# The delegated Lucas tree*

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**Abstract**

We analyze the effects of the observed increased share of delegated capital for trading strategies and equilibrium prices.

We introduce delegation into a standard Lucas exchange economy, where in equilibrium some investors trade on their own account, but others decide to delegate trading to professional fund managers. Flow-performance incentive functions describe how much capital clients provide to funds at each date as a function of past performance. Convex flow-performance relations imply that the average fund outperforms the market in recessions and underperforms in expansions. When the share of capital that is delegated is low, all funds follow the same strategy. However, when the equilibrium share of delegated capital is high funds with identical incentives employ heterogeneous trading strategies. A group of managers borrow to take on a levered position on the stock. Thus, fund returns are dispersed in the cross-section and the outstanding amounts of borrowing and lending increase. The relation between the share of delegated capital and the Sharpe ratio typically follows an inverse U-shape pattern.

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# 1 Introduction

Over the last 30 years there has been a gradual but profound change in the way money is invested in financial markets. While almost 50% of US equities were held directly in 1980, by 2007 this proportion decreased to around 20% (see French (2008)). What are the equilibrium implications of this shift? In particular, how does the increased presence of delegation affect trading strategies, and prices?[1]

To analyze the link between the incentives of financial institutions and asset prices, we introduce financial intermediaries into a Lucas exchange economy. Rather than study an optimal contracting problem, we rely on empirical regularities in flows and assume a convex relation between flows and performance relative to the market, as documented for example in Chevalier and Ellison (1997) Then we study how the risky and risk free asset are traded both by fund managers and by traders holding the assets directly (direct traders).

Based on the Jensen and Meckling (1976) risk-shifting argument that convex incentives induce gambling, naive intuition would suggest that in our model managers should leverage up, taking on more exposure to market risk than direct traders and, consequently, the presence of fund managers should lower the Sharpe ratio. Interestingly, this is not what we find. In equilibrium, the average manager has smaller exposure to market risk than direct traders. When the equilibrium share of delegated capital is small all managers follow the same strategy. However, when the equilibrium share is high a group of managers emerges that levers up, taking more exposure to the market risk than unity, and trading against the rest of the managers who hold a positive share of their capital in bonds. Thus, in equilibrium ex ante identical traders take positions against one another increasing open interest in leverage. Both the size of this latter group and the leverage of each member typically increases with the larger share of delegated capital. We connect this finding with the increased use of levered strategies and the large increase of the size of the repo market during the last decades before the financial crisis. Finally, the effect of delegation on the Sharpe ratio is non-monotonic. There is an inverse U-shape relation between the share of capital that is delegated and the Sharpe ratio of the market portfolio.

We study an exchange economy where the endowment process is represented by a Lucas tree paying a stochastic dividend each period. The dividend growth follows a two-state i.i.d. process with a larger chance for the high state. There are two financial assets: a stock which is a claim on the endowment process, and a riskless bond which is in zero net supply. The economy consists of two type of agents, both with log utility: investors and fund managers. Investors are the owners of the capital. Investors, arrive and die according to independent Poisson processes with constant intensity, while managers live forever. Newborn investors decide for life whether to be clients of managers or to trade directly in financial markets. Trading directly imposes on investors a utility

[1] See, for example, the presidential address of Allen (2001) for an elaborate discussion on the importance of the role of financial intermediaries.

cost. This utility cost represents the cost of acquiring the knowledge to understand how capital markets work, as well as the utility cost imposed by making regular time consuming investment decisions. Investors can avoid this cost by becoming a client and delegating the determination of their portfolio to a fund. However, when they delegate they need to pay fund managers a fee each period that is determined by the fund. The fee is consumed by the fund manager.

Clients' allocate capital to funds to manage each period depending on funds' past relative performance, where the relation between last period's return compared to the market and new capital flow is described by each manager's *incentive function*. We interpret the incentive function as a short-cut for an unmodeled learning process by clients on managers' talent. Its empirical counterpart is the flow-performance relation. We are agnostic as to whether the learning process is rational or not.[2]

We approximate the convex relation between flows and excess returns by a function which is piece-wise linear in logs. The combination of log utility with incentive functions of this particular functional form is the key methodological contribution that allows us to derive analytical formulas for the trading pattern and asset prices under various incentive functions. This combination results in a locally concave, but globally non-concave portfolio problem for managers. The first property keeps the framework tractable, while the second property ensures that we do not lose the general insight connected to convex incentives.

We present a stationary equilibrium where the equilibrium share of delegated capital is constant. In this equilibrium, most objects are given by simple, closed form expressions. As the main focus of this paper is the effect of the increasing share of delegated capital for equilibrium strategies and prices, we construct a range of economies as follows. We fix all other parameters and vary only the cost of direct trading, in a way that the equilibrium share of delegated capital varies along the full range of $(0,1)$. Then we compare strategies and prices across these economies.

We show that convex incentives lead the average fund to choose a smaller than one market-beta: implying that consistent with evidence in Moskowitz (2000), Kosowski (2006), Lynch and Wachter (2007), Kacperczyk, Van Nieuweburgh and Veldkamp (2010), and Glode (2010) the average fund overperforms the market in recessions and underperforms in expansions. To understand the intuition behind this result, consider the case when financial markets are only populated by direct traders and the first fund manager enters. She can decide whether to take a sufficiently contrarian position to overperform and get high capital flows in the recession, or to take a sufficiently levered position to get high capital flows in the expansion. Recall that the probability of an expansion is larger than the probability of a recession. Thus, the relative overperformance implied by her optimal contrarian position must be larger than the one implied by her optimal levered position because the earlier realizes in the low probability state. Because convex flows reward overperformance

[2] For example Berk and Green (2004) provides microfoundation for convex flow-performance relationship in a setting with incomplete information about fund managers' talents.

disproportionally, she values this larger overperformance and picks the contrarian position.

While the average fund always overperforms in recessions, the cross sectional distribution of fund returns depends on the equilibrium share of delegation in the economy. At low levels of delegation, all funds choose the same portfolio. However, as the share of delegation increases, there is a threshold above which fund managers follow heterogeneous strategies, even though funds are identical ex ante and all have the same incentive function. In particular, above the threshold as the share of delegation increases a group of decreasing size still follows a "contrarian strategy" of smaller than one market beta, while a group with increasing size follows a leveraged strategy by borrowing up and investing more than 100% of their assets under management in the stock. This is a consequence of the interaction of the shape of the flow-performance relationship and the larger share of total delegation. The idea is that when the market is dominated by fund managers, if each manager followed the same strategy, they could not beat the market in any states of the world. Thus, they could not profit from the convexity of the flow-performance relationship. Instead in equilibrium, the group of managers who leverage up beat the market and receive large capital inflow in the high state, while the other group beats the market and obtain large capital flows in the low state. Thus, there are gains from trade. The size of these two groups are determined in equilibrium so that prices make each manager indifferent between the two strategies.

Accounting for the fact that over the last three decades the share of delegation has increased considerably (Allen (2001) and French (2008)), this result is consistent with observations on the increased use of leveraged strategies across financial intermediaries in the last two decades before the 2007/2008 financial crisis.[3] Relatedly, a central contribution of the paper is to link increases in delegated portfolio management to increased amounts of borrowing and lending in equilibrium. As to lever up, managers have to borrow from the rest of the agents, the equilibrium is consistent with the observed large increase in the size of the repo market in the last decades before the 2008 financial crisis (Gorton-Metrick (2010)). Consistent with evidence in Kacperczyk, Van Nieuweburgh and Veldkamp (2010) on the return dispersion of mutual funds, we also show that the implied cross-sectional dispersion in returns among managers is typically larger in recessions than in booms.

We show that typically the Sharpe ratio follows an inverted U-pattern as the share of delegation increases. This is the outcome of the change in relative strength of two effects. First, an extra unit of return is appreciated more when it increases capital flows through the incentive function for the marginal agent. We call this the capital-flow effect. Second, an extra unit of return is appreciated more when the wealth of the marginal agent is lower. This is the standard wealth effect. The first effect is increasing in the share of delegation in the region when each fund follows the same contrarian strategy and is constant in the region with heterogeneous strategies. The second effect changes little in the first region and typically decreases in the second region. Thus, the capital

[3]See Adrian and Shin (2008) on the leverage in investment banks and Lo and Patel (2007) on the increased role of leveraged mutual funds and leveraged ETFs.

flow effect dominates for low levels of delegation and the wealth effect dominates for large levels of delegation implying the result.

Using parameters implied by the data, we calculate a simple numerical example to investigate the magnitude of these effects. Because of the structure of our model, we can directly compare our results to the ones implied by the standard Lucas economy. We find that even small convexity leads to large effect on managers' strategies. Relatedly, the increasing share of delegated capital radically increases the lending and borrowing activity. Furthermore, under reasonable parameter values for the incentive function, delegation has the potential to significantly increase the Sharpe ratio relative to the case without delegation: in the example the Sharpe ratio is up to 2.5 times higher than in the Lucas economy.

To our knowledge, our paper is the first to study the effect of the interaction between the increasing share of delegated capital and nonconcave incentives on fund managers strategies and implied asset prices. We are also the first to show that although this interaction is consistent with a smaller-than-1 beta portfolio for the average manager, it also leads to levered portfolios for a small group of increasing size. Still, our paper is related to at least three main branches of the literature. First, it is related to papers that study the effects of delegated portfolio management on asset prices (e.g. Shleifer and Vishny (1997), Vayanos (2003), Cuoco and Kaniel (2010), Dasgupta and Prat(2006)(2008), Guerrieri and Kondor(2010), Vayanos and Woolley(2008), Malliaris and Yan (2010)). Both the used framework and the focus of all these papers differ significantly from ours. Among many others, studied questions in this literature include the effect of delegation on limited arbitrage, on trading volume on price discovery, on procyclicality in premiums and on momentum. The closest to our exercise is He and Krishnamurthy (2008) who also studies the effect of delegation in a standard Lucas economy. However, in He and Krishnamurthy (2008) managers are not directly motivated by flows because they do not receive fees after their capital under management. Its main focus is on the amplification of bad shocks through the incentive constraint of managers.

Second, starting with the seminal paper of Jensen and Meckling (1976), there is a large literature on the effect on nonconcave objectives on fund managers strategies either by taking incentives as given (e.g. Dow and Gorton (1997), Basak, Pavlova and Shapiro (2007), Basak and Makarov (2010), Carpenter (2000), Ross (2004)) or by deriving them endogenously (Biais and Casamatta (2000), Cadenillas et al. (2007), Diamond (2001), Hellwig (2009), Ou-Yang (2003) and Palomino and Prat (2003), Makarov and Plantin (2010)). The starting point that nonconcave incentives induce gambling is the connection between our paper and this literature. While, the first group of papers focuses on the optimal portfolios for given prices, the second group focuses on optimal contracts to avoid risk-shifting. In contrast, we focus on the interaction of prices and portfolios under fixed contracts.

Third, our framework is also related to the literature on consumption based asset pricing with heterogeneous risk aversion (e.g. Dumas (1989), Wang (1996), Chan and Kogan (2002), Bhamra and Uppal (2007), Longstaff and Wang (2008)). Unlike in our work, in these papers identical agents follow identical strategies, less risk-averse agents always borrow from more risk averse agents which, typically, decreases the price of risk. This is true even when utility depends on consumption relative to others such as in Chan and Kogan (2002). The main reason for the different results is that this literature does not allow for convexities in incentives.

The structure of the paper is as follows. In the next section we present the general model. We discuss the general set up, our equilibrium concept and the main properties of the equilibrium. In Section 3, we present and discuss the derived implications. In Section 4, we present a simple calibrated example. Finally, we conclude.

# 2 The general model

In this section, we introduce professional fund managers into a standard Lucas exchange economy. Our main focus is effect of the increasing share of delegated asset management on the equilibrium strategies and asset prices. In what follows, we introduce our framework, define our equilibrium concept and present sufficient conditions for the existence of such an equilibrium and its basic properties.

## 2.1 The Economy

We consider a discrete-time, infinite-horizon exchange economy with complete financial markets and a single perishable consumption good. There is only one source of uncertainty and participants trade in financial securities to share risk.

The aggregate endowment process is described by the binomial tree

$$\delta_{t+1} = y_t \delta_t$$

where the growth process $y_t$ has two i.i.d. states: $s_t = H, L$. The dividend growth is either high $y_H$ or low $y_L$, with $y_H > y_L$. The probability of the high and the low states are $p > \frac{1}{2}$ and $1 - p$ respectively.[4] Investment opportunities are represented by a one period riskless bond and a risky stock. The riskless bond is in zero net supply. The stock is a claim to the dividend stream $\delta_t$ and is in unit supply. The price of the stock and the interest rate on the bond are $q_t$ and $R_t$ respectively. The return on a portfolio with portfolio weights of $\alpha$ in the stock and $1 - \alpha$ in the risk free bond is denoted by

$$\rho_{t+1}(\alpha) \equiv \alpha \left( \frac{q_{t+1} + \delta_{t+1}}{q_t} - R_t \right) + R_t. \tag{1}$$

[4]We focus on $p > \frac{1}{2}$ because the consumption growth process is negatively skewed empirically.

The economy is populated by investors and fund managers[5]. Investors own the stock, but, initially, only fund managers know how to trade assets.The mass of each group is normalized to one, all agents derive utility from inter-temporal consumption, and have *log* utility. At the beginning of each period $1-\lambda$ fraction of investors die and the same fraction is born. We assume that the aggregate capital of those who died is inherited by newborn investors in equal shares. Each living investor in any given period belongs to one of three groups: *Newborn investors* $(I)$, *direct traders* $(D)$ and *clients* $(C)$. Newborn investors can choose whether to trade directly, or delegate their trading decisions to fund managers $(M)$. This decision is made once at birth and is irreversible. Trading directly imposes a one time utility cost, $f$, on investors but gives them the free choice over their consumption and portfolio decisions in every subsequent period. We think of $f$ as the cost of acquiring the knowledge to understand how capital markets work. If they choose to trade directly, they belong to group of direct traders in all subsequent periods. If they choose to delegate, they will be assigned to a particular manager $(m \in M)$ randomly and for life. In this case they belong to the group of clients in all subsequent periods. A client doesn't suffer the utility cost she would bear if she traded directly, but gives up the flexibility to determine her consumption and stock to bond mix. As we will explain, her consumption-investment choice depends on the past performance of managers and is determined by an exogenously specified flow-performance relationship, while her portfolio is chosen by her fund manager for a fee.[6] Note that although there are four groups of agents in this economy: newly born investors, clients, direct traders and fund managers, financial assets are traded only by two of these groups: fund managers and direct traders. In what follows, we first describe the problem of each of the four groups in detail then present our specification for the flow-performance relationship. We write each problem in recursive form.

We conjecture and later verify that we have to keep track of only two state variables to fully describe the aggregate state of the economy in period $t$. The first is the dividend shock realized at the end of the last period, $s_t = H, L$, while the second is the share of aggregate investment of managers compared to total investment at the beginning of last period

$$\Omega_{t-1} \equiv \frac{\int_{m\in M}\left[w_{t-1}(m) - c_{t-1}(m)\right]dm}{\int_{i\in D}\left[w_{t-1}(i) - c_{t-1}(i)\right]di + \int_{m\in M}\left[w_{t-1}(m) - c_{t-1}(m)\right]dm}$$

where $c_{t-1}(m)$, $w_{t-1}(m)$ are the consumption and assets under management of a particular man-

[5]Conceptually, we think of fund managers as a group representing all type of institutional traders who actively participate in the equity market. That is, actively managed mutual funds, hedge funds, proprietary trading desks of investment banks, pension funds, etc. Still, when we confront our findings to empirical work, we often have to rely on observations about mutual funds only as the majority of empirical results are on this segment of the sector. Presumably, this is so because of data availability.

[6]It is apparent that in our model investors not "paying" the utility cost, $f$, delegate their trading decision by assumption: both trading and producing fruit from the tree requires a degree of sophistication that is obtained by bearing the utility costs $f$. This precludes clients from holding the tree passively, since their lack of sophistication implies that if they hold it passively it will not generate any fruit. A similar assumption is made in He and Krishnamurthy (2008).

ager $m \in M$, and $c_{t-1}(i)$, $w_{t-1}(i)$ are the consumption and wealth level of a particular direct trader investor $i \in D$. With slight abuse of notation when we refer to a general direct trader or a general manager, we write $w_{t-1}^D$ instead of $w_{t-1}(i)$, $i \in D$ and $w_{t-1}^M$ instead of $w_{t-1}(m)$, $m \in M$. We follow the same convention for all variables. We refer to $\Omega_{t-1}$ as the share of delegated capital.

*Fund Managers.* In period, $t$, each manager with assets under management $w_t^M$ chooses the fraction $\psi_t^M$ she will receive as a fee. We assume the manager must consume her fee $\psi_t^M w_t^M$.[7] She then invests the remaining $(1-\psi_t^M)w_t^M$ in a portfolio with $\alpha_t^M$ share in the stock and $\left(1-\alpha_t^M\right)$ share in the bond. Her value function is given by

$$V^M\left(w_t^M, s_t, \Omega_{t-1}\right) = \max_{\psi_t^M, \alpha_t^M} \ln \psi_t^M w_t^M + \beta E\left(V^M\left(w_{t+1}^M, s_{t+1}, \Omega_t\right)\right) \tag{2}$$

$$s.t.\ w_{t+1}^M = \Gamma_t g\left(\upsilon_{t+1}^M\right) w_{t+1,-}^M \tag{3}$$

$$w_{t+1,-}^M \equiv \rho_{t+1}\left(\alpha_t^M\right)\left(1-\psi_t^M\right) w_t^M \tag{4}$$

Note that assets under management at beginning of a period, $w_{t+1}^M$, are proportional to assets under management at the end of the previous period , $w_{t+1,-}^M$. This proportion depends on three quantities. First, the share of wealth each existing client delegates to the manager which depends on the past realized performance of this manager and given by $g(\upsilon_{t+1}^M)$ where

$$\upsilon_{t+1}^M \equiv \frac{\rho_{t+1}\left(\alpha_t^M\right)}{\frac{q_{t+1}+\delta_{t+1}}{q_t}} \tag{5}$$

is a fund's return relative to the market portfolio.[8] We specify the shape of this function below. Second, the total wealth of a manager's existing clients. Third, the total wealth of the fraction of newborn investors who decide to be clients and who are assigned to this particular manager. The second and third elements are combined into $\Gamma_t$, a state dependent scaling factor that is endogenously determined in equilibrium and which the manager takes as given. For simplicity, we refer to this variable as the size of the client-base. It impacts all funds similarly, and depends positively on the overall capital of clients in that state.

If more then one portfolio $\alpha_t^M$ solves (2)-(4), we will allow managers to mix between these portfolios. This will be useful, as sometimes the equilibrium portfolio profile requires a subset of

[7] The assumption that managers cannot invest their fees is a major simplification allowing us to not keep track of fund managers private wealth. Note also that on one hand, we are allowing $\psi_t$ to be conditional on any variable in the managers' information set in $t$. That is, we do not constrain our attention to proportional fees ex ante. On the other hand, our assumptions imply that fees are proportional in equilibrium, managers effectively maximize capital under management and fees do not play any role in the portfolio decision.

[8] In a previous version, we consider the possibility to allow the incentive function $g(\cdot)$ to depend non linearly on the fees $\psi_t^M$ charged by the fund, but this change has very little effects on the result. Thus, we omit this treatment here.

managers to follow a different strategy than other managers, and we implement this by allowing mixed strategies.

*Clients.* The utility going forward of an investor that decided to be a client, was matched with a particular manager, and has time $t$ wealth of $w_t$ is

$$\begin{aligned} V^C\left(w_t, \upsilon_t^M, s_t, \Omega_{t-1}\right) &= \ln w_t^C\left(1-g\left(\upsilon_t^M\right)\right) \\ &+\beta^I E V^C\left(\rho_{t+1}\left(\alpha_t^M\right)\left(1-\psi_t^M\right) g\left(\upsilon_t^M\right) w_t^C, \upsilon_{t+1}^M, s_{t+1}, \Omega_t\right) \end{aligned} \tag{6}$$

where $\beta^I \equiv \lambda\beta$ is the effective discount factor of investors, and $\alpha_t^M$ and $\upsilon_t^M$ are chosen portfolio and the relative return of the assigned manager in period $t$. Note that if the manager follows a mixed strategy than both $\alpha_{t+1}^M$ and $\upsilon_{t+1}^M$ are random variables from the client's point of view. Instead of deriving the incentive function, $g(\cdot)$ from first principles, we take it exogenously in the spirit of Shleifer and Vishny (1997). Below, we motivate the form of this function by empirical observations. We think of this function as a reduced form description how a client matched to the manager decides how much she"trusts" the manager's abilities to outperform the market in the next period based on past performance.

*Direct traders.* Direct traders solve a standard asset allocation problem. Denoting by $\psi_t^D$ the optimal fraction of time $t$ wealth $w_t$ a direct investor consumes, we have

$$V^D\left(w_t, s_t, \Omega_{t-1}\right) = \max_{\psi_t^D, \alpha_t^D} \ln \psi_t^D w_t^D + \beta^I E V^D\left(w_{t+1}, s_{t+1}, \Omega_t\right) \tag{7}$$

$$s.t. \;\; w_{t+1}^D \equiv \rho_{t+1}\left(\alpha_t^D\right)\left(1-\psi_t^D\right) w_t^D$$

*Newborn investors.* The expected lifetime utility of a newborn investor entering in period $t$ with wealth $w_t$ is given by

$$\begin{aligned} V^I\left(w_t, s_t, \Omega_{t-1}\right) &= \max_{\chi\in\{0,1\}, \psi_t^I, \alpha_t^I} \ln w_t^I \psi_t^I \\ &\quad +\chi\beta^I E V^C\left(\rho_{t+1}\left(\alpha_t^M\right)\left(1-\psi_t^M\right) w_t^I\left(1-\psi_t^I\right), s_{t+1}, \Omega_t\right) \\ &\quad +(1-\chi)\beta^I\left(E V^D\left(\rho_{t+1}\left(\alpha_t^I\right) w_t^I\left(1-\psi_t^I\right), s_{t+1}, \Omega_t\right)-f\right). \end{aligned}$$

where $\chi$ is her decision whether to be a client or a direct trader, $\psi_t^I$ is her consumption share, $\alpha_t^I$ is her first portfolio decision given that she chooses to be a direct trader.

*Relative Performance Incentive Functions.* Our key assumption is to model clients' share of delegated capital by a reduced form incentive function. The empirical counterpart of the incentive function is the flow-performance relationship. The incentive function $g(\cdot)$ describes how existing clients respond to the performance of a given manager. We assume it belongs to the following piece-wise constant elasticity class:[9]

$$g(\upsilon) \equiv \begin{cases} Z_B\, \upsilon^{n_B - 1} & if \quad \upsilon < \kappa \\ Z_A\, \upsilon^{n_A - 1} & if \quad \upsilon \geq \kappa \end{cases}. \tag{8}$$

The function is parameterized by the kink $\kappa \geq 1$, the scalers $Z_A, Z_B > 0$ and the elasticity parameters, $n_A \geq n_B > 1$. The subscripts refer to the cases when the relative return is above $(A)$ the kink, so managers are compensated at the higher-elasticity segment of the incentive function, and when the relative return is below $(B)$ the kink, so managers are compensated at the low-elasticity segment of the incentive function. We assume that the $g$ is continuous by imposing the restriction

$$Z_A = Z_B \kappa^{n_B - n_A}.$$

For a more intuitive form, using (4) and (8) we have

$$\ln \frac{w_{t+1}^M}{w_{t+1,-}^M} = \ln \frac{w_{t+1}^M}{\rho_{t+1}\left(\alpha_t^M\right)\left(1 - \psi_t^M\right) w_t^M} = \tag{9}$$

$$= \ln \Gamma_t Z_B + 1_{\upsilon_t \geq \kappa} \ln \kappa^{n_B - n_A} + \left[(n_B - 1)\, 1_{\upsilon_t < \kappa} + (n_A - 1)\, 1_{\upsilon_t \geq \kappa}\right] \left( \ln \rho_{t+1}\left(\alpha_t^M\right) - \ln \frac{q_{t+1} + \delta_{t+1}}{q_t} \right).$$

By choosing the appropriate parameters, this specification is a piecewise linear approximation of any convex relationship between log of capital flows and log of excess returns of funds. This is consistent with the well documented empirical convex flow-performance relation for a wide range of financial intermediaries.[10] We chose this particular approximation, because it both keeps our model analytically tractable and consistent with empirical specifications.[11]

[9] Allowing the incentive function to be a combination of more than two segments does not pose any conceptual difficulty for our method. However, as it does not add to the economic intuition either, we omit this treatment.

[10] There is a large empirical literature exploring the relationship between past performance and future fund flows. With the notable exception of Grossman, Ingressol and Ross (2002), most papers find a positive relationship for varies type of financial intermediaries. Also, Chevalier and Ellison (1997), Sirri and Tufano (1998), and Chen et al. (2003) find that the relationship is convex for mutual funds, while Agarwal, Daniel and Naik (2003) finds similar convexity for hedge funds. Kaplan and Schoar (2004) finds a positive but concave relationship for private equity partnerships.

Anecdotal evidence suggests that the capital at the disposal of top traders at investment banks and hedge funds also increases significantly as response to their stellar performance (e.g. WSJ 09/06/06 A1 on Brian Hunter of Amaranth, and WSJ 02/06/09 A1 on Boaz Weinstein of Deutsche Bank). This should lead to similar incentives to our specification.

[11] In Section 4, we estimate the parameters of 9 on a sample of mutual fund flows and returns.

Finally, we conjecture and later verify that the value functions of different types take the form

$$V^C\left(w_t^C, \upsilon_t^M, s_t, \Omega_{t-1}\right) = \frac{1}{1-\beta^I}\ln w_t^C + \Lambda^C\left(\upsilon_t^M, s_t, \Omega_{t-1}\right) \tag{10}$$

$$V^D\left(w_t^D, s_t, \Omega_{t-1}\right) = \frac{1}{1-\beta^I}\ln w_t^D + \Lambda^D\left(s_t, \Omega_{t-1}\right) \tag{11}$$

$$V^M\left(w_t^M, s_t, \Omega_{t-1}\right) = \frac{1}{1-\beta}\ln w_t^M + \Lambda^M\left(s_t, \Omega_{t-1}\right). \tag{12}$$

## 2.2 The equilibrium

In this part, we show that under weak parameter restrictions, we can always find a competitive equilibrium where the share of delegated capital is constant over time, $\Omega_t = \Omega^*$. More formally, we are looking for a stationary competitive equilibrium defined as below.

