Partial Identification in Asymmetric Second-Price Auctions in the Absence of Independence

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Abstract

This paper examines identification in second-price auctions within the private-values framework. The first part of the paper considers an arbitrary type of dependence of bidders' values and analyzes identification under several observational scenarios, in which the highest bid is never observed. In a basic scenario, only the winner's identity and the winning price are observed. The most informative is the scenario in which all the identities and all the bids except for the highest bid are known. Using results from Athey and Haile (2002), the joint distribution of bidders' values in these scenarios is not identified. The paper uses the information available in auctions' outcomes to construct bounds on the joint distribution of values for any subset of bidders. The second part of the paper takes a different tack by showing how bounds can be improved under different types of positive dependence of bidders' values.

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1 Introduction

In auctions, researchers are often interested in learning model's economic primitives, particularly the joint distribution of bidders' values. The identification problem is the problem of learning this underlying distribution from data available from the auction's outcomes.

This paper examines identification in second-price auctions within the private-values framework. It is well known that in such auctions, a weakly dominant strategy for bidders entails submitting their true value.[1] Throughout the paper I consider an equilibrium where bidders employ this strategy.

The identification analysis cannot be conducted without specifying what data are available from the auctions' outcomes. I consider four observational scenarios. In the basic scenario, the only available data pertain to the winner's identity and the winning price. In the second scenario, in addition to the winner's identity and the winning price an econometrician observes the identity of the highest losing bidder. In the third scenario, in addition to the winner's identity and the winning price an econometrician observes the starting bid and the identity of the bidder that submitted it. Finally, in the fourth observational scheme, all the identities and all the bids except for the highest bid are known. Athey and Haile (2002) provide a non-identification result, which, in particular, implies that in second-price auctions within the private values framework and without the assumption of independence of bidders' values, the joint distribution of values is not identified when only a subset of bids is observed. In particular, even if all the bids except for the highest bid are known, there are many distributions consistent with the data. Athey and Haile (2002) and Komarova (2009) discuss the point identification of the joint distribution when bidders' values are independent.

The main goal of this paper is to investigate the identification issue in the absence of independence. I exploit the idea that even though the distribution of values is not identified, the data are informative and allow finding bounds on the distribution. These bounds can be used in the analysis of counterfactuals and other applications. Because throughout the paper I assume that the highest bid is never observed, all the results of this paper apply to ascending auctions where price rises continuously and exogenously.

Many of the results obtained in the paper can be extended to dependent generalized competing risks models. In reliability theory, a classical competing risks model is a situation in which a machine breaks down as soon as one of its components reaches a failure state; the observed data pertain to the machine's lifetime and the component that caused the failure. Generalized competing risks models relax this assumption and consider cases in which an object fails because of the cumulative failure of some of its elements rather than a single one. As noticed by Athey and Haile (2002), second-price auctions within the private-values framework is a special case of these models.

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[1] See, for instance, Vickrey (1961).

For classical competing risks models the study of identification in the absence of independence was initiated by Peterson (1976), who obtained tight pointwise bounds on the joint and marginal survival functions in classical competing risks. Crowder (1991) and Bedford and Meilijson (1997) obtained new results on bounds for those functions. For generalized competing risks, several results for bounds on survival functions are established by Deshpande and Karia (1997).

Section 2 considers an arbitrary type of dependence of bidders' values. For each of the four observational scenarios described above I derive bounds on the joint distribution of values for any subset of bidders. Because the second and the third scenarios contain more information than the first scenario, and the fourth scenario has more information that the second and the third ones one, this allows me to analyze how the bounds change and how much tighter they become when more data become available – data on the identities or the bids. For the first scenario, I also show that functions that can serve as distribution functions of bidders' values necessarily satisfy a certain monotonicity condition. I also explain why in the scenario the equilibrium condition can be replaced with the conditions on bounded rationality introduced in Haile and Tamer (2003). In all four scenarios the lower bound on the joint distribution of values of *all* bidders is trivial – it is 0. This happens because the absence of the knowledge of the highest bid does not allow us to bound the value of the auction's winner.

Section 3 analyzes bounds on distributions from a different perspective. In auctions, it is not uncommon to think about the valuations of bidders as *positively dependent* random variables. There are several notions of positive dependence. For instance, types of positive dependence such as *affiliation* and *association* are discussed in detail in Milgrom and Weber (1982). For simplicity, I consider only the first observational scenario and show how several types of positive dependence allow improving the bounds on the distributions of bidders' values. In particular, they allow me to construct a non-trivial lower bound on the joint distribution of values of all bidders.

Proofs are collected in the Appendix.

2 Bounds under different scenarios

A single object is up for sale, and d buyers, $d \geq 3$, are bidding on it. The set of all bidders is known. Bids are submitted in sealed envelopes. The highest bidder wins and pays the value of the second-highest bid; thus, in these auctions, the second-highest bid is the winning price. Suppose that the bidders have private values and that they are aware of their value. It is known that in this setting, a weakly dominant strategy for bidders is to submit their true value – and this is the equilibrium considered in this paper.

Notation

Denote bidders' private values as X_1, ..., X_d. Suppose that these values have continuous marginal distributions on a common support $[t_0, T]$. It is also assumed that $P(X_i = X_j) = 0$, $i \neq j$, so that the probability of a tie is 0.[2]

Let $D = \{i_1, \ldots, i_r\}$ denote a subset that consists of bidders $i_1, \ldots, i_r$. Define Q_D as the distribution function of valuations of bidders in this subset:

$$Q_D(t_{i_1}, \ldots, t_{i_r}) = P(X_{i_1} \leq t_{i_1}, \ldots, X_{i_r} \leq t_{i_r}).$$

Let Q denote Q_D for $D = \{1, \ldots, d\}$, Q_{-m} denote Q_D for $D = \{1, \ldots, m-1, m+1, \ldots, d\}$, and F_j denote Q_D for $D = \{j\}$. The complement of set D is denoted as CD.

2.1 Basic scenario: Only the winner's identity and the winning price are observed

I first consider a scenario in which only the winner's identity and the winning price are observed in an auction's outcome. Then the probability of the event $\{price \leq t, i \text{ wins}\}$ is known for any $t \in [t_0, T]$ and any $i = 1, \ldots, d$. So, for each bidder i, we observe the following function G_i on $[t_0, T]$:

$$G_i(t) = P(price \leq t, i \text{ wins}).$$

Knowing functions G_i is not enough to identify the joint distribution or even marginal distributions of bidders' values unless some additional assumptions on the joint distribution are imposed. Athey and Haile (2002) establish a non-identification result which implies, in particular, that if the highest bid is not observed and the bidders' values are not independent, then there is an infinite number of joint distributions that rationalize observable functions. Komarova (2009) proves that the joint distribution is identified when bidders' values are independent. Here the independence of values is not assumed, and even though the distributions are not identified, the data are informative and allow us to obtain bounds on the distributions of interest.

