A BEHAVIORAL FOUNDATION FOR ENDOGENOUS SALIENCE*

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Abstract. We develop choice-theoretic foundations for the salient thinking model of Bordalo et al. [2013b] (BGS) with an endogenous reference point. We provide simple properties of a decision maker's choices that are necessary and sufficient for representation by the model even when the reference point is determined by the menu. Our results highlight the model's strong predictions and provide a new characterization of one of its key innovations, the salience function. To better understand the salient thinking model's approach, a generalization, *the regional preference model (RPM)*, is introduced that is more flexible while retaining its key ingredients. RPM provides a novel connection between three prominent models, as it also nests the loss aversion model of Tversky and Kahneman [1991] and the status quo bias model of Masatlioglu and Ok [2005].

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1. Introduction

Psychologists have long known that the context in which a decision takes place plays a significant role in decision making.[1] The context selects features of the alternatives to highlight and makes them more *salient* in the agent's subsequent evaluation. Salience affects choice because "when one's attention is differentially directed to one portion on the environment rather than to others, the information contained in that portion will receive disproportionate weighing in subsequent judgments," as described by Taylor and Thompson [1982]. This paper provides choice-theoretic foundations for studying salience's influence on economic decisions, and analyzes novel connections between some prominent models thereof.

In economics, the most prominent model of salience is the *salient thinking model* of Bordalo et al. [2013b] (henceforth, BGS). The model features an endogenous reference point, and the decision maker (DM) puts more weight on the attribute that stands out most relative to this reference. The BGS model is intuitive, tractable, and accounts for a number of empirical anomalies for the neoclassical model of choice. Despite its popularity, it can be difficult to understand *all* of the implications of BGS's model for behavior: its new components are unobservable and its functional form rather involved.

We provide the first complete characterization of the observable choice behavior equivalent to the salient thinking model, clarifying and identifying the nature of the assumptions used in the model. Our primitive is a choice correspondence describing the DM's choices. As in BGS, each alternative has a pair of observed attributes, such as price and quality or height and weight.

The first crucial step towards understanding the model is getting a handle on its novel *salience function* that determines which attribute stands out for a given reference point. While one can work out the implications of a particular salience function, this exercise is not fruitful since the particular function that applies to a given agent is

[1] See e.g. Tversky and Kahneman [1981], Slovic et al. [1982], Fischer et al. [1986], Rowe and Puto [1987], Frisch [1993], Levin et al. [1998].

unobservable. Moreover, it is not clear how the model changes when the underlying salience function changes.

We study the salience function based on a simple observation: while it influences which attribute is salient, the weight given to each attribute is independent of its magnitude. Therefore, its role is simply to divide the domain into distinct regions, each associated with a particular attribute being most salient. We study the salience function by focusing on these regions. We characterize the properties of these regions equivalent to their generation by a salience function. Surprisingly, any specification of the salience function leads to the same regions, and thus the same choice behavior.

The main remaining challenge is that the reference point is endogenously determined by the set of available options. Since the salience of each alternative depends on the reference point in addition to the salience function, varying the budget set affects the salience of, and so the DM's willingness to choose, a given alternative. Armed with our observation about the salience function, one can nevertheless infer the salience of every alternative in a menu, as the model hypothesizes that the reference point is the average of the choice set. For a given context, we can first infer the reference point and then apply our analysis of the salience function to determine the collection of alternatives that have each attribute as their most salient.

From there, six simple axioms capture the behavior content of salient thinking. The key property is *Salience-SARP*, which relaxes the classical Strong Axiom of Revealed Preference so that it applies only when the salient attribute does not change for the chosen alternatives. It requires that choice is rational when the same alternative with a different salient attribute is treated as a distinct (and possibly unavailable) option.

Our characterization allows us to test whether a DM conforms to the salient thinking model and to identify the components of the model from behavior. The characterization breaks down the BGS model into simple, easily understood behavioral regularities. Splitting them behaviors apart suggests several directions for generalizations.

For instance, we use our results to provide a natural extension of BGS to allow for non-linear aggregation of attributes.

Our second goal is to understand the key intuition behind the BGS approach to salience and how it differs from other approaches taken in the literature, such as Gabaix and Laibson [2006], Kőszegi and Szeidl [2013], Bhatia and Golman [2013], Gabaix [2014], Bushong et al. [2015]. To do so, we introduce the *regional preference model (RPM)* of salience that retains the key ingredients of the BGS approach but makes fewer functional form assumptions.[2] To make our results comparable with previous work, we assume that we observe a family of reference-dependent preference relations that describe the DM's choices for each reference point.[3] This model reveals a sharp distinction between BGS and that of other models sharing similar psychological intuitions. Moreover, it provides an unexpected connection between BGS and other behavioral models that rely on completely different psychological intuitions, as it nests several other prominent models including Tversky and Kahneman [1991] and Masatlioglu and Ok [2005].

In the BGS model, each reference point maps the alternatives into two regions according to their salience, and alternatives with the same salience are evaluated consistently. RPM allows for an arbitrary number of regions but maintains the hypothesis that goods are evaluated consistently within a region. Each region has its own utility function, and as in BGS, the context in which the decision takes place determines the reference point, which in turn then divides the alternatives into the regions. Differences in salience affect the DM's trade-offs across attributes and alter her choices. Again paralleling BGS, the DM evaluates each object by a weighted sum of their attributes where the weights depend on the region in which the object lies.

Despite its generality, RPM makes testable predictions and excludes certain types of modeling choices. Of the models cited above that attempt to capture salience, only

[2]We provide an axiomatization of RPM in Appendix B.

[3]A subset of these choices can be inferred from a choice correspondence with the BGS hypothesis that the reference point is the average using standard revealed preference argument.

BGS is an RPM. In other words, even the most general version of BGS excludes these models, so BGS offers a completely different method of modeling salience. Perhaps more surprisingly, RPM reveals a connection between BGS and several other seemingly unrelated models that have received a great deal of attention in economics. In particular, it includes the constant loss aversion model of Tversky and Kahneman [1991] (TK) and the linear status quo bias model of Masatlioglu and Ok [2005] (MO).[4] The loss aversion model is one of the most influential models of behavioral economics. MO is one of the first papers to the study status quo bias phenomenon by utilizing a choice theoretic approach. As such, RPM can account for many phenomena that the traditional rational model cannot, including asymmetric price elasticities, insensitivity to bad news, endowment effects, and buying-selling price gaps, decoy and compromise effects, and context dependent willingness to pay (Camerer [2004], Bordalo et al. [2012, 2013a,b], Crawford and Meng [2011], Ericson and Fuster [2011]).

Our results highlight trade-offs between the different modeling approaches. For instance, BGS maintains a stronger consistency condition across reference points than does TK, but TK, unlike BGS, satisfies Monotonicity across regions. The strong consistency property is normatively appealing – as long as neither alternative's salience changes, their relative ranking should not change – and ideally we would like a model that satisfies both. However, the trade-off between the two is a general property of RPM. Any RPM that exhibits salient thinking and satisfies the strong consistency condition violates Monotonicity. This finding does not favor one model over the other, but makes their differences clear. If the strong consistency property is desired in an application, then BGS could be a better fit than TK. If Monotonicity is a necessary property, then TK would be ahead of BGS.

Our analysis is closely related to the literature which studies how a reference point affects choices. The earliest strand of literature assumes that the reference point is exogenous (*e.g.* Tversky and Kahneman [1991], Munro and Sugden [2003], Sugden [2003],

[4]In this paper, for comparison purposes, we consider a straight-forward extension of their model in which the reference point might be unavailable.

Masatlioglu and Ok [2005], Sagi [2006], Salant and Rubinstein [2008], Apesteguia and Ballester [2009], Masatlioglu and Nakajima [2013], Masatlioglu and Ok [2014], Dean, Kıbrıs, and Masatlioglu [2017]). The second wave of this literature proposes models where the reference point is endogenously determined (*e.g.* Bodner and Prelec [1994], Kivetz, Netzer, and Srinivasan [2004], Orhun [2009], Bordalo, Gennaioli, and Shleifer [2012], Tserenjigmid [2015]). In these models, the reference point is a function of the contents of the choice set, and is identical for all feasible alternatives. Finally, Köszegi and Rabin [2006], Ok, Ortoleva, and Riella [2015], Freeman [2017] and Kıbrıs et al. [2018] study models where the endogenous reference point is determined by what the agent chooses but is otherwise independent of the choice set.[5] Among these papers studying endogenous reference points, similar to ours, Ok, Ortoleva, and Riella [2015], Freeman [2017], Kıbrıs, Masatlioglu, and Suleymanov [2018] and Tserenjigmid [2015] provide behavioral characterizations for their models.

Interpreting salience as arising from differential attention to attributes, RPM has a close relationship with the literature studying how limited attention affects decision making. Masatlioglu et al. [2012] and Manzini and Mariotti [2014] study a DM who has limited attention to the alternatives available. The DM maximizes a fixed preference relation over the consideration set, a subset of the alternatives actually available. In contrast, in RPM the DM the considers all available alternatives but maximizes a preference relation distorted by her attention. Caplin and Dean [2015], Oliveira et al. [2017] and Ellis [2018] study a DM who has limited attention to information. In contrast to RPM, attention is chosen rationally to maximize ex ante utility, rather determined by the framing of the decision, and choice varies across states of the world. The most related interpretation considers attributes as payoffs in a fixed state. In addition to choices varying across states, each alternative has the same weights on each attribute, similar to Kőszegi and Szeidl [2013].

[5]Maltz [2017] is the only model of which we are aware that combines an exogenous reference point with endogenous reference-point formation.

2. Overview of the BGS Model

BGS propose an intuitive and descriptive behavioral model based on salience; we refer to their model as the salient thinking model or the BGS model. We begin by providing a short summary. The DM chooses from a finite set of alternatives, and each alternative has distinct and easily observable attributes. We follow BGS in focusing on the two-attributes case. We let X denote the set of all possible alternatives, and assume that $X = \mathbb{R}^2_{++}$.[6]

In the BGS model, the salience of an attribute depends on the value of the product's attribute and the reference level of that attribute.[7] The average level of each attribute serves as the reference point, which subsequently determines the salient attribute for the product. More formally, let $S \subset X$ be a finite budget set and $A(S)_k$ be the average level of attribute k in S. That is, the alternative

$$A(S) = (A(S)_1, A(S)_2) = \left(\frac{\sum\limits_{x \in S} x_1}{|S|}, \frac{\sum\limits_{x \in S} x_2}{|S|} \right)$$

acts as the reference point for S.

The magnitude of salience is determined by a *salience function*, $\sigma := \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$. Given a reference (r_1, r_2), attribute 1 is salient for good x if $\sigma(x_1, r_1) > \sigma(x_2, r_2)$, and attribute 2 is salient for good x if $\sigma(x_1, r_1) < \sigma(x_2, r_2)$; we refer to such goods as 1-salient and 2-salient, respectively. That is, the salient attribute is the one that differs the most from the reference according to the salience function. Different attributes are salient for different goods. Observe that the salience of an object might change as the choice set changes, since the choice set determines the reference which determines the salience of each attribute.

[6] We can generalize representation results beyond two dimensions but the multi-dimension analog of BGS is unclear.

[7] In the original paper, BGS illustrate their model in an environment where one attribute is desirable (quality) and the other is undesirable (price). We provide an graphical illustration for such cases in the Appendix.

The salience function σ must satisfy the following properties. First, it increases in contrast, i.e. for $\epsilon > 0$ and $a > b$, $\sigma(a+\epsilon, b) > \sigma(a, b)$ and $\sigma(a, b-\epsilon) > \sigma(a, b)$. Second, it is Homogeneous of Degree Zero, i.e. for all $\alpha > 0$, $\sigma(\alpha a, \alpha b) = \sigma(a, b)$. Third, it is continuous. Finally, it is symmetric: $\sigma(a, b) = \sigma(b, a)$.

For each product, the DM puts higher importance on the salient attribute. For example, if attribute 2 is salient for product x, then attribute 2 attracts more attention than attribute 1 and receives greater decision weight for the valuation of x. In particular, they are represented by the function

$$V_{BGS}(x|r) = \begin{cases} wx_1 + (1-w)x_2 & \text{if } \sigma(x_1, r_1) > \sigma(x_2, r_2) \\ (1-w)x_1 + wx_2 & \text{if } \sigma(x_2, r_2) > \sigma(x_1, r_1) \end{cases} \tag{1}$$

where $w \in (0.5, 1)$ increases in the severity of salient thinking.[8]

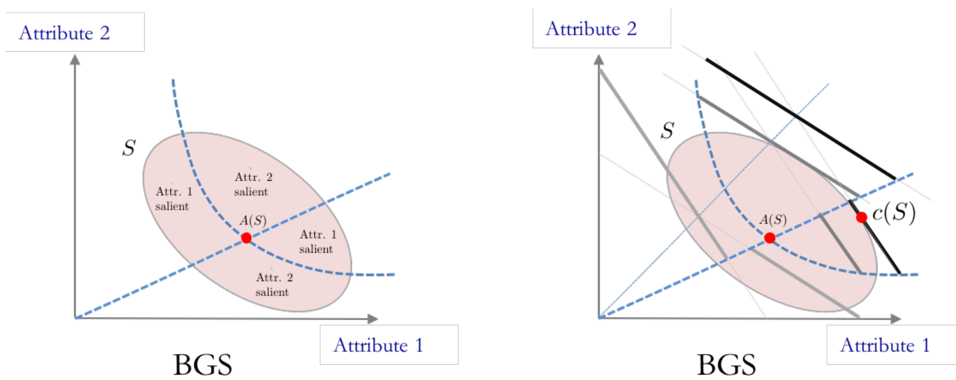

Figure 1. Salient Thinking Model

To illustrate this model, consider the salience function proposed by BGS:

$$\sigma(x_k, r_k) = \frac{|x_k - r_k|}{x_k + r_k}.$$

Based on it, the left panel in Figure 1 shows the salience of each product. The entire product space is divided into four distinct areas by two curves which intersect at the reference point. The areas lying the north and south of the reference point cover the 2-salient products. Similarly, 1-salient products lie east and west of the reference point.

[8]BGS parametrize by $\delta \in (0, 1]$, so their utility function is $2V_{BGS}(x|r)$ for $w = \frac{1}{1+\delta}$.

The right panel of Figure 1 incorporates indifference curves as well, holding fixed the reference point. There are two potential sets of indifference curves, illustrated by dotted lines. Depending on the region, one of the two is utilized to determine agent's choice. When attribute 1 is salient, the steeper one becomes the indifference curve since it puts higher weight on the first attribute. Conversely, the flatter one is the indifference curves when attribute 2 is salient. We draw three different indifference curves, where the darker color corresponds to higher utility.

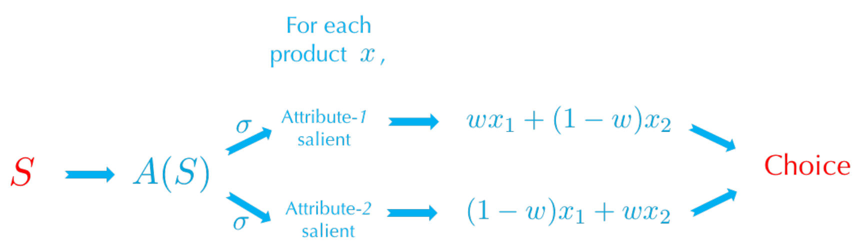

Figure 2. The choice procedure of the BGS model

We summarize the choice procedure of the BGS model. The choice context, through the reference good, determines the salience of each product. In the main analysis, the choice context and the reference good coincide with the choice set and the average levels in the choice set, respectively. Given a choice problem S, the average attribute levels $A(S)$ becomes the reference point. Next, the reference point divides the product space into two distinct regions according to which attribute is more salient; in menu S, if $\sigma(x_1, A(S)_1) > \sigma(x_2, A(S)_2)$, then the product is 1-salient. Finally, the DM evaluates each product using the utility function corresponding to the region to which it belongs. For example, $V_{BGS}(x|A(S)) = wx_1 + (1-w)x_2$ since x is 1-salient.

