# ON DYNAMIC CONSISTENCY IN AMBIGUOUS GAMES

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Abstract. I consider static, incomplete information games where players may not be ambiguity neutral. Every player is one of a finite set of types, and each knows her own type but not that of the other players. Ex ante, players differ only in their taste for outcomes. If every player is dynamically consistent with respect to her own information structure and respects consequentialism, then players act as if expected utility for uncertainty about types. The result extends to $\alpha$-maxmin expected utility preferences where players have the same perception of uncertainty but different attitudes towards it.

## 1. Objectives

Recently, the theory of incomplete information games (Harsanyi, 1967-8) has been extended to allow players to exhibit uncertainty averse behavior and applied to economic settings such as auctions, mechanism design and voting. This paper formalizes a modeling trade-off for such games, showing that uncertainty aversion poses difficulties for some of expected utility's particularly appealing properties, including dynamic consistency (DC). The following three properties imply no player is ambiguity averse with respect to the uncertainty about her own and other players' types: (i) DC, i.e. a strategy optimal conditional on each of a given player's types is also ex ante optimal for that player; (ii) consequentialism, i.e. each player updates her preferences in a manner independent of other players' strategy choices and ignores counter-factual signal realizations; and (iii) common ex ante behavior,

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i.e. each player has the same preferences (up to tastes) before observing her type, a generalization of common priors.

The main result is general, in that it applies to all standard models incorporating uncertainty, and has several immediate implications for mechanism design under ambiguity. While many of these games do not explicitly model an ex ante stage, their interpretation typically admits an ex ante stage with (at least) a participation decision. The result suggests, for instance, that satisfaction of the ex post participation constraints need not imply ex ante participation constraint. Furthermore, renegotiation may be part of the optimal contract, even when contracts are complete. In games whose interpretation does not admit any ex ante decisions, DC is particularly important for welfare comparisons. Violations of DC imply ex post welfare may differ substantially from ex ante.

Moreover, it implies that much of the literature that focuses on the strategic interaction of ambiguity averse agents with incomplete information studies agents that violate one of the three properties. For instance, Theorem 1 shows that at least one of these properties fails in (discretized versions of) nearly all of the literature on auctions and multi-agent mechanism design with ambiguity aversion, including Salo and Weber (1995), Lo (1998), Bose et al. (2006), Chen et al. (2007), Bose and Daripa (2009), Bodoh-Creed (2012), Bose and Renou (2014), Ellis (2016), and Renou (2015).[1] In other contexts, various authors have explored agents that explicitly violate either DC or consequentialism; for instance, Baliga et al. (2013) study polarization and ambiguity by maintaining DC but relaxing consequentialism. The results also provide restrictions on the ex ante behavior of agents in non-strategic interactions with uncertainty, such as the general equilibrium model of Condie and Ganguli (2011).

While it is well-known that unrestricted DC poses difficulties for ambiguity sensitive models, the approach pioneered by Epstein and Schneider (2003b) requires DC

[1]While non-discretized versions of the above papers are not covered by the result, it is easy to see that DC fails for the common sets of priors implicitly or explicitly assumed for the agents in those papers as well. The sets of priors in these papers have a product structure that fails rectangularity, as noted by Epstein and Schneider (2003a).

to hold only for the information structure in the game.[2] In a single player decision problem, this allows for ambiguity to play a non-trivial role, and the additivity obtained here need not hold. This assumption is also appealing in a strategic setting with a fixed information structure, but the results herein show that it is much more restrictive in an interactive setting. On the one hand, DC can be maintained by allowing the update rule to depend on the equilibrium strategy profile, e.g. Hanany and Klibanoff (2007). Hanany et al. (2016) develop an equilibrium concept based on this idea for games when players have preferences a la Klibanoff et al. (2005), for which there is no issue with ex ante versus ex post for welfare, equilibrium, or choice. In the games considered by the present paper, their solution concept reduces to solving the (ex ante) normal form.[3] On the other hand, while DC has very strong normative appeal, violations thereof are well documented. Modeling agents who fail to satisfy DC requires assumptions regarding sophistication (see e.g. Siniscalchi (2011)), especially if the interpretation of the game allows for (potentially unmodeled) ex ante decisions or requires ex ante welfare evaluation. Of course, under the assumption that agents are completely naive, violations of DC can be ignored.

