{1-q} < \frac{Pr(Piv_A|b,\sigma)}{Pr(Piv_A|a,\sigma)}$, and abstains with probability $\bar{a}(\sigma)$ if $\frac{q}{1-q} > \frac{Pr(Piv_A|b,\sigma)}{Pr(Piv_A|a,\sigma)}$. She is willing to abstain with any probability between $[\bar{a}(\sigma), 1]$ if $\frac{q}{1-q} = \frac{Pr(Piv_A|b,\sigma)}{Pr(Piv_A|a,\sigma)}$. This establishes the best response correspondence when $p < b(b,\sigma) < q$. □

Lemma 5. *Suppose that the worst case scenario is not independent of the strategy picked given σ and that the expected winner is correct in each state. If σ is an equilibrium and $b(b,\sigma) \leq p_t$, then $\sigma(t) \in BR_t(\sigma)$ where*

$$BR_t(\sigma) = \begin{cases} \{(1,0)\} & if\ \frac{p_t}{1-p_t} > \frac{Pr(Piv_A|b,\sigma)}{Pr(Piv_A|a,\sigma)} \\ \{1\} \times [0, \bar{s}(\sigma)] & if\ \frac{p_t}{1-p_t} = \frac{Pr(Piv_A|b,\sigma)}{Pr(Piv_A|a,\sigma)} \\ \widetilde{BR_{A,t}}(\sigma) & if\ \frac{p_t}{1-p_t} < \frac{Pr(Piv_A|b,\sigma)}{Pr(Piv_A|a,\sigma)} \end{cases}$$

and

$$\widetilde{BR_{A,t}}(\sigma) = \begin{cases} = \begin{cases} \{(0,1)\} & if\ \frac{q_t}{1-q_t} < \frac{Pr(Piv_A|b,\sigma)}{Pr(Piv_A|a,\sigma)} \\ \{1\} \times [\bar{s}(\sigma), 1] & if\ \frac{q_t}{1-q_t} = \frac{Pr(Piv_A|b,\sigma)}{Pr(Piv_A|a,\sigma)} \\ \{(1, \bar{s}(\sigma))\} & if\ \frac{q_t}{1-q_t} > \frac{Pr(Piv_A|b,\sigma)}{Pr(Piv_A|a,\sigma)} \end{cases} & if\ \mathbb{E}[U|b,\sigma] > \mathbb{E}[U|a,\sigma] \\ \{(0,1)\} & otherwise \end{cases}$$

Proof. She votes for A conditional on voting because $b(b,\sigma)$ is low enough relative to her priors. She never abstains if $\frac{p_t}{1-p_t} > \frac{Pr(Piv_A|b,\sigma)}{Pr(Piv_A|a,\sigma)}$. If $\frac{p_t}{1-p_t} = \frac{Pr(Piv_A|b,\sigma)}{Pr(Piv_A|a,\sigma)}$, she's indifferent between abstaining and voting for B and so is willing to play any mixture between voting and abstaining. She abstains at least enough that she can't affect the outcome with her vote if $\frac{p_t}{1-p_t} < \frac{Pr(Piv_A|b,\sigma)}{Pr(Piv_A|a,\sigma)}$. If she abstains more than $\bar{s}(\sigma)$, she acts as if she's an SEU voter who assigns probability p_t to a if $\mathbb{E}[U|a,\sigma] > \mathbb{E}[U|b,\sigma]$ and q_t to a if $\mathbb{E}[U|b,\sigma] > \mathbb{E}[U|a,\sigma]$. Her best response correspondence is exactly as in Bouton and Castanheira [2009], establishing the result. □

Lemma 6. *Suppose that the worst case scenario is not independent of the strategy picked given σ and that the expected winner is correct in each state. If σ is an equilibrium and $b(B,\sigma) \geq q_t$, then $\sigma(t) \in BR_t(\sigma)$ where*

$$BR_t(\sigma) = \begin{cases} \{(0,0)\} & if\ \frac{q_t}{1-q_t} < \frac{Pr(Piv_B|b,\sigma)}{Pr(Piv_B|a,\sigma)} \\ \{0\} \times [0, \bar{s}(\sigma)] & if\ \frac{q_t}{1-q_t} = \frac{Pr(Piv_B|b,\sigma)}{Pr(Piv_B|a,\sigma)} \\ \widetilde{BR_{B,t}}(\sigma) & if\ \frac{q_t}{1-q_t} > \frac{Pr(Piv_B|b,\sigma)}{Pr(Piv_B|a,\sigma)} \end{cases}$$

and

$$\widetilde{BR_{B,t}}(\sigma) = \begin{cases} = \begin{cases} \{(0,1)\} & if\ \frac{p_t}{1-p_t} > \frac{Pr(Piv_B|b,\sigma)}{Pr(Piv_B|a,\sigma)} \\ \{0\} \times [\bar{s}(\sigma),1] & if\ \frac{p_t}{1-p_t} = \frac{Pr(Piv_B|b,\sigma)}{Pr(Piv_B|a,\sigma)} \\ \{(0,\bar{s}(\sigma))\} & if\ \frac{p_t}{1-p_t} < \frac{Pr(Piv_B|b,\sigma)}{Pr(Piv_B|a,\sigma)} \end{cases} & if\ \mathbb{E}[U|a,\sigma] > \mathbb{E}[U|b,\sigma] \\ \{(0,1)\} & otherwise \end{cases}$$