**Definition 1** *An $\Omega^*$ **equilibrium** is a price process $q_t$ for the stock and $R_t$ for the bond, a relative investment by fund managers compared to all investment $\Omega^*$, consumption and strategy profiles for newborn investors, direct investors, and managers such that*

1. *given the equilibrium prices*
   - *the initial consumption choice of newborn investors $\psi_t^I$ and the decision on whether to become a direct trader or a client are optimal for each newborn investor,*
   - *fee choice $\psi_t^M$ and trading strategies $\mathcal{A}_t$, $\mathcal{M}_t$ are optimal for each manager,*
   - *consumption choices $\psi_t^D$ and trading strategies $\alpha_t^D$ are optimal for direct traders,*
2. *prices $q_t$, and $R_t$ clear both good and asset markets,*
3. *the relative investment by fund managers compared to all investment is constant overtime at the level $\Omega_t = \Omega^*$.*

As the main focus of this paper is the interaction between the increasing share of delegation and the effect of a convex flow-performance relation to equilibrium strategies and prices, we construct a range of economies as follows. We fix all other parameters and change only $f$, the cost of trading directly, in a way that the implied equilibrium implies a different share of delegation, $\Omega^*$, for each economy. Then we compare strategies and prices across these economies.[12] The following result makes sure that this method works for any $\Omega^*$.

[12] Formally, $\Omega^*$ is an equilibrium variable depending on $f$. Thus, we should define a function which gives an $f$ for every $\Omega^*$. Then, to analyze the effect of increasing $\Omega^*$, we should change $f$ along the values of this function. Instead, to keep things simple, we analyze "comparative statics" with respect to $\Omega^*$ allowing $f$ to adjust in the background accordingly.

**Proposition 1** *For any set of other model parameters there is a* $\hat{Z}$, $\hat{\lambda}$, *and an interval* $\left[\underline{f}, \bar{f}\right]$ *such that if* $Z_B < \hat{Z}, \lambda \leq \hat{\lambda}$ *then*

1. *For any* $f \in \left[\underline{f}, \bar{f}\right]$ *there exists an* $\Omega^*$ *equilibrium for some* $\Omega^* \in (0, 1)$,
2. *for any* $\Omega^* \in (0, 1)$ *there is a corresponding* $f \in \left[\underline{f}, \bar{f}\right]$ *that with that choice there is an* $\Omega^*$ *equilibrium.*

Before highlighting the most critical steps of the proof, we discuss the methodology of equilibrium construction. The key is how to deal with the convex flow-performance relation. Convexity in incentives imply that our problem is globally non-concave, so that local conditions for the equilibrium will not be sufficient. However, the interaction of log utility and a piecewise constant-elasticity incentive function imply that the problem of the manager is locally concave almost everywhere in the portfolio choice $\alpha$, even though it is globally non-concave. The dashed line on Figure 1 demonstrates this by depicting the expected utility of a manager for various $\alpha$s in a particular case when all other traders hold the market. It is apparent that the curve can be divided to three segments in a way that the curve is concave within each of these segments. Portfolios in a given segment differ from portfolios in other segments in which dividend state, if at all, the manager receives the extra capital flows implied by the high elasticity segment of her incentive function. In particular, *Contrarian portfolios* have smaller than unity exposure to market risk; overperforming the market in the low state. This overperformance in the low state is sufficiently high to generate the extra capital flows implied by the high elasticity segment of the incentive function. *Moderate portfolios* are close to the market portfolio, they generate moderate over- or under-performance, and thus do not generate extra capital flows in any state. *Aggressive portfolios* have larger than unity exposure to market risk; overperforming the market in the high state. This overperformance is sufficiently high to generate the extra capital flows in the high state. Because of local-concavity, within each of these segments there is a single optimal portfolio. Consequently, for a given set of prices managers effectively compare three possible strategies: the locally optimal contrarian, moderate and aggressive portfolios. The relative ranking of these three choices depend on equilibrium prices.

Our treatment of convex incentive functions helps to reduce the construction of an $\Omega^*$ equilibrium to the following steps.

1. We fix a given $\Omega^*$ and conjecture an equilibrium profile of portfolios for managers and direct traders. In this profile each portfolio is one of the three types of locally optimal portfolios. We verify the conjecture by showing that the profile is indeed globally optimal under the set of relative prices consistent with this profile. Sometimes in equilibrium a group of managers have to hold a different portfolio than other managers. We implement such asymmetries by allowing managers to mix between portfolios. Importantly, the equilibrium strategy profile is independent from the utility cost $f$ and the client base $\Gamma_t$.

2. By calculating the values of a client and a direct trader under the equilibrium strategies, we find the utility cost $f$ of trading directly which implies that each newborn investor is indifferent whether to be a client or a direct trader. Thus, any fraction of newborn investors choosing to be clients is consistent with the equilibrium strategies and prices for this $f$.

3. We pick the fractions of newborn investors choosing to be clients in a way that the implied total client base $\Gamma_t$ gives exactly $\Omega^*$ as the share of delegated capital. This has to be true regardless of the dividend state, $s_t = H, L$. We show that this implies that the client-base $\Gamma_t = \Gamma_H, \Gamma_L$ depends only on the dividend state.

4. Finally, we calculate the equilibrium price of the assets implied by the consumption and portfolio decisions of each group of agents.

In the rest of this section, we characterize the $\Omega^*$ equilibrium by following the structure provided by the above steps. We show that our method gives simple analytical expressions for most equilibrium objects.

### 2.2.1 Equilibrium portfolios

We start by finding the optimal consumption and portfolio decisions of direct traders and managers for fixed prices. The case of direct traders is standard. Given their log utility, the optimal consumption share is

$$\psi_t^D = \left(1 - \beta^I\right),$$

while the optimal share in stocks is given by the first order condition

$$p \frac{\frac{q_{t+1}(H)+\delta_{t+1}(H)}{q_t} - R_t}{\alpha_t^D \left(\frac{q_{t+1}(H)+\delta_{t+1}(H)}{q_t} - R_t\right) + R_t} = (1-p) \frac{R_t - \frac{q_{t+1}(L)+\delta_{t+1}(L)}{q_t}}{\alpha_t^D \left(\frac{q_{t+1}(L)+\delta_{t+1}(L)}{q_t} - R_t\right) + R_t}. \tag{13}$$

Consider the decision problem of a manager in period $t$. First we want to find the locally optimal contrarian/aggressive/moderate portfolios. For now, conjecture that the locally optimal portfolios are in the interior of the corresponding segments, just as it is in the case depicted in Figure 1. The corresponding optimization problems, given (2) and conjecture (12), are given by

$$\begin{aligned}
&\max_{\alpha_t^M,\psi_t^M} \ln \psi^M w_t^M + \\
&+\frac{\beta}{1-\beta} p \ln \Gamma_t Z_h \left(\frac{\rho_{t+1}\left(\alpha_t^M, H\right)}{\frac{q_{t+1}(H)+\delta_{t+1}(H)}{q_t}}\right)^{n_h} \frac{q_t}{q_{t+1}(H)+\delta_{t+1}(H)} \beta w_t^M + \\
&+\frac{\beta}{1-\beta}(1-p) \ln \Gamma_t Z_l \left(\frac{\rho_{t+1}\left(\alpha_t^M, L\right)}{\frac{q_{t+1}(L)+\delta_{t+1}(L)}{q_t}}\right)^{n_l} \frac{q_t}{q_{t+1}(H)+\delta_{t+1}(H)} \beta w_t^M \\
&+\beta\left(p \Lambda\left(H, \Omega_t\right)+(1-p) \Lambda\left(L, \Omega_t\right)\right),
\end{aligned} \tag{14}$$

where the indices $l, h = A, B$ refer to whether for the given exercise the performance relative to the market has to be above ($A$) or below ($B$) the kink in the low state ($l$) and the high state ($h$), respectively. For the problem of searching for the locally optimal contrarian portfolio $l = A$ and $h = B$, as by definition a contrarian portfolio performs above the market in the low state and below the market in the high state. Similarly, for the locally optimal aggressive portfolio, $l = B$ and $h = A$, while for the moderate portfolio, $l = h = B$. Note that the logarithmic terms are the new levels of assets under management, $\ln w_{t+1}^M$, in each state, $s_{t+1} = H, L$.

It is easy to see the optimal fees are a constant proportion of capital under management,

$$\psi_t^M = (1-\beta).$$

It is apparent that the decision on fees is independent from the portfolio decision. Also, $\alpha_t^M$ is the solution of

$$\begin{aligned}
& p\left(\frac{n_h}{p n_h+(1-p) n_l}\right) \frac{\frac{q_{t+1}(H)+\delta_{t+1}(H)}{q_t}-R_t}{\alpha_t^M\left(\frac{q_{t+1}(H)+\delta_{t+1}(H)}{q_t}-R_t\right)+R_t} \\
= & \left(1-p \frac{n_h}{p n_h+(1-p) n_l}\right) \frac{R_t-\frac{q_{t+1}(L)+\delta_{t+1}(L)}{q_t}}{\alpha_t^M\left(\frac{q_{t+1}(L)+\delta_{t+1}(L)}{q_t}-R_t\right)+R_t} .
\end{aligned} \tag{15}$$

Comparing this expression to (13), observe that the incentive function affects the problem only to the extent that it changes the weights of the marginal utilities in the two states. While the direct trader weights the marginal utility in the high state by $p$, its probability, the manager uses the *individual shape-adjusted probability,*

$$\xi_{lh} \equiv p \frac{n_h}{p n_h+(1-p) n_l}. \tag{16}$$

$\xi_{lh}$ is the probability of a high state adjusted to the relative elasticity of the incentive function in the two states. Just as before, the indices $l, h = A, B$ refer to whether the performance relative to the market is above ($A$) or below ($B$) the kink in the low ($l$) and high ($h$) states respectively. For fixed parameters, $\xi_{lh}$ depends only on whether the manager chooses a contrarian portfolio ($\xi_{lh} = \xi_{BA}$), a moderate portfolio ($\xi_{lh} = \xi_{BB} = p$) or an aggressive portfolio ($\xi_{lh} = \xi_{AB}$). We rewrite the first order condition, (15), as

$$\alpha_t^M = \alpha_{lh} = \frac{1-\xi_{lh}}{1-\frac{\frac{q_{t+1}(H)+\delta_{t+1}(H)}{q_t}}{R_t}} + \frac{\xi_{lh}}{1-\frac{\frac{q_{t+1}(L)+\delta_{t+1}(L)}{q_t}}{R_t}} \tag{17}$$

and pick $lh = BA, BB, AB$ to get the locally optimal contrarian, moderate and aggressive portfolios, respectively. Observe that $n_A > n_B$ implies that $\xi_{BA} < p < \xi_{AB}$. That is, a manager choosing the locally optimal contrarian (aggressive) portfolio acts as if she would distort downwards (upwards) the probability of the high state. When managers compare the three locally optimal portfolios they act as if deciding in which way to distort the probabilities.

Recall that (17) describes the locally optimal portfolios if and only if the specified $\alpha_{lh}$ portfolio is inside the corresponding segment. The next Proposition implies that it is sufficient to consider this case.

**Proposition 2** *Suppose that $\alpha^*$ is a globally optimal portfolio for a given manager for some set of prices which clear the stock market. Then*

$$\frac{\rho_{t+1}(\alpha^*, L)}{\frac{q_{t+1}(L)+\delta_{t+1}(L)}{q_t}} \neq \kappa, \quad \frac{\rho_{t+1}(\alpha^*, H)}{\frac{q_{t+1}(H)+\delta_{t+1}(H)}{q_t}} \neq \kappa,$$

*i.e., the optimal portfolio is never at the kink.*

The following proposition summarizes our findings.

**Proposition 3** *In an $\Omega^*$ equilibrium,*

1. *the optimal consumption rules of investors, direct traders and managers are given by*

$$\psi_t^I = \psi_t^D = \left(1-\beta^I\right), \quad \psi_t^M = (1-\beta). \tag{18}$$

2. *direct traders optimal trading strategy is*

$$\alpha_t^I = \alpha_t^D = \frac{1-p}{1-\frac{\frac{q_{t+1}(H)+\delta_{t+1}(H)}{q_t}}{R_t}} + \frac{p}{1-\frac{\frac{q_{t+1}(L)+\delta_{t+1}(L)}{q_t}}{R_t}} \tag{19}$$

3. *fund managers optimal trading strategies have positive weight on a maximum of two of the following three portfolios*

- *Contrarian:*

$$\alpha_{AB} = \frac{1-\xi_{AB}}{1-\frac{\frac{q_{t+1}(H)+\delta_{t+1}(H)}{q_t}}{R_t}} + \frac{\xi_{AB}}{1-\frac{\frac{q_{t+1}(L)+\delta_{t+1}(L)}{q_t}}{R_t}} \tag{20}$$

- *Aggresive:*

$$\alpha_{BA} = \frac{1-\xi_{BA}}{1-\frac{\frac{q_{t+1}(H)+\delta_{t+1}(H)}{q_t}}{R_t}} + \frac{\xi_{BA}}{1-\frac{\frac{q_{t+1}(L)+\delta_{t+1}(L)}{q_t}}{R_t}} \tag{21}$$

- *Moderate* :

$$\alpha_{BB} = \frac{1-p}{1-\frac{\frac{q_{t+1}(H)+\delta_{t+1}(H)}{q_t}}{R_t}} + \frac{p}{1-\frac{\frac{q_{t+1}(L)+\delta_{t+1}(L)}{q_t}}{R_t}} \tag{22}$$

Which locally optimal portfolio is the globally optimal one? A very convenient property of our structure is that to answer this question, we do not have to know the level of equilibrium prices. To see why, first observe that the answer critically depends on the relative returns a manager can achieve with each of the different locally optimal portfolios. Indeed, from (14), the difference between the value of choosing the optimal contrarian and the optimal aggressive strategy, for any given prices, is

$$\frac{\beta}{1-\beta}\left(p\ln\frac{Z_B\left(\frac{\rho_{t+1}(\alpha_{AB},H)}{\frac{q_{t+1}(H)+\delta_{t+1}(H)}{q_t}}\right)^{n_B}}{Z_A\left(\frac{\rho_{t+1}(\alpha_{BA},H)}{\frac{q_{t+1}(H)+\delta_{t+1}(H)}{q_t}}\right)^{n_A}} + (1-p)\ln\frac{Z_A\left(\frac{\rho_{t+1}(\alpha_{AB},L)}{\frac{q_{t+1}(L)+\delta_{t+1}(L)}{q_t}}\right)^{n_A}}{Z_B\left(\frac{\rho_{t+1}(\alpha_{BA},L)}{\frac{q_{t+1}(L)+\delta_{t+1}(L)}{q_t}}\right)^{n_B}}\right), \tag{23}$$

which is proportional to the expected log difference between the assets under management generated by relative returns of the two portfolios. Comparing other pairs of locally optimal portfolios gives similar expressions. Second, any set of prices clearing the asset market imply that relative returns take a very simple form. To be more specific, let $\mu_{lh} = \mu_{AB}, \mu_{BB}, \mu_{BA}$ be the equilibrium fraction of managers whose realized portfolio is the locally optimal contrarian, moderate and aggressive portfolios, where, just as above, the index pair $lh$ refers to whether the performance of the manager is below $(B)$ or above $(A)$ the kink, $\kappa$, after a low $(l)$ and high $(h)$ shock. Then, the *aggregate shape-adjusted probability of a high state* is

$$\tilde{\xi}(\Omega^*) \equiv \Omega^*(\mu_{AB}\xi_{AB} + \mu_{BA}\xi_{BA} + \mu_{BB}p) + (1-\Omega^*)p, \tag{24}$$

which is the weighted average of the individual shape adjusted probabilities. The next Lemma shows that relative returns generated by locally optimal portfolios are given by the proportion of individual shape adjusted probabilities to their aggregate counterpart.

**Lemma 1** *In a $\Omega^*$ equilibrium, for any set of prices for which the stock market clears, that is,*

$$\Omega^* \left(\mu_{AB}\alpha_{AB} + \mu_{BA}\alpha_{BA} + \mu_{BB}\alpha_{BB}\right) + \left(1 - \Omega^*\right)\alpha_t^D = 1$$

*the relative return implied by a locally optimal portfolio is*

$$\frac{\rho_{t+1}\left(\alpha_{lh}, H\right)}{\frac{q_{t+1}(H)+\delta_{t+1}(H)}{q_t}} = \frac{\xi_{lh}}{\tilde{\xi}\left(\Omega^*\right)}, \tag{25}$$

*in the high state and*

$$\frac{\rho_{t+1}\left(\alpha_{lh}, L\right)}{\frac{q_{t+1}(L)+\delta_{t+1}(L)}{q_t}} = \frac{1 - \xi_{lh}}{1 - \tilde{\xi}\left(\Omega^*\right)} \tag{26}$$

*in the low state where $lh = AB, BB, BA$ for the locally optimal contrarian, moderate and aggressive portfolios, respectively.*

Thus, to figure out the equilibrium strategy profile of managers, we just have to use (23)-(26) to find fractions $\mu_{AB}, \mu_{BB}, \mu_{BA}$ such that $\mu_{AB} + \mu_{BB} + \mu_{BA} = 1$ and any positive $\mu_{lh}$ corresponds to a globally optimal portfolio. We show in the Appendix, that there are four different equilibria types depending on equilibrium fund managers portfolios:

***Cont-Agg:*** some managers hold the locally optimal contrarian portfolio and others hold the locally optimal aggressive portfolio,

***Cont-Mod:*** some managers hold the locally optimal contrarian portfolio and others hold the locally optimal moderate portfolio,

***Cont:*** all managers hold the locally optimal contrarian portfolio.

***Mod:*** all managers hold the locally optimal moderate portfolio..

The following proposition matches four subsets of the relevant parameter space to the four possible types of equilibria and pins down the unique corresponding fractions $\mu_{AB}, \mu_{BB}, \mu_{BA}$.

**Proposition 4** *Suppose that $Z_B < \hat{Z}, \lambda \leq \hat{\lambda}$. There are critical values $\hat{\kappa}_{high}, \hat{\kappa}_{low}$, $\hat{p}, \bar{p} \in \left(\frac{1}{2}, 1\right)$ and $\hat{\Omega} \in (0,1)$ that*

1. *if $\kappa > \hat{\kappa}_{high}$, there is a unique interior equilibrium and it is a Moderate (Mod) equilibrium where each agent holds the market: $\alpha^D = \alpha^M = 1$,*

2. *if $\hat{\kappa}_{low} < \kappa < \hat{\kappa}_{high}$, there is a unique interior equilibrium and its type depends on $p$ as follows:*

| | $p \in \left(\frac{1}{2}, \hat{p}\right)$ | $p \in \left(\hat{p}, 1\right)$ |
|---|---|---|
| $\Omega^* \leq \hat{\Omega}$ | $Mod$ | $Cont$ |
| $\Omega^* > \hat{\Omega}$ | $Mod$ | $Cont - Mod$ |

.

*3. if $\kappa < \hat{\kappa}_{low}$,there is a unique interior equilibrium and its type depends on $p$ as follows:*

| | $p \in \left(\frac{1}{2}, \bar{p}\right)$ | $p \in (\bar{p}, 1)$ |
|---|---|---|
| $\Omega^* \leq \hat{\Omega}$ | $Cont$ | $Cont$ |
| $\Omega^* > \hat{\Omega}$ | $Cont - Agg$ | $Cont - Mod$ |

*$\hat{\kappa}_{high}, \hat{\kappa}_{low}$ are functions of $n_A, n_B$ only, while $\hat{p}, \bar{p}$ are functions of $n_A, n_B, \kappa$. These functions are given in the Appendix.*

*$\hat{\Omega}$ is a function of $n_A, n_B, \kappa$, $p$ and, together with the probability a manager hold the locally optimal contrarian portfolio, $\mu_{AB}(\Omega^*)$, are given by*

$$\hat{\Omega} \equiv \frac{p - \bar{\xi}}{p - \xi_{AB}}$$

*and*

$$\mu_{AB}(\Omega^*) = \left\{ \begin{array}{cc} 1 & if \quad \Omega^* \leq \hat{\Omega} \\ \frac{\xi_2 - \bar{\xi}}{\xi_2 - \xi_{AB}} + \left(\frac{1}{\Omega^*} - 1\right) \frac{p - \bar{\xi}}{\xi_2 - \xi_{AB}} & if \quad \Omega^* > \hat{\Omega} \end{array} \right\} \tag{27}$$

*where $\bar{\xi}$ is the solution of*

$$p \ln \kappa^{n_B - n_A} \frac{\left(\frac{\xi_{BA}}{\xi}\right)^{n_A}}{\left(\frac{\xi_{AB}}{\xi}\right)^{n_B}} = (1-p) \ln \kappa^{n_B - n_A} \frac{\left(\frac{1-\xi_{AB}}{1-\xi}\right)^{n_A}}{\left(\frac{1-\xi_{BA}}{1-\xi}\right)^{n_B}} \tag{28}$$

*and*

$$p \ln \frac{\left(\frac{p}{\xi}\right)^{n_A}}{\left(\frac{\xi_{AB}}{\xi}\right)^{n_B}} = (1-p) \ln \kappa^{n_B - n_A} \frac{\left(\frac{1-\xi_{AB}}{1-\xi}\right)^{n_A}}{\left(\frac{1-p}{1-\xi}\right)^{n_B}} \tag{29}$$

*in the parameter regions corresponding to a $Cont - Agg$ and in a $Cont - Mod$ equilibrium respectively, and $\xi_2 = \xi_{BA}$ in a $Cont - Agg$ equilibrium and $\xi_2 = p$ in a $Cont - Mod$ equilibrium.*

From the proposition it is apparent that when reaching the high elasticity segment of the incentive function would require sufficiently high relative performance (high $\kappa$) then the moderate strategy is the global optimum, since the portfolio distortions required to achieve relative returns above $\kappa$ in one of the states are too large and are suboptimal. Observe that in this case direct traders and managers follow the same strategy, which also implies that they all hold the market. This is why we sometimes refer to a moderate equilibrium as the indexed equilibrium. Outside of this range of parameters, managers always choose a contrarian strategy as long as their capital share is small (i.e., $\Omega^*$ is small). That is, they lend to direct traders, have a smaller-than-1 exposure to the market risk, and overperform the market only in the low state. From the point that the capital share of delegated management reaches a given threshold ($\Omega^* \geq \hat{\Omega}$), managers are indifferent between the contrarian portfolio and either the moderate or the aggressive portfolio. Thus, they mix between the two globally optimal portfolios. By the law of large numbers, the mixing probabilities are given by $\mu_{lh}$ as they must be identical to the fraction of managers ending up with a given portfolio. As $\Omega^*$ increases, the mixing probabilities adjust in a way to keep managers indifferent between the two strategies. This is how expressions (27)-(29) are constructed. We analyze further the properties of the equilibrium strategies in Sections 3 and 4.

#### 2.2.2 Newborn investors' decision and the client-base

Relative returns in (25)-(26) directly imply the consumption-investment decision of clients. For example, from the definition of the incentive function, a client whose manager held a contrarian portfolio in the previous period consumes

$$1 - g\left(\frac{\xi_{AB}}{\tilde{\xi}\left(\Omega^*\right)}\right), 1 - g\left(\frac{1-\xi_{AB}}{1-\tilde{\xi}\left(\Omega^*\right)}\right)$$

share of her capital in the high and low states respectively, and invests the rest with her fund manager. Given the consumption and portfolio decision of direct traders, and the equilibrium strategies of managers, we can compare directly newborn traders value if they decide to be direct traders or managers. Thus, for a given $\Omega^*$, we can find a cost of trading directly, $f$, that implies that newborn managers are indifferent which role to choose. We derive the exact expression in the Appendix.

The equilibrium distribution of relative returns given in (25),(26) and (27) also directly imply the aggregate capital clients delegate to managers at the beginning of the period and the capital managers return to clients at the end of the period. For example, in a Cont-Agg equilibrium in the high state, the total share of capital returned to those clients with managers following the contrarian strategy must be

$$\Omega^* \mu_{AB}\left(\Omega^*\right) \frac{\xi_{AB}}{\tilde{\xi}\left(\Omega^*\right)},$$

the product of the total share of invested capital by managers, the fraction holding the contrarian portfolio and the relative return corresponding to the contrarian strategy. Following this logic, the total share of capital returned to all clients is

$$\begin{aligned}
\Upsilon_H &\equiv \Omega^* \left(\mu_{AB}\left(\Omega^*\right) \frac{\xi_{AB}}{\tilde{\xi}\left(\Omega^*\right)} + \left(1 - \mu_{AB}\left(\Omega^*\right)\right) \frac{\xi_{BA}}{\tilde{\xi}\left(\Omega^*\right)}\right) \\
\Upsilon_L &\equiv \Omega^* \left(\mu_{AB}\left(\Omega^*\right) \frac{1-\xi_{AB}}{1-\tilde{\xi}\left(\Omega^*\right)} + \left(1 - \mu_{AB}\left(\Omega^*\right)\right) \frac{1-\xi_{BA}}{1-\tilde{\xi}\left(\Omega^*\right)}\right)
\end{aligned}$$

in the high and low states respectively. Similarly, in a Cont-Agg equilibrium, in the high state the total share of capital received by managers who followed the contrarian strategy in the previous period is

$$\Omega^* \mu_{AB}(\Omega^*)\,\Gamma_t g\left(\frac{\xi_{AB}}{\tilde{\xi}(\Omega^*)}\right)$$

Thus, the total share of capital delegated to managers is $\Gamma_t \bar{g}_s$ where

$$\begin{aligned}
\bar{g}_H &\equiv \Omega^* \left(\mu_{AB}(\Omega^*)\, g\left(\frac{\xi_{AB}}{\tilde{\xi}(\Omega^*)}\right) + (1-\mu_{AB}(\Omega^*))\, g\left(\frac{\xi_{BA}}{\tilde{\xi}(\Omega^*)}\right)\right) \\
\bar{g}_L &\equiv \Omega^* \left(\mu_{AB}(\Omega^*)\, g\left(\frac{1-\xi_{AB}}{1-\tilde{\xi}(\Omega^*)}\right) + (1-\mu_{AB}(\Omega^*))\, g\left(\frac{1-\xi_{BA}}{1-\tilde{\xi}(\Omega^*)}\right)\right),
\end{aligned}$$

in the high and low states respectively. Note that from the total share $\Gamma_t \bar{g}_s$, $\lambda \bar{g}_s$ comes from those clients who survived from the previous period and the rest comes from newborn investors choosing to be clients. That is,

$$\Gamma_t \bar{g}_s = \lambda \bar{g}_s + (1-\lambda)\,\beta^I \bar{\chi}_t \tag{30}$$

has to hold, where $\bar{\chi}_t$ is the aggregate share of newborn investors choosing to be clients. By allocating a given fraction of indifferent newborn investors to the group of clients, we can pick a $\Gamma_t$ which keeps the share of delegated capital, $\Omega^*$ fixed. In particular,

$$\Omega^* = \frac{\beta \Gamma_t \bar{g}_s}{\beta \Gamma_t \bar{g}_s + \beta^I \lambda (1-\Upsilon_s) + \beta^I (1-\lambda)(1-\bar{\chi}_t)}$$

has to hold in both states, $s = H, L$ where the numerator is the total invested capital share of managers while the denominator is the total invested capital share of all groups. In the denominator, the second term correspond to the invested share of aggregate capital of direct traders: $(1-\Upsilon_s)$ is the wealth share of direct traders, of which a fraction $\lambda$ survives and invests $\beta^I$ share in the asset market. The third terms corresponds to the invested share of newborn investors deciding to be direct traders. It is easy to see that we can pick the client-base and the fraction of newborns deciding to be clients in a way that they both depend only on the dividend state; i..e., $\Gamma_t = \Gamma_H, \Gamma_L$ and $\bar{\chi}_t = \bar{\chi}_H, \bar{\chi}_L$.