3. A Characterization of the BGS Regions

One of the key innovations of the BGS model is the salience function. In Figure 1, we illustrated its indifference curves for a fixed reference point and a particular specification of the salience function. However, this exercise does not reveal how the picture changes when one alters the underlying salience function. We begin our analysis by exploring this aspect of the model.

Observe that while the salience function determines which attribute is salient, it does not affect how much weight that attribute gets. Therefore, the entire effect of the salience function is captured by the set of alternatives that have each attribute salient. Thus we study the salience function by studying these sets for each reference point.

For this section, we directly assume that the reference good is $r = (r_1, r_2)$. Since there are two attributes, there are two types of products: 1-salient and 2-salient. We let $R_k(r)$ be the set of products that are k-salient for the reference point r, and treat each as a function mapping reference points to subsets of X. We sometimes refer to $R_k(r)$ as the *k-salient region.*

Theorem 1 shows that the following properties of these functions characterize the regions generated by a salience function. Surprisingly, it also shows that any two salience functions generate the same regions and thus the same choices from menu. Throughout, we adopt the usual notation that $x_{-1} = x_2$ and $x_{-2} = x_1$.

S0: (Basic) For any $r \in X$: $R_1(r) \bigcap R_2(r) = \emptyset$ and $R_1(r) \bigcup R_2(r)$ is dense in X.
S1: (Moderation) For any $\lambda \in [0, 1]$and $r \in X$:
if $x \in R_k(r)$, $y_k = x_k$, and $y_{-k} = \lambda x_{-k} + (1 - \lambda) r_{-k}$, then $y \in R_k(r)$.
S2: (Equal Salience) For any $x, r \in X$: if $\frac{x_1}{r_1} = \frac{x_2}{r_2}$ or $\frac{x_1}{r_1} = \frac{r_2}{x_2}$, then $x \notin R_k(r)$ for $k = 1, 2$.
S3: (Regular regions) For all $r \in X$ and $k = 1, 2$: $R_k(r)$ is a regular open set.[9]

The properties have natural interpretations. **S0** says no bundle is both 1-salient and 2-salient, and almost every bundle is either one or the other. **S1** indicates that making a bundle's less salient attribute closer to the reference point does not change the salience of the bundle. That is, when x and y differ only in attribute l, and y is closer to the reference in that attribute, if x is k-salient, then so is y. **S2** reads that if

[9]Recall that a set A is regular open if $A = int(cl(A))$.

every attribute of x differs from the reference point by the same percentage, then none of the attributes stands out. More formally, if the percentage difference between x_k and r_k is the same across attributes, then x is not k-salient for any $k \in \{0, 1\}$. Finally, **S3** states that regions are regular open sets. That is, there is no bundle completely surrounded by k-salient bundles that is not an k-salient bundle itself.

Theorem 1. *The following are equivalent:*

(i) The functions R_1 *and* R_2 *satisfy S0-S3,*

(ii) There exists a salience function σ *s.t.* $x \in R_k(r) \iff \sigma(x_k, r_k) > \sigma(x_{-k}, r_{-k})$*,*

(iii) For any salience function σ*,* $x \in R_k(r) \iff \sigma(x_k, r_k) > \sigma(x_{-k}, r_{-k})$*.*

This theorem provides a characterization for BGS's salience function. In other words, Theorem 1 translates the functional form assumptions on the salience function in terms properties on the salience regions. The result implies that *any* specification of the salience function leads to the same regions. Combining with our earlier observation, it implies that Figure 1 is independent of the salience function.

4. A Foundation for Salient Thinking

This section provides a behavioral foundation for the salient thinking model. That is, which properties of a DM's choices allow her behavior to represented by the functional form of BGS? Theorem 1 is a key building block, since it characterizes the (unique) set of k-salient alternatives given a reference good r. Specifically, it shows that we can write the k-salient region as

$$R_k^{BGS}(r) = \left\{ x \in X : \max\left\{\frac{x_k}{r_k}, \frac{r_k}{x_k}\right\} > \max\left\{\frac{x_{-k}}{r_{-k}}, \frac{r_{-k}}{x_{-k}}\right\}\right\}$$

for $k = 1, 2$. Throughout this section, we drop the superscript and denote $R_k^{BGS}(r)$ by simply $R_k(r)$. BGS also hypothesize that the reference for the (finite) set S is the average of each attribute, $A(S)$. Thus, given a menu of alternatives S to choose from, we can infer that the reference point equals $A(S)$, and from that, the salience of each alternative, i.e. the regions $R_k(A(S))$.

Let $\mathcal{X}$ be the set of finite and non-empty subsets of X such that $S \in \mathcal{X}$ only if $S \subset R_1(A(S)) \bigcup R_2(A(S))$. We call them menus for short. The requirement ensures that each alternative in the choice set is either 1-salient or 2-salient given the reference point $A(S)$. We leave open how the DM chooses when alternatives that have no salient attribute belong to the choice set. By leaving the choice from this small set of menus ambiguous, we can more clearly state the properties of choice implied by the model.[10]

We summarize the DM's choices by a choice correspondence $c : \mathcal{X} \rightrightarrows X$ with $c(S) \subseteq S$ and $c(S) \neq \emptyset$ for each $S \in \mathcal{X}$. The choice correspondence c has a *salient thinking representation* if there exists $w \in (1/2, 1)$ and salience function σ so that

$$c(S) = \arg\max_{x \in S} V_{BGS}(x|A(S))$$

for all $S \in \mathcal{X}$. We now state several properties of c that are implied when c has such a representation. Theorem 2, the main result of this section, shows that they are also sufficient for the representation. To state them simply, for $x, y \in X$ and $\alpha \in [0, 1]$, let $\alpha x + (1 - \alpha)y$ denotes the coordinate-by-coordinate mixture of x and y, i.e. $[\alpha x + (1 - \alpha)y]_k = \alpha x_k + (1 - \alpha)y_k$ for all $k \in \{1, 2\}$; similarly, for $S, T \subset X$, $\alpha S + (1 - \alpha)T$ denotes the set of all alternatives $\alpha x + (1 - \alpha)y$ where $x \in S$ and $y \in T$.

The first postulate resembles the classical Strong Axiom of Revealed Preference (SARP), which states that the revealed preference has no cycles. The axiom roughly requires that the DM does not violate SARP unless the salience of one of the chosen alternatives changes in different menus in the cycle.

Axiom 1 (Salience-SARP)**.** For any finite sequences of pairs $(x^i, S^i)_{i=1}^n$ such that $x^i \in c(S^i)$, $x^{i+1} \in S^i$, and $x^{i+1} \in R_k(A(S^i)) \cap R_k(A(S^{i+1}))$ for some $k \in \{1, 2\}$ for

[10] One can, of course, extend the model to account for these choices using the BGS hypothesis that these alternatives are evaluated according to their sum. Complications arise because $\left[R_1^{BGS}(r) \bigcup R_2^{BGS}(r)\right]^c$ is sparse.

every $i = 1, \ldots, n-1$:
if $x^n \in c(S^n)$, $x^1 \in S^n$, and $x^1 \in R_k(A(S^1)) \bigcap R_k(A(S^n))$ for some k, then $x^1 \in c(S^n)$.

We first illustrate in a simple two menu setting, analogous to a test case for the Weak Axiom of Revealed Preference (WARP). Consider two menus S^1 and S^2 and two chosen products $x^1 \in c(S^1)$ and $x^2 \in c(S^2)$ where both products have the same salient attribute in both menus. For example, x^1 is 1-salient in both menus, and x^2 is 2-salient in both menus. The observation $x^1 \in c(S^1)$ reveals that the valuation of x^1 is at least as high as that of x^2 when x^1 is 1-salient and x^2 is 2-salient. Since the salience of products does not change when the menu changes from S^1 to S^2, their relative valuation stays the same as well. Hence, if x^2 is chosen from S^2, then x^1 must be chosen too. Since neither products' salience has changed, the DM should obey WARP for these two menus. However, the axiom leaves open the possibility of a WARP violation when the salience of either changes.

The axiom extends this logic to sequences of choices in much the same way that SARP does to WARP. A finite sequence of choices, where the choice from the next menu is available in the current one and has the same salience in both, does not lead to a choice reversal. Since salience does not change along the sequence of choices, the choices do not exhibit a reversal.

Salience-SARP limits the effect of unchosen alternatives. Modifying them can alter the DM's choice, but only insofar as changing them changes the reference point and thus the salience of alternatives. It states that these unchosen options do not alter the relative ranking of two alternatives, unless they change the region to which the alternatives belong. That is, when comparing the same two alternatives in different menus, the DM's relative ranking does not change when neither's salience changes. This property greatly limits the effect of the reference point. In fact, a sufficiently small change in the reference never leads to a preference reversal.

The next postulate states that the indifference curves are straight and parallel lines for a given region (see Figure 1).

Axiom 2 (Regional Linearity)**.** For $\alpha \in (0,1]$, take S and y such that $S \subset R_k(A(S))$ and $\alpha S + (1-\alpha)\{y\} \subset R_k(A(\alpha S + (1-\alpha)\{y\}))$ for some k. Then, $x \in c(S)$ if and only if $\alpha x + (1-\alpha)y \in c(\alpha S + (1-\alpha)\{y\})$.

This axiom states that the choice from a linear combination of a set with a fixed alternative is the linear combination of the choices from that set with the alternative, provided that both the sets lie in the same region. Thus provided all alternatives have the same salient attribute in both menus, choice obeys the usual linearity axiom. The justification is similar to that axiom, or of independence, its analog in the risk case. The key implication is that linearity is violated only if salience changes. It implies separability across dimensions, and also that these dimensional utilities do not have diminishing sensitivity.

The next axiom says that both attributes are desirable.

Axiom 3 (Regional Monotonicity)**.** For any $x, y \in S$ with $x \neq y$, if $x \geq y$ and $x, y \in R_k(A(S))$ for some k, then $y \notin c(S)$.

Since both attributes are "goods" as opposed to "bads", Monotonicity means that if a product x contains more of some or all attributes, but no less of any, than another product y, then x is preferred to y. The postulate requires that choice respects Monotonicity for alternatives within the same salient region. However, it does not require that this comparison holds when the goods belong to different regions, and we shall see later that salience can distort comparisons enough to cause Monotonicity violations.

To interpret regions as reflecting salience, the indifference curves in Region 1 (attribute 1 is salient) should be steeper than in Region 2 (attribute 2 is salient). The next axiom guarantees that.

Axiom 4 (Salient Dimension Overvalued (SDO))**.** For $x, y \in S \bigcap S'$ with $x_k > y_k$ and $y_{-k} > x_{-k}$, if $x, y \in R_k(A(S))$, $x, y \in R_{-k}(A(S'))$, and $y \in c(S)$, then $x \notin c(S')$.

This axiom requires that regions correspond to the dimension that gets the most weight. That is, the DM is more willing to choose an alternative whose "best" attribute is k when it is k-salient. To illustrate, consider alternatives x, y with $x_1 > y_1$ and $y_2 > x_2$. Because x is relatively strong in attribute 1, x should benefit more than y from a focus on it. If x is chosen over y when attribute 2 stands out for both, then this advantage in the first dimension is so strong that even a focus on the other one does not offset it. Hence, the DM should surely choose x over y for sure when attribute 1 stands out for it.

Because both salience and preference treat attributes symmetrically, permuting the attributes of all objects in the same way does not change rankings. We say that the menu S is the *reflection* of the menu of T if $(a, b) \in S$ if and only if $(b, a) \in T$.

Axiom 5 (Reflection)**.** For any $S \in \mathcal{X}$, if $(a, b) \in c(S)$ and T is the reflection of S, then $(b, a) \in c(T)$.

Thus, preference "reflects" about the 45 degree line. This is a strong property. It can be relaxed to obtain a generalization of the BGS model. The most important feature that we use it for in the proof is to ensure that there exists a set where the DM is indifferent between choosing an item from region 1 and one from region 2. To see why, let $x = (a, b)$ and $y = (b, a)$. Then the reflection of $S = \{x, y\}$ is S itself, and Reflection implies that $\{x, y\} = c(\{x, y\})$.

Finally, we need a continuity condition.

Axiom 6 (Regional Continuity)**.** Let $y^n \to y$, $x^n \to x$, and $x, y \notin S$. Then
i) if $x_n \in c(S \cup \{x_n\})$ for all n, then $x \in c(S \cup \{x\})$, and
ii) if $z \in c(S \cup \{y_n\})$ for all n, then $z \in c(S \cup \{y\})$

Observe that a small enough change in y does not change $A(S \cup \{y\})$ much, and so does not change salience. So similarly, WARP and continuity should hold for "small" changes in y, i.e. for $S' \in \{S \cup \{y'\} : y' \in B_\epsilon(y)\}$. Note that $A(S \cup \{x_n\}) \to A(S \cup \{x\})$ whenever $x^n \to x$ and $x \notin S$.

We now state our main result.

Theorem 2. *The choice correspondence $c(\cdot)$ satisfies Axioms 1-6 if and only if it has a salient thinking representation.*

This theorem lays out the behavioral postulates that characterize the BGS model with endogenous reference point formation. Most importantly, it connects the (unobserved) components of the model to observed choice behavior. The full proof of Theorem 2 can be found in Appendix A.2. We provide an outline here. Since the reference point is endogenous, the main challenge is to elicit the agent's preferences within each region. We first show that this can be overcome by carefully constructing appropriate menus that reveal how the agent trades off attributes when choosing between alternatives with a given salient attribute. Then we establish that preferences in each region admits a linear utility representation by utilizing the mixture-space theorem. In the next step, we relate preference across the regions. Finally, we show that these properties together give exactly the BGS functional form.

4.1. **Extension: Non-linear BGS.** In the above, we provide a characterization for the original salient thinking model in which indifference curves are linear within each region. By utilizing our behavioral characterization, we now present a version of the model where indifference curves are not necessarily linear. It is clear that to do so we must relax the Regional Linearity axiom. We replace it with Regional Double Cancellation axiom, an adaptation of a standard property that yields a "separable" utility function.

Axiom 7 (Regional Double Cancellation (RDC))**.** For all $x_1, y_1, z_1, x_2, y_2, z_2 \in \mathbb{R}_{++}$ and $k \in \{1, 2\}$; if $(x_1, z_2) \in c(S^1)$, $(z_1, y_2) \in S^1$, $(z_1, x_2) \in c(S^2)$, $(y_1, z_2) \in S^2$, $(x_1, x_2), (y_1, y_2) \in S^3$ and $S^i \subset R_k(A(S^i))$ for $i \in \{1, 2, 3\}$, then $(x_1, x_2) \in c(S^3)$ whenever $(y_1, y_2) \in c(S^3)$.

RDC adapts the well-known double cancellation axioms to our setting, differing in its requirement that the menus are all subsets of the same region. This necessary condition for an additive representation appears in Krantz et al. [1971] and Tversky and Kahneman [1991], among others. Ours applies only when all compared alternatives belong to a given region. Replacing Regional Linearity with RDC guarantees an additive representation within the region.

Proposition 1. *The choice correspondence $c(\cdot)$ satisfies RDC and Axioms 1, 3, 4, 5 and 6 if and only if there exist two strictly increasing continuous utility indexes $u, v : \mathbb{R}_{++} \to \mathbb{R}$ such that*

$$c(S) = \arg\max_{x \in S} \begin{cases} u(x_1) + v(x_2) & \text{if } x \in R_1(A(S)) \\ v(x_1) + u(x_2) & \text{if } x \in R_2(A(S)) \end{cases}$$

where $u(a) - u(b) > v(a) - v(b)$ for any $a > b > 0$.

