B Online Appendix for He and Kondor (2015): Proofs and Derivations

In this Online Appendix we provide proofs for Lemma 1 and Proposition 1, the second part of Proposition 5, and Propositions 7, 8 and 9.

B.1 Proof of Lemma 1 and Proposition 1

We construct the proof in steps. In particular, we separate Proposition 1 into the following four Lemmas. These four lemmas are sufficient to prove Proposition 1.

Lemma B.2 *If the equation system (12)-(13), (7)-(9) has a solution where $c_h^* < R_K$, and both $v(c)$ and $q(c)$ are increasing in the range $c \in [c_l^*, c_h^*]$, then Proposition 1 holds.*

Lemma B.3 *The system (12)-(13), (7)-(9) always has at least one solution.*

Lemma B.4 *If $h - l$ is sufficiently small, then $c_h^* < R_K$.*

Lemma B.5 *$q(c)$ is decreasing in c. If $h - l$ is sufficiently small, then $v(c)$ is increasing for $c \in [c_l^*, c_h^*]$.*

B.1.1 Step 1: Proof of Lemma 1 and Lemma B.2

Denote the dollar share of capital in the firm's asset holdings by ψ_t^i, so that $\psi_t^i = K_t^i p_t / w_t^i$. According to our conjecture, the value function can be written as (recall the aggregate cash-to-capital ratio $c = C/K$)

$$J\left(K_t, C_t, K_t^i, C_t^i\right) = w_t^i \left[\left(1 - \psi_t^i\right) q(c_t) + \frac{\psi_t^i}{p_t} v(c)\right] = J\left(K_t, C_t, w_t^i\right),$$

is linear in w_t. This is equivalent to $J\left(C, K, K_t^i, C_t^i\right) = K_t^i v(c) + C_t^i q(c)$ stated in the Lemma. Also, we have the wealth dynamics, expressed in terms of capital share ψ_t^i, as

$$dw_t^i = -d\alpha_t^i - \theta dK_t^i + \psi_t^i w_t^i \frac{1}{p_t}\left(dp_t + \sigma dZ_t\right).$$

And, $q(c) \geq 1$ has to hold as firms can consume cash at the final date (and there is no discounting), which implies $d\alpha_t^i = 0$, i.e., firms do not consume in the aggregate stage.

As the firm is choosing capital share ψ_t^i, and the capital to build or dismantle dK_t^i, the Hamiltonian-Jacobi-Bellman (HJB) of problem (3) can be written as:

$$0 = \max_{d\psi_t^i, dK_t^i} d\alpha_t^i + J_C \mathbb{E}_t\left[dC_t\right] + \frac{1}{2} J_{CC} \mathbb{E}_t\left[dC_t^2\right] + J_w \mathbb{E}_t\left(dw_t\right) + J_K' dK_t^i + J_{w,C} \mathbb{E}_t\left[dw_t dC_t\right].$$

The endogenous price dynamics (using Ito's Lemma) is

$$dp_t = \frac{1}{2}\sigma^2 p''(c)\, dt + \sigma p'(c)\, dZ_t + dB_t^p - dU_t^p,$$

where dB_t^p (dU_t^p) reflects p at $p(c_l^*) = l$ ($p(c_h^*) = h$). This is because in any market equilibrium firms will create (dismantle) capital if $p_t = h$ ($p_t = l$), and keep doing it until the price adjusts. We derived the boundary conditions in the main text. Also, by risk neutrality and the initial homogeneity of firms, before the final date the price of the capital has to make firms indifferent whether to hold capital or cash. Otherwise markets could not clear. We also explained that $\hat{p}_\tau = c_\tau$.

Thus, inside the reflection boundary (c_l^*, c_h^*) the above HJB equation is (we drop i from now on)

$$0 = \max_{\psi_t} \left\{ \begin{array}{c} \frac{\sigma^2}{2} w_t q_c'' (c_t) + q(c_t) \psi_t w_t \frac{\frac{1}{2}\sigma^2 p''(c_t)}{p_t} + q'(c_t) \left(\left(\psi_t w_t \frac{\sigma}{p_t} (\sigma + p'(c_t)\sigma) \right) \right) \\ + \xi w_t \left[\frac{1}{2} \left(\frac{\psi_t R_K}{p_t} + (1-\psi_t) \frac{R_K}{c_t} \right) + \frac{1}{2} \left(\frac{\psi_t}{p_t} R_C c_t + (1-\psi_t) R_C \right) - q(c_t) \right] \end{array} \right\}.$$

Since the problem is linear in ψ_t, in equilibrium firms must be indifferent in their choice of ψ_t. Thus, we can calculate the dynamics of the cash (capital) value by choosing $\psi_t = 0$ ($\psi = 1$). Setting $\psi_t = 0$ directly implies (10). Choosing $\psi_t = 1$ gives

$$0 = \frac{\sigma^2}{2} q''(c) + q(c) \frac{\frac{1}{2}\sigma^2 p''(c)}{p} + q'(c) \left(\frac{1}{p} (\sigma + p'\sigma) \sigma \right) + \frac{1}{p} \left(\frac{\xi}{2} (R_K + R_C c) - q(c) p \right).$$

Since $v(c) = p(c) q(c)$, $v' = q'p + p'q$, and $v'' = q''p + 2p'q' + p''q$, we can rewrite the above equation as (11). Given that the ODEs for $v(c)$ and $q(c)$ were derived by substituting in $\psi_t = 1$ and $\psi_t = 0$, it is easy to see that these functions can be interpreted as the value of a capital and that of a unit of cash. This implies that

$$J\left(C, K, w_t^i\right) = \left(w_t^i \left(1 - \psi_t^i\right) q(c) + \frac{\psi_t^i}{p_t} w_t^i v(c) \right) = q(c) w_t$$

verifying both Lemma 1 and our conjecture on the form of $J\left(C, K, w_t^i\right)$.

B.1.2 Step 2: Proof of Lemma B.3

First, note that for any arbitrary c_h and c_l from (9), we can express A_1-A_4 in (12)-(13) as functions of c_h and c_l only. Substituting back to (12)-(13) we get our functions parameterized by c_h and c_l which we denote as $v(c; c_l, c_h)$ and $q(c; c_l, c_h)$. Evaluating these functions at $c = c_l$ and $c = c_h$, we get the following expressions. Define

$$\begin{aligned}
f_l(c_l, c_h) &\equiv \frac{e^{-\gamma c_h} (\text{Ei}[c_h\gamma] - \text{Ei}[c_l\gamma]) + e^{\gamma c_h} (\text{Ei}[-c_h\gamma] - \text{Ei}[-c_l\gamma])}{e^{\gamma(c_h - c_l)} - e^{-\gamma(c_h - c_l)}}, \\
g_l(c_l, c_h) &\equiv \frac{e^{-\gamma c_h} (\text{Ei}[c_h\gamma] - \text{Ei}[c_l\gamma]) + e^{\gamma c_h} (\text{Ei}[-\gamma c_l] - \text{Ei}[-\gamma c_h])}{e^{\gamma(c_h - c_l)} - e^{-\gamma(c_h - c_l)}}, \\
f_h(c_l, c_h) &\equiv \frac{e^{-\gamma c_l} (\text{Ei}[c_h\gamma] - \text{Ei}[c_l\gamma]) + e^{\gamma c_l} (\text{Ei}[-\gamma c_h] - \text{Ei}[-\gamma c_l])}{e^{\gamma(c_h - c_l)} - e^{-\gamma(c_h - c_l)}}, \\
g_h(c_l, c_h) &\equiv \frac{e^{-\gamma c_l} (\text{Ei}[c_h\gamma] - \text{Ei}[c_l\gamma]) + e^{\gamma c_l} (\text{Ei}[-\gamma c_l] - \text{Ei}[-\gamma c_h])}{e^{\gamma(c_h - c_l)} - e^{-\gamma(c_h - c_l)}}, \text{ and} \\
m(c_l, c_h) &\equiv \frac{e^{\gamma(c_h - c_l)} - 1}{1 + e^{\gamma(c_h - c_l)}} \in (0, 1).
\end{aligned}$$

Then the cash and capital values can be rewritten as

$$\begin{aligned}
q(c_l; c_l, c_h) &= \frac{R_C}{2} + \frac{R_K\gamma}{2} f_l(c_l, c_h), \ q(c_h; c_l, c_h) = \frac{R_C}{2} + \frac{R_K\gamma}{2} f_h(c_l, c_h), \\
v(c_l; c_l, c_h) &= R_K + \frac{c_l R_C}{2} + \frac{R_C}{2\gamma} m(c_l, c_h) + \frac{R_K\gamma}{2} \left(\frac{g_l(c_l, c_h)}{\gamma} - c_l f_l(c_l, c_h) \right), \text{ and} \\
v(c_h; c_l, c_h) &= R_K + \frac{c_h R_C}{2} - \frac{R_C}{2\gamma} m(c_l, c_h) + \frac{R_K\gamma}{2} \left(\frac{g_h(c_l, c_h)}{\gamma} - c_h f_h(c_l, c_h) \right).
\end{aligned}$$

For any c_h, define the function $H(c_h)$ implicitly as the corresponding lower threshold c_l so that at $c = c_h$ the market price is just h, i.e.,

$$p(c_h; c_l = H(c_h), c_h) = \frac{v(c_h; c_l = H(c_h), c_h)}{q(c_h; c_l = H(c_h), c_h)} = h.$$

Similarly, define $L(c_h)$ is defined implicitly by

$$p(c_l; c_l = L(c_h), c_h) \equiv \frac{v(c_l; c_l = L(c_h), c_h)}{q(c_l; c_l = L(c_h), c_h)} = l,$$

which makes the market price to be l at $c = c_l$. Obviously, once we find such c_h that $H(c_h) = L(c_h)$, then this particular c_h and the corresponding $c_l = H(c_h) = L(c_h)$ is a solution of (7)-(9), (12)-(13). To show that this solution exists, we first establish properties of $L(c_h)$ then we proceed to the properties of $H(c_h)$.

