Efficient Cheap Talk in Directed Search: On the Non-essentiality of Commitment in Market Games*

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Abstract

Directed search models are market games in which each firm announces a wage commitment to attract a worker. Miscoordination among workers generates search frictions, yet in equilibrium more productive firms post more attractive wage commitments to fill their vacancies faster, which yields constrained efficient outcomes. We show that commitment is not essential: Exactly the same efficient allocation can be sustained when announcements are pure cheap talk followed by a suitable subsequent wage-formation stage. The insights from existing commitment models extend unchanged to such a cheap-talk environment, even when workers differ in outside opportunities or observable common productivity.

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1 Introduction

Directed search models of the labor market have become popular because they combine unemployment due to search frictions with classical notions of competition. In such market games, firms commit to the terms of trade in order to attract workers, and workers decide where to apply after observing the firms' announcements. Search frictions and unemployment arise because workers cannot coordinate their job search strategies and, therefore,

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multiple workers may apply for the same job, although only one of them can get it. Nevertheless, the fact that firms can commit to wages before the search frictions arise allows more productive firms to attract workers with higher probability and induces constrained efficient market outcomes. Since previous models where workers meet firms at random and wages are determined ex post through (Nash-) bargaining failed to generate efficiency except in special circumstances, this efficiency implication of directed search has received special attention (see, e.g., Hosios (1990), Moen (1997), Peters (1997a), Acemoğlu and Shimer (1999), Shi (2001), Mortensen and Wright (2002), Shimer (2005), and Eeckhout and Kircher (2010a)).

One major concern in this context is the commitment assumption. It is the driving force that allows more productive firms to attract more workers, yet only very few real-life job announcements feature binding wage commitments. They often include information, but concrete wage announcements are rare. More common are statements that specify wage flexibility ("salary based on experience") or wage ranges ("£30000-£40000 per annum").[1] Wage ranges are particularly interesting, as they illustrate the desire to communicate some information about the willingness to pay without committing to a clear wage policy. A plausible explanation may be that firms would like to adjust the wage to the characteristics of the worker, but commitment to a wage policy may be costly, especially if workers differ in ways that are easy to identify by the firm but hard to verify to an outside party. If posting a message without commitment could sustain the overall efficient and individually most profitable outcome, firms might prefer to use such messages.

In this work we ask whether wage commitment is essential to achieve constrained efficient labor market outcomes, that is, whether efficient outcomes can be achieved without commitment. We investigate this in a single-period environment without reputational considerations. Since workers have no information about the jobs in the economy, firms need to be able to communicate with workers to sustain differential hiring at different firms. In particular, each firm can post a cheap-talk message. We analyze whether there is a subsequent wage-determination game such that an equilibrium of the overall economy supports the constrained efficient allocation and the expected payoffs as in standard commitment models.

We demonstrate that commitment is not essential since efficiency can be indeed achieved in the cheap-talk environment. It can be supported, for example, through the following plausible extensive-form game: Each worker observes all the cheap-talk messages, decides where to apply for a job, and makes a take-it-or-leave-it wage demand. Firms retain power because they can select a worker as well as refuse to hire any worker. We show that there exists a fully revealing equilibrium where all firms truthfully reveal their productivity, and

[1] These cites are common to announcements in many occupations in the U.S. (former) and U.K. (latter).

the resulting market outcome is constrained efficient and coincides with the commitment outcome. It follows as a corollary that no firm could unilaterally improve its profits by using a commitment device. Commitment is therefore not essential for the key insights of the directed search literature. All the insights that have been obtained for the commitment case carry over unchanged to this cheap-talk environment. We prove our results for the wage-demand case, but other wage-determination mechanisms might deliver similar results.

We focus in most of our exposition on the most common specification where workers are homogeneous and risk neutral and the market is large (see, e.g., Hosios (1990), McAfee (1993), Moen (1997), and Peters (1997a, 2000) for the commitment case). This setup is analytically most tractable and contains many of the important insights. We then discuss the implications of various changes to the setup, including the wage-formation stage, risk-aversion and finiteness. Most notably we derive the same insights for a finite but large economy, which has a long tradition in this literature (see, e.g., Peters (1984, 1997b, 2000) and Burdett, Shi, and Wright (2001) for the commitment case).

Finally, we analyze the important case where workers are also heterogeneous. We consider both heterogeneity in non-market opportunities and heterogeneity in a common productivity that each worker adds on top of firms' productivity. These cases are interesting because the full commitment contract does not simply feature a single wage, but there is mismatch in equilibrium where different worker types approach the same firm and firms need to post a full wage schedule specifying the compensation for each worker type. If one believes that commitment to wage schedules is difficult and costly, cheap talk might be an attractive alternative. We show that there is, again, a fully revealing equilibrium that supports the constrained efficient outcome. It might be worth noting here that there are many ways in which firms can communicate their types, and announcing a wage range is clearly one of them.

While the mathematics seems in parts more involved, the following two intuitive steps are at the heart of the argument and hopefully apply in similar form to larger classes of market games with cheap-talk communication. First, consider a fully revealing equilibrium where firms truthfully reveal their productivity. The only consideration left is to provide workers with the right search incentives. For this we use insights on the incentive properties of the (ex-post) worker-optimal core: Once workers decided which firm to apply to, they find themselves in a meeting with one firm and possibly several other workers that applied for the same job. The worker-optimal point in the core of this meeting gives *all* the surplus of the match to the worker when the worker is the only applicant, otherwise the worker obtains exactly as much as his *marginal* increment to the surplus above and beyond what the other applicants could generate. When workers are identical, this means that a worker gets all surplus when he is the only applicant, and nothing otherwise. This turns out to generate expected payoffs according to the well-known condition in Mortensen (1982) and Hosios (1990).[2] It gives the right ex ante incentives for workers to make efficient search decisions because they get paid exactly as much as their marginal contribution and, therefore, internalize the efficiency considerations. If workers play a second-price (procurement) auction for the job against an unknown number of other applicants, it will generate exactly the same ex-post outcome as the worker-optimal point in the core. When workers simply make wage demands (i.e., they play a first-price auction for the job), the outcomes, although may different ex post, are revenue-equivalent to those of the corresponding second-price auctions, which is sufficient for efficient search decisions by workers. Even though firms cannot commit to wages, the bargaining procedure gives the correct job search incentives when firms tell the truth.

The second insight is more subtle and concerns deviations by firms away from the equilibrium path. A firm that sends a message claiming that it is more efficient than it actually is would attract relatively more workers, since workers expect that they can make higher wage demands. The wage demands must increase sufficiently fast in order to deter a firm's deviation. If the firm would have to accept at least one wage demand whenever at least one worker is present, it is in fact not difficult to show that the wage demands increase so fast that deviations are not profitable. The subtlety of our argument arises because the firm can refuse to hire any worker. This limits the exposure of the firm to very high wage demands. But on the other hand it also means that some of the additional applicants cannot be hired. It turns out that when workers place wage demands all additional applicants demand too high wages, so that pretending to be a more productive firm yields no additional suitable applications to a firm.[3] The reason for this is workers' indifference condition across the wages that they demand, which should apply to many market games.

It might be worthwhile to point out the subtlety of our observations by stressing that many of the plausible market structures fail to support efficient outcomes with cheap-talk communication. To begin with, if firms make wage offers, then the first insight fails because

[2]Mortensen (1982) illustrates that "if the agent responsible for each event is allocated its capital value less a compensation paid to every other agent whose play is ended by the event that is equal to the foregone value of continued play," then the non-cooperative equilibrium outcome is efficient in a class of games, including the innovation race and the mating game. Hosios (1990) analyzes firm entry in a model with homogeneous workers and firm. With an urn-ball matching function, his condition has exactly the interpretation that we give here: A worker get the full surplus whenever he is the only applicant. In the present setting with firm heterogeneity, the condition extends in a straightforward fashion if we require it separately for each firm type.

[3]This statement is somewhat loose, given that all workers play a symmetric wage-demand strategy. In exact words, when a firm announces a higher type all workers shift their wage demands in terms of first order stochastic dominance, but the ex-ante probability of hiring at any wage below the true productivity remains exactly constant.

firms would always extract too much of the surplus. Now suppose that workers' wage demands are resolved by each firm according to a second-price auction, rather than a first-price auction.[4] In this case, the first insight regarding on-the-equilibrium-path play continues to hold, and workers would apply efficiently if firms truthfully reveal their type. However, the second insight fails because firms would always have an incentive to overstate their type: A deviating firm would reap the upside that more workers apply but refuse to hire a worker if he is alone and demands the entire exaggerated productivity. In fact, under such a wage-setting game only random search can be sustained. Finally, even in our setup, by the nature of cheap-talk communication, there are other types of equilibria, including random search (babbling),[5] although the efficient, fully revealing equilibrium is uniquely selected by a theoretical criterion in the cheap talk literature. Nevertheless, the setup that we chose seems plausible, is probably not the only market arrangement that sustains efficient outcomes, and highlights the ability of cheap-talk communication to facilitate the sorting of workers to firms in an efficient manner.

Regarding *related literature*, directed search goes back to the market games by Peters (1984, 1991, 1997a, 1997b, 2000) and Burdett, Shi, and Wright (2001). Nearly all existing work in this literature assumes commitment. An important exception is the introduction of cheap-talk communication in Menzio (2007), which most inspired our current investigation.[6] While we ask whether cheap talk can implement the efficient outcome and, thus, exactly replicates the allocation of the commitment economy, he asks a more basic question: Can cheap talk transmit any information at all in a directed search economy and allow more productive firms to hire with larger probability? It is related in the sense that he also shows that even in a pure cheap-talk environment some information can be transmitted and more productive firms attract more workers. But it differs substantially in the sense that in his setup only very limited information can be transmitted and the outcome is significantly different from the efficient outcome of the commitment economy.[7] The difference in the results is mainly due to the difference in ex post wage-determination mechanisms. Menzio

[4]The outcome of a second-price auction is not well-defined when there is only one applicant, because the reserve price is private information to each firm. For the case, one may assume that the sole applicant can make a take-it-or-leave-it wage demand. Alternatively, one may assume that workers first observe the other workers that are present in the same meeting and then engage in Bertrand competition.

[5]In this sense, our setup captures the full range from random to directed search, even though our focus is on the more surprising element that the directed search outcome can be sustained.

[6]Another exception is a model of directed search with limited commitment by Albrecht, Gautier, and Vroman (2010). In their setup, a seller is committed to her asking price, because she must accept any price above, but not fully committed, because she is free to accept any price below.

[7]In the efficient commitment economy each productivity type has a different hiring probability, while in the cheap-talk economy of Menzio (2007), at most two different messages and, therefore, at most two different hiring probabilities can be supported for the standard urn-ball matching function.

(2007) considers a bilateral alternating offer bargaining game: If more than one worker applies to a firm, then the firm randomly selects a worker and plays an alternating offer bargaining game with the worker. In contrast, we consider a setting in which each firm runs an informal first-price procurement auction (equivalently, each applicant makes a take-it-or-leave-it wage demand to a firm). In Menzio's setup, competition among workers for jobs only affects workers' and firms' matching probabilities, while in ours it plays two crucial roles. First, it enables wages to be related to the ex-post core at the state where the firm and the workers meet, which is an essential ingredient for the optimality of the fully revealing equilibrium.[8] Second, it induces workers' wage demands to respond to a firm's productivity announcement in a way to deter any firm's strategic manipulation.

It is fairly well-known that job auctions yield the efficient market outcome. Kultti (1999) shows the equivalence between second-price auctions without reserve price and directed search with commitment. Julien, Kennes, and King (2000, 2005) and Shimer (1999) prove the efficiency of second-price and first-price auctions, respectively. Each firm's productivity is public information in all of these studies[9] and, therefore, there is no role for ex ante communication. Interestingly, when coupled with cheap-talk communication, first-price and second-price auctions yield very different predictions: Only fist-price auctions support efficient equilibria. As we elaborate later, this has to do with how workers' wage demands vary according to a firm's productivity announcement, which crucially affects a firm's incentive at the communication stage.

In the literature on market games, recent work has asked which endogenous price schedules would bring about efficient outcomes even if not all worker and firm attributes are specified (see, e.g., Mailath, Postlewaite, and Samuelson (2010)). The difference from our setup is that the monetary payoffs are commitments but do not necessarily constitute complete contracts. In our setting, the main challenge of the second point is that in the ex-post bargaining the transfers will change if a worker faces a deviant firm.

In the following, we first derive our main results from the most common specification with homogeneous workers. We then discuss the subtlety as well as the robustness of our findings. We also analyze the case with heterogeneous workers. Since the case requires slightly more elaborate techniques but confirms the basic results, we present only the intuitions behind the generalization, relegating the formal analysis to Appendix A. We conclude with a discussion

[8] In Menzio (2007), the outcome is not efficient even in the absence of asymmetric information. This problem raises a question on what makes search not fully directed in his model. One corollary of our result is that search is partially directed in Menzio, not because of lack of commitment by firms, but because of the particular wage-determination mechanism considered.

[9] Kultti (1999), Julien, Kennes, and King (2000), and Shimer (1999) consider models with homogeneous firms. Julien, Kennes, and King (2005) study small markets with heterogeneous firms, but assume that each firm's productivity is publicly known.

of the breath of the result within the directed search environment and for market games in general.