I extend the main result to a model that permits a parametric distinction between perception of ambiguity and the agents attitude thereto, the $\alpha$-maxmin expected utility model. This allows relaxation of the assumption of Common Ex Ante Behavior. Such a relaxation may be desirable because the assumption implies both perception and attidudes are identical. While one can justify common perception of ambiguity as in Harsanyi (1967-8), common attitude towards uncertainty requires additional justification. Under expected utility, Common Ex Ante Behavior is equivalent to assuming a common prior, which is ubiquitous in the literature. I show that my main result remains true when players have the same perception of uncertainty but differ in their attitude towards it. A full study of how to best relax Common Ex Ante Behavior is beyond the scope of this paper.

[2]See e.g. Epstein and Le Breton (1993), Ghirardato (2002), and the debate between Al-Najjar and Weinstein (2009) and Siniscalchi (2009).

[3]Their setting is richer in that they allow sequential moves, and in these cases, the reduction to normal form need not obtain. In particular, they consider the appropriate analog to sequential equilibrium for these setting.

## 2. Incomplete Information Games

Following Harsanyi (1967-8), I model incomplete information games as follows; for related formulations with ambiguity aversion, see e.g. Lo (1999), Kajii and Ui (2005), or Azrieli and Teper (2011). Let $I = \{1, ..., n\}$ with $n \geq 2$ be the set of players, $T_0$ be a finite set of states of nature, $T_i$ be a finite set of types of player $i$ where $\#T_i \geq 2$ for all $i > 0$, and $T = T_0 \times T_1 \times ... \times T_n$ be the set of states of the world. As is standard, denote $T_{-i} = T_0 \times ... \times T_{i-1} \times T_{i+1} \times ... \times T_n$ and $(t_i, t'_{-i})$ for the state where Player $i$'s type is $t_i$ and the type of Player $j \neq i$ is $t'_j$. The space of outcomes of the game is a convex set $X$, such as the set of lotteries over action profiles. It will sometimes be convenient to write $\Delta Z$ for the set of all finite support probability measures on a set $Z$. As standard, Player $i$ learns her type before choosing a strategy.[4]

I abstract away from the formal details of the game and equilibrium. Instead, I consider each player's preference over acts that map $T$ to $X$. Every pair of a strategy profile and a mechanism or game with the above information structure maps to one such an act. Denote player $i$'s ex ante preference over acts by $\succeq^i_0$ and her preference conditional on learning that her type is $t_i \in T_i$ by $\succeq^i_{t_i}$. Each $\succeq^i_{t_i}$ and $\succeq^i_0$ are complete and transitive, and $\succeq^i_0$ has a utility representation $U^i_0(\cdot)$.

### 2.1. Assumptions.

I impose three assumptions on the each player's preference.

**Assumption 1** (Consequentialism)**.** For every $i \in I$ and every $t_i \in T_i$, if $f(t_i, t_{-i}) = g(t_i, t_{-i})$ for all $t_{-i} \in T_{-i}$, then $f \sim^i_{t_i} g$.

**Assumption 2** (Dynamic Consistency)**.** For every $i \in I$, if $f \succeq^i_{t_i} g$ for all $t_i \in T_i$, then $f \succeq^i_0 g$. If in addition there exists $t'_i \in T_i$ such that $f \succ^i_{t'_i} g$, then $f \succ^i_0 g$.

**Assumption 3** (Full Support)**.** For all $i \in I$, any $f \in \mathcal{F}$ and any $t \in T$, there exist outcomes $x, y \in X$ such that $f_x \succ^i_0 f_y$ where $f_w(t')$ equals $w$ when $t' = t$ and equals $f(t')$ otherwise for $w = x, y$.