Proof. She votes for B conditional on voting because $b(b,\sigma)$ is high enough. She never abstains if $\frac{q_t}{1-q_t} < \frac{Pr(Piv_B|b,\sigma)}{Pr(Piv_B|a,\sigma)}$. If $\frac{q_t}{1-q_t} = \frac{Pr(Piv_B|b,\sigma)}{Pr(Piv_B|a,\sigma)}$, she's indifferent between abstaining and voting for B and so is willing to play any mixture between voting and abstaining. She abstains at least enough that she can't affect the outcome with her vote if $\frac{q_t}{1-q_t} > \frac{Pr(Piv_B|b,\sigma)}{Pr(Piv_B|a,\sigma)}$. If she abstains more than $\bar{s}(\sigma)$, she acts as if she's an SEU voter who assigns probability p_t to a if $\mathbb{E}[U|a,\sigma] > \mathbb{E}[U|b,\sigma]$ and q_t to a if $\mathbb{E}[U|b,\sigma] > \mathbb{E}[U|a,\sigma]$. Her best response correspondence is exactly as in Bouton and Castanheira [2009], establishing the result. □

Now, focus on the specific conditions at equilibrium. Because posteriors that respect likelihood ratios and voters lack of confidence, $p_2 \leq p_1 < \frac{1}{2} < q_2 \leq q_1$ (perhaps after relabeling). These values partition $[0,1]$ into regions where the best response correspondence of the voters has similar properties when $b(\cdot)$ is within that region. Proceed by analyzing these regions separately.

Suppose now that $b(b,\sigma_n) \in (p_2,p_1]$. By assumption, σ_n is so that $\sigma_n(1)(\emptyset) < 1$. Since $b(b,\sigma_n) < p_1$, Lemma 5 gives that $\sigma_n(1)(B) = 0$.

First, consider the case where $\sigma_n(1)(\emptyset) < \bar{s}(\sigma_n)$ so $\frac{p_1}{1-p_1} \geq \frac{Pr(Piv_A|b,\sigma)}{Pr(Piv_A|a,\sigma)}$. Since $\tau(B|b,\sigma_n) > \tau(A|b,\sigma_n)$, it must be that $\sigma_n(2)(A) = 0$ and $\sigma_n(1)(\emptyset) < 1$. By Lemma 4, $\mathbb{E}[U|b,\sigma_n] > \mathbb{E}[U|a,\sigma_n]$ and $\frac{q_2}{1-q_2} \leq \frac{Pr(Piv_B|b,\sigma)}{Pr(Piv_B|a,\sigma)}$. Because $\tau(B|b,\sigma_n) > \tau(A|b,\sigma_n)$ and $\tau(A|a,\sigma_n) > \tau(B|a,\sigma_n)$, it follows that $Pr(Piv_A|b,\sigma_n) > Pr(Piv_B|b,\sigma_n)$ and $Pr(Piv_A|a,\sigma) < Pr(Piv_B|a,\sigma)$ and

$$\frac{Pr(Piv_A|b,\sigma)}{Pr(Piv_A|a,\sigma)} > \frac{Pr(Piv_B|b,\sigma)}{Pr(Piv_B|a,\sigma)}.$$

However, $p_1 < q_2$ so these are mutually impossible.

Now, consider the case where $1 > \sigma_n(1)(\emptyset) \geq \bar{s}(\sigma_n)$. Since $b(b,\sigma_n) \leq p_1$, Lemma 5 gives that $\sigma_n(1)(B) = 0$ and $\sigma_n(1)(A) > 0$ implies that $\mathbb{E}[U|b,\sigma_n] > \mathbb{E}[U|a,\sigma_n]$. Since $\sigma_n(1)(B) = 0$, for the expected winner in state b to be correct it must hold that $\sigma_n(2)(B) > 0$. But because $\mathbb{E}[U|b,\sigma_n] > \mathbb{E}[U|a,\sigma_n]$ and $p_2 < b(b,\sigma_n) < q_2$, Lemma 4 gives that $\sigma_n(1)(B) = 0$, a contradiction.

Now suppose that $b(b,\sigma_n) \in [q_2,q_1)$. By assumption, $\sigma_n(2)(\emptyset) < 1$. Since $b(B,\sigma_n) > q_2$, Lemma 6 gives that $\sigma_n(2)(A) = 0$.

First, consider the case where $\sigma_n(2)(\emptyset) < \bar{s}(\sigma_n)$. From Lemma 6, $\frac{q_2}{1-q_2} \leq \frac{Pr(Piv_B|b,\sigma)}{Pr(Piv_B|a,\sigma)}$. By assumption, it must be that $\sigma_n(1)(A) > 0$ so $\sigma_n(1)(\emptyset) < 1$. Because $b(b,\sigma_n) \in (p_1,q_1)$, Lemma 4 requires that $\mathbb{E}[U|a,\sigma_n] > \mathbb{E}[U|b,\sigma_n]$ and $\frac{p_1}{1-p_1} \geq \frac{Pr(Piv_A|b,\sigma_n)}{Pr(Piv_A|a,\sigma_n)}$. Because $\tau(B|b,\sigma_n) > \tau(A|b,\sigma_n)$ and $\tau(A|a,\sigma_n) > \tau(B|a,\sigma_n)$, it follows that $Pr(Piv_A|b,\sigma_n) > Pr(Piv_B|b,\sigma_n)$ and

$$\frac{Pr(Piv_B|b,\sigma)}{Pr(Piv_B|a,\sigma)} < \frac{Pr(Piv_A|b,\sigma_n)}{Pr(Piv_A|a,\sigma_n)},$$

which is impossible since $p_1 < q_2$.

Now, consider the case where $1 > \sigma_n(2)(\emptyset) \geq \bar{s}(\sigma_n)$. By assumption and Lemma 6, $\sigma_n(2)(B) > 0$. From Lemma 5, $\mathbb{E}[U|a,\sigma_n] > \mathbb{E}[U|b,\sigma_n]$. But since $p_1 < b(b,\sigma_n) < q_1$, Lemma 4 yields that $\sigma_n(1)(A) = 0$, a contradiction.