The above characterization is very similar to the one provided in Theorem 1 except that the indifference curves are not linear. The condition on u and v echoes the logic in the original salient thinking model, as it implies she is willing to give up more units of attribute 2 in exchange for an additional unit of attribute 1 when the good is 1−salient. To see why, observe that

$$\frac{u(a+\frac{1}{n}) - u(a)}{v(b+\frac{1}{n}) - v(b)} \geq \frac{v(a+\frac{1}{n}) - v(a)}{u(b+\frac{1}{n}) - u(b)}.$$

Provided that u and v are differentiable, these tend to $u'(a)/v'(b)$ and $v'(a)/u'(b)$ as $n \to \infty$. Thus, the marginal rate of substitution of attribute 2 for attribute 1 is larger if the good is 1-salient.

4.2. **Extension: Incomplete Data.** The above results have to two limitations for their applicability. First, they require a very large data set, namely that the analyst observes the DM's choice from almost every possible menu. Second, the analyst must observe not only the DM's choice from every possible menu but also the full set of options she is willing to choose. In many cases, such rich data may not be available.

However, the ideas behind the approach extend to cases with more limited data. Indeed, in Appendix C we show that if one is only concerned with testing the model, then one can do so with a sparser data set. The model is testable using appropriate, but less elegant, modifications of the above axioms. Specifically, provided that one has access to a data set consisting of a finite number of menus and one of the alternatives chosen from each, we give necessary and sufficient conditions for rationalizing the DM's choices via the BGS model.

5. Regional Preferences

We now turn to the question of how the choices implied by the BGS model relate to others in the literature. The *regional preference model (RPM)* generalizes the BGS model to make fewer functional form assumptions. However, RPM retains many of the other key features of BGS: the reference divides the space of alternatives into regions and indifference curves are linear within each region. RPM nests BGS as well as the models of Tversky and Kahneman [1991] (TK) and Masatlioglu and Ok [2005] (MO), revealing their similar underlying structure. We analyze the behavior that specializes RPM into BGS as well as into TK, and consider examples that distinguish the three models.

To aid in comparison with the existing literature and to separate the effects of reference point formation, we follow Tversky and Kahneman [1991] by taking as given a family of reference-dependent preference relations. We continue to assume that the space of alternatives is $X = \mathbb{R}^2_{++}$, and so for each reference point $r \in X$, the DM maximizes a complete and transitive preference relation, denoted by $\succsim_r$, over X. As usual, $\succ_r$ denotes strict preference and $\sim_r$ indifference. The primitive of the model is thus a family of such preferences indexed by the set of reference points, $\{\succsim_r\}_{r\in X}$.

The first ingredient of RPM is a mapping from the reference point r to n different regions. These regions divide the product space according their salience. Each region corresponds to a different saliency and changes as the reference changes. We require that the regions are open, non-empty sets, contain almost every alternative, meet at the reference point, do not overlap, and vary continuously with the reference point. Otherwise, they can have a very general structure. Formally, we impose the following properties.

Definition 1. A vector-valued function $\mathcal{R} = (R_1, R_2, \ldots, R_n)$ is a *regional function* if each $R_k : X \to 2^X$ satisfies the following properties:

(1) $R_k(r)$ is a non-empty, open set, and $cl(R_k(r))$ is connected,
(2) $\bigcup_{k=1}^n R_k(r)$ is dense,
(3) $r \in \bigcap_{k=1}^n bd(R_k(r))$,
(4) $R_k(r) \bigcap R_l(r) = \emptyset$ for all $k \neq l$, and
(5) $R_k(\cdot)$ is continuous.[11]

The second ingredient is a region-dependent utility function. This allows the consumer to value each good in a way that depends not only the product's identity, as in the standard neoclassical model, but also on the region to which the product belongs. Suppose the alternative x lies in the region k when the reference is r, that is, $x \in R_k(r)$. Then, the value of x is represented by $u_k(x|r)$. To stay close to BGS, we require that each alternative's utility is a weighted sum of its attributes, but allow these weights to vary with the region; formally, $u_k(\cdot|r)$ is an affine function. We also require that the reference point does not affect the utility trade-off within a region. In other words, changing the reference does not alter the relative ranking of two alternatives as long as both alternatives lie in the same region. To capture this feature, we assume that $u_k(\cdot|r)$ is a positive and affine transformation of $u_k(\cdot|r')$ for any references r and r'. We formally define the regional preference model for a regional function $\mathcal{R}$ as follows.

Definition 2. The family $\{\succsim_r\}_{r\in X}$ conforms to the *regional preference model (RPM)* under a regional function $\mathcal{R}$ if there exist n families $\{u_k(\cdot|r)\}_{r\in X}$ of strictly increasing, affine functions, where $u_k(\cdot|r)$ is a positive affine transformation of $u_k(\cdot|r')$ for every

[11]That is, each R_k is both upper and lower semicontinuous when viewed as a correspondence.

$r, r' \in X$, such that for all $x \in R_k(r)$ and $y \in R_l(r)$,

$$x \succsim_r y \text{ if and only if } u_k(x|r) \geq u_l(y|r).$$

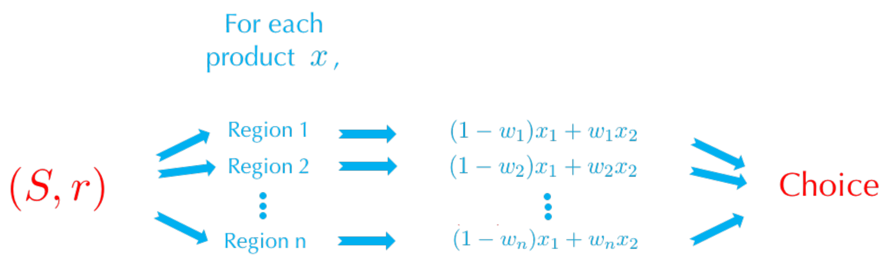

FIGURE 3. The choice procedure of the RPM model

In the appendix, we provide behavioral foundations for the RPM model. Our characterization relies on three main properties, of which a formal statement and discussion is deferred. First, if two alternatives belong to the same region for two distinct reference points, then the DM ranks them the same regardless of what the reference point is. That is, the reference may shift preferences between alternatives that belong to different regions, but within each region the preference is independent of the reference. Second, Monotonicity holds when comparing alternatives in the same region. This leaves open the possibility that the usual Monotonicity property fails across regions. Third, the DM does not violate linearity provided that the salience of alternatives does not change. This reflects that the DM evaluates objects as a weighted sum of their attributes.

As noted, one can easily verify that BGS, TK, and MO are all special cases of RPM. We next discuss the behavior that specializes RPM into BGS and TK, and the behavior that distinguishes the three. Surprisingly, a number of other papers that study salience fail to be RPM, including Gabaix and Laibson [2006], Kőszegi and Szeidl [2013], Bhatia and Golman [2013], Gabaix [2014], Bushong et al. [2015]. We demonstrate this in Appendix B.1. Roughly, if they were RPM, then they would have only a single region. But single region RPM coincides with the neoclassical model of choice.

5.1. **Special cases of RPM.** Our goal is to relate various models studied in the literature through the lens of RPM. First, we provide an alternative characterization for BGS in this new domain. This result first establishes the fact that BGS is a special case of RPM. Second it identifies the additional structures imposed by the functional form of the salient thinking model. We then introduce models of Tversky and Kahneman [1991] and Masatlioglu and Ok [2005], and discuss an unexpected connection between BGS and these models that rely on completely different psychological intuitions. Their indifference curves and regions are illustrated in Figure 4.

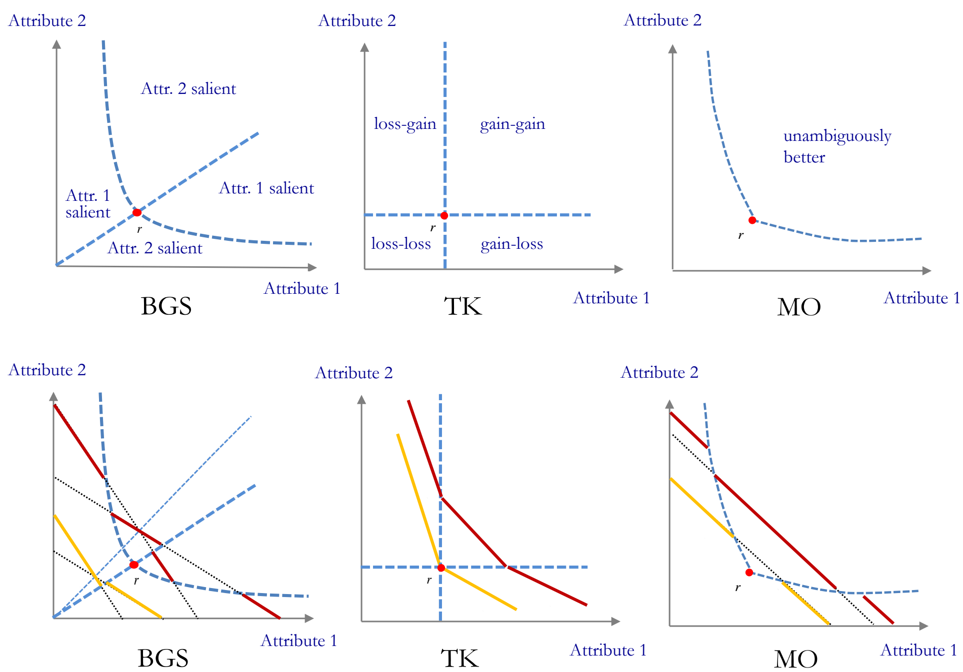

FIGURE 4. Three RPM Models

BGS. BGS is the special case of RPM with two regions, where the utility function is

$$V_{BGS}(x|r) = \begin{cases} wx_1 + (1-w)x_2 & \text{if } x \in R_1^{BGS}(r) \\ (1-w)x_1 + wx_2 & \text{if } x \in R_2^{BGS}(r) \end{cases} . \tag{2}$$

Then, this is a special case of RPM when $u_1(x|r) = wx_1 + (1-w)x_2$ and $u_2(x|r) = (1-w)x_1 + wx_2$ for any bundle x and reference r.

In addition to the particular form of regions, BGS satisfies several properties that distinguish it from other RPMs. These properties are natural adaptations of those shown by Theorem 2 to characterize BGS in the choice correspondence setting.[12]

First, changing reference point does not reverse the ranking of two products unless it also changes their salience. This is the analog of Salience-SARP from Section 4.

Axiom (Reference Irrelevance)**.** *For any $x, y, r, r' \in X$:*
If $x \in R_k(r) \bigcap R_k(r')$ and $y \in R_l(r) \bigcap R_l(r')$, then $x \succsim_r y$ if and only if $x \succsim_{r'} y$.

In RPM, the reference point can influence choice through two channels: salience and valuation. The axiom eliminates the latter. When comparing two alternatives across different reference points, the DM's relative ranking does not change when neither's salience changes. This property greatly limits the effect of the reference point. In fact, a sufficiently small change in the reference never leads to a preference reversal between a given pair of alternatives.

The remaining two properties are the analogs of SDO and reflection from Section 4.

Axiom (Salient Dimension Overweighted*)**.** *For any $x, y, r, r' \in X$:*
if $y \in R_k(r) \bigcap R_k(r')$, $x \in R_k(r) \bigcap R_l(r')$, $x \succsim_r y$ and $x_l > x_k$, then $x \succ_{r'} y$.

Axiom (Reflection*)**.** *For any $x, y, r, r' \in X$:*
$(x_1, x_2) \succsim_{(r_1,r_2)} (y_1, y_2)$ if and only if $(x_2, x_1) \succsim_{(r_2,r_1)} (y_2, y_1)$.

The salient attribute gets a higher weight in the utility. The preference "reflects" about the 45 degree line. Their justification is very similar to their analogs'.

[12]While the nature of these properties is the same, they live in a different domain. Hence, to highlight this difference, we use asterisks (*) while we name the properties.

These three properties, plus those of the regions collected in Theorem 1, specialize RPM to BGS.

Theorem 3. *The family $\{\succsim_r\}_{r\in X}$ has a BGS representation if and only if it conforms to RPM under a regional function $\mathcal{R} = (R_1, R_2)$ satisfying **S0-S3**, and Salient Dimension Overweighted*, Reference Irrelevance, and Reflection* all hold.*

This result provides us an alternative characterization of the BGS model in this domain. It also provides guidance for comparing it with other models in the RPM class.

Constant Loss Aversion Model. Tversky and Kahneman [1991] provides a reference-dependent model that extends Prospect Theory to riskless consumption bundles. Each bundle is evaluated relative to reference point r, and losses loom larger than gains. In the absence of losses, the DM values each product bundle with a linear utility function, $(x_1 - r_1) + (x_2 - r_2)$, which attaches equal weights to each attribute. If she experiences a loss in attribute i, then she inflates the weight attached to that attribute by $\lambda_i > 0$.When $\lambda_i > 1$, this captures the phenomenon of loss aversion. The linear utility implies that the sensitivity to a given gain (or loss) on dimension i does not depend on whether the reference bundle is distant or near in that dimension. There are four different regions in the TK formulation: (i) gain in both dimensions, (ii) gain in the first dimension and loss in the second dimension, (iii) loss in the first dimension and gain in the second dimension, and (iv) loss in both dimensions (see Figure 4). We model this as $\mathcal{R}^{GL} = (R_1, R_2, R_3, R_4)$ where $R_1(r) = \{x : x \gg r\}$, $R_2(r) = \{x : x_1 < r_1 \text{ and } x_2 > r_2\}$, $R_3(r) = \{x : x_1 > r_1 \text{ and } x_2 < r_2\}$, and $R_4(r) = \{x : x \ll r\}$; call this the *gain-loss regional function.* Then the utility function is

$$V_{TK}(x|r) = \begin{cases} (x_1 - r_1) + (x_2 - r_2) & \text{if } x \in R_1(r) \\ \lambda_1(x_1 - r_1) + (x_2 - r_2) & \text{if } x \in R_2(r) \\ (x_1 - r_1) + \lambda_2(x_2 - r_2) & \text{if } x \in R_3(r) \\ \lambda_1(x_1 - r_1) + \lambda_2(x_2 - r_2) & \text{if } x \in R_4(r) \end{cases}$$

This is again clearly an RPM.

We can restate Tversky and Kahneman [1991]'s classic characterization theorem in the language of RPM. Two new axioms apply.

Axiom (Monotonicity). *For all $x, y, r \in X$: if $y \geq x$ and $y \neq x$, then $y \succ_r x$.*

Axiom (Double Cancellation). *For all $x_1, y_1, z_1, x_2, y_2, z_2 \in \mathbb{R}_+$, and $r \in X$, if $(x_1, z_2) \succsim_r (z_1, y_2)$ and $(z_1, x_2) \succsim_r (y_1, z_2)$, then $(x_1, x_2) \succsim_r (y_1, y_2)$.*

Monotonicity requires that if x exceeds y in every dimension, then the DM chooses x over y for any reference point. Double Cancellation guarantees an additive structure on preference and appears in Krantz et al. [1971] and Tversky and Kahneman [1991], among other places.[13]

Theorem 5, in the appendix, shows that every regional preference satisfies versions of Monotonicity and Double Cancellation, but only within regions. The above, more standard versions are more demanding as they apply across regions. Theorem 5 leaves open the possibility that RPM violates either.

Theorem 4. *A family of preferences $\{\succsim_r\}_{r \in X}$ has a TK representation if and only if it is an RPM with a gain-loss regional function that satisfies Monotonicity, Double Cancellation and each $\succsim_r$ is continuous.*

Tversky and Kahneman [1991, p. 1053] provide an alternative axiomatic characterization of the model, and our result makes heavy use of their theorem. However, TK assume two additional properties, "Sign Dependence" and "Reference Interlocking." Both properties are implied by RPM: the analog of the former is called *Weak Reference Irrelevance* in the appendix, and the latter is implied by a combination of that property and another that we call "regional linearity".

