C Appendix: Strategic trading with heterogenous horizon and dispersed information

In this Appendix, I consider the effect of public information on strategies and prices in modified versions of a standard Kyle (1989) model of strategic trading. I show that as the number of agents grows without bound the analyzed version of the model converges to the standard competitive REE model with dispersed information analyzed in the main text.[25]

I deviate from Kyle (1989) along two main dimensions. First, I consider the general information structure. Second, I allow for the interaction of two groups who consume at different time points.

C.1 Set-up

I consider two groups of traders, A and B trading the same risky asset and a riskless bond, where A-traders and B-traders are indexed by $i = 1, ...N$ and $j = 1, ...N$ respectively. The return on the bond is normalized to 1. There are three periods, $t = 1, 2, 3$. The uncertain fundamental value θ is realized in period 3. The equilibrium price of the risky asset is denoted p_t^* in period t. Each agent has CARA utility over final wealth with the identical risk-aversion parameter γ. The total supply of assets,u_1, in period 1 and $u_2 \equiv u_1 + \Delta u_2$ in period 2 are normally and independently distributed.[26] The strategy of a trader in period t, is to submit a demand schedule $d_t^i(p_t, \mathcal{I}_t^i)$ or $d_t^j\left(p_t, \mathcal{I}_t^j\right)$, her desired gross holding of the risky asset conditional on the price, to a "Walrasian auctioneer" given her information set, $\mathcal{I}_t^i$ or $\mathcal{I}_t^j$. The auctioneer picks an equilibrium price which equals the total demand of agents to the random supply.[27] We denote the market clearing rule as

$$p_t = \tilde{p}_t(D_t) \tag{71}$$

[25] As it is explained in Kyle (1989), the limiting properties of the model depends on how the limit is exactly taken. In particular, whether the total precision of private information or the standard deviation of per capita noise in the economy is held constant. In our specification we choose the latter which ensures that our Nash equilibrium in demand schedules converges to the REE equilibrium.

[26] The independence of u_1 and u_2 implies that the additional supply in period 2, Δu_2 has to be perfectly negatively correlated with u_1. This is clearly a stark assumption, but leads to the simplest analysis. The model can be generalized to include any correlation structure across the noise terms. The main results are robust to this treatment.

[27] Just as Kyle (1989), we allow any convex-valued upper-hemicontinous correspondences mapping from prices to $[-\infty, \infty]$ as demand shedules and we require the same market clearing rules. These rules specify what the auctioneer should do in case of infinite orders for a given price and in case of multiple or non-existing finite market clearing price. Although these rules are part of the description of the game, they do not come into play in our equilibrium, so, to safe space we refer the reader to Kyle (1989) for the details.

where D_t is the vector of strategies of all traders active in period t.

In this Appendix, I consider only Case 1, where A-traders trade in period 1 and sell their portfolio to B traders in period 2 and consume the proceeds, while B traders liquidate their portfolio in period 3 for the fundamental value θ and consume the proceeds. In period 1, only A-traders trade. B-traders arrive and trade in period 2 only. Thus the utility of traders and the market clearing conditions are determined as follows.

Case 3 *Each A-trader solves*

$$\begin{aligned} & \max_{d_1^i\left(p_1,\mathcal{I}_1^i\right)} E\left[-e^{-\gamma W_A^i}|\mathcal{I}_1^i\right] \\ W_A^i &= d_1^i\left(p_1,\mathcal{I}_1^i\right)\left(p_2-p_1\right) \end{aligned} \tag{72}$$

and each B-trader solves

$$\begin{aligned} & \max_{d_2^i\left(p_2,\mathcal{I}_2^i\right)} E\left[-e^{-\gamma W_B^i}|\mathcal{I}_2^j\right] \\ W_B^j &= d_2^j\left(p_2,\mathcal{I}_2^j\right)\left(\theta-p_2\right). \end{aligned} \tag{73}$$

Components of the random supply, u_1 and u_2 are drawn independently from the distributions

$$u_1 \sim \boldsymbol{N}\left(0,\frac{N^2}{\delta_1^2}\right), u_2 \sim \boldsymbol{N}\left(0,\frac{N^2}{\delta_2^2}\right).$$

The information structure is defined in the main text in equations (1)-(5). The information sets of agents are

$$\begin{aligned} \mathcal{I}_1^i &= \left\{x^i, y\right\} \\ \mathcal{I}_1^j &= \left\{z^j, y\right\} \\ \mathcal{I}_2^j &= \left\{z^j, y, p_1\right\}. \end{aligned}$$

We are looking for a Perfect Bayesian Equilibrium in demand schedules defined as follows.