If $b(b,\sigma_n) \in [0,p_1] \cup [p_2,q_1] \cup [q_2,1]$ it follows from Lemmas 4-6 that all voters will vote for the same candidate whenever they do not abstain, a contradiction. □

Assume WLOG that $r(1|a) + r(1|b) \geq 1$ and that $r(1|a) \geq r(1|b)$ (otherwise, relabel candidates and types).

Define $\hat{\tau} : \{A,B\} \times \Omega \to [0,1]$ by

$$\begin{aligned}
\hat{\tau}(A|a) &= \Big(\frac{\sqrt{r(2|a)} + \sqrt{r(2|b)}}{\sqrt{r(1|a)} + \sqrt{r(1|b)}}\Big)^2 r(1|a) \\
\hat{\tau}(B|a) &= r(2|a) \\
\hat{\tau}(A|b) &= \Big(\frac{\sqrt{r(2|a)} + \sqrt{r(2|b)}}{\sqrt{r(1|a)} + \sqrt{r(1|b)}}\Big)^2 r(1|b) \\
\hat{\tau}(B|b) &= r(2|b)
\end{aligned}$$

which would be the limiting vote shares for each candidate in each state if voters were expected utility.

Theorem 6. *Fix any sequence $(\Gamma_n)_{n=1}^{\infty}$ of AVGAs with voters who lack confidence and posteriors that respect the likelihood ratio. If the inequalities*

$$2 + \sqrt{\frac{\hat{\tau}(B|b)}{\hat{\tau}(A|b)}} + \sqrt{\frac{\hat{\tau}(B|a)}{\hat{\tau}(A|a)}} > 2\Big(\frac{\hat{\tau}(A|b)\hat{\tau}(B|b)}{\hat{\tau}(A|a)\hat{\tau}(B|a)}\Big)^{\frac{1}{4}} - 1\Big)\frac{\sqrt{\frac{\hat{\tau}(A|b)}{\hat{\tau}(B|b)}}}{1 - \sqrt{\frac{\hat{\tau}(A|b)}{\hat{\tau}(B|b)}}} \tag{12}$$

and

$$2 + \sqrt{\frac{\hat{\tau}(B|b)}{\hat{\tau}(A|b)}} + \sqrt{\frac{\hat{\tau}(B|a)}{\hat{\tau}(A|a)}} > \frac{\sqrt{\frac{\hat{\tau}(A|b)}{\hat{\tau}(B|b)}}}{1 - \sqrt{\frac{\hat{\tau}(A|b)}{\hat{\tau}(B|b)}}} - \frac{\frac{\hat{\tau}(B|a)}{\sqrt{\hat{\tau}(A|b)\hat{\tau}(B|b)}}}{1 - \frac{\hat{\tau}(B|a)}{\sqrt{\hat{\tau}(A|b)\hat{\tau}(B|b)}}} \tag{13}$$

both hold and σ_n is an equilibrium for Γ_n where the expected votes in each state goes to infinity and the expected winners are correct given σ_n, then for n sufficiently high, the worst case scenario for all voters is independent of their vote in σ_n.

Proof. The proof will be by contradiction.

Without loss of generality, suppose that $r(1|a)+r(1|b) \geq r(2|a)+r(2|b)$ and that $\frac{r(1|a)}{r(1|b)} > \frac{r(2|a)}{r(2|b)}$ (which implies that $r(1|a) > r(1|b)$), so that $p_2 \leq p_1 < \frac{1}{2} < q_2 \leq q_1$. Define $\sigma(1)(\emptyset) = \bar{a} = 1 - (\frac{\sqrt{1-r(1|a)}+\sqrt{1-r(1|b)}}{\sqrt{r(1|a)}+\sqrt{r(1|b)}})^2$, $\sigma(1)(A) = 1-\bar{a}$ and $\sigma(2)(B) = 1$.

Lemma 7. *Suppose that* $(\sigma^*_{n_k})$ *is a convergent sub-sequence of equilibrium strategy profiles to* Γ_{n_k} *so that the worst case scenario for every voter is independent of her strategy for every* $\sigma^*_{n_k}$*. Then* $\sigma_{n_k} \to \sigma$*. Moreover,* $\tau(A|a,\sigma) \leq \tau(B|b,\sigma)$ *and* $\frac{\tau(A|b,\sigma)}{\tau(B|b,\sigma)} \geq \frac{\tau(B|a,\sigma)}{\tau(A|a,\sigma)}$, with equality only if $r(1|a)+r(1|b) = r(2|a)+r(2|b)$.

Proof. This follows follows from Bouton and Castanheira [2009] Lemma 1 and Theorem 1, noting that when the strategy profile is played all voters act as if SEU with posterior p_t or q_t. At the limit, it must be that

$$\mu(a) = \mu(b) \iff (\sqrt{\tau(A|a)} - \sqrt{\tau(B|a)})^2 = (\sqrt{\tau(B|b)} - \sqrt{\tau(A|b)})^2.$$

Rewriting,

$$\sqrt{(1-\bar{a})r(1|a)} - \sqrt{r(2|a)} = \sqrt{r(2|b)} - \sqrt{(1-\bar{a})r(1|b)}$$

where $\bar{a} = \sigma(1)(\emptyset)$. Solving for $\bar{a}$ yields

$$1 - (\frac{\sqrt{1-r(1|a)}+\sqrt{1-r(1|b)}}{\sqrt{r(1|a)}+\sqrt{r(1|b)}})^2.$$

The remaining results follow from algebra. □

Note that $\hat{\tau}(c|\omega) = \tau(c|\omega,\sigma)$.