Properties of $L(c_h)$ It is useful to observe that

$$\begin{aligned}
\frac{\partial f_l}{\partial c_l} &= \frac{\left(e^{2\gamma c_h} + e^{2\gamma c_l}\right)}{\left(e^{2\gamma c_h} - e^{2\gamma c_l}\right)}\left(\gamma f_l - \frac{1}{c_l}\right), \frac{\partial f_l}{\partial c_h} = 2\frac{\frac{1}{c_h} - \gamma f_h}{e^{\gamma(c_h - c_l)} - e^{\gamma(c_l - c_h)}} \\
\frac{\partial g_l}{\partial c_l} &= \frac{1}{c_l} + \frac{\left(e^{2\gamma c_h} + e^{2\gamma c_l}\right)}{\left(e^{2\gamma c_h} - e^{2\gamma c_l}\right)}\gamma g_l, \frac{\partial g_l}{\partial c_h} = -\frac{2\gamma g_h}{e^{\gamma(c_h - c_l)} - e^{\gamma(c_l - c_h)}}, \\
\lim_{c_l \to c_h} f_l &= \frac{1}{\gamma c_h}, \lim_{c_l \to c_h} g_l = 0, \lim_{c_l \to c_h} m = 0.
\end{aligned}$$

1. We show that $f_l(c_h, c_l)$ is monotonically decreasing in c_l. Its slope in c_l is

$$\frac{\partial f_l}{\partial c_l} = \frac{\left(e^{2\gamma c_h} + e^{2\gamma c_l}\right)}{\left(e^{2\gamma c_h} - e^{2\gamma c_l}\right)}\left(\gamma f_l(c_h, c_l) - \frac{1}{c_l}\right), \tag{B.12}$$

and the second derivative is

$$\begin{aligned}
&\frac{\partial^2 f_l}{\partial^2 c_l} = \\
&= -\left(4\gamma e^{2\gamma c_h}\frac{e^{2\gamma c_l}}{\left(e^{2\gamma c_h} - e^{2\gamma c_l}\right)^2} - \frac{\left(e^{2\gamma c_h} + e^{2\gamma c_l}\right)^2}{\left(e^{2\gamma c_h} - e^{2\gamma c_l}\right)^2}\gamma\right)\left(\frac{1}{c_l} - \gamma f_l(c_h, c_l)\right) - \frac{\left(-\frac{1}{c_l^2}\right)\left(e^{2\gamma c_h} + e^{2\gamma c_l}\right)}{\left(e^{2\gamma c_h} - e^{2\gamma c_l}\right)} = \\
&\qquad = \gamma\left(\frac{1}{c_l} - \gamma f_l(c_h, c_l)\right) + \frac{\left(e^{2\gamma c_h} + e^{2\gamma c_l}\right)}{\left(e^{2\gamma c_h} - e^{2\gamma c_l}\right)}\frac{1}{c_l^2}
\end{aligned}$$

Note that if the first derivative is zero, then the second derivative is positive implying that $f_l(c_h, c_l)$ can have only local minima, but no local maxima in c_l. At the limit one can check that

$$\lim_{c_l \to c_h}\frac{\partial f_l}{\partial c_l} = \lim_{c_l \to c_h}\left(\frac{1}{c_l}\frac{\left(e^{2\gamma c_h} + e^{2\gamma c_l}\right)}{\left(e^{2\gamma c_h} - e^{2\gamma c_l}\right)}\left(\gamma c_l f_l(c_h, c_l) - 1\right)\right) = \frac{1}{c_h}\left(-\frac{1}{2\gamma c_h}\right) < 0.$$

Thus, $f_l(c_h, c_l)$ is decreasing at $c_h = c_l$. Suppose that it is not monotonic over the range of $c_l < c_h$ in c_l. Then the largest $\hat{c}_l$ where the first derivative is 0, would be a local maximum. But we have just ruled out the existence of a local maximum. Thus $f_l(c_h, c_l)$ monotonically decreasing over the whole range of $c_l < c_h$ in c_l. This statement is equivalent to $\gamma f_l(c_h, c_l) - \frac{1}{c_l} < 0$ for $c_l < c_h$, for any fixed c_h.

2. We show that $X(c_l) \equiv f_l(c_h, c_l) - \frac{1}{\gamma c_l}$ is increasing in c_l. We would like to show that

$$X'(c_l) = \gamma \frac{\left(e^{2\gamma c_h} + e^{2\gamma c_l}\right)}{\left(e^{2\gamma c_h} - e^{2\gamma c_l}\right)} X(c_l) + \frac{1}{\gamma c_l^2} > 0. \tag{B.13}$$

Clearly, we have

$$X(c_l = c_h) = 0,\ X'(c_l = c_h) = f_l'(c_h, c_h) + \frac{1}{\gamma c_h^2} = \frac{1}{2\gamma c_h^2} > 0.$$

We know that when $c_l \to 0$, $f(c_h, c_l)$ has the order of $\mathrm{Ei}(\gamma c_l)$ which is $O(\ln c_l)$; this implies that $X(c_l) \to -\infty$ when $c_l \to 0$. Then, if $X(c_l)$ is not monotone, we must have two points $x_1 < x_2$ closest to (but below) c_h so that

$$0 > X(x_1) > X(x_2),\ X'(x_1) = X'(x_2) = 0.$$

Setting (B.13) to be zero, we have (because $0 < x_1 < x_2$)

$$X(x_1) = -\frac{\left(e^{2\gamma c_h} - e^{2\gamma x_1}\right)}{\gamma^2 x_1^2 \left(e^{2\gamma c_h} + e^{2\gamma x_1}\right)} < -\frac{\left(e^{2\gamma c_h} - e^{2\gamma x_2}\right)}{\gamma^2 x_2^2 \left(e^{2\gamma c_h} + e^{2\gamma x_2}\right)} = X(x_2),$$

in contradiction with $X(x_1) > X(x_2)$. Thus (B.13) holds always.

3. We show that the function $\frac{g_l(c_h, c_l)}{\gamma} - c_l f_l(c_h, c_l)$ is monotonically increasing in c_l. Its first derivative is (all the derivatives in this part are with respect to c_l)

$$\begin{aligned}\left(\frac{g_l}{\gamma} - c_l f_l\right)' &= \frac{1}{\gamma c_l} + \frac{\left(e^{2\gamma c_h} + e^{2\gamma c_l}\right)}{\left(e^{2\gamma c_h} - e^{2\gamma c_l}\right)} g_l(c_l, c_h) - \left(\frac{\left(e^{2\gamma c_h} + e^{2\gamma c_l}\right)}{\left(e^{2\gamma c_h} - e^{2\gamma c_l}\right)} \left(c_l \gamma f_l(c_l, c_h) - 1\right) + f_l(c_l, c_h)\right) \\ &= \frac{1}{\gamma c_l} + \gamma \frac{\left(e^{2\gamma c_h} + e^{2\gamma c_l}\right)}{\left(e^{2\gamma c_h} - e^{2\gamma c_l}\right)} \left(\frac{g_l}{\gamma} - c_l f_l\right) + \frac{\left(e^{2\gamma c_h} + e^{2\gamma c_l}\right)}{\left(e^{2\gamma c_h} - e^{2\gamma c_l}\right)} - f_l\end{aligned}$$

Whenever the first derivative is zero, at that point we have

$$\frac{g_l}{\gamma} - c_l f_l = \frac{f_l - \frac{1}{\gamma c_l}}{\gamma \frac{\left(e^{2\gamma c_h} + e^{2\gamma c_l}\right)}{\left(e^{2\gamma c_h} - e^{2\gamma c_l}\right)}} - \frac{1}{\gamma}. \tag{B.14}$$

We also know that

$$\lim_{c_l \to c_h} \left(\frac{g_l}{\gamma} - c_l f_l\right)' = 0, \text{ and } \lim_{c_l \to c_h} \left(\frac{g_l}{\gamma} - c_l f_l\right)'' = -\frac{1}{3\gamma c_h^2} < 0;$$

so for any fixed c_h, $c_l = c_h$ is a local maximum. Thus to show that $\frac{g_l}{\gamma} - c_l f_l$ is monotone, it suffices to rule out the case of a local minimum $\hat{c}_l < c_h$ so that $\left(\frac{g_l}{\gamma} - c_l f_l\right)' = 0$ and $\left(\frac{g_l}{\gamma} - c_l f_l\right)'' > 0$. In general

$$\left(\frac{g_l}{\gamma} - c_l f_l\right)'' = -\frac{1}{\gamma c_l^2} + \gamma \frac{\left(e^{2\gamma c_h} + e^{2\gamma c_l}\right)}{\left(e^{2\gamma c_h} - e^{2\gamma c_l}\right)} \left(\frac{g_l}{\gamma} - c_l f_l\right)' - f_l' + \frac{4 e^{2\gamma c_h} e^{2\gamma c_l}}{\left(e^{2\gamma c_h} - e^{2\gamma c_l}\right)^2} \gamma^2 \left(\left(\frac{g_l}{\gamma} - c_l f_l\right) + \frac{1}{\gamma}\right).$$

Thus, if there were a $\hat{c}_l$ that $\left(\frac{g_l}{\gamma} - c_l f_l\right)' = 0$, using (B.12) and (B.14) we have $\left(\frac{g_l}{\gamma} - c_l f_l\right)''$ to be equal to

$$-\frac{1}{\gamma\hat{c}_l^2}-f_l'+\frac{4\gamma^2 e^{2\gamma c_h}e^{2\gamma\hat{c}_l}}{\left(e^{2\gamma c_h}-e^{2\gamma\hat{c}_l}\right)^2}\left(\frac{f_l-\frac{1}{\gamma\hat{c}_l}}{\gamma\frac{\left(e^{2\gamma c_h}+e^{2\gamma\hat{c}_l}\right)}{\left(e^{2\gamma c_h}-e^{2\gamma\hat{c}_l}\right)}}-\frac{1}{\gamma}+\frac{1}{\gamma}\right)=-\frac{1}{\gamma\hat{c}_l^2}-\gamma\frac{\left(e^{2\gamma c_h}-e^{2\gamma\hat{c}_l}\right)}{e^{2\gamma c_h}+e^{2\gamma\hat{c}_l}}\left(f_l-\frac{1}{\gamma\hat{c}_l}\right).$$

But from (B.13) we know the above term is strictly negative, which proves the contradiction.

4. We show that $q\left(c_l;c_l,c_h\right)$ is also decreasing in c_l for any $c_l<c_h$. Given that $\left(\frac{g_l}{\gamma}-c_l f_l\right)'>0$ and $\partial\left(\frac{c_l R_C}{2}+\frac{R_C\left(e^{-\gamma(c_h-c_l)}+e^{\gamma(c_h-c_l)}-2\right)}{2\gamma\left(e^{\gamma(c_h-c_l)}-e^{-\gamma(c_h-c_l)}\right)}\right)/\partial c_l=\frac{1}{2}R_C\frac{e^{-2\gamma c_h+2\gamma c_l}+1}{\left(e^{-\gamma c_h+\gamma c_l}+1\right)^2}>0$, $v\left(c_l;c_l,c_h\right)$ is increasing in c_l. Thus, $p\left(c_l;c_l,c_h\right)$ is increasing in c_l for any $c_l<c_h$. Also one can show that $\lim_{c_l\downarrow 0}=p\left(c_l;c_l,c_h\right)=-\frac{\tanh(\gamma c_h)}{\gamma}<0$, and

$$\lim_{c_l\to c_h}p\left(c_l;c_l,c_h\right)=\frac{R_K+c_h\frac{R_C}{2}+\frac{R_K\gamma}{2}\left(-c_h\frac{1}{\gamma c_h}\right)}{\frac{R_C}{2}+\frac{R_K\xi}{\gamma\sigma^2}\frac{1}{\gamma c_h}}=\frac{R_K+c_h\frac{R_C}{2}-\frac{R_K}{2}}{\frac{R_C}{2}+\frac{R_K}{2c_h}},$$

which is larger than l as long as $c_h>l$. Thus, as long as $c_h>l$, $\lim_{c_l\to c_h}p\left(c_l;c_l,c_h\right)\geq l$ and there is a unique solution c_l for any c_h of $p\left(c_l;c_l,c_h\right)=l$. Therefore $L\left(c_h\right)$ exist. From the monotonicity in c_l, and continuity of $p\left(c_l;c_l,c_h\right)$ we also know that $L\left(c_h\right)$ is continuous.