Our findings are summarized in the following Lemma.

**Proposition 5** *In an $\Omega^*$ equilibrium*

1. *both the fraction of newborn investors choosing to delegate $\bar{\chi}_t$,*

2. *and the client-base $\Gamma_t$ depend only on the state and are*

$$\begin{aligned}
\bar{\chi}_s &= \bar{g}_s \frac{\Gamma_s - \lambda}{(1-\lambda)\,\beta^I} \\
\Gamma_s &= \Omega^* \frac{\beta^I (1-\lambda \Upsilon_s) + \bar{g}_s \lambda}{\bar{g}_s (\beta (1-\Omega^*) + \Omega^*)}
\end{aligned}$$

*for* $s = H, L$*, where* $\Upsilon_s, \bar{g}_s$ *is given by*

$$\begin{aligned}
\Upsilon_H &= \Omega^* \left( \mu_{AB}(\Omega^*) \frac{\xi_{AB}}{\tilde{\xi}(\Omega^*)} + (1 - \mu_{AB}(\Omega^*)) \frac{\xi_2}{\tilde{\xi}(\Omega^*)} \right) \\
\Upsilon_L &= \Omega^* \left( \mu_{AB}(\Omega^*) \frac{1 - \xi_{AB}}{1 - \tilde{\xi}(\Omega^*)} + (1 - \mu_{AB}(\Omega^*)) \frac{1 - \xi_2}{1 - \tilde{\xi}(\Omega^*)} \right)
\end{aligned}$$

*and*

$$\begin{aligned}
\bar{g}_H &= \Omega^* \left( \mu_{AB}(\Omega^*) g\left( \frac{\xi_{AB}}{\tilde{\xi}(\Omega^*)} \right) + (1 - \mu_{AB}(\Omega^*)) g\left( \frac{\xi_2}{\tilde{\xi}(\Omega^*)} \right) \right) \\
\bar{g}_L &= \Omega^* \left( \mu_{AB}(\Omega^*) g\left( \frac{1 - \xi_{AB}}{1 - \tilde{\xi}(\Omega^*)} \right) + (1 - \mu_{AB}(\Omega^*)) g\left( \frac{1 - \xi_2}{1 - \tilde{\xi}(\Omega^*)} \right) \right),
\end{aligned}$$

*where* $\xi_2 = \xi_{BA}, p$ *in a* $Cont - Agg$ *equilibrium and a* $Cont - Mod$ *equilibrium respectively.*

### 2.2.3 Equilibrium prices

Given all equilibrium actions, we can determine equilibrium prices by market clearing conditions. Instead of tracking the stock price $q$, and the stock price next period $q_{s'}$, it is more convenient to track the price-dividend ratio

$$\pi = \frac{q}{\delta}.$$

and the price dividend ratio next period

$$\pi_{s'} = \frac{q_{s'}}{\delta'}. \tag{31}$$

First, taking the price-dividend ratios $\pi_H, \pi_L$ and a strategy profile of portfolios (20)-(22) as given, and imposing the market clearing condition that all stock holdings have to sum up to 1 gives the following result on the implied interest rate.

**Proposition 6** *In a* $\Omega^*$ *equilibrium, the interest rate is*

$$R = \frac{\theta}{\pi}$$

*where* $\theta$ *solves*

$$\tilde{\xi}(\Omega^*) \frac{1}{y_H (1 + \pi_H(\Omega^*))} + \left(1 - \tilde{\xi}(\Omega^*)\right) \frac{1}{y_L (1 + \pi_L(\Omega^*))} = \frac{1}{\theta(\Omega^*)}, \tag{32}$$

Second, the equilibrium wealth level of all agents, their consumption share and the market clearing condition for the good market that requires that aggregate consumption has to be equal to the dividend gives the equilibrium price dividend ratios.

**Proposition 7** *In an $\Omega^*$ equilibrium, the price dividend ratio is*

$$\pi_s = \frac{\beta^I (1 - \lambda \Upsilon_s) + \lambda \bar{g}_s - (1 - \beta) \Gamma_s \bar{g}_s}{1 - \beta^I (1 - \lambda \Upsilon_s) - \lambda \bar{g}_s + (1 - \beta) \Gamma_s \bar{g}_s}$$

*in state $s = H, L$.*

# 3 Implications

In this section, we discuss the equilibrium and analyze its implications. We focus on the interaction of non-concave incentives and the increased level of delegation in financial markets. We contrast our findings with existing empirical work and present additional testable implications. We start the discussion with briefly presenting two benchmarks. Then we discuss our results connected to the distribution of relative returns and strategies, then proceed to the Sharpe ratio and managers' exposure to market risk. Finally, we discuss implications related to the gross amount of borrowing and lending and the dispersion of portfolios.

## 3.1 Benchmark cases: no delegation and constant-elasticity incentive functions

A natural benchmark is a market with only direct traders, so that the share of delegation $\Omega^*$ is zero. For example, this is the case when the utility cost of direct trading is zero. It is simple to check that our model reduces to the standard Lucas economy where all traders hold the market and the price-dividend ratio and riskfree rate are constant:

$$\pi_H = \pi_L = \frac{\beta^I}{1 - \beta^I},$$

$$R = \frac{1}{\beta^I} \frac{y_H y_L}{p y_L + (1 - p) y_H},$$

and the Sharpe ratio is constant as well and given by

$$S = \frac{p^{\frac{1}{2}} (1 - p)^{\frac{1}{2}} \| y_H - y_L \|}{p y_L + (1 - p) y_H}.$$

A second benchmark is when the utility cost is in the range which implies a positive share of delegation, but managers' incentive function has a constant elasticity. That is, $n_A = n_B$. In the following proposition we characterize the main properties of the equilibrium in this case.

**Proposition 8** *If the incentive function has constant elasticity, $n_A = n_B$, and $Z_B < 1$ and $\frac{\beta + Z_B}{\beta + 1} > \lambda$ then for any $\Omega^*$ there is an $f$ implying that an $\Omega^*$-equilibrium exists. Furthermore, in any implied $\Omega^*$ equilibrium*

*1. both direct traders and managers hold the market,*

*2. the Sharpe ratio is equal to the Sharpe ratio with no delegation.*

The proposition illustrates that delegation has little effect on the equilibrium if the incentive functions have constant elasticity. For any $\Omega^*$, each agent holds the market and the Sharpe ratio is unaffected by delegation.[13] To see this, note that with constant elasticity incentives both the individual and the aggregate shape adjusted probability is $p$, $\xi^i_{lh} = \tilde{\xi}(\Omega^*) = p$. Thus, (22) implies the same strategy for all managers. Market clearing implies that this strategy must be that each manager holds the market, and relative returns are always 1. This implies that we can create examples, where the incentive function $g(\cdot)$ a group of traders is convex in levels (i.e., $n_A = n_B = n > 2$) while the other group has standard incentives (direct traders) and still they do not take positions against each other. This is because of the interaction of log utility and constant elasticity incentive functions. The marginal utility from a dollar linearly increases in elasticity parameter $n$. Given that $n$ is the same across states, the marginal rate of substitution is not affected by $n$. Thus, the marginal rate of substitution is the same for both agents. Hence, there are no gains from trade. This illustrates well why we specify convexity of the flow-performance relationship in logs and not in levels.[14]

### 3.2 Managers' excess log-return and heterogeneity in strategies

Propositions (3)-(4) describe the trading strategies in equilibrium. We can see immediately that when reaching the larger elasticity segment of the incentive function would be sub-optimal, because the kink $\kappa$ is too large, then in equilibrium both direct traders and fund managers hold the market. The resulting indexed equilibrium has the same properties as our second benchmark: when fund managers have a constant elasticity incentive function. Because in this equilibrium delegation has little effect, in the rest of the paper, we restrict our attention to the segment of parameter space when the equilibrium is not of this type (i.e., $\kappa < \kappa_{low}$ or $\kappa_{low} < \kappa < \kappa_{high}$ and $p > \hat{p}$.).

For all remaining set of parameters, Proposition 4 implies that when share of delegation is low, $\Omega^* < \hat{\Omega}$, all fund managers follow a contrarian strategy in equilibrium. To see the intuition behind the equilibrium choice of managers, consider the first fund manager who enters a market which is populated only by direct traders, $\Omega^* \approx 0$. The manager has three choices. She can hold a moderate portfolio, but then she will never outperform the market sufficiently to get the extra capital flows in any of the states. Or she can hold the locally optimal aggressive portfolio leading to gains and extra capital flow in the high state and losses in the low state, or she can hold the

[13]The level of delegation, $\Omega^*$, still effects the level of the price dividend ratio. This is so, because managers and direct traders consume a different constant fraction of their wealth.

[14]The importance of considering the interaction of utility function and the incentives was also pointed out by Ross (2006).

locally optimal contrarian portfolio leading to gains and extra capital flow in the low state and losses in the high state. How do these two compare? Managers choose the contrarian portfolio because of the interaction of left skewed consumption growth ($p > \frac{1}{2}$) and convex flow-performance function. First, the fact that the high state has higher probability to occur implies that the size of the gain or loss compared to market return in the high state are small relative to the size of relative gain or loss compared to market return in the low state under any locally optimal strategy. For example, in the locally optimal contrarian strategy, large gains with small probability in the low state compensate for small losses with large probability in the high state. Formally, as the average relative returns under the two portfolios are equal,

$$p\frac{\xi_{AB}}{\tilde{\xi}(0)} + (1-p)\frac{1-\xi_{AB}}{1-\tilde{\xi}(0)} = p\frac{\xi_{BA}}{\tilde{\xi}(0)} + (1-p)\frac{1-\xi_{BA}}{1-\tilde{\xi}(0)} = 1,$$

$p > \frac{1}{2}$ implies that

$$\frac{\xi_{AB}}{\tilde{\xi}(0)} - 1 < 1 - \frac{1-\xi_{AB}}{1-\tilde{\xi}(0)} \quad \text{and} \quad \frac{\xi_{BA}}{\tilde{\xi}(0)} - 1 > 1 - \frac{1-\xi_{BA}}{1-\tilde{\xi}(0)}.$$

Second, the fact that the flow-performance relationship is convex implies that capital-flow rewards for gains are larger than penalties for losses of similar magnitude. As a consequence of the two effects, the manager prefers the contrarian strategy, because the implied larger gain is rewarded more by the convex flow-performance relationship.

Note that our argument is the classic idea of risk-shifting, but with a slight twist. Risk shifting implies that agents with globally non-concave incentives might prefer to take on larger variance, that is, they gamble. However, in our case this not necessarily implies a levered position. Because managers have non-concave incentives in relative instead of absolute return, in this particular case, the contrarian strategy is the larger gamble.[15]

As the share of delegation $\Omega^*$ increases, prices increasingly work against fund managers and they find the contrarian strategy less attractive. At some threshold $\hat{\Omega}$, managers become indifferent between the optimal contrarian strategy and, depending on the parameter values, either the optimal moderate strategy or the optimal aggressive strategy. For market clearing, as the market share of fund managers grows above this threshold a decreasing set of managers has to choose the contrarian strategy. Thus, the heterogeneity in strategies increases with $\Omega^*$ in this sense. The idea is simple. As managers start to dominate the market, the only way they can overperform the market in some state is if they bet against each other.

Consider now the relative return of the average manager as the share of delegation increases. We show that despite the increasing group of managers following an aggressive strategy when the

[15] A similar point regarding funds increasing tracking error volatility in the presence of benchmarks has been made in Cuoco and Kaniel (2010), and Basak, Pavlova and Shapiro (2007).

share of delegation is large, the average manager remains contrarian for any $\Omega^* < 1$. Also, over the whole range of $\Omega^*$, both managers' overperformance in the low state and underperformance in the high state becomes less extreme. For small share of delegation ($\Omega^* < \hat{\Omega}$) this is a consequence of the fact that as prices move against managers, each one chooses a portfolio which result in less extreme relative returns. For larger share of delegation ($\hat{\Omega} < \Omega^*$), the relative return of each individual manager is constant. However, as the proportion of managers choosing the aggressive portfolio increases, the relative return of the average manager has to increase in the high state and decrease in the low state. Given this monotonicity and the fact that at $\Omega^* = 1$ the average manager has to hold the market, the average manager must have a portfolio which overperforms in the low state and underperforms in the high state for any $\Omega^* < 1$.

To translate our findings to testable implications, let us define some descriptive statistics. In particular, we consider the excess log return of the average fund manager,

$$\int_{m\in M} \ln \rho_{t+1}\left(\alpha_t^m, s_{t+1}\right) dm - \ln \frac{q_{t+1}\left(s_{t+1}\right) + \delta_{t+1}\left(s_{t+1}\right)}{q_t},$$

the volatility of the excess log-return of a given fund manager is,

$$\sqrt{p\left(1-p\right)}\left| \begin{array}{r} \left(\ln \rho_{t+1}\left(\alpha_t^M, H\right) - \ln \frac{\left(q_{t+1}(H)+\delta_{t+1}(H)\right)}{q_t}\right) \\ -\left(\ln \rho_{t+1}\left(\alpha_t^M, L\right) - \ln \frac{q_{t+1}(L)+\delta_{t+1}(L)}{q_t}\right) \end{array} \right|,$$

and the cross sectional dispersion across fund managers' excess log–returns in state $s_{t+1}$,

$$\int_{n\in M} \left| \ln \rho_{t+1}\left(\alpha_t^n, s_{t+1}\right) - \int_{m\in M} \ln \rho_{t+1}\left(\alpha_t^m, s_{t+1}\right) dm \right| dn.$$

As we show in the Appendix, the intuition discussed above translates to the following statements.

**Proposition 9** *1. For any $\Omega^* < 1$, the average fund's exposure to the market is always smaller than 1, so it overperforms the market in recessions and underperforms in booms.*

*2. For $\Omega^* > \hat{\Omega}$, funds follow heterogeneous strategies. In each period, a fraction of managers, $1 - \mu_{AB}\left(\Omega^*\right)$, levers up and invests more than 100% of their capital in stocks. This fraction increases in the share of delegated capital $\Omega^*$.*

*3. For $\Omega^* > \hat{\Omega}$, fund managers' cross-sectional dispersion of log-returns is larger in the low state than in the high state when the equilibrium is Cont-Agg. When the equilibrium is Cont-Mod, this is also the case if and only if*

$$p > \frac{\sqrt{\frac{n_A}{n_B}}}{1+\sqrt{\frac{n_B}{n_A}}}. \tag{33}$$

4. *As $\Omega^*$ increases, the excess log return of the average manager increases in the high state and decreases in the low state. That is, both the overperformance in the low state and the underperformance in the high state is less severe.*

5. *The volatility of the excess log-return of each manager is decreasing in the share of delegation as long as $\Omega^* < \hat{\Omega}$.*

Consistently with statement 1, evidence shows that mutual funds perform better in recessions than in booms (e.g., Moskowitz (2000), and Glode (2010), Kacperczyk, Van Nieuweburgh and Veldkamp (2010), Kosowski (2006), Lynch and Wachter (2007)). For example, Moskowitz (2000) notes that the absolute performance of the average fund manager is 6% higher in recessions than in booms.[16]

Regarding statement 2, there is some evidence that the heterogeneity in strategies in the money management industry has been indeed increasing over the last decades. As argued by Adrian and Shin (2008), one sign of this is that the total balance sheet of investment banks[17], typically using leveraged strategies, was around 40% compared to bank holding companies in 1980 and increased over 160% by 2007. Indeed, by 2009, it has become a widely held view among policy makers that the excessive leverage of investment banks contributed to the financial crisis (see FSA (2009), FSB (2009)). Although we believe that our result has the potential to provide a simple and insightful explanation of the emergence of highly leveraged financial intermediaries over the last decade and their coexistence with more conservative institutions, we have to point out two caveats to this interpretation. First, our framework cannot distinguish between two possible interpretations of aggressive portfolio. An aggressive strategy can be interpreted as levered strategy, but it can be equally interpreted as a strategy of picking stocks with higher than 1 market-beta. Second, in our equilibrium there is no persistence in portfolios. That is, a manager who held an aggressive portfolio in one period, might hold a conservative portfolio in the next one. This does not map directly to the interpretation that managers holding different portfolios correspond to different type of financial intermediaries. However, we consider this a mainly technical issue.[18]

[16]Note however, that Kosowski (2006), Lynch and Wachter (2007) and Glode (2010) find overperformance in recessions in terms of Jensen-alpha as opposed to in terms of total returns. Given that in our model funds cannot generate alpha, only the results in Moskowitz (2000), and Kacperczyk, Van Nieuweburgh and Veldkamp (2010) translate to our proposition one-to-one.

[17]Although mutual funds typically do not use leverage, interestingly, Lo and Patel (2007) notes a large increase of leveraged mutual funds and leveraged ETFs in the last decade before the crisis.

[18]There could be persistence in managers' portfolios if we were to implement the equilibrium by heterogeneous pure strategies as opposed to mixed strategies. The problem is that in this case, we could not find a constant share of delegation, $\Omega^*$, which, regardless of the dividend state, would make newborn investors indifferent between being clients or direct traders. As a result, the share of delegation would fluctuate in equilibrium in a non-stationary way. Thus, we could analyze the effect of the increasing share of delegation only if we provided a micro-foundation for this increasing share. This is beyond the scope of this paper. However, in any specification the intuition that convex incentives and large share of delegated capital results in heterogeneous strategies and, consequently, in a fraction of managers following levered strategies ex post, should go through.

If its conditions are satisfied, statement 3 is consistent with Kacperczyk, Van Nieuweburgh and Veldkamp (2010) who find that both the dispersion in mutual funds return is larger in recessions. Interestingly, Kacperczyk, Van Nieuweburgh and Veldkamp (2010) present this result as an implication of optimal attention allocation across the business cycle by fund managers. Our model suggests that this result is consistent with a set-up where information does not play any explicit role. Instead, it is driven by competition of managers for extra capital inflows and negative skewness of the consumption growth process. Note also that while (33) tends to be satisfied when the consumption growth process is relatively skewed (large $p$), Proposition 4 shows that a Cont-Agg equilibrium typically arises when the consumption growth process $p$ is close to half. Thus, we should expect to get larger dispersion in recessions for a wide range of parameters.[19]

Because of the lack of systematic evidence on the time-series pattern of managers' return volatility, relative returns and return dispersion we think of results 4 and 5 as testable predictions for the future.

### 3.3 Exposure to market risk and the Sharpe ratio

In the previous part we characterized the distribution of the average and individual excess returns as the share of delegation increases. However, the change in relative returns does not map one to one to the change in the exposure to market risk.This mapping also depends how the relative return of the stock and the bond, that is, the price of risk changes. In this part, we focus on the change in agents' exposure to market risk and on the change of the Sharpe ratio, a particular measure of the price of risk, as share of delegated capital increases.

We find that typically, as the share of delegation increases, the Sharpe ratio follows an inverse U pattern. It increases as long as $\Omega^* < \hat{\Omega}$, and decreases for $\Omega^* > \hat{\Omega}$. As $\Omega^* = 0$ corresponds to the standard Lucas model, this also implies that the presence of delegation increases the Sharpe ratio at least as long as $\Omega^* < \hat{\Omega}$. In the same time, direct trader's exposure to market risk, $\alpha^D$, increases monotonically. Managers' exposure holding a contrarian portfolio, $\alpha_{AB}$, decreases for $\Omega^* > \hat{\Omega}$, and typically increases for $\Omega^* < \hat{\Omega}$. Managers' exposure holding an aggressive portfolio, $\alpha_{BA}$, monotonically increases in the only relevant range $\Omega^* > \hat{\Omega}$. The left column of Figure 6 and Figure 4 illustrates this for a wide range of parameters. With the only exception of the monotonicity of $\alpha_{AB}$ when $\Omega^* < \hat{\Omega}$, we find all these observations robust to all the parameter variations we experimented with.[20] However, analytically, we prove only the following weaker statements.

**Proposition 10** *For $n_a > n_b \geq 2$, the Sharpe ratio is increasing in the region $\Omega^* \leq \hat{\Omega}$.*

[19]Measuring dispersion as the ratio of relative returns implies a higher dispersion in the low state always for both $Cont - Agg$ and $Cont - Mod$ equilibria.

[20]Typically for high $Z_B$, $\frac{\partial \alpha_{AB}}{\partial \Omega^*}|_{\Omega^* < \hat{\Omega}} < 0$.

**Proposition 11** *The Sharpe ratio in the region $\Omega^* > \hat{\Omega}$ is monotone. If $Z_B$ is small or*

$$\beta^I (\xi_2 - \xi_{AB}) \left( \frac{1}{1-\bar{\xi}} + \frac{1}{\bar{\xi}} \right) - (g\left( \frac{1-\xi_{AB}}{1-\xi(\Omega^*)} \right) - g\left( \frac{\xi_{AB}}{\xi(\Omega^*)} \right)) + (g\left( \frac{1-\xi_2}{1-\xi(\Omega^*)} \right) - g\left( \frac{\xi_2}{\xi(\Omega^*)} \right)) > 0$$

*and*

$$(g\left( \frac{1-\xi_{AB}}{1-\xi(\Omega^*)} \right) - g\left( \frac{\xi_{AB}}{\xi(\Omega^*)} \right)) + (g\left( \frac{1-\xi_2}{1-\xi(\Omega^*)} \right) - g\left( \frac{\xi_2}{\xi(\Omega^*)} \right)) \frac{\bar{\xi} - \xi_{AB}}{\xi_2 - \bar{\xi}} \geq 0$$

*where $\xi_2 = \xi_{BA}, p$ in a $Cont - Agg$ equilibrium and a $Cont - Mod$ equilibrium respectively, then it is decreasing.*

**Proposition 12** *In the region $\Omega^* > \hat{\Omega}$, whenever the Sharpe ratio is decreasing in $\Omega^*$, the exposure to market risk of direct traders and managers holding an aggressive portfolio, $\alpha^D$, $\alpha_{BA}$ is increasing, while the exposure of managers holding a contrarian portfolio, $\alpha_{AB}$ is decreasing as $\Omega^*$ increases.*

To help to understand the intuition and the connection between the equilibrium Sharpe ratio and the exposure to market risk, the following lemma decomposes the Sharpe ratio in an intuitive way.

**Lemma 2** *The state price of the low state relative to the high state is*

$$\frac{y_H}{y_L} X(\Omega^*),$$

*and the Sharpe ratio is*

$$S(\Omega^*) = \frac{p^{\frac{1}{2}} (1-p)^{\frac{1}{2}} \| y_H X(\Omega^*) - y_L \|}{p y_L + (1-p) y_H X(\Omega^*)}. \tag{34}$$

*where $X(\Omega^*)$ is the product of the capital-flow effect and the wealth effect defined as follows.*

$$X(\Omega^*) \equiv \underbrace{\frac{\frac{1-\tilde{\xi}(\Omega^*)}{1-p}}{\frac{\tilde{\xi}(\Omega^*)}{p}}}_{\text{capital-flow effect}} \underbrace{\frac{1+\pi_H}{1+\pi_L}}_{\text{wealth effect}}. \tag{35}$$

*The capital flow effect is 1 at $\Omega^* = 0$ and larger than 1 for any $\Omega^* > 0$. Furthermore, it is monotonically increasing in $\Omega^*$ for any $\Omega^* < \hat{\Omega}$ and constant for any $\Omega^* > \hat{\Omega}$.*

The Lemma shows that both the deviation of relative state prices and the Sharpe ratio from the standard model is driven by the term $X(\Omega^*)$. In the standard model, the relative state prices is $\frac{y_H}{y_L}$, that is, $X \equiv 1$. The term $X(\Omega^*)$ is determined by the relative size of two effects: the capital flow effect and the wealth effect.

The capital-flow effect is similar to the classic cash-flow effect in asset pricing. Depending on the shape of the incentive function, a dollar return in a given state might attract more or less future capital flows. The first term in (35) shows the relative capital-flow generating ability of a dollar in the low state versus the high state for the average manager. As discussed in the previous section, for any non-trivial equilibrium the representative manager has a market exposure smaller than 1, and her incentive function is relatively more sensitive in the low state. This implies that she finds an additional unit of return more valuable in that state which pushes the relative state price and the Sharpe ratio up. The comparatives statics of the capital-flow effect in $\Omega^*$ are a direct consequence of the fact that the aggregate shape-adjusted probability $\tilde{\xi}\left(\Omega^*\right)$ increases in the share of delegation $\Omega^*$. We can conclude that the capital-flow effect always pushes the Sharpe ratio up compared to the standard model, and it is non-decreasing in the share of delegation.

The second effect is the wealth effect. When the price dividend ratio is higher in the low state, it increases the relative wealth of the marginal agent in the low state, which pushes both the relative state price and the Sharpe ratio down. The comparative statics on the price dividend ratio is non-trivial. Still, our numerical simulations show that some qualitative properties of the wealth effect are robust across most parameterization. Namely, the wealth effect is decreasing in the share of delegation $\Omega^*$ when $\Omega^* > \hat{\Omega}$ and its change is small when when $\Omega^* < \hat{\Omega}$. Thus, relative state price and the Sharpe ratio increases in $\Omega^*$ when $\Omega^* < \hat{\Omega}$ because of the capital-flow effect and decreases in $\Omega^*$ when $\Omega^* > \hat{\Omega}$, because of the wealth effect.

