Tutorial

The Pairwise Likelihood Method for Structural Equation Modelling with ordinal variables and data with missing values using the R package lavaan

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Contents

1 Introduction

The aim of this tutorial is twofold: first, to introduce the theory of the pairwise likelihood (PL) method of estimation and inference for factor analysis models (FAM) and structural equation models (SEM) with ordinal variables and data either completely observed or with missing values; and second, to demonstrate the PL method by analyzing real data from the European Social Survey (ESS) (http://www.europeansocialsurvey.org/) using the R package lavaan (R Core Team, 2013; Rosseel, 2017; Rosseel et al., 2016; Rosseel, 2012). It assumes basic knowledge of factor analysis and structural equation modelling with ordinal variables. Some resources for more detailed presentations of these models are Bartholomew et al. (2011); Jöreskog (2002); Muthén (1984). Familiarity with R software and the R package lavaan is a bonus but it is not necessary.

The structure of the document is as follows: Section 2 presents briefly FAM and SEM with ordinal variables and how these models are extended to accommodate a multi-group analysis. Section 3 presents the PL estimation and inference theory for single-group and multi-group analysis including the case of data with missing values. The inference tools discussed are the z-test, the Wald test, the pairwise likelihood ratio test (PLRT) for testing the overall fit of a model and for testing nested models, and the model selection criteria, PL-AIC and PL-BIC. Section 4 mentions the conventional estimation method for FAM and SEM with ordinal variables, that is the three-stage weighted least squares (3S-WLS); provides a brief comparison between the latter and the PL; and explains why the maximum likelihood (ML) method is not considered. Section 3 and 4 could be skimmed through or even skipped by those who are more interested in how to apply the PL method in lavaan and how to interpret the results. Section 5 demonstrates the PL method by analyzing real data from the ESS. The section starts by introducing the R package lavaan and the ESS data analyzed. All R commands are provided with thorough explanations. Parts of the outputs are presented and interpreted. The full outputs of all analyses are given in the Appendix. Along with this file, two more files are provided, the "ESS5Police.RData", where the ESS data are saved, and the script file "Tutorial_Rcommands.r", which contains all the R commands provided in this document.

In this document, we focus on the case where the observed variables are ordinal because the PL has been so far implemented in lavaan for this case. It is only matter of extending the lavaan code to the case of mixed type of variables, ordinal and continuous. Otherwise, the PL theory presented here is applicable to SEM with mixed type of variables.

The core literature this document is based on is: Varin et al. (2011) who provide an overview of the composite likelihood (CL) methods which is the family the PL belongs to; Katsikatsou et al. (2012) who apply the PL estimation to FAM with ordinal variables and study its finite-sample properties and its performance in comparison to that of the 3S-WLS method; Pace et al. (2011) who present a general framework of PLRT; Varin & Vidoni (2005) and Gao & Song (2009) who, respectively, extend the model selection criteria AIC and BIC to the CL framework; Katsikatsou & Moustaki (2016) who develop the PLRT for testing the overall fit of SEM and for testing nested SEM with ordinal variables, and study its finite-sample properties and its performance in comparison to the fit statistics used under the 3S-WLS method; Molenberghs et al. (2011) who provide a thorough discussion of pseudo-likelihood estimation methods under missing at random (MAR) data; and Katsikatsou & Moustaki (2017) who develop the PL estimation for FAM with ordinal variables and MAR data and study its finite-sample properties. Also, Katsikatsou (2013) presents the PL estimation for SEM with mixed type of variables, ordinal and continuous, and for SEM with ranking data, and provides all the related formulae. The above references and the references therein indicate the attention the PL has attracted as a viable alternative to ML when the latter is intractable or computationally infeasible.

2 Structural Equation Models (SEM) with ordinal variables

2.1 The model

FAM and SEM are used in behavioral sciences when the variables of interest are constructs such as skills, attitudes, beliefs, status, etc. These variables are often referred to as factors or latent variables and are denoted by η below. The latent variables cannot be measured directly due to lack of a natural measurement instrument, but they can measured indirectly through variables that represent different aspects / nuances of the construct of interest. They are referred to as indicators or items and are denoted by y below. In Social Sciences, they usually take the form of a question or a statement respondents are asked to present their level of agreement. An example of a latent variable is “environmentally friendly behavior” and possible indicators could be recycling, preference to biological products, etc. The modelling framework used for measuring the latent variables through their indicators and studying the associations between the latent variables is given below.

Let $\mathbf{y}$ be a p-dimensional random vector of ordinal variables, including binary ones, $\mathbf{y} = (y_1, \ldots, y_p)'$. A SEM with ordinal variables adopts the underlying response variable (URV) approach (Jöreskog, 2002; Muthén, 1984), where an ordinal (or binary) variable y_i is assumed to be the manifestation of an underlying continuous variable $y_i^\star$ and y_i falls into the response category a, $a = 1, \ldots, c_i$, if and only if the underlying continuous $y_i^\star$ is between the thresholds, $\tau_{i,a-1}$ and $\tau_{i,a}$, i.e.

$$y_i = a \Longleftrightarrow \tau_{i,a-1} < y_i^\star < \tau_{i,a} \, , \tag{1}$$

where $-\infty = \tau_{i,0} < \tau_{i,1} < \ldots < \tau_{i,c_i-1} < \tau_{i,c_i} = +\infty$. Since only ordinal information is available, the distribution of $y_i^\star$ is determined only up to a monotonic transformation. In practice, it is usually assumed that $y_i^\star$ follows a standard normal distribution, $y_i^\star \sim \mathcal{N}(0,1)$, and the thresholds are free parameters to be estimated. An alternative parameterisation where the mean and variance of a $y_i^\star$ are free to be estimated is possible as long as two of its thresholds are fixed to 0 and 1 to define its scale (Jöreskog, 2002). The p-dimensional vector of the underlying continuous variables $\mathbf{y}^\star$ is typically assumed to follow a multivariate normal distribution, where

$$\mathbf{y}^\star \sim \mathcal{N}_p(\mathbf{0}, P) \, , \tag{2}$$

with P being a correlation matrix. The correlations between $y^\star$'s are often referred to as polychoric correlations or tetrachoric correlations if $y^\star$'s are binary. The model defined by (1) and (2) is often called the unconstrained model because no parametric structure is imposed on P and thresholds. Let $\boldsymbol{\vartheta}$ be the parameter vector of the unconstrained model which includes the non-redundant elements of P, denoted by $\boldsymbol{\rho}$, and the thresholds of all variables, denoted by $\boldsymbol{\tau}$; i.e. $\boldsymbol{\vartheta}' = (\boldsymbol{\rho}', \boldsymbol{\tau}')$.

A structural equation model (SEM) imposes a parametric structure on P and in specific applications on $\boldsymbol{\tau}$. Along with (1), it is defined by the following equations as well:

$$\mathbf{y}^\star = \Lambda\boldsymbol{\eta} + \boldsymbol{\varepsilon}, \tag{3}$$

$$\boldsymbol{\eta} = \mathrm{B}\boldsymbol{\eta} + \boldsymbol{\zeta}, \tag{4}$$

where $\boldsymbol{\eta}$ is a q-dimensional vector of continuous latent variables, Λ is a $p \times q$ matrix of factor loadings, B is a $q \times q$ regression coefficient parameter matrix, $I - \mathrm{B}$ is a non-singular matrix with I being the identity matrix, $\boldsymbol{\varepsilon}$ and $\boldsymbol{\zeta}$ are the vectors of error terms with $\boldsymbol{\varepsilon} \sim \mathcal{N}_p(\mathbf{0}, \Theta_\varepsilon)$ and $\boldsymbol{\zeta} \sim \mathcal{N}_q(\mathbf{0}, \Psi)$, and $Cov(\boldsymbol{\eta}, \boldsymbol{\varepsilon}) = Cov(\boldsymbol{\eta}, \boldsymbol{\zeta}) = Cov(\boldsymbol{\varepsilon}, \boldsymbol{\zeta}) = \mathbf{0}$. The elements of vector $\boldsymbol{\varepsilon}$ are often referred to as measurement errors and the elements of vector $\boldsymbol{\zeta}$ are often referred to as residuals. Equation (3) is often referred to as the measurement part of the model, while Equation (4) as the structural part of the model. The matrix Λ provides information on how well an item measures the intended latent variable and is the primal focus of the measurement part. The matrices B and Ψ provide information on the associations between the latent variables and are the primal focus of the structural part. The model parameter vector $\boldsymbol{\theta}$ to be estimated includes the free parameters in Λ, B, Θ_ε, Ψ, and $\boldsymbol{\tau}$. The parametric structure implied by the model on P is

$$P = \Lambda (I - B)^{-1} \Psi \left[\Lambda (I - B)^{-1}\right]' + \Theta_\varepsilon \,. \tag{5}$$

To identify the model, the scale for each η needs to be defined. This is usually done by fixing the mean of η to 0 and the variance of the corresponding residual ζ to 1. Alternative constraints are possible (e.g. instead of fixing the variance of the corresponding ζ to 1, to fix the loading of one of the indicators of η to 1). Moreover, the number of the free SEM parameters, i.e. the dimension of vector $\boldsymbol{\theta}$, should be smaller than the number of of the free parameters of the unconstrained model, i.e. the dimension of vector $\boldsymbol{\vartheta}$. This identification requirement is necessary but not sufficient to identify a SEM (Bollen, 1989).

When Equation (4) reduces to $\boldsymbol{\eta} = \boldsymbol{\zeta}$, the model is referred to as a factor analysis model (FAM). In this model, the associations between the latent variables are summarized by the matrix Ψ.

2.2 The model for multi-group analysis

Let x be an observed grouping variable with G groups in total. A typical example of such variable is country, gender, age group, education level, etc. When the research interest lies on studying possible differences in the distribution of $\boldsymbol{\eta}$ and / or $\mathbf{y}^\star$ between the G groups, a multi-group SEM can be employed. The model is exactly the same as above with the difference that a superscript g, $g = 1, \ldots, G$, is added to all variables and parameters (Jöreskog, 2002; Muthén, 1984). The multi-group analysis can be seen as an extension of a SEM with covariates added to equations (3) and (4). The main difference is that adding covariates in these equations we can only account for mean differences in $\mathbf{y}^\star$ and $\boldsymbol{\eta}$ for different values of the covariates, while the multi-group analysis can easily accommodate differences in both the means and the variances of the variables. On the other hand, multi-group analysis requires that the grouping variable is a categorical one while covariates added to (3) and (4) can be both categorical and continuous.

2.3 The model with mixed type of variables (continuous and ordinal)

A SEM with mixed type of indicators, continuous and ordinal including binary ones, and covariates is described by Muthén (1984). As said in the Introduction Section, in this document we focus on SEM with ordinal indicators only because the lavaan code for the PL has not yet been written to cover the more general case.

3 Pairwise likelihood (PL) estimation and inference for SEM with ordinal variables

3.1 Definition of the PL estimator and its asymptotic properties

To define the PL estimator, let $f(\mathbf{y};\boldsymbol{\theta})$ be the density function of the variable vector $\mathbf{y}$, where $\boldsymbol{\theta}$ is an s-dimensional parameter vector. The pairwise likelihood function, $\mathcal{PL}$, is defined as the weighted product of the bivariate (second-order) likelihood functions over all pairs of variables. In particular, the contribution of a single observation n to $\mathcal{PL}$ is defined as:

$$\mathcal{PL}_n(\boldsymbol{\theta};\mathbf{y}_n) = \prod_{i=1}^{p-1}\prod_{j=i+1}^{p}\left[f(y_{ni},y_{nj};\boldsymbol{\theta})\right]^{w_{ij}} ,$$

where w_{ij} is a non-negative weight to be chosen for the bivariate likelihood $f(y_i,y_j;\boldsymbol{\theta})$. The weights can be ignored if they are equal. Unequal weights may be chosen to improve efficiency and rise naturally in certain applications such as in the analysis of clustered data, time series, and spatial statistics (Varin, 2008; Varin et al., 2011). For a simple random sample of N observations of $\mathbf{y}$, the $\mathcal{PL}$ is defined as:

$$\mathcal{PL}(\boldsymbol{\theta};(\mathbf{y}_1,\ldots,\mathbf{y}_N)) = \prod_{n=1}^{N}\mathcal{PL}_n(\boldsymbol{\theta};\mathbf{y}_n) .$$

Similarly to the ML method, the value of $\boldsymbol{\theta}$ that maximizes $\mathcal{PL}$ is defined to be the PL estimator, $\hat{\boldsymbol{\theta}}_{PL}$. To facilitate the maximization, the objective function considered is the pairwise log-likelihood, pl. For a simple random sampling of N observations, this is:

$$\begin{aligned} pl(\boldsymbol{\theta};(\mathbf{y}_1,\ldots,\mathbf{y}_N)) &= \sum_{n=1}^{N} pl_n(\boldsymbol{\theta};\mathbf{y}_n) \\ &= \sum_{n=1}^{N}\sum_{i=1}^{p-1}\sum_{j=i+1}^{p}\left[\log f(y_{ni},y_{nj};\boldsymbol{\theta})\right]^{w_{ij}} . \end{aligned} \tag{6}$$

The asymptotic properties of the PL estimator are derived from the composite likelihood (CL) theory since the PL is a member of this general family (Varin et al., 2011). Actually, the ML is a special case of CL methods and the ML results can be generalized to CL estimators. The shared asymptotic properties between ML and CL are those of consistency and normality. The main difference is that the ML estimator is the most efficient or at least as efficient as a CL estimator (Lindsay, 1988; Mardia et al., 2009; Pagui et al., 2015; Varin et al., 2011).

Applying the theory of CL methods to the PL, it holds that

$$\sqrt{N}\left(\hat{\boldsymbol{\theta}}_{PL}-\boldsymbol{\theta}\right) \to \mathcal{N}_s\left(0, G^{-1}(\boldsymbol{\theta})\right) , \tag{7}$$

where $G(\boldsymbol{\theta})$ is the Godambe information matrix, also known as the sandwich information matrix. The latter is defined as $G(\boldsymbol{\theta}) = H(\boldsymbol{\theta})J^{-1}(\boldsymbol{\theta})H(\boldsymbol{\theta})$, where $H(\boldsymbol{\theta}) = E\left\{-\frac{\partial^2}{\partial\boldsymbol{\theta}'\partial\boldsymbol{\theta}}pl(\boldsymbol{\theta};\mathbf{y})\right\}$ and $J(\boldsymbol{\theta}) = Var\left\{\frac{\partial}{\partial\boldsymbol{\theta}'}pl(\boldsymbol{\theta};\mathbf{y})\right\}$. Sample estimates of $H(\boldsymbol{\theta})$ and $J(\boldsymbol{\theta})$ are

$$\hat{H}(\hat{\boldsymbol{\theta}}_{PL}) = -\frac{1}{N}\left.\frac{\partial^2}{\partial\boldsymbol{\theta}'\partial\boldsymbol{\theta}}pl(\boldsymbol{\theta};(\mathbf{y}_1,\ldots,\mathbf{y}_N))\right|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}_{PL}} \tag{8}$$

and

$$\hat{J}(\hat{\boldsymbol{\theta}}_{PL}) = \frac{1}{N}\sum_{n=1}^{N}\left(\frac{\partial}{\partial\boldsymbol{\theta}'}pl_n(\boldsymbol{\theta};\mathbf{y}_n)\bigg|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}_{PL}}\right)\left(\frac{\partial}{\partial\boldsymbol{\theta}'}pl_n(\boldsymbol{\theta};\mathbf{y}_n)\bigg|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}_{PL}}\right)', \tag{9}$$

respectively. Thus a sample estimate of $G(\boldsymbol{\theta})$ is $\hat{G}(\hat{\boldsymbol{\theta}}_{PL}) = \hat{H}(\hat{\boldsymbol{\theta}}_{PL})\left[\hat{J}(\hat{\boldsymbol{\theta}}_{PL})\right]^{-1}\hat{H}(\hat{\boldsymbol{\theta}}_{PL})$.

The direct implication of (7) is that hypothesis testing for $\boldsymbol{\theta}$ (or elements of $\boldsymbol{\theta}$) can be done in a similar way as in ML, using a z-test or a Wald test. For example, let θ_k be the kth element of the parameter vector $\boldsymbol{\theta}$, $k = 1, \ldots, s$, for which we want to test the hypothesis

$$H_0 : \theta_k = 0$$
$$H_1 : \theta_k \neq 0\,.$$

Let $\hat{\theta}_{PL,k}$ be the PL estimate of θ_k (i.e. be the kth element of $\hat{\boldsymbol{\theta}}_{PL}$ estimate) and let $se(\hat{\theta}_{PL,k})$ be its standard error, where $se(\hat{\theta}_{PL,k}) = \frac{1}{\sqrt{N}}\left[\hat{G}^{-1}(\hat{\boldsymbol{\theta}}_{PL})\right]_{kk}$ and $\left[\hat{G}^{-1}(\hat{\boldsymbol{\theta}}_{PL})\right]_{kk}$ is the element of $\hat{G}^{-1}(\hat{\boldsymbol{\theta}}_{PL})$ at the kth row and kth column. To test the null hypothesis, a z-test statistic can be computed as

$$\frac{\hat{\theta}_{PL,k}}{se(\hat{\theta}_{PL,k})}$$

the value of which is compared to the standard normal distribution. The p-value of the test statistic is $\Pr\left(Z > \left|\frac{\hat{\theta}_{PL,k}}{se(\hat{\theta}_{PL,k})}\right|\right)$, where $|.|$ denotes the absolute value. The $a\%$ confidence interval for θ_k is $\hat{\theta}_{PL,k} \pm z_{1-a/2} * se(\hat{\theta}_{PL,k})$, where $z_{1-a/2}$ is the $1 - \frac{a}{2}$ quantile of the standard normal distribution.

The Wald test can be used to simultaneously test a hypothesis for more than one parameters. For example, let $\boldsymbol{\theta}_{km}$ be the 2×1 subvector of $\boldsymbol{\theta}$ that includes the kth and mth elements of $\boldsymbol{\theta}$, $k \neq m = 1, \ldots, s$. Let the hypothesis be

$$H_0 : \boldsymbol{\theta}_{km} = \mathbf{0}$$
$$H_1 : \boldsymbol{\theta}_{km} \neq \mathbf{0}\,.$$

The Wald test is defined as

$$\frac{1}{\sqrt{N}}\hat{\boldsymbol{\theta}}'_{PL,km}\left[\hat{G}^{-1}(\hat{\boldsymbol{\theta}}_{PL})\right]_{km}\hat{\boldsymbol{\theta}}_{PL,km}\,, \tag{10}$$

where $\hat{\boldsymbol{\theta}}_{PL,km}$ is the PL estimate of $\boldsymbol{\theta}_{km}$ and $\left[\hat{G}^{-1}(\hat{\boldsymbol{\theta}}_{PL})\right]_{km}$ is the 2×2 estimated asymptotic covariance matrix of $\hat{\boldsymbol{\theta}}_{PL,km}$ (consisting of the elements of $\hat{G}^{-1}(\hat{\boldsymbol{\theta}}_{PL})$ located at the kth row and kth column, mth row and mth column, and kth row and mth column). The value of the Wald test statistic in (10) is compared to a χ^2_2. Recall that, in general, the degrees of freedom is equal to the number of parameters tested. The p-value is $\Pr\left(X > \frac{1}{\sqrt{N}}\hat{\boldsymbol{\theta}}'_{PL,km}\left[\hat{G}^{-1}(\hat{\boldsymbol{\theta}}_{PL})\right]_{km}\hat{\boldsymbol{\theta}}_{PL,km}\right)$, where $X \sim \chi^2_2$.

3.2 PL estimation for SEM with ordinal variables

This section describes how the PL estimation described above can be applied to SEM with ordinal variables. For SEM with cross-sectional data, there is no obvious reason to weigh the bivariate likelihood functions differently and the weights can be dropped from Equation (6). For ordinal indicators, the exact form of $f\left(y_{ni}, y_{nj}; \boldsymbol{\theta}\right)$ in (6) is:

$$\log f\left(y_{ni}, y_{nj}; \boldsymbol{\theta}\right) = \sum_{a=1}^{c_i} \sum_{b=1}^{c_j} I\left(y_{ni} = a, y_{nj} = b\right) \ln \pi\left(y_{ni} = a, y_{nj} = b; \boldsymbol{\theta}\right) \ , \tag{11}$$

where $I\left(y_{ni} = a, y_{nj} = b\right)$ is an indicator variable indicating whether y_{ni} and y_{nj} fall into categories a and b, respectively, and $\pi\left(y_{ni} = a, y_{nj} = b; \boldsymbol{\theta}\right)$ is the corresponding model probability. The latter, based on (1) and (2), is written as:

$$\begin{aligned}\pi\left(y_{ni} = a, y_{nj} = b; \boldsymbol{\theta}\right) &= \int_{\tau_{i,a-1}}^{\tau_{i,a}} \int_{\tau_{j,b-1}}^{\tau_{j,b}} f\left(y^{\star}_{ni}, y^{\star}_{nj}; \boldsymbol{\theta}\right) dy^{\star}_{ni} dy^{\star}_{nj} \\ &= \Phi_2\left(\tau_{i,a}, \tau_{j,b}; \rho_{ij}\right) - \Phi_2\left(\tau_{i,a-1}, \tau_{j,b}; \rho_{ij}\right) - \Phi_2\left(\tau_{i,a}, \tau_{j,b-1}; \rho_{ij}\right) + \Phi_2\left(\tau_{i,a-1}, \tau_{j,b-1}; \rho_{ij}\right) \ ,\end{aligned} \tag{12}$$

where ρ_{ij} is the polychoric correlation between $y^{\star}_i$ and $y^{\star}_j$, and $\Phi_2\left(\tau_1, \tau_2; \rho\right)$ is the bivariate cumulative normal distribution with correlation ρ evaluated at the point (τ_1, τ_2).

For a multi-group analysis with G groups and independent group samples, the pairwise log-likelihood function is written as:

$$pl\left(\boldsymbol{\theta}; \left(\mathbf{y}^{(1)}_1, \ldots, \mathbf{y}^{(1)}_{N_1}, \mathbf{y}^{(G)}_1, \ldots, \mathbf{y}^{(G)}_{N_G}\right)\right) = \sum_{g=1}^{N} pl_g\left(\boldsymbol{\theta}; \left(\mathbf{y}^{(g)}_1, \ldots, \mathbf{y}^{(g)}_N\right)\right) \ , \tag{13}$$

where $pl_g\left(\boldsymbol{\theta}; \left(\mathbf{y}^{(g)}_1, \ldots, \mathbf{y}^{(g)}_N\right)\right)$ is defined in the same way as in (6) with the difference that we need to add a superscript to y's and a subscript to N, both of value g.

3.3 Pairwise Likelihood Ratio Test (PLRT)

The pairwise likelihood ratio test (PLRT) statistic is an extension of the standard likelihood ratio test (LRT) derived under ML to the case of PL estimation (Katsikatsou & Moustaki, 2016; Pace et al., 2011). Here we present the PLRT for testing the overall fit of a SEM and the PLRT for testing nested SEMs. A detailed presentation on how the tests are derived can be found in Katsikatsou & Moustaki (2016), whose simulation study indicates that the PLRT has a satisfactory performance with respect to type I and type II errors and is competitive to the test statistics derived under the 3S-WLS method.