The first two assumptions are adapted from Epstein and Schneider (2003b), and the third is ubiquitous in the work on updating to avoid conditioning on a null event. Consequentialism implies that each player cares only about the outcomes in states

[4]Formally, the information of Player $i$ is modeled as a filtration $\mathcal{F}^i = \{\mathcal{F}_0, \mathcal{F}^i_1\}$ where $\mathcal{F}_0 = \{T\}$ and $\mathcal{F}^i_1 = \{T_0 \times T_1 \times ... \times T_{i-1} \times \{t\} \times T_{i+1} \times ... \times T_n : t \in T_i\}$.

that agree with her information.[5] DC says that if player $i$ knows she will prefer $f$ to $g$ regardless of the signal she receives, then she prefers $f$ to $g$ ex ante as well. In contrast to Ghirardato (2002) or Epstein and Le Breton (1993), this version of DC applies only to the particular information structure in the game considered. By themselves, these assumptions do not restrict the perception of ambiguity perceived by player $i$ about her own type. In fact, the results of Epstein and Schneider (2003b) imply that for any set of marginal probability distributions over $T_i$, there exists a DM satisfying the above assumptions for set of priors with that set of marginals.

Our next assumption explicitly uses the structure imposed by the game.

**Assumption 4** (Common Ex Ante Behavior)**.** There exists an interval $B \subseteq \mathbb{R}$, a continuous $U_0 : B^T \to \mathbb{R}$, and a family of continuous, onto functions $\{u_i : X \to B\}_{i=1}^n$ so that $U_0^i(f) = U_0(u_i \circ f)$ for all $i$ and all $f$.

This assumption generalizes the common prior assumption typical of expected utility. In fact, whenever $U_0(\cdot)$ is an expected utility function, Common Ex Ante Behavior holds if and only if the priors are identical and the range of each $u_i$ is the same. The normalization that the range of every $u_i$ is the same is harmless if $U_0$ is maxmin expected utility, but may entail some loss of generality with other models of ambiguity aversion. Note also that every paper in mechanism design or auction theory cited above has quasi-linear utility, implying that the range of each $u_i(\cdot)$ equals $\mathbb{R}$.

It is easy to verify that Common Ex Ante Behavior allows for players whose ex ante preference accommodates all standard models of ambiguity aversion, including:

- $U_0(\psi) = \int \psi d\pi$ (Savage, 1954),
- $U_0(\psi) = \min_{\pi \in \Pi} \int \psi d\pi$ (Gilboa and Schmeidler, 1989, henceforth, MEU),
- $U_0^i(\psi) = \alpha \min_{p \in C} \int \psi dp + (1-\alpha) \max_{p \in C} \int \psi dp$ ($\alpha$-MEU) and
- $U_0(\psi) = \int \phi(\int \psi d\pi)\mu(d\pi)$, $\mu \in \Delta(\Delta T)$ (Klibanoff et al., 2005).

In any of these models, one can easily state Common Ex Ante Behavior solely in terms of the preference relation.

[5] Because of this assumption, one could have defined ex post preference on acts that depend only on the states consistent with received information. I opt not to do so for notational ease.

2.2. **Result.** The main result shows that under the assumptions, agents do not exhibit Ellsberg behavior when the actions considered depend only on the type profile. To state it, say that an act $f$ does not depend on Nature's type if $f$ is $f(\tau_0, t_{-i}) = f(\tau_0', t_{-i})$ for all $\tau_0, \tau_0' \in T_0$ and $t_{-i} \in T_{-i}$.[6] Then, the type of every player suffices to determine its outcome. The result shows that the DM acts as if expected utility over these acts.

**Theorem 1.** *Under Assumptions 1-4, for uncertainty about players' types, each player is additive. That is, for any $i \in I$ there are continuous functions $\{v_t\}_{t\in T}$ such that for any acts $f, g$ that do not depend on Nature's type:*

$$f \succeq_0^i g \iff \sum_{t\in T} v_t(f(t)) \geq \sum_{t\in T} v_t(g(t)).$$

Additivity immediately rules out violations of the sure thing principle (P2 of Savage (1954)) and other anomalies for expected utility. Hence, DC and common ex-ante behavior greatly limit the scope for ambiguity in games. No ambiguity is perceived about the types of players, only about Nature's type.