[13]Tversky and Kahneman [1991] call it simply Cancellation.

Linear Status Quo Bias Model. In the model of Masatlioglu and Ok [2005], individuals may experience some form of psychological discomfort when they have to abandon their status quo option. This discomfort imposes an additional utility cost. Of course, if an alternative is unambiguously superior to the status quo, the DM does not feel any psychological discomfort to forgo the status quo; in such cases there will be no cost. Formally, $Q(r)$ is a closed set denoting the alternatives that are unambiguously superior to the default option r (see Figure 4). If an alternative does not belong to this set, then the DM pays a cost $c(r) > 0$, which may depend on the reference point, to move away from the status quo. In this model, there are two regions: $R_1(r) = \{x|\ x \in Q(r)\}$ and $R_2(r) = \{x|\ x \notin Q(r)\}$. For any $x \neq r$, we have

$$V_{MO}(x|r) = \begin{cases} x_1 + x_2 & \text{if } x \in R_1(r) \\ x_1 + x_2 - c(r) & \text{if } x \in R_2(r) \end{cases}$$

5.2. **Comparison.** TK, BGS, MO, and the neoclassical model all belong to RPM, and Theorem 5 in the appendix describes the behavior that they have in common. However, the analysis so far, as well as the functional forms of the models, leaves open the question of what behavior distinguishes the them. To differentiate models, one might look for examples that are consistent with one but not the other. However, these models have rich parameter spaces, so determining whether an example is ruled out by all possible parametrizations is difficult. Instead, we make use of Theorems 3 and 4 to derive distinct predictions.

Table 1 compares the four models in terms of Reference Irrelevance, Monotonicity and Double Cancellation, when BGS, TK and MO do not coincide with the neoclassical model. Only the neoclassical model satisfies all conditions; none of the other three do. On the one hand, BGS satisfies Reference Irrelevance but violates Monotonicity. On the other, TK maintains Monotonicity but violates Reference Irrelevance. Finally, MO satisfies both of them.[14]

[14]Theorems 3 and 4 give the ✓'s of the table for BGS and TK. It is routine to verify that MO satisfies Monotonicity and Reference Irrelevance. We provide examples showing the other properties are violated in Appendix A.7.

	Neoclassical	BGS	TK	MO
RPM	✓	✓	✓	✓
Monotonicity	✓	✗	✓	✓
Reference Irrelevance	✓	✓	✗	✓
Double Cancellation	✓	✗	✓	✗

Table 1. Comparisons of Models

We provide a plausible example violating the Double Cancellation axiom, and hence behavior inconsistent with TK. Then, we illustrate BGS can accommodate this example even without requiring a shift in the reference point. While the example is one simple test to distinguish BGS from TK, it is also powerful as it works for a fixed reference point.

Example 1. *Consider a consumer who visits the same wine bar regularly. The bartender occasionally offers promotions. The customer prefers to pay* \$8 *for a glass of French Syrah rather than* \$2 *for a glass of Australian Shiraz. At the same time, she prefers to pay* \$2 *for a bottle of water rather than* \$10 *for the glass of French Syrah. However, without any promotion in the store, she prefers paying* \$10 *for Australian Shiraz to paying* \$8 *for water.*

The behavior in this example is both intuitively and formally consistent with the salient thinking model of BGS.[15] Without any promotion, the consumer expects to pay a high price for a relatively low quality selection. When choosing between Syrah or Shiraz, the consumer focuses on the French wine's sublime quality, and she is willing to pay at least \$6 more for it. When choosing between water and Syrah, the low price of water stands out and she reveals that the gap between wine and water is less than \$8. However, when there is no promotion, she focuses again on the quality, and she is willing to pay an additional \$2 for even her less-preferred Australian Shiraz over water.

[15]Implicitly, the example reveals that the quality of French Syrah is higher than Australian Shiraz which is in turn higher than water. The numerical value of quality assigned to each beverage is irrelevant to the violation of Double Cancellation. For examples of qualities so that choice can be represented by the BGS model, one can calculate that $(-8, q_{fs}) \succ_r (-2, q_{as})$, $(-2, q_w) \succ_r (-10, q_{fs})$ and $(-10, q_{as}) \succ_r (-8, q_w)$ for $q_{fs} = 8$, $q_{as} = 6.9$, $q_w = 5.1$, and the reference point $r = (\frac{1}{2}(-10 + -8), \frac{1}{2}(q_w + q_{as}))$ when $w = 0.6$.

Notice that this explanation does not require that the reference points are different. Since the consumer visits this bar regularly, intuitively, her reference point should be fixed and stable.

5.3. **Monotonicity and Reference Irrelevance.** While Theorems 2 and 3 highlight the fact that Monotonicity is not necessarily satisfied in RPM, it does not inform us when it is violated, nor which features of regional preferences are responsible for the violation. Moreover, Theorem 5 does not communicate how serious the Monotonicity violations are in this class. For example, there may be only a small fraction of regional preference violates Monotonicity.

The remainder of our analysis highlights a general tension between Reference Irrelevance and Monotonicity. Under Reference Irrelevance, $u_i(\cdot|r)$ does not vary with r. Salient thinking occurs when the region to which the alternative belongs alters the trade-off between the dimensions. To capture the implications of both, we follow the BGS model in requiring that the weights on attributes always add up to 1 and do not vary with r. Formally, we say that an RPM $(\{\succsim_r\}_{r\in X}, \mathcal{R})$ has *salience utilities* if it has a representation where for each region i, there is a distinct $w^i \in (0,1)$ such that $u_i(x|r) = w^i x_1 + (1-w^i)x_2$ for all r. Intuitively, the trade-off between attributes differs across regions, and objects with the same level of each attribute have the same evaluation, regardless of the region to which they belong. BGS is an example of salience utilities where $w_1 = 1 - w_2$.

Proposition 2. *If $(\{\succsim_r\}_{r\in X}, \mathcal{R})$ is an RPM with at least two regions that has salience utilities, then $\succsim_r$ violates Monotonicity for some r.*

Given salience utilities, the proposition states that Monotonicity fails. This holds no matter how one specifies the regions or the weights on utilities. Hence, there is a clash between salient thinking and Monotonicity in RPM. This is emblematic of a more general trade-off when salient thinking, more broadly defined, is exhibited: either the reference point affects the agent's valuation or the DM does not respect Monotonicity.

5.4. **Relationship between other models of salience and RPM.** A number of alternative models of salience have been proposed in the literature. Appendix B.1 describes some of their functional forms and how they differ from RPM. In this subsection, we demonstrate that the prominent and closely related models of focusing (Kőszegi and Szeidl [2013], KS) and relative thinking (Bushong et al. [2015], BRS) fail to be RPM. Neither of the two features a reference point, but they do feature a consideration set that can play the same role.[16] Letting the consideration set take on that role, a KS or BRS DM can be represented as RPM only if it reduces to the neoclassical model.

Example 2. Adapted to a linear utility index, the models of KS and BRS assume that the DM maximizes

$$U(x, C) = g_1(C)x_1 + g_2(C)x_2$$

where C is the comparison set, fixed per decision problem, and

$$g_i(C) = g\left(\max_{y \in C} y_i - \min_{y' \in C} y'_i\right)$$

where g is a strictly monotone function.[17] Fixing a consideration set C, the slopes of the indifference curve are the same everywhere, so if the DM could be represented by RPM, then each region would correspond to this same set of indifference curves. Under RPM, a change in C causes the regions to change, but the indifference curves within each do not. Therefore, a DM represented by the KS or BRS models also conforms to RPM only if each $g_i(\cdot)$ is constant. This implies that she is also neoclassical: the comparison set does not affect her choice.

6. Conclusion

This paper applies the tools of decision theory to a model of behavioral economics. Completely identifying the behavior that corresponds to the salient thinking model helps us to understand, identify, evaluate, and test its implied behavior. This is crucial

[16] Indeed, one can easily adapt the RPM model to a family of preference relations indexed by an appropriate family of subsets rather than X.

[17] Specifically, KS require $g'(x) > 0$ for all $x > 0$ and BRS require $g'(x)x + g(x) > 0$ and $g'(x) < 0$ for all $x > 0$.

for this model because its functional form is rather complicated. We highlight which aspects of the functional form should be thought of as convenient simplifications, and which are crucial to the notion of salience. We also demonstrate new connections between the behavior implied by the salient-thinking model of Bordalo et al. [2013b] and that of other widely-used models relying on different psychological intuitions, such as the loss aversion model of Tversky and Kahneman [1991].

The approaches pioneered by behavioral economics and decision theory complement each other. Both are a means to identify and to organize the implications of economic models. The former investigates the implications of a particular model in concrete economic problems. While it is able to highlight particularly interesting behaviors in important applications, it necessarily provides an incomplete picture of the overall choice patterns across settings. Often, it leaves open the question of what particular features give rise to this behavior, and how to test the model without imposing functional form assumptions. The latter identifies the complete behavioral implications of the model under consideration. While it finds specific regularities that hold in all settings and provides choice-based tests of the model, the generality may obfuscate the interesting patterns in important economic application. We think collaboration across these fields will continue to lead to fruitful insights into economic decision making.

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Appendix A. Omitted Proofs

To economize on space, we write $x\alpha y$ and $S\alpha T$ in lieu of $\alpha x + (1-\alpha)y$ and $\alpha S + (1-\alpha)T$ for $x, y \in X$ and $S, T \subset X$ throughout the appendix.

A.1. **Proof of Theorem 1.** First, we show (i) $\implies$ (ii). Set $\sigma(a,b) = \max\{a/b, b/a\}$. Clearly σ is a salience function, and we show that σ generates R_1 and R_2. Fix $r \in \mathcal{F}$ and set $A = \{x : \sigma(x_1, r_1) > \sigma(x_2, r_2)\}$. We show $A = R_1(r)$.

Claim $A \bigcap R_2(r) = \emptyset$. If not, pick $x \in A \bigcap R_2(r)$. $x \in A$ implies either (a) $x_1/r_1 > x_2/r_2$ and $x_1/r_1 > r_2/x_2$ or (b) $r_1/x_1 > x_2/r_2$ and $r_1/x_1 > r_2/x_2$. If (a) and $x_2 \leq r_2$, then

$$x_1/r_1 > r_2/x_2 \geq x_2/r_2 \text{ implies } x_1 > r_1 r_2/x_2 \geq r_1,$$

so there exists $\lambda \in [0,1)$ such that $(\lambda x_1 + (1-\lambda) r_1, x_2) = (r_1 r_2/x_2, x_2) = x'$. If (a) and $x_2 > r_2$, then

$$x_1 > r_1 x_2/r_2 > r_1,$$

so there exists $\lambda \in (0,1)$ such that $(\lambda x_1 + (1-\lambda) r_1, x_2) = (r_1 x_2/r_2, x_2) = x'$. By Moderation and $x \in R_2(r)$, $x' \in R_2(r)$. However, we have either $x'_1 x'_2 = r_1 r_2$ or $x'_1/x'_2 = r_1/r_2$ so $x' \notin R_2(r)$ by Equal Salience, a contradiction. A similar contradiction obtains if (b) holds.

Now, since $A \bigcap R_2(r) = \emptyset$ and $R_1(r) \bigcup R_2(r)$ is dense, $A \subset cl(R_1(r))$. By Regular Regions, $R_1(r) = int(cl(R_1(r)))$. Since A is an open set contained in $cl(R_1(r))$, $A \subseteq R_1(r)$. Similarly, for $B = \{x : \sigma(x_1, r_1) < \sigma(x_2, r_2)\}$, $B \subseteq R_2(r)$. But

$$(A \bigcup B)^c = \{x : x_1 x_2 = r_1 r_2 \text{ or } x_1/x_2 = r_1/r_2\},$$

and by Equal Salience, $(A \bigcup B)^c \bigcap R_k(r) = \emptyset$ for $k = 1, 2$. Thus $A = R_1(r)$ and $B = R_2(r)$, completing the proof.

Now, we show (ii) implies (i). First, Moderation follows from ordering. Second, Equal Salience follows from symmetry and homogeneity of degree zero. Third, Regular Regions follows from continuity.

Finally, to see (iii) if and only if (ii), fix any salience function s. Observe $s(a,b) > s(c,d)$ if and only if $s(a/b, 1) > s(c/d, 1)$ by homogeneity if and only if $s(\max(a/b, b/a), 1) > s(\max(c/d, d/c), 1)$ by symmetry if and only if $\max(a/b, b/a) > \max(c/d, d/c)$ by ordering. Thus if one salience function generates the regions, every other salience function does as well. ∎

A.2. **Proof of Theorem 2.** Necessity is easily verified. The proof of sufficiency proceeds as a series of claims, assuming that c satisfies Axioms 1-6.

Claim 1. *For any x and y, there exists a set $S_{xy} \in \mathcal{X}$ such that $x, y \in S_{x,y}$, $S_{xy} \subset R_1(A(S_{xy}))$ and each $z \in S_{xy} \setminus \{x, y\}$ is dominated by either x or y.*

Proof. Fix $x, y \in X$ and define

$$\alpha := \frac{1}{2} \min \left\{ \frac{x_1}{x_2}, \frac{y_1}{y_2}, x_1, x_2, y_1, y_2, 1 \right\} > 0.$$

We are going to choose the set S_{xy} so that the average $(A(S_{xy}))$ is (α^2, α), denoted by r. Consider $\bar{x} = \left(0, \alpha - (\frac{x_2-\alpha}{x_1-\alpha^2})\alpha^2\right)$, the point where the line passing through r and x intersects the $y-$axis. Note that $\alpha - (\frac{x_2-\alpha}{x_1-\alpha^2})\alpha^2$ is positive. Then the set $T_{\bar{x}x} = \{\beta\bar{x} + (1-\beta)x : \beta \in (0,1)\} \setminus \{r\}$ is the line segment connecting x and $\bar{x}$ excluding r. Now, for any $z \in T_{\bar{x}x}$, either $z \gg r$ or $r \gg z$. If $z \gg r$, then $z \in R_1(r)$ since $z_1/z_2 \in (\alpha, x_1/x_2)$ and $z_1 z_2 > \alpha^3$. If $r \gg z$, then we also have that $z \in R_1(r)$ since $z_1/z_2 \in (0, \alpha)$ and $z_1 z_2 < \alpha^3$.

We show first there exists a set S_x so that $x \in S_x$, $S_x \subset R_1(r)$, $A(S_x) = r$, and $x \gg z$ for every $z \in S_x \setminus x$. . To see this, set β^* so that $r = \beta^*\bar{x} + (1-\beta^*)x$. If $\beta^* < \frac{1}{2}$, then clearly there exists $a \in T_{\bar{x}x}$ such that $\frac{1}{2}a + \frac{1}{2}x = r$. In this case, set $S_x = \{a, x\}$.

Otherwise, pick $\gamma \in (\beta^*, 1)$ and set $a = \gamma\bar{x} + (1-\gamma)x$. Define β_a s.t. $\beta_a a + (1-\beta_a)x = r$. Since $\beta_a > \beta^*$, there exists $n > 1$ such that $\beta_a \in [\frac{n-1}{n}, \frac{n}{n+1})$. Pick any distinct $a^1, \ldots, a^{n-1} \in T_{\bar{x}x}$ such that $\frac{1}{n-1}\sum a^i = a$. If $\beta_a = \frac{n-1}{n}$, then set $S_x = \{x, a^1, \ldots, a^{n-1}\}$.

Now if $\beta_a > \frac{n-1}{n}$, define $\lambda = \beta_a(n+1) - (n-1)$, which is greater than zero since $\beta_a > \frac{n-1}{n} > \frac{n-1}{n+1}$. Since $1 - \beta_a > \frac{1}{n+1}$, we have $n+1-\beta_a(n+1)-1 > 0$, hence $1-\lambda > 0$. Take $a^n = \lambda a + (1-\lambda)x$. This belongs to $T_{\bar{x}x}$ since the weights are positive, sum to 1, and $\lambda = \beta_a$ only if $\beta_a = \frac{n-1}{n}$. Call $S_x = \{x, a^1, \ldots, a^n\}$.