To obtain a lower bound on Q_D, I use the fact that if bidder $j \notin D = \{i_1, \ldots, i_r\}$ wins and the price does not exceed t, then the values of $X_{i_1}, \ldots, X_{i_r}$ do not exceed t either. In other words, functions G_j, $j \notin D$, provide information about a lower bound on Q_D. On the other hand, if bidder i_k wins, then it is not known how large the value X_{i_k} is and, consequently, G_{i_k} is not helpful in finding a lower bound on Q_D.

To obtain an upper bound on Q_D, I exploit the fact that if the value of X_{i_k} does not exceed t_{i_k}, then the price does not exceed t_{i_k} either when bidder i_k wins. If we know the bounds on the values of $X_{i_1}, \ldots, X_{i_r}$ from above, and bidder $j \notin D = \{i_1, \ldots, i_r\}$ wins,

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[2] Instead, it can be assumed that the joint distribution of bidders' values is absolutely continuous, which implies zero probabilities of ties.

then no non-trivial conclusion can be made about the price unless CD contains at most one bidder.

The theorem below formalizes this discussion and presents bounds on Q_D.

Theorem 2.1. *Suppose the bidders play their weakly dominant strategy by submitting their true values. Also suppose that only the winner's identity and the winning price are observed. then*

(a) Function Q_D is bounded from below as follows:

$$Q_D(t_{i_1},\dots,t_{i_r}) \geq \sum_{j\in CD} G_j(\min_{k=1,\dots,r} t_{i_k}).$$

(b) Function Q is tightly bounded from above as follows:

$$Q(t_1,\dots,t_d) \leq \sum_{j=1}^{d} G_j(\min\{t_j, \max_{i\neq j} t_i\}). \tag{2.1}$$

A tight upper bound for Q_D, where $D=\{i_1,\dots,i_r\}$, is obtained from (2.1) by substituting $t_j = T$ for any $j\notin D$.

For instance, for any $m=1,\dots,d$, the distribution function Q_{-m} is tightly bounded from above as follows:

$$Q_{-m}(t_1,\dots,t_{m-1},t_{m+1},\dots,t_d) \leq \sum_{j\neq m} G_j(t_j) + G_m(\max_{i\neq m} t_i).$$

If D contains at most $d-2$ elements, then Q_D is tightly bounded from above as follows:

$$Q_D(t_{i_1},\dots,t_{i_r}) \leq \sum_{i_k\in D} G_{i_k}(t_{i_k}) + \sum_{j\in CD} G_j(T).$$

The bounds in this theorem can be improved at the border of the support:

$$Q_D(t_{i_1},\dots,t_{i_r}) \geq \sum_{l=1}^{r} 1(t_{i_l}=T) G_{i_l}(\min_{k=1,\dots,r} t_{i_k}) + \sum_{j\in CD} G_j(\min_{k=1,\dots,r} t_{i_k}),$$

$$Q(t_1,\dots,t_d) \leq 1(\min_i t_i > t_0) \sum_{j=1}^{d} G_j(\min\{t_j, \max_{i\neq j} t_i\}).$$

This improvement is obvious but not helpful in the analysis on bounds. Also, it is worth noting that the lower bound on Q is simply 0 and, thus, completely uninformative.

Example 2.1. *Consider the auction with four buyers. Let $\tilde{X}_1$, $\tilde{X}_2$, $\tilde{X}_3$, $\tilde{X}_4$ and A be independent random variables distributed on $[0,1]$ with distribution functions $\tilde{F}_1(t)=t^2$,*

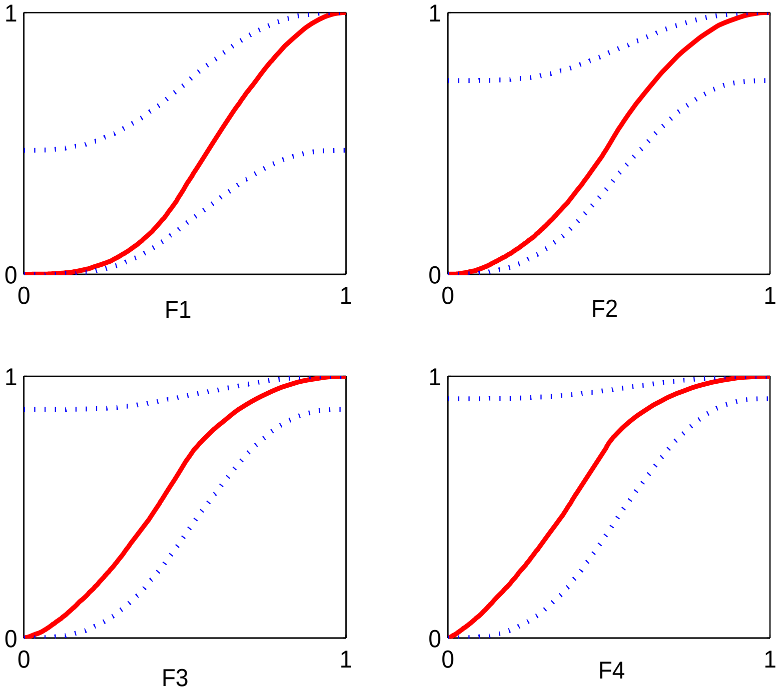

Figure 1. Bounds on the marginal distribution functions in the basic scenario.

$\tilde{F}_2(t) = t$, $\tilde{F}_3(t) = \sqrt{t}$, $\tilde{F}_4(t) = t^4$ and $\tilde{F}_A(t) = t$. Let private values X_1, X_2, X_3 and X_4 of the buyers be

$$
\begin{aligned}
X_1 &= 0.5\tilde{X}_1 + 0.5A \\
X_2 &= 0.5\tilde{X}_2 + 0.5A \\
X_3 &= 0.5\tilde{X}_3 + 0.5A \\
X_4 &= 0.5\tilde{X}_4 + 0.5A
\end{aligned}
$$

Figure 1 shows the the marginal distribution functions F_1, F_2, F_3 and F_4 an the upper and lower bounds on these functions from Theorem 2.1. The upper bounds are wide for low valuations because low bids are not observed in auction's outcomes.

Bounds in Theorem 2.1 are necessary conditions on distribution functions. Other necessary conditions can be obtained.

Consider, for instance, bounds on the joint distributions of values of $(d-1)$-element subsets of bidders:

$$G_m(\min_{i \neq m} t_i) \leq Q_{-m}(t_1, \ldots, t_{m-1}, t_{m+1}, \ldots, t_d) \leq \sum_{j \neq m} G_j(t_j) + G_m(\max_{i \neq m} t_i), \tag{2.2}$$

for $m = 1, \ldots, d$. Not not every joint distribution function of $d-1$ random variables, bounded as in (2.2), can serve as the joint distribution function of X_1, …, X_{m-1}, X_{m+1}, …, X_d.

Consider $t_1 = \ldots = t_{m-1} = t_{m+1} = \ldots = t_d = t$. Then

$$G_m(t) \leq Q_{-m}(t, \ldots, t) \leq \sum_{j=1}^{m} G_j(t).$$

A necessary condition on Q_{-m} is

The function $Q_{-m}(t, \ldots, t) - G_m(t)$ is non-decreasing in t.