To understand the change in exposure to market risk and its interaction with the Sharpe ratio consider the portfolio of a manager following a contrarian strategy. We rewrite the share a contrarian manager lends as

$$1-\alpha_{AB}=\frac{1-\frac{\xi_{AB}}{\tilde{\xi}(\Omega^*)}}{1-\frac{\theta(\Omega^*)}{y_H(1+\pi_H(\Omega^*))}}. \tag{36}$$

The numerator is the relative loss of this manager in the high state. To understand the denominator, we write (32) as

$$\tilde{\xi}\left(\Omega^*\right)+\left(1-\tilde{\xi}\left(\Omega^*\right)\right)\frac{y_H\left(1+\pi_H\right)}{y_L\left(1+\pi_L\right)}=\frac{y_H\left(1+\pi_H\left(\Omega^*\right)\right)}{\theta\left(\Omega^*\right)}. \tag{37}$$

It shows that the term $\frac{y_H(1+\pi_H(\Omega^*))}{\theta(\Omega^*)}$ is a weighted average of 1 and $\frac{y_H}{y_L}$ times the wealth effect. As $\frac{y_H(1+\pi_H(\Omega^*))}{\theta(\Omega^*)}>1$, this term behaves similarly to the relative state-price of the low state, $\frac{y_H}{y_L}X\left(\Omega^*\right)$, as it decreases in $\tilde{\xi}\left(\Omega^*\right)$ and increases in $\frac{y_H(1+\pi_H)}{y_L(1+\pi_L)}$.

Thus, we can interpret (36) along the following lines. A manager choosing the contrarian portfolio wants to enter a bet providing a relative gain in the low state and a relative loss in the high state. By borrowing more, she trades off a larger loss in the high state for larger gain in the low state. The numerator and the denominator of (36) can be seen as the terms of the available bet. In particular, as the numerator decreases, both the gain and the loss relative to the market decrease. While, if the denominator increases, as $\frac{y_H(1+\pi_H(\Omega^*))}{\theta(\Omega^*)}$ behaves similarly to the relative state price in the low state, the lottery gets more costly in utility terms. Note also, that as $\tilde{\xi}(\Omega^*)$ is decreasing in the share of delegated capital, this lottery behaves as a club good: for larger share of managers buying it in equilibrium, the less desirable it gets for each manager. In equilibrium, the increase of the capital share of managers typically induces adjustments along each margins. The cost, $\frac{y_H(1+\pi_H(\Omega^*))}{\theta(\Omega^*)}$, goes up, the desirability, $1-\frac{\xi_{AB}}{\tilde{\xi}(\Omega^*)}$, goes down, and size of the position, $1-\alpha_{AB}$, goes down. As cost, $\frac{y_H(1+\pi_H(\Omega^*))}{\theta(\Omega^*)}$, behaves analogously to the relative state price in which the Sharpe ratio is monotonic, the Sharpe ratio also increases. This explains the dynamics of the exposure and the Sharpe ratio as long as $\Omega^* < \hat{\Omega}$.

In the region $\Omega^* > \hat{\Omega}$ there is a new margin of adjustment, the fraction of managers, $1-\mu(\Omega^*)$ who take the opposite side of the available bet. As we described in the previous part, as the share of delegated capital increases, an increasing fraction of managers are leveraging up by holding the locally optimal moderate or aggressive portfolios, while the relative return of each group in each state remains constant. The new margin of adjustment implies that the increase in $\Omega^*$ does not have to imply an increase in the cost of the contrarian bet and a corresponding increase in the Sharpe ratio. Indeed, the Sharpe ratio decreases in this region. Consequently, as Proposition 12 states, the equilibrium exposures to market risk has to move away from 1 for each portfolio to keep the corresponding relative returns constant. For example, managers following an aggressive strategy have to increase their leverage to get the same relative yield.

### 3.4 Borrowing and lending, repo, derivative markets and gambling

As opposed to standard representative agent models, in our model traders typically do not hold the market portfolio. Agents buy or sell bonds in order to gain different exposure to market risk. In this section, we quantify the extent of this activity. We show that the gross amount of borrowing and lending compared to the size of the economy typically increases with an increase in the share of delegation.

Before we proceed to the formal results, it is useful to consider the empiricial counterpart of our concepts. In our framework, buying or selling the risk-free asset is the only way agents can change their exposure to market risk. In reality, financial intermediaries use various instruments for this purpose. As repo agreements are one of the most frequently used tools for a large group of financial intermediaries to manage their leverage ratio (see Adrian and Shin, 2008), one possibility is to connect gross amount of borrowing and lending in our model with the size of repo markets. Alternatively, as most financial intermediaries would use derivatives like S&P futures and options to change their exposure to market risk, we can connect the amount of borrowing and lending risk-free bonds in our model to the open interest in derivative markets.

To measure gross amount of lending and borrowing positions, we use the fact that in any equilibrium, the only group of traders who lend are managers who follow a contrarian strategy.

We define relative bond market size as the total long bond holding of this group compared to the value of the economy, $q_t + \delta_t$. Plugging in (32) into (20) and some simple algebra shows that this measure is

$$\Omega^* \mu_{AB}(\Omega^*)(1 - \alpha_{AB}) = \Omega^* \mu_{AB}(\Omega^*) \frac{1 - \frac{\frac{pn_B}{pn_B + (1-p)n_A}}{\tilde{\xi}(\Omega^*)}}{1 - \frac{\theta(\Omega^*)}{y_H(1 + \pi_H(\Omega^*))}}. \tag{38}$$

The following Lemma describes the relationship between the portfolio of managers, relative bond market size and the Sharpe ratio whenever $\Omega^* > \hat{\Omega}$.

**Lemma 3** *When the share of delegation is larger than $\hat{\Omega}$, whenever the Sharpe ratio is decreasing in $\Omega^*$, amount of bond long positions relative to the size of the economy (38) increases as the share of delegation increases.*

Together with our observation that the Sharpe ratio is typically decreasing in the share of delegation when $\Omega^* > \hat{\Omega}$, the lemma implies that relative bond market size also increases with $\Omega^*$. To interpret this result, note that risk-free bonds serve a double purpose in our economy. First, direct traders and managers have different incentives, which implies that they prefer to share risk. As we saw before, this leads direct traders to hold a portfolio with a larger than one exposure to the market. We call the part of holdings explained by this motive as the risk-sharing amount of bond holdings. Second, when the share of delegated capital is sufficiently large managers start to trade against each other. By selling or buying bonds they increase or decrease their exposure to the market in order to beat the market at least in one of the states. We call this part the gambling share of bond holdings.

As direct traders hold bonds only because of risk-sharing motives, we can decompose the total size of the bond market by comparing (38) to the total bond holding of direct traders relative to the value of the economy defined as

$$(1 - \Omega^*) \left| \frac{1 - \frac{p}{\tilde{\xi}(\Omega^*)}}{1 - \frac{\theta(\Omega^*)}{y_H(1 + \pi_H(\Omega^*))}} \right|. \tag{39}$$

The following lemma shows that the ratio of the gambling share, (39), to the total size of the credit market, (38) is increasing in the share of delegation whenever $\Omega^* > \hat{\Omega}$.

**Lemma 4** *For $\Omega^* > \hat{\Omega}$ in both Cont-Mod and Cont-Agg equilibria direct traders fraction of total borrowing decreases in $\Omega^*$ at a rate proportional to $\frac{1}{(1-\Omega^*)^2}$, where the constant of proportionality is larger in the Cont-Agg equilibrium.*

To complete our analytical results, we will argue in Section 4 that under reasonable parameter values the value of long bond holdings relative to the size of the economy monotonically increases in the share of delegated capital for any $\Omega^*$, and the implied increase in the borrowing and lending activity is quantitatively large. Furthermore, almost all of the increase is explained by gambling share. Thus, our model suggests that financial intermediaries increased competition for fund flows might explain the multiple fold increase of the repo market and derivative markets like S&P futures and options during the last decades before the financial crisis in 2007/2008.[21]

### 3.5 Systematic versus Idiosyncratic Risk

Similar to the Lucas economy, our model has systematic risk, but no idiosyncratic risk. Incorporating a source of idiosyncratic risk into the dynamic model is beyond the scope of the current paper. However, to understand the qualitative implications of introducing a source of idiosyncratic risk we have considered a stripped down two period example where we add the ability to take idiosyncratic risk by entering into a zero net supply futures contract with a futures price of zero. The future is a derivative on a sunspot: whether the long or the short positions pay-off depends on the flip of a coin. For direct traders it is obviously sub-optimal to take a position in the futures.

When the share of delegation is small, and assuming that $\kappa$ is not too large, all funds follow a contrarian strategy and do not use the futures. The pure contrarian strategy dominates since to increase tracking error relative to the market it takes advantage of the negative skewness in returns: increasing tracking error by exposure to idiosyncratic risk is less efficient.

As the size of the fund industry increases funds' trading begins to impact prices in equilibrium, reducing the attractiveness of the contrarian strategy and skewness becomes less negative. When the share of delegation becomes sufficiently large heterogeneity in fund strategies emerges. Some managers start using a less conservative yet still contrarian strategy combined with a position in the futures.[22] Such a strategy allows them in addition to being above the kink in the low state, although admittedly to a lesser extent than the pure contrarian strategy, to also be half of the time above the kink in the high state. As the share of delegation continues to grow more and more funds migrate to this strategy, and open interest in the futures increases.

At a higher share of delegation threshold some funds start following the aggressive strategy.[23] Keeping in mind that due to the negative skewness in returns the aggressive strategy considerably underperforms the market in the low state, the aggressive strategy does not add a position in the futures contract. The amount of exposure to idiosyncratic risk required in order sometimes be above the kink in the low state distorts the returns too much. As the share of delegation grows further the fraction of funds that follow the aggressive strategy increases, and the amount of borrowing and

[21] See Gorton-Metrick (2010) for estimation on the change of the size of the repo market or institutional details on this market.

[22] If the absolute value of skewness is small this strategy may be the one that is already used when the share of delegation is zero.

[23] Similar to our model in intermediate levels of $\kappa$ or if $p$ is large instead of an aggressive strategy the moderate strategy is used.

lending grows. Similar to our model, for large shares of delegation in equilibrium there is always heterogeneity in strategies.

# 4 Numerical examples

In this section we present some simple calibrated examples to show that the magnitude of the effects we discuss, especially trading strategies distortions and the impact on bond markets, can be quantitatively large.

Before proceeding with the examples it is important to keep in mind that we have constructed our model to highlight the potential important role that delegated portfolio management has on the equilibrium size of bond markets, and the link between the size of these markets and the endogenous emergence of heterogeneous strategies within the money management industry. To obtain a parsimonious and tractable setup we have made three important assumptions. First, we use logarithmic utility. Second, we assume a piece-wise constant elasticity incentive function. The combination of the two is helpful in delivering a tractable model. Third, to allow us to focus on a stationary equilibrium with a constant share of delegation we impose a specific structure of periodically reborn investors choosing to be clients or direct traders. We conjecture that our insights paired with a more flexible model with habit formation, Epstein-Zin preferences or more complex consumption processes might be useful in the quantitative dimension, but such an exercise is outside of the scope of this paper.

We experiment with two sets of parameters for the consumption growth process (Table1) and two sets of parameters for the incentive function (Table 2). Then we conduct a sensitivity analysis with regards to the latter.

The difference between our two sets of consumption growth parameters is that the first is implied by the full post-war sample, 1946-2008, while the second one is implied by the second half of the full sample 1978-2008. We consider the moments from the shorter sample also to entertain the possibility that the distribution of the consumption growth process has changed over time. Using consumption growth data from Shiller's website we estimate the mean, standard deviation and skewness of consumption growth, and then solve for $p$, $y_H$, and $y_L$ to match these three moments. It is apparent that the biggest impact of the change on the sample is on the skewness of the process. This implies a different value of $p$ in our model.

For the incentive function we consider two specifications. The first is a minimal deviation from a constant elasticity incentive function. As shown in Proposition 8, a constant elasticity incentive function implies that managers and direct traders all hold the market portfolio and that the Sharpe ratio is identical to the one in the Lucas model benchmark. This illustrate the effect of small convexities in incentives. The second is based on estimating the incentive function using

Table 1: Consumption Growth

| Data | | |
|---|---|---|
| Period | 1946-2008 | 1978-2008 |
| Mean | 0.0216 | 0.0210 |
| Standard deviation | 0.0175 | 0.0154 |
| Skewness | -0.220 | -0.605 |
| **Model Estimated Parameters** | | |
| $p$ | 0.555 | 0.645 |
| $y_H$ | 1.038 | 1.033 |
| $y_L$ | 1.002 | 1 |

data on mutual funds.[24] [25]

Using Equation (9) We can rewrite the flows as

$$\begin{aligned} FL_t &= \ln \frac{w^M_{t+1}}{\rho_{t+1}\left(\alpha^M_t\right)\left(1-\psi^M_t\right)w^M_t} = \\ &= \ln \Gamma_t Z_B + 1_{R_{e,t}\geq \ln\kappa} \ln \kappa^{n_B - n_A} + \left[(n_B-1)\, 1_{\upsilon_t < \ln\kappa} + (n_A-1)\, 1_{R_{e,t}\geq \ln\kappa}\right] R_{e,t} \\ &= \ln \Gamma_t Z_B + (n_B-1)\, R_{e,t} + (n_A - n_B)\, 1_{R_{e,t}\geq\ln\kappa}\left(R_{e,t} - \ln\kappa\right) \end{aligned}$$

where $R_{e,t}$ is the excess log return above the market:

$$R_{e,t} = \left(\ln \rho_{t+1}\left(\alpha^M_t\right) - \ln \frac{q_{t+1}+\delta_{t+1}}{q_t}\right).$$

To estimate $n_A, n_B \ln\kappa$ we therefore estimate the model

$$FL_t = \alpha_t + \beta_1 R_{e,t} + \beta_2 1_{R_{e,t}\geq\ln\kappa}(R_{e,t} - \ln\kappa).$$

Our strategy is to run a large number of panel regressions with a different fix $\ln\kappa$ in each and search for the best fit. Details on the procedure and the results are in Appendix D [26].

In all specifications we set the discount rate to $\beta = 0.98$, which implies a reasonable 2% annual management fee for managers. We set $\lambda = 0.5$ and $Z_B = 0.01$ to make sure that the equilibrium

[24] In our model managers should represent the whole financial intermediary sector including mutual funds, commercial banks, hedge funds, retirement funds etc. Our choice to use mutual fund data is based on data availability, and the fact that most empirical work on the estimation of flow-performance relationships is on mutual funds.

[25] We would like to thank Dong Lou for providing us with mutual fund performance and flow data and Eszter Nagy for providing research assistance.

[26] Appendix D is for the publication on web only, and it is provided as a separate file.

Table 2: Incentive Function

| | Minimal Deviation | Estimated from Fund Data |
|---|---|---|
| $n_A$ | 1.01 | 1.9 |
| $n_B$ | 1 | 1.4 |
| $\kappa$ | 1 | 1.05 |

exists under all sets of parameters.

For the interpretation of the figures as implied time-series, note that the $\Omega^*$ values corresponding to the share of direct equity holdings in 1960, 1980 and 2007 would be $\Omega^*_{60} = 0.15, \Omega^*_{80} = 0.52, \Omega^*_{07} = 0.78$.

### 4.1 Quantitative results

Consider first the minimal deviation scenario. As shown in the first row of Figure 2, even slight convexities lead to the emergence of heterogeneous fund strategies: 50% of managers hold 85% of their capital under management in stocks, while the other 50% hold 115%. Given these strategies, naturally the size of the bond market relative to total investment increases as the share of delegation increases. Even if, as shown in the top row of Figure 3 this increase seems small (from zero to 7%), considering that we deviate only slightly from linear incentives it is still a significant effect. The impact on the Sharpe ratio relative to the one in the Lucas economy is negligible. Why strategies react so strongly to little convexity? The reason is that for managers the cost of gambling is second order as they come from risk aversion, while the benefits in flows in the good state are first order because of the kink in the incentive function.

The second (third) row of Figure 2 display the strategies for the incentive function implied by the data, for the long (short) sample. It is apparent that the large convexity imply very large absolute positions in bonds. Focusing on the consumption process from the long sample,when the share of delegated capital is close to zero managers following the contrarian strategy invest 9 times their capital into the bond and short-sell the stock. They decrease this ratio to 7 as the share of delegated capital reaches $\hat{\Omega}$, and then increase it again to 9 as the share of delegated capital approaches one. Managers following the aggressive strategy exist in the market only if the share of delegation exceeds $\hat{\Omega}$. At $\Omega = \hat{\Omega}$ they borrow up to 10 times the size of their capital under management to invest in stocks and increase this ratio to over 11 when they approach the point that only managers populate the market. The right panel shows that the fraction of managers following the contrarian strategy decreases from 100% below $\hat{\Omega}$ to 50% when the share of delegation is close to 1. Using moments from the short sample, strategy patterns are similar but more extreme. However, aggressive managers start entering the market at a higher share of delegation and the rate at which they enter, as a function of the share of delegation, is slower. While these numbers are perhaps unrealistic at the industry level, they illustrate well the strengths of incentive to deviate from the market portfolio induced by convexities in the flow-performance relationship.

Corresponding to these extreme positions, the left panels of the bottom two rows of Figure 3 show that the size of the bond market increases considerably as the share of delegation increases. The gross amount of long bond positions is around 100% of total net investment in the economy when the share of delegation is 25%, increases to about 2 times total net investment in the economy when the share of delegation is around 40%, and increases considerably as the share of delegation increases further. The initial small increase in the region $\Omega^* < \hat{\Omega}$ is due to non-gambling positions. Beyond $\hat{\Omega}$ managers start to utilize heterogeneous strategies, and gambling positions start to emerge as an important contributing factor that increases the size of bond markets as the share of delegation increases. The fact that the percentage of managers following the aggressive strategy increases from zero to 50% throughout this region combined with the fact that in this region both contrarian and aggressive fund strategies become more extreme as the share of delegation increases amplify the expansion of bond markets even further.

Interestingly, as is evident in the figure with these parameter values the effect of delegation on the Sharpe ratio relative to the level in the Lucas model is significant.[27] Considering the share of delegation in 1960, 1980 and 2007, our model suggests that the Sharpe ratio should have maxed between 1960 and 1980 and should have decreased since then.

The graphs in the right hand side panels show the skewness of equity returns. When the share of delegation is low the skewness of market returns is close to that of the consumption growth process. Keeping in mind that contrarian managers strategies take advantage of negative skewness, in the region where all managers follow contrarian strategies an increase in the share of delegation increases skewness, as a result of the price impact of their trades. Above $\hat{\Omega}^*$ aggressive funds with aggressive trading strategies start to emerge, and the impact of their trades more than offsets that of the contrarian funds leading to a decline in skewness.[28]

Finally, we conduct sensitivity analysis of our results to the parameters by considering a wide range around the benchmark parameters $k, n_A$ and $p$ under the short sample scenario. The corresponding graphs are in Figures 4-6. Consider first an increase in the convexity by increasing $n_A$. As expected, an increase in the convexity leads to more extreme strategies both for managers holding a contrarian portfolio and those holding the levered portfolio. This also leads to a sharper increase in the amount of outstanding bonds as the share of delegation increases. The effect of increasing $\kappa$ is a bit more subtle. On one hand, for large enough changes in $\kappa$, the system moves between different type of equilibria. This is why we see a break around $\kappa = 1.1$ in all of the graphs in the second

[27]The state price of the low state relative to the high state increases by 2% (5%) between a share of delegation of 0 and $\Omega^* < \hat{\Omega}$ for the long (short) sample, where 70% (30%) is due to the capital-flow (wealth) effects; results not shown.It then declines to 1% (3.5%) in the long (short) sample as the share of delegation approaches one.

[28]In the region $\Omega^* > \hat{\Omega}$ and increase in the share of delegation always implies an increase in the difference between the capital delegated to non-contrarian and contrarian managers. That is, $\Omega^*(1-\mu_{AB}(\Omega^*))-\Omega^*\mu_{AB}(\Omega^*)$ is increasing in $\Omega^*$.

row of Figures 4-6. The system moves at that point from a Cont-Agg equilibrium to a Cont-Mod equilibrium, because reaching the high-elasticity segment by an aggressive portfolio becomes too costly. On the other hand, note from (17) that the locally optimal portfolios are effected by $\kappa$ only through prices. As we see on the corresponding plots, a higher $\kappa$ increases the return on risk in a Cont-Agg equilibrium which leads to less extreme portfolios and smaller increase in gross amount of bond borrowing and lending. Finally, making the consumption process even more skewed by increasing $p$, leads to complex comparative statics. It is so, because $p$ effects portfolios both directly and indirectly through prices. The plots suggest that a larger $p$ typically decreases the amount of outstanding long bond positions. Also, it increases the Sharpe ratio and makes the contrarian portfolio less extreme whenever the aggressive portfolio is held in equilibrium and has an opposite effect otherwise.

## 5 Conclusion

In this paper we have introduced delegation into a standard Lucas exchange economy, where in equilibrium some investors trade on their own account, but others (clients) decide to delegate trading in financial assets to funds. Flow-performance incentive functions describe how much capital fund clients provide to funds at each date as a function of past performance.

Given the significantly increased fraction of capital that is managed by delegated portfolio management intermediaries over the past 30 year, our analysis has focused on the interactions of the increased share of delegated capital and the empirically observed convex flow-performance relationship. We have been especially interested on the effects of this interaction on asset prices and on agents' optimal portfolios. The basic setup of our economy is intentionally close to the original Lucas model, allowing us a clear comparison of how delegation changes equilibrium dynamics in the Lucas economy.

Our model implies that with convex flow-performance relationship the average fund outperforms the market in recessions and underperforms in expansions; consistent with empirical evidence. When the share of capital that is delegated is low, all funds follow the same strategy. However, when the equilibrium share of delegated capital is high funds with identical incentives utilize heterogeneous trading strategies, trade among themselves, and fund returns are dispersed in the cross-section. As the share of delegated capital increases, so does the fraction of managers holding levered portfolios. Thus, the gross amount of borrowing and lending increases. We connect this fact to the sharpe increase in the size of repo markets and outstanding open interest in futures markets over the last decades. We also show that delegation affects the Sharpe ratio through two channels: discount rate and capital flow. The two work in opposite directions leading in general to an inverse U-shape relation between the share of capital that is delegated and the Sharpe ratio.

Our methodological contribution is to simplify the flow-performance relationship into a piece-wise constant elasticity function. The combination of log utility and piece-wise constant elasticity enables us to derive explicit expression for different model quantities. Arguably, we do not use our modelling framework to its full potential, because we impose a structure which implies a constant share of delegated capital for a given set of parameters. Although, we consider this framework a natural first step, our framework is well suited for the analysis of a truly dynamic structure where the share of delegated capital is a time-varying state variable. With such an extension we could investigate the changing role of different financial intermediaries over the business cycle. This extension is left for future work.

# References


Adrian, T., H. S. Shin, 2010, "Liquidity and Leverage", *Journal of Financial Intermediation*, 19, 418-437.

Allen, F., 2001, "Presidential Address: Do Financial Institutions Matter?", *Journal of Finance*, 56(4), 1165-1175.

Agarwal, V., N.D. Daniel and N.Y. Naik, 2004, "Flows, Performance, and Managerial Incentives in Hedge Funds", London Business School, mimeo.

Basak, S., A. Pavlova and A. Shapiro, 2007, "Optimal Asset Allocation and Risk Shifting in Money Management" *Review of Financial Studies* 20, 1583-1621.

Basak, S., D. Makarov, 2010, "Strategic Asset Allocation in Money Management",Working paper, London Business School.

Berk, J.B. and R.C. Green, 2004, "Mutual Fund Flows and Performance in Rational Markets", *Journal of Political Economy* 112(6), 1269-1295.

Biais B., and Casamatta, C., 1999. "Optimal Leverage and Aggregate Investment," *The Journal of Finance*, 54 (4), Papers and Proceedings, Fifty-Ninth Annual Meeting, American Finance Association, New York, New York, January 4-6 1999,1291-1323.

Bhamra, H. and R. Uppal, 2007, "The Effect of Improved Risk Sharing on Stock-Market Return Volatility When Agents Differ in Risk Aversion," *The Review of Financial Studies,* forthcoming.

Cadenillas, A., Cvitanic, J., and Zapatero, F., 2007. "Optimal Risk-sharing with Effort and Project Choice," *Journal of Economic Theory*, 133(1), 403-440.

Carpenter, J.N., 2000. "Does Option Compensation Increase Managerial Risk Appetite?," *Journal of Finance*, 55(5), 2311-2331

Chan, Y., and L. Kogan, 2002, "Catching Up with the Joneses: Heterogeneous Preferences and the Dynamics of Asset Prices", *Journal of Political Economy* 110, 1255-1285.

Cuoco, D. and R. Kaniel, 2010, "Equilibrium Prices in the Presence of Delegated Portfolio Management", forthcoming *Journal of Financial Economics*.

Chevalier, J., and G. Ellison, 1997, "Risk Taking by Mutual Funds as a Response to Incentives", *Journal of Political Economy*, 105, 1167-1200.

Dasgupta, A. and A. Prat, 2006, "Financial Equilibrium with Career Concerns", *Theoretical Economics*, 1, 67-93.

Dasgupta, A. and A. Prat, 2008, "Information Aggregation in Financial Markets with Career Concerns", *Journal of Economic Theory*, 143, 83-113, December 2008.

Dow, J. and G. Gorton, 1997, "Noise Trading, Delegated Portfolio Management, and Economic Welfare", *Journal of Political Eoconomy,* 105(5), 1024-1050.

Dumas, B., 1989, "Two-Person Dynamic Equilibrium in the Capital Market", *Review of Financial Studies* 2, 157-188.

FSA (U.K. Financial Services Authority). 2009, "The Turner Review: A Regulatory Response to the Global Banking Crisis", London.

FSB (Financial Stability Board). 2009, "Report of the Financial Stability Forum on Addressing Procyclicality in the Financial System", Basel.

Glode, V., 2010, "Why Mutual Funds Underperform?,' *Journal of Financial Economics*, forthcoming.

Goetzmann W. N., J. E. Ingersoll and S. A. Ross, 2003, "High-Water Marks and Hedge Fund Management Contracts," *Journal of Finance*,58(4),1685-1718.

Gorton, G., A. Metrick, 2010, "Securitized Banking and the Run on Repo,' Working Paper, Yale.

Gromb, D. and D. Vayanos, 2002, "Equilibrium and Welfare in Markets with Financially Constrained Arbitrageurs", *Journal of Financial Economics* 66, 361-407.

Guerrieri, V. and P. Kondor, 2009, "Fund Managers, Career Concerns, and Asset Price Volatility", NBER w14898.

He, Z., and A. Krishnamurthy, 2008. "Intermediary Asset Pricing", Northwestern University, mimeo.

Hellwig, M., 2009. "A Reconsideration of the Jensen-Meckling Model of Outside Finance," *Journal of Financial Intermediation*, 18(4), 495-525.