2 The Model

Physical environment. The labor market consists of a continuum of firms and a continuum of workers. The measure of firms is normalized to 1 and the measure of workers is given by $\beta > 0$.[10] Each worker seeks one job and each firm has one vacancy. The productivity of each vacancy is drawn from a distribution function G with support $\mathcal{Y} \subset \mathbb{R}$ with finite minimal element $\underline{y}$ and maximal element $\overline{y}$. The support can be finite or infinite. Productivity y is private information to each firm. In this section we assume that workers are homogeneous. In particular, they have the same opportunity cost of working, $c \geq 0$. To avoid a trivial case, assume that $c < \overline{y}$. All agents are risk-neutral: If a worker is hired by a productivity y firm at a wage w, then the worker receives utility $w - c$ and the firm obtains profit $y - w$.

Labor market interaction. We consider three forms of labor market interaction. The first form is the problem of a social planner who wishes to maximize gross surplus (social planner's problem). The social planner observes each firm's productivity and instructs workers to apply to each firm with a certain probability. The second form is the canonical wage-posting game (directed search implementation). Each firm posts a wage. Workers observe all posted wages and decide where to apply for a job. If more than one worker applies for the same job, then one worker is chosen at random and paid the promised page. The last form is a market communication game, which is our main focus. Each firm announces a cheap-talk message from an arbitrarily large set $\mathcal{M}$.[11] Workers observe all announced messages and apply for one job with a wage demand. Firms that receive at least one application choose whether to hire a worker and, if so, which worker to hire.

Search frictions due to anonymity. Following the spirit of the directed search literature, we impose anonymity on all the three forms of labor market interaction.[12] Precisely, we restrict our attention to the outcomes in which all workers play an identical strategy. In the social planner's problem, the social planner designs an application strategy commonly used by all workers. If a worker applies to a firm with a certain probability, then all other workers

[10] None of our results depends on the assumption that the ratio of workers to firms β is exogenously given. See Remark 1.

[11] Since messages are cheap talk, their exact form is unimportant. A message can be, for example, a negotiable wage offer, a feasible wage range (as is common in U.K.), or an advertisement about the firm's productivity.

[12] See Burdett, Shi, and Wright (2001) and Shimer (2005) for excellent discussions on the issue.

apply to the firm with the same probability. In the directed search setting, all workers apply to firms of an identical wage with equal probability. In other words, workers' application strategy conditions on posted wages, but not on their own or firms' identities. In the cheap talk setting, all workers apply to firms of an identical message with equal probability. In addition, they employ an identical wage-demand strategy for firms of an identical message. In other words, workers condition only on announced messages, neither on their own nor on firms' identities, in both application and wage-demand stages.

3 Social Planner's Problem and Directed Search Implementation

This section analyzes the social planner's problem and shows that the social optimum can be decentralized as a directed search equilibrium. Since this is a familiar exercise in the literature, we keep our discussion to a minimum.

3.1 Social Planner's Problem

Planner's objective. A worker's application strategy can be described by a distribution function $P : \mathcal{Y} \to [0, 1]$ where $P(y)$ is the probability that the worker applies to a firm whose productivity is less than or equal to y. Due to anonymity, all workers follow the same application strategy. Then, each productivity y firm receives on average

$$\lambda(y) \equiv \frac{\beta dP(y)}{dG(y)} \tag{1}$$

applications. As common in the literature, we refer to this ratio as the queue length for a productivity y firm.

Anonymity implies that these applications are distributed randomly to the firms of this productivity. If a finite number of m applications are sent to $n = m/\lambda$ firms with equal probability, the probability that a given firm does not receive an application is the joint probability that all workers apply to other firms $\left(\frac{n-1}{n}\right)^m$, which converges to $e^{-\lambda}$ as the number of applications and associated firms gets large. A productivity y firm fills its vacancy whenever at least one worker applies to it, which in a large market has probability $1-e^{-\lambda(y)}$.[13] Since a pair of a worker and a productivity y firm produces social surplus $y - c$, the social

[13]See also Burdett, Shi, and Wright (2001) for a simple derivation and insightful discussions.

planner faces the following maximization problem:

$$\max_{P:\mathcal{Y}\to[0,1]} \int_{\underline{y}}^{\overline{y}} (1-e^{-\lambda(y)})(y-c)dG(y).$$

Since $\lambda(\cdot)$ and $dP(\cdot)$ are one-to-one, we can rewrite this problem as follows:

$$\max_{\lambda:\mathcal{Y}\to\mathcal{R}_+} \int_{\underline{y}}^{\overline{y}} (1-e^{-\lambda(y)})(y-c)dG(y),$$

subject to

$$\int_{\underline{y}}^{\overline{y}} \lambda(y)dG(y) \leq \beta.$$

The constraint is due to (1) and the fact that $P(\overline{y}) \leq 1$. Intuitively, there are measure $dG(y)$ of productivity y firms in the market. Since each productivity y firm receives on average $\lambda(y)$ applications, the measure of applications made to productivity y firms is $\lambda(y)dG(y)$. The constraint states that the total number of applications cannot be larger than the measure of workers present in the market.

Characterization. The objective function is strictly concave, while the constraint is linear. Therefore, the solution to the social planner's problem is always unique and fully characterized by first-order conditions. Let μ be the multiplier attached to the constraint. Then, by the standard results for constrained maximization, a pair $(\lambda(\cdot),\mu)$ is the solution if and only if

$$\mu \geq e^{-\lambda(y)}(y-c), \text{ with the equality holding if } \lambda(y)>0, \tag{2}$$

and

$$\int_{\underline{y}}^{\overline{y}} \lambda(y)dG(y) = \beta. \tag{3}$$

A worker contributes to social surplus if and only if he applies to a firm who otherwise would not receive any application. Since the probability that a productivity y firm does not receive any application is $e^{-\lambda(y)}$, the marginal social contribution of a worker who applies to a productivity y firm is equal to $e^{-\lambda(y)}(y-c)$. The complementary slackness condition (2) states that the planner wants a worker to apply to a productivity y firm only when his marginal social contribution is at least as much as his shadow value to the planner and a worker's marginal social contribution must be constant across all the firms he might apply to. The binding market feasibility condition (3) states that the planner wants to utilize all workers. To sum up, we obtain:

Proposition 1 *There exists a unique solution to the social planner's problem, which is characterized by a pair* $(\lambda(\cdot), \mu)$ *that satisfy the complementary slackness condition (2) and the binding feasibility condition (3).*

3.2 Directed Search Implementation

The planner's problem describes the constrained efficient allocation, but does not specify the payoffs that firms and workers obtain. To obtain a benchmark for this, we consider a canonical wage-posting game with commitment: Each firm posts a wage w and then each worker decides which job to apply for. If multiple workers apply for the same job, then a worker is chosen at random and hired at the promised wage.

Strategies. Each firm decides which wage to post. Denote by $w(y)$ a wage posted by productivity y firms. A common application strategy by workers is described by a distribution function $P : \mathcal{R}_+ \to [0, 1]$ where $P(w)$ is the probability that each worker applies for a wage less than or equal to w.

Submarket outcomes. The firms who post an identical wage w and the workers who apply to them essentially constitute a *submarket*. In the submarket, each firm receives on average

$$\lambda(w) \equiv \frac{\beta dP(w)}{\int_{\{y \in \mathcal{Y}: w(y)=w\}} dG(y)}. \tag{4}$$

applications. Denote by $U(w)$ a worker's expected utility and by $V(y, w)$ a productivity y firm's expected profit in the submarket. A firm fills its vacancy whenever at least one worker applies, which has probability $1 - e^{-\lambda(w)}$. Using the fact that the measure of the workers who successfully find a job is identical to that of the firms who successfully fill their vacancy, we also find that a worker gets a job with probability $\frac{1-e^{-\lambda(w)}}{\lambda(w)}$. It then follows that

$$V(y, w) = (1 - e^{-\lambda(w)})(y - w), \text{ and} \tag{5}$$

$$U(w) = \frac{1 - e^{-\lambda(w)}}{\lambda(w)}(w - c). \tag{6}$$

Definition of directed search equilibrium. To formally define a directed search equilibrium, following the literature, we impose the *market utility condition* on out-of-equilibrium wages. The condition states that if a firm deviates and offers an out-of-equilibrium wage w, unless w is sufficiently small, the expected number of applications for the deviator is such that a worker is indifferent between applying to the firm and obtaining his market utility-the highest expected utility he could obtain by applying to any other firm. Formally, let u be workers' market utility. Then, for any out-of-equilibrium wage w, it must be that $U(w) = \frac{1-e^{-\lambda(w)}}{\lambda(w)}(w-c) \leq u$, with equality holding if $\lambda(w) > 0$. We are now ready to define a directed search equilibrium.

Definition 1 *A directed search equilibrium is a tuple $\{P(\cdot), w(\cdot)\}$ such that there exists $\lambda(\cdot)$ that satisfies the following conditions:*

1. *Optimal application: For any w in the support of P, $U(w)$ is determined by (6) and*

$$U(w) = u \equiv \max_{w'} U(w').$$

2. *Profit maximization: For any $y \in \mathcal{Y}$, $V(y,w)$ is determined by (5) and*

$$w(y) = argmax_w V(y,w).$$

3. *Consistency: $\lambda(w)$ satisfies (4) whenever w is offered by some firms (there exists $y \in \mathcal{Y}$ such that $w(y) = w$) and fulfills the market utility condition otherwise.*

Characterization. Denote by u workers' market utility. From the market utility condition we obtain the following. If $w - c \leq u$, then a worker receives less than u even if he finds a job with the wage with probability 1. Since no worker would apply for the wage, $\lambda(w) = 0$. If $w - c > u$, then $\lambda(w)$ satisfies

$$\frac{1-e^{-\lambda(w)}}{\lambda(w)}(w-c) = u. \tag{7}$$

The function $\lambda(\cdot)$ is strictly increasing. Intuitively, relatively more workers apply for a higher wage.

Each firm chooses a wage that maximizes its profits. It can always achieve zero profits by choosing $w = c + u$, but must offer strictly more to achieve a strictly positive hiring probability. The problem facing a productivity y firm is

$$\max_{w \geq c+u} (1 - e^{-\lambda(w)})(y - w),$$

where $\lambda(w)$ satisfies (7). It is straightforward to show that $w(y)$ is the optimal solution if and only if

$$e^{-\lambda(w(y))}(y-c) - u \leq 0, = 0 \text{ if } \lambda(w(y)) > 0. \tag{8}$$

The firm optimality condition (8) implies that all and only the firms whose productivity is above $c+u$ offer a wage above workers' reservation utility $(c+u)$ and, therefore, hire a worker. Moreover, higher productivity firms offer higher wages and fill their vacancy with greater probabilities.

Lastly, since all workers obviously apply to some firm, $P(w(\overline{y}))=1$, which is equivalent to

$$\int_{\underline{y}}^{\overline{y}} \lambda(w(y))dG(y)=\beta. \tag{9}$$

In other words, the total number of applications must be equal to the measure of workers in the market.

Observe that the two conditions (8) and (9) coincide with the conditions for the social optimum (2) and (3), respectively. Therefore, the induced queue length for a productivity y firm $\lambda(w(y))$ is identical to that of the social optimum $\lambda(y)$. The following result is then immediate.

Proposition 2 *There is a unique directed search equilibrium, which is characterized by a tuple $(w(\cdot),\lambda(\cdot),u)$ that satisfy the market utility condition (7), the firm optimality condition (8), and the binding market feasibility condition (9). The directed search equilibrium implements the social optimum.*

4 Cheap Talk Implementation

This section considers the following market communication game: Each firm announces a message from an arbitrarily large set $\mathcal{M}$. Workers observe all announced messages, select a firm, and make a wage demand.[14] We assume that workers do not observe the number of other workers who apply to the same firm.[15] Firms that receive at least one application decide whether and, if so, which worker to hire.

4.1 Strategies

Firms' communication strategies are described by a function $m:\mathcal{Y}\to\mathcal{M}$ where $m(y)$ is the message sent by productivity y firms. For simplicity, and without loss of generality, we

[14] We assume that each firm's productivity remains private until workers submit their wage demands. If firms' productivity is revealed to workers before wage demand, then the result is trivial: there does not exist any informative equilibrium. The reason is that by mimicking a higher productivity firm, a firm could attract relatively more workers, but workers would adjust their wage demands after application. Therefore, there would be a positive benefit of deviation but no corresponding cost.

[15] This assumption is important. See Section 5.

assume that $\mathcal{M} = \mathcal{Y}$. A common application strategy by workers is a function $P : \mathcal{M} \to [0,1]$ where $P(m)$ is the probability that a worker applies to a firm who announces a message below m. A common wage-demand strategy by workers is a function $H(\cdot,\cdot) : \mathcal{R}_+ \times \mathcal{M} \to [0,1]$ where $H(w,m)$ is the probability that a worker demands less than or equal to w to a firm with message m. Each firm's optimal hiring strategy is straightforward, so its formal description is omitted: Each firm hires a worker who demands the lowest wage, provided that the wage is not greater than its own productivity.

We are mainly interested in the fully revealing equilibrium of this communication game. In order to save notation as well as provide a direct comparison to the wage-posting game, we first characterize the outcome of the submarket in which all firms have the same productivity. At this stage, we solve for workers' optimal wage-demand strategy. We then provide a formal definition of the fully revealing equilibrium, focusing on firms' communication and workers' application strategies. Finally, we establish our main substantive results: We first show that the fully revealing equilibrium implements the social optimum, by solving for workers' optimal application strategy, and then prove that a fully revealing equilibrium always exists, by solving for firms' optimal communication strategies.