Definition 2 *A Perfect Bayesian Equilibrium in demand schedules is given by the strategy profiles D_1, D_2 in which individual strategies are best responses given the equilibrium strategies of all other players and expectations are formed according to Bayes' rule. That is, for any given i and j, and any realization of the information sets $\mathcal{I}_t^i$ or $\mathcal{I}_t^j$, $d_1^i\left(p_1,\mathcal{I}_1^i\right)$ and $d_2^j\left(p_2,\mathcal{I}_2^j\right)$ solve problems (72)-(73) subject to (71) , respectively.*

Note that in this equilibrium, each trader not only takes into account her direct impact on prices due to the market clearing mechanism, but also her indirect impact through the information content of prices.

C.1.1 Equilibrium

In the conjectured equilibrium strategies are

$$\begin{aligned} d_2^j &= \mathbf{b}_2 z^j + \mathbf{c}_2 y + \mathbf{g}_2 q_1 - \mathbf{e}_2 p_2 & (74) \\ d_1^i &= \mathbf{a}_1 x^i + \mathbf{c}_1 y - \mathbf{e}_1 p_1 & (75) \end{aligned}$$

for B and A traders, respectively, where q_1 is the price signal corresponding to period 1

$$q_1 \equiv \frac{\mathbf{e}_1 p_1 - \mathbf{c}_1 y}{\mathbf{a}_1} = \bar{x} - \frac{u_1}{\mathbf{a}_1 N} \tag{76}$$

with $\bar{x} \equiv \Sigma_i x^i$ and its conditional precision is

$$\tau_1^2 \equiv \frac{1}{var\left(q_1|\bar{x}\right)} = \frac{1}{\delta_1^2 \mathbf{a}_1^2}. \tag{77}$$

It is easy to see that p_1 and y are informationally equivalent to y and the price signal q_1. For the definition of q_1 I used the market clearing condition in the first period. Also, from the market clearing condition for the second period, we define the price signal q_2 as

$$q_2 \equiv \frac{\mathbf{e}_2 p - \mathbf{c}_2 y - \mathbf{g}_2 q_1}{\mathbf{b}_2} = \bar{x} - \frac{u_2}{\mathbf{b}_2 N}$$

with a conditional precision

$$\tau_2^2 \equiv \frac{1}{var\left(q_2|\bar{z}_2\right)} = \frac{1}{\delta_2^2 \mathbf{b}_2^2 N^2}. \tag{78}$$

Finally, we define b_2, c_2, e_2, g_2 and a_1, c_1, e_1as the linear coefficients of the conditional expectations

$$\begin{aligned} E\left(\theta|z^j, y, q_1, q_2\right) &= b_2 z^j + c_2 y + e_2 q_2 + g_2 q_1 & (79) \\ E\left(q_2|x^i, y, q_1\right) &= a_1 x^i + c_1 y + e_1 q_1 \end{aligned}$$

and

$$\tau_{\theta}^{2} \equiv \frac{1}{var\left(\theta|z^{j},y,q_{1},q_{2}\right)}$$
$$\tau_{q}^{2} \equiv \frac{1}{var\left(q_{1}|x^{i},y,q_{1}\right)}$$

as the corresponding precisions. Note that all the expectational coefficients and precisions are functions of the primitive parameters and the equilibrium values of τ_1, τ_2.

I will prove the following proposition.