The worst case scenarios is independent of the strategy chosen given σ is played if and only if either

$$\mathbb{E}[U|b,\sigma] - \frac{1}{2}Pr(Piv_A|b,\sigma) \geq \mathbb{E}[U|a,\sigma] + \frac{1}{2}Pr(Piv_A|a,\sigma) \tag{14}$$

or

$$\mathbb{E}[U|a,\sigma] - \frac{1}{2}Pr(Piv_B|a,\sigma) \geq \mathbb{E}[U|b,\sigma] + \frac{1}{2}Pr(Piv_B|b,\sigma) \tag{15}$$

as in Lemma 1. If $r(1|a) = r(2|b)$, then it's clear that neither of the equalities are satisfied because $\mathbb{E}[U|b,\sigma,n] = \mathbb{E}[Ua,\sigma,n]$. Therefore, suppose $r(1|a) \neq r(2|b)$ an Consider the limiting equilibrium strategy profile. At this strategy profile, neither of these equations holds for n large enough.

Lemma 8. *If*

$$2+\sqrt{\frac{\tau(B|b,\sigma)}{\tau(A|b,\sigma)}}+\sqrt{\frac{\tau(B|a,\sigma)}{\tau(A|a,\sigma)}}>2(\frac{\tau(A|b,\sigma)\tau(B|b,\sigma)}{\tau(A|a,\sigma)\tau(B|a,\sigma)})^{\frac{1}{4}}-1)\frac{\sqrt{\frac{\tau(A|b,\sigma)}{\tau(B|b,\sigma)}}}{1-\sqrt{\frac{\tau(A|b,\sigma)}{\tau(B|b,\sigma)}}}$$

then $\mathbb{E}[U|b,\sigma]-\mathbb{E}[U|a,\sigma]<\frac{1}{2}(Pr(Piv_A|a,\sigma,n)+Pr(Piv_A|b,\sigma,n))$ *for* n *large enough.*

Proof. For notational purposes, drop the dependence on σ. Lemma 7 shows that

$$\tau(B|b)\tau(A|b)>\tau(A|a)\tau(B|a) \tag{16}$$

and

$$\frac{\tau(A|b)}{\tau(B|b)}>\frac{\tau(B|a)}{\tau(A|a)} \tag{17}$$

whenever the above conditions are satisfied. Set

$$\mu(\omega)=-\tau(A|\omega)-\tau(B|\omega)+2\sqrt{\tau(B|\omega)\tau(A|\omega)}$$

noting that $\mu(A)=\mu(B)=\mu\in[-1,0]$.

Since

$$\mathbb{E}[U|a,\sigma] \;=\; 1-e^{-(\tau(A|a)+\tau(B|a))n}(\sum_{k=1}^{\infty}(\frac{\tau(B|a)}{\tau(A|a)})^{\frac{k}{2}}I_k(2n\sqrt{\tau(A|a)\tau(B|a)})-\frac{1}{2}I_0(2n\sqrt{\tau(A|a)\tau(B|a)})$$

and

$$\mathbb{E}[U|b,\sigma]=1-e^{-(\tau(A|b)+\tau(B|b))n}(\sum_{k=1}^{\infty}\sqrt{\frac{\tau(A|b)}{\tau(B|b)}}^{k}I_k(2n\sqrt{\tau(A|b)\tau(B|b)})-\frac{1}{2}I_0(2n\sqrt{\tau(A|b)\tau(B|b)})$$

(where $I_k(\cdot)$ is a modified Bessel function of the first kind (see Myerson [2000], p. 27)), the conclusion is equivalent to

$$e^{-(\tau(A|a)+\tau(B|a))n}\sum_{k=1}^{\infty}\sqrt{\frac{\tau(B|a)}{\tau(A|a)}}^{k}I_k(2n\sqrt{\tau(A|a)\tau(B|a)})-e^{-(\tau(A|b)+\tau(B|b))n}\sum_{k=1}^{\infty}\sqrt{\frac{\tau(A|b)}{\tau(B|b)}}^{k}I_k(2n\sqrt{\tau(A|b)\tau(B|b)}) \tag{18}$$

is less than

$$\frac{1}{2}(Pr(Piv_A|b)+Pr(Piv_A|a)+I_0(2n\sqrt{\tau(A|b)\tau(B|b)})-I_0(2n\sqrt{\tau(A|a)\tau(B|a)})).$$

Let $\phi(n)$ be the value of (18).

By Baricz [2010] equation (2.6) we have that if $y>x>0$ and $k>0$ is an integer then

$$I_k(x)<e^{x-y}(\frac{y}{x})^{\frac{1}{2}}I_k(y). \tag{19}$$

Using equations (16) and (19), we have that

$$e^{-(\tau(A|a)+\tau(B|a))n}(\sum_{k=1}^{\infty}\sqrt{\frac{\tau(B|a)}{\tau(A|a)}}^{k} I_k(2n\sqrt{\tau(A|a)\tau(B|a)})$$

$$< e^{(\mu-2\sqrt{\tau(B|b)\tau(B|b)})n}(\frac{\tau(A|b)\tau(B|b)}{\tau(A|a)\tau(B|a)})^{\frac{1}{4}}\sum_{k=1}^{\infty}\sqrt{\frac{\tau(B|a)}{\tau(A|a)}}^{k} I_k(2n\sqrt{\tau(A|b)\tau(B|b)})$$