4.2 Submarket Outcomes

Similarly to the wage-posting game, the firms who announce an identical message and the workers who apply to them constitute a submarket. Consider a submarket in which all firms (are believed to) have the same productivity y and the queue length is given by λ. In the submarket, each worker randomly selects a firm and submits a wage demand. Each firm decides whether to accept the lowest wage demand or not. Denote by $U(y,\lambda)$ a worker's expected utility and by $V(y',y,\lambda)$ a productivity y' firm's expected profit in the submarket.

Workers' expected utility. A worker's expected utility is at least as much as $e^{-\lambda}(y-c)$, because he can always demand y, which would be accepted at least when there is no other applicant, which has productivity $e^{-\lambda}$. We will show that, in fact, a worker's expected utility is exactly equal to $e^{-\lambda}(y-c)$. The result follows from the following two intermediate results. First, there is no atom in the support of workers' wage-demand strategy: Suppose there is an atom. Then instead of putting positive probability on the wage with the atom, a worker can deviate and demand slightly below that wage. This decreases his wage only slightly but yields a discrete upward jump in his employment probability because he avoids the positive probability of being tied with another worker. This yields a contradiction to the requirement that the mass point must be optimal. Second, since there is no atom, a worker who demands the highest wage is hired only when there is no other applicant at the same firm. Therefore, his wage demand must be optimal conditional on being the only applicant and, therefore, be equal to y. Since workers are indifferent between all wage demands, their utility must be equal to that of a worker who demands the highest wage y. We therefore obtain:

Lemma 1 *(Workers' expected utility)*

$$U(y, \lambda) = e^{-\lambda}(y - c).$$

Workers' wage-demand strategy. Consider a worker who demands w to a firm of productivity y, i.e., to a firm that posts message $m = y$ which he interprets as the true productivity. He is hired if and only if no other worker who demands less than w applies to the same firm.[16] Since the ratio of the workers who demand less than w to firms is $\lambda H(w, y)$, the worker is hired with probability $e^{-\lambda H(w,y)}$ and his expected utility is $e^{-\lambda H(w,y)}(w - c)$. If w is in the support of $H(\cdot, y)$, then a worker must be indifferent between w and y, and thus

$$e^{-\lambda H(w,y)}(w - c) = e^{-\lambda}(y - c) = U(y, \lambda).$$

It follows that the lowest wage demand is equal to $c + U(y, \lambda)$.

Lemma 2 *(Workers' wage-demand strategy) In the submarket where all firms have productivity y and the queue length is λ, the probability that each worker demands less than w (equivalently, the proportion of the workers who demand less than w) $H(w, y)$ satisfies*

$$e^{-\lambda H(w,y)} = \frac{U(y, \lambda)}{w - c}, \text{for any } w \in [c + U(y, \lambda), y]. \tag{10}$$

Firms' equilibrium expected profits. A firm hires a worker at a wage below w if and only if at least one worker who demands less than w applies to the firm. Since the ratio of the workers who demand less than w to firms is $\lambda H(w, y)$, the probability is equal to $1 - e^{-\lambda H(w,y)}$. A productivity y firm's expected profit is

$$V(y, y, \lambda) = \int_{c+U(y,\lambda)}^{y} (y - w) d\left(1 - e^{-\lambda H(w,y)}\right).$$

Applying Lemma 2 and arranging terms, we obtain:

Lemma 3 *(A productivity y firm's equilibrium expected profit)*

$$V(y, y, \lambda) = (1 - e^{-\lambda} - \lambda e^{-\lambda})(y - c).$$

[16] Within a submarket, search is random. Therefore, the probability that no applicant to a firm demands less than w is equal to the probability that no worker who would demand less than w applies to the firm.

Understanding the payoffs through revenue equivalence. One can intuitively understand the payoffs through the revenue equivalence theorem.[17] Our complete information wage-demand game can be interpreted as each firm running a first-price procurement auction with reserve price equal to its productivity level. Instead, suppose each firm runs a second-price procurement auction with the same reserve price. Then a worker gets the entire surplus if and only if he is the only applicant. Therefore, a worker's expected utility is $e^{-\lambda}(y-c)$. A firm would get the entire surplus if and only if there are at least two applicants. Since the probability that no worker applies is $e^{-\lambda}$ and the probability that one worker applies is $\lambda e^{-\lambda}$, a firm's expected profit is $(1-e^{-\lambda}-\lambda e^{-\lambda})(y-c)$. By revenue equivalence, these payoffs coincide with those of our game.

Deviating firms' expected profits. Consider a productivity y' firm who deviated and joined the submarket of productivity y firms. The deviating firm faces the same queue length and wage-demand strategy as a productivity y firm. The only difference is that a productivity y' accepts a wage demand only when it does not exceed y'. The deviating productivity y' firm's expected profit is

$$V(y',y,\lambda)=\int_{c+U(y,\lambda)}^{\min\{y',y\}}(y'-w)d\left(1-e^{-\lambda H(w,y)}\right).$$

We skip the derivation of a closed-form expression. Although it is straightforward, the resulting expression is rather complicated and turns out not necessary for any of the subsequent developments.

4.3 Fully Revealing Equilibrium

Given firms' communication strategies $m(\cdot)$ and workers' application strategy $P(\cdot)$, the queue length associated with message m is equal to

$$\lambda(m)=\frac{\beta dP(m)}{\int_{\{y\in\mathcal{Y}:m(y)=m\}}dG(y)}. \tag{11}$$

Denote by $\tilde{U}(m)$ the (highest) expected utility a worker can obtain by applying to a firm who announced message m. Also, denote by $\tilde{V}(y,m)$ the expected profit of a productivity y firm when it announces message m.

Building upon the previous results, we characterize a fully revealing equilibrium of our

[17] Revenue equivalence continues to hold with a stochastic number of bidders, provided that bidders are risk neutral. See McAfee and McMillan (1987).

communication game, which is formally defined below. Without loss of generality, we assume that a productivity y firm uses message y to truthfully reveal its productivity.

Definition 2 *A fully revealing equilibrium is a tuple* $(m(\cdot), P(\cdot), H(\cdot,\cdot))$ *such that there exists* $\lambda : \mathcal{M} \to \mathcal{R}_+$ *that satisfies the following conditions:*

1. *Optimal and truthful communication: It is optimal for any productivity y' firm to truthfully reveal its productivity, that is, $\tilde{V}(y', y) = V(y', y, \lambda(y))$ and*

$$m(y') = y' \in argmax_y \tilde{V}(y', y).$$

2. *Optimal application and wage demand: For any y' in the support of P and any w in the support of $H(\cdot, y')$, (10) has to hold, $\tilde{U}(y) = U(y, \lambda(y))$ and*

$$\tilde{U}(y') = \max_y \tilde{U}(y).$$

3. *Consistency: $\lambda(m)$ satisfies (11) for any $m \in \mathcal{Y}$.*

The conditions are stated deliberately in an analogous manner to those for directed search equilibrium. One notable difference is the consistency condition. It does not specify workers' beliefs on firms' types following out-of-equilibrium messages. Since we restrict attention to the fully revealing equilibrium, there are no off-equilibrium messages.[18]

4.4 Fully Revealing Equilibrium and Optimality

We first solve for workers' optimal application strategy in the subgame where each firm's productivity is publicly known. Let u be the highest expected utility a worker can achieve in the subgame. This utility is analogous to the market utility in directed search equilibrium. By the definition of u, if $\lambda(y) > 0$, then a worker must obtain the same utility u by applying to a productivity y firm. If $\lambda(y) = 0$, then he must obtain less than u. Therefore,

$$u \geq U(y, \lambda(y)) = e^{-\lambda(y)}(y - c), \text{ with equality holding if } \lambda(y) > 0. \tag{12}$$

In addition, since all workers would obviously apply to some firm, $P(\overline{y}) = 1$, which is equivalent to

$$\int_{\underline{y}}^{\overline{y}} \lambda(y) dG(y) = \beta. \tag{13}$$

[18]Of course, one could use a message space larger than $\mathcal{Y}$, in which case the same outcome can be supported by any beliefs that assign a unique type to each additional message (since firms are clearly indifferent between announcing their type and using a different message that also fully reveals their type).

Observe that the two conditions are identical to the ones for the social optimum and for directed search equilibrium. The following result is then straightforward.

Proposition 3 *The fully revealing equilibrium of the market communication game is unique and implements the social optimum.*

The underlying reason for the optimality is that agents' *interim* payoffs in our game coincide with those obtained when allocations are determined at the worker-optimal points in the ex-post core and, therefore, the Hosios condition holds for each firm type. At the worker-optimal point in the ex post core, each worker receives his marginal contribution: If there is only one worker in a meeting, he extracts the entire surplus. If there are at least two workers, all workers receive 0. Therefore, if the final allocation is determined at the worker-optimal core, then all search externalities are internalized. Agents' *ex-post* payoffs in our game differ from those of the worker-optimal core. However, by revenue equivalence, their *interim* payoffs are identical, which is sufficient for efficient search decisions by workers. Two additional remarks are important relating to entry of firms and to situations where some firms commit while others use cheap-talk messages.

Remark 1 (Entry efficiency) Our efficiency result continues to hold even if firms' entry decisions are incorporated. The reason is that the payoffs are equal to those in the commitment models, and we know that they induce efficient entry. Formally, suppose there is a large continuum of firms and each firm can create a vacancy at a cost $k > 0$. The productivity of each vacancy is drawn from the distribution function G. Then, in the fully revealing equilibrium, the ratio of workers to *active* firms β will be adjusted so that a firm's expected profit by creating a vacancy is equal to the entry cost,[19] that is,

$$\int_{\underline{y}}^{\overline{y}} V(y, y, \lambda(y))dG(y) = \int_{\underline{y}}^{\overline{y}} (1 - e^{-\lambda(y)} - \lambda(y)e^{-\lambda(y)})(y - c)dG(y) = k.$$

Observe that the left-hand side coincides with the value of an additional vacancy to the planner: The productivity of a vacancy is less than y with probability $G(y)$ and a vacancy of productivity y increases social surplus as much as $(1 - e^{-\lambda(y)} - \lambda(y)e^{-\lambda(y)})(y - c)$.[20] Since the equilibrium entry condition is identical to the planner's optimal entry condition, it follows

[19] Since the matching technology exhibits constant return to scale, we only need to determine the ratio of workers to firms.

[20] One can use the following direct argument to obtain the value of a productivity y vacancy. Suppose a vacancy of productivity y is pulled out from the market. Then, social surplus decreases as much as $(1 - e^{-\lambda(y)})(y - c)$. However, the workers who might have applied to the firm would apply to other firms who otherwise might not have received any application. This redirecting of workers has a positive effect on social surplus. Since the probability that i workers apply to a firm is $\frac{e^{-\lambda(y)}\lambda(y)^i}{i!}$ and the probability that

that the optimal number of firms would enter. This result is driven by the fact that firms also receive as much as their social marginal contribution when allocations are determined at the worker-optimal points in the ex-post core.

Remark 2 (Partial Commitment) Firms are indifferent between using a commitment announcement and a truthful cheap-talk announcement when workers believe that firms report the truth. This follows immediately from the observation that expected payoffs are the same. It is also straightforward that in any settings where some firms use commitment and the other firms use messages, in the fully-revealing equilibrium exactly the same payoffs arise, and each individual firm is indifferent between using a message or commitment. If there is any small cost to commitment, this might explain why the market adopts cheap-talk messages.

4.5 Existence of Fully Revealing Equilibrium

We now consider firms' incentives at the communication stage and examine whether each firm would be willing to truthfully reveal its productivity. Suppose a productivity y' firm unilaterally deviates and pretends to have productivity y (by announcing the same message as productivity y firms). The firm faces a trade-off between hiring probability and wage. If $y > y'$, then relatively more workers would apply to the firm, but each applicant would demand a relatively higher wage, believing that there is more surplus to extract. In order to avoid a trivial case, we consider only the case where $y, y' > c + u$,[21] which implies that $\lambda(y), \lambda(y') > 0$.

To compare the productivity y firm's equilibrium and deviation profits, first consider the probability that the firm hires a worker at a wage below $w \leq \min\{y, y'\}$. Since the queue length of the workers who would demand less than w for a firm who announces message m is equal to $\lambda(m)H(w, m)$, the probability is $1 - e^{-\lambda(y')H(w,y')}$ if the firm truthfully reveals its productivity and $1 - e^{-\lambda(y)H(w,y)}$ if the firm deviates to y. Although $\lambda(y) \neq \lambda(y')$ and $H(w, y) \neq H(w, y')$, the two probabilities are identical: By the optimal application condition in the equilibrium definition, a worker has to obtain the same utility whether he applies to

each worker is the only applicant to a firm is $e^{-\lambda(y)}$, the positive effect amounts to

$$\sum_{i=1}^{\infty} \frac{e^{-\lambda(y)}\lambda(y)^i}{i!} i e^{-\lambda(y)}(y - c) = \lambda(y)e^{-\lambda(y)}(y - c) \sum_{i=1}^{\infty} \frac{e^{-\lambda(y)}\lambda(y)^{i-1}}{(i-1)!} = \lambda(y)e^{-\lambda(y)}(y - c).$$

Therefore, the value of a productivity y vacancy to the planner is equal to $(1 - e^{-\lambda(y)} - \lambda(y)e^{-\lambda(y)})(y - c)$.