Sirri, E., and P. Tufano, 1998, "Costly Search and Mutual Fund Flows", *Journal of Finance* 53, 1589-1622.

Kacperczyk, M., S. V. Nieuwerburgh, L. Veldkamp,2010, "Attention Allocation over the business cycle", Working Paper, NYU.

Kaplan, S.N. and A. Schoar,2005, "Private Equity Performance: Returns, Persistence, and Capital Flows", *Journal of Finance* 60(4), 1791-1823.

Kosowski, R., 2006, "Do mutual funds perform when it matters most to investors? US mutual fund performance and risk in recessions and expansions", Working Paper, Imperial College.

Jensen, M. and Meckling, W, 1976. "Theory of the Firm: Managerial Behavior, Agency Costs and Ownership Structure", *Journal of Financial Economics*, 3(4),

305-360.

LONGSTAFF, F.A., AND J. WANG, 2008, "Asset pricing and the credit markets", UCLA Anderson School and MIT Sloan School, mimeo.

LYNCH, A. W., AND J. WACHTER, 2007, "Does mutual fund performance vary over the business cycle?,' Working Paper, New York University.

MAKAROV, I. AND G. PLANTIN, 2010, "Rewarding Trading Skills Without Inducing Gambling", Working Paper, London Business School.

MALLIARIS, S.G., AND H. YAN, 2010, "Reputation Concerns and Slow-Moving Capital", Working paper, Yale University.

OU-YANG, H., 2003. "Optimal Contracts in a Continuous-Time Delegated Portfolio Management Problem", *Review of Financial Studies*, 16(1), 173-208.

PALOMINO, F. AND PRAT, A., 2003. "Risk Taking and Optimal Contracts for Money Managers", *Rand Journal of Economics*, 34(1), 113-137.

ROSS, S., 2004, "Compensation, Incentives, and the Duality of Risk Aversion and Riskiness", *Journal of Finance*,59(1).

SHLEIFER, A., AND R.W. VISHNY, 1997. "The Limits of Arbitrage", *Journal of Finance*, 52(1), 35-55.

VAYANOS, D., 2003, "Flight to quality, flight to liquidity, and the pricing of risk", LSE, mimeo.

VAYANOS, D., AND P. WOOLLEY, 2009, "An Institutional Theory of Momentum and Reversal", LSE, mimeo.

WANG, J., 1996, "The Term Structure of Interest Rates in a Pure Exchange Economy with Heterogeneous Investors", *Journal of Financial Economics* 41, 75-110.

WERMERS, R., 2000, "Mutual Fund Performance: An Empirical Decomposition into Stock-picking Talent, Style, Transactions Costs, and Expenses,' *Journal of Finance*, 55, 1655-1703.

# 6 Appendix

# A Existence and characterization of equilibria

In this part we prove all statements in section 2. We largely follow the logic described in the main text.

## A.1 Equilibrium portfolios

First, we focus on the consistency of the described equilibrium portfolios and the market clearing interest rate. In particular, we show that for any given $\Omega^* \in (0,1)$ and a pair of price-dividend ratios $\pi_H, \pi_L > 0$ defined in (31) Propositions 3-4, Proposition 6 and Lemmas 2-1 describe a set of equilibrium strategies for managers and direct traders and prices which are consistent with the problem of managers and direct traders and the market clearing condition of the bond market.

We separate the proof in two parts. First, we consider a modified economy where managers' incentives are smooth. In particular, for the given parameters consider the set of $\mu_{lh} = \mu_{BA}, \mu_{AB}, \mu_{BB}$ described in Proposition 3-4 . We allocate managers into groups of size $\mu_{lh}$ and call each group *the group lh*. We replace the actual incentive function, (8), of each manager in group $lh$ by

$$g_{lh}(v) = \left\{ \begin{array}{ccc} Z_h(v)^{n_h - 1} & if & s_{t+1} = H \\ Z_l(v)^{n_l - 1} & if & s_{t+1} = L \end{array} \right\}.$$

We show that in this modified economy, the expressions in Propositions 3-4, Proposition 6 are consistent with an equilibrium.

Second, we show that the statements also hold in the original economy.

### A.1.1 Equilibrium portfolios (Proposition 6, Lemma 1 Proposition 3) in the modified problem

First we show that the market clearing conditions of the asset markets

$$\Omega^* \alpha^1 + (1 - \Omega^*)\alpha^2 = 1. \tag{40}$$

imply that in the modified economy **Proposition 6** has to hold. By simple substitution

$$\frac{\Omega^* \sum\limits_{lh} \mu_{lh}\xi_{lh} + (1-\Omega^*)p}{1 - \frac{y_L(1+\pi_L)}{\theta}} + \frac{\Omega^* \sum\limits_{lh} \mu_{lh}\left(1-\xi_{lh}\right) + (1-\Omega^*)p}{1 - \frac{y_H(1+\pi_H)}{\theta}} =$$

$$= \tilde{\xi}(\Omega^*) \frac{1}{1 - \frac{y_L(1+\pi_L)}{\theta}} + \left(1 - \tilde{\xi}(\Omega^*)\right) \frac{1}{1 - \frac{y_H(1+\pi_H)}{\theta}} = 1$$

which gives (32).

Second, we show that in the modified economy **Lemma 1** holds. We show the statement for a low shock. The proof for the high shock is analogous.

The return of a manager holding portfolio $\alpha_{lh}$ at the end of the period is

$$\begin{aligned}\rho_{t+1}(\alpha_{lh}, L) &= \alpha_{lh}\left(\frac{\delta_{t+1}+q_{t+1}}{q_t} - R_t\right) + R_t = R_t\left((1-\xi_{lh})\frac{\left(\frac{y_L(1+\pi_L)}{\theta}-1\right)}{1-\frac{y_H(1+\pi_H)}{\theta}} + (1-\xi_{lh})\right) = \\ &= R_t\left((1-\xi_{lh})\left(\frac{y_H(1+\pi_H)-y_L(1+\pi_L)}{y_H(1+\pi_H)-\theta}\right)\right)\end{aligned}$$

where we used the definition of $\pi_H, \pi_L$, $\theta$ and $\alpha_{lh}$ and that (32) implies

$$\alpha_{lh} = 1 - \frac{1-\frac{\xi_{lh}}{\tilde{\xi}(\Omega^*)}}{1-\frac{\theta(\Omega^*)}{y_H(1+\pi_H(\Omega^*))}}.$$

. Using (32) we can rewrite this as

$$\frac{(1-\xi_{lh})}{1-\tilde{\xi}(\Omega)}\frac{\delta_t}{q_t}y_L(1+\pi_L) = \frac{(1-\xi_{lh})}{1-\tilde{\xi}(\Omega)}\frac{q_{t+1}+\delta_{t+1}}{q_t}. \tag{41}$$

This gives (26).

Finally, we show that in the modified economy, prices implied by $\pi_H, \pi_L$ and (32) imply that any manager or direct trader has a value function of the form (11)-(12) and her consumption and portfolio choices are as described in **Proposition 3**.

We show the proof for a manager. The case for the direct trader follows very similarly. For any $t \geq 1$, conjecture that the value function for a manager in group $lh$ has the form of

$$V^{lh}\left(w_t^M, s_t, \Omega^*\right) = \frac{1}{1-\beta}\ln w_t^M + \Lambda^{lh}(s_t, \Omega^*).$$

Under our conjecture we can write problem as

$$\begin{aligned}V^{lh}\left(w^M, s_t, \Omega^*\right) = \max_{\alpha,\psi^M} \ln\psi^M w^M + \frac{\beta}{1-\beta}p\ln\Gamma_t Z_h(\upsilon)^{n_h-1} w_{t+1,-}^M + (1-p)\ln\Gamma_t Z_l(\upsilon)^{n_l-1} w_{t+1,-}^M \\ + \beta E\left(\Lambda_{lh}^M(s_{t+1}, \Omega^*)\right)\end{aligned}$$

for the given group $lh$. Let us fix an arbitrary $\alpha$. The first order condition in $\psi^M$ gives

$$1-\psi^M = \beta.$$

We rewrite the problem as

$$
\begin{aligned}
V^{lh}\left(w^M, s_t, \Omega^*\right) = & \max_{\alpha} \ln\left(1-\beta\right) w^M + \\
& + \frac{\beta}{1-\beta} p \ln Z_h \left( \frac{\rho_{t+1}\left(\alpha, H\right)}{\frac{q_{t+1}(H)+\delta_{t+1}(H)}{q_t}} \right)^{n_h - 1} \rho_{t+1}\left(\alpha, H\right) \beta w^M + \\
& + \frac{\beta}{1-\beta} \left(1-p\right) \ln Z_l \left( \frac{\rho_{t+1}\left(\alpha, L\right)}{\frac{q_{t+1}(L)+\delta_{t+1}(L)}{q_t}} \right)^{n_l - 1} \rho_{t+1}\left(\alpha, L\right) \beta w^M \\
& + \beta \left( p \Lambda^{lh}\left(\Omega_t^*, H\right) + \left(1-p\right) \Lambda^{lh}\left(\Omega_t^*, L\right) \right)
\end{aligned}
$$

Note that this problem is strictly concave in $\alpha$.in the modified economy. The first order condition is

$$
p n_h \frac{\frac{q_{t+1}(H)+\delta_{t+1}(H)}{q_t} - R_t}{\alpha\left(\frac{q_{t+1}(H)+\delta_{t+1}(H)}{q_t} - R_t\right) + R_t} + \left(1-p\right) n_l \frac{\left(\frac{q_{t+1}(L)+\delta_{t+1}(L)}{q_t} - R_t\right)}{\alpha\left(\frac{q_{t+1}(L)+\delta_{t+1}(L)}{q_t} - R_t\right) + R_t} = 0
$$

which is equivalent to

$$
\begin{aligned}
& \xi_{lh} \frac{\frac{q_{t+1}(H)+\delta_{t+1}(H)}{q_t} - R_t}{\alpha^i\left(\frac{q_{t+1}(H)+\delta_{t+1}(H)}{q_t} - R_t\right) + R_t} + \\
& + \left(1-\xi_{lh}\right) \frac{\left(\frac{q_{t+1}(L)+\delta_{t+1}(L)}{q_t} - R_t\right)}{\alpha^i\left(\frac{q_{t+1}(L)+\delta_{t+1}(L)}{q_t} - R_t\right) + R_t} = 0.
\end{aligned} \tag{42}
$$

Solving for $\alpha$ gives $\alpha_{lh}$. Substituting back $\alpha_{lh}$ and $\psi^M$ into the value function implies that our conjecture is correct with the choice of function $\Lambda^{lh}\left(s_t, \Omega^*\right)$ solving

$$
\begin{aligned}
\Lambda^{lh}\left(s_t, \Omega^*\right) = & \ln\left(1-\beta\right) + \\
& + \beta \frac{1}{1-\beta} p \ln Z_h \left( \frac{\xi_{lh}}{\tilde{\xi}\left(\Omega^*\right)} \right)^{n_h} \frac{1}{\pi_{s_t}} y_H \left(1+\pi_H\right) \beta + \\
& + \beta \frac{1}{1-\beta} \left(1-p\right) \ln Z_l \left( \frac{1-\xi_{lh}}{1-\tilde{\xi}\left(\Omega^*\right)} \right)^{n_l} \frac{1}{\pi_{s_t}} y_L \left(1+\pi_L\right) \beta \\
& + \beta \left( p \Lambda^{lh}\left(H, \Omega^*\right) + \left(1-p\right) \Lambda^{lh}\left(L, \Omega^*\right) \right)
\end{aligned} \tag{43}
$$

which has the conjectured form.

### A.1.2 Equilibrium portfolios in the original problem (Proposition 4 and Lemma 2)

In this part, we show that the strategies described by Propositions 3-4 for given price dividend ratios $\pi_H, \pi_L$ and interest rate given in Proposition 6 which we have found to be optimal in the modified economy, they are still optimal in the original economy.

To make sure that the prescribed portfolios remain optimal in the original economy, we prove the following statements.

1. Whenever $\mu_{l_1h_1} > 0$ for a given $lh = l_1h_1$ in Proposition 4 then

$$V^{l_1h_1}\left(w_t^M, s_t, \Omega^*\right) \geq V^{l_2h_2}\left(w_t^M, s_t, \Omega^*\right) \tag{44}$$

   for any $l_2h_2$ with strict equality if $\mu_{l_2h_2} > 0$. That is, deviation to an other locally-optimal portfolio from the equilibrium portfolios is suboptimal.

2. Whenever $\mu_{l_1h_1} > 0$ for a given $lh = l_1h_1$ in Proposition 4 then

$$\frac{\xi_{l_1h_1}}{\tilde{\xi}\left(\Omega^*\right)} > (<)\kappa \tag{45}$$

   if $h_1 = A(B)$ and

$$\frac{1-\xi_{l_1h_1}}{1-\tilde{\xi}\left(\Omega^*\right)} > (<)\kappa \tag{46}$$

   if $l_1 = A\left(B\right).$ This ensures that the prescribed portfolios remain locally optimal in the original economy. This also implies Lemma 2.

Thus, first we introduce the analytical formulas for deviations from the prescribed equilibrium portfolios. Second, we show that condition (44) holds for $\Omega^* = 0$. Third, we show that condition (44) holds for $\Omega^* > 0$. Finally, we show that conditions (45)-(46) holds for any $\Omega^*$.

We also show that in Proposition 4,

$$\hat{\kappa}_{high} \equiv \exp\left(\frac{\ln\frac{n_A}{n_B}}{\left(1-\frac{n_B}{n_A}\right)} + 1\right) \tag{47}$$

$$\hat{\kappa}_{low} = \exp\left(\frac{n_B \ln n_B + n_A \ln n_A - (n_A+n_B)\ln\frac{n_B+n_A}{2}}{n_A - n_B}\right) \tag{48}$$

and $\hat{p}$ is given by the unique solution in $\left[\frac{1}{2}, 1\right]$ of

$$\Delta^{AB-BB}\left(\hat{p}\right) = 0 \tag{49}$$

where

$$\Delta^{l_1h_1-l_2h_2}(p) \equiv p \ln \frac{Z_{h_1}}{Z_{h_2}} \frac{\left(\frac{\xi_{l_1h_1}}{p}\right)^{n_{h_1}}}{\left(\frac{\xi_{l_2h_2}}{p}\right)^{n_{h_2}}} + (1-p) \ln \frac{Z_{l_1}}{Z_{l_2}} \frac{\left(\frac{1-\xi_{l_1h_1}}{1-p}\right)^{n_{l_1}}}{\left(\frac{1-\xi_{l_2h_2}}{1-p}\right)^{n_{l_2}}},$$

while $\bar{p}$ is given by the unique solution in $\left[\frac{1}{2}, 1\right]$ of

$$\bar{p} \exp\left(\frac{\Delta^{BA-BB}(\bar{p})}{\bar{p}(n_A - n_B)}\right) + (1-\bar{p}) \exp\left(\frac{\Delta^{AB-BB}(\bar{p}_{BA-AB})}{(n_A - n_B)(1-\bar{p})}\right) = 1. \tag{50}$$

**Useful expressions for comparing value functions** Define $\tilde{V}^{l_1h_1-l_2h_2}(\Omega^*)$ as

$$\begin{aligned}\tilde{V}^{l_1h_1-l_2h_2}(\Omega^*) &\equiv \frac{1-\beta}{\beta}\left(V^{l_1h_1}\left(w_t^M, s_t, \Omega^*\right) - V^{l_2h_2}\left(w_t^M, s_t, \Omega^*\right)\right) = \\ & p \ln \frac{Z_{h_1}}{Z_{h_2}} \frac{\left(\frac{\xi_{l_1h_1}}{\tilde{\xi}(\Omega^*)}\right)^{n_{h_1}}}{\left(\frac{\xi_{l_2h_2}}{\tilde{\xi}(\Omega^*)}\right)^{n_{h_2}}} + (1-p) \ln \frac{Z_{l_1}}{Z_{l_2}} \frac{\left(\frac{1-\xi_{l_1h_1}}{1-\tilde{\xi}(\Omega^*)}\right)^{n_{l_1}}}{\left(\frac{1-\xi^1_{l_2h_2}}{1-\tilde{\xi}(\Omega^*)}\right)^{n_{l_2}}} = \\ &= \Delta^{l_1h_1-l_2h_2}(p) + p(n_{h_1} - n_{h_2}) \ln \frac{p}{\tilde{\xi}(\Omega^*)} + (1-p)(n_{l_1} - n_{l_2}) \ln \frac{1-p}{1-\tilde{\xi}(\Omega^*)},\end{aligned} \tag{51}$$

the difference in the value of following the locally optimal $l_1h_1$ and $l_2h_2$ strategies.

Note also that both in a Cont-Mod and a Cont-Agg equilibrium, we can rewrite the second part of the above expression as

$$\begin{aligned} & p(n_{h_1} - n_{h_2}) \ln \frac{p}{\tilde{\xi}(\Omega^*)} + (1-p)(n_{l_1} - n_{l_2}) \ln \frac{1-p}{1-\tilde{\xi}(\Omega^*)} = \\ &= \left\{ \begin{array}{cc} \begin{array}{c} (n_{h_1} - n_{h_2}) p \ln\left(\frac{\Omega^* n_B}{(1-p)n_A + pn_B} + (1-\Omega^*)\right) \\ -(n_{l_1} - n_{l_2})(1-p) \ln\left(\frac{n_A}{(1-p)n_A + pn_B}\Omega^* + (1-\Omega^*)\right) \end{array} & for \quad \Omega^* < \hat{\Omega} \\ \begin{array}{c} (n_{h_1} - n_{h_2}) p \ln\left(\frac{\hat{\Omega} n_B}{(1-p)n_A + pn_B} + \left(1-\hat{\Omega}\right)\right) \\ -(n_{l_1} - n_{l_2})(1-p) \ln\left(\frac{n_A}{(1-p)n_A + pn_B}\hat{\Omega} + \left(1-\hat{\Omega}\right)\right) \end{array} & otherwise \end{array} \right\} \end{aligned} \tag{52}$$

However, the value of $\hat{\Omega}$ depends on the type of the equilibrium. Denoting the type of the equilibrium in the subscript, $\hat{\Omega}_{Cont-Mod}$ and $\hat{\Omega}_{Cont-Agg}$ are defined as the solution of

$$\Delta^{AB-BB}(p) = (1-p)(n_A - n_B) \ln\left(\hat{\Omega}_{Cont-Mod} \frac{n_A}{(1-p)n_A + pn_B} + \left(1-\hat{\Omega}_{Cont-Mod}\right)\right). \tag{53}$$

and

$$\Delta^{AB-BA}(p) = (n_A - n_B)(1-p)\ln\left(\hat{\Omega}_{Cont-Agg}\frac{n_A}{(1-p)n_A + pn_B} + \left(1-\hat{\Omega}_{Cont-Agg}\right)\right) \quad (54)$$

$$-(n_A - n_B)p\ln\left(\frac{n_B\hat{\Omega}_{Cont-Agg}}{(1-p)n_A + pn_B} + \left(1-\hat{\Omega}_{Cont-Agg}\right)\right), \quad (55)$$

respectively.

**Global optimality (Proposition 4) when $\Omega^* = 0$** In this part, we show that under the classification in Proposition 4, condition (44) holds at least when $\Omega^* = 0$.Note that $\tilde{\xi}(0) = p$ by definition, so 51 implies that

$$\tilde{V}^{l_1h_1-l_2h_2}(0) = \Delta^{l_1h_1-l_2h_2}(p).$$

The following Lemmas characterize $\Delta^{l_1h_1-l_2h_2}(p)$, thus, together with expressions (47)-(48), imply the result.

**Lemma 5** $\Delta^{BA-AB}(p) < 0$.

**Proof.** Consider that

$$\Delta^{BA-AB}(p) \equiv (n_A - n_B)((1-p)-p)\ln\kappa + \ln\frac{\left(\frac{n_B}{((1-p)n_B+pn_A)}\right)^{(1-p)n_B}\left(\frac{n_A}{((1-p)n_B+pn_A)}\right)^{pn_A}}{\left(\frac{n_A}{((1-p)n_A+pn_B)}\right)^{(1-p)n_A}\left(\frac{n_B}{((1-p)n_A+pn_B)}\right)^{pn_B}}.$$

Observe that

$$\Delta^{BA-AB}(1) = -(n_A - n_B)\ln\kappa < 0$$
$$\Delta^{BA-AB}\left(\frac{1}{2}\right) = 0.$$

and

$$\frac{\partial^2\Delta^{BA-AB}(p)}{\partial^2 p} = (n_A - n_B)^3\frac{2p-1}{((1-p)n_A + pn_B)(n_B(1-p) + pn_A)} > 0.$$

Thus, there cannot be a maximum in the range $\left(\frac{1}{2}, 0\right)$. As $\Delta^{BA-AB}(p)$ must decrease at some range by continuity, its slope cannot be positive at any point. Thus, $\Delta^{BA-AB}(p) < 0$ for all $p$. ■

**Lemma 6** $\Delta^{AB-BB}(p) < 0$ *for all* $p$*, if*

$$\kappa > \hat{\kappa}_{high}.$$

*If*

$$\hat{\kappa}_{low} > \kappa.$$

*then* $\Delta^{AB-BB}(p) > 0$ *for all* $p > \frac{1}{2}$. *If* $\hat{\kappa}_{low} < \kappa < \hat{\kappa}_{high}$ *then there is* $\hat{p} > \frac{1}{2}$ *that* $\Delta^{AB-BB}(p) < 0$ *iff* $p < \hat{p}$ *and* $\Delta^{AB-BB}(\hat{p}) = 0$.

**Proof.** Note that

$$\Delta^{AB-BB}(p) \equiv -(1-p)(n_A - n_B)\ln\kappa - (pn_B + (1-p)n_A)\ln(pn_B + (1-p)n_A) + \\ + pn_B \ln n_B + (1-p)n_A \ln n_A.$$

The statement comes from simple analysis observing that

$$\begin{aligned}
\Delta^{AB-BB}(0) &= -(n_A - n_B)\ln\kappa < 0 \\
\Delta^{AB-BB}(1) &= 0 \\
\frac{\partial\Delta^{AB-BB}(p)}{\partial p} &= (n_A - n_B)(\ln\kappa + 1) + (n_A - n_B)\ln(pn_B + (1-p)n_A) + n_B \ln n_B - n_A \ln n_A \\
\frac{\partial\Delta^{AB-BB}(p)}{\partial p}|_{p=1} &= (n_A - n_B)(\ln\kappa + 1) + n_A \ln\frac{n_B}{n_A} \\
\frac{\partial^2\Delta^{AB-BB}(p)}{\partial^2 p} &= -\frac{(n_B - n_A)^2}{(pn_B + (1-p)n_A)} < 0
\end{aligned}$$

and that

$$\Delta^{AB-BB}\left(\frac{1}{2}\right) = -\frac{1}{2}(n_A - n_B)\ln\kappa + \left[\frac{1}{2}(n_B \ln n_B + n_A \ln n_A) - \frac{(n_B + n_A)}{2}\ln\frac{n_B + n_A}{2}\right]$$

where the term in the bracket is positive as $x\ln x$ is a convex function. ■

**Lemma 7** *If*

$$\kappa > \hat{\kappa}_{high}$$

*then* $\Delta^{BA-BB}(p) < 0$ *for all* $p$. *If* $\hat{\kappa}_{low} < \kappa < \hat{\kappa}_{high}$ *then* $\Delta^{BA-BB}(p) > 0$ *iff* $p < \hat{p}_{BA-BB}$, *where* $\hat{p}_{BA-BB}$ *is given by*

$$\Delta(\hat{p}_{BA-BB}) = 0$$

*and it is in the range* $\left[0, \frac{1}{2}\right]$. *If* $\hat{\kappa}_{low} > \kappa$ *then* $\Delta^{BA-BB}(p) > 0$ *iff* $p < \hat{p}_{BA-BB}$ *and* $\hat{p}_{BA-BB} \in \left[0, \frac{1}{2}\right]$.

**Proof.** Note that

$$\Delta^{BA-BB}(p) = -(n_A - n_B)p\ln\kappa - n_A p \ln\frac{((1-p)n_B + pn_A)}{n_A} - (1-p)n_B \ln\frac{((1-p)n_B + pn_A)}{n_B}.$$

The Lemma comes from the following observations.

$$\begin{aligned}
\Delta^{BA-BB}(0) &= 0 \\
\Delta^{BA-BB}(1) &= -(n_A - n_B)\ln\kappa \\
\frac{\partial^2 \Delta^{BA-BB}(p)}{\partial^2 p} &= \frac{-(n_A - n_B)^2}{((1-p)\,n_B + p n_A)} < 0
\end{aligned}$$

and

$$\frac{\partial \Delta^{BA-BB}(p)}{\partial p} = -(n_A - n_B)\ln\kappa - (n_A - n_B)\ln((1-p)\,n_B + p n_A) +$$
$$+ n_A \ln n_A - n_B \ln n_B - (n_A - n_B)$$

and

$$\frac{\partial \Delta^{BA-BB}(p)}{\partial p}|_{p=0} = -(n_A - n_B)(\ln\kappa + 1) + n_A \ln \frac{n_A}{n_B}.$$

Finally,

$$\Delta^{BA-BB}\left(\frac{1}{2}\right) = -(n_A - n_B)\frac{1}{2}\ln\kappa + \left[\frac{1}{2}(n_B \ln n_B + n_A \ln n_A) - \frac{(n_A + n_B)}{2}\ln\frac{n_B + n_A}{2}\right]$$

where the term in the bracket is positive as the function $x \ln x$ is convex . ■

**Global optimality (Proposition 4) when $\Omega^* > 0$** In this part, we prove that condition (44) holds for any $\Omega^* > 0$ under the classification in Proposition 4. We start with two Lemmas.