Proposition 10 *1. For any $\hat{N} > 2$, there are $\hat{\delta}_1, \hat{\delta}_2$ thresholds that for every $N > \hat{N}, \delta_1 < \hat{\delta}_1$ and $\delta_2 < \hat{\delta}_2$ there is a symmetric linear equilibrium,where*

$$\mathbf{b}_2 = \tau_{\theta}^{2}\frac{b_2\left(N-2\right)-e_2}{\gamma\left(N-1\right)} \tag{80}$$

$$\mathbf{c}_2 = \tau_{\theta}^{2}\frac{\left(N-2\right)b_2-e_2}{\gamma\left(N-1\right)}\frac{c_2}{b_2+e_2} \tag{81}$$

$$\mathbf{e}_2 = \tau_{\theta}^{2}\frac{b_2\left(N-2\right)-e_2}{\gamma\left(N-1\right)\left(b_2+e_2\right)}. \tag{82}$$

$$\mathbf{g}_2 = \tau_{\theta}^{2}\frac{\left(N-2\right)b_2-e_2}{\gamma\left(N-1\right)}\frac{g_2}{e_2+b_2} \tag{83}$$

and

$$\mathbf{a}_1 = \frac{\tau_q^2}{\gamma}\frac{a_1}{\left(e_2+b_2\right)\left(1+\frac{\left(a_1+e_1\right)}{\left(\left(N-2\right)a_1-e_1\right)}\right)} \tag{84}$$

$$\mathbf{c}_1 = \frac{\tau_q^2}{\gamma}\frac{\left(\left(b_2+e_2\right)c_1+c_2\right)\frac{a_1}{\left(e_2+b_2\right)\left(b_1+e_1\right)+g_2}}{\left(e_2+b_2\right)\left(1+\frac{\left(a_1+e_1\right)}{\left(\left(N-2\right)a_1-e_1\right)}\right)} \tag{85}$$

$$\mathbf{e}_1 = \frac{\tau_q^2}{\gamma}\frac{\left(N-2\right)a_1-e_1}{\left(N-1\right)\left(e_2+b_2\right)\left(\left(e_2+b_2\right)\left(a_1+e_1\right)+g_2\right)} \tag{86}$$

Furthermore, all coefficients and equilibrium constants are calculated at $\tau_1 = \tau_1^*\left(N\right)$ and $\tau_2 = \tau_2^*\left(N\right)$ where τ_2^*, τ_1^* are the fixed point of

$$\delta_2\tau_{\theta}^{2}\frac{b_2\left(N-2\right)-e_2}{\gamma\left(N-1\right)} = \tau_2 \tag{87}$$

$$\delta_1\frac{\tau_q^2}{\gamma}\frac{a_1}{\left(e_2+b_2\right)\left(1+\frac{\left(a_1+e_1\right)}{\left(\left(N-2\right)a_1-e_1\right)}\right)} = \tau_1 \tag{88}$$

respectively.

2. *For any parameters, as $N \to \infty$, the equilibrium converges to a symmetric linear equilibrium where the equilibrium objects converges to their counterparts in Proposition 2 of the main text.*

First I derive the equilibrium objects for given τ_1 and τ_2.

The problem of B traders is very similar to the corresponding problem in Kyle (1989). The main difference is that now they use the price of the first period as as an additional, endogenous public signal on the fundamental value. From conjecture (21) and the market clearing condition, agent j faces the residual demand curve

$$p_2 = \tilde{p}_2 - \lambda_2 d_2^j$$

where

$$\lambda_2 \equiv \frac{1}{(N-1)\,\mathbf{e}_2}. \tag{89}$$

Thus, a B trader maximizes

$$\begin{aligned}\left(E\left(\theta|z^j, y, \tilde{p}_2, q_1\right) - \tilde{p}_2 - \lambda_2 d_2^j\right) d_2^j - \left(d_2^j\right)^2 \left(\frac{\gamma}{2\tau_\theta^2}\right) &= \\ = \left(E\left(\theta|z^j, y, \tilde{p}_2, q_1\right) - \tilde{p}_2\right)\left(d_2^j\right) - \left(d_2^j\right)^2 \left(\lambda_2 + \frac{\gamma}{2\tau_\theta^2}\right)\end{aligned}$$

which gives

$$\left(E\left(\theta|z^j, y, \tilde{p}_2, q_1\right) - \tilde{p}_2\right) - d_2^j \left(2\lambda_2 + \frac{\gamma}{\tau_\theta^2}\right) = 0$$

implying

$$d_2^j = \frac{E\left(\theta|z^j, y, \tilde{p}_2, q_1\right) - p_2 + \lambda_2 d_2^j}{2\lambda_2 + \frac{\gamma}{\tau_\theta^2}}$$

and

$$d_2^j = \frac{E\left(\theta|z^j, y, q_2, q_1\right) - p_2}{\lambda_2 + \frac{\gamma}{\tau_\theta^2}}. \tag{90}$$