[21] If $y - c \leq u$, then, by (12), $\lambda(y) = 0$, and thus a productivity y' firm obviously has no incentive to deviate. Independently of a firm's productivity, the lowest wage demand made by a worker is $c + u$. Therefore, if $y' - c \leq u$, then a productivity y' firm would never be able to hire a worker whether it deviates or not.

a productivity y firm or a productivity y' firm. In addition, conditional on selecting a particular firm, a worker is indifferent over all wages in the support of his wage-demand strategy. Therefore, for any wage demand $w \in [c+u, \min\{y, y'\}]$, a worker has the same employment probability, whether he applies to a productivity y firm or a productivity y' firm. Since a worker who demands w to a firm who announced message m is hired with probability $e^{-\lambda(m)H(w,m)}$, it follows that $e^{-\lambda(y)H(w,y)} = e^{-\lambda(y')H(w,y')}$ for any $w \leq \min\{y, y'\}$, which formally follows from (10).

The above result shows that a firm's deviation affects its expected payoff only through the change of the probability that the lowest wage demand lies between y and y'. If $y' > y$, then the deviating firm loses the opportunity to hire a worker at a wage $w \in (y, y']$. Therefore, the firm strictly prefers truthfully revealing its productivity to deviating to a lower productivity. If $y' < y$, then the lowest demand would be $w \in (y, y']$ with a positive probability. However, a productivity y firm does not accept such a wage and, therefore, it has no consequence on the firm's expected profit. It follows that the firm is indifferent over all submarkets weakly above its productivity.[22] We conclude that no firm has an incentive to misreport its productivity and, therefore, there always exists a fully revealing equilibrium.

Proposition 4 *In the market communication game, there always exists a fully revealing equilibrium (which implements the social optimum).*

5 Discussion

Our demonstration of the non-essentiality of commitment in directed search is based on two insights. First, if the allocation in each meeting is determined according to the worker-optimal point in the ex post core, then workers make socially optimal application decisions. Second, workers' wage demands can vary depending on a firm's productivity, so that no firm has an incentive to misreport its productivity. In this section, we show that our two insights are distinct, and also discuss the extent of the robustness of our insights.

[22] More directly, as derived in 4.2, a productivity y''s expected payoff when it deviates to y is

$$\tilde{V}(y', y) = V(y', y, \lambda(y)) = \int_{c+u}^{\min\{y', y\}} (y' - w) d\left(1 - e^{-\lambda(y)H(w,y)}\right).$$

Since $e^{-\lambda(y)H(w,y)} = e^{-\lambda(y')H(w,y')}$ whenever $w \leq \min\{y, y'\}$,

$$\tilde{V}(y', y') - \tilde{V}(y', y) = \int_{\min\{y,y'\}}^{y'} (y' - w) d\left(1 - e^{-\lambda(y')H(w,y')}\right).$$

Therefore, $\tilde{V}(y', y') > \tilde{V}(y', y)$ for any $y < y'$ and $\tilde{V}(y', y') = \tilde{V}(y', y)$ for any $y \geq y'$.

5.1 The Role of First-price Auctions with Uncertain Number of Competitors

We assumed that workers submit their wage demands, without knowing how many other workers applied to the same firm. Clearly, a firm would always like to pretend that there are other applicants to depress the wage demand from any given worker, which means that workers would not trust any such statements from the firm. But if they could truly observe the actual number of competing applicants before making their wage demand, then our first insight still holds, but the second one fails. In other words, if firms' types were fully revealed, the market interaction would induce the efficient outcome, but firms would not be willing to reveal their productivity, so there does not exist a fully revealing equilibrium.

If workers can condition on the number of competitors for their wage demands, then the fully revealing outcome coincides with the one obtained when each firm runs a second-price procurement auction with reserve price equal to its productivity. If there is only one worker, then the worker would demand the entire surplus. If more than one worker apply to the same firm, they would engage in Bertrand competition and, therefore, a worker would be hired at a wage equal to c. Again, by revenue equivalence, the resulting (interim) payoffs coincide with those of our game and, therefore, the resulting market outcome would be efficient.

However, there does not exist a fully revealing equilibrium. Suppose it does exist. Consider a productivity y' firm who pretends to have productivity $y(> y')$. If at least two workers apply, then the firm hires a worker at c. If only one worker applies, then the worker demands y, and the productivity y' firm would reject the wage demand. Since the queue length for the deviating firm is $\lambda(y)$, the firm's expected profit is $(1 - e^{-\lambda(y)} - \lambda(y)e^{-\lambda(y)})(y' - c)$. Since $\lambda(y) > \lambda(y')$, this payoff is strictly larger than the firm's equilibrium payoff, which is a contradiction. This unraveling argument in fact extends to any informative equilibrium, so there does not exist any informative equilibrium in this alternative setup.

The difference from our original communication game lies in how the distribution of workers' wage demands varies according to a firm's productivity. In both cases, relatively more workers apply, and workers demand relatively higher wages, to a higher productivity firm.

If workers observe the number of competitors, workers demand either c (when there are competitors) or y (when there is no competitor). In this case, the deviation benefit comes from the increase of the probability that at least two workers apply to the firm $(1-e^{-\lambda}-\lambda e^{-\lambda})$, while the deviation cost is related to the increase of the probability that one worker applies to the firm $(\lambda e^{-\lambda})$. However, when only one worker applies, the firm extracts no surplus, whether it deviated or not. Therefore, there is a positive benefit, but no corresponding cost, of mimicking a higher productivity firm.

If workers do not observe the number of competitors, by definition, the increase of workers' wage demands is independent of the number of workers. In this case, a deviating firm faces a real deviation cost. All workers would demand more than y' with a positive probability, so the firm may not fill its vacancy even if several workers apply. Furthermore, conditional on hiring a worker, the firm would pay a relatively higher wage. The characterization in Section 4 showed that this cost exactly cancels out the benefit of attracting more workers (the upward shift of the distribution of the number of workers) and, therefore, no firm has an incentive to deviate.

This discussion illuminates the subtle difference between our communication setup and the second-price auction setup by Peters (1997b) (and Peters and Severinov (1997)). Peters considers the setup in which each firm runs a second-price auction and commits to a reserve wage.[23] He shows that it is an equilibrium that each firm commits to its own productivity level, so in some sense there is a fully revealing equilibrium. The equilibrium outcome is exactly identical to the fully revealing outcome that arises when workers observe the number of competitors and, therefore, implements the social optimum. The difference from ours lies in firms' incentive off the equilibrium path, in particular, the source of firms' deviation cost. When firms commit to a reserve wage, a productivity y' firm that deviated to y must accept the wage demand y, higher than its own productivity level, in the event that only one worker applies. What Peters shows is that this cost (that the firm must pay a wage strictly above its own productivity with a positive probability) outweighs the benefit of attracting more workers (the increase of $1 - e^{-\lambda} - \lambda e^{-\lambda}$). As explained above, without firms' commitment to the terms of trade, the change of workers' wage demands may not be sufficient to deter firms from misreporting their productivity. Wage demands must vary in a particular way according to a firm's productivity. What we showed is that our simple communication game generates the necessary features.

Finally, we can relate our results to the literature that uses ex-post bilateral bargaining to determine wages. Suppose wages are determined through ex-post bilateral bargaining. Let $\eta(\lambda)$ be a firm's hiring probability, which is $1 - e^{-\lambda}$ in our setting, and $W(\lambda)$ be a worker's expected wage when the queue length is λ. For this bargaining game to yield the same payoffs as our original game, a worker's employment probability $\frac{\eta(\lambda(y))}{\lambda(y)}$ times his expected wage $W(\lambda(y))$ when he applies to a productivity y firm must coincide with workers' market

[23] His actual model is more general in that sellers (firms) can choose and commit to any incentive-compatible mechanism.

utility u. Since we know from Lemma 2 that $u = e^{-\lambda(y)}(y-c) = \eta'(\lambda)(y-c)$,

$$W(\lambda(y)) = \frac{\lambda(y)\eta'(\lambda)}{\eta(\lambda(y))}(y-c).$$

The second term gives the surplus of the match, while the first (multiplicative) term can be interpreted as the "bargaining weight." Our wage-determination mechanism delivers this weight endogenously. This is the key to sustain the efficient outcome. A bilateral bargaining protocol in general gives a fixed bargaining weight to the worker. This is a major problem for generating efficient outcomes, since the optimal "bargaining weight" is in general not constant but varies with the queue length. Therefore, it might not be surprising that the efficient outcome cannot be supported. Menzio (2007) investigates an incomplete information alternating offer bargaining process, and shows formally the limits of communication that can be sustained when the bargaining weight is constant.

5.2 Risk Averse Workers

With risk-averse workers, opposite to the previous case, our first insight fails (the fully revealing equilibrium is not efficient), but our second insight holds (the fully revealing equilibrium exists). Suppose a worker who is hired at a wage w obtains utility $\phi(w-c)$, where $\phi(\cdot)$ is a strictly increasing and concave function with $\phi(0) = 0$. With this change, the fully revealing equilibrium is characterized by a tuple $(\lambda(\cdot), H(\cdot,\cdot), u)$ such that

$$u \geq e^{-\lambda(y)}\phi(y-c), \text{ with equality holding if } \lambda(y) > 0,$$

$$e^{-\lambda(y)H(w,y)}\phi(w-c) = u, \text{ whenever } w \in [c+\phi^{-1}(u), y], \tag{14}$$

and

$$\int_{\underline{y}}^{\overline{y}} \lambda(y)dG(y) = \beta.$$

The suboptimality of the fully revealing equilibrium is most evident in the homogeneous firms case (F is degenerate at some y). With homogeneous firms, as is well-known, there is no wage dispersion in the wage-posting game. Our communication game, however, exhibits wage dispersion, because workers still play a mixed wage-demand strategy. With risk-averse workers, this yields clear welfare losses.

Nevertheless, the fully revealing equilibrium always exists. It directly follows from (14): The lowest wage demand to a firm is less than w with probability $1 - e^{-\lambda(y)H(w,y)}$, which is, via (14), essentially independent of a firm's productivity. By the same reasoning as before, each firm strictly prefers revealing its productivity to deviating to a lower productivity and is indifferent over all productivity levels above its own.

The fact that the fully revealing equilibrium always exists implies that the non-essentiality of commitment in directed search is robust in the limit as workers' risk aversion vanishes. As workers become less risk averse, the fully revealing equilibrium converges to the social optimum. Therefore, if workers are nearly risk neutral, then our communication game would induce a nearly efficient market outcome.

5.3 Finite Markets

In the literature there is a long tradition to build up the economy with an infinite number of players as the limit of games with finite numbers of players (see, e.g., Peters (1991, 2000), Burdett, Shi, and Wright (2001), and Galenianos and Kircher (2012)). Here we will do the same and consider the incentives for truthful revelation and efficiency in a finite market. We show that both of our insights hold for finite markets.

Suppose there are m workers and n firms. Again, each firm posts a cheap-talk message. Then, workers decide which firm to apply to and submit a wage demand, and each firm hires the worker with the lowest wage demand below its productivity. We analyze sequential equilibria in symmetric strategies by the workers.

Consider $n > 2$ firms, and let the number of workers m be the smallest integer number above $\beta n > 2$, so that the ratio of workers to firms is similar to the continuum economy. Formally, let $m = \lceil \beta n \rceil > 2$. As before, firms draw their productivity from G, while workers are identical. For convenience, assume throughout this subsection that the lower bound of the support of productivities $\mathcal{Y}$ is above the disutility of working c, that is, $\underline{y} > c$. The formal analysis of the general existence of the fully revealing equilibrium is rather involved. We will therefore analyze the case where the support of the productivity draws $\mathcal{Y}$ is finite, although we allow it to be arbitrarily fine. For this case, we will show that for sufficiently large n and associated m, a fully revealing equilibrium exists and implements the efficient allocation.

Let y_i denote the productivity draw of firm i and p_i denote the probability that each worker applies to firm i. To provide a benchmark, consider a planner who knows all productivity draws and chooses the application probabilities so as to maximize total output. Since output is produced when a firm has at least one applicant, whose probability is $1-(1-p_i)^m$, constrained efficiency is the solution to

$$\max_{(p_1,\ldots,p_n)} \sum_i \left(1-(1-p_i)^m\right)(y_i - c) \ s.t. \ \sum_i p_i = 1.$$

Considering the first order conditions, Montgomery (1991) shows that application probabilities $(p_1, ..., p_n)$ are optimal if and only if the probabilities sum to one and

$$p_i > 0 \;\Rightarrow\; (1-p_i)^{m-1}(y_i - c) \geq (1-p_j)^{m-1}(y_j - c) \text{ for all } j, \tag{15}$$

which implies equality of marginal product across all firms that may receive applications.

Now consider the decentralized equilibrium. We focus on a fully revealing equilibrium. We show that it exists if the market is sufficiently large, and that it is efficient. To analyze the incentives for truth-telling, consider a single firm, say firm n. We want to show that it is willing to report its type truthfully when the other firms report their types truthfully. Since n was chosen arbitrarily, this establishes truth-telling for all firms. Firm n does not know which draws the other firms have received when making its announcements, but knows that they report truthfully. We will show that for any possible draw by the other firms, even if firm n knows exactly their types, it is optimal for the firm to truthfully reveal its own type. Since truth-telling is optimal for any draw of the other firms' types, it is optimal also when the firm does not know which draw the other firms have taken.

To establish that truth-telling is optimal even if the types of the other firms are known, assume that $y_1, ..., y_{n-1}$ are publicly known. Suppose workers believe that firm n's announcement would always be truthful. The expected utility that the workers obtain in the ensuing subgame depends on the productivities of the firms, and in particular on y_n.