so we find that

$$\begin{aligned}\frac{\phi(n)}{e^{-2\sqrt{\tau(B|A)\tau(B|B)})n}} &< e^{\mu n}(\frac{\tau(A|b)\tau(B|b)}{\tau(A|a)\tau(B|a)})^{\frac{1}{4}}\sum_{k=1}^{\infty}\sqrt{\frac{\tau(B|a)}{\tau(A|a)}}^{k} I_k(2n\sqrt{\tau(A|b)\tau(B|b)}) - \\ &\quad -e^{\mu n}\sum_{k=1}^{\infty}\sqrt{\frac{\tau(A|b)}{\tau(B|b)}}^{k} I_k(2n\sqrt{\tau(A|b)\tau(B|b)}) \\ &< (\frac{\tau(A|b)\tau(B|b)}{\tau(A|a)\tau(B|a)})^{\frac{1}{4}}\sum_{k=1}^{\infty}\sqrt{\frac{\tau(B|b)}{\tau(A|b)}}^{k} I_k(2n\sqrt{\tau(A|b)\tau(B|b)}) - \\ &\quad -e^{\mu n}\sum_{k=1}^{\infty}\sqrt{\frac{\tau(A|b)}{\tau(B|b)}}^{k} I_k(2n\sqrt{\tau(A|b)\tau(B|b)})\end{aligned}$$

since $\frac{\tau(B|a)}{\tau(A|a)} < \frac{\tau(A|b)}{\tau(B|b)}$. Setting

$$\bar{K}(n) = \sum_{k=1}^{\infty}\sqrt{\frac{\tau(A|b)}{\tau(B|b)}}^{k} I_k(2n\sqrt{\tau(A|b)\tau(B|b)}) > 0$$

and

$$\theta = (\frac{\tau(A|b)\tau(B|b)}{\tau(A|a)\tau(B|a)})^{\frac{1}{4}} - 1 > 0$$

yields that

$$\mathbb{E}[U|b,\sigma] - \mathbb{E}[U|a,\sigma] < \theta e^{\mu n}\bar{K}(n)e^{-2n\sqrt{\tau(A|b)\tau(B|b)}}$$

Note that

$$
\begin{aligned}
\bar{K}(n) &= \sum_{k=1}^{\infty} \sqrt{\frac{\tau(A|b)}{\tau(B|b)}}^{k} I_k(2n\sqrt{\tau(A|b)\tau(B|b)}) \\
&< \sum_{k=1}^{\infty} \sqrt{\frac{\tau(A|b)}{\tau(B|b)}}^{k} I_0(2n\sqrt{\tau(A|b)\tau(B|b)}) \\
&\approx \sum_{k=1}^{\infty} \sqrt{\frac{\tau(A|b)}{\tau(B|b)}}^{k} \frac{e^{\sqrt{(2n\sqrt{\tau(A|b)\tau(B|b)})^2}}}{\sqrt{2\pi\sqrt{(2n\sqrt{\tau(A|b)\tau(B|b)})^2}}} \\
&= \frac{\sqrt{\frac{\tau(A|b)}{\tau(B|b)}}}{1-\sqrt{\frac{\tau(A|b)}{\tau(B|b)}}} \frac{e^{2n\sqrt{\tau(A|b)\tau(B|b)}}}{2\sqrt{\pi n\sqrt{\tau(A|b)\tau(B|b)}}}
\end{aligned}
$$

by Abramowitz and Stegun [1972] equations (9.7.1) and (9.7.7) and that when $k \geq 0$ it follows that $I_k(x) > I_{k+1}(x)$. Therefore, for n large enough

$$
\phi(n) < \frac{\sqrt{\frac{\tau(A|b)}{\tau(B|b)}}}{1-\sqrt{\frac{\tau(A|b)}{\tau(B|b)}}} \theta e^{\mu n} I_0(2n\sqrt{\tau(A|b)\tau(B|b)}) e^{-2n\sqrt{\tau(A|B)\tau(B|B)}}
$$

Since

$$
\begin{aligned}
&\frac{1}{2}[Pr(Piv_A|b) + Pr(Piv_A|a) + e^{-(\tau(A|b)+\tau(B|b))n} I_0(2n\sqrt{\tau(A|b)\tau(B|b)}) - \\
&\quad -e^{-(\tau(A|a)+\tau(B|a))n} I_0(2n\sqrt{\tau(A|a)\tau(B|a)})] \\
= &\frac{1}{2}[e^{-(\tau(A|b)+\tau(B|b))n}(2I_0(2n\sqrt{\tau(A|b)\tau(B|b)}) + I_1(2n\sqrt{\tau(A|b)\tau(B|b)})\sqrt{\frac{\tau(B|b)}{\tau(A|b)}}) + \\
&\quad +e^{-(\tau(A|a)+\tau(B|a))n} I_1(2n\sqrt{\tau(A|a)\tau(B|a)})\sqrt{\frac{\tau(B|a)}{\tau(A|a)}}] \\
\approx &\frac{e^{\mu n}}{4\sqrt{\pi n}}\left(\frac{\sqrt{\frac{\tau(B|a)}{\tau(A|a)}}}{(\tau(A|a)\tau(B|a))^{\frac{1}{4}}} + \frac{2+\sqrt{\frac{\tau(B|b)}{\tau(A|b)}}}{(\tau(A|b)\tau(B|b))^{\frac{1}{4}}}\right),
\end{aligned}
$$

it suffices to show that

$$
\left[\frac{\sqrt{\frac{\tau(B|a)}{\tau(A|a)}}}{(\tau(A|a)\tau(B|a))^{\frac{1}{4}}} + \frac{2+\sqrt{\frac{\tau(B|b)}{\tau(A|b)}}}{(\tau(A|b)\tau(B|b))^{\frac{1}{4}}}\right] > \frac{\sqrt{\frac{\tau(A|b)}{\tau(B|b)}}}{1-\sqrt{\frac{\tau(A|b)}{\tau(B|b)}}} \frac{2(\frac{\tau(A|b)\tau(B|b)}{\tau(A|a)\tau(B|a)})^{\frac{1}{4}}-1)}{(\tau(A|b)\tau(B|b))^{\frac{1}{4}}}.
$$