Repeat the above with y taking the place of x to get S_y with $y \in S_y$, $A(S_y) = r$, $S_y \subset R_1(r)$, and $y \gg z$ for every $z \in S_y \setminus y$. We have two sets whose average is exactly (α^2, α), so for $S_{xy} = S_x \cup S_y$, the average is r, $S_{xy} \subset R_1(A(S_{xy}))$, and for any $z \in S_{xy} \setminus \{x, y\}$, either $z \ll x$ or $z \ll y$. $\square$

We now define a binary relation, which will represent preference of agent in region 1. Define

$$x \succsim_1 y \text{ if there exists } S \text{ such that } x \in c(S), y \in S, \text{ and } x, y \in R_1(A(S)).$$

Claim 2. $\succsim_1$ *admits a linear utility representation.*

Proof. Claim 1 implies that $\succsim_1$ is a complete binary relation. By Salience-SARP, it is routine to show that $\succsim_1$ is transitive. We now establish that $\succsim_1$ is linear. To see this, pick x, y, z and $\beta^* \in (0, 1]$. Suppose that $x \succsim_1 y$, so we need to show that $x\beta^*z \succsim_1 y\beta^*z$.

Define the menu $S_\alpha = S_{x\alpha z y\alpha z}$ and then let $S_{\alpha\beta} = \beta S_\alpha + (1-\beta)\{z\}$ for any $\alpha \in [0,1]$ and $\beta > 0$, the Minkowski sum of the menus viewed as points in $\mathbb{R}^n$. In particular, $\beta > 1$ is allowed. Let $O(\alpha) = \{\alpha\beta : S_{\alpha\beta} \subset R_1(A(S_{\alpha\beta}))\}$ if $\gamma \in O(\alpha)$, then $x\gamma z, y\gamma z \in S_{\alpha\beta}$ for some β and either $x\gamma z$ or $y\gamma z$ is chosen from $S_{\alpha\beta}$ by Regional Monotonicity and construction of S_{xy}. Moreover, $O(\alpha)$ is open in $[0,1]$ by the openness of $R_1(\cdot)$ and $\alpha \in O(\alpha)$ for every α. Now, $[\beta^*, 1]$ is compact and $\{O(\alpha) : \alpha \in [\beta^*, 1]\}$ is an open cover. Hence, there exists a finite subcover $\{O(\alpha_1), \ldots, O(\alpha_n)\}$, where WLOG $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ and no element is contained in the union of the others.

Then, $1 \in O(\alpha_1)$ and $\beta^* \in O(\alpha_n)$. Choose $\gamma_i \in O(\alpha_i) \cap O(\alpha_{i+1})$ for $i = 1, \ldots, n$. Write $\gamma_0 = 1$ and $\gamma_{n+1} = \beta^*$, and observe that we have $x\gamma_0 z \succsim_1 y\gamma_0 z$. Proceed by recursion, given that $x\gamma_i z \succsim_1 y\gamma_i z$. In the menu $S_{\alpha_i \frac{\gamma_i}{\alpha_i}}$, $x\gamma_i z$ and $y\gamma_i z$ are available, and by construction of S_{α_i}, nothing other than the two is chosen. Since they all belong to region 1, $x\gamma_i z \in c(S_{\alpha_i \frac{\gamma_i}{\alpha_i}})$ by Salience-SARP. By Regional Linearity, $x\gamma_{i+1} z \in c(S_{\alpha_i \frac{\gamma_{i+1}}{\alpha_i}})$ since both $\gamma_i, \gamma_{i+1} \in O(\alpha_i)$. Hence, $x\gamma_{i+1} z \succsim_1 y\gamma_{i+1} z$. Conclude

$$x\gamma_{n+1} z = x\beta^* z \succsim_1 y\beta^* z = y\gamma_{n+1} z.$$

This establishes that $\succsim_1$ is linear.

To establish continuity, suppose that $x_n \succsim_1 y$ for all n and $x_n \to x$. Let $S = S_{xy} \setminus \{x\}$ and observe $A(S \cup \{x_n\}) \to A(S)$ and $x_n \in R_1(A(S \cup \{x_n\}))$ for all n large enough. For any such n, either x_n or y dominates all other bundles in the menus. Then, only the two can be chosen, and hence x_n must be since $x_n \succsim_1 y$ by Salient-SARP. Then, Regional Continuity gives that $x \in c(S_{xy})$ and thus $x \succsim_1 y$. Similarly, if $x \succsim_1 y_n$ for all n and $y_n \to y$ then $x \succsim_1 y$. Hence, $\succsim_1$ is continuous.

Using Herstein-Milnor (1952), there exists an affine $u_1 : X \to \mathbb{R}$ that represents $\succsim_1$. □

For any alternative $x = (x_1, x_2)$, write $x' = (x_2, x_1)$ for its reflection.

Claim 3. *For any x and $k \in \{1, 2\}$, there exists a subset S such that (i) both x and its reflection are chosen; (ii) x belongs to region k; and (iii) the reflection belongs to the other region. (i.e. $x, x' \in c(S)$, $x \in R_k(A(S))$, $x' \in R_l(A(S))$, and $k \neq l$).*

Proof. Write $x = (a, b)$, and WLOG, $a > b$. Then $x \in R_2(A(\{x, x'\}))$ and $x' \in R_1(A(\{x, x'\}))$. By reflection, $x, x' \in c(\{x, x'\})$. It remains to be show for $x \in R_1(A(S))$ and $x' \in R_2(A(S))$. For any n, consider $S_n = \{x, x'\} \cup \bigcup_{i=1}^n \{(3^{-i}b, 2^{-i}b), (2^{-i}b, 3^{-i}b)\}$. Observe $A(S_n)_1 = A(S_n)_2$, and $A(S_n)_1 < b$ for n large enough. For this n, we have $x \in R_1(A(S_n))$ and $x' \in R_2(A(S_n))$. Moreover, S_n is its own reflection, and either x or x' is chosen by Regional Monotonicity. Hence, by Reflection, $x, x' \in c(S_n)$. □

Define now define a utility function on $X \times \{1, 2\}$, where $u(x, k)$ represents the value of x when x belongs to Region k. Define $u(x, 1) = u_1(x)$ and $u(x, 2) = u_1(x')$ where x' is the reflection of x. Define the function k so that $\mathrm{k}(x, S) = k$ if and only if $x \in R_k(A(S))$.

Claim 4. *u represents c. That is, $x \in c(S)$ if and only if $x \in \arg\max_{y \in S} u\,(y, \mathrm{k}(y, S))$.*

Proof. Take $x \in c(S)$, WLOG by reflection, assume $x \in R_1(A(S))$. For all $y \in R_1(A(S))$, $x \in c(S_{xy})$ by Salient-SARP, hence $u(x, 1) \geq u(y, 1)$. Now, if $y \in R_2(A(S))$, then there exists S' by Claim 3 with $y, y' \in c(S')$, $y \in R_2(A(S'))$, and $y' \in R_1(A(S'))$. Consider $S_{xy'}$, recalling that $x, y' \in R_1(A(S_{xy'}))$ and either $x \in c(S_{xy'})$ or $y' \in c(S_{xy'})$. In the former case, we obtain immediately that $u(x, 1) \geq u(y', 1) = u(y, 2)$. In the latter case, using $x = x^1$, $y = x^2$, and $y' = x^3$ with $S^1 = S$, $S^2 = S'$ and $S^3 = S_{xy'}$, we obtain that $x \in c(S_{xy'})$ using Salient-SARP, implying that $u(x, 1) \geq u(y', 1) = u(y, 2)$. Hence, x maximizes $u\,(y, \mathrm{k}(y, S))$ in S.

Let x be a maximizer of $u\left(y, \mathrm{k}(y, S)\right)$ in S and y is chosen from S. WLOG, assume x belongs to region 1 in S. If $y \in R_1(A(S))$, $u(x,1) \geq u(y,1)$ implies $x \in c(S_{xy})$. By Salient-SARP, x must be chosen from S. Now, if $y \in R_2(A(S))$, then we have $u(x,1) \geq u(y,2) = u(y',1)$. Then there exists S' by Claim 3 with $y, y' \in c(S')$, $y \in R_2(A(S'))$, and $y' \in R_1(A(S'))$. Consider $S_{xy'}$, recalling that $x, y' \in R_1(A(S_{xy'}))$ and $x \in c(S_{xy'})$. Using $x = x^1$, $y' = x^2$, and $y = x^3$ with $S^1 = S_{xy'}$, $S^2 = S'$ and $S^3 = S$, we obtain that $x \in c(S)$ using Salient-SARP. $\square$

Regional Monotonicity and taking an appropriate affine transformation, there exists w such that $u(x,1) = wx_1 + (1-w)x_2$ and $u(x,2) = (1-w)x_1 + wx_2$. We now show that $w > \frac{1}{2}$, hence the salient attribute gets higher weight in the final evaluation. After Claim 4, we have $u_1(x) = wx_1 + (1-w)x_2$ and $u_2(x) = wx_2 + (1-w)x_1$. Pick x, y with $x_1 > y_1$ and $y_2 > x_2$ so that $wx_1 + (1-w)x_2 = wy_1 + (1-w)y_2$. Pick x, y with $x_1 > y_1$ and $y_2 > x_2$ so that $wx_1+(1-w)x_2 = wy_1+(1-w)y_2$. By Claim 1, there exists S containing x, y where $x, y \in R_1(A(S))$ and either x or y is chosen. But then by Claim 4, both are chosen, $x, y \in c(S)$. Let S' be the set as above but for region 2, and by SDO, x is not chosen. Let $\epsilon = x_1 - y_1$, $\delta = y_2 - x_2$ so $\epsilon, \delta > 0$ and $x = y + (\epsilon, -\delta)$. Hence $w\epsilon - (1-w)\delta = 0$ and $(1-w)\epsilon - (w)\delta < 0$, and $\frac{w}{1-w} > \frac{\epsilon}{\delta} = \frac{1-w}{w}$. But this requires $w > \frac{1}{2}$. $\blacksquare$

A.3. **Proof of Proposition 1.** The claims establishing Theorem 2 go through, word-for-word, except Claim 2. The proof of Claim 2 is easily adapted using standard arguments, e.g. Theorem 2 of Chapter 6 of Krantz et al. [1971], by applying Salient-SARP to get that $\succsim_1$ has an additive representation, and similarly for $\succsim_2$. This, coupled with reflection, establishes that that $u(x,1) = u(x_1)+v(x_2)$ and $u(x,2) = v(x_1)+u(x_2)$ for strictly increasing continuous utility indexes $u, v : \mathbb{R}_{++} \to \mathbb{R}$.

We must also show that $u(a) - u(b) > v(a) - v(b)$ for all $a > b$. Suppose not, and consider some $a > b$ where $u(a) - u(b) \leq v(a) - v(b)$. Let $x = (a, b)$ and $y = (b, a)$. Follow Claim 1 to construct $S_{xy} \subset R_1(A(S_{xy}))$. By hypothesis, $u(y,1) \geq u(x,1)$ and $y \in c(S_{xy})$. Now, construct S'_{xy} in the same manner as Claim 1, but reversing the role of dimensions so that $S'_{xy} \subset R_2(A(S'_{xy}))$. By SDO and construction of S'_{xy}, $\{y\} = c(S'_{xy})$. But then $u(y,2) > u(x,2)$, which is incompatible with $u(a)-u(b) \leq v(a)-v(b)$. Hence $u(a) - u(b) > v(a) - v(b)$ for all $a > b$. $\blacksquare$

A.4. **Proof of Theorem 3.** Necessity is easily verified, so we show only sufficiency. Assume the family $\{\succsim_r\}_{r \in X}$ satisfies the axioms in Theorem 3. The next two lemmas show these axiom imply useful properties in establishing the representation.

Lemma 1. *For any r, there exists $x^k \in R_k(r)$ for $k = 1, 2$ with $x^1 \sim_r x^2$.*

Proof. We first show the result for $r = (z, z)$. Let $R_k^z = \{x \in R_k(z) : x_1 x_2 > z^2\}$, the points in $R_k(r)$ above the Cobb-Douglas line passing through r. Pick $a \in R_1^z$ and $b \in R_2^z$. The result is immediate if $a \sim_r b$. If $a \succ_r b$, then by Reflection and S1-S3, $a' \succ_r b'$ when $a' = (a_2, a_1) \in R_2^z$ and $b' = (b_2, b_1) \in R_1^z$. If $b \succ_r b'$, then $a \succ_r b \succ_r b'$.

Continuity of u_1 and that R_1^z is open and connected yields existence of $c \in R_1^z(r)$ with $c \sim_r b$. Otherwise, $b' \succsim_r b$, so $a' \succ_r b' \succsim_r b$. R_2^z is also open and connected and u_2 is also continuous, so there exists $c \in R_2^z(r)$ with $c \sim_r b'$. A similar proof holds if $b \succ_r a$.

Note the above arguments are true for $R_k^{z'}$ when $z' < z$, since $R_k^{z'} \subset R_k^z \subset R_k(r)$. Within each R_k^z, indifference curves are linear, parallel and downward sloping. For z' to be close 0, the x^1, x^2 we find must lie on indifference curves that intersect the boundary without leaving R_k^z. Thus for any r', we can find y^1 and y^2 so that $x^k \sim_r y^k$ and $x^k, y^k \in R_k(r) \bigcap R_k(r')$ for $k = 1, 2$. But then by Reference Irrelevance and Transitivity, $y^1 \sim_{r'} y^2$. $\square$

Lemma 2. *There is an RPM of $\{\succsim_r\}_{r\in X}$ where for each k, $u_k(\cdot|r) = u_k(\cdot|r')$ for all $r, r' \in X$.*

Proof. By taking appropriate affine transformations of the entire utility function, there is no loss of generality in assuming $u_1(\cdot|r) = u_1(\cdot|r')$ for all r, r'. For contradiction, suppose that $u_2(\cdot|r) \neq u_2(\cdot|r')$ for some r, r'. Let $A = \{\alpha \in [0,1] : u_2(\cdot|r\alpha r') \neq u_2(\cdot|r)\}$ be the set of references between r and r' that have a different utility function for R_2. Set $\alpha^* = \sup A$, noting it is finite since $0 \in A$ and $1 \notin A$. Take $\hat{\alpha}_n$ to be a sequence in A and α_n to be a sequence in $[0,1] \setminus A$ that both approach α^*. Then, for $\hat{r}_n = r\hat{\alpha}_n r'$ and $r_n = r\alpha_n r'$, $u_2(\cdot|r_n) = u_2(\cdot|r)$ and $u_2(\cdot|\hat{r}_n) \neq u_2(\cdot|r)$. Moreover, r_n and $\hat{r}_n$ have the same limit $\hat{r} = r\alpha^* r'$.

By Lemma 1, there exists $x \in R_1(\hat{r})$ and $y \in R_2(\hat{r})$ so that $x_2 \sim_{\hat{r}} x_1$. Since $R_1(r)$ and $R_2(r)$ are open for each r, there exists $\epsilon > 0$ so that $B_\epsilon(x_2) \subset R_2(\hat{r})$, $B_\epsilon(x_1) \subset R_1(\hat{r})$. By continuity of the regional function, $B_\epsilon(x_2) \subseteq R_2(\hat{r}_n) \bigcap R_2(r_n)$ and $B_\epsilon(x_1) \subseteq R_1(\hat{r}_n) \bigcap R_1(r_n)$ for n large enough. For every z close enough to x_2, there exists $y(z) \in B_\epsilon(x_1)$ such that $z \sim_{\hat{r}} y(z)$. But then by Reference Irrelevance, $z \sim_{r_n} y(z)$ and $z \sim_{\hat{r}_n} y(z)$. Thus $u_i(z|r_n) = u_1(y(z)|r_n) = u_1(y(z)|\hat{r}_n) = u_i(z|\hat{r}_n)$ for all z close enough to x_i, implying that $u_i(\cdot|r_n) = u_i(\cdot|\hat{r}_n)$, a contradiction. Conclude $u_i(\cdot|r) = u_i(\cdot|r')$ for all r, r'. $\square$

By the above lemmas, there are u_1, u_2 such that if $x \in R_k(r)$ and $y \in R_l(r)$, then $x \succsim_r y \iff u_k(x) \geq u_l(y)$. Now, $u_k(x) = a_k x_1 + b_k x_2 + c_k$, and $c_1 = 0$ WLOG. Regional monotonicity implies $a_k, b_k > 0$.