Properties of $H\left(c_h\right)$ First, we show that for any $c_h\in\left[l,R_K\right]$, $H\left(c_h\right)$ is a continuous function and $H\left(c_h\right)\in\left[0,c_h\right]$. Again, the notation $'$ means we are taking the derivative with respect to c_l. We use the following facts:

$$\begin{aligned}\frac{\partial f_h}{\partial c_l}&=\frac{2\left(\gamma f_l\left(c_h,c_l\right)-\frac{1}{c_l}\right)}{\left(e^{\gamma(c_h-c_l)}-e^{-\gamma(c_h-c_l)}\right)},\frac{\partial g_h}{\partial c_l}=\frac{2\gamma g_l\left(c_h,c_l\right)}{\left(e^{\gamma(c_h-c_l)}-e^{-\gamma(c_h-c_l)}\right)}\\ \frac{\partial f_h}{\partial c_h}&=\frac{\left(e^{2\gamma c_h}+e^{2\gamma c_l}\right)}{\left(e^{2\gamma c_h}-e^{2\gamma c_l}\right)}\left(\frac{1}{c_h}-\gamma f_h\left(c_h,c_l\right)\right),\frac{\partial g_h}{\partial c_h}=\frac{1}{c_h}-\frac{\left(e^{2\gamma c_h}+e^{2\gamma c_l}\right)}{\left(e^{2\gamma c_h}-e^{2\gamma c_l}\right)}\gamma g_h\left(c_l,c_h\right)\\ \lim_{c_l\to c_h}f_h&=\frac{1}{\gamma c_h},\quad\lim_{c_l\to c_h}g_h=0.\end{aligned}$$

1. The result of $\frac{\partial f_h}{\partial c_l}=\frac{2\left(\gamma f_l(c_h,c_l)-\frac{1}{c_l}\right)}{\left(e^{\gamma(c_h-c_l)}-e^{-\gamma(c_h-c_l)}\right)}<0$ follows from the step 1 in the previous subsection.

2. We show $\left(\frac{g_h}{\gamma}-f_h c_h\right)'>0$ for $c_l<c_h$. We have $\left(\frac{g_h}{\gamma}-f_h c_h\right)'=2\frac{g_l-c_h\gamma f_l+c_h\frac{1}{c_l}}{e^{\gamma(c_h-c_l)}-e^{-\gamma(c_h-c_l)}}$ and

$$\frac{\partial^2\left(\frac{g_h}{\gamma}-f_h c_h\right)}{\partial^2 c_l}=\frac{2g_l'-c_h 2\gamma f_h'-2\frac{c_h}{c_l^2}}{e^{\gamma(c_h-c_l)}-e^{-\gamma(c_h-c_l)}}+\gamma e^{-\gamma(c_h-c_l)}\frac{e^{2(-\gamma(c_h-c_l))}+1}{\left(e^{-2\gamma(c_h-c_l)}-1\right)^2}\left(2g_l-c_h 2\gamma f_l+\frac{2c_h}{c_l}\right).$$

If the first derivative is zero at a point $c_h>c_l$, then the second derivative is

$$\frac{2\frac{1}{c_l}+2\gamma\frac{\left(e^{2\gamma c_h}+e^{2\gamma c_l}\right)}{\left(e^{2\gamma c_h}-e^{2\gamma c_l}\right)}\left(g_l\left(c_l,c_h\right)-c_h\gamma f_l\left(c_h,c_l\right)+\frac{c_h}{c_l}\right)-c_h 2\frac{1}{c_l^2}}{\left(e^{\gamma(c_h-c_l)}-e^{-\gamma(c_h-c_l)}\right)}=\frac{-2\frac{c_h-c_l}{c_l^2}}{\left(e^{\gamma(c_h-c_l)}-e^{-\gamma(c_h-c_l)}\right)}<0.$$

for any $c_h > c_l$, which implies that it can have no minimum in that range. Also

$$\lim_{c_l \to c_h} \frac{\partial \left(\frac{g_h}{\gamma} - f_h c_h \right)}{\partial c_l} = 0, \quad \lim_{c_l \to c_h} \frac{\partial^2 \left(\frac{g_h}{\gamma} - f_h c_h \right)}{\partial^2 c_l} = -\frac{1}{3\gamma c_h^2}$$

so $c_l = c_h$ must be the unique maximum in the range $c_h \geq c_l$, and the result follows.

3. Consequently, $q\left(c_h; c_h, c_l\right)$ is monotonically decreasing and $v\left(c_h; c_h, c_l\right)$ is monotonically increasing in c_l. Thus, $p\left(c_h; c_h, c_l\right)$ is monotonically increasing in c_l.

4. Observe that the following hold

$$\lim_{c_l \to c_h} p\left(c_h; c_l, c_h\right) = \lim_{c_l \to c_h} \frac{v\left(c_h; c_l, c_h\right)}{q\left(c_h; c_l, c_h\right)} = \frac{R_K c_h + c_h^2 \frac{R_C}{2} - \frac{R_K}{2} c_h}{\frac{R_C}{2} c_h + \frac{R_K}{2}} = \frac{c_h^2 R_C + R_K c_h}{R_C c_h + R_K} = c_h.$$

Because $\lim_{c_l \to 0} p\left(c_h; c_l, c_h\right) = -c_h$, hence we know that for any $c_h > h$ there is a unique $c_l \in [0, c_h]$ which solves $p\left(c_h; c_l, c_h\right) = h$. From the monotonicity of $p\left(c_h; c_h, c_l\right)$ in c_l and the continuity in c_h, the resulting function $H\left(c_h\right)$ is continuous in c_h.

Intercept of $H\left(c_h\right)$ and $L\left(c_h\right)$

1. Here we show that $H\left(h\right) > L\left(h\right)$. We know that $H\left(h\right) = h$ because

$$\lim_{c_l \to h} = \frac{v\left(c_h; c_l, c_h\right)}{q\left(c_h; c_l, c_h\right)} = \frac{R_K + h\frac{R_C}{2} + \frac{R_K \xi}{\gamma\sigma^2}\left(-h\frac{1}{\gamma h}\right)}{\frac{R_C}{2} + \frac{R_K \xi}{\gamma\sigma^2}\frac{1}{\gamma h}} = \frac{R_K + h\frac{R_C}{2} + \frac{R_K}{2}\gamma\left(-h\frac{1}{\gamma h}\right)}{\frac{R_C}{2} + \frac{R_K}{2}\gamma\frac{1}{\gamma h}} = h.$$

However, note that

$$\lim_{c_l \to h} \frac{v\left(c_l; c_l, c_h\right)}{q\left(c_l; c_l, c_h\right)} = \frac{R_K + h\frac{R_C}{2} + \frac{R_K\gamma}{2}\left(-h\frac{1}{\gamma h}\right)}{\frac{R_C}{2} + \frac{R_K}{2h}} = h,$$

and $\frac{v(c_l; c_l, c_h)}{q(c_l; c_l, c_h)}$ is increasing in c_l. Since $L\left(h\right)$ is defined by $\frac{v(c_l; L(h), h)}{q(c_l; L(h), h)} = l < h$, $L\left(h\right) < h = H\left(h\right)$ must hold.

2. Now we show that $\lim_{c_h \to \infty} H\left(c_h\right) = 0 < \lim_{c_h \to \infty} L\left(c_h\right)$. It is easy to check that

$$\lim_{c_h \to \infty} f_l = \frac{-\operatorname{Ei}[-c_l\gamma]}{e^{\gamma(-c_l)}}, \lim_{c_h \to \infty} g_l = \frac{\operatorname{Ei}\left[-\gamma c_l\right]}{e^{\gamma(-c_l)}}, \lim_{c_h \to \infty} f_h = 0, \lim_{c_h \to \infty} g_h = 0$$

Thus, $\lim_{c_h \to \infty} \frac{v(c_l; c_l, c_h)}{q(c_l; c_l, c_h)}$ takes the value of

$$\begin{aligned} &\lim_{c_h \to \infty} \frac{R_K + \frac{c_l R_C}{2} + \frac{R_C m(c_l, c_h)}{2\gamma} + \frac{R_K \gamma}{2}\left(\frac{g_l(c_l, c_h)}{\gamma} - c_l f_l\left(c_l, c_h\right)\right)}{\frac{R_C}{2} + \frac{R_K\gamma}{2} f_l\left(c_l, c_h\right)} \\ = \quad &\frac{R_K + \frac{c_l R_C}{2} + \frac{R_C}{2\gamma} + \frac{R_K\gamma}{2}\left(\frac{\operatorname{Ei}[-\gamma c_l]}{\gamma e^{\gamma(-c_l)}} - c_l \frac{-\operatorname{Ei}[-c_l\gamma]}{e^{\gamma(-c_l)}}\right)}{\frac{R_C}{2} - \frac{\operatorname{Ei}[-c_l\gamma]}{e^{\gamma(-c_l)}}}. \end{aligned}$$

Thus, $\lim_{c_h \to \infty} L\left(c_h\right)$ is the finite positive solution of

$$\frac{R_K + \frac{c_l R_C}{2} + \frac{R_C}{2\gamma} + \frac{R_K\gamma}{2}\left(\frac{\operatorname{Ei}[-\gamma c_l]}{\gamma e^{\gamma(-c_l)}} - c_l \frac{-\operatorname{Ei}[-c_l\gamma]}{e^{\gamma(-c_l)}}\right)}{\frac{R_C}{2} - \frac{\operatorname{Ei}[-c_l\gamma]}{e^{\gamma(-c_l)}}} = l.$$

In contrast, $\lim_{c_h\to\infty}\frac{v(c_h;c_l,c_h)}{q(c_h;c_l,c_h)}$ takes the value of

$$\begin{aligned}
&\lim_{c_h\to\infty}\frac{R_K+\frac{c_hR_C}{2}-\frac{R_C}{2\gamma}m\left(c_l,c_h\right)+\frac{R_K\gamma}{2}\left(\frac{g_h(c_l,c_h)}{\gamma}-c_hf_h\left(c_l,c_h\right)\right)}{\frac{R_C}{2}+\frac{R_K\gamma}{2}f_h\left(c_l,c_h\right)}\\
=&\lim_{c_h\to\infty}\frac{\frac{R_K}{c_h}+\frac{R_C}{2}-\frac{R_C}{c_h2\gamma}+\frac{R_K\gamma}{2}\left(\frac{g_h(c_l,c_h)}{c_h\gamma}-f_h\left(c_l,c_h\right)\right)}{\frac{R_C}{2c_h}+\frac{R_K\gamma}{2}\frac{f_h(c_l,c_h)}{c_h}}=\lim_{c_h\to\infty}\frac{\frac{R_C}{2}+\frac{R_K\gamma}{2}\left(\frac{g_h(c_l,c_h)}{c_h\gamma}\right)}{\frac{R_K\gamma}{2}\frac{f_h(c_l,c_h)}{c_h}}=\infty,
\end{aligned}$$

Hence, $\frac{v(c_h;c_l,c_h)}{q(c_h;c_l,c_h)}$ grows without bound for any fixed c_l, and $\frac{v(c_h;c_l,c_h)}{q(c_h;c_l,c_h)}$ is monotonically increasing in c_l. As a result, in order to have a solution of $\lim_{c_h\to\infty}\frac{v(c_h;c_l,c_h)}{q(c_h;c_l,c_h)}=l$, c_l has to go to zero, implying $\lim_{c_h\to\infty}H\left(c_h\right)=0$.

The two results imply that there is always an intercept $c_h\in(h,\infty)$ that $H\left(c_h\right)=L\left(c_h\right)$. This concludes the step proving that (7)-(9), (12)-(13) has a solution.