Using (18) and the definition of q_2, rewrite this as

$$d_2^j = \frac{b_2 z^j + c_2 y + e_2 \frac{\mathbf{e}_2 p_2 - \mathbf{c}_2 y - \mathbf{g}_2 q_1}{\mathbf{b}_2} + g_2 q_1 - p_2}{\lambda_2 + \frac{\gamma}{\tau_\theta^2}}.$$

In a symmetric equilibrium, the coefficient of z^j, y, q_1 and p_2 has to be equal $\mathbf{b}_2, \mathbf{c}_2, \mathbf{g}_2$ and $\mathbf{e}_2$ respectively. Using also (89) gives (80)-(83).

The problem of each A trader is more subtle. Each A trader has to liquidate her assets before the fundamental value, θ, realizes. Thus, she has to form expectations about the price in period 2. As (90) shows for this she have to form expectations about the expectations of B traders. That is, second order expectations matter. Furthermore, when she forms expectations on p_2, she has to take into account the effect of her own trades on period 2 prices through period 1 prices which B traders use as signals.

In particular, market clearing in period 1 implies that given the conjectured strategies of B traders,A agents face the residual demand curve

$$p_1 = \tilde{p}_1 - \lambda_1 d_1^j$$

in period 1 where

$$\lambda_1 \equiv \frac{1}{(N-1)\,\mathbf{e}_1}.$$

Market clearing in period 2 implies that

$$p_2 = \frac{\mathbf{b}_2 \bar{z} + \mathbf{c}_2 y + \mathbf{g}_2 q_1 - \frac{u_2}{N}}{\mathbf{e}_2}.$$

By the definition of q_1 we can rewrite this as

$$p_2 = \tilde{p}_A - \lambda_A d_1^j \tag{91}$$

where

$$\begin{aligned} \lambda_A &\equiv \frac{\mathbf{g}_2}{\mathbf{e}_2}\frac{\mathbf{e}_1}{\mathbf{a}_1}\lambda_1 \\ \tilde{p}_A &\equiv \frac{\mathbf{b}_2 \bar{z} + \left(\mathbf{c}_2 - \mathbf{g}_2 \frac{\mathbf{c}_1}{\mathbf{a}_1}\right) y - \frac{u_2}{N} + \mathbf{g}_2 \frac{\mathbf{e}_1}{\mathbf{a}_1} \tilde{p}_1}{\mathbf{e}_2}. \end{aligned}$$

Thus, problem (72) is equivalent to maximizing

$$\begin{aligned} &\left(E\left(\tilde{p}_A | \mathcal{I}_1^j, p_1\right) - \lambda_A d_1^i - \left(\tilde{p}_1 - \lambda_1 d_1^i\right)\right) d_1^i - \left(d_1^i\right)^2 \left(\frac{\gamma}{2} var\left(p_2 | x^i, y, p_1\right)\right) = \\ &\qquad = \left(E\left(\tilde{p}_A | x^i, y, \tilde{p}_A\right) - \tilde{p}_1\right) d_1^i - \left(d_1^i\right)^2 \left(\left(\lambda_1 - \lambda_A\right) + \frac{\gamma}{2} var\left(p_2 | x^i, y, p_1\right)\right) \end{aligned}$$

with the second order condition

$$\left(\lambda_1 - \lambda_A\right) + \frac{\gamma}{2} var\left(p_2 | x^i, y, p_1\right) > 0.$$

From the first order condition of A traders this gives

$$d_1^i = \frac{E\left(\tilde{p}_A|\mathcal{I}_1^j, q_1\right) - \tilde{p}_1}{2\left(\lambda_1 - \lambda_A\right) + \gamma var\left(p_2|x^i, y, p_1\right)}$$

or, equivalently,

$$d_1^i = \frac{E\left(p_2|x^i, y, q_1\right) - p_1}{\lambda_1 - \lambda_A + \gamma var\left(p_2|x^i, y, q_1\right)} = \frac{E\left(\frac{1}{N}\sum_j E\left(\theta|z^j, y, q_2, q_1\right) - \left(\lambda_2 + \frac{\gamma}{\tau_\theta^2}\right)\frac{u_2}{N}|x^i, y, q_1\right) - p_1}{\lambda_1 - \lambda_A + \gamma var\left(p_2|x^i, y, q_1\right)}. \tag{92}$$