Fix arbitrary $r = (r_1, r_2)$ and let $r' = (r_2, r_1)$. For any distinct $x, y \in R_1(r)$ such that $x \sim_r y$, $a_1 x_1 + b_1 x_2 = a_1 y_1 + b_1 y_2$. By reflection and the structure of $\mathcal{R}$, we have $(x_2, x_1), (y_2, y_1) \in R_2(r')$ and $(x_2, x_1) \sim_{r'} (y_2, y_1)$. Hence, $a_2 x_2 + b_2 x_1 + c_2 = a_2 y_2 + b_2 y_1 + c_2$. Then we have $\frac{b_2}{a_2} = \frac{x_2 - y_2}{y_1 - x_1}$ and $\frac{b_1}{a_1} = \frac{x_1 - y_1}{y_2 - x_2}$, so there exists $\alpha > 0$ such that $b_2 = \alpha a_1$ and $a_2 = \alpha b_1$.

For contradiction, suppose $\alpha \neq 1$. Pick $(x, y) \in R_1(1,1)$ and $(d, e) \in R_2(1,1)$ so that $(x, y) \sim_{(1,1)} (d, e)$. Note $(e, d) \in R_1(1,1)$ and $(y, x) \in R_2(1,1)$. By Reflection,

$(y,x) \sim_{(1,1)} (e,d)$. Then

$$\begin{aligned}
a_1x + b_1y &= \alpha[a_1e + b_1d] + c_2 \\
a_1e + b_1d &= \alpha[a_1x + b_1y] + c_2 \\
a_1e + b_1d &= \alpha[\alpha[a_1e + b_1d] + c_2] + c_2 \\
a_1e + b_1d &= \alpha^2[a_1e + b_1d] + (1+\alpha)c_2 \\
(1-\alpha^2)[a_1e + b_1d] &= (1+\alpha)c_2
\end{aligned}$$

Since we can find similar indifferences for any good in a small enough neighborhood of (d,e), this requires $\alpha = 1$ and $c_2 = 0$.

After a normalization, we have $u_1(x) = wx_1 + (1-w)x_2$ and $u_2(x) = (1-w)x_1 + wx_1$ for some $w \in (0,1)$. To conclude, we must show $w > \frac{1}{2}$. Pick any $y > 0$. Consider the line $L(y) = \{x' \in X : u_2(x') = u_2(y,y)\}$. This is the line with slope $\frac{-w}{1-w}$ that intersects the 45-degree line at (y,y). For any $x \in L(y)$ such that $x_1 > y > x_2$, we can find r, r' so that $(y,y) \in R_2(r) \bigcap R_2(r')$, $(x_1,x_2) \in R_2(r) \bigcap R_1(r')$. By construction $(x_1,x_2) \sim_r (y,y)$, so since $x_1 > x_2$, by SDO* we have $(x_1,x_2) \succ_{r'} (y,y)$ and $wx_1 + (1-w)x_2 > y = (1-w)x_1 + wx_2$. Letting $(x_1,x_2) \to (y,y)$ gives that $w > \frac{1}{2}$. ■

A.5. **Proof of Theorem 4.** Necessity follows from the discussion above and TK's theorem. To show sufficiency, we rely on TK's theorem, which states that any monotone, continuous family of preference relations that satisfies cancellation, sign-dependence and reference interlocking has a TK representation. Given our assumptions, we need to show that $\{\succsim_r\}$ satisfies sign-dependence and reference interlocking.

TK say that $\{\succsim_r\}$ satisfies sign-dependence if "for any $x, y, r, s \in X$, $x \succsim_r y \iff x \succsim_s y$ whenever x and y belong to the same quadrant with respect to r and with respect to s, and r and s belong to the same quadrant with respect to x and with respect to y." This happens if and only if $x \in R_k^{GL}(r) \bigcap R_k^{GL}(s)$ and $y \in R_k^{GL}(r) \bigcap R_k^{GL}(s)$ for some $k \in \{1,2,3,4\}$. Then, this is exactly an implication of RPM, since $u_k(\cdot|r) = \alpha u_k(\cdot|s) + \beta$ for $\alpha > 0$.

TK say that $\{\succsim_r\}$ satisfies reference interlocking if "for any $w, w', x, x', y, y', z, z'$ that belong to the same quadrant with respect to r as well as with respect to s, $w_1 = w_1'$, $x_1 = x_1', y_1 = y_1', z_1 = z_1'$ and $x_2 = z_2, w_2 = y_2, x_2' = z_2', w_2' = y_2'$, if $w \sim_r x$, $y \sim_r z$, and $w' \sim_s x'$ then $y' \sim_s z'$." The assumptions on quadrants imply that $w, w', x, x', y, y', z, z' \in R_k^{GL}(r) \bigcap R_l^{GL}(s)$. By affinity there exist α_k, β_k, c_k so that $u_k(a|r) = \alpha_k a_1 + \beta_k a_2 + c_k$ for each $a \in X$ and $u_l(a|s) = \alpha_l a_1 + \beta_l a_2 + c_l$ for each $a \in X$. Therefore, $w \sim_r x$ implies $\alpha_k(w_1 - x_1) = \beta_k(x_2 - w_2)$, $y \sim_r z$ implies $\alpha_k(y_1 - z_1) = \beta_k(z_2 - y_2) = \beta_k(x_2 - w_2)$ and $w' \sim_s x'$ implies $\alpha_l(w_1' - x_1') = \beta_l(x_2' - w_2')$. This last equality holds if and only if $\alpha_l(w_1 - x_1) = \beta_l(z_2' - y_2')$. Now, the above equalities imply

$$\alpha_l(y_1' - z_1') = \alpha_l(y_1 - z_1) = \alpha_l \frac{\beta_k}{\alpha_k}(x_2 - w_2) = \alpha_l(w_1 - x_1).$$

Hence, $\alpha_l(y_1' - z_1') = \beta_l(z_2' - y_2')$ and we have $y' \sim_s z'$.

A.6. **Proof of Proposition 2.** By Salience Utilities, drop the dependence on r and write $u_k(\cdot)$ instead of $u_k(\cdot|r)$. Let $U_k = \{x \in X : u_k(x) > u_l(x) \text{ for all } l \neq k\}$ and $L_k = \{x \in X : u_k(x) < u_l(x) \text{ for all } l \neq k\}$, the set of alternatives whose utility is highest and lowest, respectively, in region k. Define $R_{-k}(r) = \bigcup_{l \neq k} R_l(r)$.

Lemma 3. *If $\succsim_r$ satisfies Monotonicity and $z \in U_k \bigcap R_k(r)$, then $R_{-k}(r) \bigcap \{x : x \gg z\} \bigcap U_k = \emptyset$.*

Proof. Suppose not, so there is $z \in U_k \bigcap R_k(r)$ such that $A = R_{-k}(r) \bigcap \{x : x \gg z\} \bigcap U_k \neq \emptyset$. Then let y_n be a sequence of points in A approaching as close as possible to z. WLOG, $y_n \to \bar{y}$ (since y_n must eventually belong to the compact set $\bar{B}_\epsilon(z)$ for some $\epsilon > 0$). Then we can pick $x_n \in R_k(r) \bigcap \{y : y \leq \bar{y}\}$ that converges to $\bar{y} \in bd(R_k(r))$.[18] Noting $u_{-k}(x) = \max_{l \neq k} u_l(x)$ is continuous,

$$\lim u_k(x_n) = u_k(\bar{y}) > u_{-k}(\bar{y}) = \lim u_{-k}(y_n)$$

and so $x_n \succ_r y_m$ for n, m large enough, but $y_m \geq x_n$ by taking n large enough that $d(x_n, \bar{y})) < d(y_m, \bar{y})$. This contradicts Monotonicity. □

Lemma 4. *If $\succsim_r$ satisfies Monotonicity and $r \in U_k$, then $R_{-k}(r) \bigcap \{x : x \gg r\} \bigcap U_k = \emptyset$.*

Proof. Follows from applying Lemma 3 to a sequence x_n in $R_k(r)$ that converges to r. □

Lemma 5. *If $\succsim_r$ satisfies Monotonicity and $z \in L_k \bigcap R_k(r)$ then $R_{-k}(r) \bigcap \{x : x \ll z\} \bigcap L_k = \emptyset$.*

Proof. Dual to Lemma 3. □

Lemma 6. *If $\succsim_r$ satisfies Monotonicity and $r \in L_k$, then $R_{-k}(r) \bigcap \{x : x \ll r\} \bigcap L_k = \emptyset$.*

Proof. Follows from applying Lemma 5 to a sequence x_n in $R_k(r)$ that converges to r. □

Since the RPM has salience utilities, there are regions k and l so that $U_k = L_l = \{(x_1, x_2) : x_2 < x_1\}$ and $U_l = L_k = \{(x_1, x_2) : x_2 > x_1\}$; w^k is the largest weight on attribute 1 and w^l is the lowest. Without loss of generality, let $k = 1$ and $l = 2$.

Pick $r_0 \in X$ so that $u_1(r_0) = u_2(r_0)$. Note that $\{x : u_1(x) = u_2(x)\}$ is an upward sloping line, and for $\epsilon > 0$, $u_1(r_0 - (0, \epsilon)) > u_2(r_0 - (0, \epsilon))$ and $u_2(r_0 + (0, \epsilon)) > u_1(r_0 + (0, \epsilon))$. Noting $U_1 = L_2$ and $U_2 = L_1$, by the Lemmas and that the regions are dense:

$$\begin{aligned}
cl(R_1(r_0 + (\epsilon, 0))) &\supseteq \{x \in U_1 : x \gg r_0 + (\epsilon, 0)\} \\
cl(R_2(r_0 + (0, \epsilon))) &\supseteq \{x \in U_2 : x \gg r_0 + (0, \epsilon)\} \\
cl(R_1(r_0 + (0, \epsilon))) &\supseteq \{x \in L_1 = U_2 : x \ll r_0 + (0, \epsilon)\} \\
cl(R_2(r_0 + (\epsilon, 0))) &\supseteq \{x \in L_2 = U_1 : x \ll r_0 + (\epsilon, 0)\}
\end{aligned}$$

[18] If $\bar{y} = r$, then take $x_n = r$ for all n; otherwise, $d(x', r) < d(\bar{y}, r)$ implies $x' \in R_k(r)$.

Set $A_k^+ = \{x \in U_k : x \gg r_0\}$ and $A_k^- = \{x \in U_1 : x \ll r_0\}$ for $k = 1, 2$. Letting $\epsilon \to 0$ and applying continuity of each R_i, $cl(R_1(r_0)) \supseteq A_1^+ \bigcup A_2^-$ and $cl(R_2(r_0)) \supseteq A_1^- \bigcup A_2^+$.

For $k = 1, 2$, let O_k^+ and O_k^- be open neighborhoods contained in, but not equal to, $A_k^+ \bigcap [R_1(r_0) \bigcup R_2(r_0)]$ and $A_k^- \bigcap [R_1(r_0) \bigcup R_2(r_0)]$, respectively. For all r sufficiently close to r_0, $R_1(r) \supseteq O_1^+ \bigcup O_2^-$ and $R_2(r) \supseteq O_1^- \bigcup O_2^+$ using continuity of the regional function. Choose such an r for which $u_1(r) \neq u_2(r)$ that also belongs to U_1.

The set $R_1 = R_1(r) \bigcup \{r\}$ is connected, R_1 intersects U_1 and U_2 by above, and clearly $U_1 \bigcap U_2 = \emptyset$, so there exists $z \in R_1 \setminus [U_1 \bigcup U_2]$. Since $r \in U_1$, $z \in R_1(r)$ and since $R_1(R)$ is open, there exists $\epsilon > 0$ so that $A = \{x : z + (\epsilon, \epsilon) \gg x \gg z - (\epsilon, \epsilon)\} \subset R_1(r)$. Moreover, by Lemmas 3 and 5, we have that $R_2(r) \bigcap A' = \emptyset$ when

$$A' = \bigcup_{x \in A \cap L_1} \{y : y \ll x\} \cup \bigcup_{x \in A \cap U_1} \{y : y \gg x\} \cup A.$$

Let $A^* = A'^c \bigcap U_1^c$ be the points above A' and $A_* = A'^c \bigcap L_1^c$ be the points below A'. Note A' is open while A^* and A_* are closed.

Now, $A^* \bigcap R_2(r) \neq \emptyset$ and $A_* \bigcap R_2(r) \neq \emptyset$ since $R_2(r) \supseteq O_1^- \bigcup O_2^+$. Since $R_2(r) \bigcup \{r\}$ is connected, and $A^* \bigcup A_* \supset R_2(r) \bigcup \{r\}$, $A^* \bigcap A_*$ must be non-empty. But $A^* \bigcap A_* = \emptyset$ by construction so we have a contradiction. ■

A.7. **Examples from Table 1.** It is clear that BGS violates Monotonicity from Section 5.3, and Example 1 shows that BGS violates cancellation. It remains to show that TK violates Reference Irrelevance and that MO violates Cancellation. This is established by the following two examples.

Example 3 (TK violates Reference Irrelevance)**.** Consider a TK model with $\lambda_1 = \lambda_2 = 2$. Then, for $r = (10, 10)$, $x = (12, 12)$ and $y = (9, 16)$, $y \succsim_r x$ since $(12 - 10) + (12 - 10) = 2(9 - 10) + (16 - 10)$. For $r' = (11, 11)$, $x \succ_r y$ since $(12 - 11) + (12 - 11) > 2(9 - 11) + (16 - 11)$. But $x \in R_1^{GL}(r) \bigcap R_1^{GL}(r')$ and $r \in R_2^{GL}(r) \bigcap R_2^{GL}(r')$, so the family violates Reference Irrelevance.

Example 4 (MO violates Cancellation)**.** Let $Q(r) = \{x \in X : x_1/2 + x_2 > r_1/2 + r_2\}$ and $c(r) = 1$. Then, let $x = (2, 1)$, $y = (1, 2)$, $z = (4, 4)$, and $r = (0.9, 1.9)$. Since $(x_1, z_2) = (2, 4) \succsim_r (4, 2) = (z_1, y_2)$ and $(z_1, x_2) = (4, 1) \succsim_r (1, 4) = (y_1, z_2)$ because all four points belong to $Q(r)$, cancellation requires that $x \succsim_r y$. However, $x \notin Q(r)$, so $y \succ_r x$, so cancellation does not hold.

Appendix B. Behavioral Characterization for RPM

We provide a set of behavioral postulates characterizing RPM. This behavior represents the key features of the model. The first postulates states that changing the reference point does not alter the relative ranking of two alternatives as long as both of the alternatives lie in the same region.

Axiom (Weak Reference Irrelevance)**.** *For any $r, r' \in X$: if $x, y \in R_k(r) \bigcap R_k(r')$, then*

$$x \succsim_r y \iff x \succsim_{r'} y.$$

The next postulate states that the indifference curves are straight and parallel lines for a given region.

Axiom (Regional Linearity*)**.** *For any $r \in X$, $\alpha \in (0,1]$, $x, x\alpha y, y \in R_k(r)$, and $a, a\alpha b, b \in R_l(r)$: if $x \succsim_r a$ and $y \succsim_r b$, then $x\alpha y \succsim_r a\alpha b$, strictly whenever $x \succ_r a$.*

For a given region, Regional Linearity is equivalent to the usual linearity axiom (a close relative of the independence axiom) when preference is complete and transitive.