B.1.3 Step 3: Proof of Lemma B.4

We have shown that $H\left(h\right)=h$. Note also that if $c_h=c_l$ then $\frac{v_h}{q_h}=\frac{v_l}{q_l}$. This, and the continuity of $H\left(\cdot\right)$ and $L\left(\cdot\right)$ in l, implies that at the limit $l\to h$, there is a solution of the system (7)-(9), (12)-(13) that $c_l^*-c_h^*\to 0$ and $c_h^*,c_l^*\to h$. Then, the statement comes from $h<hR_C<R_K$ (as $R_C>1$).

B.1.4 Step 4: Proof of Lemma B.5

First we show that $q\left(c\right)$ is always deceasing, and there exists a critical value $\widehat{c}\in\left(c_l,c_h\right)$ so that $q''\left(c\right)<0$ for $c\in\left(c_l,\widehat{c}\right)$ and $q''\left(c\right)>0$ for $c\in\left(\widehat{c},c_h\right)$. Moreover, for $c\in\left(c_l,\widehat{c}\right)$ where $q''\left(c\right)<0$, we have that $q'''\left(c\right)>0$.

1. To show that $q'<0$, we differentiate the ODE $0=\frac{\sigma^2}{2}q''+\frac{\xi}{2}\left(R_C+\frac{R_K}{c}\right)-\xi q$ again to reach

$$0=\frac{\sigma^2}{2}q'''-\frac{\xi}{2}\frac{R_K}{c^2}-\xi q'. \tag{B.15}$$

Due to boundary conditions, we have at both ends c_l^* and c_h^*, the function $q'\left(c\right)$ equals zero and its second derivative $\frac{\sigma^2}{2}q'''=\frac{\xi}{2}\frac{R_K}{c^2}>0$. Suppose to the contrary that $q'\left(\widetilde{c}\right)>0$ for some point $\widetilde{c}\in\left(c_l,c_h\right)$; then we can pick $\widetilde{c}$ so that $q'\left(\widetilde{c}\right)>0$ and $q'''\left(\widetilde{c}\right)=0$ (otherwise the function $q'\left(\cdot\right)$ is zero at one end, is convex globally, and thus never comes back to zero at the other end). But because $\frac{\sigma^2}{2}q'''\left(\widetilde{c}\right)=\frac{\xi}{2}\frac{R_K}{\widetilde{c}^2}+\xi q'\left(\widetilde{c}\right)>0$, contradiction. This proves that $q'<0$.

2. We know that $q''\left(c_l\right)<0$ and $q''\left(c_h\right)>0$, and therefore there exists $\widehat{c}$ so that $q''\left(\widehat{c}\right)=0$. We show this point is unique. Because $0=\frac{\sigma^2}{2}q''+\frac{\xi}{2}\left(R_C+\frac{R_K}{c}\right)-\xi q$, we have $0=\frac{\sigma^2}{2}q'''-\frac{\xi}{2}\frac{R_K}{c^2}-\xi q'$, and

$$0=\frac{\sigma^2}{2}q''''+\frac{\xi R_K}{c^3}-\xi q''. \tag{B.16}$$

Suppose we have multiple solutions for $q''\left(\widehat{c}\right)=0$. Clearly, it is impossible to have $q''\left(\widehat{c}\right)=0$ but $q''\left(\widehat{c}-\right)>0$ and $q''\left(\widehat{c}+\right)>0$; otherwise $q''''\left(\widehat{c}\right)>0$ which contradicts with (B.16). Then there must exist two points $c_1>\widehat{c}$ and $c_2>c_1>\widehat{c}$ that $q''\left(c_1\right)=0$, $q''\left(c_2\right)<0$ and $q''''\left(c_2\right)>0$, but $q''\left(c\right)<0$ for $c\in\left(c_1,c_2\right)$. This implies that $\frac{\sigma^2}{2}q''''\left(c_1\right)=-\frac{\xi R_K}{c_1^3}+\xi q''\left(c_1\right)<0$. As a result, there exists another point $c_3\in\left(c_1,c_2\right)$ so that $q''''\left(c_3\right)=0$ with $q''\left(c_3\right)<0$. But this contradicts with (B.16).

3. Now we show that for $c\in\left(c_l,\widehat{c}\right)$ with $q''\left(c\right)<0$, we have $q'''\left(c\right)>0$, i.e., $q''\left(c\right)$ is increasing. Suppose not. Since $q'''\left(c_l\right)>0$ so that $q''\left(c\right)$ is increasing at the beginning, there must exist some reflecting point c_4 for the function q'' so that $q''''\left(c_4\right)=0$. But because $q''\left(c_4\right)<0$, it contradicts with (B.16).

Second, we show that $v(c)$ is increasing if $h-l$ is sufficiently small.

1. We show that if $v''(c_l)>0$, then $v(c)$ is increasing in c. Let $F(c)\equiv v'(c)$, so that

$$0=q''\sigma^2+\frac{\sigma^2}{2}F''+\frac{\xi}{2}R_C-\xi F$$

with boundary conditions that $F(c_l)=F(c_h)=0$. The assumption $v''(c_l)>0$ implies that $F'(c_l)>0$. Thus, if there are some points with $F(c)<0$ in the range of (c_l,c_h) then we can find two points c_1 and c_2 (a maximum and a minimum) so that $c_1<c_2$ but $F''(c_1)<0$ $F''(c_2)>0$, $F'(c_1)=F'(c_2)=0$ and $F(c_1)>0>F(c_2)$. We can apply the ODE to these two points:

$$\begin{aligned}0&=q''(c_1)\sigma^2+\frac{\sigma^2}{2}F''(c_1)+\frac{\xi}{2}R_C-\xi F(c_1),\\0&=q''(c_2)\sigma^2+\frac{\sigma^2}{2}F''(c_2)+\frac{\xi}{2}R_C-\xi F(c_2).\end{aligned}$$

The second equation implies that $q''(c_2)<0$, which implies that $c_1<c_2<\widehat{c}$. However, the above two equations also imply that

$$q''(c_1)\sigma^2>\frac{\xi}{2}R_C>q''(c_2)\sigma^2$$

contradiction with the previous lemma which shows that q'' is increasing over $[c_l,\widehat{c}]$.

2. Now we show that if $h-l$ is sufficiently small, then $v''(c_l)>0$; with the first result we obtain our claim. From our ODE,

$$v''(c_l)=-\frac{\xi}{\sigma^2}2\left(\frac{(R_Cc_l+R_K)}{2}-v(c_l)\right)=\frac{\xi}{\sigma^2}2\left(\frac{R_K}{2}+\frac{R_C}{2\gamma}h(c_l,c_h)+\frac{R_K\xi}{\gamma\sigma^2}\left(\frac{g_l(c_l,c_h)}{\gamma}-c_lf_l(c_l,c_h)\right)\right).$$

We know that as $h-l\to 0$, $c_h-c_l\to 0$. We will prove the statement by showing that (1) $\lim_{c_l\to c_h}\left(\frac{(R_Cc_l+R_K)}{2}-v(c_l)\right)=0$, because $\lim_{c_l\to c_h}\left(\frac{(R_Cc_l+R_K)}{2}-v(c_l)\right)$ equals

$$\lim_{c_l\to c_h}\left(\frac{R_K}{2}+\frac{R_C}{2\gamma}h(c_l,c_h)+\frac{R_K\gamma}{2}\left(\frac{g_l(c_l,c_h)}{\gamma}-c_lf_l(c_l,c_h)\right)\right)=\frac{R_K}{2}+0+\frac{R_K\xi}{\gamma\sigma^2}\left(0-\frac{1}{\gamma}\right)=0$$

and (2) $\lim_{c_l\to c_h}\frac{\partial\left(\frac{(R_Cc_l+R_K)}{2}-v(c_l)\right)}{\partial c_l}=\lim_{c_l\to c_h}\frac{\partial\left(\frac{R_C}{2\gamma}h(c_l,c_h)+\frac{R_K\gamma}{2}\left(\frac{g_l(c_l,c_h)}{\gamma}-c_lf_l(c_l,c_h)\right)\right)}{\partial c_l}<0$, because it equals

$$\begin{aligned}&\lim_{c_l\to c_h}\left(-\frac{R_Ce^{\gamma(c_h-c_l)}}{\left(e^{\gamma(c_h-c_l)}+1\right)^2}+\frac{R_K\gamma}{2}\left(\frac{1}{\gamma c_l}+\frac{\left(e^{2\gamma c_h}+e^{2\gamma c_l}\right)}{\left(e^{2\gamma c_h}-e^{2\gamma c_l}\right)}g_l-\frac{\left(e^{2\gamma c_h}+e^{2\gamma c_l}\right)}{\left(e^{2\gamma c_h}-e^{2\gamma c_l}\right)}(c_l\gamma f_l-1)\right)\right)\\=&\;-R_C\frac{1}{(1+1)^2}+\frac{R_K\gamma}{2}\left(\frac{1}{\gamma c_h}-\frac{1}{2\gamma c_h}-\frac{1}{2\gamma c_h}\right)=-\frac{R_C}{4}<0.\end{aligned}$$

These two statements imply that if c_h-c_l is small enough then $v''(c_l)>\lim_{c_l\to c_h}v''(c_l)=0$.

B.2 Proof of the Second Part of Proposition 5

The result $c_h^*>h$ is a consequence of the fact that we defined $H(c_h)$ as the unique c_l solving $\frac{v_h(c_l,c_h)}{q_h(c_l,c_h)}=h$ when $c_h>h$. (see part 4 in section B.1.2.)

For the result $c_l^*\le l$, consider the possibility that $c_l^*>l$. The following lemma states that in this case $p''(c_l^*)<0$. This implies that this is not an equilibrium. To see this, we have $p'(c_l^*)=0$ by the boundary conditions $v'(c_l^*) = q'(c_l^*) = 0$. Thus $p''(c_l^*) < 0$, combined with $p(c_l^*) = l$ and $p'(c_l^*) = 0$, would imply that $p(c) < l$ for c sufficiently close to c_l^*.

Lemma B.6 *The sign of $p''(c_l^*)$ is the same as that of $l - c_l^*$.*

Proof. Simple algebra implies that

$$\begin{aligned}
p''(c_l^*) &= \left(\frac{v'q - q'v}{q^2}\right)' = \frac{(v''q + v'q' - (q''v + v'q'))}{q^2} - 2q^{-3}(v'q - q'v) \\
&= \frac{v''q - q''v}{q^2} = \frac{\left(-\frac{\xi}{2}(R_C c_l^* + R_K) + \xi l q(c_l^*)\right)\frac{2}{\sigma^2}q - \left(-\frac{\xi}{2}(R_C c_l^* + R_K) + \xi c_l^* q(c_l^*)\right)\frac{2}{\sigma^2 c_l^*}v}{q^2} \\
&= \frac{\left(-\frac{\xi}{2}(R_C c_l^* + R_K) + \xi l q(c_l^*) + \xi c_l^* q(c_l^*) - \xi c_l^* q(c_l^*)\right)\frac{2}{\sigma^2}q - \left(-\frac{\xi}{2}(R_C c_l^* + R_K) + \xi c_l^* q(c_l^*)\right)\frac{2}{\sigma^2 c_l^*}v}{q^2} \\
&= \frac{\left(-\frac{\xi}{2}(R_C c_l^* + R_K) + \xi c_l^* q(c_l^*)\right)\frac{2}{\sigma^2}\left(q - \frac{v}{c_l^*}\right) + (l - c_l^*)\xi q(c_l^*)\frac{2}{\sigma^2}q}{q^2} \\
&= (l - c_l^*)\frac{\frac{1}{c_l^*}\left(\frac{\xi}{2}(R_C c_l^* + R_K) - \xi c_l^* q(c_l^*)\right)\frac{2}{\sigma^2} + \xi q(c_l^*)\frac{2}{\sigma^2}}{q}
\end{aligned}$$

which gives the lemma by noticing that q is decreasing in c and the boundary $q'(c_l^*) = 0$ implies that

$$-\frac{\xi}{2}(R_C c_l^* + R_K) + \xi c_l^* q(c_l^*) \propto q''(c_l^*) < 0.$$

■

The third statement is a consequence of the following Lemma.