By the same arguments as for large markets, the workers' wage-demand strategy for a given firm has no mass points. Since workers are indifferent over all wage demands, we can focus on the worker with the highest demand, which equals the firm's productivity (assuming that the firm truthfully reveals it). The expected payoff of a worker who applies to firm i is equal to the probability that he is the only applicant to the firm ($(1-p_i)^{m-1}$) times the total surplus $(y_i - c)$. Observe that the payoff coincides with the worker's marginal social contribution, because a worker increases social surplus only when he applies to a firm who otherwise would not receive any application, which is the driving force behind the following efficiency result.

The fully revealing equilibrium is characterized by the equilibrium application probabilities $(p_1(y_n), ..., p_n(y_n))$ and workers' market utility $u(y_n)$ such that

$$u(y_n) \geq (1 - p_i(y_n))^{m-1}(y_i - c), \text{ with equality holding if } p_i(y_n) > 0. \tag{16}$$

Note that this holds for all firms, so that (15) holds and efficiency is obtained in the fully-revealing equilibrium.[24] The utility $u(y_n)$ that workers obtain is determined uniquely from

[24]For observable types and finite markets, a similar efficiency result has been shown for second-price

equalization of the return to sending an application across firms. In contrast to the infinite economies studied previously, this utility is no longer independent of the firm's type. If the firm has a higher type and reveals it truthfully, each individual worker reaps a higher utility in the subgame, while in a large economy the effect of a single announcement is negligible.

To avoid a trivial case, consider only the case where $p_n(y_n) \in (0,1)$. Denote by $H(w, y_n)$ the probability that each worker demands less than or equal to w from firm n. Then, for any w in the support of $H(\cdot, y_n)$,

$$(1 - p_n(y_n)H(w, y_n))^{m-1}(w - c) = u(y_n).$$

Using this, we find that the probability that the lowest wage demand to firm n is less than w is equal to

$$1 - (1 - p_n(y_n)H(w, y_n))^m = 1 - \left(\frac{u(y_n)}{w - c}\right)^{\frac{m}{m-1}}. \tag{17}$$

Therefore, firm n's expected profit when its true productivity is y'_n but it announces y_n is $\tilde{V}(y'_n, y_n, u(y_n))$ where

$$\tilde{V}(y'_n, y_n, u) = \int_{c+u}^{\min\{y'_n, y_n\}} (y'_n - w) d\left(1 - \left(\frac{u}{w - c}\right)^{\frac{m}{m-1}}\right).$$

$\tilde{V}$ is continuous in its three arguments. If the workers' market utility $u(y_n)$ were independent of firm n's announcement, then the same argument as for large markets would apply: Firm n would strictly prefers truthfully revealing its productivity to deviating to a lower productivity and would be indifferent over all productivity levels above its own. That is: $\tilde{V}(y'_n, y_n, u) < \tilde{V}(y'_n, y'_n, u)$ if $y'_n > y_n$ and $\tilde{V}(y'_n, y'_n, u) = \tilde{V}(y'_n, y'_n, u)$ if $y'_n < y_n$.[25]

In a finite market, the workers' utility $u(\cdot)$ is not constant but strictly increasing. As y_n increases, each worker is more willing to apply to firm n, which increases the value of their outside options (applying to any other firm) and raises their utility. Therefore, the distribution of the lowest wage demand facing firm n changes according to firm n's announcement. In particular, as y_n increases, the distribution increases in the sense of first-order stochastic dominance (see (17)), which negatively affects firm n's profit. Now firm n has strictly less incentive to deviate to a higher productivity. Since firm n would be indifferent over all productivity levels above its own in the absence of this indirect effect, firm n now strictly prefers truthfully revealing its productivity to deviating to a higher productivity.

auctions in Julien, Kennes, and King (2005). Efficiency fails for simple price posting as shown, e.g., in Montgomery (1991) or Galenianos, Kircher, and Virag (2011).

[25]This follows by direct analogy to Footnote 22.

Formally, for $y'_n < y_n$ we have $\tilde{V}(y'_n, y'_n, u(y'_n)) = \tilde{V}(y'_n, y_n, u(y'_n)) > \tilde{V}(y'_n, y_n, u(y_n))$, where the equality follows from the preceding paragraph and the inequality follows from the fact that $u(.)$ is increasing and $\tilde{V}$ is decreasing in its third argument. Therefore, there are strict disincentives to pretend to be a higher type.

Without this indirect effect, firm n strictly prefers truthfully revealing its productivity to deviating to a lower productivity. Since there is a finite number of types, pretending to be a lower type means a discrete change in type, which means that the size of this effect is bounded away from zero for any market size.[26] Therefore, firm n would still not have an incentive to deviate to a lower productivity, provided that the indirect effect through a change in $u(y_n)$ is sufficiently small. It is relatively easy to see that the impact of y_n on $u(y_n)$ is low when the market is large: It is bounded by $\bar{y}/m$ since the productivity of the firm has to be distributed across workers. This term vanishes for large m. Therefore, since the third derivative of $\tilde{V}$ is bounded the indirect effects vanish in a large market, and pretending to be a lower type is strictly unprofitable.[27]

Proposition 5 *Let the set of productivities $\mathcal{Y}$ be finite, let the lowest productivity be above cost ($\underline{y} > c$), and consider a fixed worker-firm ratio of β (so that $m = \lceil \beta n \rceil$). When n is sufficiently large, there exists a fully revealing equilibrium, which is constrained efficient.*

Proof. See the preceding proof. ■

Standard arguments can be applied to show that the allocation in our finite economy approaches the limit economy as the market becomes large. It might be instructive to note two further things. First, the restriction that $\underline{y} > c$ is made for convenience. It ensures that even if all other firms have low draws the workers get strictly positive market utility. Otherwise, it could be that all other firms draw productivities below c, and so all workers approach the last firm at any announcement, in which case a low announcement would be a profitable deviation. While our assumption ensures that even if the firm knew all productivities truth-telling is optimal, we conjecture that $\bar{y} > c$ will be sufficient to ensure in a large market that a firm that does not know the other productivities reveals its type. The reason is that with very high likelihood the workers do have strictly positive utility since some firms are likely to have high draws, and deviations are then not optimal.

[26] For any number $\delta > 0$, as long as $u \geq c + \delta$, there exists a number $\varepsilon > 0$ such that $\tilde{V}(y'_n, y'_n, u) - \tilde{V}(y'_n, y_n, u) \geq \varepsilon$ for any $y'_n \in \mathcal{Y}$, any $y_n \in \mathcal{Y}$ with $y_n < y'_n$, any $n > 2$ and $m = \lceil \beta n \rceil > 2$, and any fixed set of productivity draws of the other players. It follows from the fact that the first derivative of $\tilde{V}(y'_n, y, u)$ is strictly bounded away from zero when y is bounded away from y'_n, and since y_n is a discrete distance away from y'_n since the support $\mathcal{Y}$ has a finite number of elements, these incremental improvements aggregate. Note the $u \geq c + \delta$ in the subgame since even the firms' worst draws $\underline{y}$ are still strictly above cost c.

[27] The third derivative of $\tilde{V}$ is bounded if u is bounded away from c. The assumption that $\underline{y} > c$ ensures the property for any draws of the other firms.

Second, we can relax the assumption that the support of productivities is finite, but only at the cost of much additional analytic complexity. In particular, assume that the productivities are drawn from an interval. Then even in large but finite markets full revelation cannot be sustained, because there would always be a lower type sufficiently close that the firm would prefer to deviate to that one's announcement. Nevertheless, we can show that we can partition the productivities in intervals such that all firms with productivities in the same interval announce the same message, and the width of these intervals shrinks to zero as the market becomes large. This induces perfect revelation and efficiency in the limit. We omit the derivation due to space constraints.

5.4 Other Equilibria and Equilibrium Selection

Unlike the wage-posting game, the communication game has multiple equilibria. To begin with, as in any other communication game, there always exists a babbling equilibrium: If all firms babble (uniformly randomize over all messages), then workers would ignore all messages and randomly select a firm. This, in turn, makes firms indifferent over all messages. The following proposition characterizes the set of all partitional equilibria under a mild condition on the distribution function G.[28]

Proposition 6 *Suppose that* $(1 - G(y))(y - c)$ *is strictly quasi-concave.*[29] *Then, for any* $\tilde{y} \in [\underline{y}, \overline{y}]$*, there is an equilibrium in which all firms below* $\tilde{y}$ *fully reveal their productivity, while all firms above* $\tilde{y}$ *constitute exactly one submarket. There does not exist any other partitional equilibrium that yields a different outcome from any of these.*

Proof. See Appendix B. ■

This is a consequence of the fact that each firm strictly prefers revealing its productivity to deviating to a lower productivity, while it is indifferent over all submarkets above its productivity. A firm below $\tilde{y}$ has no incentive to deviate for the same reason as in the fully revealing equilibrium. A firm above $\tilde{y}$ can deviate only to below $\tilde{y}$, but the deviation is not profitable, again, as in the fully revealing equilibrium. The interval above $\tilde{y}$ cannot be segmented, because if so the highest firms in the lower segment would deviate.

[28]We say an equilibrium is partitional if firms' communication strategies can be characterized by a partition of $\mathcal{Y}$ such that firms in the same partition element announce an identical message (in other words, the set of firms' types in each submarket is convex). There may exist non-partitional equilibria in which the set of firms in a submarket is not convex, even though we are not aware of any. We are also not aware of a way to systematically analyze such equilibria.

[29]A sufficient condition is the usual regularity assumption that the inverse hazard ratio of G, $\frac{dG(y)}{1-G(y)}$, is strictly decreasing.

Nevertheless, the fully revealing equilibrium is more appealing than the others at least for two reasons. First, it yields the socially efficient outcome and, therefore, would be a natural focal point. Second, it is the only equilibrium that satisfies NITS (no incentive to separate) defined by Chen, Kartik, and Sobel (2008). Although the condition is defined in the context of the standard two-player communication game, it is directly applicable to our market setup. NITS can be justified in several ways. For example, NITS arises if players can be non-strategic with small probabilities or face a convex cost of lying.

Proposition 7 *Among all the partitional equilibria, only the fully revealing equilibrium satisfies NITS.*

Proof. See Appendix B. ∎

6 Heterogeneous Workers

Our insights also generalize into the environment with heterogeneous workers. We consider two most common specifications where workers differ in terms of either their opportunity cost of working or observable skills that add the same value across firms.[30] These settings are important because the complete commitment contract must specify the wage for each worker type, as in Shi (2002) and Shimer (2005). Rather than implementing a commitment to a full wage policy, firms might opt for cheap-talk messages if these deliver the same results. We focus on our central insights, relegating all the formalities to Appendix A.

6.1 Heterogeneous Opportunity Costs of Working

Setup. Suppose each worker independently and identically draws his opportunity cost of working from the set $\mathcal{C} \equiv [\underline{c}, \overline{c}]$ according to a distribution function F with continuous and everywhere-positive density f.[31] All the other configurations of the physical environment are identical to those in Section 2. It will be also notationally convenient to restrict attention to productivity draws from distribution G with convex support and every-where positive density g, although none of our results hinge on that.

[30] See, e.g., Peters (1997b) and Mortensen and Wright (2002) for the first specification and Shi (2002) and Shimer (2005) for the second one.

[31] Alternatively, each worker may have a predetermined opportunity cost of working. By the law of large numbers, the setup yields the same outcome.

Optimality of the fully revealing equilibrium. Consider the same communication game as in Section 4,[32] and suppose firms' types were fully revealed. Since revenue equivalence continues to hold, for simplicity, suppose each firm runs a second-price procurement auction with reserve wage equal to its productivity. Then a type c worker is hired if and only if there is no other applicant whose type is below c. Denote by $\lambda(y,c)$ the queue length of type c workers for a productivity y firm, and let $\Lambda(y,c) \equiv \int_{\underline{c}}^{c} \lambda(y,c')dc'$ (the queue length of the workers whose types are below c for a productivity y firm). Then, a type c worker is hired by a productivity y firm with probability $e^{-\Lambda(y,c)}$. The wage paid to a type c worker is determined by the next most efficient (second lowest c) type. The next most efficient type is above c' if and only if no worker below c' applies to the same firm, whose probability is $e^{-\Lambda(y,c')}$. It then follows that a type c worker's expected payoff by applying to a productivity y firm is

$$e^{-\Lambda(y,c)}(y-c) - \int_{c}^{\overline{c}} (y-c')d\left(1-e^{-\Lambda(y,c')}\right). \tag{18}$$

Observe that this is identical to a type c worker's marginal social contribution. The first term is the expected social surplus he creates by applying to a productivity y firm. The second term is the negative externality by the worker to the workers above his type: If a type c worker did not apply, then the firm could have hired a worker whose type is above c. Since each worker always receives his marginal social contribution, workers make socially efficient application decisions and, therefore, the fully revealing equilibrium is efficient.

Existence of the fully revealing equilibrium. Denote by $u(c)$ a type c worker's market utility. In the fully revealing equilibrium, $u(c)$ coincides with (18) for any y that a type c might apply to. Notice that

$$u'(c) = -e^{-\Lambda(y,c)}.$$

Since $e^{-\Lambda(y,c)}$ is the employment probability of a type c worker who applies to a productivity y firm and $u'(c)$ is obviously independent of y, this implies that a type c worker's employment probability is essentially independent of a firm's productivity. This result, in turn, implies that with first-price auctions, a type c worker demands the same wage to all firms he might apply to. Let $w(c)$ denote a type c worker's wage demand.