We assume that each product consists of desirable attributes. Monotonicity means that if a product x contains more of some or all attributes, but no less of any, than another product y, then x is preferred to y. The next postulate assumes that Monotonicity is maintained within a given region and reference point.

Axiom (Regional Monotonicity*)**.** *For any $r \in X$ and $x, y \in R_k(r)$: if $y \geq x$, then $y \succsim_r x$, strictly whenever $y \neq x$.*

The final two properties are continuity conditions. The first requires that, for a fixed reference point r, two sequences in the same region with the same limit behave similarly far enough along the sequence. That is, if $w \succ_r x_n \succ_r z$ for all large n, then $w \succ_r y_n \succ_r z$ as well for all large n. The second requires that for an unbounded regions have unbounded utilities. This is an implication of the affine increasing utility.

Axiom (Regional Continuity*)**.** *For any $w, x \in \bigcup_{l=1}^n R_l(r)$ and sequences $(y_n), (z_n) \in R_k(r)$: if (y_n) and (z_n) have the same limit and there exists N so that both $w \succ_r y_n$ and $y_n \succ_r x$ hold for all $n > N$, then for any $w', x' \in \bigcup_{j=1}^n R_l(r)$ with $w' \succ_r w$ and $x \succ_r x'$ there exists m so that $w' \succ_r z_m$ and $z_m \succ_r x'$.*

Axiom (Unbounded)**.** *For any $r \in X$: if $R_ki(r)$ is unbounded, then for any $x \in \bigcup_{l=1}^n R_l(r)$, there exists $x^* \in R_k(r)$ so that $x^* \succ_r x$.*

Our following result states that these five postulates characterize RPM.

Theorem 5. *For a regional function $\mathcal{R}$, $(\{\succsim_r\}_{r\in X}, \mathcal{R})$ satisfies Weak Reference Irrelevance, Regional Linearity*, Regional Monotonicity*, Regional Continuity* and Unbounded if and only if $\{\succsim_r\}_{r\in X}$ conforms to RPM under $\mathcal{R}$.*

Proof. First, we show the regional affine representation for each reference r. Second, we extend it across references. To save notation, until Lemma 14, we fix r and write R_k instead of $R_k(r)$ and $\succsim$ instead of $\succsim_r$.

Lemma 7. *For each R_k, there is an affine and increasing $\hat{v}_k : R_k \to \mathbb{R}$ so that for $x, y \in R_k$, $x \succsim y \iff \hat{v}_k(x) \geq \hat{v}_k(y)$.*

Proof. For each R_k, pick arbitrary $x^k \in R_k$ and $\epsilon_k > 0$ s.t. $B_{\epsilon_k}(x^k) \subset R_k$ and observe that $B_{\epsilon_k}(x^k)$ is a mixture space . By Regional Linearity* (RL) and Regional Continuity* (RC), $\succsim$ satisfies the mixture space axioms when restricted to $B_{\epsilon_k}(x^k)$, so let it have the representation v_k, normalized so that $v_k(x^k) = 0$. We now extend v_k outside of $B_{\epsilon_k}(x^k)$. For all $x \in R_k$, define $\hat{v}_k$ by $\hat{v}_k(x) = \frac{1}{\alpha} v_k(x\alpha x^k)$ for any $\alpha \in (0, 1]$ so that $x\alpha x^k \in B_{\epsilon_k}(x^k)$. To see $\hat{v}_k$ is well defined, suppose $x\alpha x^k, x\beta x^k \in B_{\epsilon_k}(x^k)$ and (WLOG) $\beta < \alpha$. Then, $x\beta x^k = (x\alpha x^k)\frac{\beta}{\alpha}x^k$, and since v_k is affine, $\frac{1}{\beta}v_k(x\beta x^k) = \frac{1}{\alpha}v_k(x\alpha x^k)$. □

Lemma 8. *If $x^k \in R_k$, $x^l \in R_l$, and $x^k \sim x^l$, then there is $\alpha > 0, \beta \in \mathbb{R}$ such that for $x \in R_k$ and $y \in R_l$, $x \succsim y \iff \hat{v}_k(x) \geq \alpha\hat{v}_l(y) + \beta$.*

Proof. WLOG, take $\hat{v}_k(x^k) = 0$. As above, there is $\epsilon_k > 0$ such that $B_{2\epsilon_k}(x^k) \subset R_k$. By Regional Monotonicity* (RM) and RC , there is $\epsilon_l > 0$ such that $B_{\epsilon_l}(x^l) \subset R_l$ and for all $y \in B_{\epsilon_l}(x^l)$, $x^* = x^k + \epsilon_k \succ y \succ x^k - \epsilon_k = x_*$. For any $y \in R_l$ and α such that $y\alpha x^l \in B_{\epsilon_l}(x^l)$, there exists $\beta \in (0, 1)$ such that $x^*\beta x_* \sim y\alpha x^l$ by RC, RM, and that $\succsim$ is a weak order. Let $V_l(y) = \alpha^{-1}\hat{v}_k(x^*\beta x_*)$. This is well defined for the same reason as above, and is also affine, increasing, and ranks alternatives in the same way as $\hat{v}_l$. Thus, $V_l(y) = a\hat{v}_l(y) + b$ for $a > 0$ and $b \in \mathbb{R}$.

For any $x \in R_k$ and $y \in R_l$, pick α such that $x\alpha x^k \in B_{\epsilon_k}(x^k)$ and $y\alpha x^l \in B_{\epsilon_l}(x^l)$. By construction, $y\alpha x^l \sim y'$ when $y' \in B_{\epsilon_k}(x^k)$ and $\hat{v}_k(y') = \alpha V_l(y)$. Thus, $x\alpha x^k \succsim y' \sim y\alpha x^l$ holds if and only if $\hat{v}_k(x) \geq V_l(y)$ and $x \succsim y \iff x\alpha x^k \succsim y\alpha x^l$ by RL since $x^k \sim x^l$, completing the proof. □

Definition 3. A finite sequence $(Q_1, ..., Q_{m+1})$ with each $Q_i \in \{R_1, ..., R_n\}$ is an *indifference sequence* (IS) if there exists $x^1, ..., x^m, y^1, ..., y^m$ with $x^k \in Q_k$, $y^k \in Q_{k+1}$ and $x^k \sim y^k$. The function v is a *utility for the indifference sequence* $(Q_1, ..., Q_m)$ if v is affine and increasing on each Q_k and for all k, $x, y \in Q_k \bigcup Q_{k+1}$: $x \succsim y \iff v(x) \geq v(y)$.

For an indifference sequence $(Q_1, ..., Q_m)$ with utility v, we label the range of utilities as $v(Q_k) = (l_k, u_k)$ where $l_k \leq u_k$. Note that we allow $Q_k = Q_l$ for $k \neq l$.

Lemma 9. *For an indifference sequence $(Q_1, ..., Q_m)$, there is an affine, increasing utility v for it.*

Proof. The proof is by induction. We claim that there is a utility $v^k : X \to \mathbb{R}$ that is utility for the IS $(Q_1, \ldots, Q_k)$ for any k. When $k = 1$ or $k = 2$, this is true by the above lemmas. The induction hypothesis (IH) is that the claim is true for $k = N$. Consider $k = N + 1$ and let v^N be the utility for $(Q_1, \ldots, Q_N)$ be index that exists by the IH. If $Q_N \subseteq \bigcup_{i=1}^N Q_i$, then we are done. If not, then let $\alpha_{N+1}, \beta_{N+1}$ be the scalars claimed to exist by Lemma 8 so that $\hat{v}_j = \alpha_{N+1}\hat{v}_l + \beta_{N+1}$ for $R_l = Q_N$ and $R_k = Q_{N+1}$. Restricted to Q_N, $\hat{v}_l$ agrees with v^N and is also affine, so there also exists α', β' so that

$$\hat{v}_l = \alpha' v^N + \beta'.$$

Define $v^{N+1}(x) = v^N(x)$ if $x \in \bigcup_{i=1}^N Q_i$ and

$$v^{N+1} = \alpha_{N+1}\alpha'\hat{v}^{N+1}(\cdot) + \alpha_{N+1}\beta' + \beta_{N+1}$$

otherwise. Then, if $l < N$ and $x, y \in Q_l \bigcup Q_{l+1}$, then we are done by the IH, since $v^{N+1}(x) \geq v^{N+1}(y) \iff v^N(x) \geq v^N(y)$. If $x, y \in Q_N \bigcup Q_{N+1}$, then Lemma 8 and construction implies the result. The claim then holds by induction.

□

Lemma 10. *Fix an indifference sequence* $(Q_1, ..., Q_n)$ *with utility* v. *If* $x^k \in Q_k$ *for* $k = i, i+1, i+2$ *with* $x^i \sim x^{i+1} \sim x^{i+2}$, *then* $(Q_1, ..., Q_i, Q_{i+2}, ..., Q_n)$ *is an indifference sequence (after relabeling) with utility* v.

Proof. The Lemma is vacuously true for any 1 or 2-element IS. Fix an IS $(Q_1, ..., Q_n)$ with $n \geq 3$ and v as above, and suppose $x^k \in Q_k$ for $k = i, i+1, i+2$ with $x^i \sim x^{i+1} \sim x^{i+2}$. By transitivity $x^i \sim x^{i+2}$, so $(Q_1, ..., Q_i, Q_{i+2}, ..., Q_n)$ is an IS; it remains to be shown that v is a utility for it. There is an $\epsilon > 0$ s.t. $B = B_\epsilon(v(x^i))) \subset (l_k, u_k)$ for $k = i, i+1, i+2$. Let $v^{-1}(u) : B \to Q_{i+1}$ be an arbitrary point in Q_{i+1} such that $v[v^{-1}(u)] = u$. Now, fix $x \in Q_i$ and $y \in Q_{i+2}$. For α small enough, $v(x\alpha x^i), v(y\alpha x^{i+2}) \in B$. Then $x\alpha x^i \sim v^{-1}(v(x\alpha x^i))$ and $y\alpha x^{i+2} \sim v^{-1}(v(y\alpha x^{i+2}))$. So

$$\begin{aligned} x \succsim y \quad &\iff \quad x\alpha x^i \succsim y\alpha x^{i+2} \\ &\iff \quad v^{-1}(v(x\alpha x^i)) \succsim v^{-1}(v(y\alpha x^{i+2})) \\ &\iff \quad v[v^{-1}(v(x\alpha x^i))] \geq v[v^{-1}(v(y\alpha x^{i+2}))] \\ &\iff \quad \alpha v(x) + (1-\alpha)v(x^i) \geq \alpha v(y) + (1-\alpha)v(x^{i+2}) \\ &\iff \quad v(x) \geq v(y) \end{aligned}$$

This establishes the Lemma. □

Lemma 11. *Fix an indifference sequence* $(Q_1, ..., Q_n)$ *with utility* v. *If* $(l_1, u_1) \bigcap (l_n, u_n) \neq \emptyset$, *then there exists* i *and* $x^k \in Q_k$ *for* $k = i, i+1, i+2$ *with* $x^i \sim x^{i+1} \sim x^{i+2}$.

Proof. If there is i with $(l_i, u_i) \bigcap (l_{i+2}, u_{i+2}) \neq \emptyset$, then there is $u \in \bigcap_{j=i,i+1,i+2} (l_j, u_j)$ so there exists $x_j \in Q_j$ with $v(x_j) = u$ for $j = i, i+1, i+2$ and thus by hypothesis, $x_i \sim x_{i+1} \sim x_{i+2}$. We show there exists such an i by contradiction. If $l_{i+2} > u_i$ for all i or $l_i > u_{i+2}$ for all i, then $(l_1, u_1) \bigcap (l_n, u_n) = \emptyset$, a contradiction. So there must exist i such that $[l_{i+2} > u_i$ and $l_{i+2} > u_{i+4}]$ or $[u_{i+2} < l_i$ and $u_{i+2} < l_{i+4}]$. In the first case, $l_{i+2} \in (l_{i+1}, u_{i+1}) \bigcap (l_{i+3}, u_{i+3})$; in the second, $u_{i+2} \in (l_{i+1}, u_{i+1}) \bigcap (l_{i+3}, u_{i+3})$. In either case, we have a contradiction. □

Lemma 12. *Fix an indifference sequence* $(Q_1, ..., Q_n)$ *with utility* v. *Then for all* $x, y \in \bigcup_i Q_i$, $x \succsim y \iff v(x) \geq v(y)$.

Proof. This is clearly true if $n = 1$. (IH) Suppose the claim is true for any IS with $m < n$ elements. Fix an IS $(Q_1, ..., Q_n)$ with utility v. If $x \notin Q_1 \bigcup Q_n$ or $y \notin Q_1 \bigcup Q_n$, then the claim immediately follows from the IH, and clearly holds if $x, y \in Q_i$ for some i. So it suffices to consider arbitrary $x \in Q_1$ and $y \in Q_n$. By Lemmas 10 and 11, if $(u_1, l_1) \bigcap (l_n, u_n) \neq \emptyset$, we can form a shorter IS from Q_1 to Q_n and the claim then follows from the IH.

There are two cases to consider: $l_n > u_1$ and $u_n < l_1$. If $l_n > u_1$, then there exists $y' \in Q_{n'}$ for $n' < n$ with $v(y') = l_n$. This follows from the construction of the indifference sequence. The range of v restricted to $\bigcup_{i=1}^{n-1} Q_i$ is $\bigcup_{i=1}^{n-1} (l_i, u_i)$, which is an open interval $(\bar{l}, \bar{u})$ since $(l_{t-1}, u_{t-1}) \bigcap (l_t, u_t) \neq \emptyset$ for each t. Because $x_{n-1} \sim y_n$, $(l_{n-1}, u_{n-1}) \bigcap (l_n, u_n) \neq \emptyset$. If $l_n \notin (\bar{l}, \bar{u})$, then $l_n > \bar{u} \geq u_1$ and the intersection above must be empty. Hence, $l_n \in (\bar{l}, \bar{u})$ and we can find $x \in Q_{n'}$ with $v(x) = l_n$ and $n' < n$.

By the IH, that $(Q_1, ..., Q_{n'})$ is an IS and $v(y') > v(x)$, $y' \succ x$. Similarly, since $(Q_{n'}, ..., Q_n)$ is an IS and $v(y) > v(y')$, $y \succ y'$. By transitivity, $y \succ x$ and since $v(y) > l_n > u_1 > v(x)$, the claim holds. Similar arguments obtain the desired conclusion when $u_n < l_1$. $\square$

Define the relation $\cong$ by $x \cong y \iff$ there exists an indifference sequence $(Q_1, ..., Q_m)$ with $x \in Q_1$ and $y \in Q_m$. It is easy to see that $\cong$ is an equivalence relation (reflexive and transitive). Let $[x]$ denote the $\cong$ equivalence class of x.

Lemma 13. *If $y \notin [x]$ and $x \succ y$, then $x' \succ y'$ for all $x' \in [x]$ and $y' \in [y]$.*

Proof. For regions R_i and R_j, either (i) there exists $x_i \in R_i$ and $x_j \in R_j$ so that $x_i \sim x_j$; or (ii) $x_i \succ x_j$ for all $x_i \in R_i$ and $x_j \in R_j$; or (iii) $x_j \succ x_i$ for all $x_i \in R_i$ and $x_j \in R_j$.

To see this, we first show that for any $x \in X$ and region i: the sets $\{y \in R_i : y \succ x\}$ and $\{y \in R_i : x \succ y\}$ are open. Consider $U_i(x) = \{y \in R_i : y \succ x\}$. If $U_i(x)$ is not open, then there exists a point $y_0 \in U_i(x)$ and a sequence of points y_n in R_i with $x \succsim y_n$ and $y_n \to y_0$ since R_i is open. Since R_i is open, there is $\epsilon > 0$ such that $y'_n = y_0 + \frac{\epsilon}{n} \in R_i$ for all $n \geq 1$. Observing that $y'_n \succ y_0 \succ x \succsim y_n$, since $y'_n \to y_0$, $y_n \to y_0$ and $y'_n \succ y_0$ for all n, RC implies that $y_m \succ x$ for some m, a contradiction. Similar arguments hold for the lower contour set.