3.3.1 Testing the overall fit of a SEM

To test the overall fit, we compare the hypothesized SEM defined by equations (1), (3), and (4) against the unconstrained model for $\mathbf{y}^{\star}$ defined by (1) and (2). Essentially, we test the parametric structure imposed on the polychoric correlation matrix P by the hypothesized SEM given in (5). Usually no parametric structure is imposed on thresholds and then they are treated as nuisance parameters. For this, let the partition of $\boldsymbol{\theta}$, $\boldsymbol{\theta} = (\boldsymbol{\varphi}', \boldsymbol{\tau}')'$, where $\boldsymbol{\varphi}$ is the vector of all SEM parameters excluding the thresholds. Let $\boldsymbol{\varphi}$ be of dimension t, $t < s$, where is s is the dimension of $\boldsymbol{\theta}$. Also, let the partition of $\boldsymbol{\vartheta}$ (the parameter vector of the unconstrained model), $\boldsymbol{\vartheta} = (\boldsymbol{\rho}', \boldsymbol{\tau}')'$ with $\boldsymbol{\rho}$ being of dimension $\tilde{p} = p(p-1)/2$. The hypothesis for overall fit can be written as:

$$\begin{aligned}&H_0 : \boldsymbol{\rho} = g(\boldsymbol{\varphi}) \\ &H_1 : \boldsymbol{\rho} \text{ unconstrained}\end{aligned}$$

where g is a model-dependent function and $g : \mathbb{R}^t \to \mathbb{R}^{\tilde{p}}$. The PLRT statistic for overall fit, $PLRT_{SEM}$, is defined in a similar way as the standard LRT under ML; that is

$$PLRT_{SEM} = 2\left(pl\left(\hat{\boldsymbol{\vartheta}}_{PL}\right) - pl\left(\hat{\boldsymbol{\theta}}_{PL}\right)\right), \tag{14}$$

where $\hat{\boldsymbol{\vartheta}}_{PL}$ and $\hat{\boldsymbol{\theta}}_{PL}$ are the PL estimates under H_1 and H_0, respectively. Under H_0, the asymptotic distribution of $PLRT_{SEM}$ is a chi-squared distribution the degrees of freedom of which can only be approximated. Using the Satterthwaite approximation (also referred to as the first and second moment adjustment), under H_0, it holds that:

$$\alpha\left(\boldsymbol{\theta}\right) PLRT_{SEM} \overset{\text{app}}{\to} \chi^2_{df(\boldsymbol{\theta})} \,. \tag{15}$$

We will refer to $\alpha\left(\boldsymbol{\theta}\right) PLRT_{SEM}$ as the adjusted $PLRT_{SEM}$, the adjustment is $\alpha\left(\boldsymbol{\theta}\right) = \frac{\alpha_1(\boldsymbol{\theta})}{0.5*\alpha_2(\boldsymbol{\theta})}$, and the degrees of freedom are $df\left(\boldsymbol{\theta}\right) = \frac{[\alpha_1(\boldsymbol{\theta})]^2}{0.5*\alpha_2(\boldsymbol{\theta})}$, where

$$\alpha_1\left(\boldsymbol{\theta}\right) = \mathrm{tr}\left\{G^{\rho\rho}\left(\boldsymbol{\vartheta}\right)\left[H^{\rho\rho}\left(\boldsymbol{\vartheta}\right)\right]^{-1}\right\} - \mathrm{tr}\left\{G^{\varphi\varphi}\left(\boldsymbol{\theta}\right)\left[H^{\varphi\varphi}\left(\boldsymbol{\theta}\right)\right]^{-1}\right\} \,,$$

$$\begin{aligned}\alpha_2\left(\boldsymbol{\theta}\right) = {} & 2\mathrm{tr}\left\{G^{\rho\rho}\left(\boldsymbol{\vartheta}\right)\left[H^{\rho\rho}\left(\boldsymbol{\vartheta}\right)\right]^{-1} G^{\rho\rho}\left(\boldsymbol{\vartheta}\right)\left[H^{\rho\rho}\left(\boldsymbol{\vartheta}\right)\right]^{-1}\right\} + \\ & 2\mathrm{tr}\left\{G^{\varphi\varphi}\left(\boldsymbol{\theta}\right)\left[H^{\varphi\varphi}\left(\boldsymbol{\theta}\right)\right]^{-1} G^{\varphi\varphi}\left(\boldsymbol{\theta}\right)\left[H^{\varphi\varphi}\left(\boldsymbol{\theta}\right)\right]^{-1}\right\} - \\ & 4\mathrm{tr}\left\{\left[M\left(\boldsymbol{\varphi}\right)\right]'\left[H^{\rho\rho}\left(\boldsymbol{\vartheta}\right)\right]^{-1} M\left(\boldsymbol{\varphi}\right) G^{\varphi\varphi}\left(\boldsymbol{\theta}\right)\left[H^{\varphi\varphi}\left(\boldsymbol{\theta}\right)\right]^{-1} G^{\varphi\varphi}\left(\boldsymbol{\theta}\right)\right\} \,, \text{ and}\end{aligned}$$

$M\left(\boldsymbol{\varphi}\right) = \frac{\partial}{\partial \boldsymbol{\varphi}} g\left(\boldsymbol{\varphi}\right)$. The G and H matrices are the ones defined in Section 3.1. When the matrices are functions of $\boldsymbol{\vartheta}$, they refer to the unconstrained model and the superscript $\rho\rho$ denotes that only the parts of the matrices referring to the polychoric correlations $\boldsymbol{\rho}$ are used. The G and H matrices which are functions of $\boldsymbol{\theta}$ refer to the hypothesized SEM and the superscript $\varphi\varphi$ denotes that only the part of the matrices referring to all parameters except thresholds are involved in the computation. Both $\alpha_1\left(\boldsymbol{\theta}\right)$ and $\alpha_2\left(\boldsymbol{\theta}\right)$ are written as functions of $\boldsymbol{\theta}$ because under H_0, $\boldsymbol{\vartheta} = (g(\boldsymbol{\varphi}), \boldsymbol{\tau})$, i.e. $\boldsymbol{\vartheta}$ is a function of $\boldsymbol{\theta}$. In practice, since we do not know the value of $\boldsymbol{\theta}$ we replace it by its PL estimate $\hat{\boldsymbol{\theta}}_{PL}$. Thus, in the formulae above, we use estimates of both $\alpha\left(\boldsymbol{\theta}\right)$ and $df\left(\boldsymbol{\theta}_0\right)$ which are subject to sample variability and the estimate of $df\left(\boldsymbol{\theta}_0\right)$ is not an integer.

In the case of a SEM which imposes a parametric structure both on P and the thresholds, the thresholds are not nuisance parameters anymore. The null hypothesis is modified to $H_0 : \boldsymbol{\vartheta} = g(\boldsymbol{\theta})$ versus $H_1 : \boldsymbol{\vartheta}$ unconstrained. The results for the PLRT remain the same as above with the difference that, in the expressions of $\alpha_1\left(\boldsymbol{\theta}\right)$ and $\alpha_2\left(\boldsymbol{\theta}\right)$, we drop the superscripts from all matrices as the complete matrices should be used, and $M\left(\boldsymbol{\varphi}\right)$ should be replaced by $M\left(\boldsymbol{\theta}\right)$ where $M\left(\boldsymbol{\theta}\right) = \frac{\partial}{\partial \boldsymbol{\theta}} g\left(\boldsymbol{\theta}\right)$.

3.3.2 Testing nested SEMs

For the comparison of two nested SEMs, the hypothesis can be written as:

$$\begin{aligned} H_0 &: g\left(\boldsymbol{\theta}\right) = \mathbf{0} \\ H_1 &: g\left(\boldsymbol{\theta}\right) \neq \mathbf{0} \end{aligned}$$

where $g\left(\boldsymbol{\theta}\right)$ is a function of $\boldsymbol{\theta}$, $g : \mathbb{R}^s \to \mathbb{R}^r$, where s is the dimension of $\boldsymbol{\theta}$ and r is the number of constraints imposed on $\boldsymbol{\theta}$. The constraints can be both equality constraints between some parameters and constraints where some parameters are set equal to specific values. The PLRT statistic is defined as:

$$PLRT = 2\left(pl\left(\hat{\boldsymbol{\theta}}_{PL}\right) - pl\left(\tilde{\boldsymbol{\theta}}_{PL}\right)\right) \ , \tag{16}$$

where $\hat{\boldsymbol{\theta}}_{PL}$ and $\tilde{\boldsymbol{\theta}}_{PL}$ are the PL estimates of the model parameters under H_1 and H_0, respectively. Under H_0, the asymptotic distribution of $PLRT$ is a chi-squared distribution the degrees of freedom of which need to be approximated. Using the Satterthwaite approximation, under H_0, it holds that:

$$\alpha\left(\boldsymbol{\theta}\right) PLRT \overset{\text{app}}{\rightarrow} \chi^2_{df(\boldsymbol{\theta})} \, . \tag{17}$$

We will refer to $\alpha\left(\boldsymbol{\theta}\right) PLRT$ as the adjusted $PLRT$, the adjustment is $\alpha\left(\boldsymbol{\theta}\right) = \frac{\alpha_1(\boldsymbol{\theta})}{0.5*\alpha_2(\boldsymbol{\theta})}$, and the degrees of freedom are $df\left(\boldsymbol{\theta}\right) = \frac{[\alpha_1(\boldsymbol{\theta})]^2}{0.5*\alpha_2(\boldsymbol{\theta})}$, where

$$\alpha_1\left(\boldsymbol{\theta}\right) = \text{tr}\left\{B(\boldsymbol{\theta})[A\left(\boldsymbol{\theta}\right)]^{-1}\right\} \ ,$$
$$\alpha_2\left(\boldsymbol{\theta}\right) = 2\text{tr}\left\{B(\boldsymbol{\theta})[A\left(\boldsymbol{\theta}\right)]^{-1}B(\boldsymbol{\theta})[A\left(\boldsymbol{\theta}\right)]^{-1}\right\} \ .$$

$B(\boldsymbol{\theta}) = M\left(\boldsymbol{\theta}\right) G^{-1}\left(\boldsymbol{\theta}\right)\left[M\left(\boldsymbol{\theta}\right)\right]'$, $A\left(\boldsymbol{\theta}\right) = M\left(\boldsymbol{\theta}\right) H^{-1}\left(\boldsymbol{\theta}\right)\left[M\left(\boldsymbol{\theta}\right)\right]'$, and $M\left(\boldsymbol{\theta}\right) = \frac{\partial}{\partial \boldsymbol{\theta}'} g\left(\boldsymbol{\theta}\right)$. In practice, since we do not know the value of the true parameter $\boldsymbol{\theta}$, we evaluate all the above quantities at the PL estimate under H_0, $\tilde{\boldsymbol{\theta}}_{PL}$, and for this, the estimate of $df\left(\boldsymbol{\theta}\right)$ is not an integer.

3.4 PL-AIC and PL-BIC model selection

Based on the results of Varin & Vidoni (2005) and Gao & Song (2009), the PL version of Akaike information criterion, AIC_{PL}, is defined as:

$$AIC_{PL} = -pl\left(\hat{\boldsymbol{\theta}}_{PL}; (\mathbf{y}_1, \ldots, \mathbf{y}_N)\right) + \text{tr}\left[\hat{J}\left(\hat{\boldsymbol{\theta}}_{PL}\right)\hat{H}^{-1}\left(\hat{\boldsymbol{\theta}}_{PL}\right)\right], \tag{18}$$

and the PL version of Bayesian information criterion, BIC_{PL}, is defined as:

$$BIC_{PL} = -2pl\left(\hat{\boldsymbol{\theta}}_{PL}; (\mathbf{y}_1, \ldots, \mathbf{y}_N)\right) + \log N * \text{tr}\left[\hat{J}\left(\hat{\boldsymbol{\theta}}_{PL}\right)\hat{H}^{-1}\left(\hat{\boldsymbol{\theta}}_{PL}\right)\right] \ , \tag{19}$$

where $\hat{\boldsymbol{\theta}}_{PL}$ is the PL estimate under the hypothesized model. As in the case of AIC and BIC, the model with the smallest AIC_{PL} or BIC_{PL} is selected. Katsikatsou & Moustaki (2016) applied the formulae to SEM with ordinal variables and, via a simulation study, found that both perform very well for nested models with BIC_{PL} performing better than AIC_{PL}.

3.5 PL estimation for SEM with ordinal variables and data with missing values

Here, we consider the case of item non-response, where at least one of the p indicators is observed for each sample unit, and not that of unit non-response, where there may be sample units with missing values in all p indicators. For item non-response, the most practical adaptations of the PL are the complete-pairs PL (CP) and the available-cases PL (AC). Both require only a model for the observed data, the estimation is done in a single step, and the computation of standard errors is straightforward.

3.5.1 Definition of complete-pairs PL (CP) and available-cases PL (AC)

In the CP, each sample unit contributes to the pairwise likelihood function with the bivariate likelihood functions for which both variables are observed while in the AC, additionally to these bivariate likelihoods, the sample unit also contributes with the univariate likelihood functions of the observed variables derived from those pairs of variables where one variable is observed and the other is missing. Let $pl^{CP}\left(\boldsymbol{\theta};(\mathbf{y}_1,\ldots,\mathbf{y}_N)\right)$ and $pl^{AC}\left(\boldsymbol{\theta};(\mathbf{y}_1,\ldots,\mathbf{y}_N)\right)$ denote, respectively, the complete-pairs pairwise log-likelihood function and the available-cases pairwise log-likelihood function for a sample of N observations. Maximizing the functions over $\boldsymbol{\theta}$, we obtain the CP estimator, $\hat{\boldsymbol{\theta}}_{CP}$, and the AC estimator, $\hat{\boldsymbol{\theta}}_{AC}$, respectively. For a random sample of observations, each log-likelihood function is equal to the sum of the N individual contributions, the exact form of which is given below.

For the sample unit n, let $\tilde{p}_n$ and m_n be the number of items with observed values and the number of items with missing values, respectively, where $\tilde{p}_n + m_n = p$, and $\tilde{p}_n > 0$. Also, let $\mathbf{y}_n^o$ and $\mathbf{y}_n^m$ denote, respectively, the $\tilde{p}_n$-dimensional vector of observed variables and the m_n-dimensional vector of missing variables. The contribution of the sample unit n to $pl^{CP}\left(\boldsymbol{\theta};(\mathbf{y}_1,\ldots,\mathbf{y}_N)\right)$ is defined as:

$$pl_n^{CP}\left(\boldsymbol{\theta};\mathbf{y}_n\right) = \sum_{i=1}^{\tilde{p}_n-1}\sum_{j=i+1}^{\tilde{p}_n} \log f\left(y_{ni}^o, y_{nj}^o;\boldsymbol{\theta}\right) \ , \tag{20}$$

where $\log f\left(y_{ni}^o, y_{nj}^o;\boldsymbol{\theta}\right)$ is defined in (11), and the contribution to $pl^{AC}\left(\boldsymbol{\theta};(\mathbf{y}_1,\ldots,\mathbf{y}_N)\right)$ is defined as:

$$pl_n^{AC}\left(\boldsymbol{\theta};\mathbf{y}_n\right) = pl_n^{CP}\left(\boldsymbol{\theta};\mathbf{y}_n\right) + m_n \sum_{i=1}^{\tilde{p}_n} \log f\left(y_{ni}^o;\boldsymbol{\theta}\right) \ , \tag{21}$$

where, for ordinal indicators,

$$\log f\left(y_{ni}^o;\boldsymbol{\theta}\right) = \sum_{a=1}^{c_i} I\left(y_{ni}^o = a\right) \ln \pi\left(y_{ni}^o = a;\boldsymbol{\theta}\right) \ , \tag{22}$$

$I\left(y_{ni}^o = a\right)$ is an indicator whether y_{ni}^o falls into category a, and the corresponding model probability $\pi\left(y_{ni}^o = a;\boldsymbol{\theta}\right)$ following equations (1) and (2) is

$$\pi\left(y_{ni}^o = a;\boldsymbol{\theta}\right) = \int_{\tau_{i,a-1}}^{\tau_{i,a}} f\left(y_{ni}^\star;\boldsymbol{\theta}\right) dy_{ni}^\star = \Phi_1\left(\tau_{i,a}\right) - \Phi_1\left(\tau_{i,a-1}\right) \tag{23}$$

with $\Phi_1(\tau)$ being the univariate cumulative standard normal distribution evaluated at point τ.

As in the case of data with no missing values, the Godambe information matrix is used to compute the standard errors of $\hat{\boldsymbol{\theta}}_{CP}$ and $\hat{\boldsymbol{\theta}}_{AC}$ and to estimate it, the expressions (8) and (9) can be used with the difference that pl is replaced by pl^{CP} and pl^{AC}, and $\hat{\boldsymbol{\theta}}_{PL}$ is replaced by $\hat{\boldsymbol{\theta}}_{CP}$ and $\hat{\boldsymbol{\theta}}_{AC}$, respectively.

For a multi-group analysis with independent group samples, the CP and AC log-likelihood functions are defined in the same way as in Equation (13) with the difference that we add the superscript CP and AC to pl, respectively.

3.5.2 Performance of the CP and the AC for SEM with ordinal variables and missing at random (MAR) data

In general, the CP and the AC estimators yield biased estimates when the missing data are missing at random (MAR) (Molenberghs et al., 2011) but a simulation study indicates that this is not the case when the CP and the AC are applied to FAM with ordinal variables (Katsikatsou & Moustaki, 2017). Both the CP and the AC are found to have close-to-zero standardized bias for loading and factor correlation estimates which is comparable to the bias of the PL applied to completely observed data and is decreasing with a sample-size increase. Also, the CP and the AC coverage rate of 95% confidence interval for loadings and factor correlations is satisfactory and improves with a sample size increase. It is only the thresholds for which the performance of CP and AC is not found satisfactory. Although the AC yields thresholds estimates with acceptable level of standardized bias, much less than the bias of the CP threshold estimates, it underestimates their standard errors while the CP does not. However, the simulation study indicates that the quality of the estimation of thresholds does not affect the quality of the estimation of loadings and factor correlations. Thus, as long as the thresholds are not parameters of interest, the CP and the AC are recommended.

Whenever the general result, that the CP and the AC yield biased estimates for MAR data, applies, the doubly-robust (DR) adaptation of PL is recommended instead, which in general yields unbiased estimators under MAR (Molenberghs et al., 2011). Along with a model for the observed data, the DR requires a predictive model for the bivariate and univariate log-likelihood functions for those variables with missing values. The predictive model needs to be rather computationally practical so that it does not defeat the computationally simplicity of the PL. Despite that both the CP and the AC are found to perform very satisfactorily in the case of FAM, they have also been compared to the DR (Katsikatsou & Moustaki, 2017). The obvious candidates for the predictive model within the framework of FAM with ordinal variables render the DR very demanding computationally and impractical when the number of indicators p gets rather large (e.g. larger than 30). Besides, for a smaller number of indicators, the performance of the DR is only better than that of CP and AC in the case of thresholds. The DR estimation needs to be carried out in two steps, which renders the computation of the DR standard errors rather complicated. For all these reasons, the CP and the AC are recommended to the DR and the DR is not presented in this tutorial. Note though that these findings do not imply that the DR should not be explored as an alternative in specific applications where computationally simple predictive models may be motivated by the application and/or the data at hand.

3.5.3 Performance of the CP and the AC for SEM with ordinal variables and missing not at random (MNAR) data

The performance of the CP and the AC for MNAR data is under research. Some first simulation results indicate that there may be occasions that the CP and the AC may perform satisfactorily in terms of estimates bias when applied to FAM / SEM with MNAR data.

4 Alternative methods for SEM with ordinal variables

4.1 The three-stage weighted least squares method (3S-WLS)

The conventional estimation method for SEM with ordinal variables is the three-stage weighted least squares (3S-WLS) (Muthén, 1984; Jöreskog, 2002). Similarly to the PL, it is a limited-information estimation method using the univariate and bivariate likelihood functions only. Contrary to the PL estimation which is carried out in one step, the 3S-WLS is conducted in three steps. In the first step, the thresholds are estimated and in the second step, given the threshold estimates, the polychoric correlations are estimated. In the third step, based on the parametric structure the hypothesized SEM imposes on the polychoric correlation matrix, a least squares fit function is minimized to provide estimates of the model parameters. To compute the correct standard errors, an estimate of the asymptotic covariance matrix of the estimated polychoric correlations is involved in the procedure.

Based on simulation studies with finite samples and data completely observed (no missing values), the 3S-WLS and the PL are found to be competitive and none of them is uniformly better than the other (Katsikatsou et al., 2012; Liu, 2007; Xi, 2011). The main advantage of the PL is that the well-established results of the ML theory can be extended to apply to the case of PL. As already seen above, the PLRT, the PL-AIC, and the PL-BIC are extensions of the standard LRT, AIC, and BIC derived under the ML.

In the case of MAR data, both the CP and the AC appear more attractive than the adaptation of the 3S-WLS, where multiple imputation (MI) needs to precede before fitting the hypothesized model using the 3S-WLS (Asparouhov & Muthén, 2010). While the CP and the AC require only a model for the observed data, the "MI followed by 3S-WLS" approach requires a model for the imputation as well. Besides, MI gets more complicated in the case of multi-group analysis; it should be conducted with caution so that the data imputation will not distort possible interaction effects between the grouping variable and the remaining variables.

4.2 Why not Maximum Likelihood (ML)?

ML estimation for SEM with ordinal indicators, despite being possible theoretically, is not computationally feasible with a large number of ordinal variables (Lee et al., 1990; Poon & Lee, 1987). More specifically, the log-likelihood function for a single observation is:

$$\begin{aligned} l(\boldsymbol{\theta}; \mathbf{y}) &= \log f(y_1, \ldots, y_p; \boldsymbol{\theta}) \\ &= \sum_{a=1}^{c_1} \cdots \sum_{b=1}^{c_p} I(y_1 = a, \ldots, y_p = b) \ln \pi(y_1 = a, \ldots, y_p = b; \boldsymbol{\theta}) \ , \end{aligned}$$

where

$$\pi(y_1 = a, \ldots, y_p = b; \boldsymbol{\theta}) = \int_{\tau_{1,a-1}}^{\tau_{1,a}} \cdots \int_{\tau_{p,b-1}}^{\tau_{p,b}} f\left(y_1^\star, \ldots, y_p^\star; \boldsymbol{\theta}\right) dy_1^\star \ldots dy_p^\star .$$

Thus, a p-dimensional integral over a p-variate normal distribution, which cannot be written in a closed form as in (12), needs to computed for each sample unit. This is computationally feasible for a small p (e.g. equal to 6) but the computation time increases rapidly as p increases rendering the computation impractical for a large p. For example, the R package mnormt, also used internally in lavaan, computes up to 30-dimensional normal probabilities.

5 PL for SEM with ordinal variables in the R package lavaan

5.1 The R package lavaan

The R package lavaan has been developed to fit any model that can be expressed using the structural equation modelling framework. A detailed presentation of the package along with related material and resources are given in the website http://lavaan.org. The PL methodology for SEM with ordinal variables presented in Section 3 has been implemented in lavaan version 0.6-1.1157 or higher.

To install lavaan in R, follow the same procedure as installing any other R package. In the R menu, click on "Packages" and then select "Install package(s)". In the pop-up window, choose the CRAN mirror nearest to you and click OK. In the next pop-up window, choose "lavaan" from the list of packages and click OK. If the lavaan version released on CRAN is not yet at least the 0.6-1.1157 one, install the latest version (soon to be released) giving the command below.

```
install.packages("lavaan", repos = "http://www.da.ugent.be", type = "source")
```

Once lavaan has been installed, it needs to be loaded to use its functions whenever an R session is started. To do so, give the command below.

```
library(lavaan)
```

For those already familiar with lavaan, we summarize here the main functions that can be used in order to apply the PL to SEM with ordinal variables. Detailed examples for all these functions are given in the remaining document. To fit a SEM using the PL, employ the lavaan fit functions `sem` or `lavaan` or `cfa` as usual, but state explicitly that the estimation procedure is the PL by specifying that the functions' input argument `estimator` equal to `‘‘PML’’`. If there are missing values in the data the default setting of the fit functions is listwise deletion (i.e. the cases where at least one variable has missing value are omitted from the data used to fit the model). To apply the PL adaptations, CP or AC, additionally to the specification `estimator = ‘‘PML’’`, set the input argument `missing` equal to `‘‘pairwise’’` for the CP and `‘‘available.cases’’` for the AC. The PLRT value for overall fit and its p-value when thresholds are nuisance parameters are given on the top of the standard output of a fitted model provided that the model has been fitted using the PL. Alternatively, the function `lavaan:::ctr_pml_plrt()` with input a model fitted using the PL can be employed. When the fitted model imposes parametric structure on both polychoric correlations and thresholds, the function `lavaan:::ctr_pml_plrt2()` needs to be used to obtain the correct PLRT for overall fit. To compare nested models using the PLRT, the standard lavaan function `lavTestLRT()` with input the two nested models fitted with the PL can be used. The function also provides the PL-AIC and PL-BIC values of the compared models. To obtain the PL-AIC and PL-BIC values for a model fitted with PL, use the function `lavaan:::ctr_pml_aic_bic()` with input the fitted model. Note that the PLRT, the PL-AIC, and the PL-BIC are not yet available for the PL adaptations, CP and AC, which deal with missing values. When a model is fitted using the CP or the AC, in the fit functions, `sem` or `lavaan` or `cfa`, add the argument `test = ’’none’’` to save computational time. Otherwise, a version of the PLRT, not yet fully studied, is computed.