Clearly, we can find distributions of $d-1$ random variables, bounded as in (2.2), that do not satisfy this monotonicity condition. Without a loss of generality, let us prove this monotonicity for $m = 1$. Let $\tau > t$.

$$\begin{aligned} Q_{-1}(\tau, \ldots, \tau) - Q_{-1}(t, \ldots, t) &= Pr(t < \max_{j=2,\ldots d} X_i \leq \tau) = Pr(t < \max_{j=2,\ldots d} b_i \leq \tau) \\ &\geq Pr(t < \max_{j=2,\ldots d} b_i \leq \tau, 1 \text{ wins }) \\ &= Pr(\max_{j=2,\ldots d} b_i \leq \tau, 1 \text{ wins }) - Pr(\max_{j=2,\ldots d} b_i \leq t, 1 \text{ wins }) \\ &= G_1(\tau) - G_1(t). \end{aligned}$$

From here,

$$Q_{-1}(\tau, \ldots, \tau) - G_1(\tau) \geq Q_{-1}(t, \ldots, t) - G_1(t).$$

Thus, if distribution functions bounded as in Theorem 2.1, can serve as the distribution functions for $(d-1)$-element subsets of bidders, they necessarily satisfy the monotonicity condition of $Q_{-m}(t, \ldots, t) - G_m(t)$, $m = 1, \ldots, d$.

Another issue worth mentioning is that Theorem 2.1 relies on the fact that bidders submit their true values. This condition can be relaxed if we employ the following two assumptions.

Assumption I (AI). Bidders do not bid more than they are willing to pay.

Assumption II (AII). Bidders do not allow an opponent to win at a price they are willing to beat.

These assumptions on *bounded rationality* were introduced in Haile and Tamer (2003). The authors were the first ones to relax equilibrium conditions in the econometric analysis of auctions and allow other types of bidders' behavior. One of their contributions is the construction of bounds on the distributions of interest for this limited structure in certain auction models.

The proposition below shows that when the equilibrium condition is replaced with AI and AII, the bounds on the distribution functions remain the same as in Theorem 2.1. In addition, it is assumed that the probability of a tie is 0.

Proposition 2.2. *Suppose that only the winner's identity and the transaction price are observed and* $P(b_i = b_j) = 0$ *for* $i \neq j$.
(a) If AI holds, then Q_D *are tightly bounded from above as in Theorem 2.1.*
(b) If AII holds, then Q_D *are bounded from below as in Theorem 2.1.*

Below I analyze how the bounds change when more information from the auction outcomes becomes available.

2.2 The winner's identity, the winning price and the identity of the highest losing bidder are observed

In this scenario, there is additional information about the identity of the bidder with the highest losing bid. This is the bidder that submitted the bid equal to the winning price. Observed are the following $d(d-1)$ functions:

$$G_{ij}(t) = Pr(i \text{ wins}, \textit{price} \leq t, j \text{ submits highest losing bid}), \quad i \neq j, \quad i, j = 1, \dots, d.$$

Clearly,

$$G_i(t) = \sum_{j \neq i} G_{ij}(t).$$

The theorem below shows that this additional information allows improving upper bounds on the distributions of bidders' values. The lower bounds remain the same.

For simplicity, improved bounds are presented only for the joint distribution of values of all bidders and for the marginal distribution functions. Such bounds however can be obtained for an arbitrary subset of bidders.

Theorem 2.3. *Suppose that bidders play their weakly dominant strategy by submitting their true values. Then the joint distribution and the marginal distributions of bidders' values are bounded from above as follows:*

$$Q(t_1, \dots, t_d) \leq \sum_{i=1}^{d} \sum_{j \neq i} G_{ij}(\min\{t_i, t_j\}).$$

$$F_i(t) \leq G_i(t) + \sum_{j \neq i} G_{ji}(t) + \sum_{j \neq i} \sum_{h \neq i, h \neq j} G_{jh}(T).$$

In Theorem 2.1, the marginal distributions are bounded from above as follows:

$$F_i(t) \leq G_i(t) + \sum_{j \neq i} G_j(T).$$

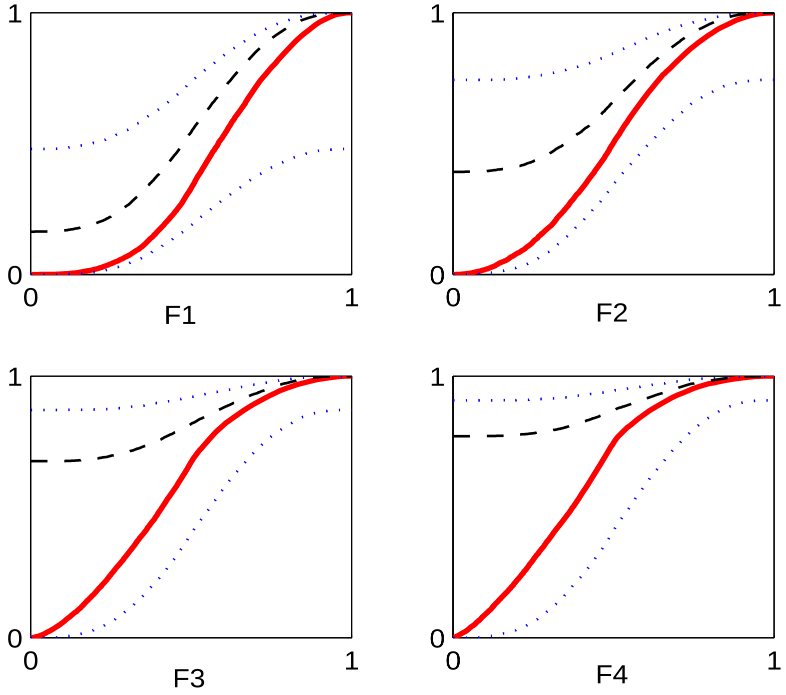

Figure 2

Since $G_{ji}(T) \geq G_{ji}(t)$, then the upper bound on F_i in Theorem 2.3 is indeed in general tighter than that in Theorem 2.1.

In Figure 2, the thick lines depict F_1, F_2, F_3 and F_4 in Example 2.1, the dotted lines depict the bounds on these functions in the basic scenario and the dashed lines show the improved upper bounds from Theorem 2.3.

The equilibrium condition in this theorem can be replaced with the assumption AI.

2.3 The winner's identity, the winning price, the lowest bid and the identity of the corresponding bidder are observed

This scenario contains additional information about the starting bid in the auction and the identity of the bidder that submitted it. Observed are the following $d(d-1)$ functions of two variables:

$$\tilde{G}_{ij}(t,s) = Pr(i \text{ wins}, \, price \leq t, \, b_j = \min_m b_m, \, \min_m b_m \leq s), \quad i \neq j, \quad i,j = 1,\ldots,d.$$

These functions have the following property:

$$\forall s \geq t \quad \tilde{G}_{ij}(t,s) = \tilde{G}_{ij}(t,t),$$

$$G_i(t) = \sum_{j \neq i} \tilde{G}_{ij}(t,t).$$

Also, $\tilde{G}_{ij}(t,s)$ is nondecreasing in t and s.