Note that

$$\frac{\sqrt{\frac{\tau(B|a)}{\tau(A|a)}}}{(\tau(A|a)\tau(B|a))^{\frac{1}{4}}} + \frac{2+\sqrt{\frac{\tau(B|b)}{\tau(A|b)}}}{(\tau(A|b)\tau(B|b))^{\frac{1}{4}}} > \frac{2+\sqrt{\frac{\tau(B|b)}{\tau(A|b)}}+\sqrt{\frac{\tau(B|a)}{\tau(A|a)}}}{(\tau(A|b)\tau(B|b))^{\frac{1}{4}}}$$

because $\tau(B|b)\tau(A|b) > \tau(B|a)\tau(A|a)$. Therefore, if

$$2+\sqrt{\frac{\tau(B|b)}{\tau(A|b)}}+\sqrt{\frac{\tau(B|a)}{\tau(A|a)}} > 2(\frac{\tau(A|b)\tau(B|b)}{\tau(A|a)\tau(B|a)})^{\frac{1}{4}}-1)\frac{\sqrt{\frac{\tau(A|b)}{\tau(B|b)}}}{1-\sqrt{\frac{\tau(A|b)}{\tau(B|b)}}}$$

then the claim holds. □

Lemma 9. *If*

$$2+\sqrt{\frac{\tau(B|b,\sigma)}{\tau(A|b,\sigma)}}+\sqrt{\frac{\tau(B|a,\sigma)}{\tau(A|a,\sigma)}} > \frac{\sqrt{\frac{\tau(A|b,\sigma)}{\tau(B|b,\sigma)}}}{1-\sqrt{\frac{\tau(A|b,\sigma)}{\tau(B|b,\sigma)}}} - \frac{\frac{\tau(B|a,\sigma)}{\sqrt{\tau(A|b,\sigma)\tau(B|b,\sigma)}}}{1-\frac{\tau(B|a,\sigma)}{\sqrt{\tau(A|b,\sigma)\tau(B|b,\sigma)}}}$$

then $\mathbb{E}[U|a,\sigma]-\mathbb{E}[U|b,\sigma] < \frac{1}{2}Pr(Piv_B|a,\sigma)+\frac{1}{2}Pr(Piv_B|b,\sigma)$ *for* n *large enough.*

Proof. For notational purposes, drop the dependence on σ. As in Lemma 8, we can write the claim as

$$e^{-(\tau(A|b)+\tau(B|b))n}\sum_{k=1}^{\infty}\sqrt{\frac{\tau(A|b)}{\tau(B|b)}}^{k} I_k(2n\sqrt{\tau(A|b)\tau(B|b)})- \tag{20}$$

$$-e^{-(\tau(A|a)+\tau(B|a))n}\sum_{k=1}^{\infty}\sqrt{\frac{\tau(B|a)}{\tau(A|a)}}^{k} I_k(2n\sqrt{\tau(A|a)\tau(B|a)})$$

is less than

$$\frac{1}{2}(Pr(Piv_A|b)+Pr(Piv_A|a)-I_0(2n\sqrt{\tau(A|b)\tau(B|b)})+I_0(2n\sqrt{\tau(A|a)\tau(B|a)})).$$

Write $\phi(n)$ to be the value of (20). Note that

$$e^{-(\tau(A|a)+\tau(B|a))n}\sum_{k=1}^{\infty}\sqrt{\frac{\tau(B|a)}{\tau(A|a)}}^{k} I_k(2n\sqrt{\tau(A|a)\tau(B|a)})$$

$$> e^{-(\tau(A|a)+\tau(B|a))n}\sum_{k=1}^{\infty}\sqrt{\frac{\tau(B|a)}{\tau(A|a)}}^{k}\sqrt{\frac{\tau(A|a)\tau(B|a)}{\tau(A|b)\tau(B|b)}}^{k} I_k(2n\sqrt{\tau(A|b)\tau(B|b)})e^{2n\sqrt{\tau(A|a)\tau(B|a)}-2n\sqrt{\tau(A|b)\tau(B|b)}}$$

$$= e^{\mu n-2n\sqrt{\tau(A|b)\tau(B|b)}}\sum_{k=1}^{\infty}\frac{\tau(B|a)}{\sqrt{\tau(A|b)\tau(B|b)}}^{k} I_k(2n\sqrt{\tau(A|b)\tau(B|b)})$$

since whenever $k > \frac{1}{2}$ and $y > x$ we have

$$I_k(x) > (\frac{x}{y})^k e^{x-y} I_k(y) \tag{21}$$

by equation (2.2) of Baricz [2010].