Moreover, if players are MEU, then ex ante preference is classical subjective expected utility for acts that do not depend on Nature's type.[7]

**Corollary 1.** *If players satisfy Assumptions 1-4 and $U_0$ is MEU with set of priors $\Pi$, then for any $t_{-0} \in T_{-0}$ and any $\pi, \pi' \in \Pi$, $\pi(T_0 \times \{t_{-0}\}) = \pi'(T_0 \times \{t_{-0}\})$.*

Not all ambiguity about Nature's type is permitted. The assumptions imply a stark structure on the set of priors: there exists a probability measure $q \in \Delta T_{-0}$ and a closed, convex $\Pi_{t_{-0}} \subset \Delta(T_0 \times \{t_{-0}\})$ for every $t_{-0} \in T_{-0}$ so that $\Pi$ equals

$$\left\{ \sum_{t_{-0}\in T_{-0}} q(t_{-0})\pi_{t_{-0}}(\cdot) : \pi_{t_{-0}} \in \Pi_{t_{-0}} \forall t_{-0} \in T_{-0} \right\}.$$

Not only is the set of priors rectangular (Epstein and Schneider, 2003b) with respect to the partition generated by players' types, agents have a common prior, $q$, about the distribution of players' types. Indeed, if Nature has no types, i.e. $T_0$ is a singleton, then agents are actually (state independent) subjective expected utility.

[6] Equivalently, $f$ is measurable with respect to the algebra $\{\emptyset, T_0\} \times \prod_{i=1}^n 2^{T_i}$.

[7] The result extends to any $U_0$ satisfying $U_0(f + k) = U_0(f) + k$ whenever $k \in \mathbb{R}$ and $f, f + k \in B^T$. This condition is analogous to Maccheroni et al. (2006)'s weak certainty independence axiom.

The proof relies on tools developed by Gorman (1968) that have had numerous applications in economics.[8] The closest of these to the present is Epstein and Seo (2011), which exploits the connection of DC and Consequentialism with a property of events that Gorman calls Separability.[9] They show that exchangeable models have separable singletons, implying an additive representation. In the present paper, no such exchangeability exists, and singletons need not be separable when $T_0$ is not a singleton. Instead, I rely on a second result of Gorman to show that the representation must be additive on "components" of the state space which include all sets of the form $T_0 \times \{t_{-0}\}$.

*Proof of Theorem 1.* Suppose Assumptions 1-4 are satisfied. By construction, $U_0$ satisfies Gorman (1968)'s P1 and P2. Full support implies Gorman's P4 (which implies his P3). For any $E \subset T$ and $f, g \in B^T$, define $fEg$ by the element of $B^T$ so that $fEg(t) = f(t)$ if $t \in E$ and $fEg(t) = g(t)$ if $t \notin E$. A set $E \subset T$ is *separable* if for any $x, y, z, z' \in B^T$,

$$U_0(xEz) \geq U_0(yEz) \iff U_0(xEz') \geq U_0(yEz').$$

Consequentialism and DC imply that every set of the form $\{t : t_i = \tau\}$ for any $\tau \in T_i$ is separable.

Say that $A, E \subseteq T$ *overlap* if they intersect and neither contains the other. Gorman's Theorem 1 states that if $A, E \subseteq T$ overlap and are separable, $A \bigcup E$ is separable. For each $t \in T_{-0}$, define

$$Q_t = \{\tau \in T : t_i \neq \tau_i \text{ for all } i > 0\}.$$

For any $\hat{t} \in T_{-0}$, I claim $Q_{\hat{t}}$ is separable. Fix arbitrary $\hat{t} \in T_{-0}$ and set $N = \prod_{i=1}^{n}(\#T_i - 1)$ and order the player-type pairs via $n : \{1, ..., N\} \to I$ and $\tau : \{1, ..., N\} \to \cup_{i \in I} T_i$ with the following properties: (i) for $i \in I$, $n(i) = i$; (ii) for all $k$, $\tau(k) \in T_{n(k)}$; and (iii) for every $i$, $\{\tau(k) : n(k) = i\} = T_i \setminus \hat{t}_i$. That is: (i) the first $I$ players listed by $n$ are all distinct, (ii) $\tau(k)$ is a possible type for player $n(k)$, and (iii) for player $i$, all types of $i$ except $\hat{t}_i$ occur exactly once.

[8]See e.g. Wakker (1989), Epstein and Seo (2011) and Mongin and Pivato (2015).

[9]To keep the paper self-contained, the appendix includes statements of Gorman's theorems used in the proof.