For simplicity, bounds are presented only for the joint distribution of values of all bidders and for the marginal distribution functions. They can be obtained however for an arbitrary subset of bidders.

Theorem 2.4. *Suppose that bidders play their weakly dominant strategy by submitting their true values. Then*

$$0 \leq Q(t_1, \ldots, t_d) \leq \sum_{i=1}^{d} \sum_{j \neq i} \tilde{G}_{ij}(\min\{t_i, \max_{m \neq i, m \neq j} t_m\}, \min_{m=1,\ldots,d} t_m).$$

For the marginal distribution functions,

$$\sum_{j \neq i} (\tilde{G}_{ji}(T,t) + \sum_{m \neq i, m \neq j} \tilde{G}_{jm}(t,t)) \leq F_i(t) \leq G_i(t) + \sum_{j \neq i} \sum_{m \neq j} \tilde{G}_{jm}(T,t).$$

In Theorem 2.1, the marginal distributions are bounded as follows:

$$\sum_{j \neq i} G_j(t) \leq F_i(t) \leq G_i(t) + \sum_{j \neq i} G_j(T).$$

Since $\tilde{G}_{ji}(T,t) \geq \tilde{G}_{ji}(t,t)$, then

$$\tilde{G}_{ji}(T,t) + \sum_{m \neq i, m \neq j} \tilde{G}_{jm}(t,t) \geq \sum_{m \neq j} \tilde{G}_{jm}(t,t) = G_j(t).$$

Taking into account that in general $\tilde{G}_{ji}(T,t)$ is strictly greater than $\tilde{G}_{ji}(t,t)$, we can conclude that the lower bounds on F_i's in Theorem 2.4 are indeed tighter than those in Theorem 2.1.

Since $\tilde{G}_{jm}(T,t) \leq \tilde{G}_{jm}(T,T)$, then

$$\sum_{j \neq i} \sum_{m \neq j} \tilde{G}_{jm}(T,t) \leq \sum_{j \neq i} \sum_{m \neq j} \tilde{G}_{jm}(T,T) = \sum_{j \neq i} G_j(T).$$

Taking into account that in general $\tilde{G}_{jm}(T,t)$ is strictly less than $\tilde{G}_{jm}(T,T)$, we obtain that the upper bound on F_i's in Theorem 2.4 are indeed tighter than those in Theorem 2.1.

In Figure 3, the thick lines depict F_1, F_2, F_3 and F_4 in Example 2.1, the dotted lines depict the bounds on these functions in the basic scenario and the dashed lines show the improved upper and lower bounds from Theorem 2.4. As can be see, observing the lowest

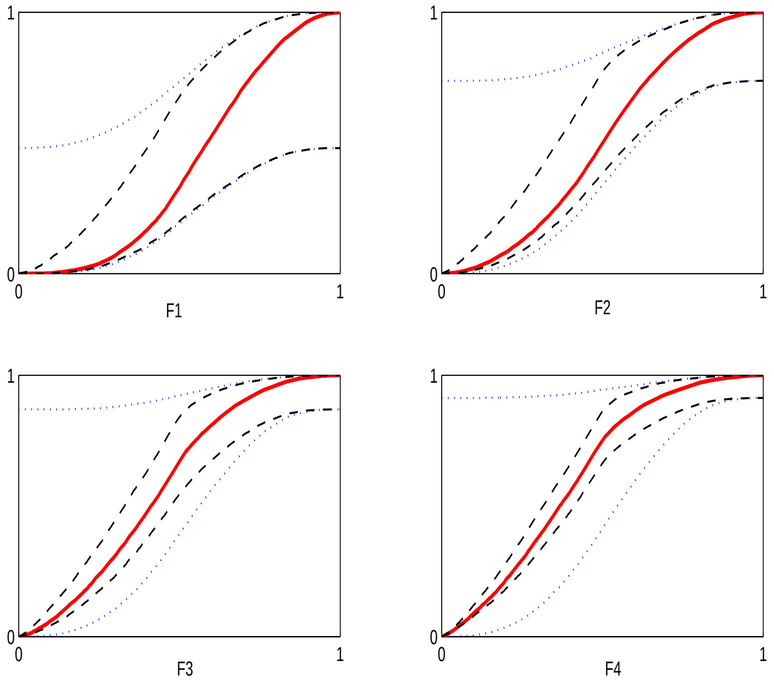

Figure 3

bids significantly improves the bounds around the lower support point, and the upper bound takes value 0 at t_0.

2.4 Bounds when all the identities and all the bids except for the highest bid are observed

This scenario represents an even more detailed observational scheme – everything except for the highest bid is observed. In reliability, this corresponds to so-called continuous monitoring models.

Let Π_d denote the set of all the permutations of set $\{1, \dots, d\}$ and $\rho \in \Pi_d$. Let $\rho(i)$ stand for the ith element of permutation ρ. In the auction context, bidder $\rho(i)$ is the ith highest bidder.

The following $d!$ functions are observed:

$$W_\rho(s_2, \dots, s_d) = P(\cap_{i \neq 1}(b_{\rho(i)} \leq s_i), b_{\rho(1)} > b_{\rho(2)} > \dots > b_{\rho(d)}).$$

Notice that

$$G_j(t) = \sum_{\rho \in \Pi_d : \rho(1) = j} W_\rho(t, \dots, t).$$

To avoid sophisticated technical expressions, I show bounds only for the joint distribution function and the marginal distribution functions.

Theorem 2.5. *Suppose that bidders play their weakly dominant strategy by submitting their true values. Also suppose all the identities and all the bids except for the highest bid are observed. Then Q and F_i are bounded from above and below as follows:*

$$0 \leq Q(t_1, \dots, t_d) \leq \sum_{\rho \in \Pi_d} W_\rho(\min\{t_{\rho(1)}, t_{\rho(2)}\}, \dots, \min_{m=1,\dots,l} t_{\rho(m)}, \dots, \min_{i=1,\dots,d} t_i).$$

$$\begin{aligned} F_i(t) \leq G_i(t) + \sum_{j \neq i} G_{ji}(t) + \sum_{\rho \in \Pi_d : \rho(3)=i} W_\rho(T, t, t, \dots, t, t) + \sum_{\rho \in \Pi_d : \rho(4)=i} W_\rho(T, T, t, \dots, t, t) \\ + \dots + \sum_{\rho \in \Pi_d : \rho(d)=i} W_\rho(T, T, T, \dots, T, t) \end{aligned}$$

$$\begin{aligned} F_i(t) \geq \sum_{j \neq i} G_{ji}(t) + \sum_{\rho \in \Pi_d : \rho(3)=i} W_\rho(T, t, t, \dots, t, t) + \sum_{\rho \in \Pi_d : \rho(4)=i} W_\rho(T, T, t, \dots, t, t) \\ + \dots + \sum_{\rho \in \Pi_d : \rho(d)=i} W_\rho(T, T, T, \dots, T, t) \end{aligned}$$

In Figure 4, the thick lines depict F_1, F_2, F_3 and F_4 in Example 2.1, the dotted lines depict the bounds on these functions in the basic scenario and the dashed lines show the improved upper and lower bounds from Theorem 2.5.