To prove the existence, suppose a productivity y' firm deviates and announces y. The firm hires a worker at a wage below $w \leq \min\{y, y'\}$ if and only if at least one worker who demands

[32] The social planner's problem does not change, except that now the planner instructs workers to follow an application strategy as a function of their type and firms to follow a hiring strategy as a function of their type and the types of applicants. The wage-posting game changes rather significantly. Now each firm posts a type-contingent wage contract, that is, a wage for each worker type. See Appendix A for the formal descriptions.

less than w applies, whose probability is $1-e^{-\Lambda(y,c)}$ where c is such that $w(c)=w$. As shown above, this probability is independent of y. Therefore, as in the homogeneous workers case, a productivity y firm's deviation to y does not affect the firm's probability of hiring a worker at a wage below $w \leq \min\{y, y'\}$. Given this, it is straightforward that a productivity y' firm strictly prefers truthfully revealing its productivity to deviating to $y < y'$, because the deviation takes away the firm's chance of hiring a worker at a wage between y and y', and is indifferent over all productivity levels above its productivity, because the firm does not accept a wage demand above y' and, therefore, the chance of facing the lowest wage demand between y' and y is irrelevant to the firm's expected profit.

6.2 Heterogeneous Skills

Setup. Now suppose workers have the same opportunity cost of working c, but differ in terms of their observable skills. For simplicity, normalize c to 0. Each worker's skill, denoted by x, is independently and identically drawn from the set $\mathcal{X} = [\underline{x}, \overline{x}]$ according to a distribution function F with continuous and everywhere-positive density f. If a type x worker is hired by a productivity y firm at a wage w, then the worker receives utility w and the firm obtains profit $x + y - w$.[33] Since the analysis of this specification is essentially identical to that of the previous one, we illustrate only the necessary changes for this specification.

Optimality of the fully revealing equilibrium. As before, suppose firms' types were fully revealed and each firm runs a second-price auction. In the current specification, a firm's reserve wage depends on a worker' type. Specifically, a productivity y firm's reserve wage for a type x worker is equal to $x + y$. Denote by $\lambda(y, x)$ the queue length of type x workers for a productivity y firm, and let $\Lambda(y, x) \equiv \int_x^{\overline{x}} \lambda(y, x)dx$ (the queue length of the workers who are more efficient than x for a productivity y firm). Since a type x worker is hired only when no other worker above x applies to the same firm and his wage is determined by the

[33]Our analysis is restricted to the case where there is no complementarity between firms' and workers' types. The most general form of this specification would allow complementarity as well as two-sided heterogeneity, as in Shimer (2005). We do know that the fully revealing equilibrium continues to be efficient. Unfortunately, we do not know whether the fully revealing equilibrium exists or not. We have no counter-example, but our proof technique for the existence no longer applies, because with complementarity a worker's employment probability *does* depend on a firm's productivity. A worker demands different wages to different productivity firms, so a firm's incentive problem becomes significantly more complicated. Let us note that, even if our communication game does not work in a more general environment, it does not necessarily invalidate our central insights as there may exist other kinds of wage-determination mechanism that induce truth-telling and efficiency. We also note that our analysis does go beyond most work in the directed search literature, which focusses at most on one-sided heterogeneity.

next most efficient worker, his expected payoff by applying to a productivity y firm is

$$e^{-\Lambda(y,x)}(y+x) - \int_{\underline{x}}^{x} (y+x')d\left(1 - e^{-\Lambda(y,x')}\right).$$

This payoff is, again, identical to the worker's marginal social contribution, with the first term representing the expected social surplus he creates by applying to a productivity y firm and the second term the negative externality to the workers below his type. All workers apply efficiently and, therefore, the fully revealing equilibrium implements the social optimum.

Existence of the fully revealing equilibrium. Letting $u(x)$ be a type x worker's market utility, as before, we get

$$u'(x) = e^{-\Lambda(y,x)}.$$

From here, again, we find that a worker's employment probability is essentially independent of a firm's productivity and, therefore, when firms run first-price auctions, each worker demands the same wage to all the firms he might apply to. Then, by the same reasoning as before, the fully revealing equilibrium exists because a firm strictly prefers revealing its productivity to deviating to a lower productivity and is indifferent over all productivity levels above its own.

7 Conclusion

Most directed search models follow the usual structure of market games: Firms make binding commitments to attract workers, and then workers decide where to look for jobs. Market frictions arise because workers use anonymous strategies which prevents them from coordinating where to apply. The commitments sustain constrained efficient outcomes, taking the miscoordination as the constraint.

We show that commitment is not essential to sustain the constrained efficient outcome of the commitment economy. It is crucial that firms can communicate with workers. It is also important that firms who pretend to have a higher willingness to pay end up paying a higher wage in expectation. Yet commitment is not necessary to sustain this, since simple wage determination games such as wage demands by workers generate such outcomes.

The result is fairly general in this class of models, spanning the case where only firms are heterogeneous to the case where workers also differ in their disutility of leisure and their observable common productivity. The latter might also indicate why firms do not simply post a wage commitment: With worker heterogeneity firms would commit to auction-like contracts in the case of leisure differences (McAfee (1993), Peters (1997b)) and to wages that condition on the entire skill set (Shi (2002), Shimer (2005)), which tend to be complex objects. Under plausible conditions on the market structure, cheap-talk communication will induce the right compensation, and more elaborate schemes are not necessary.

In our analysis we only consider one round of market interaction. We strongly conjecture that our results carry over to repeated settings, and even to settings with business-cycle shocks and on-the-job-search as in Menzio and Shi (2011) as long as at the time when the worker and the firm meet they can write a fully contingent contract on their future actions. In such a setting workers essentially ask for parts of the ensuing surplus, and the rest of our results should apply. We also conjecture that similar insights would arise in a setting such as Kircher (2009) where workers are allowed to apply to several firms and firms allocate their slots according to a stable matching. For other environments, we have less clear intuition. In particular, when worker productivity interacts non-additively with firm productivity, the analysis becomes substantially more intricate and we do not have any results for this case. We conjecture that other forms of decentralized competition such as all-pay bids by workers might still implement the efficient outcome, but might lack the intuitive appeal that of a simple wage-demand game.

Finally, the key insights for the results are the connection between the ex-post core and the ex-ante incentives on the equilibrium path, and the requirement that after a deviation workers' payoffs must remain close to those that they would have received had they faced the original type rather than the deviant. The latter part is the key requirement, and we show that not all wage-demand games have this structure even if they deliver the Hosios condition on the equilibrium path. But some plausible games do implement it, and constrained efficiency through cheap-talk communication can be sustained. While we have focussed on the class of directed search market games, we hope that they contribute to our understanding more broadly of market games and that extensions of these insights can be applied to larger classes.

Appendix A: Formal Analysis of Worker Heterogeneity

In this appendix, we provide a formal analysis of the model with heterogeneous workers introduced in Section 6. Since the two specifications are essentially equivalent, we focus on the first specification where workers differ in their opportunity cost of working.

As for the homogeneous workers case, we consider three forms of labor market interaction and show that all the three forms induce the same allocation. For notational simplicity, through this appendix, we assume that the distribution function G has a continuous and everywhere-positive density g.

Social Planner's Problem

Planner's choice set. Consider the social planner who instructs workers to follow an application strategy as a function of their type and firms to follow a hiring strategy as a function of their type and the types of applicants. The latter is straightforward: Each firm hires the most efficient worker (lowest c), as long as the worker's opportunity cost of working does not exceed the firm's productivity. For workers, the social planner designs a function $P : \mathcal{Y} \times \mathcal{C} \to [0, 1]$ where $P(y, c)$ is the probability that a type c worker applies to a firm whose productivity is below y. For expositional simplicity, we assume that $P(\cdot, c)$ has a well-defined density $p(\cdot, c)$. It will turn out that the optimal solution indeed has well-defined densities for almost all c.

Planner's objective. Given workers' common application strategy $P(\cdot, \cdot)$, the queue length of type c workers for a productivity y firm is equal to

$$\lambda(y, c) \equiv \frac{\beta f(c) p(y, c)}{g(y)}, \tag{19}$$

and the queue length of workers whose types are below c for a productivity y firm is

$$\Lambda(y, c) \equiv \int_{\underline{c}}^{c} \lambda(y, c') dc'.$$

Since the probability that a productivity y firm hires a worker whose type is below c is equal to $1 - e^{-\Lambda(y,c)}$, the social planner faces the following maximization problem:

$$\max_{P(\cdot,\cdot)} \int_{\underline{y}}^{\overline{y}} \left(\int_{\underline{c}}^{\overline{c}} (y - c) d \left(1 - e^{-\Lambda(y,c)}\right) \right) dG(y).$$

Using (19) and the fact that $P(\overline{y}, c) \le 1$ for any $c \in \mathcal{C}$, the problem can be rewritten as follows:

$$\max_{\lambda(\cdot,\cdot)} \int_{\underline{y}}^{\overline{y}} \left(\int_{\underline{c}}^{\overline{c}} (y - c) d \left(1 - e^{-\Lambda(y,c)}\right) \right) dG(y)$$

subject to

$$\int_{\underline{y}}^{\overline{y}} \lambda(y, c) dG(y) \le \beta f(c), \forall c \in \mathcal{C}.$$

As in the homogeneous workers case, the constraint is due to the fact that the total number of application made by type c workers cannot be larger than the number of type c workers.

Characterization. Observe that the constraints are linear, while the objective function is concave in $\lambda(\cdot, \cdot)$.[34] Therefore, the solution to the planner's problem is unique and can be characterized only by first-order conditions. Letting $\mu(c)$ be the multiplier on the constraint

[34] See the proof of Proposition 1 in Shimer (2005) for the finite type case.

for c, the Lagrangian is given by

$$\begin{aligned}\mathcal{L}(\lambda,\mu) &= \int_{\underline{y}}^{\overline{y}}\left(\int_{\underline{c}}^{\overline{c}} e^{-\Lambda(y,c)}\lambda(y,c)(y-c)dc\right)dG(y)+\int_{\underline{c}}^{\overline{c}}\mu(c)\left(\beta f(c)-\int_{\underline{y}}^{\overline{y}}\lambda(y,c)dG(y)\right)dc\\ &= \int_{\underline{y}}^{\overline{y}}\left(\int_{\underline{c}}^{\overline{c}}\lambda(y,c)\left(e^{-\Lambda(y,c)}(y-c)-\mu(c)\right)dc\right)dG(y)+\int_{\underline{c}}^{\overline{c}}\mu(c)\beta f(c)dc.\end{aligned}$$

By the standard results for constrained maximization, a pair $(\lambda(\cdot,\cdot),\mu(\cdot))$ is the solution if and only if

$$\mu(c)\geq e^{-\Lambda(y,c)}(y-c)-\int_{c}^{\overline{c}}(y-c')d\left(1-e^{-\Lambda(y,c')}\right), \tag{20}$$

with equality holding if $\lambda(y,c)>0$, and

$$\int_{\underline{y}}^{\overline{y}}\lambda(y,c)dG(y)=\beta f(c),\forall c\in\mathcal{C}. \tag{21}$$

Intuitively, the left-hand side in (20) is the planner's opportunity cost of sending a type c worker to a firm (the shadow value of a type c worker to the planner), while the right-hand side is the corresponding benefit. A type c worker is hired when no worker below his type applies to the same firm, whose probability is $e^{-\Lambda(y,c)}$. If the worker is hired by a productivity y firm, then they create social surplus $y-c$. The second term represents the negative externality by a type c worker to the workers above his type: If a type c worker did not apply to a firm, then the firm could have hired a worker whose type is below c. It is trivial that the feasibility condition (21) must bind for any worker type.

Proposition 8 *A pair $(\lambda(\cdot,\cdot),\mu(\cdot))$ is the unique solution to the social planner's problem if and only if it satisfies the complementary slackness condition (20) and the binding feasibility constraints (21).*

Directed Search Implementation

We now show that the social optimum can be decentralized as a directed search equilibrium.

Strategies. We consider the same wage-posting game as Shi (2002) and Shimer (2005). Each firm posts a type-contingent wage contract $w:\mathcal{Y}\times\mathcal{C}\rightarrow\mathbb{R}_+$ where $w(y,c)$ is the wage promised to a type c worker by a productivity y firm. Workers observe all announced contracts and apply to one firm: Let $\mathfrak{W}$ be the set of all feasible functions $w(\cdot)$. Workers' common application strategy is a function $P:\mathfrak{W}\times\mathcal{C}\rightarrow[0,1]$ where $P(W,c)$ is the probability that a type c worker applies to a firm who posts a wage schedule $w(\cdot)$ in a set $W\subset\mathfrak{W}$. Firms that receive at least one application decide which applicant to hire. We restrict attention to the case where each firm posts a wage contract that is increasing and continuous in worker type. This implies that a firm hires the lowest worker type whenever there are multiple applicants. This must be the case if a directed search equilibrium implements the social optimum, and will turn out to be true in equilibrium.[35]

Submarket outcome. Consider a submarket where all firms post an increasing wage schedule $w : \mathcal{C} \to \mathbb{R}_+$, where $w(c)$ is a wage offered to a type c worker. Denote by $\lambda(c)$ the queue length of type c workers for a firm, and let $\Lambda(c) \equiv \int_{\underline{c}}^{c} \lambda(c')dc'$ be the queue length of workers whose types are below c for a firm. Since, by assumption, $w(\cdot)$ is increasing, a type c worker is hired when no worker whose type is below c applies to the same firm, whose probability is $e^{-\Lambda(c)}$. Therefore, his expected utility is

$$U(c, w(\cdot), \lambda(\cdot)) = e^{-\Lambda(c)}(w(c) - c). \tag{22}$$

A productivity y firm hires a worker at a wage below $w(c)$ when at least one worker whose type is below c applies, whose probability is $1 - e^{-\Lambda(c)}$. Therefore, a productivity y firm's expected profit is

$$V(y, w(\cdot), \lambda(\cdot)) = \int_{\underline{c}}^{\overline{c}} (y - w(c))d\left(1 - e^{-\Lambda(c)}\right).$$

Definition of directed search equilibrium. Consider a submarket that features a wage schedule $w(\cdot)$. In the submarket, the queue length of type c workers for a firm is equal to

$$\lambda(w(\cdot), c) = \frac{\beta f(c) P(w(\cdot), c)}{\int_{\{y \in \mathcal{Y} : w(y,\cdot) = w(\cdot)\}} dG(y)}. \tag{23}$$

The queue length for workers whose types are below c for a firm is

$$\Lambda(w(\cdot), c) = \int_{\underline{c}}^{c} \lambda(w(\cdot), c')dc'.$$

The queue lengths associated with a wage contract $w(\cdot)$ are well-defined whenever some firms offer the wage contract. To formally define a directed search equilibrium, as in the homogeneous workers case, the queue lengths for out-of-equilibrium wage contracts $w(\cdot)$ must be also specified. We again invoke the market utility condition. Since now workers are heterogeneous, we generalize the condition and impose it for each worker type. Formally, let $u(c)$ be a type c worker's market utility - the highest utility a type c worker can achieve in the market. For any $w(\cdot)$, we require that

$$U(c, w(\cdot), \lambda(w(\cdot), \cdot)) = e^{-\Lambda(w(\cdot), c)}(w(c) - c) \leq u(c), \tag{24}$$

with equality holding if $\lambda(w(\cdot), c) > 0$.[36]

[35] See Shimer (2001, 2005) for more detailed discussion.