Let $A_0 = \emptyset$ and for every $k \in \{1, ...., N\}$, define

$$E_k = \{t : t_{n(k)} = \tau(k)\}$$

and

$$A_k = \cup_{k'=1}^{k} E_k = A_{k-1} \cup E_k.$$

Every $A_k$ is separable. Clearly $A_1 = E_1$ is separable by the above. Suppose that $A_{k-1}$ is separable for $k > 1$. If $A_{k-1}$ and $E_k$ overlap, then Gorman's Theorem 1 implies that $A_k$ is separable. $A_{k-1} \cap E_k \neq \emptyset$ since it contains $t^1$ where $t^1_{n(k)} = \tau(k)$ and $t^1_i = \tau(i)$ for all $i \in I \setminus n(k)$, $t^1 \in A_{k-1} \cap E_k$ for $t_0 \in T_0$. $E_k \not\subset A_{k-1}$ because $t^2$ with $t^2_{n(k)} = \tau(k)$ and $t^2_i = \hat{t}_i$ for all $i \in I \setminus n(k)$ belongs to $E_k$ but not $A_{k-1}$. Finally, $A_{k-1} \not\subset E_k$ since $t^3$ with $t^3_{n(k)} = \hat{t}_{n(k)}$ and $t^3_{\bar{i}} = \tau(\bar{i})$ for $\bar{i} = \min I \setminus \{n(k)\}$ belongs to $A_{k-1}$ but not $E_k$. Thus the sets overlap, and successive application yields that $A_N = Q_{\hat{t}}$ is separable.

Say that $A \subset T$ is a *top element* if $A \neq T$, $A$ is separable, and the only separable $E \supset A$ without equality is $T$ itself. Let $\mathcal{B} = \{C_1, ..., C_m\}$ be the set of top elements. Fix distinct $t, t' \in T_{-0}$ and let $C^t, C^{t'} \in \mathcal{B}$ contain $Q_t, Q_{t'}$ respectively. Note $Q_t \bigcup Q_{t'} = T$ so if $Q_{t'} \subset C^t$, then $C^t = T$, a contradiction. Hence $C^t \neq C^{t'}$. Now, $C^t \bigcap C^{t'} \neq \emptyset$ so, combining with $\#T_{-0} \geq 4$, $\#\mathcal{B} \geq 4$, conclude $\mathcal{B}$ satisfies the hypothesis of Gorman's Theorem 2. Applying the result and noting $T_0 \times \{t\} = \bigcup \{T \setminus C_i : Q_t \subseteq C_i,\ C_i \in \mathcal{B}\}$ obtains the desired representation. $\square$

## 3. Discussion

All of our assumptions are important for the result to hold. For instance, if consequentialism is relaxed, Hanany et al. (2016) provide a framework compatible with assumptions 2-4. Many models are compatible with all assumptions except either DC or Common Ex Ante Behavior. Less obviously, full-support is important for the result. Its role is to rule out that one can reduce a multi-agent problem to a single agent one.[10] For instance, suppose that $T_1 = T_2 = \{R, B\}$ and that $U_0$ is MEU. If the set of priors equals

$$\Pi = \{\pi(RR) = p \text{ and } \pi(BB) = 1 - p : p \in [\underline{p}, \bar{p}]\}$$

[10] Of course, one can make assumptions weaker than full support that rule such behavior out.

then all players satisfy Assumptions 1, 2 and 4. However, she violates full support.

The most compelling assumption to relax is Common Ex Ante Behavior. As noted in the introduction, this assumption implies that both perception of and attitude towards uncertainty are the same for all players. In this subsection, I consider maintaining the same perception of uncertainty while allowing players to have different attitudes said uncertainty. I focus on the $(\alpha, C)$-MEU model because it provides a separation of the two.[11] A player has a $(\alpha_i, C)$-MEU representation for $\alpha_i \in [0,1]$ and a closed, convex $C \subseteq \Delta(T)$ if her preference is represented by

$$U_0^i(u_i \circ f) = \alpha_i \min_{p \in C} \int u_i \circ f dp + (1-\alpha_i) \max_{p \in C} \int u_i \circ f dp.$$

Intuitively, $C$ represents perception of ambiguity and $\alpha$ represents the agents attitude towards that ambiguity, with a lower $\alpha$ reflecting more uncertainty averse behavior.

I relax Common Ex Ante Behavior as follows.