3 Bounds under positive dependence

As we have seen in section 2, additional information about the identities of the bidders and the bids up to the second-highest bid does not improve the trivial lower bound on the joint distribution of values of *all bidders*, that is, on $Q(t_1, \dots, t_d)$. This is because the absence of the knowledge of the highest bid does not allow us to bound from above the value of the auction's winner.

In auctions, it is not uncommon to think about the valuations of bidders as *positively dependent* random variables. There are several notions of positive dependence, and the necessary relations among different notions of positive dependence are well established in the statistical literature.[3] Milgrom and Weber (1982) describe in detail two types of positive dependence – *affiliation*, which is called the *multivariate total positivity of order 2* in statistics, and *association*. For instance, if bidders' values X_i, $i = 1, \dots, d$, are defined

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[3] See, for instance, Karlin and Rinott (1980). A nice review of types of dependence can be found in de Castro (2009).

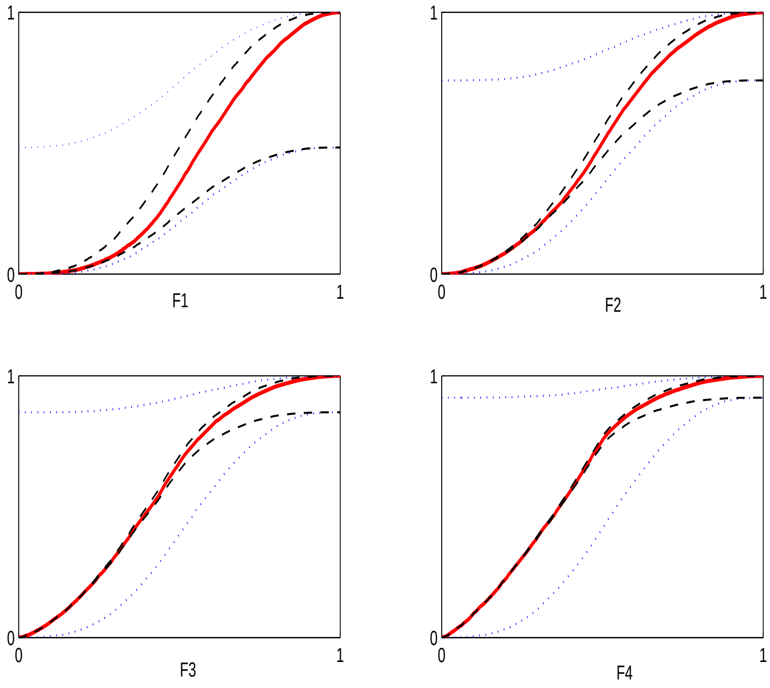

Figure 4

as

$$X_i = \widetilde{X}_i + A \quad i = 1, \ldots, d,$$

where A, $\widetilde{X}_i$, $i = 1, \ldots, d$, are independent of each other, then X_i, $i = 1, \ldots, d$, are affiliated if the densities of $\widetilde{X}_i$, $i = 1, \ldots, d$, are log-concave. Barlow and Proschan (1975) discuss total positivity properties in the context of reliability models, and second-price auctions within the private values framework are isomorphic to a subclass of such models.

Affiliation is a strong notion of positive dependence, which implies, in particular, such properties as association, positive upper orthant dependence, positive lower orthant dependence and nonnegative correlation. Nonnegative correlation is a weak type of positive dependence implied by almost any other type of positive dependence. The example below illustrates the situations of affiliation and nonnegative correlation.

Example 3.1. *Consider the case of 3 bidders whose private values X_1, X_2 and X_3 are*

$$X_1 = Z_1 + A,$$
$$X_2 = Z_2 + A,$$
$$X_3 = Z_3 + A,$$

where Z_1, Z_2, Z_3 are mutually independent random variables. For now, let us ignore the *assumption of the finite lower support point and suppose that*

$$Z_1 \sim \mathcal{N}(0,1), \quad Z_1 \sim \mathcal{N}(0,1), \quad Z_3 \sim \mathcal{N}(0,1), \quad A \sim \mathcal{N}(0,1),$$

$$corr(Z_1, A) = \rho_1, \quad corr(Z_2, A) = \rho_2, \quad corr(Z_3, A) = \rho_3.$$

Then

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2+2\rho_1 & 1+\rho_1+\rho_2 & 1+\rho_1+\rho_3 \\ 1+\rho_1+\rho_2 & 2+2\rho_2 & 1+\rho_2+\rho_3 \\ 1+\rho_1+\rho_3 & 1+\rho_2+\rho_3 & 2+2\rho_3 \end{pmatrix}\right),$$

and obviously the variables are non-negatively correlated iff

$$1+\rho_1+\rho_2 \geq 0, \quad 1+\rho_1+\rho_3 \geq 0, \quad 1+\rho_2+\rho_3 \geq 0. \tag{3.1}$$

As indicated by Karlin and Rinott (1980), the variables in a random vector with multivariate normal distribution $\mathcal{N}(\mu, \Sigma)$ are affiliated iff the off-diagonal elements of the matrix Σ^{-1} are non-positive.

In our case, the necessary and sufficient conditions for affiliation take the form

$$\begin{aligned} 1+\rho_1+\rho_2 &\geq (\rho_1-\rho_3)(\rho_2-\rho_3) \\ 1+\rho_1+\rho_3 &\geq (\rho_1-\rho_2)(\rho_3-\rho_2) \\ 1+\rho_2+\rho_3 &\geq (\rho_2-\rho_1)(\rho_3-\rho_1) \end{aligned} \tag{3.2}$$

It can be verified that conditions (3.2) imply (3.1). Importantly, (3.2) are strictly stronger than (3.1). Indeed, consider $\rho_1 = -\frac{1}{2}$, $\rho_2 = -\frac{1}{2}$, $\rho_3 = 0$. Conditions (3.1) are satisfied while (3.2) are not because

$$1+\rho_1+\rho_2 < (\rho_1-\rho_3)(\rho_2-\rho_3).$$

Bounds on distributions in section 2 are obtained for *any* type of dependence of bidders' values. This section shows how assumptions about types of positive dependence can improve bounds on distribution functions, especially the lower bound on function $Q(t_1, \ldots, t_d)$. For simplicity, I only consider the first scenario, in which the winner's identity and the winning price are observed.