5.2 The data

The data analyzed in this tutorial come from the European Social Survey (ESS) Round 5 (2010/2011) that took place in 27 European countries (ESS, 2010, 2014). In particular, the data refer to the

Trust in police effectiveness (TrEf) is measured by:
D12. Based on what you have heard or your own experience how successful do you think the police are at preventing crimes in [country] where violence is used or threatened? (plcpvcr)
D13. How successful do you think the police are at catching people who commit house burglaries in [country]? (plccbrg)
D14. If a violent crime were to occur near to where you live and the police were called, how slowly or quickly do you think they would arrive at the scene? (plcarcr)
The response scale for D12 - D13 is an 11-point Likert scale, from 0 to 10, with 0 being labelled as "Extremely unsuccessful" and 10 as "Extremely successful". The response scale for D14 is the same 11-point Likert scale with 0 being labelled as "Extremely slowly" and 10 as "Extremely quickly" plus an additional response category "Violent crimes never occur near to where I live". The latter is treated as missing value in this tutorial.

Trust in police procedural fairness (TrFa) is measured by:
D15. Based on what you have heard or your own experience how often would you say the police generally treat people in [country] with respect? (plcrspc)
D16. About how often would you say that the police make fair, impartial decisions in the cases they deal with? (plcfrdc)
D17. When dealing with people in [country], how often would you say the police generally explain their decisions and actions when asked to do so? (plcexdc)
The response scale for D15 - D17 is a 4-point Likert scale, from 1 to 4, with 1 being labelled as "Not at all often", 2 "Not very often", 3 "Often", and 4 "Very often".

Felt obligation to obey the police (ObOb) is measured by:
To what extent is it your duty to...
D18. ...back the decisions made by the police even when you disagree with them? (bplcdc)
D19. ...do what the police tell you even if you don't understand or agree with the reasons? (doplcsy)
D20. ... do what the police tell you to do, even if you don't like how they treat you? (dpcstrb)
The response scale for D18 - D20 is an 11-point Likert scale, from 0 to 10, with 0 being labelled as "Not at all my duty" and 10 as "Completely my duty".

Moral alignment with the police (MoAl) is measured by:
D21. The police generally have the same sense of right and wrong as I do. (plcrgwr)
D22. The police stand up for values that are important to people like me. (plcipvl)
D23. I generally support how the police usually act. (gsupplc)
The response scale for D21 - D23 is a 5-point Likert scale from 1 to 5 being labelled, respectively, as "Agree strongly", "Agree", "Neither agree nor disagree", "Disagree", and "Disagree strongly".

Willingness to cooperate with the police (WiCo) is measured by:
D40. Imagine that you were out and saw someone push a man to the ground and steal his wallet. How likely would you be to call the police? (caplcst)
D41. How willing would you be to identify the person who had done it? (widprsn)
D42. And how willing would you be to give evidence in court against the accused? (wevdct)
The response scale for D40 - D42 is a 4-point Likert scale from 1 to 4 being labelled, respectively, as "Not at all likely", 2 "Not very likely", 3 "Likely", and 4 "Very likely".

Table 1: The indicators of the latent variables along with their response scales of the analyzed ESS data; inside parentheses, the labels of the variables

ESS section “Trust in the Police and Courts” (section D of the ESS questionnaire). We focus on fifteen questions, D12 – D23 and D40 – D42, which measure five latent variables. Table 1 presents the latent variables (printed in bold italic), the questions measuring them, and the response scales. The labels of the variables are given inside parentheses. The data are saved in the file “ESS5Police.RData”[2]. After downloading the file, load the data in R, by clicking on “File” in the R menu and then selecting “Load workspace”. In the window that pops up, select the folder where you have saved “ESS5Police.RData”, then the file, and click on “Open”. Alternatively, open R and use the command `load()`. For example, if you have saved the file in the folder C:\PL, give the command below.

```
load("C:\\PL\\ESS5Police.RData")
```

Note that a path inside the function `load()` needs to be written with double backslash or forward slash. To get an idea how the data set looks like, you can print the first rows of the data on the R session by giving the command below.

```
head(PoliceDataRc)
```

5.3 The hypothesized SEM

The path diagram of the hypothesized SEM considered in the remaining document is displayed in Figure 1. The model is a part of a larger model discussed in Jackson et al. (2012), where the research interest lies on the structural part of the model, i.e. on the associations between the latent variables. For the hypothesized SEM we consider here, both “Trust in police effectiveness” (TrEf) and “Trust in police procedural fairness” (TrFa) are hypothesized to have an effect on “Felt obligation to obey the police” (ObOb) and on “Moral alignment with the police” (MoAl). “Willingness to cooperate with the police” (WiCo) is hypothesized to be affected by TrEf, ObOb, and MoAl. In the current model, TrEf and TrFa are exogenous latent variables and the remaining three latent variables are endogenous. TrEf and TrFa are allowed to be correlated as well as the residuals of ObOb and MoAl. The equations composing the structural part of the model are given in Table 2. Note that, in order to define the unit of the factor scales, we fix the variances of TrEf and TrFa as well as the variances of the residuals of the endogenous latent variables equal to 1.

[2]The file “ESS5Police.RData” has been created as follows. After downloading the full ESS5 Stata data file from the ESS website (http://www.europeansocialsurvey.org/data/), we kept only the 15 variables listed in Table 1 plus the variables “idno” and “cntry” which are the ID and the country of a respondent, respectively. The modified Stata file was imported in R using the function `read.dta()` of the R package “foreign”. The codes of the responses for questions D21-D23 were reversed so that 1 denotes “Disagree strongly”, 2 “Disagree”, and so on. This way, the responses to all questions are coded in such a way that higher-numbered response options indicate more positive attitudes toward the police and the criminal justice system.

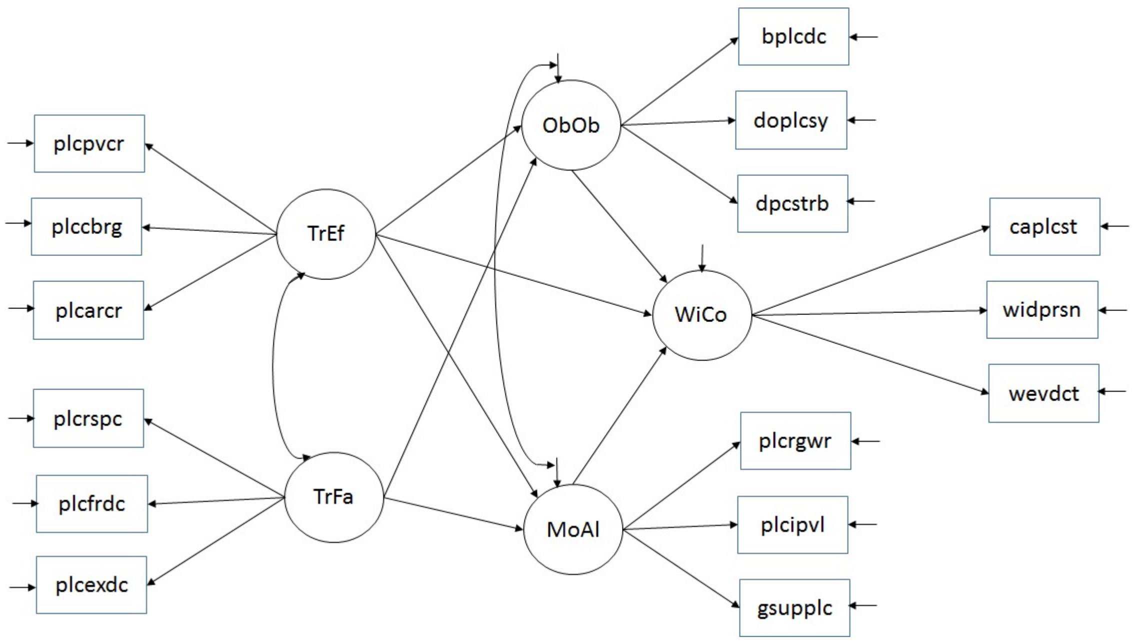

Figure 1: The path diagram of the hypothesized SEM

$$\begin{aligned}
&\text{ObOb} = \beta_{31}\text{TrEf} + \beta_{32}\text{TrFa} + \zeta_3\\
&\text{MoAl} = \beta_{41}\text{TrEf} + \beta_{42}\text{TrFa} + \zeta_4\\
&\text{WiCo} = \beta_{51}\text{TrEf} + \beta_{53}\text{ObOb} + \beta_{54}\text{MoAl} + \zeta_5\ ,
\end{aligned}$$

where $Var\,(\text{TrEf}) = Var\,(\text{TrFa}) = Var\,(\zeta_3) = Var\,(\zeta_4) = Var\,(\zeta_5) = 1$;
TrEf and TrFa are uncorrelated with ζ_3, ζ_4, ζ_5; ObOb and MoAl are uncorrelated with ζ_5; and ζ_3 and ζ_4 are allowed to be correlated.

Table 2: The equations of the structural part of the hypothesized SEM

5.4 Single-group analysis

In this section, we consider only the data of Great Britain (GB) from the whole data set.

5.4.1 Fitting the model

To fit the hypothesized SEM described in Section 5.3 to the GB data, use the commands below. Explanations for each command are preceded by the # sign. Recall that # is used to denote a comment in R.

```
# Extract the GB data from all-country data and save it under the name
# 'PoliceDataRcGB' by giving the command below.
PoliceDataRcGB <- PoliceDataRc[PoliceDataRc$cntry == "GB", ]

# Note that lavaan requires the data to be saved as data frame which is the
# case for 'PoliceDataRcGB'. To confirm use the command below.
is.data.frame(PoliceDataRcGB)
```

```r
# If a data set you wish to analyze with lavaan is not data frame, use the
# command as.data.frame().

# Specify that all variables are ordinal except of course for the variables
# 'idno' and 'cntry' in the first two columns of the data. Save the new
# format of the data under the name 'PoliceDataRcGBOrd'.
PoliceDataRcGBOrd <- PoliceDataRcGB
PoliceDataRcGBOrd[, 2:17] <- lapply(PoliceDataRcGBOrd[, 2:17], ordered)

# Specify the model to be fitted and save it as 'Ex1Model'.
Ex1Model <- "
 #Measurement part of the model
  TrEf =~ plcpvcr + plccbrg + plcarcr
  TrFa =~ plcrspc + plcfrdc + plcexdc
  ObOb =~ bplcdc  + doplcsy + dpcstrb
  MoAl =~ plcrgwr + plcipvl + gsupplc
  WiCo =~ caplcst + widprsn + wevdct

 #Structural part of the model
  ObOb ~ TrEf + TrFa
  MoAl ~ TrEf + TrFa
  WiCo ~ TrEf + ObOb + MoAl

  TrEf ~~ TrFa #Cov(TrEf, TrFa) to be estimated
  ObOb ~~ MoAl #Cov(zeta3, zeta4) to be estimated
"

# Fit the model using the function sem. Specify PL as the estimation method
# by setting the input argument 'estimator' equal to 'PML'. The argument
# 'std.lv = TRUE' fixes the variances of TrEf and TrFa, and the variances of
# the residuals of ObOb, MoAl, and WiCo to 1 to define the unit of the
# factor scales. The argument 'verbose = TRUE' prints the progress of the
# computations on the R session. If you do not wish so, you can omit it.
Ex1FittedModel <- sem(model = Ex1Model, data = PoliceDataRcGBOrd, estimator = "PML",
    std.lv = TRUE, verbose = TRUE)

# To print the output of the fitted model in the R session give the command
# below. The output is given in the Appendix. The most important parts of
# the output are explained in the subsections that follow.
summary(Ex1FittedModel)

# To save the parameter estimates, standard errors, p-values, and their 95%
# confidence interval as a data frame, use the command below. This enables
# you to select certain values for further process.
Ex1FittedModel_ParEst <- parameterEstimates(Ex1FittedModel)
```

Recall that, regarding missing values, the default setting of the function `sem` is listwise deletion. The second line of the output of the fitted model explicitly states the total number of observations in the data and the number of observations used to fit the model. In the case of the GB data, there are 2422 observations but only 1805 are used to fit the model, as you will see in the output of the fitted model given in the Appendix.

5.4.2 Parameter estimates, standard errors, z-tests, and 95% confidence intervals

The table below is a part of the output of the fitted model and reports the estimates ("est") and their standard errors ("se"), the z-test values and their p-values, as well as the 95% confidence interval (CI) for the loadings and the parameters of the structural part of the model. In the table, "LL_95CI" and "UL_95CI" denote the lower and upper limit of the 95% CI, respectively. For all of the loadings, the p-value of the z-test is smaller than 0.001 indicating that they all are statistical significant at any conventional significance level. Their signs are in the expected direction. For all factors, higher values represent more positive attitudes toward the police and higher levels of obeying, moral alignment, and willingness to cooperate with the police. The 95% confidence intervals of the loading of "plcpvcr", for example, is (0.728, 0.816) implying a rather strong indicator for TrEf.

The p-values of the z-tests for the regression coefficients are all smaller than 0.001 except for the coefficient of TrEf in the WiCo regression, where the p-value is 0.140. This means that all hypothesized paths between the latent variables are found statistically significant at any conventional significance level except for the path from TrEf to WiCo. All statistically significant regression coefficients are estimated to be positive. In particular, both TrEf and TrFa have positive effect on ObOb and MoAl. In other words, the levels of trust in police effectiveness and fairness are positively associated with the levels of feeling the obligation to obey the police and being morally aligned with the police. The willingness to cooperate with the police is positively associated with feeling obliged to obey the police and the moral alignment with the police.

The correlation between the exogenous latent variables TrEf and TrFa is estimated to be 0.62 indicating that the trust in police effectiveness and the trust in police fairness are rather highly correlated. The correlation of the residuals for MoAl and ObOb is estimated 0.256 implying that the correlation between MoAl and ObOb cannot be fully explained by TrEf and TrFa.

```
Loadings_Example1

           est    se z_value p_value LL_95CI UL_95CI
plcpvcr  0.772 0.022  34.383   0.000   0.728   0.816
plccbrg  0.741 0.020  36.467   0.000   0.701   0.781
plcarcr  0.633 0.022  28.779   0.000   0.590   0.676
plcrspc  0.830 0.019  44.540   0.000   0.794   0.867
plcfrdc  0.858 0.018  47.060   0.000   0.823   0.894
plcexdc  0.723 0.019  37.535   0.000   0.685   0.760
bplcdc   0.614 0.014  44.173   0.000   0.586   0.641
doplcsy  0.817 0.018  46.159   0.000   0.782   0.851
dpcstrb  0.775 0.015  50.661   0.000   0.745   0.805
plcrgwr  0.464 0.024  19.189   0.000   0.417   0.512
plcipvl  0.549 0.026  20.736   0.000   0.497   0.601
gsupplc  0.528 0.024  22.436   0.000   0.481   0.574
```

```
caplcst  0.734 0.018  41.024   0.000   0.699   0.769
widprsn  0.960 0.014  70.502   0.000   0.933   0.987
wevdct   0.875 0.013  65.589   0.000   0.849   0.902

Regression_Coefficients_Example1

                est    se z_value p_value LL_95CI UL_95CI
ObOb ON TrEf  0.276 0.047   5.939   0.000   0.185   0.368
ObOb ON TrFa  0.285 0.047   6.116   0.000   0.194   0.377
MoAl ON TrEf  0.379 0.067   5.676   0.000   0.248   0.510
MoAl ON TrFa  0.976 0.088  11.034   0.000   0.803   1.150
WiCo ON TrEf -0.072 0.049  -1.477   0.140  -0.168   0.024
WiCo ON ObOb  0.115 0.033   3.469   0.001   0.050   0.180
WiCo ON MoAl  0.141 0.033   4.324   0.000   0.077   0.204

Covariance_Exogenous_Factors_Example1

                  est    se z_value p_value LL_95CI UL_95CI
TrEf WITH TrFa  0.620 0.027  23.095   0.000   0.568   0.673

Covariance_Residuals_Example1

                  est    se z_value p_value LL_95CI UL_95CI
ObOb WITH MoAl  0.256 0.041   6.271   0.000   0.176   0.336
```

5.4.3 PLRT for overall fit (where thresholds are nuisance parameters)

To test the overall fit of the model, we can use the PLRT. One way to obtain its value and the p-value is to type the name under which the fitted model is saved, which is "Ex1FittedModel" in our example. The returned output is presented below.

```
Ex1FittedModel

lavaan (0.6-1.1179) converged normally after  98 iterations

                                                  Used       Total
  Number of observations                          1805        2422

  Estimator                                        PML      Robust
  Model Fit Test Statistic                     422.379    1116.759
  Degrees of freedom                                81     131.241
  P-value                                           NA       0.000
  Scaling correction factor                                  0.378
    for the mean and variance adjusted correction
```

The PLRT value is given under the column "PML" in the line "Model Fit Test Statistic" but it is not correct to compare it to a χ^2_{81}. Instead, we use the adjusted PLRT given under the column

"Robust" in line "Model Fit Test Statistic" the value of which is 1116.759 for our example. The value of the adjustment is given in the line "Scaling correction factor". The value of the adjusted PLRT can be compared to a $\chi^2_{131.241}$ giving a p-value smaller than 0.001. Thus, based on the PLRT, the null hypothesis the model fits the data is rejected at any conventional significance level. (Details on the PLRT and the adjusted PLRT for overall fit, and why the chi-squared distribution has non-integer degrees of freedom are given in Section 3.3.1.)

The output with the PLRT results is also given in the beginning of the full output obtained by the following command.

```
summary(Ex1FittedModel)
```

An alternative way to obtain the PLRT results is by giving the command

```
lavaan:::ctr_pml_plrt(Ex1FittedModel)
```

which produces the following output.

```
$PLRTH0Sat
[1] 422.3793

$PLRTH0Sat.group
[1] 422.3793

$stat
[1] 1116.759

$df
[1] 131.2406

$p.value
[1] 0

$scaling.factor
[1] 2.643972
```

Note that the model fitted to the GB data does not impose parametric structure on the thresholds. If a model imposes parametric structure on both polychoric correlations and thresholds, as in the analysis of the next section, the correct PLRT can be obtained by only using the function `lavaan:::ctr_pml_plrt2()` and not by typing the name under which the fitted model is saved or using the function `summary()`.

5.5 Multi-group analysis

5.5.1 Fitting the model

In this section, we conduct a two-group analysis using the data for GB and Ireland (IE). The hypothesized model for both countries is the one presented in Section 5.3. In multi-group analysis, the aim is usually to compare the distributions of the latent variables between the countries, i.e. the means, variances, and correlations of the latent variables. In this section, we assume full measurement equivalence for all indicators, i.e. cross-country equality constraints on all loadings and thresholds. This implies that all indicators operate in the same way as measurement instruments of the latent variables they measure in both countries. A detailed discussion on measurement equivalence (alternatively referred to as measurement invariance) and how it can be tested in latent variable modelling can be found in Millsap (2012). To identify a multi-group SEM where full measurement equivalence is adopted and at the same time allow for meaningful comparisons of the latent variable distributions between the groups, the following constraints are usually employed. The mean and the variance of each latent variable are fixed to 0 and 1, respectively, in one group and are free to be estimated in the other groups. The variance of each underlying variable is also fixed to 1 in one group and is free to be estimated in the other groups.

The model syntax in lavaan for our example is given below.

```
Ex2_MeasEquivModel <- "
 #Measurement part of the model
  TrEf =~ plcpvcr + plccbrg + plcarcr
  TrFa =~ plcrspc + plcfrdc + plcexdc
  ObOb =~ bplcdc  + doplcsy + dpcstrb
  MoAl =~ plcrgwr + plcipvl + gsupplc
  WiCo =~ caplcst + widprsn + wevdct

 #Structural part of the model
  ObOb ~ TrEf + TrFa
  MoAl ~ TrEf + TrFa
  WiCo ~ TrEf + ObOb + MoAl

  TrEf ~~ TrFa
  ObOb ~~ MoAl

  TrEf ~~ c(1,NA)*TrEf
  TrFa ~~ c(1,NA)*TrFa
  ObOb ~~ c(1,NA)*ObOb
  MoAl ~~ c(1,NA)*MoAl
  WiCo ~~ c(1,NA)*WiCo
"

# Vector c(1,NA) specifies that the variance of each factor is fixed to 1 in
# the first group, here GB, and is free to be estimated in the other group,
# here IE. In general, the vector c() has as many elements as the number of
# groups which are ordered in alphabetical order. We have added this
# specification in the model syntax because, when we fit the model below
# using the function sem, we specify 'std.lv = TRUE' which fixes the
# variances of the factors to 1 in both groups. So, we override this
# specification by adding the vector c(1,NA) above. In turn, the argument
# 'std.lv = TRUE' needs to be added in the sem function below to override
# the default setting which is to fix the loading of one indicator of each
```

```
# factor to 1 (used to determine the factor scale unit). No need to add such
# a vector for the factor means and the variances of the underlying
# variables because these parameters are fixed to 0 and 1, respectively,
# only in the first group by default.
```

To fit the model to the GB and IE data give the following commands.

```
# First extract the GB and IE data from the full data set and save them
# under the name PoliceDataRcGBIE, for example.
PoliceDataRcGBIE <- PoliceDataRc[PoliceDataRc$cntry == "GB" | PoliceDataRc$cntry ==
    "IE", ]

# Recall that the data is already saved as data frame. To confirm:
is.data.frame(PoliceDataRcGBIE)

# Specify that all the variables, except for 'idno' and 'cntry', are
# ordinal. Save the new format of the data under the name
# PoliceDataRcGBIEOrd.
PoliceDataRcGBIEOrd <- PoliceDataRcGBIE
PoliceDataRcGBIEOrd[, 3:17] <- lapply(PoliceDataRcGBIEOrd[, 3:17], ordered)

# Fit the model and save the output as Ex2FittedMeasEquivModel. Note that
# here we use two more input arguments, 'group' and 'group.equal'. The first
# one is to specify the grouping variable, here 'cntry', and this way, state
# that a multi-group SEM should be fitted. With the second argument, we
# determine the parameters which cross-country equality constraints will be
# imposed on; here, loadings and thresholds.
Ex2FittedMeasEquivModel <- sem(model = Ex2_MeasEquivModel, data = PoliceDataRcGBIEOrd,
    std.lv = TRUE, estimator = "PML", group = "cntry", group.equal = c("loadings",
        "thresholds"), verbose = TRUE)

# To print the output give the command below. The output is given in the
# Appendix, where the results for the GB data are presented first followed
# by those for the IE data. Parameters constrained to be equal are marked by
# the same label printed in parentheses next to the parameters.
summary(Ex2FittedMeasEquivModel)
```

5.5.2 Parameter estimates, standard errors, z-tests, and 95% confidence interval

Below we present the results for the structural part of the fitted model for both countries with the left part of the table referring to GB, the right part to IE, "est" denoting estimate, and "se" standard error.

```
Factor_Means_Example2

     group      est    group     est     se p_value
```

```
TrEf    GB 0(fixed) |    IE -0.143 0.059   0.016
TrFa    GB 0(fixed) |    IE  0.200 0.061   0.001
ObOb    GB 0(fixed) |    IE -0.205 0.058   0.000
MoAl    GB 0(fixed) |    IE  0.197 0.085   0.020
WiCo    GB 0(fixed) |    IE -0.482 0.058   0.000

Factor_Variances_Example2

     group      est   group    est    se p_value
TrEf    GB 1(fixed) |    IE  1.282 0.126   0.000
TrFa    GB 1(fixed) |    IE  1.370 0.137   0.000
ObOb    GB 1(fixed) |    IE  1.087 0.107   0.000
MoAl    GB 1(fixed) |    IE  1.548 0.275   0.000
WiCo    GB 1(fixed) |    IE  0.829 0.097   0.000

Covariance_Exogenous_Factors_Example2

               group    est    se p_value   group    est    se p_value
TrEf WITH TrFa    GB  0.626 0.038   0.000 |    IE  0.923 0.090   0.000

Regression_Coefficients_Example2

             group    est    se p_value   group    est    se p_value
ObOb ON TrEf    GB  0.288 0.067   0.000 |    IE  0.223 0.074   0.003
ObOb ON TrFa    GB  0.282 0.066   0.000 |    IE  0.420 0.070   0.000
MoAl ON TrEf    GB  0.393 0.094   0.000 |    IE  0.573 0.114   0.000
MoAl ON TrFa    GB  0.964 0.122   0.000 |    IE  1.034 0.127   0.000
WiCo ON TrEf    GB -0.069 0.070   0.324 |    IE  0.060 0.067   0.372
WiCo ON ObOb    GB  0.114 0.047   0.014 |    IE  0.023 0.042   0.584
WiCo ON MoAl    GB  0.139 0.046   0.002 |    IE  0.157 0.038   0.000

Covariance_Residuals_Example2

               group    est    se p_value   group    est    se p_value
ObOb WITH MoAl    GB  0.256 0.057   0.000 |    IE  0.332 0.078   0.000
```

We see that, at 5% significance level, the factor means of IE are all statistically significant and thus statistically different from those of GB. However, this is not the case for the means of TrEf and MoAl if 1% significance level is considered. On average, TrEf is higher in GB, while the opposite is true for TrFa which is higher in IE. Given zero values for TrEf and TrFa, the average level of ObOb is higher in GB while the average of MoAl is higher in IE. Given zero value for TrEf, ObOb, and MoAl, the average level of WiCo is higher in GB. The absolute differences in the factor means/ intercepts between the two countries are smaller than 0.5 standard deviations. Regarding the factor variances and the variances of the residuals, they are all slightly larger in IE except for WiCo. The absolute differences though are maximum 0.55 standard deviations. The estimated correlation of TrEf and TrFa is 0.626 for GB, and, for IE, is equal to $0.923/\sqrt{1.282 * 1.370} = 0.696$. Thus, TrEf and TrFa exhibit very similar correlation in both countries. The associations between the remaining latent variables appear to be in the same direction and of similar magnitude in both countries. The only exception is the association between WiCo and TrEf given ObOb and MoAl which is estimated though very close to 0 in both countries and is not statistically significant at any conventional significance level in either countries (the p-values are 0.324 and 0.372 for GB and IE, respectively). Also, the regression coefficient of ObOb in the regression of WiCo is nearly 0 and not statistically significant (p-value = 0.584) in IE. Finally, the estimated correlation of ObOb and MoAl residuals is estimated to be the same in both countries; 0.256 for GB and $0.332/\sqrt{1.087 * 1.548} = 0.256$ for IE. Recall that the fitted model and consequently the results are based on the assumption of full measurement equivalence. If the assumption (which can be tested) is not correct, the above parameter estimates may be biased.