We have that

$$\begin{aligned}
\phi(n) &< e^{-(\tau(A|b)+\tau(B|b))n}\sum_{k=1}^{\infty}\sqrt{\frac{\tau(A|b)}{\tau(B|b)}}^{k} I_k(2n\sqrt{\tau(A|b)\tau(B|b)}) - \\
&\quad - e^{\mu n-2n\sqrt{\tau(A|b)\tau(B|b)}}\sum_{k=1}^{\infty}\frac{\tau(B|a)}{\sqrt{\tau(A|b)\tau(B|b)}}^{k} I_k(2n\sqrt{\tau(A|b)\tau(B|b)}) \\
&= \frac{e^{\mu n}\sum_{k=1}^{\infty}(\sqrt{\frac{\tau(A|b)}{\tau(B|b)}}^{k}-\frac{\tau(B|a)}{\sqrt{\tau(A|b)\tau(B|b)}}^{k})I_k(2n\sqrt{\tau(A|b)\tau(B|b)})}{e^{2n\sqrt{\tau(A|b)\tau(B|b)}}} \\
&< \frac{e^{\mu n}I_0(2n\sqrt{\tau(A|b)\tau(B|b)}\sum_{k=1}^{\infty}(\sqrt{\frac{\tau(A|b)}{\tau(B|b)}}^{k}-\frac{\tau(B|a)}{\sqrt{\tau(A|b)\tau(B|b)}}^{k})}{e^{2n\sqrt{\tau(A|b)\tau(B|b)}}} \\
&\approx \frac{e^{\mu n}\sum_{k=1}^{\infty}(\sqrt{\frac{\tau(A|b)}{\tau(B|b)}}^{k}-\frac{\tau(B|a)}{\sqrt{\tau(A|b)\tau(B|b)}}^{k})}{2\sqrt{\pi n\sqrt{\tau(A|b)\tau(B|b)}}} \\
&= \frac{e^{\mu n}(\frac{\sqrt{\frac{\tau(A|b)}{\tau(B|b)}}}{1-\sqrt{\frac{\tau(A|b)}{\tau(B|b)}}}-\frac{\frac{\tau(B|a)}{\sqrt{\tau(A|b)\tau(B|b)}}}{1-\frac{\tau(B|a)}{\sqrt{\tau(A|b)\tau(B|b)}}})}{2\sqrt{\pi n\sqrt{\tau(A|b)\tau(B|b)}}}
\end{aligned}$$

so it suffices to show that

$$2+\sqrt{\frac{\tau(B|b)}{\tau(A|b)}}+\sqrt{\frac{\tau(B|a)}{\tau(A|a)}} > \frac{\sqrt{\frac{\tau(A|b)}{\tau(B|b)}}}{1-\sqrt{\frac{\tau(A|b)}{\tau(B|b)}}}-\frac{\frac{\tau(B|a)}{\sqrt{\tau(A|b)\tau(B|b)}}}{1-\frac{\tau(B|a)}{\sqrt{\tau(A|b)\tau(B|b)}}}$$

which is the hypothesis. □

Now, consider the specific conditions at equilibrium. Suppose that σ is an equilibrium. If the election is not close, then it must be that either

$$\mathbb{E}[U|a,\sigma]-\mathbb{E}[U|b,\sigma] > \frac{1}{2}(Pr(Piv_B|b,\sigma)+Pr(Piv_B|a,\sigma))$$

or

$$\mathbb{E}[U|b,\sigma]-\mathbb{E}[U|a,\sigma] > \frac{1}{2}(Pr(Piv_A|b,\sigma)+Pr(Piv_A|a,\sigma)).$$

By Bouton and Castanheira [2009] Lemma 1, restrict attention to profiles indexed by $\theta \in [0,1]$ defined by $\sigma_\theta(1)(\emptyset)=\theta$, $\sigma_\theta(1)(A)=1-\theta$ and $\sigma_\theta(2)(B)=1$. Let $\bar{a}$ be defined as in Lemma 1.

For n high enough, if σ_θ is an equilibrium then $\theta \in (0,1)$. Therefore, it must be the case that either

$$p_1(Pr(Piv_A|a,\sigma_\theta)) = (1-p_1)Pr(Piv_A|b,\sigma_\theta), \tag{22}$$

$$\frac{p_2}{1-p_2} < \frac{Pr(Piv_A|b,\sigma_\theta) + Pr(Piv_B|b,\sigma_\theta)}{Pr(Piv_A|a,\sigma_\theta) + Pr(Piv_B|a,\sigma_\theta)} < \frac{p_1}{1-p_1}, \tag{23}$$

and (14) all hold or

$$q_1(Pr(Piv_A|a,\sigma_\theta)) = (1-q_1)Pr(Piv_A|b,\sigma_\theta), \tag{24}$$

$$\frac{q_2}{1-q_2} < \frac{Pr(Piv_A|b,\sigma_\theta) + Pr(Piv_B|b,\sigma_\theta)}{Pr(Piv_A|a,\sigma_\theta) + Pr(Piv_B|a,\sigma_\theta)} < \frac{q_1}{1-q_1}, \tag{25}$$

and (15) all hold.

By Lemmas 8 and 9 above neither (15) nor (14) holds at $\sigma_{\bar\theta}$. The following inequalities hold given the signal structure, as long as θ is so that $\tau(A|a,\sigma_\theta) > \frac{1}{2}$ and $\tau(B|b,\sigma_\theta) > \frac{1}{2}$.

- $\frac{\partial \frac{Pr(Piv_A|b,\sigma_\theta)+Pr(Piv_B|b,\sigma_\theta)}{Pr(Piv_A|a,\sigma_\theta)+Pr(Piv_B|a,\sigma_\theta)}}{\partial\theta} < 0$
- $\frac{\partial \frac{Pr(Piv_A|b,\sigma_\theta)}{Pr(Piv_A|a,\sigma_\theta)}}{\partial\theta} < 0$
- $\frac{\partial}{\partial\theta}(\mathbb{E}[U|b,\sigma_\theta] - \frac{1}{2}Pr(Piv_A|b,\sigma_\theta)) > 0$
- $\frac{\partial}{\partial\theta}(\mathbb{E}[U|a,\sigma_\theta] + \frac{1}{2}Pr(Piv_A|a,\sigma_\theta)) < 0$
- $\frac{\partial}{\partial\theta}(\mathbb{E}[U|b,\sigma_\theta] + \frac{1}{2}Pr(Piv_B|b,\sigma_\theta)) > 0$
- $\frac{\partial}{\partial\theta}(\mathbb{E}[U|a,\sigma_\theta] - \frac{1}{2}Pr(Piv_B|a,\sigma_\theta)) < 0$