Positive lower orthant dependence (PLOD)

PLOD of random variables X_i, $i = 1, \ldots, d$, is defined as

$$P(X_i \leq t_i, i = 1, \ldots, d) \geq \prod_{i=1}^{d} P(X_i \leq t_i).$$

It means that X_i, $i = 1, \dots, d$, are more likely to simultaneously take low values than their independent copies. In terms of copulas this dependence means that the copula $C(u_1, u_2, \dots, u_d)$ corresponding to the joint distribution of X_i, $i = 1, \dots, d$, satisfies the property

$$C(u_1, u_2, \dots, u_d) \geq u_1 u_2 \dots u_d.$$

Suppose that the seller wants to introduce a reserve price r and is interested in learning the probability of the object not being sold. Using (2.1) in Theorem 2.1, the seller can find an upper bound on this probability:

$$P(\text{object not sold}) = Q(r, \dots, r) \leq \sum_{j=1}^{d} G_j(r).$$

Theorem 2.1 gives the trivial lower bound of 0 on this probability. Under PLOD, however, it is possible to find a non-trivial lower bound because

$$Q(t_1, \dots, t_d) \geq \prod_{i=1}^{d} F(t_i) \geq \prod_{i=1}^{d} \left(\sum_{j \neq i} G_j(t_j) \right),$$

and thus,

$$P(\text{object not sold}) \geq \prod_{i=1}^{d} \left(\sum_{j \neq i} G_j(r) \right).$$

Lower bounds on the joint distributions for subsets of bidders may also improve. Consider, for instance, $D = \{1, 2\}$. From Theorem 2.1,

$$P(X_1 \leq t_1, X_2 \leq t_2) \geq \sum_{j \neq 1, j \neq 2} G_j(\min\{t_1, t_2\}). \tag{3.3}$$

Using PLOD, we obtain

$$P(X_1 \leq t_1, X_2 \leq t_2) \geq F_1(t_1) F_2(t_2) \geq \left(\sum_{j \neq 1} G_j(t_1) \right) \cdot \left(\sum_{j \neq 2} G_j(t_2) \right). \tag{3.4}$$

An improved lower bound for $P(X_1 \leq t_1, X_2 \leq t_2)$ is

$$\max \left\{ \sum_{j \neq 1, j \neq 2} G_j(\min\{t_1, t_2\}),\ \left(\sum_{j \neq 1} G_j(t_1) \right) \cdot \left(\sum_{j \neq 2} G_j(t_2) \right) \right\}.$$

Such an analysis can be conducted for any subset D.

,

Positive upper orthant dependence (PUOD)

PUOD of random variables X_i, $i = 1, \dots, d$, is defined in terms of survival functions:

$$P(X_i > t_i, i = 1, \dots, d) \geq \prod_{i=1}^{d} P(X_i > t_i).$$

It means that X_i, $i = 1, \dots, d$, are more likely to simultaneously take high values than their independent copies. For survival functions it is possible to obtain results analogous to the results in Theorem 2.1. PUOD property allows us to find a non-trivial lower bound on the joint survival function of the values of all bidders.

If the seller, for instance, is interested in learning something about the probability of all bidders participating in the auction if a reserve price r is introduced, then PUOD helps to get a non-trivial lower bound on that probability:

$$\begin{aligned} P(\text{all bidders participate}) = P(X_i > r, i = 1, \dots, d) &\geq \prod_{i=1}^{d} P(X_i > r) \\ &\geq \prod_{i=1}^{d} \left(1 - G_i(r) - \sum_{j \neq i} G_j(T)\right). \end{aligned}$$

Left-tail decreasing (LTD)

Let D be a non-empty subset of $\{1, \dots, d\}$. Variable X_i, $i \in D^c$, is LTD in X_j, $j \in D$, if

$$P(X_i \leq t_i,\, i \in D^c |\, X_j \leq t_j,\, j \in D)$$

is non-increasing in t_k for any $k \in D$.

Suppose the LTD property holds for $D = \{d\}$. Then

$$\begin{aligned} P(X_i \leq t_i, i = 1, \dots, d) &= P(X_i \leq t_i, i = 1, \dots, d-1 |\, X_d \leq t_d) P(X_d \leq t_d) \\ &\geq P(X_i \leq t_i, i = 1, \dots, d-1 |\, X_d \leq T) P(X_d \leq t_d) \\ &= P(X_i \leq t_i, i = 1, \dots, d-1) P(X_d \leq t_d) \\ &\geq G_d(\min_{j \neq d} t_j) \cdot \sum_{j \neq d} G_j(t_d) \end{aligned}$$

If the LTD property holds for several different subsets D, then several lower bounds can be obtained and therefore we can bound the joint distribution of bidders' values by the maximum of all those bounds. The LTD property my also help to improve the lower bound on the joint distributions for subsets of bidders.

Right-tail increasing (RTI)

Let D be a non-empty subset of $\{1, \dots, d\}$. Variable X_i, $i \in D^c$, is RTI in X_j, $j \in D$, if

$$P(X_i > t_i,\, i \in D^c |\, X_j > t_j,\, j \in D)$$

is non-decreasing in t_k for any $k \in D$. Since this notion is formulated in terms of survival functions, this type of dependence is useful for improving lower bounds on the joint survival function for subsets of bidders, in particular, the joint survival function of all bidders' values.

Affiliation

Affiliation of random variables if defined in the following way. Suppose that the distribution of bidders is absolutely continuous. Vector $X = (X_1, \ldots, X_d)$ is *affiliated* if for any $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$

$$f(x \vee y) f(x \wedge y) \geq f(x) f(y).$$

Affiliation is a strong condition on dependence and it is difficult to use for the construction of bounds on distribution functions of bidders values. However, affiliation implies PLOD and PUOD, and also LTD and RTI. Based on the analysis above, this means that the affiliation property improves bounds on distributions. An important question is how much, for instance, affiliation improves bounds compared to the improvement obtained using PLOD, PUOD, LTD, RTI. This is a difficult question and is yet to be analyzed.

4 Appendix

Proof of Theorem 2.1.

(a) First, I prove the result for the lower bound.

$$\begin{aligned} Q_D(t_{i_1},\dots,t_{i_r}) &= P(\cap_{k=1}^{r}(X_{i_k}\le t_{i_k})) = P(\cap_{k=1}^{r}(b_{i_k}\le t_{i_k})) \\ &= \sum_{m=1}^{r} P(\cap_{k=1}^{r}(b_{i_k}\le t_{i_k}), i_m \text{ wins}) + \sum_{j\in CD} P(\cap_{k=1}^{r}(b_{i_k}\le t_{i_k}), j \text{ wins}) \\ &\ge \sum_{j\in CD} P(\text{price } \le \min_{k=1,\dots,r} t_{i_k}, j \text{ wins}) = \sum_{j\in CD} G_j(\min_{k=1,\dots,r} t_{i_k}). \end{aligned}$$

(b)

$$\begin{aligned} Q(t_1,\dots,t_d) = P(\cap_{i=1}^{d}(X_i\le t_i)) = P(\cap_{i=1}^{d}(b_i\le t_i)) &= \sum_j P(\cap_{i\neq j}(b_i\le t_i), b_j\le t_j, j \text{ wins}) \\ \le \sum_j P(\text{price } \le \min\{\max_{i\neq j} t_i, t_j\}, j \text{ wins}) &= \sum_j G_j(\min\{\max_{i\neq j} t_i, t_j\}). \end{aligned}$$

To obtain an upper bound for Q_{-m}, use the upper bound for Q and the fact that

$$Q_{-m}(t_1,\dots,t_{m-1},t_{m+1},\dots,t_d) = Q(t_1,\dots,t_{m-1},T,t_{m+1},\dots,t_d).$$

Clearly,

$$\begin{aligned} Q_{-m}(t_1,\dots,t_{m-1},t_{m+1},\dots,t_d) &\le \sum_{j\neq m} G_j(\min\{T,t_j\}) + G_m(\min\{\max_{i\neq m} t_i, T\}) \\ &= \sum_{j\neq m} G_j(t_j) + G_m(\max_{i\neq m} t_i). \end{aligned}$$

To obtain an upper bound for $D=\{t_{i_1},\dots,t_{i_r}\}$ that contains at most $d-2$ elements, use the upper bound for Q and substitute values of t_j, $j\notin D$, with the value of T.