[36] The restriction that a firm can post only an increasing wage contract is of particular help here. Without the restriction, one cannot resort to (22) for workers' expected utilities. Characterization of workers' expected utilities for any arbitrary $w(\cdot)$ is, to our knowledge, not known in the literature and certainly exceeds the scope of this paper.

Definition 3 *A directed search equilibrium is a tuple $\{P(\cdot,\cdot), w(\cdot,\cdot)\}$ such that there exists $\lambda(\cdot,\cdot)$ that satisfies the following conditions:*

1. *Optimal application: For any $c \in \mathcal{C}$, for any $w(\cdot) \in \mathfrak{W}$ in the support of $P(\cdot, c)$,*
$$u(c) = U(c, w(\cdot), \lambda(w(\cdot),\cdot)) = \max_{w'(\cdot)\in\mathfrak{W}} U(c, w'(\cdot), \lambda(w'(\cdot),\cdot)).$$
2. *Profit maximization: For any $y \in \mathcal{Y}$,*
$$w(y,\cdot) = argmax_{w(\cdot)\in\mathfrak{W}} V(y, w(\cdot), \lambda(w(\cdot),\cdot)) = \int_{\underline{c}}^{\overline{c}} (y - w(c))d\left(1 - e^{-\Lambda(w(\cdot),c)}\right).$$
3. *Consistency: $\lambda(w(\cdot), c)$ satisfies (23) whenever $w(\cdot)$ is offered by some firms (there exists $y \in \mathcal{Y}$ such that $w(y,\cdot) = w(\cdot)$) and fulfills the market utility condition otherwise.*

Characterization. Fix $w(\cdot) \in \mathfrak{W}$. By the market utility condition, if a type c worker may apply to a firm that posts $w(\cdot)$ ($\lambda(w(\cdot), c) > 0$), then the worker must be indifferent between obtaining his market utility and applying to the firm. Therefore,

$$u(c) = e^{-\Lambda(w(\cdot),c)}(w(c) - c). \tag{25}$$

This equation uniquely pins down the relationship between $u(\cdot)$ and $\lambda(w(\cdot),\cdot)$ and corresponds to the wage-queue-length equation (7) in the homogeneous workers case. The difference is that now the relationship between wage contract and queue length is defined for each worker type.

Each firm maximizes its profits, taking (25) into account. A productivity y firm's problem can be written as follows:

$$\max_{w(\cdot)} \int_{\underline{c}}^{\overline{c}} (y - w(c))d\left(1 - e^{-\Lambda(w(\cdot),c)}\right),$$

subject to

$$e^{-\Lambda(w(\cdot),c)}(w(c) - c) \geq u(c), \forall c.$$

Substituting $w(c)$ in the objective function with the binding constraint, the problem shrinks to

$$\max_{\lambda(w(\cdot),\cdot)} \int_{\underline{c}}^{\overline{c}} \lambda(w(\cdot), c)\left(e^{-\Lambda(w(\cdot),c)}(y - c) - u(c)\right) dc.$$

The optimal solution, denoted by $\lambda(w(y,\cdot),\cdot)$, satisfies

$$u(c) \geq e^{-\Lambda(w(y,\cdot),c)}(y - c) - \int_{c}^{\overline{c}} (y - c')d\left(1 - e^{-\Lambda(w(y,\cdot),c')}\right), \tag{26}$$

with equality holding if $\lambda(w(y,\cdot), c) > 0$.

Lastly, as usual, all workers apply to some firm, and thus the market feasibility condition must hold for each worker type. Formally,

$$\int_{\underline{y}}^{\overline{y}} \lambda(w(y,\cdot),c)dG(y) = \beta f(c), \forall c \in \mathcal{C}. \tag{27}$$

Optimality. The two equilibrium conditions (26) and (27) are exactly identical to those for the social optimum, (20) and (21), respectively. It then follows that the induced queue lengths of each worker type for each firm type coincide with those for the social optimum ($\lambda(w(y,\cdot),c) = \lambda(y,c)$ for all y and c), and thus a directed search equilibrium implements the social optimum.

Proposition 9 *There is a unique directed search equilibrium, which is characterized by a tuple $(\lambda(\cdot,\cdot), w(\cdot,\cdot), u(\cdot))$ that satisfy the market utility condition (25), the firm optimality condition (26), and the feasibility constraints (27). The directed search equilibrium implements the social optimum.*

Cheap Talk Implementation

We consider the same communication gams as in the homogenous workers case. Each firm announces a cheap-talk message from the set $\mathcal{M}$, which is, without loss of generality, assumed to coincide with the set of firm types, $\mathcal{Y}$. Workers observe all announced messages and apply to a firm with a wage demand. Firms that receive at least one application decide whether and, if so, which worker to hire. Again, a firm's optimal hiring strategy is straightforward: Each firm hires a worker who demands the lowest wage, if and only if the wage is not greater than its own productivity.

Strategies. To minimize notation, we directly introduce a fully revealing strategy profile and state its equilibrium conditions. In the strategy profile, each productivity y firm truthfully reveals its productivity: Denoting by $m(y)$ the message announced by a productivity y firm, $m(y) = y$ for all $y \in \mathcal{Y}$. Each type c worker applies to a firm who announces a message below y with probability $P(y,c)$ and demands $w(y,c)$ to a firm who announces y. As in the social planner's problem, for notational simplicity, we assume that $P(\cdot,c)$ has a well-defined density $p(\cdot,c)$.

Submarket outcome. Consider a submarket in which all firms have productivity y and the queue length of type c workers for a firm is given by $\lambda(c)$. Denote by $\Lambda(c) \equiv \int_{\underline{c}}^{c} \lambda(c')dc'$ the queue length of workers whose types are below c for a firm. We consider only the case where the set of worker types in a submarket is convex (if $\lambda(c), \lambda(c'') > 0$ and $c < c' < c''$, then $\lambda(c') > 0$), and the highest worker type is below y. These properties will turn out to hold in equilibrium. Lastly, denote by $U(c,y,\lambda(\cdot))$ a type c worker's expected utility and by $V(y',y,\lambda(\cdot))$ a productivity y' firm's expected profit in this submarket.

For now, suppose each firm runs a second-price auction, instead of a first-price auction, with reserve wage equal to y. In this case, a type c worker gets a job if and only if no applicant whose type is below c applies to the same firm. If there is no other applicant, whose probability is $e^{-\Lambda(y)}$, then the worker obtains $y - c$. If there are other applicants, let c' be the lowest worker type among other applicants. A type c worker obtains $c' - c$ if $c' > c$ and 0 otherwise. Since the probability that c' is less than x is equal to $1 - e^{-\Lambda(x)}$, a type c worker's expected utility is

$$U(c, y, \lambda(\cdot)) = e^{-\Lambda(y)}(y - c) + \int_c^{\overline{c}} (c' - c) d\left(1 - e^{-\Lambda(c')}\right).$$

Arranging terms,

$$U(c, y, \lambda(\cdot)) = e^{-\Lambda(c)}(y - c) - \int_c^{\overline{c}} (y - c') d\left(1 - e^{-\Lambda(c')}\right). \tag{28}$$

Intuitively, in a second-price auction, each bidder obtains his marginal social contribution. In the current context, it is equal to the expected social surplus created by his application to a productivity y firm (first term) minus the negative externality to less efficient workers (second term).

Returning to first-price auctions, let $\Lambda'(w)$ be the queue length of workers who demand less than w for a firm. By similar arguments to those in Section 4, $\Lambda'(\cdot)$ has no atom and the maximum wage demand is equal to y. A type c worker's optimal wage demand maximizes $e^{-\Lambda'(w)}(w - c)$. By a standard argument in auction theory, a lower type worker demands a strictly lower wage and, therefore, is hired prior to higher types. Since the employment probabilities of each worker type are identical to those for second-price auctions, the standard revenue equivalence applies, which implies that each worker's expected utility will be as in (28).

A type c worker's optimal wage demand, denoted by $w(y, c)$, can also be deduced from second-price auctions. For revenue equivalence, the wage demand by a type c worker for first-price auctions must be identical to his expected wage with second-price auctions. Since, with second-price auctions, the probability that the second lowest wage demand is less than c', conditional on c being the lowest bid, is equal to $(1 - e^{-\Lambda(c')})/e^{-\Lambda(c)}$,

$$w(y, c) = \int_c^{\overline{c}} c' \frac{d(1 - e^{-\Lambda(c')})}{e^{-\Lambda(c)}}.$$

For firms' expected profits, notice that since $w(y, \cdot)$ is strictly increasing, the queue length of workers who demand less than $w(y, c)$ is equal to the queue length of workers whose types are below c, that is,

$$\Lambda'(w(y, c)) = \Lambda(c).$$

Since the probability that the lowest wage demand to a firm is less than $w(y, c)$ is equal to $1 - e^{-\Lambda'(w(y,c))} = 1 - e^{-\Lambda(c)}$, a productivity y' firm's expected profit is

$$V(y', y, \lambda(\cdot)) = \int_{\underline{c}}^{\overline{c}} \max\{y' - w(y, c), 0\} d\left(1 - e^{-\Lambda(c)}\right).$$

The following lemma summarizes all the results.

Lemma 4 *Consider a submarket where all firms (are believed to) have productivity y and the queue length of type c workers for a firm is given by $\lambda(c)$. A type c worker's expected utility in the submarket is*

$$U(c, y, \lambda(\cdot)) = e^{-\Lambda(c)}(y - c) - \int_c^{\overline{c}} (y - c') d\left(1 - e^{-\Lambda(c')}\right). \tag{29}$$

A productivity y' firm's expected profit is

$$V(y', y, \lambda(\cdot)) = \int_{\underline{c}}^{\overline{c}} \max\{y' - w(y, c), 0\}\, d\left(1 - e^{-\Lambda(c)}\right), \tag{30}$$

where

$$w(y, c) = \int_c^{\overline{c}} c' \frac{d(1 - e^{-\Lambda(c')})}{e^{-\Lambda(c)}}. \tag{31}$$

Definition of fully revealing equilibrium. Given firms' communication strategies $m(y) = y$ and workers' application strategies $P(\cdot, \cdot)$, the queue length of type c workers for a productivity y firm is

$$\lambda(y, c) = \frac{\beta f(c) p(y, c)}{g(y)}. \tag{32}$$

Denote by $\tilde{V}(y', y)$ a productivity y' firm's expected profit when it announces message y and by $\tilde{U}(c, y)$ a type c worker's expected utility when he applies to a productivity y firm (with an optimal wage demand). In addition, let $u(c) \equiv \max_y \tilde{U}(c, y)$ be a type c worker's maximal utility in the market. $u(c)$ is analogous to a type c worker's market utility in directed search equilibrium.

Definition 4 *A fully revealing equilibrium is a tuple $(m(\cdot), P(\cdot, \cdot), w(\cdot, \cdot))$ such that there exists $\lambda : \mathcal{Y} \times \mathcal{C} \to \mathcal{R}_+$ that satisfies the following conditions:*

1. *Optimal and truthful communication: It is optimal for a productivity y' firm to truthfully reveal its productivity, that is,*

$$m(y') = y' \in argmax_y \tilde{V}(y', y) = V(y', y, \lambda(y, \cdot)), \forall y' \in \mathcal{Y}.$$

2. *Optimal application: For any y' in the support of $P(\cdot, c)$,*

$$u(c) = \tilde{U}(c, y') = \max_y \tilde{U}(c, y) = U(c, y, \lambda(y, \cdot)).$$

3. *Consistency: $\lambda(y, c)$ satisfies (32) for any $(y, c) \in \mathcal{Y} \times \mathcal{C}$.*

Full revealing equilibrium outcome and optimality. Given firms' truthful communication ($m(y) = y$ for all $y \in \mathcal{Y}$) and other workers' application strategies $P(\cdot, \cdot)$, let $u(c)$ be the highest expected utility (market utility) a type c worker can obtain in the market. Then, a type c worker applies to a productivity y firm ($\lambda(y,c) > 0$) only when $U(c, y, \lambda(y, \cdot)) \geq u(c)$. Since, by definition, $u(c) \geq U(c, y, \lambda(y, \cdot))$, this implies

$$u(c) \geq U(c, y, \lambda(y, \cdot)) = e^{-\Lambda(y,c)}(y - c) - \int_c^{\overline{c}} (y - c')d\left(1 - e^{-\Lambda(y,c')}\right), \tag{33}$$

with equality holding if $\lambda(y,c) > 0$. In addition, since all workers would obviously apply, the market feasibility condition binds for any worker type:

$$\int_{\underline{y}}^{\overline{y}} \lambda(y,c)dG(y) = \beta f(c), \forall c \in \mathcal{C}. \tag{34}$$

As in directed search equilibrium, these two conditions coincide with those for the social optimum (20) and (21), respectively. It then follows that the fully revealing equilibrium, if exists, implements the social optimum.