Lemma B.7 *We have the following limiting results:*

$$\begin{aligned}
\lim_{\gamma\to\infty} \gamma f_l &= \frac{1}{c_l},\ \lim_{\gamma\to\infty} \gamma f_h = \frac{1}{c_h},\ \lim_{\gamma\to\infty} g_h = 0,\ \lim_{\gamma\to\infty} g_l = 0; \\
\text{and } \lim_{\gamma\to\infty} c_h^* &= h,\ \lim_{\gamma\to\infty} c_l^* = l.
\end{aligned}$$

Proof. The first four results are based on L'Hopital rule. Take the first result for illustration:

$$\begin{aligned}
\lim_{\gamma\to\infty} \gamma f_l &= \lim_{\gamma\to\infty} \frac{\gamma(\mathrm{Ei}[-c_h\gamma] - \mathrm{Ei}[-c_l\gamma])}{e^{\gamma(-c_l)}} = \lim_{\gamma\to\infty} \frac{\mathrm{Ei}[-c_h\gamma] - \mathrm{Ei}[-c_l\gamma]}{\frac{1}{\gamma}e^{\gamma(-c_l)}} \\
&= \lim_{\gamma\to\infty} \frac{\frac{e^{-c_h\gamma}}{\gamma} - \frac{e^{-c_l\gamma}}{\gamma}}{-\frac{1}{\gamma^2}e^{\gamma(-c_l)} + \frac{(-c_l)}{\gamma}e^{\gamma(-c_l)}} = \lim_{\gamma\to\infty} \frac{-e^{-c_l\gamma}/\gamma}{\frac{(-c_l)}{\gamma}e^{\gamma(-c_l)}} = \frac{1}{c_l}.
\end{aligned}$$

These four results imply that

$$\lim_{\gamma\to\infty} \frac{v_h}{q_h} = \lim_{\gamma\to\infty} \frac{R_K + \frac{c_h R_C}{2} - \frac{R_C}{2\gamma}m(c_l, c_h) + R_K\frac{\gamma}{2}\left(\frac{g_h(c_l, c_h)}{\gamma} - c_h f_h(c_l, c_h)\right)}{\frac{R_C}{2} + R_K\frac{\gamma}{2}f_h(c_l, c_h)} = \frac{R_K + \frac{c_h R_C}{2} - R_K\frac{1}{2}}{\frac{R_C}{2} + R_K\frac{1}{2c_h}}$$

Thus, in the limit the solution of $\frac{v_h}{q_h} = h$ is the solution for the equation of

$$\frac{R_K + \frac{c_h R_C}{2} - R_K\frac{1}{2}}{\frac{R_C}{2} + R_K\frac{1}{2c_h}} = h,$$

which gives $\lim_{\gamma\to\infty} c_h^* = h$. Similarly, the following calculation implies that $\lim_{\gamma\to\infty} c_l^* = l$:

$$\lim_{\gamma\to\infty}\frac{v_l}{q_l} = \lim_{\gamma\to\infty}\frac{R_K + \frac{c_l R_C}{2} + \frac{R_C}{2\gamma} m\left(c_l, c_h\right) + R_K \frac{\gamma}{2}\left(\frac{g_l(c_l,c_h)}{\gamma} - c_l f_l\left(c_l, c_h\right)\right)}{\frac{R_C}{2} + R_K \frac{\gamma}{2} f_l\left(c_l, c_h\right)} = \frac{R_K + \frac{c_l R_C}{2} + R_K \frac{1}{2}}{\frac{R_C}{2} + R_K \frac{1}{2c_l}}.$$

■

B.3 Proof of Proposition 7

The proofs of the two statements follow the same logic. Thus, we prove the first statement in detail and explain the necessary modifications for the second statement at the end of the proof.

Consider the functions $\tilde{q}\left(c; q_0, v_0, c_h\right)$ and $\tilde{v}\left(c; q_0, v_0, c_h\right)$ of c parameterized by q_0, v_0, and c_h:

$$0 = \frac{\sigma^2}{2}\tilde{q}''\left(c\right) + \frac{\xi}{2}\left(R_C - \tilde{q}\left(c\right)\right) + \frac{\xi}{2}\left(\frac{R_K}{c} - \tilde{q}\left(c\right)\right) \tag{B.17}$$

$$0 = \tilde{q}'\left(c\right)\sigma^2 + \frac{\sigma^2}{2}\tilde{v}''\left(c\right) + \frac{\xi}{2}\left(R_C c - \tilde{v}\left(c\right)\right) + \frac{\xi}{2}\left(R_K - \tilde{v}\left(c\right)\right). \tag{B.18}$$

and the boundary conditions

$$\tilde{v}'\left(c_h\right) = \tilde{q}'\left(c_h\right) = 0, \tag{B.19}$$

$$\tilde{q}\left(c_0\right) = q_0, \tilde{v}\left(c_0\right) = v_0. \tag{B.20}$$

The general solution is

$$\widetilde{q}\left(c\right) = \frac{R_C}{2} + e^{-c\gamma}A_1 + e^{c\gamma}A_2 + \frac{R_K\gamma}{2}\frac{-e^{c\gamma}\operatorname{Ei}\left(-\gamma c\right) + e^{-c\gamma}\operatorname{Ei}\left(c\gamma\right)}{2} \tag{B.21}$$

$$\widetilde{v}\left(c\right) = R_K + \frac{cR_C}{2} + e^{c\gamma}\left(A_3 - cA_2\right) - e^{-c\gamma}\left(A_4 + cA_1\right) + \frac{cR_K\gamma}{2}\frac{e^{\gamma c}\operatorname{Ei}\left(-\gamma c\right) - e^{-c\gamma}\operatorname{Ei}\left(\gamma c\right)}{2} \tag{B.22}$$

$$= R_K + R_C c + e^{c\gamma}A_3 - e^{-c\gamma}A_4 - c\widetilde{q}\left(c\right). \tag{B.23}$$

where A_1-A_4 (may differ from those in (12) and (13)) are pinned down by (B.19)-(B.20). We have

$$\widetilde{q}'\left(c\right) = -\gamma e^{-c\gamma}A_1 + \gamma e^{c\gamma}A_2 - \frac{R_K\gamma^2\left(e^{-c\gamma}\operatorname{Ei}[c\gamma] + e^{c\gamma}\operatorname{Ei}[-c\gamma]\right)}{2},$$

$$\widetilde{v}'\left(c\right) = \frac{R_C}{2} + \frac{R_K\gamma\left(-e^{-c\gamma}\operatorname{Ei}[c\gamma] + e^{c\gamma}\operatorname{Ei}[-c\gamma]\right)}{2} + \frac{R_K c\gamma^2\left(e^{-c\gamma}\operatorname{Ei}[c\gamma] + e^{c\gamma}\operatorname{Ei}[-c\gamma]\right)}{2}$$
$$+e^{c\gamma}\left(\left(-\gamma c - 1\right)A_2 + \gamma A_3\right) + e^{-c\gamma}\left(\left(\gamma c - 1\right)A_1 + \gamma A_4\right).$$

Define the function $c_h\left(q_0, v_0\right)$ implicitly by $\tilde{v}\left(c_h; q_0, v_0, c_h\right) = h\tilde{q}\left(c_h; q_0, v_0, c_h\right)$, and we are interested in the derivatives

$$\frac{\partial c_h}{\partial q_0} = -\frac{\tilde{v}'_{q_0} - h\tilde{q}'_{q_0}}{\tilde{v}'_{c_h} - h\tilde{q}'_{c_h}}, \frac{\partial c_h}{\partial v_0} = -\frac{\tilde{v}'_{v_0} - h\tilde{q}'_{v_0}}{\tilde{v}'_{c_h} - h\tilde{q}'_{c_h}}.$$

We proceed as follows. First we show that $\frac{\partial c_h}{\partial q_0} > 0$ and $\frac{\partial c_h}{\partial v_0} < 0$. This proves that if $q_\pi\left(c_0\right) \leq q\left(c_0\right)$ and $v_\pi\left(c_0\right) \geq v\left(c_0\right)$ then $c_h^\pi < c_h^*$, that is, such policies make the overinvestment problem worse. Then we show that this is true even if $q_\pi\left(c_0\right) \leq q\left(c_0\right)$ and $v_\pi\left(c_0\right) \leq v\left(c_0\right)$, as long as the policy increases the price at c_0, i.e. $\frac{v_\pi(c_0)}{q_\pi(c_0)} > \frac{v(c_0)}{q(c_0)}$.

We start with the following lemmas.

Lemma B.8 *We have*

$$\begin{aligned}\frac{\partial \tilde{q}\left(c_h; q_0, v_0, c_h\right)}{\partial q_0} &= \frac{2}{e^{c_h\gamma}e^{-c_0\gamma}+e^{-c_h\gamma}e^{\gamma c_0}} > 0, &\text{(B.24)}\\ \frac{\partial \tilde{v}\left(c_h; q_0, v_0, c_h\right)}{\partial v_0} &= \frac{2}{e^{-\gamma(c_h-c_0)}+e^{\gamma(c_h-c_0)}} > 0, \frac{\partial \tilde{q}\left(c_h; q_0, v_0, c_h\right)}{\partial v_0} = 0. &\text{(B.25)}\end{aligned}$$

Proof. We show (B.24) first. We know that $\tilde{q}\left(c_0\right) = q_0$, which based on (B.21) can be written as $e^{-c_0\gamma}A_1 + e^{\gamma c_0}A_2 + l_q = q_0$ (where l_q is independent of q_0) which implies

$$A_1 = \frac{-l_q - e^{\gamma c_0}A_2 + q_0}{e^{-c_0\gamma}}. \tag{B.26}$$

and $\tilde{q}'\left(c_h\right) = 0$ which can be rewritten as $-e^{-c_h\gamma}\gamma A_1 + e^{c_h\gamma}\gamma A_2 + s_q = 0$ (where s_q is independent of q_0) which implies

$$A_2 = \frac{e^{-c_h\gamma}\gamma A_1 - s_q}{e^{c_h\gamma}\gamma} = \frac{e^{-c_h\gamma}\gamma\frac{-l_q-e^{\gamma c_0}A_2+q_0}{e^{-c_0\gamma}} - s_q}{e^{c_h\gamma}\gamma} \Rightarrow A_2 = \frac{e^{-c_h\gamma}\gamma\frac{-l_q+q_0}{e^{-c_0\gamma}} - s_q}{\left(1+e^{-2c_h\gamma}e^{\gamma 2c_0}\right)e^{c_h\gamma}\gamma}. \tag{B.27}$$

Thus, (B.27) and (B.26) imply that

$$\begin{aligned}\frac{\partial A_2}{\partial q_0} &= \frac{e^{-c_h\gamma}}{e^{c_h\gamma}e^{-c_0\gamma}+e^{-c_h\gamma}e^{\gamma c_0}}, &\text{(B.28)}\\ \frac{\partial A_1}{\partial q_0} &= \frac{1}{e^{-c_0\gamma}} - e^{\gamma 2c_0}\frac{e^{-c_h\gamma}}{e^{c_h\gamma}e^{-c_0\gamma}+e^{-c_h\gamma}e^{\gamma c_0}} = \frac{e^{c_h\gamma}}{e^{c_h\gamma}e^{-c_0\gamma}+e^{-c_h\gamma}e^{\gamma c_0}}. &\text{(B.29)}\end{aligned}$$

Using (B.21) we obtain our result.