**Assumption 5.** There exist sets $C, C_t \subset \Delta T$ and an $\alpha_i \in [0,1] \setminus \{\frac{1}{2}\}$ for every $i \in I$ such that for all $i \in I$:

(i) $\succeq_0^i$ has a $(\alpha_i, C)$-MEU representation, and

(ii) $\succeq_{t_i}^i$ has a $(\alpha_i, C_{t_i})$-MEU representation for each $t_i \in T_i$.

Because $C$ is constant across players, each perceives the same ambiguity ex ante. The posterior beliefs, $C_{t_i}$, must relate to $C$ through DC. A necessary, but not sufficient, condition for DC and Consequentialism is that $C_{t_i}$ results from prior-by-prior Bayesian updating of $C$ (Lemma 2). However, the attitude towards ambiguity, $\alpha_i$, may vary from player to player. Thus players have the same perception of ambiguity but differ in their attitude towards it.

To conclude, I show that replacing Common Ex Ante Behavior with the above assumption implies existence of a common prior over players' types.

**Theorem 2.** *Under Assumptions 1-3 and 5, for any $t \in T_{-0}$ and any $\pi, \pi' \in C$, $\pi(T_0 \times \{t\}) = \pi'(T_0 \times \{t\})$.*

[11] The other prominent model featuring a separation between the two is the smooth model. However, Klibanoff et al. (2009) show that the recursive version of the model has a form that depends on the information structure of the agent. Indeed, the ex ante preference of such a model does not fall into the static version of the smooth model unless ambiguity plays no role.

The result suggests that common perception of uncertainty, as opposed to common attitude towards ambiguity, drives the impossibility result. To prove the result, I show that if an event is separable, in the sense of Gorman, for an $(\alpha_i, C)$-MEU DM, then it also is separable for a $(1, C)$-MEU DM. The result then follows from applying Corollary 1 to the hypothetical game where all players are $(1, C)$-MEU and so Common Ex Ante Behavior is satisfied.

*Proof of Thoerem 2.* To save space, for any measure $p$ over all subsets of $T$ and vector $f \in \mathbb{R}^T$, write $p(f)$ for $\int f dp$. As in Theorem 1, identify acts with real vectors. For any measure $p$ and event $B$ define another measure $p_B$ by $p_B(A) = p(A \cap B)$.[12]

I first prove two Lemmas. The first shows that fixing an $\alpha_i \neq \frac{1}{2}$ (and tastes), the resulting set of priors $C$ is unique.[13]

**Lemma 1.** *If a preference $\succeq$ has both $(\alpha, \mathcal{C})$-MEU and $(\alpha, \mathcal{D})$-MEU representations for closed, convex $\mathcal{C}, \mathcal{D} \subseteq \Delta T$ and $\alpha \neq \frac{1}{2}$, then $\mathcal{C} = \mathcal{D}$.*

*Proof.* This follows from the following adaptation of the second part of the proof of Proposition 20 in GMM. If $\mathcal{C} \neq \mathcal{D}$, then one can use a separating hyperplane theorem to pick $\varphi \in \mathbb{R}^T$ such that $\bar{c} = \max_{p \in \mathcal{C}} p(\varphi) \neq \max_{p \in \mathcal{D}} p(\varphi) = \bar{d}$. As in GMM, define $\min_{p \in \mathcal{C}} p(\varphi) = \underline{c}$ and $\min_{p \in \mathcal{D}} p(\varphi) = \underline{d}$. The second (unnumbered) display equation on page 172 of GMM requires $\bar{d} + \underline{d} = \bar{c} + \underline{c}$ and GMM's Eq. (B.3) with $\alpha = \beta \neq \frac{1}{2}$ requires $\bar{d} - \underline{d} = \bar{c} - \underline{c}$. Together, this requires that $\bar{d} = \bar{c}$, contradicting that $\max_{p \in \mathcal{C}} p(\varphi) \neq \max_{p \in \mathcal{D}} p(\varphi)$. $\square$

Now, I show that $C$ is rectangular with respect to each player's partition.