Suppose that equations (22), (23) and (15) all hold for some σ_θ. It is the case that

$$\frac{Pr(Piv_A|b,\sigma_{\bar\theta})}{Pr(Piv_A|a,\sigma_{\bar\theta})} > 1$$

for n large enough (using standard formulas for pivot probabilities). Since (22) holds and $\frac{p_1}{1-p_2} < 1$, it must be that $\theta > \bar\theta$ because $\frac{\partial \frac{Pr(Piv_A|B,\sigma_{\bar\theta})}{Pr(Piv_A|A,\sigma_{\bar\theta})}}{\partial\theta} < 0$. However, this implies that

$$\mathbb{E}[U|a,\sigma_\theta] - \frac{1}{2}Pr(Piv_A|a,\sigma_\theta) < \mathbb{E}[U|a,\sigma_{\bar\theta}] - \frac{1}{2}Pr(Piv_A|a,\sigma_{\bar\theta}))$$

and

$$\mathbb{E}[U|b,\sigma_\theta] + \frac{1}{2}Pr(Piv_B|b,\sigma_\theta) > \mathbb{E}[U|b,\sigma_{\bar\theta}] + \frac{1}{2}Pr(Piv_B|b,\sigma_{\bar\theta}).$$

Note therefore that

$$\mathbb{E}[U|b,\sigma_\theta] + \frac{1}{2}Pr(Piv_B|b,\sigma_\theta) > \mathbb{E}[U|a,\sigma_\theta] - \frac{1}{2}Pr(Piv_A|a,\sigma_\theta))$$

which means that (15) cannot hold.

Now, suppose that equations (24), (25) and (14) all hold for some σ_a. It can be verified that

$$\frac{Pr(Piv_A|b,\sigma_{\bar{\theta}}) + Pr(Piv_B|b,\sigma_{\bar{\theta}})}{Pr(Piv_A|a,\sigma_{\bar{\theta}}) + Pr(Piv_B|a,\sigma_{\bar{\theta}})} \leq 1$$

for n large enough using Myerson [2000] Equation 5.5, with equality holding only if $r(1|a) = r(2|b)$. Since (25) holds and $\frac{q_2}{1-q_2} > 1$, since $\frac{\partial \frac{Pr(Piv_A|B,\sigma_\theta)+Pr(Piv_B|B,\sigma_\theta)}{Pr(Piv_A|A,\sigma_\theta)+Pr(Piv_B|A,\sigma_\theta)}}{\partial\theta} < 0$ it must be that $\theta < \bar{\theta}$ for n large enough. However, this implies that

$$\mathbb{E}[U|a,\sigma_\theta] + \frac{1}{2}Pr(Piv_A|a,\sigma_\theta)) > \mathbb{E}[U|a,\sigma_{\bar{\theta}}] + \frac{1}{2}Pr(Piv_A|a,\sigma_{\bar{\theta}}))$$

and

$$\mathbb{E}[U|b,\sigma_\theta] - \frac{1}{2}Pr(Piv_A|b,\sigma_\theta) < \mathbb{E}[U|b,\sigma_{\bar{\theta}}] - \frac{1}{2}Pr(Piv_A|b,\sigma_{\bar{\theta}}).$$

Note therefore that

$$\mathbb{E}[U|b,\sigma_\theta] - \frac{1}{2}Pr(Piv_A|b,\sigma_\theta) < \mathbb{E}[U|a,\sigma_\theta] + \frac{1}{2}Pr(Piv_A|a,\sigma_\theta))$$

which means that (14) cannot hold. Therefore, σ_θ is not an equilibrium, which is a contradiction and completes the proof. □

Proof of Proposition 1:

Proof. Given σ^*, $\mathbb{E}[U|a,\sigma^*] = \mathbb{E}[U|b,\sigma^*]$ and $Pr(Piv_c|\omega,\sigma^*) = Pr(Piv_{c'}|\omega',\sigma^*)$ for any $c, c' \in C$ and any $\omega, \omega' \in \Omega$. The logic in Proposition 1 shows that a fixed voter would prefer to randomize with equal probability between A and B rather than play any other strategy that mixes between voting for A and B. However, abstaining with probability s and voting for A and B with probability $\frac{1-s}{2}$ each induces the same distribution over outcomes by abstaining. Hence, she's indifferent between playing this strategy and any other strategy. □

Proof of Proposition 3:

Proof. Given σ^*, $\mathbb{E}[U|a,\sigma^*] = \mathbb{E}[U|b,\sigma^*]$, $Pr(Piv_A|a,\sigma^*) = Pr(Piv_B|b,\sigma^*)$ and $Pr(Piv_B|a,\sigma^*) = Pr(Piv_A|b,\sigma^*)$ because $\tau(A|a,\sigma^*) = \tau(B|b,\sigma^*)$ and $\tau(B|a,\sigma^*) = \tau(A|b,\sigma^*)$. Lemma 4 gives that 1_m and 2_m best respond by abstaining with probability 1, since $\pi > r$ implies that $p_{i_m} < \frac{1}{2} < q_{i_m}$ for $i \in \{1,2\}$. Further, Lemma 5 gives that 1_s best responds to σ^* by voting for A, and Lemma 6 gives that 2_s best responds by voting for B. Consequently, σ^* is an equilibrium. □