**Lemma 8** *If either $\hat{\kappa}_{low} < \kappa < \hat{\kappa}_{high}$ and $p > \hat{p}$ or $\kappa < \hat{\kappa}_{low}$, there is $\hat{\Omega} \in (0,1)$ which solves*

$$\Delta^{AB-BB}(p) = (1-p)(n_A - n_B)\ln\left(\hat{\Omega}\frac{n_A}{(1-p)\,n_A + p n_B} + \left(1 - \hat{\Omega}\right)\right)$$

**Proof.** We have shown in Lemma 6 that under the conditions of this Lemma $\Delta^{AB-BB}(p) > 0$. As the left hand side is zero for $\hat{\Omega} = 0$, we only have to prove that

$$\Delta^{AB-BB}(p) < (1-p)(n_A - n_B)\ln\left(\hat{\Omega}\frac{n_A}{(1-p)\,n_A + p n_B} + \left(1 - \hat{\Omega}\right)\right)|_{\hat{\Omega}=1}.$$

Substituting in the expression $\Delta^{AB-BB}(p)$ from Lemma 6 shows that the inequality is equivalent to

$$0 < (1-p)(n_A - n_B)\ln\kappa + n_B(\ln(p n_B + (1-p)\,n_A) - p \ln n_B - (1-p)\ln n_A)$$

which holds by the concavity of the logarithmic function. ■

**Lemma 9** *Consider the system in $p$ and $\Omega$*

$$\begin{aligned}
\Delta^{AB-BB}(p) &= (n_A - n_B)(1-p)\ln\left(\Omega\frac{n_A}{(1-p)n_A + pn_B} + (1-\Omega)\right) \\
\Delta^{BA-BB}(p) &= (n_A - n_B)p\ln\left(\frac{n_B\Omega}{(1-p)n_A + pn_B} + (1-\Omega)\right).
\end{aligned}$$

*It has no solution if*

$$\hat{\kappa}_{low} < \kappa \tag{56}$$

*and it has a single solution $(\bar{p}, \bar{\Omega})$ for which $\bar{p} > \frac{1}{2}$ otherwise.*

**Proof.** Note that the system is equivalent to

$$\begin{aligned}
\exp\left(\frac{\Delta^{AB-BB}(p)}{(n_A - n_B)(1-p)}\right) &\equiv \left(\Omega\frac{n_A}{(1-p)n_A + pn_B} + (1-\Omega)\right) \\
\exp\left(\frac{\Delta^{BA-BB}(p)}{(n_A - n_B)p}\right) &= \left(\frac{n_B\Omega}{(1-p)n_A + pn_B} + (1-\Omega)\right),
\end{aligned}$$

hence, any solution of the system has to satisfy

$$\tilde{\Pi}(p) \equiv (1-p)\exp\left(\frac{\Delta^{AB-BB}(p)}{(n_A - n_B)(1-p)}\right) + p\exp\left(\frac{\Delta^{BA-BB}(p)}{(n_A - n_B)p}\right) = 1.$$

From

$$\begin{aligned}
\frac{\Delta^{BA-BB}(p)}{p(n_A - n_B)} &= -\ln\kappa - \frac{n_A}{n_A - n_B}\ln\frac{((1-p)n_B + pn_A)}{n_A} \\
&\quad -\frac{(1-p)}{p}\frac{n_B}{n_A - n_B}\ln\frac{((1-p)n_B + pn_A)}{n_B} \\
\frac{\Delta^{AB-BB}(p)}{(1-p)(n_A - n_B)} &= -\ln\kappa - \frac{n_A}{n_A - n_B}\ln\frac{(pn_B + (1-p)n_A)}{n_A} \\
&\quad -\frac{p}{1-p}\frac{n_B}{n_A - n_B}\ln\frac{pn_B + (1-p)n_A}{n_B}
\end{aligned}$$

observe that this function is symmetric in the sense that if

$$\Pi(p) \equiv p\exp\left(\frac{\Delta^{BA-BB}(p)}{p(n_A - n_B)}\right)$$

then

$$\tilde{\Pi}(p) = \Pi(p) + \Pi(1-p)$$

which implies

$$\tilde{\Pi}(p) = \tilde{\Pi}(1-p).$$

Also

$$\begin{aligned}\frac{\partial \Pi(p)}{\partial p} &= e^{\frac{\Delta_{BA-BB}(p)}{p(n_A-n_B)}}\left(1+p\frac{\partial\left(\frac{\Delta_{BA-BB}(p)}{p(n_A-n_B)}\right)}{\partial p}\right)= \\ &= e^{\frac{\Delta_{BA-BB}(p)}{p(n_A-n_B)}}\frac{1}{p}\frac{n_B}{n_A-n_B}\ln\frac{((1-p)\,n_B+pn_A)}{n_B}>0.\end{aligned}$$

and

$$\lim_{p\to 0}\tilde{\Pi}(p)=\lim_{p\to 1}\tilde{\Pi}(p)=\frac{1}{\kappa}<1.$$

Thus, $\tilde{\Pi}(p)$ is increasing for $p<\frac{1}{2}$ and decreasing for $p>\frac{1}{2}$ and its maximum is at $p=\frac{1}{2}$. If $\hat{\kappa}_{low}<\kappa$ holds, then

$$\tilde{\Pi}\left(\frac{1}{2}\right)=2\Pi\left(\frac{1}{2}\right)<1,$$

which implies that $\tilde{\Pi}(p)=1$ does not have a solution. However, if $\hat{\kappa}_{low}>\kappa$ holds, then $\tilde{\Pi}(p)=1$ has two solutions. If we denote the first by $\bar{p}>\frac{1}{2}$ then the second one is $(1-\bar{p})$. Note that $n_A>n_B$ implies that a given $p'$ can be the part of the solution of our system only if $\Delta^{BA-BB}(p')<0$ and $\Delta^{AB-BB}(p')>0$. Also, by Lemmas 7-6, this is possible only if $p'>\frac{1}{2}$. Thus, the only relevant solution is $(\bar{p},\bar{\Omega})$ where $\bar{\Omega}$ solves

$$\Delta^{BA-BB}(\bar{p})=(n_A-n_B)\,\bar{p}\ln\left(\frac{n_B\bar{\Omega}}{(1-p)\,n_A+pn_B}+\left(1-\bar{\Omega}\right)\right).$$

■

To see that Proposition 4 holds, first note from (51)-(52) that

$$\tilde{V}^{AB-l_2h_2}(\Omega^*)$$

is monotonically decreasing for $l_2h_2=BA,BB$ for any $\Omega^*<\hat{\Omega}$ and constant for $\Omega^*>\hat{\Omega}$ regardless of the type of the equilibrium. This monotonicity together with Lemmas 7 and 6 imply that if either $\kappa>\hat{\kappa}_{high}$ or $\hat{\kappa}_{low}<\kappa<\hat{\kappa}_{high}$ and $p\in\left(\frac{1}{2},\hat{p}\right)$, then

$$\begin{aligned}\tilde{V}^{AB-BB}(\Omega^*) &< 0\\ \tilde{V}^{BA-BB}(\Omega^*) &< 0\end{aligned}$$

for all $\Omega^*$. Thus, the locally optimal moderate portfolio is globally optimal.

Our observations also implies that for all other cases of Proposition 4 it is sufficient to show that $\hat{\Omega}_{Cont-Mod}$ always exist in the range $(0,1)$ and whenever both $\hat{\Omega}_{Cont-Mod}<(>)\hat{\Omega}_{Cont-Agg}$

exist and Proposition 4 describes a Cont-Mod (Cont-Agg) equilibrium then

$$\hat{\Omega}_{Cont-Mod} < (>) \hat{\Omega}_{Cont-Agg}.$$

The existence of $\hat{\Omega}_{Cont-Mod}$ under the relevant parameter restrictions is ensured by Lemma 8.

To compare $\hat{\Omega}_{Cont-Mod}$ and $\hat{\Omega}_{Cont-Agg}$ consider expression

$$\Delta^{AB-BB}(p) = (1-p)(n_A - n_B) \ln \left( \Omega \frac{n_A}{(1-p) n_A + p n_B} + (1-\Omega) \right).$$

as an implicit function giving a $p$ for any given $\Omega$ whenever $\hat{\Omega}_{Cont-Mod}$ exists. Let us call this function $\mathbf{p}_1(\Omega)$. By definition, in a Cont-Mod equilibrium, $p = \mathbf{p}_1\left(\hat{\Omega}_{Cont-Mod}\right)$. Similarly,

$$\Delta^{BA-BB}(p) = (n_A - n_B) p \ln \left( \frac{n_B \Omega}{(1-p) n_A + p n_B} + (1-\Omega) \right),$$

determine a function $\mathbf{p}_2(\Omega)$ which gives a $p$ for any given $\Omega$, whenever $\hat{\Omega}_{Cont-Agg}$ exists. From (53) and (54) and the identity $\Delta^{AB-BB}(p) - \Delta^{AB-BA}(p) = \Delta^{BA-BB}(p)$ in a Cont-Agg equilibrium, $p = \mathbf{p}_2\left(\hat{\Omega}_{Cont-Agg}\right)$.

If $\hat{\kappa}_{low} < \kappa < \hat{\kappa}_{high}$ then Lemmas 6-7 imply

$$\mathbf{p}_2(0) < \frac{1}{2} < \mathbf{p}_1(0)$$

and Lemma 9 ensures that the functions $\mathbf{p}_1(\Omega), \mathbf{p}_2(\Omega)$ do not cross in the space $[0,1] \, X \, [0,1]$. That is,

$$\hat{\Omega}_{Cont-Mod} < \hat{\Omega}_{Cont-Agg}$$

for all possible $p$. This implies a Cont-Mod equilibrium when $\Omega^* > \hat{\Omega}_{Cont-Mod}$.

If $\hat{\kappa}_{low} > \kappa$ then Lemmas 6-7 imply that

$$\mathbf{p}_1(0) < \frac{1}{2} < \mathbf{p}_2(0)$$

and Lemma 9 ensures that the functions $\mathbf{p}_1(\Omega), \mathbf{p}_2(\Omega)$ cross exactly once in the space $[0,1] \, X \left[\frac{1}{2}, 1\right]$. The intersection is given by the pair $\left(\bar{p}, \bar{\Omega}\right)$. Thus, whenever $\frac{1}{2} < p < \bar{p}$,

$$\hat{\Omega}_{Cont-Agg} < \hat{\Omega}_{Cont-Mod}$$

while the relationship reverses if $p > \bar{p}$. This concludes the proof of **Proposition 4**.

**Conditions (45)-(46) (Proposition 2)** In this part we show that if a locally optimal portfolio $\alpha_{lh}$ is preferred to the locally optimal moderate portfolio, $\alpha_{BB}$, then this implies that $\alpha_{lh}$ satisfies conditions (45)-(46). Thus, the Lemmas below proves **Proposition 2**

**Lemma 10** *Suppose that* $\Omega^* > \hat{\Omega}$. *Then*$\tilde{V}^{BA-BB}(\Omega^*) > 0$ *implies*

$$\frac{\xi_{BA}}{\xi_{BA}\left(\hat{\Omega}\right)} = \frac{\frac{n_A}{pn_A+(1-p)n_B}}{\hat{\Omega}\frac{n_A}{pn_A+(1-p)n_B} + \left(1-\hat{\Omega}\right)} > \kappa.$$

**Proof.**

$$\begin{aligned}
0 < \tilde{V}^{BA-BB}(\Omega^*) = & \\
&= \tilde{V}^{BA-BB}(0) - p(n_A - n_B)\ln\left(\hat{\Omega}\frac{pn_A}{pn_A+(1-p)n_B} + p\left(1-\hat{\Omega}\right)\right) = \\
&= (n_A - n_B)\,p\ln\frac{\frac{n_A}{((1-p)n_B+pn_A)}}{\kappa} + n_B\ln\frac{n_A^p n_B^{(1-p)}}{((1-p)n_B+pn_A)} \\
&\quad - p(n_A - n_B)\ln\left(\hat{\Omega}\frac{n_A}{pn_A+(1-p)n_B} + \left(1-\hat{\Omega}\right)\right) = \\
&= (n_A - n_B)\,p\ln\frac{\frac{n_A}{((1-p)n_B+pn_A)}}{\kappa\left(\hat{\Omega}\frac{n_A}{pn_A+(1-p)n_B} + \left(1-\hat{\Omega}\right)\right)} + \\
&\quad + n_B\ln\frac{n_A^p n_B^{(1-p)}}{((1-p)n_B+pn_A)}
\end{aligned}$$

As $\frac{n_A^p n_B^{(1-p)}}{((1-p)n_B+pn_A)} < 1$ because of the inequality of arithmetic and geometric means,

$$\frac{\frac{n_A}{((1-p)n_B+pn_A)}}{\left(\hat{\Omega}\frac{n_A}{((1-p)n_B+pn_A)} + \left(1-\hat{\Omega}\right)\right)} > \kappa$$

must hold. ■

**Lemma 11** $\tilde{V}^{AB-BB}(\Omega^*) > 0$ *implies*

$$\frac{1-\xi_{AB}}{1-\xi_{AB}(\Omega^*)} > \kappa.$$

**Proof.** For $\Omega^* < \hat{\Omega}$

$$
\begin{aligned}
0 < \tilde{V}^{AB-BB}(\Omega^*) = & \\
&= \tilde{V}^{AB-BB}(0) - (1-p)(n_A - n_B)\ln\left(1 - \tilde{\xi}_{AB}(\Omega^*)\right) = \\
&= -(1-p)(n_A-n_B)\ln\kappa - pn_B \ln\frac{pn_B + (1-p)n_A}{n_B} - (1-p)n_A\ln\frac{(pn_B+(1-p)n_A)}{n_A} - \\
&\quad - (1-p)(n_A-n_B)\ln\left(\Omega^*\frac{n_A}{((1-p)n_B+pn_A)} + (1-\Omega^*)\right) = \\
&= (1-p)(n_A-n_B)\ln\frac{\frac{n_A}{(pn_B+(1-p)n_A)}}{\kappa\left(\Omega^*\frac{n_A}{((1-p)n_B+pn_A)} + (1-\Omega^*)\right)} + n_B\ln\frac{n_B^p n_A^{(1-p)}}{(pn_B+(1-p)n_A)}
\end{aligned}
$$

the second part is negative, so $\frac{\frac{n_A}{(pn_B+(1-p)n_A)}}{\left(\Omega^*\frac{n_A}{pn_A+(1-p)n_B}+(1-\Omega^*)\right)} > \kappa$ must hold. For $\Omega^* > \hat{\Omega}$ the proof is analogous by exchanging $\Omega^*$ to $\hat{\Omega}$ in the above expressions. ■

## A.2 Finding $\hat{Z}, \hat{\lambda}, f, \Gamma_s, \pi_s$ implying a $\Omega^*$ equilibrium (Proposition 5, Proposition 7, Proposition 1)

Let us conjecture that there is an $f$ which makes investors indifferent whether to be clients or direct traders for a given $\Omega^*$. We will verify this in the following part.

The proof for the expressions of **Proposition 5** are given in the main text. The expression for $\bar{\chi}_s$ is a direct consequence of (30).

We get the expression for the price-dividend ratio in **Proposition 7** by the market clearing condition for the good market

$$
\delta_t = \left((1-\lambda)\left(1-\beta^I\right) + \lambda\left(\tilde{\Upsilon}_s - \bar{g}_s\right) + \lambda\left(1-\tilde{\Upsilon}_s\right)\left(1-\beta^I\right) + (1-\beta)\Gamma_s\bar{g}_s\right)(\delta_t + q_t)
$$

where the terms in the bracket on the right hand side are the consumption share of newborns, the consumption share of old clients, the consumption share of old direct traders and the consumption share of managers respectively. Simple algebra gives $\pi_H$ and $\pi_L$.

To complete the proof of **Proposition 1**, we have to find thresholds $\hat{Z}$ and $\hat{\lambda}$. Threshold $\hat{Z}$ comes from the requirement that the delegated share of capital for any of the managers following equilibrium strategies in any of the states always have to be smaller than 1, i.e.

$$
g_{1H} \equiv g\left(\frac{\xi_{AB}}{\tilde{\xi}(\Omega^*)}\right), g_{1L} \equiv g\left(\frac{1-\xi_{AB}}{1-\tilde{\xi}(\Omega^*)}\right), g_{2H} \equiv g\left(\frac{\xi_2}{\tilde{\xi}(\Omega^*)}\right), g_{2L} \equiv g\left(\frac{1-\xi_2}{1-\tilde{\xi}(\Omega^*)}\right) < 1
$$

where $\xi_2 = \xi_{BA}, p$ in a $Cont-Agg$ equilibrium and a $Cont-Mod$ equilibrium respectively. As all these expressions are proportional to $Z_B$ such $\hat{Z}$ clearly exists. While the threshold $\hat{\lambda}$ comes from the requirement that $\bar{\chi}_s^j$ is between zero and 1. Such $\hat{\lambda}$ exists by the following arguments. For any other parameters $\Omega^* = 1$ implies $\bar{\chi}_s = 1$, and $\Omega^* = 0$ implies $\bar{\chi}_s = 0$ by simple substitution, while for any $\Omega^* \in (0,1)$

$$\lim_{\lambda \to 0} \bar{\chi}_s = \lim_{\lambda \to 0} \bar{g}_s \frac{\Gamma_s - \lambda}{(1-\lambda)\beta^I} = \frac{\Omega^*}{\beta(1-\Omega^*) + \Omega^*} \in (0,1).$$

Thus, sufficiently low $\lambda$ pushes $\bar{\chi}_s$ into $[0,1]$ for any $\Omega^*$ by continuity.

To conclude the proof, in next part we verify that for any $\Omega^*$, there is an $f$ which makes investors indifferent between being direct traders or clients and that this relationship is continuous.

### A.2.1 Equilibrium value functions

Suppose the equilibrium strategies of managers where a measure $\mu$ follows a strategy leading to relative return

$$\upsilon_{1H} \equiv \frac{\xi_1}{\tilde{\xi}(\Omega^*)}, \;\; \upsilon_{1L} \equiv \frac{1-\xi_1}{1-\tilde{\xi}(\Omega^*)}.$$

while a measure $(1-\mu)$ follow strategies leading to

$$\upsilon_{2H} \equiv \frac{\xi_2}{\tilde{\xi}(\Omega^*)}, \;\; \upsilon_{2L} \equiv \frac{1-\xi_2}{1-\tilde{\xi}(\Omega^*)}$$

Let also denote the corresponding absolute returns in period $t$ as $\rho_{2t,H}, \rho_{2t,L}, \rho_{1t,H}, \rho_{1t,L}$ respectively. Write

$$\begin{aligned}
\Lambda^C_{1s_t} &\equiv \Lambda^C\left(\upsilon_{1s_t}, \Omega^*, s_t\right) \\
\Lambda^C_{2s_t} &\equiv \Lambda^C\left(\upsilon_{2s_t}, \Omega^*, s_t\right)
\end{aligned}$$

for $s_t = H, L$ and let

$$E\Lambda^C \equiv \mu(\Omega^*)\left(p\Lambda^C_{1H} + (1-p)\Lambda^C_{1L}\right) + (1-\mu(\Omega^*))\left(p\Lambda^C_{2H} + (1-p)\Lambda^C_{2L}\right)$$

Conjecture that in equilibrium we can write the lifetime utility of a client with initial wealth $w$ as

$$\begin{aligned}
V^C(w, \upsilon_1, s) &= \frac{1}{1-\beta^I}\ln w + \Lambda^C_{1s} \\
V^C(w, \upsilon_2, s) &= \frac{1}{1-\beta^I}\ln w + \Lambda^C_{2s}
\end{aligned}$$

then for $\upsilon_s = \upsilon_{1s}, \upsilon_{2s}$ and $g_s = g_{1s}, g_{2s}$ and $\rho_{t,s} = \rho_{1t,s}, \rho_{2t,s}$

$$\begin{aligned}
V^C(w,\upsilon,s) &= \ln w(1-g_s)+\beta^I EV^C(w_{t+1},\upsilon_t,\Omega_t,s_t)= \\
&= \ln w(1-g_s)+\beta^I\frac{1}{1-\beta^I}\left(p\ln\rho_{t,H}w_t\beta g_s+(1-p)\left(\ln\rho_{t,L}w_t\beta g_s\right)\right)+\beta^I E\Lambda^C \\
&= \frac{1}{1-\beta^I}\ln w_t+\ln(1-g_s)+\beta^I\frac{1}{1-\beta^I}\ln\beta g_s+\beta^I\frac{1}{1-\beta^I}\ln\frac{1}{\pi_t} \\
&\quad +\beta^I\frac{1}{1-\beta^I}\left(\left(p\ln\pi_t\rho_{t,H}\right)+(1-p)\left(\ln\pi_t\rho_{t,L}\right)\right)+ \\
&\quad +\beta^I E\Lambda^C
\end{aligned}$$

Note that in our equilibrium

$$\pi_t\rho_{t,s}=\upsilon_s y_s(1+\pi_s).$$

Thus, the conjecture is correct if

$$\begin{aligned}
\Lambda^C_{1H} &= \ln(1-g_{1H})+\beta^I\frac{1}{1-\beta^I}\ln\beta g_{1H}+\beta^I\frac{1}{1-\beta^I}\ln\frac{1}{\pi_H}+ \\
&\quad \beta^I\frac{1}{1-\beta^I}\left(p\ln\upsilon_{1H}y_H(1+\pi_H)+(1-p)\ln\upsilon_{1L}y_L(1+\pi_L)\right)+\beta^I E\Lambda^C
\end{aligned}$$

$$\begin{aligned}
\Lambda^C_{2H} &= \ln(1-g_{2H})+\beta^I\frac{1}{1-\beta^I}\ln\beta g_{2H}+\beta^I\frac{1}{1-\beta^I}\ln\frac{1}{\pi_H}+ \\
&\quad \beta^I\frac{1}{1-\beta^I}\left(p\ln\upsilon_{2H}y_H(1+\pi_H)+(1-p)\ln\upsilon_{2L}y_L(1+\pi_L)\right)+\beta^I E\Lambda^C
\end{aligned}$$

$$\begin{aligned}
\Lambda^C_{1L} &= \ln(1-g_{1L})+\beta^I\frac{1}{1-\beta^I}\ln\beta g_{1L}+\beta^I\frac{1}{1-\beta^I}\ln\frac{1}{\pi_L}+ \\
&\quad \beta^I\frac{1}{1-\beta^I}\left(p\ln\upsilon_{1H}y_H(1+\pi_H)+(1-p)\ln\upsilon_{1L}y_L(1+\pi_L)\right)+\beta^I E\Lambda^C
\end{aligned}$$

$$\begin{aligned}
\Lambda^C_{2L} &= \ln(1-g_{2L})+\beta^I\frac{1}{1-\beta^I}\ln\beta g_{2L}+\beta^I\frac{1}{1-\beta^I}\ln\frac{1}{\pi_L}+ \\
&\quad \beta^I\frac{1}{1-\beta^I}\left(p\ln\upsilon_{2H}y_H(1+\pi_H)+(1-p)\ln\upsilon_{2L}y_L(1+\pi_L)\right)+\beta^I E\Lambda^C
\end{aligned}$$

which implies

$$E\Lambda^{C}\left(1-\beta^{I}\right)^{2}=\mu\left(\begin{array}{c} p\left(\left(1-\beta^{I}\right)\ln\left(1-g_{1H}\right)+\beta^{I}\ln\beta g_{1H}+\beta^{I}\ln\upsilon_{1H}\right)+ \\ \left(1-p\right)\left(\left(1-\beta^{I}\right)\ln\left(1-g_{1L}\right)+\beta^{I}\ln\beta g_{1L}+\beta^{I}\ln\upsilon_{1L}\right) \end{array}\right)$$
$$+\left(1-\mu\right)\left(\begin{array}{c} p\left(\left(1-\beta^{I}\right)\ln\left(1-g_{2H}\right)+\beta^{I}\ln\beta g_{2H}+\beta^{I}\ln\upsilon_{2H}\right) \\ +\left(1-p\right)\left(\left(1-\beta^{I}\right)\ln\left(1-g_{2L}\right)+\beta^{I}\ln\beta g_{1L}+\beta^{I}\ln\upsilon_{2L}\right) \end{array}\right)$$
$$+\beta^{I}\left(p\ln\frac{y_{H}\left(1+\pi_{H}\right)}{\pi_{H}}+\left(1-p\right)\ln\frac{y_{L}\left(1+\pi_{L}\right)}{\pi_{L}}\right)$$

Similarly, writing the value function of direct traders as

$$V^{D}\left(w,s_{t-1}\right)=\frac{1}{1-\beta^{I}}\ln w+\Lambda_{s_{t-1}}^{D}$$

then

$$\begin{aligned}
\Lambda_{H}^{D} &= \ln\left(1-\beta^{I}\right)+\beta^{I}\frac{1}{1-\beta^{I}}\ln\beta^{I}+\beta^{I}\frac{1}{1-\beta^{I}}\ln\frac{1}{\pi_{H}}+ \\
&\quad \beta^{I}\frac{1}{1-\beta^{I}}p\left(\ln\pi_{H}\rho_{t,H}\left(\alpha^{D}\right)+\left(1-p\right)\ln\pi_{H}\rho_{t,L}\left(\alpha^{D}\right)\right)+\beta^{I}\left(p\Lambda_{H}^{D}+\left(1-p\right)\Lambda^{D}\right) \\
\Lambda_{L}^{C} &= \ln\left(1-\beta^{I}\right)+\beta^{I}\frac{1}{1-\beta^{I}}\ln\beta^{I}+\beta^{I}\frac{1}{1-\beta^{I}}\ln\frac{1}{\pi_{L}}+ \\
&\quad \beta^{I}\frac{1}{1-\beta^{I}}p\left(\ln\pi_{L}\rho_{t,H}\left(\alpha^{D}\right)+\left(1-p\right)\ln\pi_{L}\rho_{t,L}\left(\alpha^{D}\right)\right)+\beta^{I}\left(p\Lambda_{H}^{D}+\left(1-p\right)\Lambda_{L}^{D}\right)
\end{aligned}$$

implying

$$\begin{aligned}
E\Lambda^{D}(1-\beta^{I})^{2} &= \left(p\Lambda_{H}^{D}+\left(1-p\right)\Lambda_{L}^{D}\right)\left(1-\beta^{I}\right)^{2} \\
&= \left(1-\beta^{I}\right)\ln\left(1-\beta^{I}\right)+\beta^{I}\ln\beta^{I}+\beta^{I}\left(p\ln\frac{p}{\tilde{\xi}\left(\Omega\right)}+\left(1-p\right)\ln\frac{1-p}{1-\tilde{\xi}\left(\Omega\right)}\right) \\
&\quad +\beta^{I}\left(p\ln\frac{y_{H}\left(1+\pi_{H}\right)}{\pi_{H}}+\left(1-p\right)\ln\frac{y_{L}\left(1+\pi_{L}\right)}{\pi_{L}}\right).
\end{aligned}$$

Thus, using the definitions of $\upsilon_{1s},\upsilon_{2s}$, the expected value of a new born if he becomes a client is

$$EV^C\left(\rho\left(\alpha_t^M\right)\beta\beta^I w_t,\Omega_t,s\right)=\frac{1}{1-\beta^I}\ln\beta\beta^I w_t$$
$$+\frac{1}{1-\beta^I}\begin{pmatrix}\mu\left(p\ln\frac{\xi_1}{\tilde{\xi}(\Omega)}+(1-p)\ln\frac{1-\xi_1}{1-\tilde{\xi}(\Omega)}\right)\\+(1-\mu)\left(p\ln\frac{\xi_2}{\tilde{\xi}(\Omega)}+(1-p)\ln\frac{1-\xi_2}{1-\tilde{\xi}(\Omega)}\right)\end{pmatrix}+$$
$$+\frac{1}{1-\beta^I}\left(p\ln\frac{y_H(1+\pi_H)}{\pi_H}+(1-p)\ln\frac{y_L(1+\pi_L)}{\pi_L}\right)+E\Lambda^C$$

if he becomes a direct trader it is

$$EV^D\left(\rho\left(\alpha_t^D\right)w_t\beta^I,\Omega_t,s\right)=\frac{1}{1-\beta^I}\ln\beta^I w_t+\frac{1}{1-\beta^I}\left(p\ln\frac{p}{\tilde{\xi}(\Omega)}+(1-p)\ln\frac{1-p}{1-\tilde{\xi}(\Omega)}\right)$$
$$+\frac{1}{1-\beta^I}\left(p\ln\frac{y_H(1+\pi_H)}{\pi_H}+(1-p)\ln\frac{y_L(1+\pi_L)}{\pi_L}\right)+E\Lambda^D$$

Thus,

$$\left(EV^D-EV^C\right)\left(1-\beta^I\right)^2=-\ln\beta+\beta^I\ln\beta^I+\left(1-\beta^I\right)\ln(1-\beta^I)$$
$$-\mu\begin{pmatrix}p\left(\left(1-\beta^I\right)\ln(1-g_{1H})+\beta^I\ln g_{1H}+\beta^I\ln\frac{\xi_1}{p}\right)+\\+(1-p)\left(\left(1-\beta^I\right)\ln(1-g_{1L})+\beta^I\ln g_{1L}+\beta^I\ln\frac{(1-\xi_1)}{1-p}\right)\end{pmatrix}$$
$$-(1-\mu)\begin{pmatrix}p\left(\left(1-\beta^I\right)\ln(1-g_{2H})+\beta^I\ln g_{2H}+\beta^I\ln\frac{\xi_2}{p}\right)\\+(1-p)\left(\left(1-\beta^I\right)\ln(1-g_{2L})+\beta^I\ln g_{2L}+\beta^I\ln\frac{(1-\xi_2)}{1-p}\right)\end{pmatrix}\quad(57)$$

Picking $f=\left(EV^D-EV^C\right)$ satisfies our conditions.