5.5.3 Wald test

Z-tests are used to test a hypothesis for a scalar parameter. To carry out a test for a parameter vector, we can use the Wald test. Below we demonstrate how to compute the Wald test and its p-value for testing simultaneously the null hypothesis that the means / intercepts of the five factors for IE are all equal to 0.

```
# First save the estimates of the factor means / interecepts in a vector,
# e.g. named 'est_IE_means'. Recall that all parameter estimates can be
# retrieved using the function parameterEstimates() with input the name of
# the fitted model. Giving the command
# parameterEstimates(Ex2FittedMeasEquivModel) we see that the parameters of
# interest are reported in lines 334 to 338 of the returned table. Out of
# all columns of the table, we only need the one that gives the estimates
# labelled 'est'.
est_IE_means <- parameterEstimates(Ex2FittedMeasEquivModel)[334:338, "est"]

# Next obtain the estimated variance-covariance matrix of all parameter
# estimates using the function vcov() and input the name of the fitted
# model.
VCOV_Ex2FittedMeasEquivModel <- vcov(Ex2FittedMeasEquivModel)

# From the above matrix, we only need the rows and columns which provide the
# variances and covariances of the estimated factor means / intercepts. An
# easy way to identify the index of these rows and columns is to call the
# function rownames(VCOV_Ex2FittedMeasEquivModel) which gives the labels of
# the rows. This way, we see that the row indices are 249 to 253. We save
# the submatrix of variances-covariances under the name 'VCOV_IE_means'.
VCOV_IE_means <- VCOV_Ex2FittedMeasEquivModel[249:253, 249:253]

# We use the vector of the estimates and the matrix of the estimated
# variances-covariances to compute the Wald test. Recall that the matrix of
# the estimated variances-covariances needs to be inverted and for this, we
# use the command solve().
Wald_test <- matrix(est_IE_means, nrow = 1) %*% solve(VCOV_IE_means) %*%
matrix(est_IE_means, ncol = 1)
```

```
# The value of the Wald test should be compared to a chi-squared
# distribution with 5 degrees of freedom since we test 5 parameters
# simultaneously. The p-value of the test can be computed as follows.
pvalue_Wald_test <- 1 - pchisq(Wald_test, df = 5)
```

The value of Wald test is

```
[1] 127.2896
```

and the p-value is

```
[1] 0
```

i.e. < 0.001. Thus, the null hypothesis that all the factor means / intercepts for IE are zero is rejected at any conventional significance level.

5.5.4 PLRT for overall fit (where parametric structure is imposed on thresholds)

To test the overall fit of the model, we can use the PLRT. Recall that the two-group hypothesized model not only imposes parametric structure structure on the polychoric correlations but also on the thresholds by constraining them equal between the two countries. Thus, the thresholds are not any longer nuisance parameters as in the example of the previous section. To obtain the correct PLRT, we use the function below with input the name of the fitted model.

```
lavaan:::ctr_pml_plrt2(Ex2FittedMeasEquivModel)
```

The output of the function is printed below.

```
$PLRTH0Sat
[1] 4789.878

$PLRTH0Sat.group
[1] 2384.416 2405.461

$stat
[1] 280.2645

$df
[1] 60.26033

$p.value
[1] 0

$scaling.factor
[1] 0.05851182
```

The PLRT value is given under the section "PLRTH0Sat". However, it is not correct to compare this value with a chi-squared distribution with degrees of freedom equal to the number of parameters of the unconstrained model for the underlying variables (defined by (1) and (2)) minus the number of parameters of the hypothesized model. Instead, we need to use the adjusted PLRT given in the section "stat" the value of which, 280.2645, can be compared to a chi-squared distribution with degrees of freedom given in the section "df", i.e to $\chi^2_{60.26}$. The p-value is $<$0.001 which implies that the null hypothesis the model fits the data is rejected at any conventional significance level. The value of the adjustment used to adjust the PLRT is given under "scaling.factor". The values under "PLRTH0Sat.group" give the unadjusted PLRT for each group separately. (Details on the PLRT and the adjusted PLRT for overall fit, and why the chi-squared distribution has non-integer degrees of freedom are given in Section 3.3.1.)

Note that only the function `lavaan:::ctr_pml_plrt2()` provides the correct PLRT when the hypothesized model imposes parametric structure on both the polychoric correlation matrix and the thresholds. The default PLRT values printed when the name of the fitted model is given, in our example `Ex2FittedMeasEquivModel`, or when the function `summary()` is used, in our example `summary(Ex2FittedMeasEquivModel)`, are the ones computed under the assumption that no parametric structure is imposed on the thresholds which is not the case in the current example.

5.6 PLRT for nested models, PL-AIC, and PL-BIC

5.6.1 Fitting the models to be compared

In this section we fit the same model for GB and IE data as in the previous section but with two different assumptions about the measurement equivalence. In the one specification of the model, we only impose a minimum set of constraints in order to identify the model and at the same time being able to make meaningful comparisons of the latent variable distributions between the two countries (Millsap & Yun-Tein, 2004). In the second specification of the model, we add cross-country equality constraints on all of the loadings to the minimum set of identification constraints. The two models are nested and we can test the assumption of the cross-country equality constraints on the loadings using the PLRT. We can also use the model selection criteria PL-AIC and PL-BIC to compare the models.

Let present first the model with the minimum set of constraints. In this model, cross-country equality constraints are imposed on the loading of one indicator for each factor, here on the loadings of "plcpvcr", "plcrspc", "bplcdc", "plcrgwr", and "caplcst". These indicators are sometimes referred to as "anchors". For all of the "anchor" indicators, cross-country equality constraints are imposed on their first two thresholds as well. For the remaining indicators, cross-country equality constraints are imposed only on their first threshold. The means and variances of all factors are fixed to 0 and 1, respectively, only in one group, here GB, and are free to be estimated for IE. Finally, the variances of all underlying variables are fixed to 1 only for GB and are free to be estimated for IE. The lavaan model syntax for this model is given below.

```
Ex2_MinConModel <- '
 ##Measurement part of the model
  TrEf =~   c(c1,c1)*plcpvcr + plccbrg + plcarcr
  TrFa =~   c(c4,c4)*plcrspc + plcfrdc + plcexdc
  ObOb =~   c(c7,c7)*bplcdc  + doplcsy + dpcstrb
  MoAl =~ c(c10,c10)*plcrgwr + plcipvl + gsupplc
  WiCo =~ c(c13,c13)*caplcst + widprsn + wevdct

  plcpvcr | c(p1,p1)*t1 + c(p2,p2)*t2 + t3 + t4 + t5 + t6 + t7 + t8 + t9 + t10
  plccbrg | c(p3,p3)*t1 + t2 + t3 + t4 + t5 + t6 + t7 + t8 + t9 + t10
  plcarcr | c(p4,p4)*t1 + t2 + t3 + t4 + t5 + t6 + t7 + t8 + t9 + t10

  plcrspc | c(p5,p5)*t1 + c(p6,p6)*t2 + t3
  plcfrdc | c(p7,p7)*t1 + t2 + t3
  plcexdc | c(p8,p8)*t1 + t2 + t3

  bplcdc  | c(p9,p9)*t1 + c(p10,p10)*t2 + t3 + t4 + t5 + t6 + t7 + t8 + t9 + t10
  doplcsy | c(p11,p11)*t1 + t2 + t3 + t4 + t5 + t6 + t7 + t8 + t9 + t10
  dpcstrb | c(p12,p12)*t1 + t2 + t3 + t4 + t5 + t6 + t7 + t8 + t9 + t10

  plcrgwr | c(p13,p13)*t1 + c(p14,p14)*t2 + t3 + t4
  plcipvl | c(p15,p15)*t1 + t2 + t3 + t4
  gsupplc | c(p16,p16)*t1 + t2 + t3 + t4

  caplcst | c(p17,p17)*t1 + c(p18,p18)*t2 + t3
  widprsn | c(p19,p19)*t1 + t2 + t3
  wevdct  | c(p20,p20)*t1 + t2 + t3

  #Define the unit of the scales of the underlying variables.
  plcpvcr ~*~ c(1,NA)*plcpvcr
  plccbrg ~*~ c(1,NA)*plccbrg
  plcarcr ~*~ c(1,NA)*plcarcr
  plcrspc ~*~ c(1,NA)*plcrspc
  plcfrdc ~*~ c(1,NA)*plcfrdc
  plcexdc ~*~ c(1,NA)*plcexdc
  bplcdc  ~*~ c(1,NA)*bplcdc
  doplcsy ~*~ c(1,NA)*doplcsy
  dpcstrb ~*~ c(1,NA)*dpcstrb
  plcrgwr ~*~ c(1,NA)*plcrgwr
  plcipvl ~*~ c(1,NA)*plcipvl
  gsupplc ~*~ c(1,NA)*gsupplc
  caplcst ~*~ c(1,NA)*caplcst
  widprsn ~*~ c(1,NA)*widprsn
  wevdct  ~*~ c(1,NA)*wevdct
```

```
 ##Structural part of the model
  ObOb ~ TrEf + TrFa
  MoAl ~ TrEf + TrFa
  WiCo ~ TrEf + ObOb + MoAl

  TrEf ~~ TrFa
  ObOb ~~ MoAl

  #Define the origin of the factor scales.
  TrEf ~ c(0,NA)*1
  TrFa ~ c(0,NA)*1
  ObOb ~ c(0,NA)*1
  MoAl ~ c(0,NA)*1
  WiCo ~ c(0,NA)*1

  #Define the unit of the factor scales.
  TrEf ~~ c(1,NA)*TrEf
  TrFa ~~ c(1,NA)*TrFa
  ObOb ~~ c(1,NA)*ObOb
  MoAl ~~ c(1,NA)*MoAl
  WiCo ~~ c(1,NA)*WiCo
'

#Note that the elements of the vector c() can be either a label, a number, or NA.
#Same labels imply equality constraint on the corresponding parameters. A number
#implies that the corresponding parameter is fixed to that number, and NA impies
#that the corresponding parameter is free to be estimated. Whenever there is no
#vector, it is implied that distinct parameters are to be estimated for the groups.
```

The model syntax for the second model where, additionally to the minimum set of constraints, cross-country equality constraints are imposed on all of the loadings is given below.

```
Ex2_MetricInvModel <- '
 ##Measurement part of the model
  #Only the following 5 lines differ from the previous model syntax.
  TrEf =~   c(c1,c1)*plcpvcr +   c(c2,c2)*plccbrg +   c(c3,c3)*plcarcr
  TrFa =~   c(c4,c4)*plcrspc +   c(c5,c5)*plcfrdc +   c(c6,c6)*plcexdc
  ObOb =~   c(c7,c7)*bplcdc  +   c(c8,c8)*doplcsy +   c(c9,c9)*dpcstrb
  MoAl =~ c(c10,c10)*plcrgwr + c(c11,c11)*plcipvl + c(c12,c12)*gsupplc
  WiCo =~ c(c13,c13)*caplcst + c(c14,c14)*widprsn + c(c15,c15)*wevdct

  plcpvcr | c(p1,p1)*t1 + c(p2,p2)*t2 + t3 + t4 + t5 + t6 + t7 + t8 + t9 + t10
  plccbrg | c(p3,p3)*t1 + t2 + t3 + t4 + t5 + t6 + t7 + t8 + t9 + t10
  plcarcr | c(p4,p4)*t1 + t2 + t3 + t4 + t5 + t6 + t7 + t8 + t9 + t10
```

```
 plcrspc | c(p5,p5)*t1 + c(p6,p6)*t2 + t3
 plcfrdc | c(p7,p7)*t1 + t2 + t3
 plcexdc | c(p8,p8)*t1 + t2 + t3

 bplcdc  | c(p9,p9)*t1 + c(p10,p10)*t2 + t3 + t4 + t5 + t6 + t7 + t8 + t9 + t10
 doplcsy | c(p11,p11)*t1 + t2 + t3 + t4 + t5 + t6 + t7 + t8 + t9 + t10
 dpcstrb | c(p12,p12)*t1 + t2 + t3 + t4 + t5 + t6 + t7 + t8 + t9 + t10

 plcrgwr | c(p13,p13)*t1 + c(p14,p14)*t2 + t3 + t4
 plcipvl | c(p15,p15)*t1 + t2 + t3 + t4
 gsupplc | c(p16,p16)*t1 + t2 + t3 + t4

 caplcst | c(p17,p17)*t1 + c(p18,p18)*t2 + t3
 widprsn | c(p19,p19)*t1 + t2 + t3
 wevdct  | c(p20,p20)*t1 + t2 + t3

 #Define the unit of the scales of the underlying variables.
 plcpvcr ~*~ c(1,NA)*plcpvcr
 plccbrg ~*~ c(1,NA)*plccbrg
 plcarcr ~*~ c(1,NA)*plcarcr
 plcrspc ~*~ c(1,NA)*plcrspc
 plcfrdc ~*~ c(1,NA)*plcfrdc
 plcexdc ~*~ c(1,NA)*plcexdc
 bplcdc  ~*~ c(1,NA)*bplcdc
 doplcsy ~*~ c(1,NA)*doplcsy
 dpcstrb ~*~ c(1,NA)*dpcstrb
 plcrgwr ~*~ c(1,NA)*plcrgwr
 plcipvl ~*~ c(1,NA)*plcipvl
 gsupplc ~*~ c(1,NA)*gsupplc
 caplcst ~*~ c(1,NA)*caplcst
 widprsn ~*~ c(1,NA)*widprsn
 wevdct  ~*~ c(1,NA)*wevdct

##Structural part of the model
 ObOb ~ TrEf + TrFa
 MoAl ~ TrEf + TrFa
 WiCo ~ TrEf + ObOb + MoAl

 TrEf ~~ TrFa
 ObOb ~~ MoAl

#Define the origin of the factor scales.
 TrEf ~ c(0,NA)*1
 TrFa ~ c(0,NA)*1
 ObOb ~ c(0,NA)*1
 MoAl ~ c(0,NA)*1
```

```
  WiCo ~ c(0,NA)*1

 #Define the unit of the factor scales.
  TrEf ~~ c(1,NA)*TrEf
  TrFa ~~ c(1,NA)*TrFa
  ObOb ~~ c(1,NA)*ObOb
  MoAl ~~ c(1,NA)*MoAl
  WiCo ~~ c(1,NA)*WiCo
'
```

To fit the models to the GB and IE data give the following commands.

```
# We use the data saved in 'PoliceDataRcGBIEOrd' defined by the R commands
# given in the previous section.

# To fit the model with the minimum set of constraints.
Ex2FittedMinConModel <- sem(model = Ex2_MinConModel, data = PoliceDataRcGBIEOrd,
    std.lv = TRUE, estimator = "PML", group = "cntry", verbose = TRUE)

# To fit the model with the additional cross-country equality constraints on
# the loadings.
Ex2_FittedMetricInvModel <- sem(model = Ex2_MetricInvModel, data = PoliceDataRcGBIEOrd,
    std.lv = TRUE, estimator = "PML", group = "cntry", verbose = TRUE)

# To print the outputs of the fitted models.
summary(Ex2FittedMinConModel)
summary(Ex2_FittedMetricInvModel)

# The full outputs of the fitted models are given in the Appendix.
```

5.6.2 PLRT for nested models, PL-AIC, PL-BIC

To test the null hypothesis that the loadings of all indicators are equal between the two countries, we compute the PLRT for nested models. For the model selection decision, we can also consult the values of PL-AIC and PL-BIC. To obtain all these results, the lavaan function `lavTestLRT()` can be used. The input of the function are the two nested models fitted using PL. For our example, the function is specified as follows:

```
# compare nested models
lavTestLRT(Ex2_FittedMetricInvModel, Ex2FittedMinConModel)
```

and the output is printed below.

```
Scaled Chi Square Difference Test (method = "mean.var.adjusted.PLRT")

                          Df  PL_AIC  PL_BIC  Chisq Chisq diff Df diff
Ex2FittedMinConModel     162 2209526 2226159 899.22
Ex2_FittedMetricInvModel 172 2209491 2225908 933.43     7.6249  8.0625
                         Pr(>Chisq)
Ex2FittedMinConModel
Ex2_FittedMetricInvModel     0.4774
```

The value of the (adjusted) PLRT, given in the column "Chisq diff", is 7.6249. This should be compared with a chi-squared distribution with degrees of freedom being equal to the number given in the column "Df diff", i.e. to $\chi^2_{8.0625}$. Thus, the p-value, given in column "Pr(>Chisq)", is 0.4774 and the null hypothesis is not rejected at any conventional significance level. (Details on the PLRT and the adjusted PLRT for nested models, and why the chi-squared distribution has non-integer degrees of freedom are given in Section 3.3.2.) The PL-AIC and PL-BIC values, given in the corresponding columns of the above table, are smaller for the more restricted model. Therefore, we can conclude that the data supports the cross-country equality constraints on the loadings.

Note that the lavaan function `lavTestLRT()` should be used for nested models. If the models to be compared are not nested, the PLRT is not appropriate, and only the PL-AIC and the PL-BIC can be used. To get the PL-AIC and PL-BIC values of a fitted model, use the function `lavaan:::ctr_pml_aic_bic()` with input the name of the model fitted using the PL. For example, the PL-AIC and PL-BIC values for the fitted model `Ex2FittedMinConModel` can be obtained as follows:

```
# Obtain the PL-AIC and PL-BIC values of the model.
lavaan:::ctr_pml_aic_bic(Ex2FittedMinConModel)
```

and the returned output is the following.

```
$logPL
[1] -1102068

$PL_AIC
[1] 2209526

$PL_BIC
[1] 2226159
```

The first value of the list, named "logPL", gives the value of the pairwise log-likelihood function for the fitted model. The second and third values are, respectively, the PL-AIC and PL-BIC values of the fitted model.

5.7 Dealing with missing values: the complete-pairs (CP) and the available-cases (AC) PL

As pointed out in Section 5.4 where the hypothesized model is fitted to the GB data, the default setting of the lavaan fit functions is listwise deletion for data with missing values. We saw that, out of the total 2422 observations of the GB data, only 1805 observations were used to fit the hypothesized model. To obtain this information, just type the name of the fitted model in R, `Ex1FittedModel` in our example. The returned output is presented below.

```
Ex1FittedModel

lavaan (0.6-1.1179) converged normally after  98 iterations

                                                  Used       Total
  Number of observations                          1805        2422

  Estimator                                        PML      Robust
  Model Fit Test Statistic                     422.379    1116.759
  Degrees of freedom                                81     131.241
  P-value                                           NA       0.000
  Scaling correction factor                                  0.378
    for the mean and variance adjusted correction
```

To include the cases where at least one of the indicators is observed in the estimation, we can use either of the PL adaptations, the CP or the AC. (More details about them in Section 5.7.) The commands for fitting the hypothesized model to the GB data using the CP and the AC are given below.

```
# Fitting the model using the CP. Note that the input argument missing =
# 'pairwise' determines that the CP should be used. The argument test='none'
# has been added to save computational time from calculating the default
# PLRT for overall fit. Extending the PLRT to the case of CP is still under
# research.
Ex3FittedModel_CP <- sem(model = Ex1Model, data = PoliceDataRcGBOrd, estimator = "PML",
    missing = "pairwise", std.lv = TRUE, test = "none", verbose = TRUE)

# Fitting the model using the AC. Here we specify missing =
# 'available.cases'. The argument test='none' has been added for the same
# reason as above.
Ex3FittedModel_AC <- sem(model = Ex1Model, data = PoliceDataRcGBOrd, estimator = "PML",
    missing = "available.cases", std.lv = TRUE, test = "none", verbose = TRUE)
```

To see how many observations of the GB data are used in the model estimation we type in R the names of the fitted models, `Ex3FittedModel_CP` and `Ex3FittedModel_AC` in our example. The returned outputs are presented below.

```
Ex3FittedModel_CP

lavaan (0.6-1.1179) converged normally after  90 iterations

                                                  Used       Total
  Number of observations                          2421        2422
  Number of missing patterns                       193

  Estimator                                        PML

Ex3FittedModel_AC

lavaan (0.6-1.1179) converged normally after 105 iterations

                                                  Used       Total
  Number of observations                          2421        2422
  Number of missing patterns                       193

  Estimator                                        PML
```

We see that both estimation methods use all but one observations of the data. To print the full output of the fitted models, we use as before the function `summary()`. The full outputs of the fitted models are given in the Appendix. Comparing the results to those of `Ex1FittedModel` in Section 5.4, we see that the CP and the AC standard errors are smaller for all parameter estimates. This is expected as both use more observations to fit the model than in the case of listwise deletion. Comparing the results between the CP and the AC, both methods give nearly identical estimates and standard errors for all the parameters except maybe the thresholds where the AC standard errors tend to be smaller than the CP ones. As mentioned in Section 3.5.2, for missing at random (MAR) data, a simulation study indicates that the CP and the AC perform nearly identically in the estimation of loadings and factor correlations but not in the estimation of thresholds. AC tends to yield less biased threshold estimates than CP but underestimates their standard errors, while the CP standard errors are more reliable. The simulation study also suggests that the quality of the estimation of thresholds does not have a negative impact on the estimation of the rest of the parameters and concludes that, as long as the thresholds are not the parameters of interest, for the remaining parameters, the CP and the AC provide reliable results and can be used interchangeably.