The first result in (B.25) follows similarly. The second result $\frac{\partial \tilde{q}(c_h;q_0,v_0,c_h)}{\partial v_0} = 0$ comes from the fact that (B.17) and the boundary conditions $\tilde{q}'\left(c_h\right) = 0$ and $\tilde{q}\left(c_0\right) = q_0$ are independent of v_0. ■

Lemma B.9 *We have*

$$\begin{aligned}\frac{\partial \tilde{v}\left(c_h; q_0, v_0, c_h\right)}{\partial q_0} &= 2\frac{e^{\gamma(c_h-c_0)} - e^{-\gamma(c_h-c_0)} - \gamma\left(c_h-c_0\right)\left(e^{-\gamma(c_h-c_0)}+e^{\gamma(c_h-c_0)}\right)}{\gamma\left(e^{\gamma c_0}e^{-\gamma c_h}+e^{-\gamma c_0}e^{\gamma c_h}\right)^2} < 0,\\ \frac{\partial \tilde{v}\left(c_h; q_0, v_0, c_h\right)}{\partial q_0} - h\frac{\partial \tilde{q}\left(c_h; q_0, v_0, c_h\right)}{\partial q_0} &= 2\frac{e^{\gamma(c_h-c_0)} - e^{-\gamma(c_h-c_0)} - \gamma\left(c_h+h-c_0\right)\left(e^{-\gamma(c_h-c_0)}+e^{\gamma(c_h-c_0)}\right)}{\gamma\left(e^{\gamma c_0}e^{-\gamma c_h}+e^{-\gamma c_0}e^{\gamma c_h}\right)^2} < 0\end{aligned}$$

Proof. We show the first result. We rewrite $\widetilde{v}\left(c_0\right)$ and $\widetilde{v}'\left(c_h\right)$ as (as before here l_{vq} and s_{vq} are independent of q_0)

$$\begin{aligned}\widetilde{v}\left(c_0\right) &= e^{c_0\gamma}\left(A_3 - c_0A_2\right) - e^{-c_0\gamma}\left(A_4 + c_0A_1\right) + l_{vq},\\ \widetilde{v}'\left(c_h\right) &= s_{vq} + e^{c_h\gamma}\left(\left(-\gamma c_h - 1\right)A_2 + \gamma A_3\right) + e^{-c_h\gamma}\left(\left(\gamma c_h - 1\right)A_1 + \gamma A_4\right)\end{aligned}$$

Thus, the boundary conditions $\widetilde{v}\left(c_0\right) = v_0$ and $\widetilde{v}'\left(c_h\right) = 0$ imply that

$$\begin{aligned}A_3 &= c_0A_2 + e^{-c_0\gamma}v_0 - e^{-c_0\gamma}l_{vq} + e^{-2c_0\gamma}\left(A_4 + c_0A_1\right),\\ A_4 &= -\frac{\begin{array}{c}\left(-e^{\gamma c_h}\left(\gamma c_h - \gamma c_0 + 1\right)\right)A_2 + \left(e^{-\gamma c_h}\left(\gamma c_h - 1\right) + \gamma c_0 e^{-2\gamma c_0}e^{\gamma c_h}\right)A_1\\ +\left(\gamma e^{-\gamma c_0}e^{\gamma c_h}\right)v_0 + \left(s_{qv} - \gamma e^{-\gamma c_0}e^{\gamma c_h}l_{qv}\right)\end{array}}{\gamma e^{-\gamma c_h} + \gamma e^{-2\gamma c_0}e^{\gamma c_h}}\end{aligned}$$

Thus, using the result in (B.28) and (B.29) one can derive that

$$\frac{\partial A_4}{\partial q_0} = e^{\gamma c_h} \frac{2e^{\gamma c_0} e^{-\gamma c_h} - \gamma c_0 \left(e^{\gamma c_0} e^{-\gamma c_h} + e^{-\gamma c_0} e^{\gamma c_h}\right)}{\gamma \left(e^{\gamma c_0} e^{-\gamma c_h} + e^{-\gamma c_0} e^{\gamma c_h}\right)^2}.$$

Similarly it implies that

$$\frac{\partial A_3}{q_0} = \frac{\partial A_1}{q_0} e^{-2c_0\gamma} c_0 + \frac{\partial A_2}{q_0} c_0 + \frac{\partial A_4}{q_0} e^{-2c_0\gamma} = \frac{2e^{-\gamma c_0} + \gamma c_0 \left(e^{\gamma c_0} e^{-2\gamma c_h} + e^{-\gamma c_0}\right)}{\gamma \left(e^{\gamma c_0} e^{-\gamma c_h} + e^{-\gamma c_0} e^{\gamma c_h}\right)^2}$$

Consequently, using (B.23), we have (where we have used (B.24))

$$\begin{aligned} \frac{\partial \widetilde{v}\left(c_h\right)}{\partial q_0} &= e^{c_h\gamma} \frac{\partial A_3}{q_0} - e^{-c_h\gamma} \frac{\partial A_4}{q_0} - c_h \frac{\partial \widetilde{q}\left(c_h\right)}{\partial q_0} \\ &= 2 \frac{e^{\gamma(c_h - c_0)} - e^{-\gamma(c_h - c_0)} - \gamma\left(c_h - c_0\right)\left(e^{-\gamma(c_h - c_0)} + e^{\gamma(c_h - c_0)}\right)}{\gamma \left(e^{\gamma c_0} e^{-\gamma c_h} + e^{-\gamma c_0} e^{\gamma c_h}\right)^2} < 0. \end{aligned}$$

The last inequality comes from the fact that the function $e^x - e^{-x} - x\left(e^{-x} + e^x\right)$ is negative and monotonically decreasing for all $x > 0$. The second statement comes directly from the expression for $\frac{\partial \widetilde{q}(c_h)}{\partial q_0}$. ■

Lemma B.10 *If* $\frac{v_0}{q_0} < h$, *then* $\tilde{v}\left(y; q_0, v_0, y\right) - h\tilde{q}\left(y; q_0, v_0, y\right) > 0$.

Proof. We parameterize c_h by y. The idea is that if the function $\tilde{v}\left(y; q_0, v_0, y\right) - h\tilde{q}\left(y; q_0, v_0, y\right)$ is negative at $y = c_0$ and positive as $y \to \infty$, then there is a $y = c_h$ so that this function is zero (satisfying the definition of c_h) and where the slope of this function is positive, which is the claim of our lemma.

The function $\tilde{v}\left(y; q_0, v_0, y\right) - h\tilde{q}\left(y; q_0, v_0, y\right)$ can be solved by imposing the boundary conditions

$$\tilde{v}'\left(y\right) = \tilde{q}'\left(y\right) = 0, \tilde{q}\left(c_0\right) = q_0, \tilde{v}\left(c_0\right) = v_0. \tag{B.30}$$

for all $y \geq c_0$. Thus, by setting $y = c_0$, we must have

$$\tilde{v}\left(c_0; q_0, v_0, c_0\right) - h\tilde{q}\left(c_0; q_0, v_0, c_0\right) = v_0 - hq_0 < 0,$$

by the condition of the proposition.

Now we show that $\tilde{v}\left(y; q_0, v_0, y\right) - h\tilde{q}\left(y; q_0, v_0, y\right) \to \infty$ as $y \to \infty$. We first show calculate $\lim_{y\to\infty} \tilde{q}\left(y; q_0, v_0, y\right)$ in (B.21). For this, we solve for $e^{-y\gamma} A_1$ and $e^{y\gamma} A_2$ from (B.21)-(B.22) and (B.30):

$$e^{-y\gamma} A_1 = \frac{q_0 - \frac{R_C}{2} + e^{(c_0 - y)\gamma} \frac{R_K M'(y)}{2} - \frac{R_K\gamma}{2} M\left(c_0\right)}{e^{(y - c_0)\gamma} + e^{\gamma(c_0 - y)}}, e^{y\gamma} A_2 = \frac{q_0 - \frac{R_C}{2} - e^{(y - c_0)\gamma} \frac{R_K \gamma M'(y)}{2} - \frac{R_K\gamma}{2} M\left(c_0\right)}{e^{(y - c_0)\gamma} + e^{\gamma(c_0 - y)}}.$$

where $M\left(y\right) \equiv -e^{\gamma y} \operatorname{Ei}\left[-\gamma y\right] + e^{-\gamma y} \operatorname{Ei}\left[y\gamma\right]$. Using $\lim_{y\to\infty} M'\left(y\right) = 0$, it is easy to show that $\lim_{y\to\infty} e^{y\gamma} A_2 = \lim_{y\to\infty} e^{-y\gamma} A_1 = 0$, which implies that $\lim_{y\to\infty} \tilde{q}\left(y; q_0, v_0, y\right) = \frac{R_C}{2}$ in (B.21). A similar argument implies that $\lim_{c\to\infty} \tilde{v}\left(c; q_0, v_0, c\right) = \infty$. Thus, $\tilde{v}\left(c; q_0, v_0, c\right) - h\tilde{q}\left(c; q_0, v_0, c\right) = \infty$. This prove the statement. ■

Putting together the above three lemmas, we have

$$\frac{\partial c_h}{\partial q_0} = -\frac{\tilde{v}'_{q_0} - h\tilde{q}'_{q_0}}{\tilde{v}'_{c_h} - h\tilde{q}'_{c_h}} > 0, \text{ and } \frac{\partial c_h}{\partial v_0} = -\frac{\tilde{v}'_{v_0} - h\tilde{q}'_{v_0}}{\tilde{v}'_{c_h} - h\tilde{q}'_{c_h}} < 0$$

This implies that $c_h^\pi < c_h^*$ whenever $q_\pi\left(c_0\right) \leq q\left(c_0\right)$ and $v_\pi\left(c_0\right) \geq v\left(c_0\right)$.

For the last step, as $\frac{\partial c_h}{\partial v_0} = -\frac{\tilde{v}'_{v_0} - h\tilde{q}'_{v_0}}{\tilde{v}'_{c_h} - h\tilde{q}'_{c_h}} < 0$, it suffices to show that this result holds for the worst v_0 drop to maintain p_0, i.e., v_0 and q_0 decrease proportionally so v_0/q_0 remains at constant.