For the second equation, I used the market clearing condition in period 2 in order to emphasize the role of second-order expectations in A traders demand functions. Rewrite (75) as

$$d_1^i = \frac{\frac{\mathbf{b}_2}{\mathbf{e}_2}\left(a_1 x^i + e_1\left(\frac{\mathbf{e}_1}{\mathbf{a}_1}p_1 - y\frac{\mathbf{c}_1}{\mathbf{a}_1}\right) + c_1 y\right) + \frac{\mathbf{g}_2\left(\frac{\mathbf{e}_1}{\mathbf{a}_1}p_1 - y\frac{\mathbf{c}_1}{\mathbf{a}_1}\right) + \mathbf{c}_2 y}{\mathbf{e}_2} - p_1}{\lambda_1 - \lambda_A + \left(\frac{\mathbf{b}_2}{\mathbf{e}_2}\right)^2 \frac{\gamma}{\tau_q^2}.}$$

For an equilibrium, we have to find $\mathbf{a}_1, \mathbf{c}_1, \mathbf{e}_1$, such that (75) equals to (75) for any realizations of the random variables. Equating the coefficients in the two equations give (84)-(86). Finally, substituting (80) and (84) into (77) and (78) give (87) and (30).

For the existence and convergence to the REE equilibrium, first we have to find the coefficients $b_2, e_2, a_1, e_1, \tau_q^2, \tau_\theta^2$ by Projection Theorem and substitute in to (87) and (30). This procedure gives the following fixed point problem for any fixed N

$$\begin{aligned}
\tau_2 &= F_2\left(\tau_2, \tau_1\right) \\
\tau_1 &= F_1\left(\tau_2, \tau_1, Y\right) \\
Y &= F_Y\left(\tau_2, \tau_1\right)
\end{aligned}$$

where

$$F_2 = \frac{\delta_2(\kappa+\omega)N\alpha\left(N\alpha(N-2)-\tau_2^2(N-1)\right)\left(\kappa\tau_1^2+N\alpha\kappa+N\alpha\tau_1^2\right)}{\gamma(N-1)\left(\left(N^2\alpha^2(N-1)\left(\tau_1^2+\kappa+\omega\right)+N\alpha(N-1)\left(\kappa^2+2\kappa\tau_1^2+2\omega\kappa+2\omega\tau_1^2\right)+\kappa\tau_1^2(\kappa+2\omega)(N-1)\right)\tau_2^2+\left(N^3\alpha^2\left(\kappa^2+\alpha\kappa+\alpha\omega+2\kappa\omega+\alpha\tau_1^2+\kappa\tau_1^2+\omega\tau_1^2\right.\right.\right.}$$

and

$$\begin{aligned}
F_1 &= \delta_1 \frac{1}{\gamma} \frac{N^3\alpha^3\tau_2^2\left(\kappa^2 - \beta\omega\right)}{N\alpha\left(N\alpha + (N-1)\tau_2^2\right)\left(\kappa\tau_1^2 + N\alpha\kappa + N\alpha\tau_1^2\right)(\kappa+\omega)} Y \\
F_Y &= \frac{k_1}{k_2} \frac{N\alpha\frac{(N-2)}{(N-1)} - \tau_1^2}{N\alpha}
\end{aligned}$$

where

$$
\begin{aligned}
k_1 &= (J-1)\begin{pmatrix} (\kappa+J\alpha)\left(\kappa\beta+\kappa\omega+2\beta\omega+2J\alpha\kappa+J\alpha\beta+J\alpha\omega\right)\tau_1^2 + \\ J\alpha\left(\kappa^2\beta+\kappa^2\omega+2\kappa\beta\omega+J\alpha\kappa^2+J\alpha\kappa\beta+J\alpha\kappa\omega+J\alpha\beta\omega\right)\end{pmatrix}\tau_2^2 + \\
& J^2\alpha\begin{pmatrix} \begin{pmatrix} \alpha\kappa^2+\kappa^2\beta+\kappa^2\omega+\alpha\kappa\beta+\alpha\kappa\omega+\alpha\beta\omega+2\kappa\beta\omega+J\alpha\kappa^2+2J\alpha^2\kappa \\ +J\alpha^2\beta+J\alpha^2\omega+J\alpha\kappa\beta+J\alpha\kappa\omega+J\alpha\beta\omega \end{pmatrix}\tau_1^2 \\ +J\alpha\left(\alpha\kappa^2+\kappa^2\beta+\kappa^2\omega+\alpha\kappa\beta+\alpha\kappa\omega+\alpha\beta\omega+2\kappa\beta\omega\right)\end{pmatrix}
\end{aligned}
$$