The CP and the AC can also be used to fit a multi-group SEM. Below we provide the commands for the two-group model fitted in Section 5.5.

```r
# Using the CP.
Ex4FittedMeasEquivModel_CP <- sem(model = Ex2_MeasEquivModel, data = PoliceDataRcGBIEOrd,
    std.lv = TRUE, estimator = "PML", missing = "pairwise", group = "cntry",
    group.equal = c("loadings", "thresholds"), test = "none", verbose = TRUE)

# Using the AC.
Ex4FittedMeasEquivModel_AC <- sem(model = Ex2_MeasEquivModel, data = PoliceDataRcGBIEOrd,
    std.lv = TRUE, estimator = "PML", missing = "available.cases", group = "cntry",
    group.equal = c("loadings", "thresholds"), test = "none", verbose = TRUE)
```

6 Appendix

6.1 The output of the model fitted in Section 5.4

```
summary(Ex1FittedModel)

lavaan (0.6-1.1179) converged normally after  98 iterations

                                                  Used       Total
  Number of observations                          1805        2422

  Estimator                                        PML      Robust
  Model Fit Test Statistic                     422.379    1116.759
  Degrees of freedom                                81     131.241
  P-value                                           NA       0.000
  Scaling correction factor                                  0.378
    for the mean and variance adjusted correction

Parameter Estimates:

  Information                                 Observed
  Observed information based on                Hessian
  Standard Errors                   Robust.huber.white

Latent Variables:
                   Estimate  Std.Err  z-value  P(>|z|)
  TrEf =~
    plcpvcr           0.772    0.022   34.383    0.000
    plccbrg           0.741    0.020   36.467    0.000
    plcarcr           0.633    0.022   28.779    0.000
  TrFa =~
    plcrspc           0.830    0.019   44.540    0.000
    plcfrdc           0.858    0.018   47.060    0.000
    plcexdc           0.723    0.019   37.535    0.000
  ObOb =~
    bplcdc            0.614    0.014   44.173    0.000
    doplcsy           0.817    0.018   46.159    0.000
    dpcstrb           0.775    0.015   50.661    0.000
  MoAl =~
    plcrgwr           0.464    0.024   19.189    0.000
    plcipvl           0.549    0.026   20.736    0.000
    gsupplc           0.528    0.024   22.436    0.000
  WiCo =~
    caplcst           0.734    0.018   41.024    0.000
    widprsn           0.960    0.014   70.502    0.000
    wevdct            0.875    0.013   65.589    0.000
```

```
Regressions:
                   Estimate  Std.Err  z-value  P(>|z|)
  ObOb ~
    TrEf              0.276    0.047    5.939    0.000
    TrFa              0.285    0.047    6.116    0.000
  MoAl ~
    TrEf              0.379    0.067    5.676    0.000
    TrFa              0.976    0.088   11.034    0.000
  WiCo ~
    TrEf             -0.072    0.049   -1.477    0.140
    ObOb              0.115    0.033    3.469    0.001
    MoAl              0.141    0.033    4.324    0.000

Covariances:
                   Estimate  Std.Err  z-value  P(>|z|)
  TrEf ~~
    TrFa              0.620    0.027   23.095    0.000
 .ObOb ~~
   .MoAl              0.256    0.041    6.271    0.000

Intercepts:
                   Estimate  Std.Err  z-value  P(>|z|)
   .plcpvcr           0.000
   .plccbrg           0.000
   .plcarcr           0.000
   .plcrspc           0.000
   .plcfrdc           0.000
   .plcexdc           0.000
   .bplcdc            0.000
   .doplcsy           0.000
   .dpcstrb           0.000
   .plcrgwr           0.000
   .plcipvl           0.000
   .gsupplc           0.000
   .caplcst           0.000
   .widprsn           0.000
   .wevdct            0.000
    TrEf              0.000
    TrFa              0.000
   .ObOb              0.000
   .MoAl              0.000
   .WiCo              0.000

Thresholds:
                   Estimate  Std.Err  z-value  P(>|z|)
    plcpvcr|t1       -2.171    0.075  -28.883    0.000
```

```
    plcpvcr|t2        -1.779    0.054  -32.703    0.000
    plcpvcr|t3        -1.353    0.042  -32.281    0.000
    plcpvcr|t4        -0.889    0.034  -25.943    0.000
    plcpvcr|t5        -0.477    0.031  -15.541    0.000
    plcpvcr|t6         0.060    0.029    2.045    0.041
    plcpvcr|t7         0.508    0.031   16.558    0.000
    plcpvcr|t8         1.136    0.037   30.399    0.000
    plcpvcr|t9         1.955    0.062   31.365    0.000
    plcpvcr|t10        2.438    0.099   24.745    0.000
    plccbrg|t1        -1.875    0.059  -31.890    0.000
    plccbrg|t2        -1.408    0.043  -32.640    0.000
    plccbrg|t3        -0.883    0.034  -25.808    0.000
    plccbrg|t4        -0.394    0.030  -12.997    0.000
    plccbrg|t5         0.014    0.029    0.477    0.633
    plccbrg|t6         0.505    0.031   16.461    0.000
    plccbrg|t7         0.902    0.034   26.434    0.000
    plccbrg|t8         1.465    0.044   33.113    0.000
    plccbrg|t9         2.050    0.068   30.333    0.000
    plccbrg|t10        2.532    0.109   23.212    0.000
    plcarcr|t1        -2.125    0.072  -29.629    0.000
    plcarcr|t2        -1.743    0.053  -32.830    0.000
    plcarcr|t3        -1.383    0.043  -32.357    0.000
    plcarcr|t4        -1.000    0.036  -27.922    0.000
    plcarcr|t5        -0.643    0.032  -20.111    0.000
    plcarcr|t6        -0.257    0.030   -8.634    0.000
    plcarcr|t7         0.071    0.029    2.429    0.015
    plcarcr|t8         0.596    0.031   19.080    0.000
    plcarcr|t9         1.352    0.042   32.561    0.000
    plcarcr|t10        2.046    0.067   30.362    0.000
    plcrspc|t1        -2.096    0.070  -30.053    0.000
    plcrspc|t2        -0.991    0.036  -27.768    0.000
    plcrspc|t3         0.937    0.035   27.079    0.000
    plcfrdc|t1        -2.173    0.075  -29.063    0.000
    plcfrdc|t2        -0.886    0.034  -25.795    0.000
    plcfrdc|t3         1.209    0.039   31.182    0.000
    plcexdc|t1        -1.784    0.055  -32.574    0.000
    plcexdc|t2        -0.443    0.031  -14.473    0.000
    plcexdc|t3         1.139    0.038   30.308    0.000
    bplcdc|t1         -1.508    0.045  -33.235    0.000
    bplcdc|t2         -1.297    0.040  -32.105    0.000
    bplcdc|t3         -0.998    0.035  -28.282    0.000
    bplcdc|t4         -0.636    0.032  -20.107    0.000
    bplcdc|t5         -0.351    0.030  -11.696    0.000
    bplcdc|t6          0.129    0.030    4.366    0.000
    bplcdc|t7          0.397    0.030   13.070    0.000
    bplcdc|t8          0.765    0.033   23.226    0.000
```

```
    bplcdc|t9          1.361    0.042   32.458    0.000
    bplcdc|t10         1.810    0.055   32.611    0.000
    doplcsy|t1        -1.894    0.059  -31.868    0.000
    doplcsy|t2        -1.646    0.050  -33.050    0.000
    doplcsy|t3        -1.408    0.043  -32.668    0.000
    doplcsy|t4        -1.050    0.036  -28.917    0.000
    doplcsy|t5        -0.767    0.033  -23.320    0.000
    doplcsy|t6        -0.281    0.030   -9.445    0.000
    doplcsy|t7        -0.010    0.029   -0.356    0.721
    doplcsy|t8         0.390    0.030   12.977    0.000
    doplcsy|t9         0.989    0.035   28.133    0.000
    doplcsy|t10        1.452    0.044   32.939    0.000
    dpcstrb|t1        -1.950    0.062  -31.558    0.000
    dpcstrb|t2        -1.684    0.051  -32.848    0.000
    dpcstrb|t3        -1.302    0.041  -31.830    0.000
    dpcstrb|t4        -0.970    0.035  -27.514    0.000
    dpcstrb|t5        -0.689    0.032  -21.413    0.000
    dpcstrb|t6        -0.184    0.029   -6.242    0.000
    dpcstrb|t7         0.113    0.029    3.850    0.000
    dpcstrb|t8         0.569    0.031   18.338    0.000
    dpcstrb|t9         1.107    0.037   29.951    0.000
    dpcstrb|t10        1.591    0.048   33.173    0.000
    plcrgwr|t1        -2.019    0.066  -30.628    0.000
    plcrgwr|t2        -0.997    0.036  -28.010    0.000
    plcrgwr|t3        -0.505    0.031  -16.354    0.000
    plcrgwr|t4         1.348    0.042   32.240    0.000
    plcipvl|t1        -2.162    0.074  -29.208    0.000
    plcipvl|t2        -1.134    0.038  -30.024    0.000
    plcipvl|t3        -0.529    0.031  -17.036    0.000
    plcipvl|t4         1.417    0.043   32.721    0.000
    gsupplc|t1        -2.322    0.085  -27.398    0.000
    gsupplc|t2        -1.425    0.044  -32.543    0.000
    gsupplc|t3        -0.834    0.034  -24.714    0.000
    gsupplc|t4         1.354    0.042   32.380    0.000
    caplcst|t1        -2.223    0.079  -28.235    0.000
    caplcst|t2        -1.609    0.048  -33.198    0.000
    caplcst|t3        -0.528    0.031  -17.051    0.000
    widprsn|t1        -2.054    0.068  -30.386    0.000
    widprsn|t2        -1.363    0.042  -32.512    0.000
    widprsn|t3        -0.170    0.030   -5.735    0.000
    wevdct|t1         -1.712    0.052  -32.757    0.000
    wevdct|t2         -0.983    0.035  -27.991    0.000
    wevdct|t3          0.018    0.030    0.604    0.546

Variances:
                   Estimate  Std.Err  z-value  P(>|z|)
```

```
   .plcpvcr            0.404
   .plccbrg            0.451
   .plcarcr            0.599
   .plcrspc            0.310
   .plcfrdc            0.263
   .plcexdc            0.478
   .bplcdc             0.527
   .doplcsy            0.163
   .dpcstrb            0.246
   .plcrgwr            0.449
   .plcipvl            0.230
   .gsupplc            0.288
   .caplcst            0.422
   .widprsn            0.011
   .wevdct             0.177
    TrEf               1.000
    TrFa               1.000
   .ObOb               1.000
   .MoAl               1.000
   .WiCo               1.000

Scales y*:
                   Estimate  Std.Err  z-value  P(>|z|)
    plcpvcr           1.000
    plccbrg           1.000
    plcarcr           1.000
    plcrspc           1.000
    plcfrdc           1.000
    plcexdc           1.000
    bplcdc            1.000
    doplcsy           1.000
    dpcstrb           1.000
    plcrgwr           1.000
    plcipvl           1.000
    gsupplc           1.000
    caplcst           1.000
    widprsn           1.000
    wevdct            1.000
```

6.2 The output of the model fitted in Section 5.5

```
summary(Ex2FittedMeasEquivModel)

lavaan (0.6-1.1179) converged normally after 162 iterations
```

```
                                                  Used       Total
  Number of observations per group
  GB                                              1805        2422
  IE                                              1732        2576

  Estimator                                        PML      Robust
  Model Fit Test Statistic                    4789.878    -106.679
  Degrees of freedom                               242       5.110
  P-value                                           NA          NA
  Scaling correction factor                                -44.900
    for the mean and variance adjusted correction

Chi-square for each group:

  GB                                          2384.416     -53.105
  IE                                          2405.461     -53.574

Parameter Estimates:

  Information                                 Observed
  Observed information based on                Hessian
  Standard Errors                   Robust.huber.white


Group 1 [GB]:

Latent Variables:
                   Estimate  Std.Err  z-value  P(>|z|)
  TrEf =~
    plcpvcr (.p1.)    0.766    0.026   28.923    0.000
    plccbrg (.p2.)    0.710    0.025   28.029    0.000
    plcarcr (.p3.)    0.662    0.023   28.330    0.000
  TrFa =~
    plcrspc (.p4.)    0.830    0.024   34.743    0.000
    plcfrdc (.p5.)    0.875    0.023   38.185    0.000
    plcexdc (.p6.)    0.702    0.025   27.828    0.000
  ObOb =~
    bplcdc  (.p7.)    0.634    0.019   34.077    0.000
    doplcsy (.p8.)    0.801    0.024   33.836    0.000
    dpcstrb (.p9.)    0.763    0.021   36.224    0.000
  MoAl =~
    plcrgwr (.10.)    0.471    0.032   14.881    0.000
    plcipvl (.11.)    0.550    0.035   15.526    0.000
    gsupplc (.12.)    0.514    0.032   16.073    0.000
  WiCo =~
    caplcst (.13.)    0.731    0.025   28.863    0.000
```

```
    widprsn (.14.)    0.971    0.018   53.419    0.000
    wevdct  (.15.)    0.866    0.020   43.551    0.000

Regressions:
                   Estimate  Std.Err  z-value  P(>|z|)
  ObOb ~
    TrEf              0.288    0.067    4.324    0.000
    TrFa              0.282    0.066    4.261    0.000
  MoAl ~
    TrEf              0.393    0.094    4.176    0.000
    TrFa              0.964    0.122    7.894    0.000
  WiCo ~
    TrEf             -0.069    0.070   -0.986    0.324
    ObOb              0.114    0.047    2.446    0.014
    MoAl              0.139    0.046    3.029    0.002

Covariances:
                   Estimate  Std.Err  z-value  P(>|z|)
  TrEf ~~
    TrFa              0.626    0.038   16.656    0.000
 .ObOb ~~
   .MoAl              0.256    0.057    4.468    0.000

Intercepts:
                   Estimate  Std.Err  z-value  P(>|z|)
   .plcpvcr           0.000
   .plccbrg           0.000
   .plcarcr           0.000
   .plcrspc           0.000
   .plcfrdc           0.000
   .plcexdc           0.000
   .bplcdc            0.000
   .doplcsy           0.000
   .dpcstrb           0.000
   .plcrgwr           0.000
   .plcipvl           0.000
   .gsupplc           0.000
   .caplcst           0.000
   .widprsn           0.000
   .wevdct            0.000
    TrEf              0.000
    TrFa              0.000
   .ObOb              0.000
   .MoAl              0.000
   .WiCo              0.000
```

```
Thresholds:
                   Estimate  Std.Err  z-value  P(>|z|)
    plcpv|1 (.30.)   -2.243    0.088  -25.473    0.000
    plcpv|2 (.31.)   -1.846    0.068  -27.217    0.000
    plcpv|3 (.32.)   -1.400    0.053  -26.368    0.000
    plcpv|4 (.33.)   -0.954    0.044  -21.601    0.000
    plcpv|5 (.34.)   -0.503    0.038  -13.104    0.000
    plcpv|6 (.35.)    0.045    0.036    1.246    0.213
    plcpv|7 (.36.)    0.493    0.037   13.194    0.000
    plcpv|8 (.37.)    1.078    0.044   24.240    0.000
    plcpv|9 (.38.)    1.855    0.067   27.577    0.000
    plcp|10 (.39.)    2.332    0.098   23.751    0.000
    plccb|1 (.40.)   -1.856    0.068  -27.371    0.000
    plccb|2 (.41.)   -1.386    0.052  -26.462    0.000
    plccb|3 (.42.)   -0.880    0.043  -20.622    0.000
    plccb|4 (.43.)   -0.378    0.037  -10.120    0.000
    plccb|5 (.44.)    0.047    0.036    1.309    0.191
    plccb|6 (.45.)    0.543    0.038   14.329    0.000
    plccb|7 (.46.)    0.930    0.042   22.004    0.000
    plccb|8 (.47.)    1.446    0.052   27.622    0.000
    plccb|9 (.48.)    2.091    0.081   25.684    0.000
    plcc|10 (.49.)    2.618    0.126   20.768    0.000
    plcrc|1 (.50.)   -2.194    0.086  -25.396    0.000
    plcrc|2 (.51.)   -1.812    0.069  -26.352    0.000
    plcrc|3 (.52.)   -1.406    0.056  -25.288    0.000
    plcrc|4 (.53.)   -0.940    0.044  -21.291    0.000
    plcrc|5 (.54.)   -0.567    0.038  -14.900    0.000
    plcrc|6 (.55.)   -0.200    0.035   -5.680    0.000
    plcrc|7 (.56.)    0.106    0.035    3.084    0.002
    plcrc|8 (.57.)    0.598    0.037   16.255    0.000
    plcrc|9 (.58.)    1.355    0.050   27.247    0.000
    plcr|10 (.59.)    2.025    0.075   26.914    0.000
    plcrs|1 (.60.)   -2.156    0.090  -23.958    0.000
    plcrs|2 (.61.)   -0.996    0.047  -21.092    0.000
    plcrs|3 (.62.)    0.912    0.045   20.408    0.000
    plcfr|1 (.63.)   -2.196    0.091  -24.021    0.000
    plcfr|2 (.64.)   -0.902    0.046  -19.772    0.000
    plcfr|3 (.65.)    1.179    0.050   23.393    0.000
    plcxd|1 (.66.)   -1.732    0.066  -26.433    0.000
    plcxd|2 (.67.)   -0.404    0.037  -10.766    0.000
    plcxd|3 (.68.)    1.196    0.048   25.022    0.000
    bplcd|1 (.69.)   -1.639    0.060  -27.338    0.000
    bplcd|2 (.70.)   -1.405    0.053  -26.485    0.000
    bplcd|3 (.71.)   -1.031    0.044  -23.502    0.000
    bplcd|4 (.72.)   -0.621    0.037  -16.696    0.000
    bplcd|5 (.73.)   -0.363    0.035  -10.330    0.000
```

```
    bplcd|6 (.74.)    0.089    0.034    2.581    0.010
    bplcd|7 (.75.)    0.365    0.036   10.249    0.000
    bplcd|8 (.76.)    0.706    0.039   18.157    0.000
    bplcd|9 (.77.)    1.263    0.049   25.886    0.000
    bplc|10 (.78.)    1.684    0.062   27.273    0.000
    dplcs|1 (.79.)   -1.980    0.080  -24.697    0.000
    dplcs|2 (.80.)   -1.724    0.070  -24.524    0.000
    dplcs|3 (.81.)   -1.399    0.059  -23.858    0.000
    dplcs|4 (.82.)   -0.940    0.046  -20.413    0.000
    dplcs|5 (.83.)   -0.678    0.042  -16.339    0.000
    dplcs|6 (.84.)   -0.230    0.037   -6.149    0.000
    dplcs|7 (.85.)    0.025    0.037    0.675    0.500
    dplcs|8 (.86.)    0.435    0.038   11.383    0.000
    dplcs|9 (.87.)    1.015    0.045   22.380    0.000
    dplc|10 (.88.)    1.472    0.056   26.239    0.000
    dpcst|1 (.89.)   -2.077    0.082  -25.190    0.000
    dpcst|2 (.90.)   -1.774    0.069  -25.749    0.000
    dpcst|3 (.91.)   -1.360    0.056  -24.274    0.000
    dpcst|4 (.92.)   -0.924    0.045  -20.616    0.000
    dpcst|5 (.93.)   -0.636    0.040  -15.899    0.000
    dpcst|6 (.94.)   -0.164    0.036   -4.516    0.000
    dpcst|7 (.95.)    0.126    0.036    3.487    0.000
    dpcst|8 (.96.)    0.548    0.038   14.341    0.000
    dpcst|9 (.97.)    1.089    0.046   23.666    0.000
    dpcs|10 (.98.)    1.559    0.058   26.784    0.000
    plcrg|1 (.99.)   -2.089    0.086  -24.428    0.000
    plcrg|2 (.100)   -0.983    0.046  -21.191    0.000
    plcrg|3 (.101)   -0.437    0.038  -11.500    0.000
    plcrg|4 (.102)    1.331    0.051   25.931    0.000
    plcpv|1 (.103)   -2.207    0.090  -24.434    0.000
    plcpv|2 (.104)   -1.176    0.051  -23.047    0.000
    plcpv|3 (.105)   -0.519    0.040  -12.959    0.000
    plcpv|4 (.106)    1.369    0.055   24.869    0.000
    gsppl|1 (.107)   -2.460    0.110  -22.307    0.000
    gsppl|2 (.108)   -1.508    0.062  -24.424    0.000
    gsppl|3 (.109)   -0.779    0.043  -18.039    0.000
    gsppl|4 (.110)    1.312    0.052   25.222    0.000
    cplcs|1 (.111)   -2.266    0.098  -23.008    0.000
    cplcs|2 (.112)   -1.595    0.062  -25.850    0.000
    cplcs|3 (.113)   -0.519    0.037  -14.089    0.000
    wdprs|1 (.114)   -2.174    0.099  -21.979    0.000
    wdprs|2 (.115)   -1.312    0.055  -23.895    0.000
    wdprs|3 (.116)   -0.188    0.041   -4.620    0.000
    wvdct|1 (.117)   -1.738    0.069  -25.191    0.000
    wvdct|2 (.118)   -0.964    0.044  -21.715    0.000
    wvdct|3 (.119)    0.021    0.040    0.507    0.612
```

```
Variances:
                   Estimate  Std.Err  z-value  P(>|z|)
    TrEf              1.000
    TrFa              1.000
   .ObOb              1.000
   .MoAl              1.000
   .WiCo              1.000
   .plcpvcr           0.413
   .plccbrg           0.496
   .plcarcr           0.562
   .plcrspc           0.310
   .plcfrdc           0.234
   .plcexdc           0.508
   .bplcdc            0.491
   .doplcsy           0.190
   .dpcstrb           0.263
   .plcrgwr           0.433
   .plcipvl           0.226
   .gsupplc           0.323
   .caplcst           0.427
   .widprsn          -0.012
   .wevdct            0.196

Scales y*:
                   Estimate  Std.Err  z-value  P(>|z|)
    plcpvcr           1.000
    plccbrg           1.000
    plcarcr           1.000
    plcrspc           1.000
    plcfrdc           1.000
    plcexdc           1.000
    bplcdc            1.000
    doplcsy           1.000
    dpcstrb           1.000
    plcrgwr           1.000
    plcipvl           1.000
    gsupplc           1.000
    caplcst           1.000
    widprsn           1.000
    wevdct            1.000


Group 2 [IE]:

Latent Variables:
                   Estimate  Std.Err  z-value  P(>|z|)
```

```
  TrEf =~
    plcpvcr (.p1.)    0.766    0.026   28.923    0.000
    plccbrg (.p2.)    0.710    0.025   28.029    0.000
    plcarcr (.p3.)    0.662    0.023   28.330    0.000
  TrFa =~
    plcrspc (.p4.)    0.830    0.024   34.743    0.000
    plcfrdc (.p5.)    0.875    0.023   38.185    0.000
    plcexdc (.p6.)    0.702    0.025   27.828    0.000
  ObOb =~
    bplcdc  (.p7.)    0.634    0.019   34.077    0.000
    doplcsy (.p8.)    0.801    0.024   33.836    0.000
    dpcstrb (.p9.)    0.763    0.021   36.224    0.000
  MoAl =~
    plcrgwr (.10.)    0.471    0.032   14.881    0.000
    plcipvl (.11.)    0.550    0.035   15.526    0.000
    gsupplc (.12.)    0.514    0.032   16.073    0.000
  WiCo =~
    caplcst (.13.)    0.731    0.025   28.863    0.000
    widprsn (.14.)    0.971    0.018   53.419    0.000
    wevdct  (.15.)    0.866    0.020   43.551    0.000

Regressions:
                   Estimate  Std.Err  z-value  P(>|z|)
  ObOb ~
    TrEf              0.223    0.074    3.019    0.003
    TrFa              0.420    0.070    5.979    0.000
  MoAl ~
    TrEf              0.573    0.114    5.009    0.000
    TrFa              1.034    0.127    8.143    0.000
  WiCo ~
    TrEf              0.060    0.067    0.893    0.372
    ObOb              0.023    0.042    0.547    0.584
    MoAl              0.157    0.038    4.126    0.000

Covariances:
                   Estimate  Std.Err  z-value  P(>|z|)
  TrEf ~~
    TrFa              0.923    0.090   10.277    0.000
 .ObOb ~~
   .MoAl              0.332    0.078    4.241    0.000

Intercepts:
                   Estimate  Std.Err  z-value  P(>|z|)
   .plcpvcr           0.000
   .plccbrg           0.000
   .plcarcr           0.000
```

```
   .plcrspc           0.000
   .plcfrdc           0.000
   .plcexdc           0.000
   .bplcdc            0.000
   .doplcsy           0.000
   .dpcstrb           0.000
   .plcrgwr           0.000
   .plcipvl           0.000
   .gsupplc           0.000
   .caplcst           0.000
   .widprsn           0.000
   .wevdct            0.000
    TrEf             -0.143    0.059   -2.408    0.016
    TrFa              0.200    0.061    3.300    0.001
   .ObOb             -0.205    0.058   -3.543    0.000
   .MoAl              0.197    0.085    2.322    0.020
   .WiCo             -0.482    0.058   -8.364    0.000

Thresholds:
                   Estimate  Std.Err  z-value  P(>|z|)
    plcpv|1 (.30.)   -2.243    0.088  -25.473    0.000
    plcpv|2 (.31.)   -1.846    0.068  -27.217    0.000
    plcpv|3 (.32.)   -1.400    0.053  -26.368    0.000
    plcpv|4 (.33.)   -0.954    0.044  -21.601    0.000
    plcpv|5 (.34.)   -0.503    0.038  -13.104    0.000
    plcpv|6 (.35.)    0.045    0.036    1.246    0.213
    plcpv|7 (.36.)    0.493    0.037   13.194    0.000
    plcpv|8 (.37.)    1.078    0.044   24.240    0.000
    plcpv|9 (.38.)    1.855    0.067   27.577    0.000
    plcp|10 (.39.)    2.332    0.098   23.751    0.000
    plccb|1 (.40.)   -1.856    0.068  -27.371    0.000
    plccb|2 (.41.)   -1.386    0.052  -26.462    0.000
    plccb|3 (.42.)   -0.880    0.043  -20.622    0.000
    plccb|4 (.43.)   -0.378    0.037  -10.120    0.000
    plccb|5 (.44.)    0.047    0.036    1.309    0.191
    plccb|6 (.45.)    0.543    0.038   14.329    0.000
    plccb|7 (.46.)    0.930    0.042   22.004    0.000
    plccb|8 (.47.)    1.446    0.052   27.622    0.000
    plccb|9 (.48.)    2.091    0.081   25.684    0.000
    plcc|10 (.49.)    2.618    0.126   20.768    0.000
    plcrc|1 (.50.)   -2.194    0.086  -25.396    0.000
    plcrc|2 (.51.)   -1.812    0.069  -26.352    0.000
    plcrc|3 (.52.)   -1.406    0.056  -25.288    0.000
    plcrc|4 (.53.)   -0.940    0.044  -21.291    0.000
    plcrc|5 (.54.)   -0.567    0.038  -14.900    0.000
    plcrc|6 (.55.)   -0.200    0.035   -5.680    0.000
```

```
    plcrc|7 (.56.)    0.106    0.035    3.084    0.002
    plcrc|8 (.57.)    0.598    0.037   16.255    0.000
    plcrc|9 (.58.)    1.355    0.050   27.247    0.000
    plcr|10 (.59.)    2.025    0.075   26.914    0.000
    plcrs|1 (.60.)   -2.156    0.090  -23.958    0.000
    plcrs|2 (.61.)   -0.996    0.047  -21.092    0.000
    plcrs|3 (.62.)    0.912    0.045   20.408    0.000
    plcfr|1 (.63.)   -2.196    0.091  -24.021    0.000
    plcfr|2 (.64.)   -0.902    0.046  -19.772    0.000
    plcfr|3 (.65.)    1.179    0.050   23.393    0.000
    plcxd|1 (.66.)   -1.732    0.066  -26.433    0.000
    plcxd|2 (.67.)   -0.404    0.037  -10.766    0.000
    plcxd|3 (.68.)    1.196    0.048   25.022    0.000
    bplcd|1 (.69.)   -1.639    0.060  -27.338    0.000
    bplcd|2 (.70.)   -1.405    0.053  -26.485    0.000
    bplcd|3 (.71.)   -1.031    0.044  -23.502    0.000
    bplcd|4 (.72.)   -0.621    0.037  -16.696    0.000
    bplcd|5 (.73.)   -0.363    0.035  -10.330    0.000
    bplcd|6 (.74.)    0.089    0.034    2.581    0.010
    bplcd|7 (.75.)    0.365    0.036   10.249    0.000
    bplcd|8 (.76.)    0.706    0.039   18.157    0.000
    bplcd|9 (.77.)    1.263    0.049   25.886    0.000
    bplc|10 (.78.)    1.684    0.062   27.273    0.000
    dplcs|1 (.79.)   -1.980    0.080  -24.697    0.000
    dplcs|2 (.80.)   -1.724    0.070  -24.524    0.000
    dplcs|3 (.81.)   -1.399    0.059  -23.858    0.000
    dplcs|4 (.82.)   -0.940    0.046  -20.413    0.000
    dplcs|5 (.83.)   -0.678    0.042  -16.339    0.000
    dplcs|6 (.84.)   -0.230    0.037   -6.149    0.000
    dplcs|7 (.85.)    0.025    0.037    0.675    0.500
    dplcs|8 (.86.)    0.435    0.038   11.383    0.000
    dplcs|9 (.87.)    1.015    0.045   22.380    0.000
    dplc|10 (.88.)    1.472    0.056   26.239    0.000
    dpcst|1 (.89.)   -2.077    0.082  -25.190    0.000
    dpcst|2 (.90.)   -1.774    0.069  -25.749    0.000
    dpcst|3 (.91.)   -1.360    0.056  -24.274    0.000
    dpcst|4 (.92.)   -0.924    0.045  -20.616    0.000
    dpcst|5 (.93.)   -0.636    0.040  -15.899    0.000
    dpcst|6 (.94.)   -0.164    0.036   -4.516    0.000
    dpcst|7 (.95.)    0.126    0.036    3.487    0.000
    dpcst|8 (.96.)    0.548    0.038   14.341    0.000
    dpcst|9 (.97.)    1.089    0.046   23.666    0.000
    dpcs|10 (.98.)    1.559    0.058   26.784    0.000
    plcrg|1 (.99.)   -2.089    0.086  -24.428    0.000
    plcrg|2 (.100)   -0.983    0.046  -21.191    0.000
    plcrg|3 (.101)   -0.437    0.038  -11.500    0.000
```

```
    plcrg|4 (.102)    1.331    0.051   25.931    0.000
    plcpv|1 (.103)   -2.207    0.090  -24.434    0.000
    plcpv|2 (.104)   -1.176    0.051  -23.047    0.000
    plcpv|3 (.105)   -0.519    0.040  -12.959    0.000
    plcpv|4 (.106)    1.369    0.055   24.869    0.000
    gsppl|1 (.107)   -2.460    0.110  -22.307    0.000
    gsppl|2 (.108)   -1.508    0.062  -24.424    0.000
    gsppl|3 (.109)   -0.779    0.043  -18.039    0.000
    gsppl|4 (.110)    1.312    0.052   25.222    0.000
    cplcs|1 (.111)   -2.266    0.098  -23.008    0.000
    cplcs|2 (.112)   -1.595    0.062  -25.850    0.000
    cplcs|3 (.113)   -0.519    0.037  -14.089    0.000
    wdprs|1 (.114)   -2.174    0.099  -21.979    0.000
    wdprs|2 (.115)   -1.312    0.055  -23.895    0.000
    wdprs|3 (.116)   -0.188    0.041   -4.620    0.000
    wvdct|1 (.117)   -1.738    0.069  -25.191    0.000
    wvdct|2 (.118)   -0.964    0.044  -21.715    0.000
    wvdct|3 (.119)    0.021    0.040    0.507    0.612

Variances:
                   Estimate  Std.Err  z-value  P(>|z|)
    TrEf              1.282    0.126   10.141    0.000
    TrFa              1.370    0.137    9.967    0.000
   .ObOb              1.087    0.107   10.163    0.000
   .MoAl              1.548    0.275    5.621    0.000
   .WiCo              0.829    0.097    8.529    0.000
   .plcpvcr           0.381
   .plccbrg           0.627
   .plcarcr           0.583
   .plcrspc           0.414
   .plcfrdc           0.255
   .plcexdc           0.600
   .bplcdc            0.514
   .doplcsy           0.138
   .dpcstrb           0.270
   .plcrgwr           0.505
   .plcipvl           0.196
   .gsupplc           0.318
   .caplcst           0.361
   .widprsn           0.059
   .wevdct            0.181

Scales y*:
                   Estimate  Std.Err  z-value  P(>|z|)
    plcpvcr           0.939    0.032   28.959    0.000
    plccbrg           0.886    0.030   29.158    0.000
```

```
    plcarcr           0.934    0.032   29.266    0.000
    plcrspc           0.858    0.035   24.221    0.000
    plcfrdc           0.875    0.036   24.396    0.000
    plcexdc           0.886    0.035   25.207    0.000
    bplcdc            0.935    0.033   28.732    0.000
    doplcsy           0.936    0.035   26.964    0.000
    dpcstrb           0.920    0.033   28.001    0.000
    plcrgwr           0.814    0.031   25.849    0.000
    plcipvl           0.799    0.032   25.104    0.000
    gsupplc           0.812    0.031   26.052    0.000
    caplcst           1.059    0.055   19.380    0.000
    widprsn           1.003    0.058   17.398    0.000
    wevdct            1.040    0.055   18.745    0.000
```