Tightness

To prove that the upper bound on Q is tight, we need to show that for each bound there exists a joint probability distribution, having the specified observable functions G_j, that has probability mass arbitrarily close to the mass of the bound.

For a given collection of observable functions G_j, consider a joint distribution that is the sum of d sub-distributions such that the sub-distribution j places mass at $(X_1,X_2,\dots,X_d)$ which is equal to x_i for $i\neq j$ and $\alpha\max_{i\neq j} x_i+(1-\alpha)T$ for x_j according to $G_j(\max_{i\neq j} x_i)$, where $\alpha<1$ is arbitrarily close to 1. This joint distribution has the specified G_j and approaches the upper bound arbitrarily closely because α can be taken arbitrarily close to 1.

For $D=\{1,\dots,m-1,m+1,\dots,d\}$, consider a joint distribution that is the sum of d sub-distributions such that the sub-distribution j, $j\neq m$, places mass at at $(X_1,X_2,\dots,X_d)$ which is equal to x_i for $i\neq j$ and $\alpha\max_{i\neq j,i\neq m} x_i+(1-\alpha)T$ for x_j and $\beta\max_{i\neq j,i\neq m} x_i+(1-\beta)t_0$ for x_m according to $G_j(\max_{i\neq j,i\neq m} x_i)$, where $\alpha<1$ and $\beta<1$ are arbitrarily close to 1. The sub-distribution m places mass at at $(X_1, X_2, \ldots, X_d)$ which is equal to x_i for $i \neq m$ and $\alpha \max_{i\neq m} x_i + (1-\alpha)T$ for x_m according to $G_m(\max_{i\neq m} x_i)$, where $\alpha > 0$ is arbitrarily close to 0.

Similarly, for any subset D, the upper bound on Q_D is tight.

Proof of Proposition 2.2.

(a) According to AI, for any $D = \{t_{i_1}, \ldots, t_{i_r}\}$, the event $\{\cap_{k=1}^r (X_{i_k} \le t_{i_k})\}$ implies the event $\{\cap_{k=1}^r (b_{i_k} \le t_{i_k})\}$. Therefore,

$$Q_D(t_{i_1}, \ldots, t_{i_r}) = P(\cap_{k=1}^r (X_{i_k} \le t_{i_k})) \le P(\cap_{k=1}^r (b_{i_k} \le t_{i_k})).$$

The rest of the proof for the upper bounds is the same as in Theorem 2.1.

(b) Suppose that $\max_{k=1,\ldots,r} t_{i_k} < T$. Then

$$\begin{aligned} Q_D(t_{i_1}, \ldots, t_{i_r}) &= P(\cap_{k=1}^r (X_{i_k} \le t_{i_k})) = \sum_{k=1}^r P(\cap_{k=1}^r (X_{i_k} \le t_{i_k}), i_k \text{ wins}) \\ &+ \sum_{j \in CD} P(\cap_{k=1}^r (X_{i_k} \le t_{i_k}), j \text{ wins}) \ge \sum_{j \in CD} P(\cap_{k=1}^r (X_{i_k} \le t_{i_k}), j \text{ wins}). \end{aligned}$$

According to AII, for $j \in CD$, the event $\{\text{price } \le \min_{k=1,\ldots,r} t_{i_k}, j \text{ wins}\}$ implies the event $\{\cap_{k=1}^r (X_{i_k} \le t_{i_k}), j \text{ wins}\}$. Indeed, if some X_{i_k} was larger than t_{i_k}, then bidder i_k would not allow bidder j to win at a price less or equal than $\min_{k=1,\ldots,r} t_{i_k}$. Therefore,

$$Q_D(t_{i_1}, \ldots, t_{i_r}) \ge \sum_{j \in CD} P(\text{price } \le \min_{k=1,\ldots,r} t_{i_k}, j \text{ wins}).$$

The rest of the proof for the lower bounds is the same as in Theorem 2.1.

Proof of Theorem 2.3.

$$\begin{aligned} Q(t_1, \ldots, t_d) = P(\cap_{m=1}^d (X_m \le t_m)) &= P(\cap_{m=1}^d (b_m \le t_m)) = \\ &= \sum_{i=1}^d \sum_{j \neq i} P(\cap_{m=1}^d (b_m \le t_m), i \text{ wins}, j \text{ submits highest losing bid}) \\ &\le \sum_{i=1}^d \sum_{j \neq i} P(\text{price } \le \min\{t_i, t_j\}, i \text{ wins}, j \text{ submits highest losing bid}) \\ &= \sum_{i=1}^d \sum_{j \neq i} G_{ij}(\min\{t_i, t_j\}). \end{aligned}$$

The upper bound on $F_i(t)$ is the upper bound on $Q(t_1, \ldots, t_d)$ calculated at $t_i = t$ and

$$t_1 = \ldots = t_{i-1} = t_{i+1} = \ldots = t_d = T.$$

Proof of Theorem 2.4. The lower bound of 0 on $Q(t_1, \ldots, t_d)$ is obvious. Let us derive the upper bound.

$$
\begin{aligned}
Q(t_1,\ldots,t_d) &= P(\cap_{m=1}^{d}(X_m \le t_m)) = P(\cap_{m=1}^{d}(b_m \le t_m)) = \\
&= \sum_{i=1}^{d}\sum_{j\neq i} P(price \le \max_{m\neq i, m\neq j} t_m,\ b_i \le t_i, i \text{ wins},\ b_j = \min_m b_m,\ b_j \le t_j) \\
&\le \sum_{i=1}^{d}\sum_{j\neq i} P(price \le \min\{t_i, \max_{m\neq i, m\neq j} t_m\},\ i \text{ wins},\ b_j = \min_m b_m,\ b_j \le \min_m t_m) \\
&= \sum_{i=1}^{d}\sum_{j\neq i} \tilde{G}_{ij}(\min\{t_i, \max_{m\neq i, m\neq j} t_m\}, b_j \le \min_m t_m).
\end{aligned}
$$

Let us now derive the lower on $F_i(t)$.