To this end, we consider decreasing q_0 to $\bar{q}_0 = q_0 - \varepsilon$ where ε is very small. To make sure that $\frac{\bar{v}_0}{\bar{q}_0} = \frac{v_0}{q_0}$, we need that $\bar{v}_0 = v_0 - a\varepsilon$ where $a = \frac{v_0}{q_0}$. Let us refer to all the objects after the change with the bar. Our goal is to show that $\bar{v}(c_h)/\bar{q}(c_h)$ would increase; then $\tilde{v}'_{c_h} - h\tilde{q}'_{c_h} > 0$ implies that $c_h^\pi < c_h^*$. Using the first two Lemmas above, we have (denoting $x \equiv (c_h - c_0)\gamma$)

$$\begin{aligned}\bar{q}(c_h) &= \tilde{q}(c_h) - \varepsilon\frac{2}{e^x + e^{-x}}\\ \bar{v}(c_h) &= \tilde{v}(c_h) - 2\varepsilon\frac{e^x - e^{-x} - x(e^{-x}+e^x)}{\gamma(e^x+e^{-x})^2} - \frac{v_0}{q_0}\frac{2\varepsilon}{e^x+e^{-x}}\end{aligned}$$

Hence for sufficiently small ε we have (up to the first order)

$$\begin{aligned}\frac{\bar{v}(c_h)}{\bar{q}(c_h)} &= \frac{\tilde{v}(c_h)}{\tilde{q}(c_h)} - \frac{2\varepsilon}{\tilde{q}(c_h)}\left(\frac{e^x - e^{-x} - x(e^{-x}+e^x)}{\gamma(e^x+e^{-x})^2} + \frac{v_0}{q_0}\frac{1}{e^x+e^{-x}}\right) + \frac{\tilde{v}(c_h)}{\tilde{q}^2(c_h)}\frac{2\varepsilon}{e^x+e^{-x}}\\ &= \frac{\tilde{v}(c_h)}{\tilde{q}(c_h)} - \frac{2\varepsilon}{\tilde{q}(c_h)}\left(\frac{e^x - e^{-x} - x(e^{-x}+e^x)}{\gamma(e^x+e^{-x})^2} + \frac{\left(\frac{v_0}{q_0} - \frac{\tilde{v}(c_h)}{\tilde{q}(c_h)}\right)}{e^x+e^{-x}}\right) > \frac{\tilde{v}(c_h)}{\tilde{q}(c_h)}.\end{aligned} \tag{B.31}$$

Here, the third inequality in (B.31) is because the term $e^x - e^{-x} - x(e^{-x}+e^x) < 0$ for all $x > 0$ and $\frac{v_0}{q_0} - \frac{\tilde{v}(c_h)}{\tilde{q}(c_h)}$ is strictly negative because $\frac{v_0}{q_0} < \frac{\tilde{v}(c_h)}{\tilde{q}(c_h)} = h$; hence the first order impact of decreasing q_0 is an increase in $\bar{v}(c_h)/\bar{q}(c_h)$. Because the above argument holds for any v_0 and q_0, tracing out the first-order effect implies that any intervention which lowers cash value but keeps capital price unchanged will lower $\frac{\bar{v}(c_h)}{\bar{q}(c_h)}$. Compared to that change, an increase in v_0 just decreases c_h^π further. That concludes our proof.

The second statement follows the same steps with the following modifications. Each c_h has to be changed to c_l and each h has to be changed to l at every point of the proof. Then the first lemma remains the same, the first statement in the second lemma changes to $\frac{\partial\tilde{v}(c_l;q_0,v_0,c_l)}{\partial q_0} > 0$, while the second statement does not change. Also, in the proof of the first statement we use that $e^x - e^{-x} - x(e^{-x}+e^x) > 0$ for all $x < 0$, and the proof of the second statement we use that $e^x - e^{-x} - (x+y)(e^{-x}+e^x) < 0$ for all $x < 0$ and $y > 0$. In the last part we follow the same steps, but the inequality (B.31) in the modified version is switched. This gives that $c_l^\pi > c_l^*$ under the conditions of the statement.

B.4 Solution for Price Floor Policy and Proof of Proposition 8

B.4.1 Characterizing the equilibrium with price floor policy

We first derive the solutions for price floor policy. A price floor policy $\pi(c)$ is defined as

$$0 = q'(c)\sigma^2 + \frac{\sigma^2}{2}v''(c) - v(c) + \frac{\xi}{2}(R_C c + R_K) + c\pi(c), \tag{B.32}$$

$$0 = \frac{\sigma^2}{2}q'' - q(c) + \frac{\xi}{2}\left(R_C + \frac{R_K}{c}\right) - \pi(c). \tag{B.33}$$

so that 1) for $c \in (c_0, c_h^g]$, $\pi(c) = 0$, and at the upper investment threshold $p(c_h^g) = h$; and 2) for $c \in [c_l^g, c_0]$, $v(c) = (l+\delta)q(c)$ always. Here, $v(c)$, $q(c)$, $\pi(c)$, c_0 and c_h^g are endogenous. We have the following lemma.

Lemma B.11 *Given the lower disinvestment threshold c_l^g, the solution to the price floor policy can be calculated as follows.*

1. *Given the upper investment threshold c_h^g, first calculate the welfare function $j_g(c) = R_K + R_C c +$*

$D_1 e^{-\gamma c} + D_2 e^{\gamma c}$, where the constants D_1-D_2 are given by the boundary conditions

$$j_g\left(c_l^g; c_h^g\right) = \left(c_l^g + l\right) j_g'\left(c_l^g\right), \text{ and } j\left(c_h^g; c_h^g\right) = \left(c_h^g + h\right) j_g'\left(c_h^g\right).$$

2. *For $c \in (c_0, c_h^g]$, the capital price and cash price is given by*

$$\begin{aligned} v(c) &= R_K + \frac{R_C c}{2} + e^{c\gamma}\left(A_3 - cA_2\right) - e^{-c\gamma}\left(A_4 + cA_1\right) + cR_K \frac{\gamma}{2}\frac{\left(e^{\gamma c}\operatorname{Ei}(-\gamma c) - e^{-c\gamma}\operatorname{Ei}(\gamma c)\right)}{2}, \\ q(c) &= \frac{R_C}{2} + e^{-c\gamma}A_1 + e^{c\gamma}A_2 + R_K \frac{\gamma}{2}\frac{-e^{c\gamma}\operatorname{Ei}(-\gamma c) + e^{-c\gamma}\operatorname{Ei}(c\gamma)}{2}. \end{aligned}$$

Here, $A_4 = -D_1$ and $A_3 = D_2$. The other four constants, i.e., A_1-A_2, c_0 and c_h^g, are determined by the following four boundary conditions

$$v'\left(c_h^g\right) = 0, q'\left(c_h^g\right) = 0, v\left(c_0\right) = (l + \delta)\, q\left(c_0\right), v'\left(c_0\right) = (l + \delta)\, q'\left(c_0\right)$$

3. *For $c \in [c_l^g, c_0]$, we have*

$$q(c) = \frac{j_g(c)}{l + c} \text{ and } v(c) = \frac{l + \delta}{l + c} j_g(c) \tag{B.34}$$

and the taxation is given by

$$\pi(c) = \frac{\sigma^2}{2} q'' - \xi q(c) + \frac{\xi}{2}\left(R_C + \frac{R_K}{c}\right) > 0$$

Proof. *The total welfare function $j(c) = v(c) + cq(c)$ given in the step 1 of Lemma B.11 only depends on the investment/disinvestment policies c_l^g and c_h^g (see explanations around equation (18) and (19)). For $c \in (c_0, c_h^g]$, there is not taxation and the derivation is the same as before, except that at the endogenous intervention point c_0 we are value-matching and smooth-pasting so that the price is the implemented floor price $l + \delta$. Note that by construction we have $v\left(c_h^g\right) = hq\left(c_h^g\right)$ (due to $j\left(c_h^g\right) = \left(c_h^g + h\right) j'\left(c_h^g\right)$). For $c \in [c_l^g, c_0]$, notice that $v(c) = (l + \delta)\, q(c)$ always; (B.34) follows because of $j_g(c) = v(c) + cq(c) = (l + c)\, q(c)$. The endogenous taxation $\pi(c)$ follows from (B.33).* ■

B.4.2 Proof of Proposition 8

Now we set $\delta = 0$ and prove Proposition 8. There are three steps.

Step 1. Rewrite the problem Clearly, for $c \in (c_0, c_h^g]$ the same structure solution applies without policy, with the only difference at the lower end c_0 so that $v'(c_0) = lq'(c_0)$ might not be zero. This allows us to draw connection between the equilibrium with policy and the one without. We first show that for $c_l^g < c_l^*$, the resulting slope at c_0 has to be negative, i.e.

$$v'(c_0) = lq'(c_0) < 0. \tag{B.35}$$

To show this, focus on $c \in [c_l^g, c_0]$. By $v(c) = \frac{l}{l+c} j_g(c)$ and boundary condition of $j_g(c)$, we have

$$v'\left(c_l^g\right) = \frac{l\left[j_g'\left(c_l^g\right)\left(l + c_l^g\right) - j_g\left(c_l^g\right)\right]}{\left(l + c_l^g\right)^2} = 0.$$

Moreover, since $j_g''(c) < 0$ (see Proposition 2 and its proof), we have $\left[j_g'(c)(l+c) - j_g(c)\right]' = j_g''(c)(l+c) < 0$. As a result, since $c_0 > c_l^g$, we have

$$sign\left[v'(c_0)\right] = sign\left[j_g'(c_0)(l+c_0) - j_g(c_0)\right] < 0.$$

This proves (B.35).

This suggest us to introduce $\{v(\cdot), q(\cdot), c_0, c_h^g; x\}$ indexed by x as the solution to the ODE system (10) and (11), with modified boundary conditions

$$\begin{aligned} v'(c_h^g) &= q'(c_h^g) = 0, v(c_h^g) = hq(c_h^g), \\ v'(c_0) &= -xl, q'(c_0) = -x, v(c_0) = lq(c_0). \end{aligned}$$

Here, the parameter $x > 0$ captures the negative slope of $v'(c_0) = lq'(c_0) < 0$. As shown shortly, our key result does not depend on the exact value of x, which will be determined by pre-determined lower disinvestment threshold c_l^g.