Now, if neither (ii) nor (iii) holds, then after relabeling, there exist $x \in R_i$ and $y, z \in R_j$ such that $y \succ x \succ z$. Let $U_j(x)$ and $L_j(x)$ be the strict upper and lower contour sets of x in region j. Any point in $R_j \setminus [U_j(x) \bigcup L_j(x)]$ is indifferent to x, so either (i) holds or the set is empty. If empty, then $U_j(x) \bigcup L_j(x) = R_j$. Since $cl(R_j) = cl(U_j(x) \bigcup L_j(x)) = cl(U_j(x)) \bigcup cl(L_j(x))$ is connected, there exists $y_0 \in cl(U_j(x)) \bigcap cl(L_j(x))$. Then let $y_n \in U_j(x)$ and $z_n \in L_j(x)$ be such that $y_n, z_n \to y_0$. Pick an $\epsilon > 0$ so that $B_\epsilon(x) \subset R_i$. Then, $y_n \succ x$ for all n and $x \succ x - \epsilon/2$, so there is m such that $z_m \succ x - \epsilon/2$ by RC. By construction $x \succsim z_m$ and if $x \sim z_m$ we have established (i), so assume $x \succ z_m$. Note $B_\epsilon(x)$ is connected. Let U' and L' be the R_i strict upper and lower contour sets for z_m, intersected with $B_\epsilon(x)$. They are open, non-empty since $x \in U'$ and $x - \epsilon/2 \in L'$, non-intersecting subsets of $B_\epsilon(x)$. Since they cannot cover it, pick $x' \in B_\epsilon(x) \setminus [U' \bigcup L']$ and by Weak Order it must be indifferent to z_m.

Fix $x, y \in X$ with $y \notin [x]$ and $x \succ y$, and assume $x \in R_k$. Pick any $y' \in [y]$. By definition, there is an IS $(Q_1, ..., Q_m)$ with $y' \in Q_m$ and $y \in Q_1$. Let $i = 1$ and $y_1 = y$. If there exists $y'' \in Q_i$ with $y'' \succsim x$, then $y'' \succsim x \succ y_i$, so by the above arguments, we can find $z \in Q_i$ with $z \sim x'$ for $x' \in R_k$. If that occurs, then $(R_k, Q_i, ..., Q_1)$ is an IS and $y \in [x]$, a contradiction. Thus $x \succ y''$ for all $y'' \in Q_i$.

Now, there exists $y_{i+1} \in Q_{i+1}$ with $x \succ y_{i+1}$ by transitivity and definition of IS. Hence, we can apply above logic to Q_{i+1} as well: $x \succ y''$ for all $y'' \in Q_{i+1}$. Inductively, this extends all the way to Q_m, so $x \succ y'$ in particular. Since y' is arbitrary, this extends to any $y' \in [y]$.

Similar arguments show that $x' \succ y$ for any $x' \in [x]$. Combining, $x' \succ y'$ whenever $x' \in [x]$ and $y' \in [y]$. □

Let $A_1, ..., A_n$ be the distinct equivalence classes of $\cong$. By Lemma 13, these sets can be completely ordered by $\succ$, i.e. $A_i \succ A_j \iff x \succ y$ for all $x \in A_i$ and $y \in A_j$. WLOG, $A_1 \succ A_2 \succ ... \succ A_n$.

By Lemma 12, there is v_i on A_i so that v_i is affine and increasing on region contained in A_i and $x \succsim y \iff v_i(x) \geq v_i(y)$ for all $x, y \in A_i$. By Unbounded and Lemma 13, every unbounded region is a subset of A_1, so $v_i(A_i)$ is bounded for all $i > 1$. Define $V(x) = v_1(x)$ for all $x \in A_1$. For $x \in A_i$ with $i > 1$, define $V(x)$ recursively by

$$V(x) = v_i(x) - \sup_{y \in A_i} v_i(y) + \inf_{y \in A_{i-1}} V(y) - 1.$$

Observe $V(\cdot)$ is a positive affine transformation of $v_i(\cdot)$ when restricted to A_i, and if $x \in A_i$, $y \in A_j$ and $i > j$, then $V(x) > V(y)$. Thus V represents $\succsim$ and is affine and increasing when restricted to any given region.

Up to now, we fixed $r \in X$ and constructed a representation for $\succsim_r$. Since r is arbitrary, this establishes that each $\succsim_r$ has a representation $V(\cdot|r)$ that is affine and increasing on $R_i(r)$ for $i = 1, ..., n$. Denoting $u_j(\cdot|r)$ the restriction to $R_j(r)$, we have established the existence of an RPM if $u_j(\cdot|r')$ is a positive affine transformation of $u_j(\cdot|r)$ for any r, r'.

Lemma 14. *For $i = 1, ..., n$ and all $r, r' \in X$, there are $\alpha > 0$ and $\beta \in \mathbb{R}$ so that $u_i(\cdot|r') = \alpha u_i(\cdot|r) + \beta$.*

Proof. Write $u \approx v$ if there are $\alpha > 0$ and $\beta \in \mathbb{R}$ so that $u = \alpha v + \beta$. Pick any $r \in X$ and let

$$E = \{e \in X : u_i(\cdot|e) \approx u_i(\cdot|r)\}.$$

E is closed, since if $e_n \in E$ and $e_n \to e \in X$, then for any open ball $B \subset R_i(e)$, $R_i(e_n) \bigcap B \neq \emptyset$ for n large enough by lower semicontinuity of R_i. Observe $B' = R_i(e_n) \bigcap B \subset R_i(e_n) \bigcap R_i(e)$ is a mixture space and FI plus the usual uniqueness argument for affine utility functions gives that $u_i(\cdot|e_n) \approx u_i(\cdot|e)$. Now, consider $E' = cl(E^c)$. If $E' \bigcap E \neq \emptyset$, then the same argument as above with a sequence in E^c converging to $e \in E$ yields a contradiction. Hence $E' \bigcap E = \emptyset$ and $E \bigcup E' = X$. Since X is connected, $E' = \emptyset$ and $E = X$. □

Thus $u_i(\cdot|r)$ is an affine transformation of $u_i(\cdot|r')$ for all $r, r' \in X$ and $\{\succsim_r\}_{r \in X}$ conforms to RPM under $\mathcal{R}$. □

B.1. **Other models and RPM.** In this subsection, we present the functional forms of the other models of salience we discussed (adapted to be linear, in some cases, to aid comparability), and show that they are not RPM an so not BGS.

- Gabaix and Laibson [2006] assume that consumers choose between $(-p, -p^*)$ and $(-p-e, 0)$ where $(-p, -p^*)$ indicates purchasing an add-on while $(-p-e, 0)$ indicates paying an additional e to substitute away from the add-on. True utility is $u(x,y) = x + y$. Sophisticated agents and informed myopic agents maximize u, while uninformed myopic agents maximize $u_M(x,y) = x$.

- Bhatia and Golman [2013] assume that the DM chooses the bundle x that maximizes
$$U(x|r) = \alpha_1(r_1)[V(x_1) - V(r_1)] + \alpha_2(r_2)[V(x_2) - V(r_2)]$$
given that a reference point r, where each α_i is increasing and positive.
- Gabaix [2014] assumes a rational DM would maximize $u(a,w)$ but actually maximizes
$$u\left(a, (w_1 m_1^*, \dots, w_n m_n^*)\right)$$
where
$$m^* \in \arg\min_{m\in[0,1]^n} \frac{1}{2}\sum_{i,j}(1-m_i)\Lambda_{ij}(1-m_j) + \kappa\sum_i m_i^\alpha$$
where Λ_{ij} incorporates the "variance" in the marginal utility of dimensions i and j. When n is large, m_i^* is often zero, so $(w_1 m_1^*, \dots, w_n m_n^*)$ is a "sparse" vector.

All of the above fail to be RPM as the indifference curves have the same slope everywhere for a fixed reference. If they were RPM, then they would necessarily have only a single region. Single region RPM coincides with the neoclassical model.

Appendix C. Incomplete Data and BGS

This section shows how to use Theorem 1 to test whether the BGS model explains a DM's behavior with only incomplete data. An incomplete data set takes the following form:
$$\mathcal{D} \subset \{(S,x) \mid S \in \mathcal{X} \text{ and } x \in c(S)\}$$
We assume that the cardinality of $\mathcal{D}$ is finite. The goal is to provide a simple, testable condition under which the BGS model rationalizes the data.

The key to our approach is to follow Salience-SARP by treating alternatives with different salience distinctly. We also rely heavily on reflection, using it to replace each revealed comparison with one between two products that are both attribute 1-salient. Write $x' = (x_2, x_1)$ for the reflection of $x = (x_1, x_2)$. Observe that if $x \in R_1(S)$ and $y' \in S \bigcap R_2(S)$, then $wx_1 + (1-w)x_2 \geq wy_1 + (1-w)y_2$. Hence, the DM's choice reveals that an attribute 1-salient x is revealed preferred to an attribute 1-salient y, even though y need not belong to S (see Figure 5). We term this "revealed saliently preferred," and extend it to all possible combinations as follows.

Definition 4. A vector x is *revealed saliently preferred* to a vector y from the data set $\mathcal{D}$, written $xR^{\mathcal{D}}y$, if there exists $S \in \mathcal{X}$ so that one of the following holds:
(1) $(S,x) \in \mathcal{D}$, $x \in R_1(A(S))$ and $y \in S \bigcap R_1(A(S))$
(2) $(S,x) \in \mathcal{D}$, $x \in R_1(A(S))$ and $y' \in S \bigcap R_2(A(S))$
(3) $(S,x') \in \mathcal{D}$, $x \in R_2(A(S))$ and $y \in S \bigcap R_1(A(S))$
(4) $(S,x') \in \mathcal{D}$, $x \in R_2(A(S))$ and $y' \in S \bigcap R_2(A(S))$

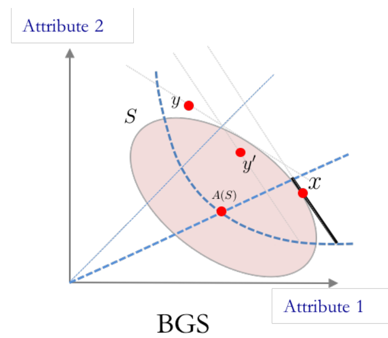

FIGURE 5. Revealed Saliently Preferred Relation

The axiom ensure that $R^{\mathcal{D}}$ has no cycles that reveal Monotonicity violations. Let $\bar{R}$ be the transitive closure of $R^{\mathcal{D}}$, i.e. $y^1 \bar{R} y^n$ only if there exist $y^2, \ldots, y^{n-1}$ so that $y^1 R^{\mathcal{D}} y^2 R^{\mathcal{D}} \ldots R^{\mathcal{D}} y^n$.

Axiom 8 (Salience GARP)**.** For any $\alpha \in [0,1]$ and $x, y, \hat{x}, \hat{y} \in X$: if $x\bar{R}y$ and $\hat{x}\bar{R}\hat{y}$, then either $\alpha x_1 + (1-\alpha)\hat{x}_1 \geq \alpha y_1 + (1-\alpha)\hat{y}_1$ or $\alpha x_2 + (1-\alpha)\hat{x}_2 \geq \alpha y_2 + (1-\alpha)\hat{y}_2$.

Also, $R^{\mathcal{D}}$ values dimension 1 more than dimension 2.

Axiom 9 (SDO)**.** If $xR^{\mathcal{D}}y$ and $x_2 \geq y_2$, then $x_2 - y_2 > y_1 - x_1$.

These two properties characterize the data sets that are rationalizable by the BGS model.

Theorem 6. *The finite data set $\mathcal{D}$ satisfies Salience GARP and SDO if and only if is rationalized by the BGS model.*

Proof of Theorem 6. Let $A = \left\{x - y : x\bar{R}y\right\}$. The goal is to find a separating hyperplane between A and the set $B = B_1 \bigcup B_2$ where $B_1 = \{x \in \mathbb{R}^2 : x_1 < 0 \leq x_2 \ \&\ x_1 \leq -x_2\}$ and $B_2 = \{x \in \mathbb{R}^2 : x_1 < 0 \ \&\ x_2 < 0\}$. Note that A intersects neither B_2 (by Salience-GARP) nor B_1 (by SDO). We must show that $co(A)$ does not intersect the interior of B.

We show that $bd(co(A)) \bigcap \mathbb{R}_{--} = \emptyset$ implies $co(A) \bigcap \mathbb{R}_{--} = \emptyset$. Suppose not, so $bd(co(A))$ does not intersect $\mathbb{R}_{--}$ but there exists $\bar{a} \in co(A) \bigcap \mathbb{R}_{--}$. By Caratheodory's thoerem, $\bar{a} = \alpha a^1 + \beta a^2 + (1-\alpha-\beta)a^3$ for a^1, a^2, a^3 extreme points (and so on the boundary) of A. If $a^i \geq 0$ for some i, an immediate contradiction is obtained. Moreover, if all three lie in the same quadrant, so is any convex combination of the three. Hence WLOG it suffices to consider the case where $a^1_1 > 0 > a^1_2$ and $a^2_2, a^3_2 > 0 > a^2_1, a^3_1$.

There exists an increasing affine function H so that $H(a_1) = H(\bar{a})$. If $H(a^2), H(a^3) > H(\bar{a})$, then $H(\bar{a}) > H(\bar{a})$, so WLOG, $H(a^2) < H(\bar{a})$. Consider $Q = \{\alpha a^1 + (1-\alpha)a^2 : \alpha \in [0,1]\}$. Since $H(x) < H(\bar{a})$ for $x \in Q \setminus \{a^1\}$ and H is increasing, $Q \bigcap \{x : x \geq \bar{a}\}$ is empty. Now, Q intersects $\{x \in X : \bar{a} \gg x\}$ since Q is a connected set and intersects both the disjoint open sets $\{x \not\gg a : x_1 > \bar{a}_1\}$ and $\{x \not\gg a : x_2 > \bar{a}_2\}$. But this contradicts our hypothesis that $bd(co(A)) \bigcap \mathbb{R}_{--} = \emptyset$.

Then, the boundary of $co(A)$ does not intersect B_2 by Salience GARP, so $co(A)$ does not intersect B_2. Suppose $co(A)$ intersects B_1. By SDO, there are $x, x' \in A$ so that $co(\{x, x'\}) \bigcap B_1 \neq \emptyset$. WLOG, $x_1 \leq 0 \leq x_2$ and $x'_1 \geq 0 \geq x'_2$. Consider the line that passes through the two points. If its first dimensional intercept is positive, then the line does not pass through B_1 since $x \notin B_1$ by SDO. So the intercept must be negative. But then it passes through B_2, contradicting Salience-GARP.

To wrap up, use the separating hyperplane theorem to get a linear function l that separates A from the interior of B. this passes through 0 since 0 belongs to boundary of both sets. Also, we can take $||l||_\infty = 1$, and $l(1,0) = w$ and $l(0,1) = 1 - w$, and $w \geq 1/2$ since $B_1 \subset B$. Then we have constructed our representation: if $(S, x) \in \mathcal{D}$ and $x \in R_1(A(S))$, then $x\bar{R}y$ when $y \in S \bigcap R_1(A(S))$ and $x\bar{R}y'$ when $y \in S \bigcap R_2(A(S))$. Then, $l(x - y) \geq 0$ and $l(x) \geq l(y)$ in the first case and $l(x) \geq l(y')$ in the second place. Similar arguments with x' replacing x cover $(S, x) \in \mathcal{D}$ and $x \in R_2(A(S))$. $\square$