Proposition 10 *The fully revealing equilibrium of the market communication game is unique and implements the social optimum.*

The intuition behind this result is the same as for the homogeneous workers case. If the allocations are determined at the worker-optimal points in the ex post core, a worker always receives his marginal social contribution. Therefore, they make socially efficient application decisions. Second-price auctions implement worker-optimal ex post core allocations. Via revenue equivalence, first-price auctions yield the same interim payoffs to workers and, therefore, also induce efficient search decisions by workers.

Existence of fully revealing equilibrium. We now prove that there always exists a fully revealing equilibrium by solving for firms' optimal communication strategies. It suffices to compare a firm's equilibrium profit $\tilde{V}(y', y') = V(y', y', \lambda(y', \cdot))$ to its deviation profit $\tilde{V}(y', y) = V(y', y, \lambda(y, \cdot))$.

As a first step, suppose $\lambda(y,c) > 0$. Then, it must be that

$$u(c) = e^{-\Lambda(y,c)}(w(y,c) - c) = \int_c^{\overline{c}} c'd\left(1 - e^{-\Lambda(y,c')}\right) - e^{-\Lambda(y,c)}c.$$

The second equality is due to Equation (31) in Lemma 4. Differentiating both sides with respect to c,[37]

$$u'(c) = -e^{-\Lambda(y,c)}.$$

Recall that $e^{-\Lambda(y,c)}$ is a type c worker's employment probability when he applies to a productivity y firm. The result states that, since $u'(c)$ is obviously independent of y, a worker's employment probability must be independent of y. Since a worker's employment probability is determined by the expected number of more efficient workers who apply to the same firm, this means that the same number of more efficient workers must apply to *all firms* a type c worker might apply to, that is, $\Lambda(y,c)$ must be constant for any y such that $\lambda(y,c) > 0$.

[37] $u(\cdot)$ is continuous and strictly decreasing. Therefore, it is differentiable almost everywhere.

The result has the following three implications. First, since $e^{-\Lambda(y,c)}$ is independent of y and $u(c) = e^{-\Lambda(y,c)}(w(y,c) - c)$ for any y such that $\lambda(y,c) > 0$, $w(y,c)$ is also independent of y. In other worker, each worker demands the same wage to any firm he might apply to. From now on, abusing notation slightly, denote by $w(c)$ a type c worker's wage demand. Second, since the result must hold for any worker type, whenever a type c worker applies to a productivity y firm, all type $c'(< c)$ workers also apply to the firm. This implies that in equilibrium a type c worker applies to all firms above $w(c)$, so the support of a type c worker's application strategy is $[w(c), \overline{y}]$. Third, again, since the result must hold for any worker type, the queue length of each worker type for each firm type $\lambda(y,c)$ must be constant for any y such that $\lambda(y,c) > 0$. We exploit these properties to establish the existence of the fully revealing equilibrium below.

Suppose a productivity y' firm deviates to y. The lowest wage demand to the firm is less than $w(c)$ whenever at least one worker whose type is below c applies, whose probability is $1 - e^{-\Lambda(y,c)}$. Suppose $w(c) \leq \min\{y', y\}$. Since $\Lambda(y,c)$ is independent of y, the probability that a productivity y firm hires a worker at a wage below $w(c)$ is independent of whether a firm deviates or not. Since a productivity y' firm does not accept a wage above y', this immediately implies that $\tilde{V}(y', y') = \tilde{V}(y', y)$ for any $y \geq y'$, that is, a productivity y' firm is indifferent over all productivity levels weakly above its own. Now suppose $y < y'$. In this case, by deviating to y, a productivity y firm loses the opportunity to hire a worker at a wage between y and y'. Therefore, the deviation strictly lowers the firm's expected profit, that is, $\tilde{V}(y', y') > \tilde{V}(y', y)$. We conclude that no firm has an incentive to deviate.

Proposition 11 *There always exists a fully revealing equilibrium.*

Appendix B: Omitted Proofs

Proof of Proposition 6: We first show that for each $\tilde{y} \in \mathcal{Y}$, it is an equilibrium that all firms below $\tilde{y}$ reveal their productivity, while all other firms constitute one submarket. We prove later that there does not exist any other partitional equilibrium.

Submarket outcome. The outcome of the submarket that consists of productivity $y(< \tilde{y})$ firms is exactly identical to the one in Subsection 4.2. For the submarket where all firms above $\tilde{y}$ participate (the *large submarket*, henceforth), the two observations in 4.2 still apply, so each worker's expected utility is once again the probability that a worker is the only applicant to a firm times the expected utility a worker obtains when he knows that he is the only applicant. For the latter part, define

$$y^*(\tilde{y}) \equiv argmax_{y \geq \tilde{y}} \int_y^{\overline{y}} (y - c) \frac{dG(y)}{1 - G(\tilde{y})} = \frac{1 - G(y)}{1 - G(\tilde{y})}(y - c).$$

$y^*(\tilde{y})$ is the optimal wage demand by a worker who knows that he is the only applicant to a firm, whose productivity is known to be above $\tilde{y}$. The strict quasi-concavity assumption guarantees that $y^*(\tilde{y})$ is uniquely determined. To sum up, a worker's expected utility is

$$e^{-\lambda}\frac{1-G(\min\{y^*(\tilde{y}),\tilde{y}\})}{1-G(\tilde{y})}(y^*(\tilde{y})-c),$$

where λ is the ratio of workers to firms in the submarket. For later use, denote by $H_{\tilde{y}}$ the distribution function that represents workers' wage-demand strategy in the large submarket. Since a worker is indifferent between $y^*(\tilde{y})$ and any y in the support of $H_{\tilde{y}}$,

$$e^{-\lambda H_{\tilde{y}}(w)}(w-c)=e^{-\lambda}\frac{1-G(\min\{y^*(\tilde{y}),\tilde{y}\})}{1-G(\tilde{y})}(y^*(\tilde{y})-c). \tag{35}$$

Workers' optimal application strategy Let $u(\tilde{y})$ be workers' market utility in the subgame in which all firms below $\tilde{y}$ revealed their productivity, while all firms above $\tilde{y}$ formed one large submarket. In addition, denote by $\lambda(y)$ the queue length for a productivity $y(<\tilde{y})$ firm and by $\tilde{\lambda}$ the queue length for a firm in the large submarket. Then, as in Subsection 4.4, the following conditions must be satisfied:

1. Market utility condition in small submarkets: For each $y<\tilde{y}$,

$$u(\tilde{y})\geq e^{-\lambda(y)}(y-c),\ \text{with equality holding if } y-c\geq u(\tilde{y}).$$

2. Market utility condition in the large submarket:

$$u(\tilde{y})\geq e^{-\tilde{\lambda}}\frac{1-G(\min\{y^*(\tilde{y}),\tilde{y}\})}{1-G(\tilde{y})}(y^*(\tilde{y})-c),\ \text{with equality holding if } \tilde{\lambda}>0.$$

3. Market feasibility:

$$\int_{\underline{y}}^{\tilde{y}}\lambda(y)dG(y)+\int_{\tilde{y}}^{\overline{y}}\tilde{\lambda}dG(y)=\beta.$$

Combining all the conditions,

$$\int_{\underline{y}}^{\tilde{y}}\max\left\{\ln\left(\frac{y-c}{u(\tilde{y})}\right),0\right\}dG(y)+\int_{\tilde{y}}^{\overline{y}}\max\left\{\ln\left(\frac{1-G(\min\{y^*(\tilde{y}),\tilde{y}\})}{1-G(\tilde{y})}\frac{y^*(\tilde{y})-c}{u(\tilde{y})}\right),0\right\}=\beta.$$

To show that this outcome is well-defined for any $\tilde{y}$, let $\overline{u}$ be workers' market utility in the fully revealing equilibrium. In addition, denote by $\underline{u}$ workers' market utility in the babbling equilibrium. Since the babbling equilibrium is the one in which $\tilde{y}=\underline{y}$,

$$\underline{u}=e^{-\beta}(1-G(y^*(\underline{y}))(y^*(\underline{y})-c).$$

Now for each pair $(\tilde{y},u)\in[\underline{y},\overline{y}]\times[\underline{u},\overline{u}]$, let

$$B(\tilde{y},u)=\int_{\underline{y}}^{\tilde{y}}\max\left\{\ln\left(\frac{y-c}{u}\right),0\right\}dG(y)+\int_{\tilde{y}}^{\overline{y}}\max\left\{\ln\left(\frac{1-G(\min\{y^*(\tilde{y}),\tilde{y}\})}{1-G(\tilde{y})}\frac{y^*(\tilde{y})-c}{u}\right),0\right\}.$$

The economic meaning of $B(\tilde{y}, u)$ is the measure of workers in the market that is necessary to provide workers with market utility u, provided that all firms below $\tilde{y}$ reveal their productivity, while all firms above $\tilde{y}$ announce an identical message.

It suffices to show that for each $\tilde{y} \in \mathcal{Y}$, there exists a unique $u(\tilde{y}) \in [\underline{u}, \overline{u}]$ such that $B(\tilde{y}, u(\tilde{y})) = \beta$. The result immediately follows from the following four properties of the function B: (1) $B(\underline{y}, \underline{u}) = \beta$ (babbling equilibrium), (2) $B(\overline{y}, \overline{u}) = \beta$ (fully revealing equilibrium), (3) $B(\tilde{y}, \cdot)$ is continuous and strictly decreasing in interior (less workers for higher u), and (4) $B(\cdot, u)$ is continuous and strictly increasing in interior (the more types were revealed, the more workers can be present in the market).

Firms' optimal communication strategy. Next, we prove that at the communication stage, no firm has an incentive to deviate. First consider a productivity $y(< \tilde{y})$ firm. The firm has no incentive to deviate to any other $y' < \tilde{y}$ for the same reason as in the fully revealing equilibrium: a firm strictly prefers truthfully revealing its productivity to deviating to a lower productivity and is indifferent over all productivity levels above its own. Suppose the firm deviates to the large submarket. Then, the firm's expected profit is

$$\int_{c+\tilde{u}(\tilde{y})}^{y} (y - w) d\left(\frac{\tilde{u}(\tilde{y})}{w - c}\right),$$

which is exactly identical to the firm's equilibrium expected profit. Therefore, the firm also does not have an incentive to deviate to the large submarket. We conclude that no firm below $\tilde{y}$ would deviate.

Now consider a productivity $y(\geq \tilde{y})$ firm. The firm can deviate only to $y' < \tilde{y}$. The firm's deviation profit is

$$\int_{c+\tilde{u}}^{y'} (y - w) d\left(\frac{\tilde{u}(\tilde{y})}{w - c}\right).$$

This deviation profit is strictly smaller than the firm's equilibrium payoff,

$$\int_{c+\tilde{u}}^{y^*(\tilde{y})} (y - w) d\left(\frac{\tilde{u}(\tilde{y})}{w - c}\right),$$

because $y' < \tilde{y} \leq y^*(\tilde{y})$. We conclude that no firm in the large submarket would deviate.

Uniqueness. Suppose there exists another form of partitional equilibrium. The equilibrium necessarily has an interval of firms $[y^*, y^{**}]$ such that all firms in the interval announce an identical message, productivity y^{**} firms hire with a positive probability, and $y^{**} < \overline{y}$. Let u be workers' expected utility in the submarket. Then the expected profit of a productivity y^{**} is strictly smaller than

$$\int_{c+u}^{y'} (y - w) d\left(\frac{u}{w - c}\right), \tag{36}$$

because the highest wage demand in the submarket would be strictly less than y^{**} (in the submarket, a worker who knows that he is the only applicant demands strictly less than y^{**}, because the probability that the firm has productivity y^{**} is zero). If a productivity y^{**}

deviates to any higher submarket, it would obtain exactly as much as in (36). Therefore, such an equilibrium cannot exist. Q.E.D.

Proof of Proposition 7: NITS requires that no type should have an incentive to reveal its type. Consider an equilibrium in which the cutoff firm productivity is given by $\tilde{y}$. A productivity $\overline{y}$ firm's expected payoff is

$$\int_{c+\tilde{u}(\tilde{y})}^{y^*(\tilde{y})} (y-w) d\left(\frac{\tilde{u}(\tilde{y})}{w-c}\right).$$

If a productivity $\overline{y}$ firm deviates and truthfully reveals its productivity, then the firm's expected payoff is

$$\int_{c+\tilde{u}(\tilde{y})}^{\overline{y}} (y-w) d\left(\frac{\tilde{u}(\tilde{y})}{w-c}\right).$$

The latter payoff is strictly larger than the former one, unless $\tilde{y} = \overline{y}$. Therefore, only the fully revealing equilibrium satisfies NITS. Q.E.D.

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[1]

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