6.3 The outputs of the models fitted in Section 5.6

```
summary(Ex2FittedMinConModel)

lavaan (0.6-1.1179) converged normally after 285 iterations

                                                  Used       Total
  Number of observations per group
  GB                                              1805        2422
  IE                                              1732        2576

  Estimator                                        PML      Robust
  Model Fit Test Statistic                     899.215     -28.269
  Degrees of freedom                               162       5.743
  P-value                                           NA          NA
  Scaling correction factor                                -31.810
    for the mean and variance adjusted correction

Chi-square for each group:

  GB                                           422.379     -13.278
  IE                                           476.836     -14.990

Parameter Estimates:

  Information                                 Observed
  Observed information based on                Hessian
  Standard Errors                   Robust.huber.white


Group 1 [GB]:
```

```
Latent Variables:
                   Estimate  Std.Err  z-value  P(>|z|)
  TrEf =~
    plcpvcr   (c1)    0.772    0.031   24.563    0.000
    plccbrg           0.741    0.028   26.051    0.000
    plcarcr           0.633    0.031   20.559    0.000
  TrFa =~
    plcrspc   (c4)    0.830    0.026   31.819    0.000
    plcfrdc           0.858    0.026   33.618    0.000
    plcexdc           0.723    0.027   26.814    0.000
  ObOb =~
    bplcdc    (c7)    0.614    0.019   31.556    0.000
    doplcsy           0.817    0.025   32.975    0.000
    dpcstrb           0.775    0.021   36.191    0.000
  MoAl =~
    plcrgwr  (c10)    0.464    0.034   13.709    0.000
    plcipvl           0.549    0.037   14.815    0.000
    gsupplc           0.528    0.033   16.029    0.000
  WiCo =~
    caplcst  (c13)    0.734    0.025   29.306    0.000
    widprsn           0.960    0.019   50.364    0.000
    wevdct            0.875    0.019   46.854    0.000

Regressions:
                   Estimate  Std.Err  z-value  P(>|z|)
  ObOb ~
    TrEf              0.276    0.065    4.242    0.000
    TrFa              0.285    0.065    4.369    0.000
  MoAl ~
    TrEf              0.379    0.094    4.055    0.000
    TrFa              0.976    0.124    7.883    0.000
  WiCo ~
    TrEf             -0.072    0.069   -1.055    0.291
    ObOb              0.115    0.046    2.478    0.013
    MoAl              0.141    0.046    3.089    0.002

Covariances:
                   Estimate  Std.Err  z-value  P(>|z|)
  TrEf ~~
    TrFa              0.620    0.038   16.498    0.000
 .ObOb ~~
   .MoAl              0.256    0.057    4.480    0.000

Intercepts:
                   Estimate  Std.Err  z-value  P(>|z|)
    TrEf              0.000
```

```
    TrFa                0.000
   .ObOb                0.000
   .MoAl                0.000
   .WiCo                0.000
   .plcpvcr             0.000
   .plccbrg             0.000
   .plcarcr             0.000
   .plcrspc             0.000
   .plcfrdc             0.000
   .plcexdc             0.000
   .bplcdc              0.000
   .doplcsy             0.000
   .dpcstrb             0.000
   .plcrgwr             0.000
   .plcipvl             0.000
   .gsupplc             0.000
   .caplcst             0.000
   .widprsn             0.000
   .wevdct              0.000

Thresholds:
                   Estimate  Std.Err  z-value  P(>|z|)
    plcpvc|1  (p1)   -2.171    0.105  -20.634    0.000
    plcpvc|2  (p2)   -1.779    0.076  -23.362    0.000
    plcpvc|3         -1.353    0.059  -23.060    0.000
    plcpvc|4         -0.889    0.048  -18.532    0.000
    plcpvc|5         -0.477    0.043  -11.102    0.000
    plcpvc|6          0.060    0.041    1.461    0.144
    plcpvc|7          0.508    0.043   11.829    0.000
    plcpvc|8          1.136    0.052   21.716    0.000
    plcpvc|9          1.955    0.087   22.406    0.000
    plcpv|10          2.438    0.138   17.677    0.000
    plccbr|1  (p3)   -1.875    0.082  -22.781    0.000
    plccbr|2         -1.408    0.060  -23.317    0.000
    plccbr|3         -0.883    0.048  -18.436    0.000
    plccbr|4         -0.394    0.042   -9.285    0.000
    plccbr|5          0.014    0.041    0.341    0.733
    plccbr|6          0.505    0.043   11.759    0.000
    plccbr|7          0.902    0.048   18.883    0.000
    plccbr|8          1.465    0.062   23.654    0.000
    plccbr|9          2.050    0.095   21.669    0.000
    plccb|10          2.532    0.153   16.582    0.000
    plcrcr|1  (p4)   -2.125    0.100  -21.166    0.000
    plcrcr|2         -1.743    0.074  -23.453    0.000
    plcrcr|3         -1.383    0.060  -23.114    0.000
    plcrcr|4         -1.000    0.050  -19.946    0.000
```

```
    plcrcr|5          -0.643    0.045  -14.366    0.000
    plcrcr|6          -0.257    0.042   -6.168    0.000
    plcrcr|7           0.071    0.041    1.735    0.083
    plcrcr|8           0.596    0.044   13.630    0.000
    plcrcr|9           1.352    0.058   23.261    0.000
    plcrc|10           2.046    0.094   21.690    0.000
    plcrsp|1  (p5)    -2.096    0.098  -21.469    0.000
    plcrsp|2  (p6)    -0.991    0.050  -19.836    0.000
    plcrsp|3           0.937    0.048   19.345    0.000
    plcfrd|1  (p7)    -2.173    0.105  -20.762    0.000
    plcfrd|2          -0.886    0.048  -18.427    0.000
    plcfrd|3           1.209    0.054   22.276    0.000
    plcxdc|1  (p8)    -1.784    0.077  -23.270    0.000
    plcxdc|2          -0.443    0.043  -10.339    0.000
    plcxdc|3           1.139    0.053   21.651    0.000
    bplcdc|1  (p9)    -1.508    0.063  -23.742    0.000
    bplcdc|2 (p10)    -1.297    0.057  -22.935    0.000
    bplcdc|3          -0.998    0.049  -20.204    0.000
    bplcdc|4          -0.636    0.044  -14.364    0.000
    bplcdc|5          -0.351    0.042   -8.355    0.000
    bplcdc|6           0.129    0.041    3.119    0.002
    bplcdc|7           0.397    0.043    9.337    0.000
    bplcdc|8           0.765    0.046   16.592    0.000
    bplcdc|9           1.361    0.059   23.187    0.000
    bplcd|10           1.810    0.078   23.296    0.000
    dplcsy|1 (p11)    -1.894    0.083  -22.766    0.000
    dplcsy|2          -1.646    0.070  -23.610    0.000
    dplcsy|3          -1.408    0.060  -23.337    0.000
    dplcsy|4          -1.050    0.051  -20.657    0.000
    dplcsy|5          -0.767    0.046  -16.659    0.000
    dplcsy|6          -0.281    0.042   -6.747    0.000
    dplcsy|7          -0.010    0.041   -0.255    0.799
    dplcsy|8           0.390    0.042    9.270    0.000
    dplcsy|9           0.989    0.049   20.097    0.000
    dplcs|10           1.452    0.062   23.531    0.000
    dpcstr|1 (p12)    -1.950    0.087  -22.544    0.000
    dpcstr|2          -1.684    0.072  -23.465    0.000
    dpcstr|3          -1.302    0.057  -22.738    0.000
    dpcstr|4          -0.970    0.049  -19.655    0.000
    dpcstr|5          -0.689    0.045  -15.296    0.000
    dpcstr|6          -0.184    0.041   -4.459    0.000
    dpcstr|7           0.113    0.041    2.750    0.006
    dpcstr|8           0.569    0.043   13.100    0.000
    dpcstr|9           1.107    0.052   21.396    0.000
    dpcst|10           1.591    0.067   23.698    0.000
    plcrgw|1 (p13)    -2.019    0.092  -21.880    0.000
```

```
    plcrgw|2 (p14)   -0.997    0.050  -20.009    0.000
    plcrgw|3         -0.505    0.043  -11.683    0.000
    plcrgw|4          1.348    0.059   23.031    0.000
    plcpvl|1 (p15)   -2.162    0.104  -20.865    0.000
    plcpvl|2         -1.134    0.053  -21.448    0.000
    plcpvl|3         -0.529    0.043  -12.170    0.000
    plcpvl|4          1.417    0.061   23.375    0.000
    gspplc|1 (p16)   -2.322    0.119  -19.572    0.000
    gspplc|2         -1.425    0.061  -23.248    0.000
    gspplc|3         -0.834    0.047  -17.655    0.000
    gspplc|4          1.354    0.059   23.131    0.000
    cplcst|1 (p17)   -2.223    0.110  -20.170    0.000
    cplcst|2 (p18)   -1.609    0.068  -23.716    0.000
    cplcst|3         -0.528    0.043  -12.181    0.000
    wdprsn|1 (p19)   -2.054    0.095  -21.706    0.000
    wdprsn|2         -1.363    0.059  -23.226    0.000
    wdprsn|3         -0.170    0.041   -4.097    0.000
    wvdct|t1 (p20)   -1.712    0.073  -23.400    0.000
    wvdct|t2         -0.983    0.049  -19.996    0.000
    wvdct|t3          0.018    0.041    0.431    0.666

Variances:
                   Estimate  Std.Err  z-value  P(>|z|)
    TrEf              1.000
    TrFa              1.000
   .ObOb              1.000
   .MoAl              1.000
   .WiCo              1.000
   .plcpvcr           0.404
   .plccbrg           0.451
   .plcarcr           0.599
   .plcrspc           0.310
   .plcfrdc           0.263
   .plcexdc           0.478
   .bplcdc            0.527
   .doplcsy           0.163
   .dpcstrb           0.246
   .plcrgwr           0.449
   .plcipvl           0.230
   .gsupplc           0.288
   .caplcst           0.422
   .widprsn           0.011
   .wevdct            0.177

Scales y*:
                   Estimate  Std.Err  z-value  P(>|z|)
```

```
    plcpvcr           1.000
    plccbrg           1.000
    plcarcr           1.000
    plcrspc           1.000
    plcfrdc           1.000
    plcexdc           1.000
    bplcdc            1.000
    doplcsy           1.000
    dpcstrb           1.000
    plcrgwr           1.000
    plcipvl           1.000
    gsupplc           1.000
    caplcst           1.000
    widprsn           1.000
    wevdct            1.000


Group 2 [IE]:

Latent Variables:
                   Estimate  Std.Err  z-value  P(>|z|)
  TrEf =~
    plcpvcr   (c1)    0.772    0.031   24.563    0.000
    plccbrg           0.732    0.060   12.183    0.000
    plcarcr           0.652    0.080    8.172    0.000
  TrFa =~
    plcrspc   (c4)    0.830    0.026   31.819    0.000
    plcfrdc           0.916    0.075   12.272    0.000
    plcexdc           0.816    0.077   10.614    0.000
  ObOb =~
    bplcdc    (c7)    0.614    0.019   31.556    0.000
    doplcsy           0.788    0.052   15.128    0.000
    dpcstrb           0.746    0.058   12.905    0.000
  MoAl =~
    plcrgwr  (c10)    0.464    0.034   13.709    0.000
    plcipvl           0.537    0.050   10.793    0.000
    gsupplc           0.468    0.056    8.394    0.000
  WiCo =~
    caplcst  (c13)    0.734    0.025   29.306    0.000
    widprsn           0.918    0.068   13.568    0.000
    wevdct            0.826    0.066   12.532    0.000

Regressions:
                   Estimate  Std.Err  z-value  P(>|z|)
  ObOb ~
    TrEf              0.191    0.097    1.965    0.049
```

```
    TrFa              0.405    0.131    3.093    0.002
  MoAl ~
    TrEf              0.502    0.193    2.597    0.009
    TrFa              1.013    0.218    4.654    0.000
  WiCo ~
    TrEf              0.053    0.061    0.883    0.377
    ObOb              0.025    0.042    0.602    0.547
    MoAl              0.149    0.050    2.992    0.003

Covariances:
                   Estimate  Std.Err  z-value  P(>|z|)
  TrEf ~~
    TrFa              0.821    0.271    3.036    0.002
 .ObOb ~~
   .MoAl              0.241    0.091    2.647    0.008

Intercepts:
                   Estimate  Std.Err  z-value  P(>|z|)
    TrEf             -0.015    0.709   -0.021    0.984
    TrFa              0.078    0.194    0.403    0.687
   .ObOb             -0.299    0.505   -0.592    0.554
   .MoAl             -0.188    0.503   -0.374    0.708
   .WiCo             -0.676    0.346   -1.952    0.051
   .plcpvcr           0.000
   .plccbrg           0.000
   .plcarcr           0.000
   .plcrspc           0.000
   .plcfrdc           0.000
   .plcexdc           0.000
   .bplcdc            0.000
   .doplcsy           0.000
   .dpcstrb           0.000
   .plcrgwr           0.000
   .plcipvl           0.000
   .gsupplc           0.000
   .caplcst           0.000
   .widprsn           0.000
   .wevdct            0.000

Thresholds:
                   Estimate  Std.Err  z-value  P(>|z|)
    plcpvc|1  (p1)   -2.171    0.105  -20.634    0.000
    plcpvc|2  (p2)   -1.779    0.076  -23.362    0.000
    plcpvc|3         -1.325    0.174   -7.596    0.000
    plcpvc|4         -0.905    0.290   -3.125    0.002
    plcpvc|5         -0.423    0.427   -0.988    0.323
```

```
    plcpvc|6            0.124    0.586    0.211    0.833
    plcpvc|7            0.563    0.714    0.789    0.430
    plcpvc|8            1.089    0.869    1.254    0.210
    plcpvc|9            1.821    1.085    1.679    0.093
    plcpv|10            2.291    1.226    1.869    0.062
    plccbr|1  (p3)     -1.875    0.082  -22.781    0.000
    plccbr|2           -1.365    0.150   -9.074    0.000
    plccbr|3           -0.838    0.287   -2.923    0.003
    plccbr|4           -0.283    0.442   -0.640    0.522
    plccbr|5            0.197    0.579    0.340    0.734
    plccbr|6            0.736    0.734    1.003    0.316
    plccbr|7            1.139    0.851    1.339    0.181
    plccbr|8            1.631    0.994    1.641    0.101
    plccbr|9            2.385    1.216    1.962    0.050
    plccb|10            2.997    1.400    2.141    0.032
    plcrcr|1  (p4)     -2.125    0.100  -21.166    0.000
    plcrcr|2           -1.750    0.128  -13.664    0.000
    plcrcr|3           -1.309    0.194   -6.747    0.000
    plcrcr|4           -0.778    0.298   -2.608    0.009
    plcrcr|5           -0.394    0.379   -1.037    0.300
    plcrcr|6           -0.051    0.455   -0.111    0.911
    plcrcr|7            0.227    0.516    0.439    0.661
    plcrcr|8            0.671    0.615    1.090    0.276
    plcrcr|9            1.416    0.785    1.805    0.071
    plcrc|10            2.050    0.934    2.196    0.028
    plcrsp|1  (p5)     -2.096    0.098  -21.469    0.000
    plcrsp|2  (p6)     -0.991    0.050  -19.836    0.000
    plcrsp|3            0.714    0.236    3.027    0.002
    plcfrd|1  (p7)     -2.173    0.105  -20.762    0.000
    plcfrd|2           -0.957    0.099   -9.691    0.000
    plcfrd|3            0.983    0.272    3.612    0.000
    plcxdc|1  (p8)     -1.784    0.077  -23.270    0.000
    plcxdc|2           -0.438    0.117   -3.755    0.000
    plcxdc|3            1.206    0.273    4.426    0.000
    bplcdc|1  (p9)     -1.508    0.063  -23.742    0.000
    bplcdc|2 (p10)     -1.297    0.057  -22.935    0.000
    bplcdc|3           -0.935    0.120   -7.826    0.000
    bplcdc|4           -0.572    0.204   -2.803    0.005
    bplcdc|5           -0.389    0.249   -1.565    0.117
    bplcdc|6           -0.056    0.330   -0.168    0.866
    bplcdc|7            0.172    0.387    0.445    0.656
    bplcdc|8            0.421    0.449    0.938    0.348
    bplcdc|9            0.837    0.553    1.515    0.130
    bplcd|10            1.160    0.634    1.829    0.067
    dplcsy|1 (p11)     -1.894    0.083  -22.766    0.000
    dplcsy|2           -1.661    0.090  -18.489    0.000
```

```
    dplcsy|3          -1.310    0.143   -9.179    0.000
    dplcsy|4          -0.837    0.245   -3.412    0.001
    dplcsy|5          -0.621    0.295   -2.105    0.035
    dplcsy|6          -0.259    0.381   -0.679    0.497
    dplcsy|7          -0.051    0.431   -0.117    0.907
    dplcsy|8           0.319    0.520    0.613    0.540
    dplcsy|9           0.805    0.638    1.261    0.207
    dplcs|10           1.198    0.735    1.629    0.103
    dpcstr|1 (p12)    -1.950    0.087  -22.544    0.000
    dpcstr|2          -1.664    0.098  -16.935    0.000
    dpcstr|3          -1.288    0.150   -8.612    0.000
    dpcstr|4          -0.841    0.237   -3.543    0.000
    dpcstr|5          -0.590    0.290   -2.036    0.042
    dpcstr|6          -0.223    0.370   -0.602    0.547
    dpcstr|7           0.016    0.423    0.037    0.971
    dpcstr|8           0.337    0.494    0.682    0.495
    dpcstr|9           0.796    0.597    1.333    0.182
    dpcst|10           1.182    0.685    1.724    0.085
    plcrgw|1 (p13)    -2.019    0.092  -21.880    0.000
    plcrgw|2 (p14)    -0.997    0.050  -20.009    0.000
    plcrgw|3          -0.475    0.097   -4.909    0.000
    plcrgw|4           0.945    0.259    3.650    0.000
    plcpvl|1 (p15)    -2.162    0.104  -20.865    0.000
    plcpvl|2          -1.272    0.089  -14.244    0.000
    plcpvl|3          -0.649    0.116   -5.576    0.000
    plcpvl|4           0.926    0.275    3.364    0.001
    gspplc|1 (p16)    -2.322    0.119  -19.572    0.000
    gspplc|2          -1.490    0.111  -13.390    0.000
    gspplc|3          -0.763    0.109   -7.010    0.000
    gspplc|4           0.847    0.228    3.712    0.000
    cplcst|1 (p17)    -2.223    0.110  -20.170    0.000
    cplcst|2 (p18)    -1.609    0.068  -23.716    0.000
    cplcst|3          -0.676    0.216   -3.136    0.002
    wdprsn|1 (p19)    -2.054    0.095  -21.706    0.000
    wdprsn|2          -1.306    0.157   -8.294    0.000
    wdprsn|3          -0.471    0.330   -1.427    0.154
    wvdct|t1 (p20)    -1.712    0.073  -23.400    0.000
    wvdct|t2          -1.046    0.154   -6.795    0.000
    wvdct|t3          -0.242    0.334   -0.726    0.468

Variances:
                   Estimate  Std.Err  z-value  P(>|z|)
    TrEf              1.278    0.767    1.667    0.096
    TrFa              1.093    0.293    3.736    0.000
   .ObOb              0.812    0.417    1.946    0.052
   .MoAl              1.211    0.374    3.235    0.001
```

```
   .WiCo              0.598    0.251    2.382    0.017
   .plcpvcr           0.331
   .plccbrg           0.780
   .plcarcr           0.556
   .plcrspc           0.352
   .plcfrdc           0.222
   .plcexdc           0.600
   .bplcdc            0.290
   .doplcsy           0.141
   .dpcstrb           0.191
   .plcrgwr           0.345
   .plcipvl           0.150
   .gsupplc           0.235
   .caplcst           0.285
   .widprsn           0.006
   .wevdct            0.145

Scales y*:
                   Estimate  Std.Err  z-value  P(>|z|)
    plcpvcr           0.957    0.282    3.387    0.001
    plccbrg           0.826    0.241    3.426    0.001
    plcarcr           0.954    0.220    4.344    0.000
    plcrspc           0.951    0.117    8.120    0.000
    plcfrdc           0.936    0.108    8.638    0.000
    plcexdc           0.868    0.095    9.179    0.000
    bplcdc            1.172    0.295    3.978    0.000
    doplcsy           1.075    0.266    4.037    0.000
    dpcstrb           1.092    0.250    4.364    0.000
    plcrgwr           0.955    0.115    8.321    0.000
    plcipvl           0.930    0.110    8.451    0.000
    gsupplc           1.000    0.106    9.476    0.000
    caplcst           1.222    0.246    4.975    0.000
    widprsn           1.284    0.293    4.380    0.000
    wevdct            1.258    0.301    4.186    0.000
```

```
summary(Ex2_FittedMetricInvModel)

lavaan (0.6-1.1179) converged normally after 265 iterations

                                                  Used       Total
  Number of observations per group
  GB                                              1805        2422
  IE                                              1732        2576

  Estimator                                        PML      Robust
  Model Fit Test Statistic                     933.430     -25.740
```

```
  Degrees of freedom                                 172       4.074
  P-value                                             NA          NA
  Scaling correction factor                                  -36.264
    for the mean and variance adjusted correction

Chi-square for each group:

  GB                                             443.138     -12.220
  IE                                             490.292     -13.520

Parameter Estimates:

  Information                                   Observed
  Observed information based on                  Hessian
  Standard Errors                     Robust.huber.white


Group 1 [GB]:

Latent Variables:
                   Estimate  Std.Err  z-value  P(>|z|)
  TrEf =~
    plcpvcr   (c1)    0.770    0.030   25.920    0.000
    plccbrg   (c2)    0.735    0.028   26.276    0.000
    plcarcr   (c3)    0.640    0.029   22.185    0.000
  TrFa =~
    plcrspc   (c4)    0.808    0.027   29.509    0.000
    plcfrdc   (c5)    0.859    0.025   33.756    0.000
    plcexdc   (c6)    0.742    0.026   29.053    0.000
  ObOb =~
    bplcdc    (c7)    0.621    0.020   30.865    0.000
    doplcsy   (c8)    0.813    0.024   33.857    0.000
    dpcstrb   (c9)    0.772    0.022   34.947    0.000
  MoAl =~
    plcrgwr  (c10)    0.477    0.033   14.578    0.000
    plcipvl  (c11)    0.558    0.037   15.050    0.000
    gsupplc  (c12)    0.515    0.033   15.728    0.000
  WiCo =~
    caplcst  (c13)    0.751    0.027   28.103    0.000
    widprsn  (c14)    0.961    0.019   49.783    0.000
    wevdct   (c15)    0.870    0.020   43.963    0.000

Regressions:
                   Estimate  Std.Err  z-value  P(>|z|)
  ObOb ~
    TrEf              0.276    0.065    4.233    0.000
```

```
    TrFa              0.287    0.066    4.375    0.000
  MoAl ~
    TrEf              0.377    0.093    4.047    0.000
    TrFa              0.970    0.123    7.891    0.000
  WiCo ~
    TrEf             -0.069    0.068   -1.002    0.316
    ObOb              0.116    0.046    2.502    0.012
    MoAl              0.138    0.045    3.047    0.002

Covariances:
                   Estimate  Std.Err  z-value  P(>|z|)
  TrEf ~~
    TrFa              0.623    0.037   16.626    0.000
 .ObOb ~~
   .MoAl              0.252    0.057    4.430    0.000

Intercepts:
                   Estimate  Std.Err  z-value  P(>|z|)
    TrEf              0.000
    TrFa              0.000
   .ObOb              0.000
   .MoAl              0.000
   .WiCo              0.000
   .plcpvcr           0.000
   .plccbrg           0.000
   .plcarcr           0.000
   .plcrspc           0.000
   .plcfrdc           0.000
   .plcexdc           0.000
   .bplcdc            0.000
   .doplcsy           0.000
   .dpcstrb           0.000
   .plcrgwr           0.000
   .plcipvl           0.000
   .gsupplc           0.000
   .caplcst           0.000
   .widprsn           0.000
   .wevdct            0.000

Thresholds:
                   Estimate  Std.Err  z-value  P(>|z|)
    plcpvc|1  (p1)   -2.174    0.101  -21.504    0.000
    plcpvc|2  (p2)   -1.778    0.075  -23.647    0.000
    plcpvc|3         -1.352    0.059  -23.115    0.000
    plcpvc|4         -0.889    0.048  -18.526    0.000
    plcpvc|5         -0.477    0.043  -11.094    0.000
```

```
    plcpvc|6          0.060    0.041    1.462    0.144
    plcpvc|7          0.508    0.043   11.832    0.000
    plcpvc|8          1.136    0.052   21.715    0.000
    plcpvc|9          1.955    0.087   22.390    0.000
    plcpv|10          2.438    0.138   17.675    0.000
    plccbr|1  (p3)   -1.878    0.080  -23.540    0.000
    plccbr|2         -1.410    0.060  -23.479    0.000
    plccbr|3         -0.884    0.048  -18.480    0.000
    plccbr|4         -0.394    0.042   -9.293    0.000
    plccbr|5          0.014    0.041    0.336    0.737
    plccbr|6          0.505    0.043   11.756    0.000
    plccbr|7          0.902    0.048   18.884    0.000
    plccbr|8          1.466    0.062   23.658    0.000
    plccbr|9          2.051    0.095   21.674    0.000
    plccb|10          2.532    0.153   16.583    0.000
    plcrcr|1  (p4)   -2.121    0.097  -21.972    0.000
    plcrcr|2         -1.741    0.073  -23.974    0.000
    plcrcr|3         -1.381    0.059  -23.383    0.000
    plcrcr|4         -0.999    0.050  -20.058    0.000
    plcrcr|5         -0.643    0.045  -14.409    0.000
    plcrcr|6         -0.257    0.042   -6.171    0.000
    plcrcr|7          0.072    0.041    1.741    0.082
    plcrcr|8          0.596    0.044   13.632    0.000
    plcrcr|9          1.351    0.058   23.273    0.000
    plcrc|10          2.045    0.094   21.725    0.000
    plcrsp|1  (p5)   -2.113    0.095  -22.220    0.000
    plcrsp|2  (p6)   -0.995    0.050  -19.884    0.000
    plcrsp|3          0.938    0.049   19.328    0.000
    plcfrd|1  (p7)   -2.172    0.100  -21.641    0.000
    plcfrd|2         -0.886    0.048  -18.403    0.000
    plcfrd|3          1.209    0.054   22.255    0.000
    plcxdc|1  (p8)   -1.772    0.073  -24.245    0.000
    plcxdc|2         -0.441    0.043  -10.327    0.000
    plcxdc|3          1.138    0.053   21.657    0.000
    bplcdc|1  (p9)   -1.501    0.060  -24.904    0.000
    bplcdc|2 (p10)   -1.292    0.054  -23.709    0.000
    bplcdc|3         -0.995    0.049  -20.517    0.000
    bplcdc|4         -0.634    0.044  -14.433    0.000
    bplcdc|5         -0.350    0.042   -8.358    0.000
    bplcdc|6          0.129    0.041    3.135    0.002
    bplcdc|7          0.397    0.043    9.348    0.000
    bplcdc|8          0.765    0.046   16.605    0.000
    bplcdc|9          1.360    0.059   23.219    0.000
    bplcd|10          1.809    0.078   23.333    0.000
    dplcsy|1 (p11)   -1.897    0.082  -23.268    0.000
    dplcsy|2         -1.648    0.070  -23.666    0.000
```

```
    dplcsy|3          -1.409    0.060  -23.307    0.000
    dplcsy|4          -1.051    0.051  -20.631    0.000
    dplcsy|5          -0.767    0.046  -16.650    0.000
    dplcsy|6          -0.281    0.042   -6.747    0.000
    dplcsy|7          -0.011    0.041   -0.258    0.796
    dplcsy|8           0.390    0.042    9.267    0.000
    dplcsy|9           0.989    0.049   20.092    0.000
    dplcs|10           1.453    0.062   23.523    0.000
    dpcstr|1 (p12)    -1.955    0.084  -23.277    0.000
    dpcstr|2          -1.687    0.071  -23.918    0.000
    dpcstr|3          -1.303    0.057  -22.958    0.000
    dpcstr|4          -0.971    0.049  -19.742    0.000
    dpcstr|5          -0.690    0.045  -15.336    0.000
    dpcstr|6          -0.184    0.041   -4.463    0.000
    dpcstr|7           0.113    0.041    2.746    0.006
    dpcstr|8           0.569    0.043   13.097    0.000
    dpcstr|9           1.107    0.052   21.398    0.000
    dpcst|10           1.591    0.067   23.698    0.000
    plcrgw|1 (p13)    -2.003    0.088  -22.685    0.000
    plcrgw|2 (p14)    -0.995    0.049  -20.153    0.000
    plcrgw|3          -0.504    0.043  -11.687    0.000
    plcrgw|4           1.347    0.058   23.038    0.000
    plcpvl|1 (p15)    -2.154    0.100  -21.545    0.000
    plcpvl|2          -1.133    0.053  -21.403    0.000
    plcpvl|3          -0.528    0.043  -12.147    0.000
    plcpvl|4           1.416    0.061   23.344    0.000
    gspplc|1 (p16)    -2.354    0.118  -19.972    0.000
    gspplc|2          -1.432    0.061  -23.345    0.000
    gspplc|3          -0.836    0.047  -17.667    0.000
    gspplc|4           1.356    0.059   23.120    0.000
    cplcst|1 (p17)    -2.216    0.111  -19.903    0.000
    cplcst|2 (p18)    -1.597    0.062  -25.824    0.000
    cplcst|3          -0.526    0.043  -12.225    0.000
    wdprsn|1 (p19)    -2.058    0.092  -22.252    0.000
    wdprsn|2          -1.363    0.059  -23.178    0.000
    wdprsn|3          -0.170    0.041   -4.100    0.000
    wvdct|t1 (p20)    -1.719    0.071  -24.291    0.000
    wvdct|t2          -0.985    0.049  -20.045    0.000
    wvdct|t3           0.017    0.041    0.414    0.679

Variances:
                   Estimate  Std.Err  z-value  P(>|z|)
    TrEf              1.000
    TrFa              1.000
   .ObOb              1.000
   .MoAl              1.000
```

```
   .WiCo              1.000
   .plcpvcr           0.407
   .plccbrg           0.460
   .plcarcr           0.591
   .plcrspc           0.346
   .plcfrdc           0.262
   .plcexdc           0.449
   .bplcdc            0.514
   .doplcsy           0.168
   .dpcstrb           0.251
   .plcrgwr           0.423
   .plcipvl           0.209
   .gsupplc           0.328
   .caplcst           0.396
   .widprsn           0.011
   .wevdct            0.188

Scales y*:
                   Estimate  Std.Err  z-value  P(>|z|)
    plcpvcr           1.000
    plccbrg           1.000
    plcarcr           1.000
    plcrspc           1.000
    plcfrdc           1.000
    plcexdc           1.000
    bplcdc            1.000
    doplcsy           1.000
    dpcstrb           1.000
    plcrgwr           1.000
    plcipvl           1.000
    gsupplc           1.000
    caplcst           1.000
    widprsn           1.000
    wevdct            1.000


Group 2 [IE]:

Latent Variables:
                   Estimate  Std.Err  z-value  P(>|z|)
  TrEf =~
    plcpvcr   (c1)    0.770    0.030   25.920    0.000
    plccbrg   (c2)    0.735    0.028   26.276    0.000
    plcarcr   (c3)    0.640    0.029   22.185    0.000
  TrFa =~
    plcrspc   (c4)    0.808    0.027   29.509    0.000
```

```
    plcfrdc   (c5)    0.859    0.025   33.756    0.000
    plcexdc   (c6)    0.742    0.026   29.053    0.000
  ObOb =~
    bplcdc    (c7)    0.621    0.020   30.865    0.000
    doplcsy   (c8)    0.813    0.024   33.857    0.000
    dpcstrb   (c9)    0.772    0.022   34.947    0.000
  MoAl =~
    plcrgwr  (c10)    0.477    0.033   14.578    0.000
    plcipvl  (c11)    0.558    0.037   15.050    0.000
    gsupplc  (c12)    0.515    0.033   15.728    0.000
  WiCo =~
    caplcst  (c13)    0.751    0.027   28.103    0.000
    widprsn  (c14)    0.961    0.019   49.783    0.000
    wevdct   (c15)    0.870    0.020   43.963    0.000

Regressions:
                   Estimate  Std.Err  z-value  P(>|z|)
  ObOb ~
    TrEf              0.179    0.084    2.139    0.032
    TrFa              0.365    0.110    3.310    0.001
  MoAl ~
    TrEf              0.459    0.150    3.055    0.002
    TrFa              0.891    0.162    5.495    0.000
  WiCo ~
    TrEf              0.049    0.057    0.857    0.391
    ObOb              0.024    0.043    0.565    0.572
    MoAl              0.157    0.051    3.090    0.002

Covariances:
                   Estimate  Std.Err  z-value  P(>|z|)
  TrEf ~~
    TrFa              0.897    0.250    3.597    0.000
 .ObOb ~~
   .MoAl              0.219    0.079    2.759    0.006

Intercepts:
                   Estimate  Std.Err  z-value  P(>|z|)
    TrEf              0.026    0.651    0.040    0.968
    TrFa              0.105    0.199    0.527    0.598
   .ObOb             -0.329    0.475   -0.692    0.489
   .MoAl             -0.259    0.435   -0.595    0.552
   .WiCo             -0.615    0.309   -1.991    0.047
   .plcpvcr           0.000
   .plccbrg           0.000
   .plcarcr           0.000
   .plcrspc           0.000
```

```
   .plcfrdc          0.000
   .plcexdc          0.000
   .bplcdc           0.000
   .doplcsy          0.000
   .dpcstrb          0.000
   .plcrgwr          0.000
   .plcipvl          0.000
   .gsupplc          0.000
   .caplcst          0.000
   .widprsn          0.000
   .wevdct           0.000

Thresholds:
                   Estimate  Std.Err  z-value  P(>|z|)
    plcpvc|1  (p1)   -2.174    0.101  -21.504    0.000
    plcpvc|2  (p2)   -1.778    0.075  -23.647    0.000
    plcpvc|3         -1.316    0.165   -7.966    0.000
    plcpvc|4         -0.889    0.269   -3.304    0.001
    plcpvc|5         -0.398    0.393   -1.012    0.311
    plcpvc|6          0.157    0.536    0.293    0.770
    plcpvc|7          0.604    0.652    0.926    0.354
    plcpvc|8          1.139    0.792    1.439    0.150
    plcpvc|9          1.884    0.987    1.909    0.056
    plcpv|10          2.361    1.114    2.120    0.034
    plccbr|1  (p3)   -1.878    0.080  -23.540    0.000
    plccbr|2         -1.360    0.143   -9.489    0.000
    plccbr|3         -0.824    0.267   -3.082    0.002
    plccbr|4         -0.258    0.409   -0.631    0.528
    plccbr|5          0.231    0.534    0.432    0.666
    plccbr|6          0.780    0.675    1.156    0.248
    plccbr|7          1.191    0.782    1.524    0.127
    plccbr|8          1.693    0.912    1.856    0.064
    plccbr|9          2.461    1.114    2.209    0.027
    plccb|10          3.084    1.283    2.403    0.016
    plcrcr|1  (p4)   -2.121    0.097  -21.972    0.000
    plcrcr|2         -1.740    0.111  -15.707    0.000
    plcrcr|3         -1.295    0.165   -7.869    0.000
    plcrcr|4         -0.759    0.261   -2.904    0.004
    plcrcr|5         -0.371    0.338   -1.098    0.272
    plcrcr|6         -0.025    0.410   -0.061    0.951
    plcrcr|7          0.255    0.468    0.544    0.586
    plcrcr|8          0.703    0.562    1.250    0.211
    plcrcr|9          1.454    0.723    2.010    0.044
    plcrc|10          2.093    0.865    2.419    0.016
    plcrsp|1  (p5)   -2.113    0.095  -22.220    0.000
    plcrsp|2  (p6)   -0.995    0.050  -19.884    0.000
```

```
    plcrsp|3           0.749     0.233     3.222     0.001
    plcfrd|1  (p7)    -2.172     0.100   -21.641     0.000
    plcfrd|2          -0.946     0.091   -10.402     0.000
    plcfrd|3           1.007     0.264     3.814     0.000
    plcxdc|1  (p8)    -1.772     0.073   -24.245     0.000
    plcxdc|2          -0.423     0.108    -3.915     0.000
    plcxdc|3           1.214     0.257     4.714     0.000
    bplcdc|1  (p9)    -1.501     0.060   -24.904     0.000
    bplcdc|2 (p10)    -1.292     0.054   -23.709     0.000
    bplcdc|3          -0.936     0.117    -8.005     0.000
    bplcdc|4          -0.578     0.198    -2.919     0.004
    bplcdc|5          -0.398     0.240    -1.655     0.098
    bplcdc|6          -0.070     0.318    -0.220     0.826
    bplcdc|7           0.154     0.372     0.414     0.679
    bplcdc|8           0.398     0.430     0.925     0.355
    bplcdc|9           0.808     0.529     1.526     0.127
    bplcd|10           1.125     0.606     1.855     0.064
    dplcsy|1 (p11)    -1.897     0.082   -23.268     0.000
    dplcsy|2          -1.668     0.087   -19.175     0.000
    dplcsy|3          -1.321     0.139    -9.483     0.000
    dplcsy|4          -0.851     0.240    -3.546     0.000
    dplcsy|5          -0.638     0.289    -2.206     0.027
    dplcsy|6          -0.278     0.373    -0.745     0.456
    dplcsy|7          -0.072     0.422    -0.170     0.865
    dplcsy|8           0.295     0.509     0.579     0.563
    dplcsy|9           0.777     0.625     1.243     0.214
    dplcs|10           1.166     0.720     1.621     0.105
    dpcstr|1 (p12)    -1.955     0.084   -23.277     0.000
    dpcstr|2          -1.672     0.091   -18.454     0.000
    dpcstr|3          -1.299     0.140    -9.294     0.000
    dpcstr|4          -0.856     0.228    -3.753     0.000
    dpcstr|5          -0.607     0.281    -2.160     0.031
    dpcstr|6          -0.242     0.362    -0.668     0.504
    dpcstr|7          -0.005     0.415    -0.012     0.990
    dpcstr|8           0.314     0.487     0.646     0.518
    dpcstr|9           0.771     0.590     1.306     0.191
    dpcst|10           1.153     0.678     1.700     0.089
    plcrgw|1 (p13)    -2.003     0.088   -22.685     0.000
    plcrgw|2 (p14)    -0.995     0.049   -20.153     0.000
    plcrgw|3          -0.485     0.094    -5.157     0.000
    plcrgw|4           0.901     0.244     3.698     0.000
    plcpvl|1 (p15)    -2.154     0.100   -21.545     0.000
    plcpvl|2          -1.276     0.084   -15.170     0.000
    plcpvl|3          -0.662     0.112    -5.909     0.000
    plcpvl|4           0.885     0.264     3.347     0.001
    gspplc|1 (p16)    -2.354     0.118   -19.972     0.000
```

```
    gspplc|2          -1.531    0.100  -15.386    0.000
    gspplc|3          -0.797    0.103   -7.769    0.000
    gspplc|4           0.835    0.238    3.513    0.000
    cplcst|1 (p17)    -2.216    0.111  -19.903    0.000
    cplcst|2 (p18)    -1.597    0.062  -25.824    0.000
    cplcst|3          -0.649    0.201   -3.232    0.001
    wdprsn|1 (p19)    -2.058    0.092  -22.252    0.000
    wdprsn|2          -1.295    0.154   -8.388    0.000
    wdprsn|3          -0.444    0.323   -1.372    0.170
    wvdct|t1 (p20)    -1.719    0.071  -24.291    0.000
    wvdct|t2          -1.039    0.157   -6.632    0.000
    wvdct|t3          -0.216    0.332   -0.651    0.515

Variances:
                   Estimate  Std.Err  z-value  P(>|z|)
    TrEf              1.334    0.659    2.023    0.043
    TrFa              1.254    0.284    4.413    0.000
   .ObOb              0.753    0.364    2.070    0.038
   .MoAl              1.052    0.282    3.734    0.000
   .WiCo              0.569    0.249    2.284    0.022
   .plcpvcr           0.339
   .plccbrg           0.802
   .plcarcr           0.571
   .plcrspc           0.344
   .plcfrdc           0.230
   .plcexdc           0.620
   .bplcdc            0.285
   .doplcsy           0.136
   .dpcstrb           0.186
   .plcrgwr           0.346
   .plcipvl           0.160
   .gsupplc           0.220
   .caplcst           0.302
   .widprsn           0.004
   .wevdct            0.148

Scales y*:
                   Estimate  Std.Err  z-value  P(>|z|)
    plcpvcr           0.941    0.247    3.815    0.000
    plccbrg           0.810    0.211    3.833    0.000
    plcarcr           0.946    0.206    4.600    0.000
    plcrspc           0.927    0.108    8.622    0.000
    plcfrdc           0.930    0.103    9.004    0.000
    plcexdc           0.873    0.090    9.668    0.000
    bplcdc            1.193    0.289    4.134    0.000
    doplcsy           1.084    0.264    4.100    0.000
```

```
    dpcstrb           1.098    0.254    4.331    0.000
    plcrgwr           0.979    0.111    8.831    0.000
    plcipvl           0.947    0.108    8.790    0.000
    gsupplc           0.985    0.105    9.385    0.000
    caplcst           1.208    0.236    5.113    0.000
    widprsn           1.258    0.275    4.576    0.000
    wevdct            1.228    0.277    4.432    0.000
```