$$
\begin{aligned}
F_i(t) = P(X_i \le t) &= P(b_i \le t) = P(i \text{ wins},\ b_i \le t) + P(b_i = \min_m b_m,\ b_i \le t) \\
&+ P(\min_m b_m < b_i,\ i \text{ does not win},\ b_i \le t) \ge P(b_i = \min_m b_m,\ b_i \le t) \\
&+ P(\min_m b_m < b_i,\ i \text{ does not win},\ b_i \le t) = \sum_{j\neq i} \tilde{G}_{ij}(T,t) \\
&+ P(\min_m b_m < b_i,\ i \text{ does not win},\ b_i \le t) \ge \sum_{j\neq i} \tilde{G}_{ij}(T,t) \\
&+ \sum_{j\neq i}\sum_{h\neq i, h\neq j} P(j \text{ wins},\ b_h = \min_m b_m,\ price \le t, b_h \le t) \\
&= \sum_{j\neq i} \tilde{G}_{ij}(T,t) + \sum_{j\neq i}\sum_{h\neq i, h\neq j} \tilde{G}_{jh}(t,t).
\end{aligned}
$$

The upper bound on $F_i(t)$ is the upper bound on $Q(t_1,\ldots,t_d)$ calculated at $t_i = t$ and

$$t_1 = \ldots = t_{i-1} = t_{i+1} = \ldots = t_d = T.$$

Proof of Theorem 2.4. The lower bound of 0 on $Q(t_1,\ldots,t_d)$ is obvious. Let us derive the upper bound.

$$
\begin{aligned}
Q(t_1,\ldots,t_d) &= P(\cap_{m=1}^{d}(X_m \le t_m)) = P(\cap_{m=1}^{d}(b_m \le t_m)) = \\
&= \sum_{i=1}^{d}\sum_{j\neq i} P(price \le \max_{m\neq i, m\neq j} t_m,\ b_i \le t_i, i \text{ wins},\ b_j = \min_m b_m,\ b_j \le t_j) \\
&\le \sum_{i=1}^{d}\sum_{j\neq i} P(price \le \min\{t_i, \max_{m\neq i, m\neq j} t_m\},\ i \text{ wins},\ b_j = \min_m b_m,\ b_j \le \min_m t_m) \\
&= \sum_{i=1}^{d}\sum_{j\neq i} \tilde{G}_{ij}(\min\{t_i, \max_{m\neq i, m\neq j} t_m\}, b_j \le \min_m t_m).
\end{aligned}
$$

Let us now derive the lower on $F_i(t)$.

$$\begin{aligned}F_i(t) &= P(X_i \leq t) = P(b_i \leq t) = P(i \text{ wins},\, b_i \leq t) + P(b_i = \min_m b_m,\, b_i \leq t)\\ &+ P(\min_m b_m < b_i,\, i \text{ does not win},\, b_i \leq t) \geq P(b_i = \min_m b_m,\, b_i \leq t)\\ &+ P(\min_m b_m < b_i,\, i \text{ does not win},\, b_i \leq t) = \sum_{j\neq i} \tilde{G}_{ij}(T,t)\\ &+ P(\min_m b_m < b_i,\, i \text{ does not win},\, b_i \leq t) \geq \sum_{j\neq i} \tilde{G}_{ij}(T,t)\\ &+ \sum_{j\neq i}\sum_{h\neq i, h\neq j} P(j \text{ wins},\, b_h = \min_m b_m,\, \mathit{price} \leq t, b_h \leq t)\\ &= \sum_{j\neq i} \tilde{G}_{ij}(T,t) + \sum_{j\neq i}\sum_{h\neq i, h\neq j} \tilde{G}_{jh}(t,t).\end{aligned}$$

The upper bound on $F_i(t)$ is the upper bound on $Q(t_1, \ldots, t_d)$ calculated at $t_i = t$ and

$$t_1 = \ldots = t_{i-1} = t_{i+1} = \ldots = t_d = T.$$

Proof of Theorem 2.5.

(a) The lower bound is obvious. The upper bound is obtained in the following way:

$$\begin{aligned}Q(t_1, \ldots, t_d) &= \sum_{\rho\in\Pi_d} P(\cap_{i=1,\ldots d}(b_i \leq t_i), b_{\rho(1)} > b_{\rho(2)} > \ldots > b_{\ \rho(d)})\\ &\leq \sum_{\rho\in\Pi_d} P(\cap_{l=2,\ldots,d}(b_{\rho(l)} \leq \min_{m=1,\ldots,l} t_{\rho(m)}), b_{\rho(1)} > b_{\rho(2)} > \ldots > b_{\rho(d)})\\ &= \sum_{\rho\in\Pi_d} W_\rho(\min\{t_{\rho(1)}, t_{\rho(2)}\}, \ldots, \min_{m=1,\ldots,l} t_{\rho(m)}, \ldots, \min_{i=1,\ldots,d} t_i).\end{aligned}$$

(b) Without a loss of generality, consider function F_1.

$$\begin{aligned}F_1(t) = P(X_1 \leq t) = P(b_1 \leq t) &= \sum_i \sum_{h\neq i} P(\max_{l\neq h, l\neq i} b_l < b_h, b_h < b_i, b_1 \leq t)\\ &= \sum_{h\neq 1} P(\max_{l\neq h, l\neq 1} b_l < b_h, b_h < b_1, b_1 \leq t) + \sum_{i\neq 1} P(\max_{l\neq 1, l\neq i} b_l < b_1, b_1 < b_i, b_1 \leq t)\\ &+ \sum_{i\neq 1}\sum_{h\neq i, h\neq 1} P(\max_{l\neq h, l\neq i} b_l < b_h, b_h < b_i, b_1 \leq t).\end{aligned}$$

For the upper bound,

$$\begin{aligned}P(b_1 \leq t) &\leq \sum_{h\neq 1} P(\max_{l\neq h, l\neq 1} b_l < b_h, b_h < b_1, b_h \leq t) + \sum_{i\neq 1} P(\max_{l\neq 1, l\neq i} b_l < b_1, b_1 < b_i, b_1 \leq t)\\ &+ \sum_{i\neq 1}\sum_{h\neq i, h\neq 1} P(\max_{l\neq h, l\neq i} b_l < b_h, b_h < b_i) = \sum_{h\neq 1} G_{1h}(t) + \sum_{i\neq 1} G_{i1}(t) + \sum_{i\neq 1}\sum_{h\neq i, h\neq 1} G_{ih}(T).\end{aligned}$$

For the lower bound,

$$\begin{aligned}
P(b_1 \leq t) &\geq 0 + \sum_{i\neq 1} P(\max_{l\neq 1, l\neq i} b_l < b_1, b_1 < b_i, b_1 \leq t) \\
&+ \sum_{i\neq 1} \sum_{h\neq i, j\neq 1} P(\max_{l\neq h, l\neq i} b_l < b_h, b_h < b_i, b_h \leq t) = \sum_{i\neq 1} G_{i1}(t) + \sum_{i\neq 1} \sum_{h\neq i, h\neq 1} G_{ih}(t) \\
&= \sum_{i\neq 1} \left(G_{i1}(t) + \sum_{h\neq i, h\neq 1} G_{ih}(t) \right) = \sum_{i\neq 1} \sum_{h\neq i} G_{ih}(t) = \sum_{i\neq 1} G_i(t).
\end{aligned}$$

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