and

$$
\begin{aligned}
k_2 &= \tau_1^2(J-1)\begin{pmatrix} (\kappa+J\alpha)\left(\kappa\beta+\kappa\omega+2\beta\omega+2J\alpha\kappa+J\alpha\beta+J\alpha\omega\right)\tau_2^2 \\ +J\alpha\left(\kappa^2\beta+\kappa^2\omega+2\kappa\beta\omega+J\alpha\kappa^2+J\alpha\kappa\beta+J\alpha\kappa\omega+J\alpha\beta\omega\right)\end{pmatrix} + \\
& J^2\alpha\begin{pmatrix} \alpha\kappa^2+\kappa^2\beta+\kappa^2\omega+\alpha\kappa\beta+\alpha\kappa\omega+\alpha\beta\omega+2\kappa\beta\omega+J\alpha\kappa^2+2J\alpha^2\kappa + \\ J\alpha^2\beta+J\alpha^2\omega+J\alpha\kappa\beta+J\alpha\kappa\omega+J\alpha\beta\omega \end{pmatrix}\tau_2^2 + \\
& J^3\alpha^2\left(\alpha\kappa^2+\kappa^2\beta+\kappa^2\omega+\alpha\kappa\beta+\alpha\kappa\omega+\alpha\beta\omega+2\kappa\beta\omega\right)
\end{aligned}
$$

Observe first, that $\tau_2 = F_2(\tau_2, \tau_1)$ gives a unique solution for any fixed τ_1 and this root is always positive. This is so, because it can be rewritten as a third-order polynomial in τ_2, with positive coefficients but negative intercept. It is easy to check that the root is finite for any $\tau_1 \in (-\infty, \infty)$. Second, $F_Y(\tau_1, \tau_2)$ is positive as long as $N\alpha\frac{(N-2)}{(N-1)} > \tau_1^2$. Finally, for any positive Y and τ_2, $\tau_1 = F_1(\tau_2, \tau_1, Y)$ gives a unique solution with the same sign as $(\kappa^2 - \beta\omega)$.

Then, the proof is constructed by the following procedure.

1. Let us fix a $\bar{N}^{(1)}$. Define $\tau_2^{\max}(N)$ as

$$
\begin{aligned}
\tau_2^{\max}(N) &= \max_{\tau_1\in(-\infty,\infty)} \tau_2 \\
s.t. F_2(\tau_2, \tau_1) &= \tau_2.
\end{aligned}
$$

   and define $\tau_2^{\max(1)} = \max_{N>\bar{N}^{(1)}} \tau_2^{\max}(N)$.

2. Consider $F_Y(\tau_1^2, \tau_2^2)$ for any τ_1^2, τ_2^2, for any $N > \bar{N}^{(1)}$, we define $Y^{\max}(N), Y^{\min}(N)$ as

$$
\begin{aligned}
Y^{\max}(N) &= \max_{\tau_2^2\in\left[0,\tau_2^{\max(1)}\right],\tau_1^2\in\left[0,N\alpha\frac{(N-2)}{(N-1)}\right]} F_Y(\tau_1, \tau_2) \\
Y^{\min}(N) &= \min_{\tau_2^2\in\left[0,\tau^{\max(1)}\right],\tau_1^2\in\left[0,N\alpha\frac{(N-2)}{(N-1)}\right]} F_Y(\tau_1, \tau_2)
\end{aligned}
$$

   also define

$$
Y^{\max(1)} = \max_{N>\bar{N}^{(1)}} Y^{\max}(N)
$$

3. Consider $F_1(\tau_1, \tau_2, Y) = \tau_1$ equation. Define $\tau_1^{\max}(N)$ as

$$\begin{aligned} \tau_1^{\max}(N) &= \max_{\tau_2 \in \left[0, \tau_2^{\max}\right], Y \in \left[Y^{\min(1)}, Y^{\max(1)}\right]} |\tau_1| \\ s.t. F_1(\tau_2, \tau_1, Y) &= \tau_1. \end{aligned}$$