6.4 The outputs of the models fitted in Section 5.7

```
summary(Ex3FittedModel_CP)

lavaan (0.6-1.1179) converged normally after  90 iterations

                                                  Used       Total
  Number of observations                          2421        2422
  Number of missing patterns                       193

  Estimator                                        PML

Parameter Estimates:

  Information                                 Observed
  Observed information based on                Hessian
  Standard Errors                   Robust.huber.white

Latent Variables:
                   Estimate  Std.Err  z-value  P(>|z|)
  TrEf =~
    plcpvcr           0.768    0.020   37.899    0.000
    plccbrg           0.736    0.019   38.153    0.000
    plcarcr           0.639    0.020   31.965    0.000
  TrFa =~
    plcrspc           0.833    0.017   48.497    0.000
    plcfrdc           0.875    0.017   52.323    0.000
    plcexdc           0.720    0.018   40.175    0.000
  ObOb =~
    bplcdc            0.619    0.013   49.002    0.000
    doplcsy           0.830    0.015   54.090    0.000
    dpcstrb           0.784    0.013   59.586    0.000
  MoAl =~
    plcrgwr           0.494    0.021   23.813    0.000
    plcipvl           0.585    0.022   26.074    0.000
    gsupplc           0.554    0.020   28.323    0.000
  WiCo =~
```

```
    caplcst           0.729    0.016   45.598    0.000
    widprsn           0.979    0.012   84.481    0.000
    wevdct            0.875    0.011   76.696    0.000

Regressions:
                   Estimate  Std.Err  z-value  P(>|z|)
  ObOb ~
    TrEf              0.284    0.043    6.626    0.000
    TrFa              0.261    0.043    6.086    0.000
  MoAl ~
    TrEf              0.364    0.057    6.437    0.000
    TrFa              0.879    0.073   12.103    0.000
  WiCo ~
    TrEf             -0.025    0.043   -0.573    0.566
    ObOb              0.099    0.029    3.360    0.001
    MoAl              0.110    0.030    3.706    0.000

Covariances:
                   Estimate  Std.Err  z-value  P(>|z|)
  TrEf ~~
    TrFa              0.614    0.025   24.302    0.000
 .ObOb ~~
   .MoAl              0.256    0.034    7.474    0.000

Intercepts:
                   Estimate  Std.Err  z-value  P(>|z|)
   .plcpvcr           0.000
   .plccbrg           0.000
   .plcarcr           0.000
   .plcrspc           0.000
   .plcfrdc           0.000
   .plcexdc           0.000
   .bplcdc            0.000
   .doplcsy           0.000
   .dpcstrb           0.000
   .plcrgwr           0.000
   .plcipvl           0.000
   .gsupplc           0.000
   .caplcst           0.000
   .widprsn           0.000
   .wevdct            0.000
    TrEf              0.000
    TrFa              0.000
   .ObOb              0.000
   .MoAl              0.000
   .WiCo              0.000
```

```
Thresholds:
                   Estimate  Std.Err  z-value  P(>|z|)
    plcpvcr|t1       -2.095    0.062  -33.988    0.000
    plcpvcr|t2       -1.777    0.048  -36.931    0.000
    plcpvcr|t3       -1.339    0.037  -36.380    0.000
    plcpvcr|t4       -0.912    0.031  -29.756    0.000
    plcpvcr|t5       -0.494    0.027  -18.081    0.000
    plcpvcr|t6        0.057    0.026    2.201    0.028
    plcpvcr|t7        0.494    0.027   18.192    0.000
    plcpvcr|t8        1.114    0.033   34.016    0.000
    plcpvcr|t9        1.915    0.053   35.819    0.000
    plcpvcr|t10       2.397    0.083   28.959    0.000
    plccbrg|t1       -1.857    0.052  -35.985    0.000
    plccbrg|t2       -1.407    0.038  -36.552    0.000
    plccbrg|t3       -0.877    0.030  -28.778    0.000
    plccbrg|t4       -0.396    0.027  -14.626    0.000
    plccbrg|t5       -0.005    0.026   -0.195    0.846
    plccbrg|t6        0.499    0.027   18.223    0.000
    plccbrg|t7        0.895    0.030   29.466    0.000
    plccbrg|t8        1.428    0.039   36.965    0.000
    plccbrg|t9        2.010    0.058   34.513    0.000
    plccbrg|t10       2.410    0.085   28.412    0.000
    plcarcr|t1       -2.071    0.061  -33.971    0.000
    plcarcr|t2       -1.721    0.047  -36.876    0.000
    plcarcr|t3       -1.367    0.038  -36.145    0.000
    plcarcr|t4       -0.995    0.032  -31.122    0.000
    plcarcr|t5       -0.649    0.029  -22.637    0.000
    plcarcr|t6       -0.268    0.027  -10.048    0.000
    plcarcr|t7        0.062    0.026    2.349    0.019
    plcarcr|t8        0.574    0.028   20.632    0.000
    plcarcr|t9        1.331    0.037   36.277    0.000
    plcarcr|t10       2.009    0.058   34.355    0.000
    plcrspc|t1       -2.102    0.062  -33.983    0.000
    plcrspc|t2       -1.004    0.032  -31.784    0.000
    plcrspc|t3        0.950    0.031   31.045    0.000
    plcfrdc|t1       -2.146    0.065  -32.765    0.000
    plcfrdc|t2       -0.898    0.031  -29.144    0.000
    plcfrdc|t3        1.201    0.034   34.880    0.000
    plcexdc|t1       -1.799    0.051  -35.394    0.000
    plcexdc|t2       -0.443    0.028  -15.808    0.000
    plcexdc|t3        1.146    0.035   33.082    0.000
    bplcdc|t1        -1.481    0.039  -37.689    0.000
    bplcdc|t2        -1.275    0.035  -36.231    0.000
    bplcdc|t3        -0.971    0.031  -31.556    0.000
    bplcdc|t4        -0.617    0.028  -22.215    0.000
    bplcdc|t5        -0.342    0.026  -12.930    0.000
```