# B Other proofs

## B.1 Proof of Proposition 8

For the existence note that in this case,

$$\Upsilon_H=\Upsilon_L=\Omega^*$$

$$\bar{g}_H=\bar{g}_L=\Omega^* Z_B<1.$$

Following the logic of the proof of finding $\hat{Z},\hat{\lambda}$ in the general case, we have to show that under our conditions as $\bar{g}_s<1$ and $\bar{\chi}_s\leq 1$. For the first condition $Z_B<1$ is sufficient. For the second condition note that

$$\bar{\chi}_s = \bar{g}_s \frac{\Gamma_s - \lambda}{(1-\lambda)\,\beta^I} = \frac{\Omega^* \frac{\beta^I(1-\lambda\Upsilon_s)+\bar{g}_s\lambda}{(\beta(1-\Omega^*)+\Omega^*)} - \bar{g}_s\lambda}{(1-\lambda)\,\beta^I} = \Omega^* \frac{1 - Z_B\,(1-\Omega^*) - \lambda\Omega^*}{(1-\lambda)\,(\beta + \Omega^*\,(1-\beta))} =$$

and

$$\begin{aligned}
\bar{\chi}_s|_{\Omega^*=1} &= \frac{1-\lambda}{1-\lambda} = 1 \\
\frac{\partial \bar{\chi}_s}{\partial \Omega^*} &= \frac{\beta\,(1-Z_B) + \Omega^*\,(Z_B - \lambda)\,(\Omega^*\,(1-\beta) + 2\beta)}{(1-\lambda)\,(-\Omega^* - \beta + \Omega^*\beta)^2}.
\end{aligned}$$

Thus, $\frac{\partial \bar{\chi}_s}{\partial \Omega^*} > 0$ would imply the result, which is guaranteed if

$$\frac{\beta\,(1-Z_B) + \Omega^*\,(Z_B)\,(\Omega^*\,(1-\beta) + 2\beta)}{\Omega^*\,(\Omega^*\,(1-\beta) + 2\beta)} > \lambda.$$

It is easy to check that the left hand side of this equation is decreasing in $\Omega^*$. Thus, choosing $\Omega^* = 1$ is a sufficient condition and it gives $\frac{\beta + Z_B}{\beta + 1} > \lambda$.

The rest of the proposition comes by direct substitution of $n_A = n_B$ into our expressions on the optimal strategies and Sharpe ratio.

## B.2 Proof of Proposition 9

Note that Lemma 1 and Proposition 3-4 imply that the average funds' excess log-return is

$$\int \ln \rho_{t+1}\,(\alpha_t^m, s_{t+1})\,dm - \ln \frac{q_{t+1}\,(s_{t+1}) + \delta_{t+1}\,(s_{t+1})}{q_t} =$$
$$= \left\{ \begin{array}{ccc}
\ln \frac{\xi_{AB}}{\bar{\xi}(\Omega^*)} & if & s_{t+1} = H \text{ and } \Omega^* < \hat{\Omega} \\
\ln \frac{\mu_{AB}(\Omega^*)\xi_{AB} + (1-\mu_{AB}(\Omega^*))\xi_2}{\bar{\xi}} & if & s_{t+1} = H \text{ and } \Omega^* \geq \hat{\Omega} \\
\ln \frac{1-\xi_{AB}}{1-\bar{\xi}(\Omega^*)} & if & s_{t+1} = L \text{ and } \Omega^* < \hat{\Omega} \\
\ln \frac{1-\mu_{AB}(\Omega^*)\xi_{AB} - (1-\mu_{AB}(\Omega^*))\xi_2}{1-\bar{\xi}} & if & s_{t+1} = L \text{ and } \Omega^* \geq \hat{\Omega}
\end{array} \right\}$$

where $\xi_2 = \xi_{BA}$ in a $Cont-Agg$ equilibrium and $\xi_2 = p$ in a $Cont-Mod$ equilibrium. This implies that the volatility of the average funds' excess log return is

$$p\,(1-p) \left\{ \begin{array}{ccc}
\left(\ln \frac{\xi_{AB}}{\bar{\xi}(\Omega^*)} - \ln \frac{1-\xi_{AB}}{1-\bar{\xi}(\Omega^*)}\right)^2 & if & \Omega^* < \hat{\Omega} \\
\mu_{AB}\,(\Omega^*) \left(\ln \frac{1-\bar{\xi}}{\bar{\xi}} \frac{\xi_{AB}}{1-\xi_{AB}}\right)^2 + (1-\mu_{AB}\,(\Omega^*)) \left(\ln \frac{\bar{\xi}}{1-\bar{\xi}} \frac{1-\xi_2}{\xi_2}\right)^2 & if & \Omega^* \geq \hat{\Omega}
\end{array} \right\}$$

while the cross-sectional dispersion of managers excess log-return is proportional to

$$\left\{ \begin{array}{ccc} \mu_{AB}\left(\Omega^{*}\right)\left(1-\mu_{AB}\left(\Omega^{*}\right)\right)\ln\frac{\xi_{2}}{\xi_{AB}} & if & s_{t+1}=H \text{ and } \Omega^{*}\geq\hat{\Omega} \\ \mu_{AB}\left(\Omega^{*}\right)\left(1-\mu_{AB}\left(\Omega^{*}\right)\right)\ln\frac{1-\xi_{AB}}{1-\xi_{2}} & if & s_{t+1}=L \text{ and } \Omega^{*}\geq\hat{\Omega} \end{array} \right\}$$

and 0 otherwise.

**Statement 1 and 4** comes directly from the facts that $\xi_{AB}<\tilde{\xi}\left(\Omega^{*}\right)$ and $\frac{\partial\mu_{AB}(\Omega^{*})}{\partial\Omega^{*}}<0$ and $\frac{1-\mu_{AB}(1)\xi_{AB}-(1-\mu_{AB}(1))\xi_{2}}{1-\xi}=1$. **Statement 2** is direct consequence of Proposition 3-4 and that $\frac{\partial\mu_{AB}(\Omega^{*})}{\partial\Omega^{*}}<0$. For **Statement 3**, it is sufficient that

$$\frac{\xi_{2}}{\xi_{AB}}<\frac{1-\xi_{AB}}{1-\xi_{2}}.$$

In a Cont-Mod equilibrium this is equivalent to

$$\frac{\left(1-p\right)n_{A}+pn_{B}}{n_{B}}<\frac{n_{A}}{\left(1-p\right)n_{A}+pn_{B}}$$

or

$$p>\frac{\sqrt{\frac{n_{A}}{n_{B}}}}{\sqrt{\frac{n_{A}}{n_{B}}}+1}.$$

Substituting for Cont-Agg equilibrium shows that the condition always holds. **Statement 5** is a consequence of $\frac{\partial\tilde{\xi}(\Omega^{*})}{\partial\Omega^{*}}<0$.

### B.3 Proof of Lemma 2

Observe that reading (32) as $\pi E\left(\phi_{s}\right)=\frac{\pi}{R}$ where $\phi_{s}$ is the state price, one can see that

$$\begin{aligned} \phi_{H} &= \frac{\tilde{\xi}\left(\Omega^{*}\right)}{p}\frac{1}{\frac{1}{\pi}y_{H}\left(1+\pi_{H}\left(\Omega_{t}^{*}\right)\right)} \\ \phi_{L} &= \frac{\left(1-\tilde{\xi}\left(\Omega^{*}\right)\right)}{1-p}\frac{1}{\frac{1}{\pi}y_{L}\left(1+\pi_{L}\left(\Omega_{t}^{*}\right)\right)}=. \end{aligned}$$

By definition, $X\left(\Omega^{*}\right)=\frac{\phi_{L}}{\phi_{H}}$ which gives our decomposition..Also, the Sharpe ratio is

$$\begin{aligned} S\left(\Omega^{*}\right) &= \frac{\sqrt{Var\left(\phi_{s}\right)}}{E\left(\phi_{s}\right)}=\frac{p^{\frac{1}{2}}\left(1-p\right)^{\frac{1}{2}}\left\|\frac{\left(1-\tilde{\xi}(\Omega^{*})\right)}{1-p}y_{H}\left(1+\pi_{H}\left(\Omega^{*}\right)\right)-\frac{\tilde{\xi}(\Omega^{*})}{p}y_{L}\left(1+\pi_{L}\left(\Omega^{*}\right)\right)\right\|}{\tilde{\xi}\left(\Omega^{*}\right)y_{L}\left(1+\pi_{L}\left(\Omega^{*}\right)\right)+\left(1-\tilde{\xi}\left(\Omega^{*}\right)\right)y_{H}\left(1+\pi_{H}\left(\Omega^{*}\right)\right)} \\ &= \frac{p^{\frac{1}{2}}\left(1-p\right)^{\frac{1}{2}}\left\|y_{H}X\left(\Omega^{*}\right)-y_{L}\right\|}{py_{L}+\left(1-p\right)y_{H}X\left(\Omega^{*}\right)}. \end{aligned}$$

## B.4 Proof of Propositions 10

$$1+\pi_s=\frac{1}{1-\beta^I\left(1-\lambda\Upsilon_s\right)-\lambda\bar{g}_s+(1-\beta)\,\Gamma_s\bar{g}_s}$$

Plugging in

$$\bar{g}_s\Gamma_s \quad = \quad \Omega^*\frac{\beta^I\left(1-\lambda\Upsilon_s\right)+\bar{g}_s\lambda}{\beta\left(1-\Omega^*\right)+\Omega^*}$$

and simplifying gives

$$1+\pi_s=\frac{\left(\beta+(1-\beta)\,\Omega^*\right)}{\left(\beta+(1-\beta)\,\Omega^*\right)-\beta\left(\beta^I\left(1-\lambda\Upsilon_s\right)+\lambda\bar{g}_s\right)}$$

In the region $\Omega^*<\hat{\Omega}$ we have

$$\Upsilon_H=\Omega^*\frac{\xi_{AB}}{\tilde{\xi}\left(\Omega^*\right)} \qquad \Upsilon_L=\Omega^*\frac{1-\xi_{AB}}{1-\tilde{\xi}\left(\Omega^*\right)}$$

and

$$\bar{g}_H=\Omega^*g\left(\frac{\xi_{AB}}{\tilde{\xi}\left(\Omega^*\right)}\right) \qquad \bar{g}_L=\Omega^*g\left(\frac{1-\xi_{AB}}{1-\tilde{\xi}\left(\Omega^*\right)}\right),$$

which implies that

$$\begin{aligned}\frac{1+\pi_H}{1+\pi_L} &= \frac{\left(\beta+(1-\beta)\,\Omega^*\right)-\beta\left(\beta^I\left(1-\lambda\Omega^*\frac{1-\xi_{AB}}{1-\tilde{\xi}(\Omega^*)}\right)+\lambda\Omega^*g\left(\frac{1-\xi_{AB}}{1-\tilde{\xi}(\Omega^*)}\right)\right)}{\left(\beta+(1-\beta)\,\Omega^*\right)-\beta\left(\beta^I\left(1-\lambda\Omega^*\frac{\xi_{AB}}{\tilde{\xi}(\Omega^*)}\right)+\lambda\Omega^*g\left(\frac{\xi_{AB}}{\tilde{\xi}(\Omega^*)}\right)\right)}\\ &= \frac{\frac{\beta(1-\beta^I)}{\Omega^*}+(1-\beta)+\beta\left(\beta^I\lambda\frac{1-\xi_{AB}}{1-\tilde{\xi}(\Omega^*)}-\lambda g\left(\frac{1-\xi_{AB}}{1-\tilde{\xi}(\Omega^*)}\right)\right)}{\frac{\beta(1-\beta^I)}{\Omega^*}+(1-\beta)+\beta\left(\beta^I\lambda\frac{\xi_{AB}}{\tilde{\xi}(\Omega^*)}-\lambda g\left(\frac{\xi_{AB}}{\tilde{\xi}(\Omega^*)}\right)\right)}\end{aligned}$$

Therefore,

$$
\begin{aligned}
X\left(\Omega^{*}\right) &= \frac{\frac{1-\tilde{\xi}(\Omega^{*})}{1-p}}{\frac{\tilde{\xi}(\Omega^{*})}{p}} \frac{\frac{\beta(1-\beta^{I})}{\Omega^{*}}+(1-\beta)+\beta\left(\beta^{I}\lambda\frac{1-\xi_{AB}}{1-\tilde{\xi}(\Omega^{*})}-\lambda g\left(\frac{1-\xi_{AB}}{1-\tilde{\xi}(\Omega^{*})}\right)\right)}{\frac{\beta(1-\beta^{I})}{\Omega^{*}}+(1-\beta)+\beta\left(\beta^{I}\lambda\frac{\xi_{AB}}{\tilde{\xi}(\Omega^{*})}-\lambda g\left(\frac{\xi_{AB}}{\tilde{\xi}(\Omega^{*})}\right)\right)} \\
&= \frac{\frac{1-\xi_{AB}}{1-p}}{\frac{\xi_{AB}}{p}} \frac{\left(\frac{\beta(1-\beta^{I})}{\Omega^{*}}+(1-\beta)\right)\frac{1-\tilde{\xi}(\Omega^{*})}{1-\xi_{AB}}+\beta^{I}\left(\beta^{I}-g\left(\frac{1-\xi_{AB}}{1-\tilde{\xi}(\Omega^{*})}\right)/\frac{1-\xi_{AB}}{1-\tilde{\xi}(\Omega^{*})}\right)}{\left(\frac{\beta(1-\beta^{I})}{\Omega^{*}}+(1-\beta)\right)\frac{\tilde{\xi}(\Omega^{*})}{\xi_{AB}}+\beta^{I}\left(\beta^{I}-g\left(\frac{\xi_{AB}}{\tilde{\xi}(\Omega^{*})}\right)/\frac{\xi_{AB}}{\tilde{\xi}(\Omega^{*})}\right)}
\end{aligned}
$$

Which can be written as

$$
\frac{\left(\frac{\beta(1-\beta^{I})}{\Omega^{*}}+(1-\beta)\right)\frac{\tilde{\xi}(\Omega^{*})}{p}+\left(\frac{\beta(1-\beta^{I})}{\Omega^{*}}+(1-\beta)\right)\left(\frac{1-\tilde{\xi}(\Omega^{*})}{1-p}-\frac{\tilde{\xi}(\Omega^{*})}{p}\right)+\frac{1-\xi_{AB}}{1-p}\beta^{I}\left(\beta^{I}-g\left(\frac{1-\xi_{AB}}{1-\tilde{\xi}(\Omega^{*})}\right)/\frac{1-\xi_{AB}}{1-\tilde{\xi}(\Omega^{*})}\right)}{\left(\frac{\beta(1-\beta^{I})}{\Omega^{*}}+(1-\beta)\right)\frac{\tilde{\xi}(\Omega^{*})}{p}+\frac{\xi_{AB}}{p}\beta^{I}\left(\beta^{I}-g\left(\frac{\xi_{AB}}{\tilde{\xi}(\Omega^{*})}\right)/\frac{\xi_{AB}}{\tilde{\xi}(\Omega^{*})}\right)} \tag{58}
$$

A couple of things to note: Since $\tilde{\xi}\left(\Omega^{*}\right)$ is decreasing in $\Omega^{*}$, and keeping in mind that $p > \tilde{\xi}\left(\Omega^{*}\right) > \xi_{AB}$ it is easy to see that for $n_a > n_b \geq 2$

- The right hand side term in the denominator is non increasing in $\Omega^{*}$ (using $n_b \geq 2$), and the right hand side term in the numerator is increasing in $\Omega^{*}$ (using $n_a > 2$). Thus, the effect of these two terms is to increase $X(\Omega^{*})$ as one increases $\Omega^{*}$.

- The middle term in the in the numerator is increasing in $\Omega^{*}$. This follows from the fact that

$$
\left(\frac{1-\tilde{\xi}\left(\Omega^{*}\right)}{1-p}-\frac{\tilde{\xi}\left(\Omega^{*}\right)}{p}\right)=\frac{p-\tilde{\xi}\left(\Omega^{*}\right)}{p(1-p)}=\frac{\Omega^{*}(p-\xi_{AB})}{p(1-p)}
$$

- The left hand side term in the denominator and in numerator is the same and is decreasing in $\Omega^{*}$

Since decreasing a numerator and denominator of a fraction that is bigger than 1 by the same amount increases the number if $X(\Omega^{*}) > 1$, the joint effect of the first term in the denominator and the first term in the numerator is to increase $X(\Omega^{*})$, if $X(\Omega^{*}) > 1$.

The above imply that if at some $\bar{\Omega}$ $X(\bar{\Omega}) > 1$, then it is above 1 for all $\Omega > \bar{\Omega}$, and is increasing in $\Omega^{*}$.

The proof follows by noting $X(0) = 1$ and showing that at $\Omega = 0$ $X(\Omega)$ is increasing in $\Omega$: $X(0) = 1$ implies that at zero the two right terms in the numerator equal the term on the right of the denominator. Since the two right hand side terms in the numerator are increasing in $\Omega$, and the right hand side term in the denominator is non increasing in $\Omega$ the proof follows.

## B.5 Proof of Propositions 11

In the region $\Omega^* > \hat{\Omega}$ the capital flow effect is constant and the wealth effect takes the form

$$\frac{\beta(1-\beta^I)+(1-\beta)\Omega^*+\beta^I\left(\beta^I\left(\frac{p-\bar{\xi}}{1-\xi}+\Omega^*\frac{1-p}{1-\xi}\right)-\left(\left(\frac{p-\bar{\xi}}{\xi_2-\xi_1}+\Omega^*\frac{\xi_2-p}{\xi_2-\xi_1}\right)g\left(\frac{\xi_1}{\xi}\right)-\left(\frac{p-\bar{\xi}}{\xi_2-\xi_1}-\Omega^*\frac{p-\xi_1}{\xi_2-\xi_1}\right)g\left(\frac{\xi_2}{\xi}\right)\right)\right)}{\beta(1-\beta^I)+(1-\beta)\Omega^*+\beta^I\left(\beta^I\left(\Omega^*\frac{p}{\xi}-\frac{p-\bar{\xi}}{\xi}\right)-\left(\left(\frac{p-\bar{\xi}}{\xi_2-\xi_1}+\Omega^*\frac{\xi_2-p}{\xi_2-\xi_1}\right)g\left(\frac{1-\xi_1}{1-\xi}\right)-\left(\frac{p-\bar{\xi}}{\xi_2-\xi_1}-\Omega^*\frac{p-\xi_1}{\xi_2-\xi_1}\right)g\left(\frac{1-\xi_2}{1-\xi}\right)\right)\right)}$$

Taking a derivative with respect to $\Omega^*$ and simplifying shows that the sign of the derivative is constant and independent of $\Omega^*$.

Define for $\Omega^* \in [0,1]$ $\overline{WEF}$ as

$$\frac{\beta(1-\beta^I)+(1-\beta)\Omega^*+\beta^I\left(\beta^I\left(\frac{p-\bar{\xi}}{1-\xi}+\Omega^*\frac{1-p}{1-\xi}\right)-\left(\left(\frac{p-\bar{\xi}}{\xi_2-\xi_1}+\Omega^*\frac{\xi_2-p}{\xi_2-\xi_1}\right)g\left(\frac{\xi_1}{\xi}\right)-\left(\frac{p-\bar{\xi}}{\xi_2-\xi_1}-\Omega^*\frac{p-\xi_1}{\xi_2-\xi_1}\right)g\left(\frac{\xi_2}{\xi}\right)\right)\right)}{\beta(1-\beta^I)+(1-\beta)\Omega^*+\beta^I\left(\beta^I\left(\Omega^*\frac{p}{\xi}-\frac{p-\bar{\xi}}{\xi}\right)-\left(\left(\frac{p-\bar{\xi}}{\xi_2-\xi_1}+\Omega^*\frac{\xi_2-p}{\xi_2-\xi_1}\right)g\left(\frac{1-\xi_1}{1-\xi}\right)-\left(\frac{p-\bar{\xi}}{\xi_2-\xi_1}-\Omega^*\frac{p-\xi_1}{\xi_2-\xi_1}\right)g\left(\frac{1-\xi_2}{1-\xi}\right)\right)\right)}$$

By construction, for $\Omega^* \in [\hat{\Omega}, 1]$ $\overline{WEF} = \frac{1+\pi_H}{1+\pi_L}$, the wealth effect component of the Sharpe ratio.[29]

Given that the Sharpe ratio is monotone it is decreasing in this region iff $\overline{WEF}(0) > \overline{WEF}(1)$.

Observe that

$$\overline{WEF}(0) = \frac{\beta(1-\beta^I)\frac{\xi_2-\xi_1}{p-\bar{\xi}} - \beta^I(g_{1L}-g_{2L}-\beta^I\frac{\xi_2-\xi_1}{1-\bar{\xi}})}{\beta(1-\beta^I)\frac{\xi_2-\xi_1}{p-\bar{\xi}} - \beta^I(g_{1H}-g_{2H}+\beta^I\frac{\xi_2-\xi_1}{\bar{\xi}})}$$

and

$$\overline{WEF}(1) = \frac{((1-\beta+\beta(1-\beta^I)+(\beta^I)^2)(\xi_2-\xi_1)-\beta^I(g_{2L}(\bar{\xi}-\xi_1)+g_{1L}(\xi_2-\bar{\xi})))}{((1-\beta+\beta(1-\beta^I)+(\beta^I)^2)(\xi_2-\xi_1)-\beta^I(g_{2H}(\bar{\xi}-\xi_1)+g_{1H}(\xi_2-\bar{\xi})))}$$

Given that $g()$ converges to 0 as $Z_B$ goes to zero and $\xi_1, \xi_2,$ and $\bar{\xi}$ are independent of $Z_B$ it is easy to see that when $Z_B$ is sufficiently small $\overline{WEF}(0) > \overline{WEF}(1)$.

Finally, the first condition implies $\overline{WEF}(0) > 1$, and the second implies $1 > \overline{WEF}(1)$.

## B.6 Proof of Proposition 12

Expression (35) shows that the Sharpe ratio is monotonic in the wealth effect for $\Omega^* > \hat{\Omega}$ given that $\tilde{\xi}(\Omega^*)$ is constant in this region. Expression (37) shows that $\frac{\partial \frac{y_H(1+\pi_H(\Omega^*))}{\theta(\Omega^*)}}{\partial \Omega^*}$ has the same sign

[29] Note that in the region $\Omega^* \in [0, \hat{\Omega}]$ $\overline{WEF} \neq \frac{1+\pi_H}{1+\pi_L}$

as $\frac{\partial \frac{y_H(1+\pi_H)}{y_L(1+\pi_L)}}{\partial \Omega^*}$. Thus, rewriting the equilibrium strategies as

$$\alpha_{lh} = 1 - \frac{1 - \frac{\xi_{lh}}{\xi}}{1 - \frac{\theta(\Omega^*)}{y_H(1+\pi_H(\Omega^*))}}$$

gives the result.

## B.7 Proof of Lemma 3

We already showed in the proof of Proposition 12 that $(1 - \alpha_{AB})$ is increasing in $\Omega^*$ in the region $\Omega^* > \hat{\Omega}$ whenever the Sharpe ratio is decreasing. Note also that using (27) in a Cont-Mod equilibrium

$$\frac{\partial \mu_{AB}(\Omega^*)\,\Omega^*}{\partial \Omega^*} = 0$$

while in a Cont-Agg equilibrium

$$\frac{\partial \mu_{AB}(\Omega^*)\,\Omega^*}{\partial \Omega^*} = \frac{\xi_{BA} - p}{\xi_{BA} - \xi_{AB}} > 0.$$

Putting together these two points gives the result.

## B.8 Proof of Lemma 4

Dividing (38) by (39), using the facts that $p > \tilde{\xi}(\Omega^*) > \xi_{AB}$ and that for $\Omega^* > \hat{\Omega}$ $\tilde{\xi}(\Omega^*) = \bar{\bar{\xi}}$, and simplifying gives

$$\frac{\Omega^* \mu_{AB}(\Omega^*)}{1 - \Omega^*} \frac{\bar{\xi} - \xi_{AB}}{p - \bar{\xi}}$$

Plugging in (27) for $\mu_{AB}$ gives

$$\frac{1}{1 - \Omega^*} \left( \frac{p - \bar{\xi}}{p - \xi_{AB}} \right) \frac{\bar{\xi} - \xi_{AB}}{p - \bar{\xi}}$$

for Cont-Mod and

$$\frac{1}{1 - \Omega^*} \left( \frac{\xi_{BA} - \bar{\xi}}{\xi_{BA} - \xi_{AB}} - (1 - \Omega^*) \frac{\xi_{BA} - p}{\xi_{BA} - \xi_{AB}} \right) \frac{\bar{\xi} - \xi_{AB}}{p - \bar{\xi}}$$

for Cont-Agg.