It is easy to show that if $c_l^g = c_l^*$, i.e., the policy sets the lower disinvestment threshold as the one in the market solution, then $x = 0$ and we have $c_0 = c_l^* = c_l^g$ and $c_h^g = c_h^*$. Given this result, the claim in Proposition 8 is equivalent to show that

$$\lim_{\gamma\to\infty} \frac{\partial c_h}{\partial x} > 0.$$

Step 2. Solve the new ODE system For simplicity, we denote c_h^g by c_h. Given c_0 and c_h, the boundary conditions $v'(c_h) = q'(c_h) = 0$ and $v'(c_0) = -xl, q'(c_0) = -x$ imply that

$$\begin{aligned} q(c_0; c_0, x, c_h) &= q(c_l; c_l, c_h)|_{c_l=c_0} + \frac{x\left(e^{2\gamma c_0} + e^{2\gamma c_h}\right)}{\gamma\left(e^{2\gamma c_h} - e^{2\gamma c_0}\right)} \\ q(c_h; c_0, x, c_h) &= q(c_h; c_l, c_h)|_{c_l=c_0} + \frac{2xe^{\gamma(c_0+c_h)}}{\gamma\left(e^{2\gamma c_h} - e^{2\gamma c_0}\right)}, \\ v(c_0; c_0, x, c_h) &= v(c_l; c_l, c_h)|_{c_l=c_0} + \frac{x\left(e^{2\gamma c_0}(\gamma l+1) + e^{2\gamma c_h}(\gamma l - 1)\right)}{\gamma^2\left(e^{2\gamma c_h} - e^{2\gamma c_0}\right)}, \\ v(c_h; c_0, x, c_h) &= v(c_h; c_l, c_h)|_{c_l=c_0} + \frac{2xe^{\gamma(c_0+c_h)}(c_0 - c_h + l)}{\gamma\left(e^{2\gamma c_h} - e^{2\gamma c_0}\right)}. \end{aligned}$$

where $q(c_l; c_l, c_h), q(c_h; c_l, c_h), v(c_l; c_l, c_h), v(c_h; c_l, c_h)$ has been defined above. Then, c_0 and c_h solve $F_h(c_0, x, c_h) = F_l(c_0, x, c_h) = 0$ where we define

$$\begin{aligned} & F_h(c_0, x, c_h) \equiv v(c_h; c_0, x, c_h) - hq(c_h; c_0, x, c_h) \\ = \; & R_K + \frac{(c_h - h)R_C}{2} - \frac{R_C}{2\gamma}m(c_0, c_h) + \frac{R_K\gamma}{2}\left(\frac{g_h(c_0, c_h)}{\gamma} - (c_h + h)f_h(c_0, c_h)\right) \\ & + \frac{2xe^{\gamma(c_0+c_h)}(c_0 - c_h + l)}{\gamma\left(e^{2\gamma c_h} - e^{2\gamma c_0}\right)} - h\frac{2xe^{\gamma(c_0+c_h)}}{\gamma\left(e^{2\gamma c_h} - e^{2\gamma c_0}\right)}, \text{ and} \\ & F_l(c_0, x, c_h) \equiv v(c_0; c_0, x, c_h) - lq(c_0; c_0, x, c_h) \\ = \; & R_K + \frac{(c_0 - l)R_C}{2} + \frac{R_C}{2\gamma}m(c_0, c_h) + \frac{R_K\gamma}{2}\left(\frac{g_l(c_0, c_h)}{\gamma} - (c_0 + l)f_l(c_0, c_h)\right) \\ & + \left(\frac{x\left(e^{2\gamma c_0}(\gamma l+1) + e^{2\gamma c_h}(\gamma l - 1)\right)}{\gamma^2\left(e^{2\gamma c_h} - e^{2\gamma c_0}\right)} - l\frac{x\left(e^{2\gamma c_0} + e^{2\gamma c_h}\right)}{\gamma\left(e^{2\gamma c_h} - e^{2\gamma c_0}\right)}\right) \end{aligned}$$

Simple derivation reveals

$$\frac{\partial F_l}{\partial c_0} = \frac{R_C}{2} - \frac{R_C}{2}\frac{2e^{(c_0+c_h)\gamma}}{\left(e^{c_0\gamma}+e^{c_h\gamma}\right)^2} + \frac{R_K\gamma}{2}\left(\frac{1}{\gamma c_0} + \frac{\left(e^{2\gamma c_h}+e^{2\gamma c_0}\right)}{\left(e^{2\gamma c_h}-e^{2\gamma c_0}\right)}g_l - f_l - (c_0+l)\frac{\left(e^{2\gamma c_h}+e^{2\gamma c_0}\right)}{\left(e^{2\gamma c_h}-e^{2\gamma c_0}\right)}\left(\gamma f_l - \frac{1}{c_0}\right)\right)$$

$$\begin{aligned}\frac{\partial F_h}{\partial c_0} &= \frac{R_C}{2}\frac{2e^{(c_0+c_h)\gamma}}{\left(e^{c_0\gamma}+e^{c_h\gamma}\right)^2} + \frac{R_K\gamma}{2}\left(\frac{2g_l\left(c_h,c_0\right)}{\left(e^{\gamma(c_h-c_0)}-e^{-\gamma(c_h-c_0)}\right)} - (c_h+h)\frac{2\left(\gamma f_l - \frac{1}{c_0}\right)}{\left(e^{\gamma(c_h-c_0)}-e^{-\gamma(c_h-c_0)}\right)}\right)\\ &\quad + \frac{2xe^{\gamma(c_0+c_h)}}{\gamma\left(e^{2\gamma c_h}-e^{2\gamma c_0}\right)}\left(\frac{\gamma(c_0-c_h+l-h)\left(e^{2\gamma c_0}+e^{2\gamma c_h}\right)}{\left(e^{2\gamma c_h}-e^{2\gamma c_0}\right)}+1\right)\end{aligned}$$

$$\frac{\partial F_l}{\partial c_h} = \frac{R_C}{2}\frac{2e^{(c_0+c_h)\gamma}}{\left(e^{c_0\gamma}+e^{c_h\gamma}\right)^2} + \frac{R_K\gamma}{2}\left(-\frac{2g_h}{e^{\gamma(c_h-c_0)}-e^{\gamma(c_0-c_h)}} - 2\left(c_0+l\right)\frac{\frac{1}{c_h}-\gamma f_h}{e^{\gamma(c_h-c_0)}-e^{\gamma(c_0-c_h)}}\right)$$

$$\begin{aligned}\frac{\partial F_h}{\partial c_h} &= \frac{R_C}{2} - \frac{R_C}{2}\frac{2e^{(c_0+c_h)\gamma}}{\left(e^{c_0\gamma}+e^{c_h\gamma}\right)^2} + \frac{R_K\gamma}{2}\left(\begin{array}{c}\frac{1}{\gamma c_h} - \frac{\left(e^{2\gamma c_h}+e^{2\gamma c_0}\right)}{\left(e^{2\gamma c_h}-e^{2\gamma c_0}\right)}g_h\left(c_0,c_h\right) - \\ (c_h+h)\frac{\left(e^{2\gamma c_h}+e^{2\gamma c_0}\right)}{\left(e^{2\gamma c_h}-e^{2\gamma c_0}\right)}\left(\frac{1}{c_h}-\gamma f_h\left(c_h,c_0\right)\right) - f_h\left(c_0,c_h\right)\end{array}\right)\\ &\quad - \frac{2xe^{\gamma(c_0+c_h)}}{\gamma\left(e^{2\gamma c_h}-e^{2\gamma c_0}\right)}\left(\frac{\gamma(c_0-c_h+l-h)\left(e^{2\gamma c_0}+e^{2\gamma c_h}\right)}{\left(e^{2\gamma c_h}-e^{2\gamma c_0}\right)}+1\right)\end{aligned}$$

Step 3. Prove the claim Now we are ready to show our desired result $\lim_{\gamma\to\infty}\frac{\partial c_h}{\partial x} > 0$. First of all, it is easy to show that when $\gamma\to\infty$, $c_h\to h$ and $c_0\to l$ are bounded. The Cramer's rule (or implicit function theorem) implies

$$\lim_{\gamma\to\infty}\frac{\partial c_h}{\partial x} = -\lim_{\gamma\to\infty}\left|\begin{array}{cc}\frac{\partial F_h}{\partial x} & \frac{\partial F_h}{\partial c_0}\\ \frac{\partial F_l}{\partial x} & \frac{\partial F_l}{\partial c_0}\end{array}\right| \Bigg/ \left|\begin{array}{cc}\frac{\partial F_h}{\partial c_h} & \frac{\partial F_h}{\partial c_0}\\ \frac{\partial F_l}{\partial c_h} & \frac{\partial F_l}{\partial c_0}\end{array}\right| = \lim_{\gamma\to\infty}\frac{-\frac{\partial F_h}{\partial x}\frac{\partial F_l}{\partial c_0}+\frac{\partial F_h}{\partial c_0}\frac{\partial F_l}{\partial x}}{\frac{\partial F_h}{\partial c_h}\frac{\partial F_l}{\partial c_0}-\frac{\partial F_l}{\partial c_h}\frac{\partial F_h}{\partial c_0}}.$$

Focus on the denominator first. It is easy to show that

$$\lim_{\gamma\to\infty}\frac{\partial F_l}{\partial c_0} = \frac{R_C}{2}+\frac{R_K}{2}\frac{l}{c_0^2},\ \lim_{\gamma\to\infty}\frac{\partial F_h}{\partial c_h} = \frac{R_C}{2}+\frac{R_K h}{2c_h^2}, \text{and } \lim_{\gamma\to\infty}\frac{\partial F_h}{\partial c_0} = \lim_{\gamma\to\infty}\frac{\partial F_l}{\partial c_h} = 0,$$

implying

$$\lim_{\gamma\to\infty}\frac{\partial c_h}{\partial x} = \frac{\lim_{\gamma\to\infty}\left(\frac{\partial F_h}{\partial c_0}\frac{\partial F_l}{\partial x}-\frac{\partial F_h}{\partial x}\frac{\partial F_l}{\partial c_0}\right)}{\left(\frac{R_C}{2}+\frac{R_K h}{2c_h^2}\right)\left(\frac{R_C}{2}+\frac{R_K}{2}\frac{l}{c_0^2}\right)} \tag{B.36}$$

For the numerator, since $\frac{\partial F_l}{\partial x} = -\frac{1}{\gamma^2}$ and $\frac{\partial F_h}{\partial x} = -\frac{2e^{(c_0+c_h)\gamma}(h-l+(c_h-c_0))}{\left(e^{\gamma 2c_h}-e^{\gamma 2c_0}\right)\gamma}$, we can show the following two limiting results:

$$\lim_{\gamma\to\infty}\gamma\left(e^{\gamma(c_h-c_0)}-e^{-\gamma(c_h-c_0)}\right)\frac{\partial F_h}{\partial x}\frac{\partial F_l}{\partial c_0} = -2\left(h-l+(c_h-c_0)\right)\left(\frac{R_C}{2}+\frac{R_K}{2}\frac{l}{c_0^2}\right); \tag{B.37}$$

and

$$
\begin{aligned}
&\lim_{\gamma\to\infty}\gamma\left(e^{\gamma(c_h-c_0)}-e^{-\gamma(c_h-c_0)}\right)\frac{\partial F_h}{\partial c_0}\frac{\partial F_l}{\partial x} \\
= \quad &\lim_{\gamma\to\infty}\frac{1}{\gamma}\left(\begin{array}{c}\frac{R_C}{2}\frac{2\left(e^{\gamma c_h}-e^{\gamma c_0}\right)}{\left(e^{c_0\gamma}+e^{c_h\gamma}\right)}+\frac{R_K\gamma}{2}\left(2g_l\left(c_h,c_0\right)-\left(c_h+h\right)2\left(\gamma f_l-\frac{1}{c_0}\right)\right)\\ +\frac{2x}{\gamma}\left(\frac{\gamma\left(c_0-c_h+l-h\right)\left(e^{2\gamma c_0}+e^{2\gamma c_h}\right)}{\left(e^{2\gamma c_h}-e^{2\gamma c_0}\right)}+1\right)\end{array}\right)=0. \qquad \text{(B.38)}
\end{aligned}
$$

Hence, applying (B.37) and (B.38) to (B.36), we have

$$
\lim_{\gamma\to\infty}\gamma\left(e^{\gamma(c_h-c_0)}-e^{-\gamma(c_h-c_0)}\right)\frac{\partial c_h}{\partial x}=\frac{2\left(h-l+\left(c_h-c_0\right)\right)\left(\frac{R_C}{2}+\frac{R_K}{2}\frac{l}{c_0^2}\right)}{\left(\frac{R_C}{2}+\frac{R_K h}{2c_h^2}\right)\left(\frac{R_C}{2}+\frac{R_K}{2}\frac{l}{c_0^2}\right)}>0.
$$

QED.