**Lemma 2.** *Assumptions 1-3 and 5 imply that for any $i \in I$ and $\tau_i \in T_i$, the event $\{t \in T : t_i = \tau_i\}$ is $\succeq'$-separable for $\succeq'$, when $\succeq'$ has a $(1, C)$-MEU representation.*

*Proof of Lemma 2.* The result is trivial if $\alpha_i \in \{0, 1\}$, so consider only $\alpha \equiv \alpha_i \neq 0, 1, \frac{1}{2}$. Fix arbitrary $i \in I$ and $t_i \in T_i$, and let $V$ represent $\succeq_0^i$, $V_{i,t_i}$ represent $\succeq_{t_i}^i$, and $E = \{\tau \in T : t_i = \tau_i\}$. Consider any $f \in \mathbb{R}_+^T$. Note $fE0 \sim_0^i V_{i,t}(f)E0$ and

[12] Note $p_B$ may not be a probability measure even if $p$ is.

[13] See e.g. Ghirardato et al. (2004) (henceforth, GMM) for discussion and interpretation. Eichberger et al. (2011) show that the $C$ obtained by GMM is unique only if $\alpha$ equals 0 or 1.

$fE0 \succeq_0^i xE0$ if and only if $V_{i,t}(f) \geq x$. In particular,

$$\begin{aligned} V(fE0) &= \alpha \min_{p\in C} p_E(f) + (1-\alpha)\max_{p\in C} p_E(f) \\ &= V(V_{i,t_i}(f)E0) = [\alpha \min_{p\in C} p(E) + (1-\alpha)\max_{p\in C} p(E)]V_{i,t}(f) \end{aligned}$$

so setting $p^* = \alpha \min_{p\in C} p(E) + (1-\alpha)\max_{p\in C} p(E)$ and defining

$$C^*_{t_i} = \{\frac{p_E(\cdot)}{p^*} : p \in C\}$$

gives an $(\alpha, C^*_{t_i})$-MEU representation of $\succeq^i_{t_i}$. Applying Lemma 1 gives that $C^*_t = C_t$ (the argument above goes through even if there exists $\mu \in C^*_t$ with $\mu(T) < 1$). This requires that $p(E) = p^*$ for all $p \in C$ and thus $C_t = \{p(\cdot|E) : p \in C\}$.

Now for an arbitrary $g$,

$$\begin{aligned} V(g) = V(V_{i,t_i}(g)Eg) =& \alpha \min_{p\in C}[p^* V_{i,t_i}(g) + (1-p^*)p(g|E^c)] \\ & + (1-\alpha)\max_{p\in C}[p^* V_{i,t_i}(g) + (1-p^*)p(g|E^c)] \\ =& p^*[\alpha \min_{q\in C} q(g|E) + (1-\alpha)\max_{q\in C} q(g|E)] + \\ & + (1-p^*)[\alpha \min_{p\in C} p(g|E^c) + (1-\alpha)\max_{p\in C} p(g|E^c) \\ =& \alpha \min_{p\in D} p(g) + (1-\alpha)\max_{p\in D} p(g) \end{aligned}$$

where

$$D = \{p^* q(\cdot|E) + (1-p^*)q'(\cdot|E^c) : q, q' \in C\}.$$

Thus $\succeq_0^i$ has both $(\alpha, C)$-MEU and $(\alpha, D)$-MEU representations. Lemma 1 implies $D = C$. Applying Epstein and Schneider (2003b) yields that $E$ is $\succeq'$-separable. □

Using Lemma 2, $C$ must be rectangular with respect to each player's filtration. We can apply Corollary 1 to get the conclusion. □

## Appendix A. Statements of Gorman's Theorems

All statements assume P1-P4, as stated in the Proof of Theorem 1. The clauses not used by the proofs are omitted.

**Theorem** (Gorman, 1968, Theorem 1)**.** *If $A, B \subseteq T$ overlap and are separable, then $A \cup B$ and $A \cap B$ are separable.*

For a vector $\psi \in B^T$ and $E \subset T$, let $\psi_E$ be its projection onto $B^E$.

**Theorem** (Gorman, 1968, Theorem 2)**.** *For $m \geq 3$, let $\mathcal{B} = \{C_1, ..., C_m\}$ be the collection of top elements and $E_j = T \setminus C_j$ for each $j$. If two distinct elements of $\mathcal{B}$ intersect, then $\{E_1, \ldots, E_m\}$ is a partition of $T$ and $U(\psi) = \sum_{j=1}^m U_j(\psi_{E_j})$, perhaps after a normalization.*

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