The result follows by taking a derivative with respect to $\Omega^*$ and noting that

$$\frac{\xi_{BA} - \bar{\xi}}{\xi_{BA} - \xi_{AB}} > \frac{p - \bar{\xi}}{p - \xi_{AB}} > 0$$

# C Figures

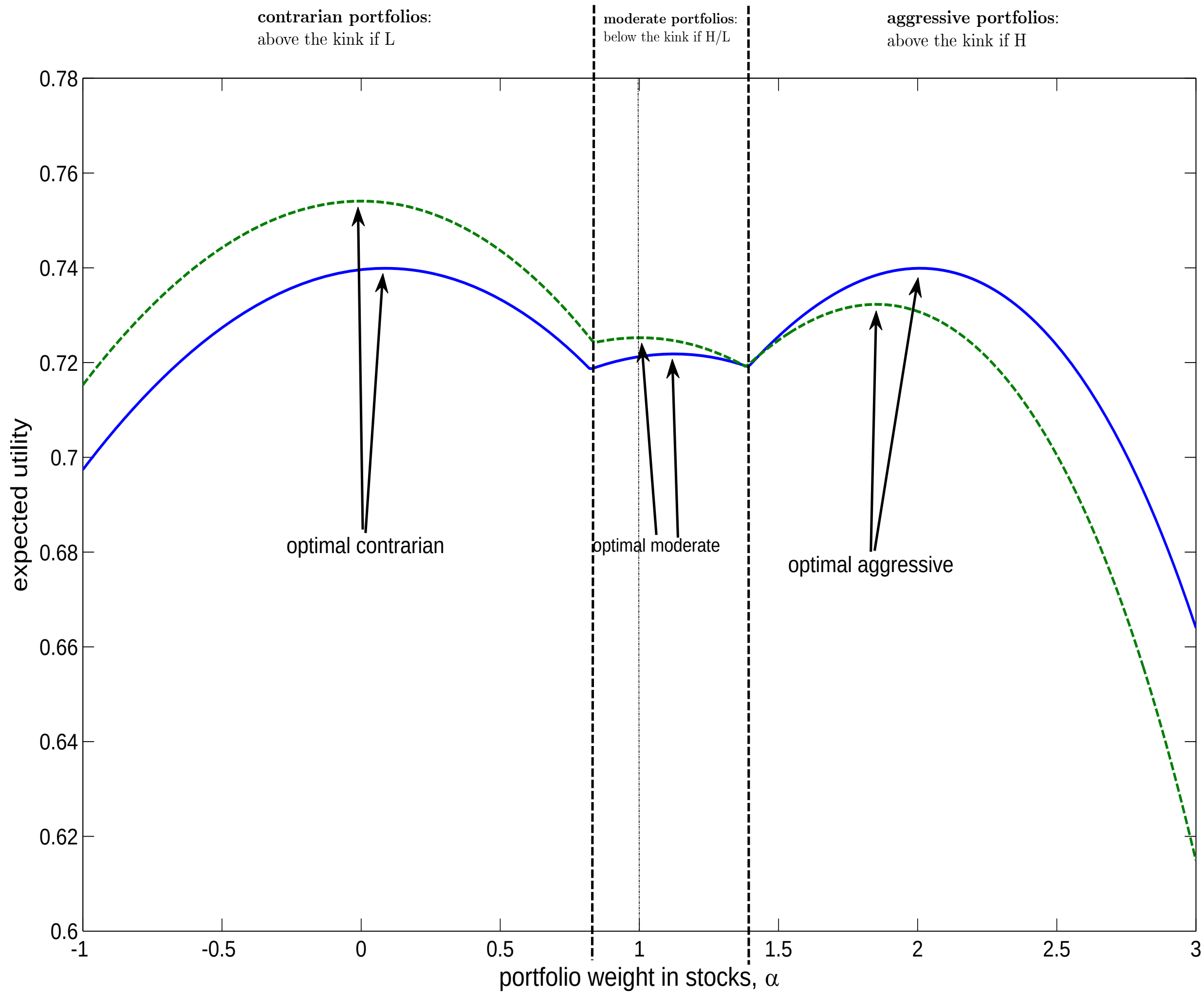


Figure 1: The graph plots the expected utility of a representative manager as function of her portfolio choice, $\alpha$, for two different set of prices. The dashed line corresponds to the case when the invested capital share of managers, $\Omega^*$, is zero. In this case all other traders hold the market. The solid line corresponds to the case when $\Omega^*$=1. The parameters are set to $\lambda = 0.5, \beta = 0.95, p = 0.7, y_H = 1.2, y_L = 0.8, Z_B = 0.3, n_A = 3,$ and $n_B = 2$.

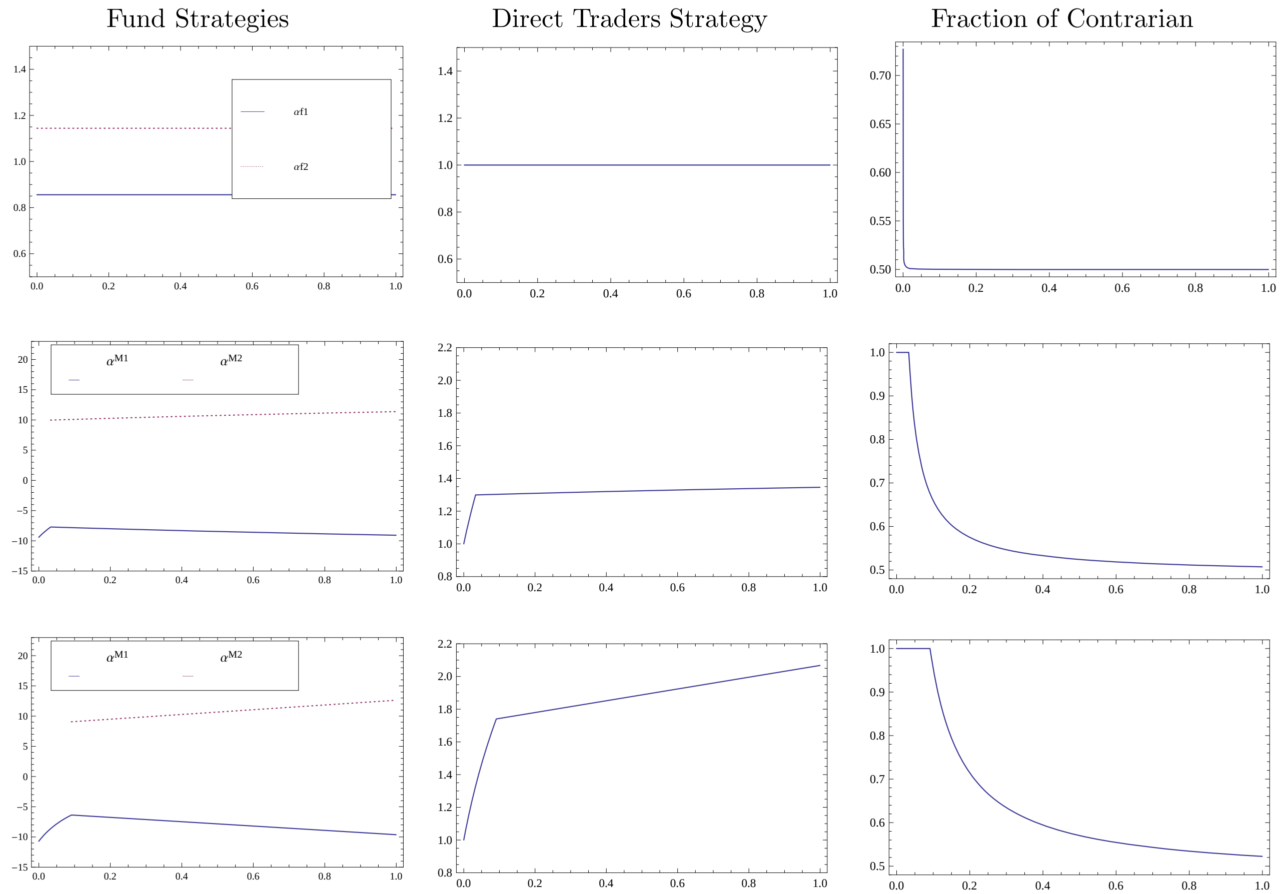


Figure 2: The graphs plot the equilibrium strategies as a function of the share of delegation. In each row the first panel plots funds' portfolios, the second direct traders' portfolio, and the third the fraction of fund managers' who are contrarian. The first row corresponds to a minimal deviation scenario where consumption parameters are taken from the full sample 1946-2008, i.e.,$p = 0.555, y_H = 1.038, y_L = 0.002, n_A = 1.01, \kappa = 1 + 10^{-13}$ and $n_B = 1$. The second and third row corresponds to the parameters implied by the Chevalier-Ellison estimation,i.e., $\kappa = 1.05, n_A = 1.9,$ and $n_B = 1.4$. The second row uses the consumption parameters from the full sample, while the third row uses the consumption parameters from the shorter sample 1978-2008 ($p = 0.645, y_H = 1.033$ and $y_L = 1$). In each example $\beta = 0.98, \lambda = 0.5$ and $Z_B = 0.01$.

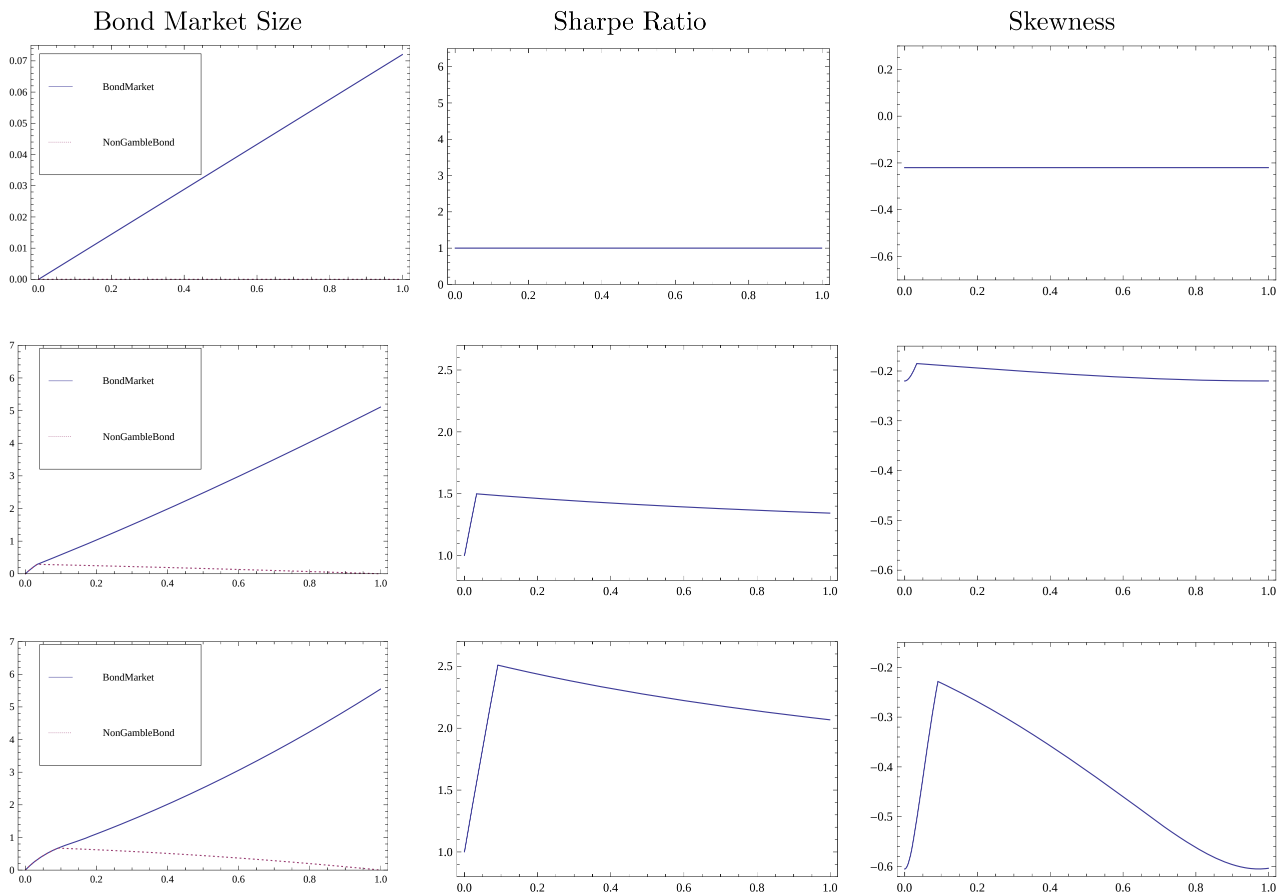


Figure 3: The graphs plot the ratio of the value of the total long bond holdings relative to the value of the economy $q_t + \delta_t$ (left panels), the Sharpe ratio (middle panels), and the equity return skewness (right panel) as a function of the share of delegation. The credit market graphs plot both the total size of the credit market (solid line), and the non-gambling component of the credit market (dotted line). The first row corresponds to a minimal deviation scenario where consumption parameters are taken from the full sample 1946-2008, i.e.,$p = 0.555, y_H = 1.038, y_L = 0.002, n_A = 1.01, \kappa = 1 + 10^{-13}$ and $n_B = 1$. The second and third row corresponds to the parameters implied by the Chevalier-Ellison estimation,i.e., $\kappa = 1.05, n_A = 1.9,$ and $n_B = 1.4$. The second row uses the consumption parameters from the full sample, while the third row uses the consumption parameters from the shorter sample 1978-2008 ($p = 0.645, y_H = 1.033$ and $y_L = 1$). In each example $\beta = 0.98, \lambda = 0.5$ and $Z_B = 0.01$.

Contrarian Funds' Strategy    Non-Contrarian Funds' Strategy

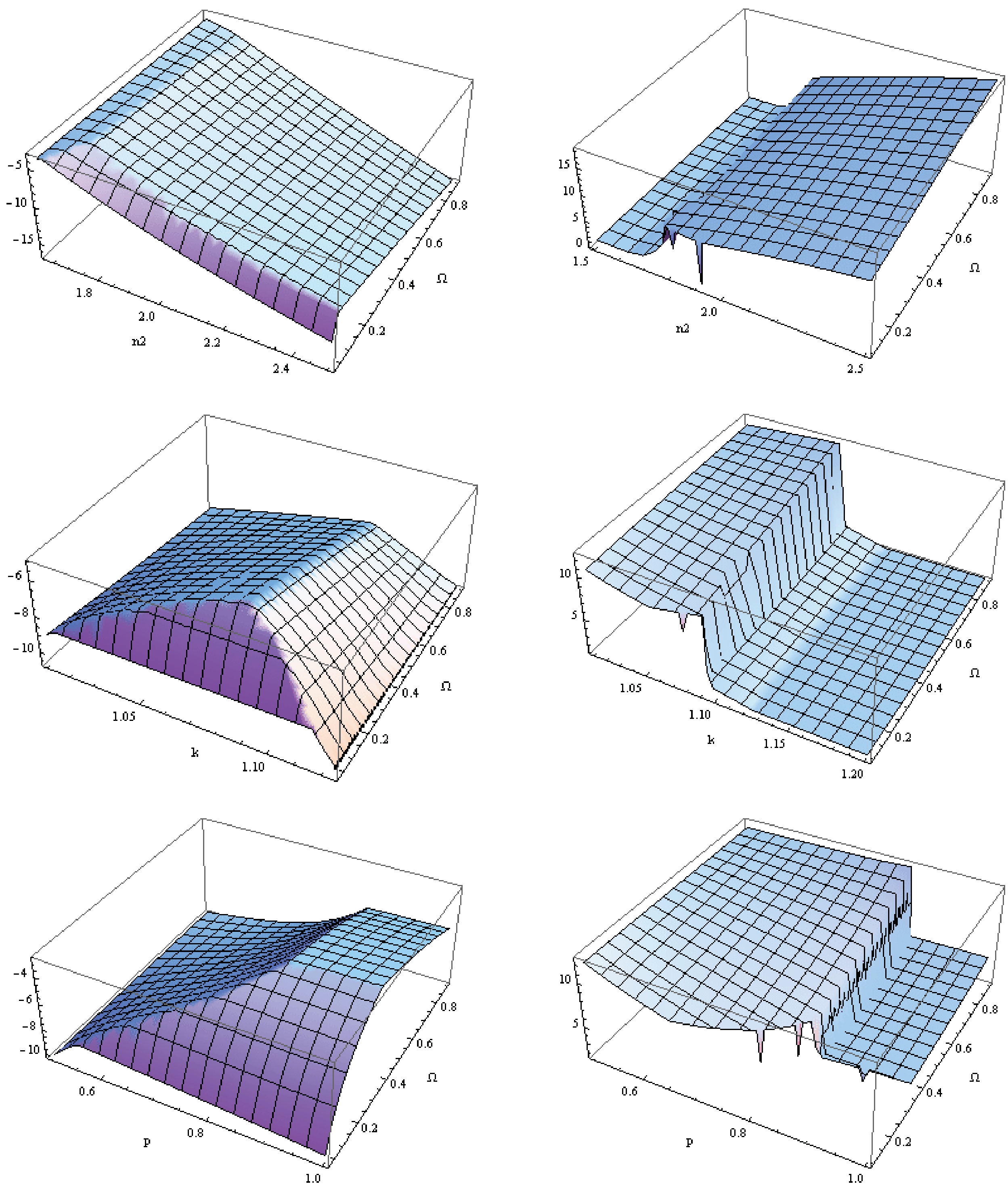


Figure 4: The graphs plot equilibrium stock position of managers following the contrarian strategy. In each of the graphs we vary the share of delegation $\Omega^*$ and one additional parameter. The parameters are set to $\lambda = 0.5, \beta = 0.98, p = 0.645, y_H = 1.033, y_L = 1, Z_B = 0.01, , n_A = 1.9,$ and $n_B = 1.4$.

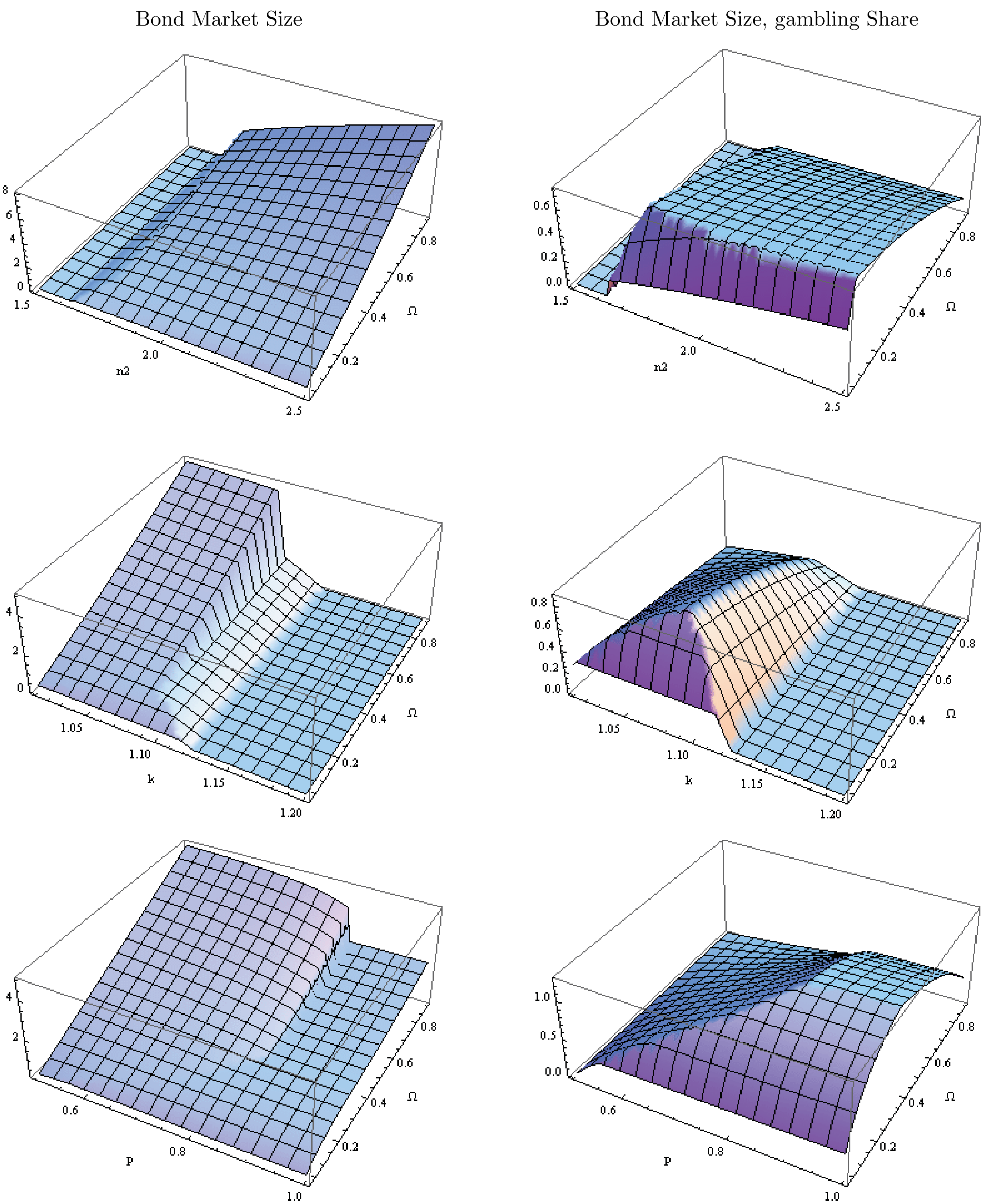


Figure 5: The graphs plot the relative size of the outstanding credit compared the value of the economy (left panels) and the risk sharing component of the outstanding credit compared to the value of the economy (right panels). In each of the graphs we vary the share of delegation $\Omega^*$ and one additional parameter. The parameters are set to $\lambda = 0.5, \beta = 0.98, p = 0.645, y_H = 1.033, y_L = 1, Z_B = 0.01, n_A = 1.9,$ and $n_B = 1.4$.

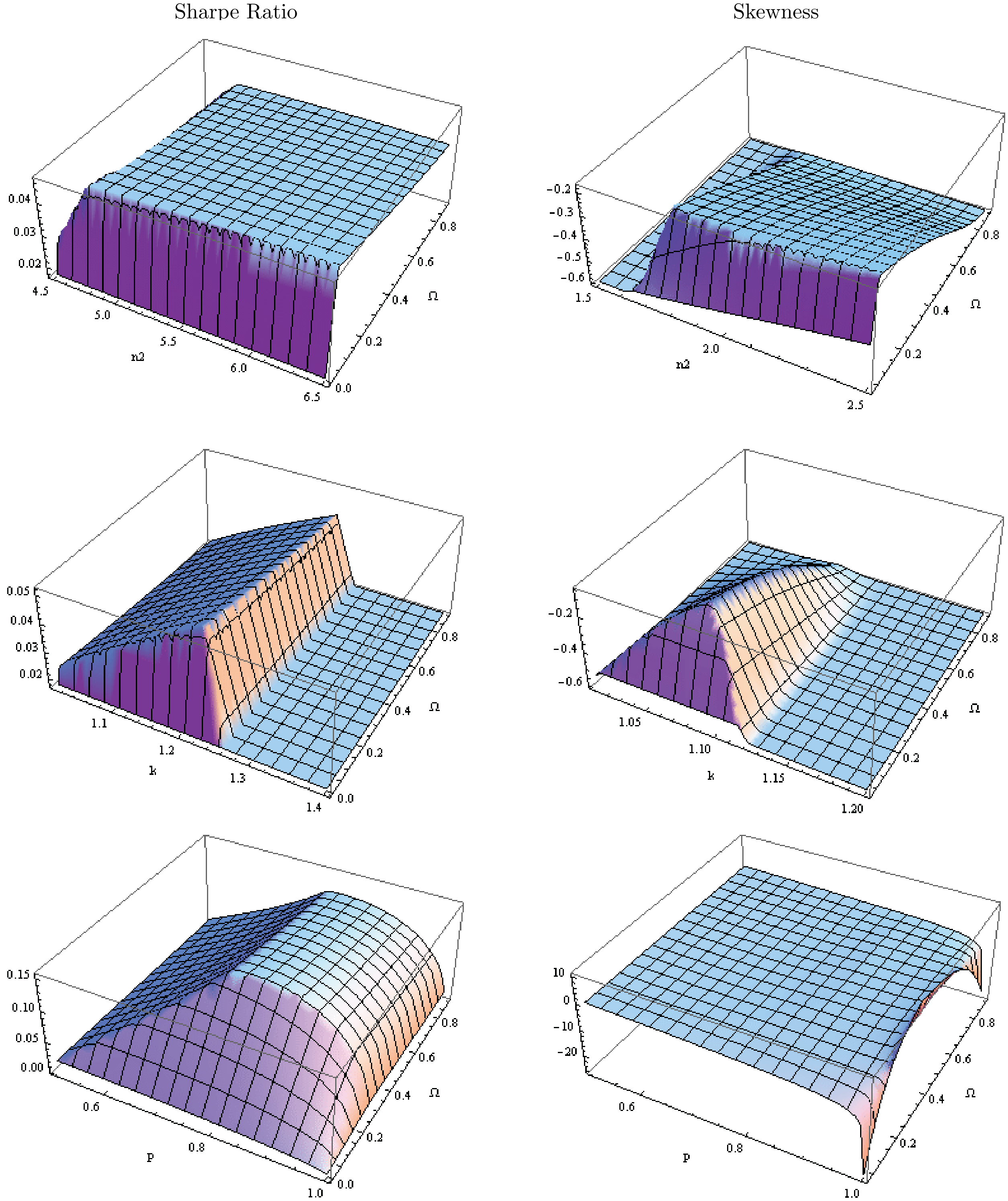


Figure 6: The graphs plot the Sharpe ratio(left panels) and Skewness of the market portfolio (right panels). In each of the graphs we vary the share of delegation $\Omega^*$ and one additional parameter. The parameters are set to $\lambda = 0.5, \beta = 0.98, p = 0.645, y_H = 1.033, y_L = 1, Z_B = 0.01, n_A = 1.9,$ and $n_B = 1.4$.