bplcdc|t6	0.131	0.026	5.038	0.000
bplcdc|t7	0.397	0.027	14.787	0.000
bplcdc|t8	0.763	0.029	26.251	0.000
bplcdc|t9	1.334	0.036	36.647	0.000
bplcdc|t10	1.747	0.046	37.581	0.000
doplcsy|t1	-1.826	0.049	-37.080	0.000
doplcsy|t2	-1.611	0.043	-37.764	0.000
doplcsy|t3	-1.371	0.037	-36.974	0.000
doplcsy|t4	-1.021	0.032	-32.360	0.000
doplcsy|t5	-0.750	0.029	-26.057	0.000
doplcsy|t6	-0.272	0.026	-10.411	0.000
doplcsy|t7	-0.001	0.026	-0.037	0.970
doplcsy|t8	0.401	0.027	15.121	0.000
doplcsy|t9	0.983	0.031	31.801	0.000
doplcsy|t10	1.408	0.038	37.330	0.000
dpcstrb|t1	-1.883	0.051	-36.639	0.000
dpcstrb|t2	-1.636	0.044	-37.593	0.000
dpcstrb|t3	-1.283	0.036	-36.050	0.000
dpcstrb|t4	-0.959	0.031	-31.042	0.000
dpcstrb|t5	-0.696	0.028	-24.465	0.000
dpcstrb|t6	-0.188	0.026	-7.222	0.000
dpcstrb|t7	0.109	0.026	4.202	0.000
dpcstrb|t8	0.561	0.027	20.494	0.000
dpcstrb|t9	1.097	0.033	33.720	0.000
dpcstrb|t10	1.541	0.041	37.747	0.000
plcrgwr|t1	-2.044	0.059	-34.580	0.000
plcrgwr|t2	-0.996	0.031	-31.977	0.000
plcrgwr|t3	-0.486	0.027	-18.068	0.000
plcrgwr|t4	1.368	0.037	36.963	0.000
plcipvl|t1	-2.149	0.064	-33.497	0.000
plcipvl|t2	-1.155	0.034	-34.450	0.000
plcipvl|t3	-0.539	0.027	-19.761	0.000
plcipvl|t4	1.421	0.038	37.367	0.000
gsupplc|t1	-2.314	0.073	-31.591	0.000
gsupplc|t2	-1.428	0.038	-37.353	0.000
gsupplc|t3	-0.822	0.029	-28.110	0.000
gsupplc|t4	1.360	0.037	37.150	0.000
caplcst|t1	-2.206	0.068	-32.540	0.000
caplcst|t2	-1.597	0.042	-38.102	0.000
caplcst|t3	-0.507	0.027	-18.850	0.000
widprsn|t1	-2.050	0.059	-34.817	0.000
widprsn|t2	-1.325	0.036	-37.051	0.000
widprsn|t3	-0.122	0.026	-4.702	0.000
wevdct|t1	-1.675	0.044	-37.773	0.000
wevdct|t2	-0.950	0.030	-31.303	0.000
wevdct|t3	0.056	0.026	2.148	0.032

```
Variances:
                   Estimate  Std.Err  z-value  P(>|z|)
   .plcpvcr           0.409
   .plccbrg           0.458
   .plcarcr           0.592
   .plcrspc           0.307
   .plcfrdc           0.234
   .plcexdc           0.481
   .bplcdc            0.524
   .doplcsy           0.146
   .dpcstrb           0.238
   .plcrgwr           0.440
   .plcipvl           0.214
   .gsupplc           0.294
   .caplcst           0.441
   .widprsn          -0.008
   .wevdct            0.196
    TrEf              1.000
    TrFa              1.000
   .ObOb              1.000
   .MoAl              1.000
   .WiCo              1.000

Scales y*:
                   Estimate  Std.Err  z-value  P(>|z|)
    plcpvcr           1.000
    plccbrg           1.000
    plcarcr           1.000
    plcrspc           1.000
    plcfrdc           1.000
    plcexdc           1.000
    bplcdc            1.000
    doplcsy           1.000
    dpcstrb           1.000
    plcrgwr           1.000
    plcipvl           1.000
    gsupplc           1.000
    caplcst           1.000
    widprsn           1.000
    wevdct            1.000
```

```
summary(Ex3FittedModel_AC)

lavaan (0.6-1.1179) converged normally after 105 iterations
```

```
                                                  Used       Total
  Number of observations                          2421        2422
  Number of missing patterns                       193

  Estimator                                        PML

Parameter Estimates:

  Information                                 Observed
  Observed information based on                Hessian
  Standard Errors                   Robust.huber.white

Latent Variables:
                   Estimate  Std.Err  z-value  P(>|z|)
  TrEf =~
    plcpvcr           0.769    0.020   38.050    0.000
    plccbrg           0.737    0.019   38.334    0.000
    plcarcr           0.639    0.020   32.093    0.000
  TrFa =~
    plcrspc           0.832    0.017   48.400    0.000
    plcfrdc           0.875    0.017   52.197    0.000
    plcexdc           0.720    0.018   40.169    0.000
  ObOb =~
    bplcdc            0.621    0.013   49.293    0.000
    doplcsy           0.830    0.015   54.178    0.000
    dpcstrb           0.784    0.013   59.713    0.000
  MoAl =~
    plcrgwr           0.493    0.021   23.815    0.000
    plcipvl           0.585    0.022   26.087    0.000
    gsupplc           0.554    0.020   28.330    0.000
  WiCo =~
    caplcst           0.729    0.016   45.774    0.000
    widprsn           0.978    0.012   84.409    0.000
    wevdct            0.875    0.011   76.666    0.000

Regressions:
                   Estimate  Std.Err  z-value  P(>|z|)
  ObOb ~
    TrEf              0.285    0.043    6.645    0.000
    TrFa              0.261    0.043    6.084    0.000
  MoAl ~
    TrEf              0.365    0.057    6.453    0.000
    TrFa              0.878    0.073   12.096    0.000
  WiCo ~
    TrEf             -0.024    0.043   -0.568    0.570
    ObOb              0.100    0.030    3.375    0.001
```

```
    MoAl              0.110    0.030    3.694    0.000

Covariances:
                   Estimate  Std.Err  z-value  P(>|z|)
  TrEf ~~
    TrFa              0.614    0.025   24.317    0.000
 .ObOb ~~
   .MoAl              0.257    0.034    7.471    0.000

Intercepts:
                   Estimate  Std.Err  z-value  P(>|z|)
   .plcpvcr           0.000
   .plccbrg           0.000
   .plcarcr           0.000
   .plcrspc           0.000
   .plcfrdc           0.000
   .plcexdc           0.000
   .bplcdc            0.000
   .doplcsy           0.000
   .dpcstrb           0.000
   .plcrgwr           0.000
   .plcipvl           0.000
   .gsupplc           0.000
   .caplcst           0.000
   .widprsn           0.000
   .wevdct            0.000
    TrEf              0.000
    TrFa              0.000
   .ObOb              0.000
   .MoAl              0.000
   .WiCo              0.000

Thresholds:
                   Estimate  Std.Err  z-value  P(>|z|)
    plcpvcr|t1       -2.086    0.059  -35.491    0.000
    plcpvcr|t2       -1.775    0.047  -38.160    0.000
    plcpvcr|t3       -1.335    0.036  -37.586    0.000
    plcpvcr|t4       -0.912    0.030  -30.682    0.000
    plcpvcr|t5       -0.496    0.027  -18.696    0.000
    plcpvcr|t6        0.058    0.025    2.296    0.022
    plcpvcr|t7        0.493    0.026   18.738    0.000
    plcpvcr|t8        1.109    0.032   35.115    0.000
    plcpvcr|t9        1.914    0.052   37.036    0.000
    plcpvcr|t10       2.387    0.078   30.439    0.000
    plccbrg|t1       -1.855    0.050  -37.058    0.000
    plccbrg|t2       -1.407    0.037  -37.610    0.000
```

plccbrg|t3	-0.876	0.030	-29.613	0.000
plccbrg|t4	-0.395	0.026	-15.037	0.000
plccbrg|t5	-0.007	0.025	-0.281	0.778
plccbrg|t6	0.498	0.027	18.752	0.000
plccbrg|t7	0.895	0.030	30.332	0.000
plccbrg|t8	1.425	0.037	38.142	0.000
plccbrg|t9	2.006	0.056	35.751	0.000
plccbrg|t10	2.401	0.081	29.735	0.000
plcarcr|t1	-2.069	0.059	-35.040	0.000
plcarcr|t2	-1.719	0.045	-38.001	0.000
plcarcr|t3	-1.364	0.037	-37.241	0.000
plcarcr|t4	-0.994	0.031	-32.007	0.000
plcarcr|t5	-0.649	0.028	-23.283	0.000
plcarcr|t6	-0.268	0.026	-10.311	0.000
plcarcr|t7	0.062	0.026	2.440	0.015
plcarcr|t8	0.572	0.027	21.168	0.000
plcarcr|t9	1.329	0.036	37.345	0.000
plcarcr|t10	2.010	0.057	35.291	0.000
plcrspc|t1	-2.105	0.060	-34.978	0.000
plcrspc|t2	-1.007	0.031	-32.915	0.000
plcrspc|t3	0.954	0.030	32.152	0.000
plcfrdc|t1	-2.148	0.064	-33.512	0.000
plcfrdc|t2	-0.901	0.030	-30.005	0.000
plcfrdc|t3	1.202	0.034	35.884	0.000
plcexdc|t1	-1.800	0.050	-36.127	0.000
plcexdc|t2	-0.444	0.027	-16.184	0.000
plcexdc|t3	1.147	0.034	33.796	0.000
bplcdc|t1	-1.474	0.038	-39.024	0.000
bplcdc|t2	-1.270	0.034	-37.419	0.000
bplcdc|t3	-0.965	0.030	-32.506	0.000
bplcdc|t4	-0.614	0.027	-22.814	0.000
bplcdc|t5	-0.340	0.026	-13.250	0.000
bplcdc|t6	0.131	0.025	5.200	0.000
bplcdc|t7	0.396	0.026	15.235	0.000
bplcdc|t8	0.761	0.028	27.041	0.000
bplcdc|t9	1.327	0.035	37.952	0.000
bplcdc|t10	1.735	0.044	39.276	0.000
doplcsy|t1	-1.812	0.047	-38.949	0.000
doplcsy|t2	-1.602	0.041	-39.330	0.000
doplcsy|t3	-1.363	0.036	-38.345	0.000
doplcsy|t4	-1.015	0.030	-33.404	0.000
doplcsy|t5	-0.746	0.028	-26.827	0.000
doplcsy|t6	-0.270	0.025	-10.660	0.000
doplcsy|t7	0.000	0.025	0.014	0.988
doplcsy|t8	0.403	0.026	15.659	0.000
doplcsy|t9	0.983	0.030	32.810	0.000

```
    doplcsy|t10       1.402    0.036   38.661    0.000
    dpcstrb|t1       -1.868    0.048  -38.525    0.000
    dpcstrb|t2       -1.625    0.041  -39.182    0.000
    dpcstrb|t3       -1.277    0.034  -37.293    0.000
    dpcstrb|t4       -0.955    0.030  -32.001    0.000
    dpcstrb|t5       -0.695    0.028  -25.209    0.000
    dpcstrb|t6       -0.188    0.025   -7.459    0.000
    dpcstrb|t7        0.108    0.025    4.316    0.000
    dpcstrb|t8        0.562    0.027   21.171    0.000
    dpcstrb|t9        1.097    0.032   34.794    0.000
    dpcstrb|t10       1.535    0.039   39.162    0.000
    plcrgwr|t1       -2.049    0.058  -35.576    0.000
    plcrgwr|t2       -0.999    0.030  -33.211    0.000
    plcrgwr|t3       -0.483    0.026  -18.676    0.000
    plcrgwr|t4        1.371    0.036   38.352    0.000
    plcipvl|t1       -2.150    0.062  -34.619    0.000
    plcipvl|t2       -1.159    0.032  -35.696    0.000
    plcipvl|t3       -0.539    0.026  -20.483    0.000
    plcipvl|t4        1.423    0.037   38.751    0.000
    gsupplc|t1       -2.313    0.070  -32.955    0.000
    gsupplc|t2       -1.432    0.037  -38.813    0.000
    gsupplc|t3       -0.820    0.028  -29.247    0.000
    gsupplc|t4        1.363    0.035   38.645    0.000
    caplcst|t1       -2.207    0.065  -33.762    0.000
    caplcst|t2       -1.590    0.040  -39.851    0.000
    caplcst|t3       -0.495    0.026  -19.232    0.000
    widprsn|t1       -2.047    0.056  -36.286    0.000
    widprsn|t2       -1.315    0.034  -38.655    0.000
    widprsn|t3       -0.109    0.025   -4.375    0.000
    wevdct|t1        -1.659    0.042  -39.830    0.000
    wevdct|t2        -0.940    0.029  -32.400    0.000
    wevdct|t3         0.066    0.025    2.640    0.008

Variances:
                   Estimate  Std.Err  z-value  P(>|z|)
   .plcpvcr           0.408
   .plccbrg           0.457
   .plcarcr           0.592
   .plcrspc           0.308
   .plcfrdc           0.235
   .plcexdc           0.482
   .bplcdc            0.522
   .doplcsy           0.144
   .dpcstrb           0.237
   .plcrgwr           0.441
   .plcipvl           0.215
```

```
   .gsupplc          0.294
   .caplcst          0.442
   .widprsn         -0.007
   .wevdct           0.194
    TrEf             1.000
    TrFa             1.000
   .ObOb             1.000
   .MoAl             1.000
   .WiCo             1.000

Scales y*:
                   Estimate  Std.Err  z-value  P(>|z|)
    plcpvcr          1.000
    plccbrg          1.000
    plcarcr          1.000
    plcrspc          1.000
    plcfrdc          1.000
    plcexdc          1.000
    bplcdc           1.000
    doplcsy          1.000
    dpcstrb          1.000
    plcrgwr          1.000
    plcipvl          1.000
    gsupplc          1.000
    caplcst          1.000
    widprsn          1.000
    wevdct           1.000
```

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