Then define $\tau_1^{\max(1)} = \max_{N > \bar{N}^{(1)}} \left|\tau_1^{\max}(N)\right|.$

4. Check that whether the above steps give

$$\begin{aligned} \left(\tau_1^{\max(1)}\right)^2 &< \bar{N}^{(1)} \alpha \frac{\left(\bar{N}^{(1)} - 2\right)}{\left(\bar{N}^{(1)} - 1\right)} \\ \left(\tau_2^{\max(1)}\right)^2 &< \bar{N}^{(1)} \alpha \frac{\left(\bar{N}^{(1)} - 2\right)}{\left(\bar{N}^{(1)} - 1\right)}. \end{aligned}$$

If not, decrease δ_1 and δ_2 until the point that repeating steps 1-3 ensures that these conditions hold. As $\delta_1 \to 0$ implies $\tau_1 \to 0$ for any τ_2 and Y, and $\delta_2 \to 0$ implies $\tau_2 \to 0$ for any τ_1, for any $N^{(1)}$ there will be a $\hat{\delta}_1^{(1)}$ and $\hat{\delta}_2^{(1)}$ that this condition holds for any $\delta_1 < \hat{\delta}_1^{(1)}$ and $\delta_2 < \hat{\delta}_2^{(1)}$ and $N \geq N^{(1)}$.

5. Thus, we can define the compact space $\left[0, \tau_1^{\max(1)}\right] X \left[0, \tau_2^{\max(1)}\right] X \left[Y^{\min(1)}, Y^{\max(1)}\right]$ for $\kappa^2 > \beta\omega$ and $\left[\tau_1^{\max(1)}, 0\right] X \left[0, \tau_2^{\max(1)}\right] X \left[Y^{\min(1)}, Y^{\max(1)}\right]$ otherwise on which are fixed point problem is a continuos self-map for every $N > \bar{N}^{(1)}$, so we will have an equilibrium for each of these points. This proves existence with $\hat{\delta}_1 = \hat{\delta}_1^{(1)}$ and $\hat{\delta}_2 = \hat{\delta}_2^{(1)}$.

6. .Now define $\psi^{(1)} = \left[\tau_1^{(1)}, \tau_2^{(1)}, Y^{(1)}\right]$ as (one of) the fixed point of the system

$$\begin{aligned} \tau_2 &= F_2(\tau_2, \tau_1) \\ \tau_1 &= F_1(\tau_2, \tau_1, Y) \\ Y &= F_Y(\tau_2, \tau_1) \end{aligned}$$

for $N = \bar{N}^{(1)}$. Following the steps above, after choosing $\bar{N}^{(n+1)} = \bar{N}^{(n)} + 1$, we can construct a series of fixed points,$\psi^{(n)}$, compact spaces, $\tau_1^{\max(n)}, \tau_2^{\max(n)}, Y^{\min(n)}, Y^{\max(n)}$and thresholds $\bar{N}^{(n)}, \hat{\delta}_1^{(n)}, \hat{\delta}_2^{(n)}$. If there are more than one fixed point in step n, let us choose the one which is closest to the fixed point $[\tau_1^*, \tau_2^*, Y^*]$ given in Proposition 2 in the Euclidean sense. By construction, the compact space is non increasing, the thresholds $\hat{\delta}_1^{(n)}, \hat{\delta}_2^{(n)}$ are non-decreasing, and $\lim_{n\to\infty} \hat{\delta}_1^{(n)}, \hat{\delta}_2^{(n)} = \infty$. As all the coefficients in polynomials defined by $F_1 = \tau_1$ and $F_2 = \tau_2$ and the coefficients in polynomials defining Y are converging to their equivalents in the limit problem, the functions must also converge to the ones in the limit problem point by point. Thus, for every ε, there is a sufficiently large n that $\left|\psi^{(n)} - \psi^{(n-1)}\right| < \varepsilon$. Thus, it is a convergent series and its limit point must be $\psi^{(\infty)} = [\tau_1^*, \tau_2^*, Y^*]$.