C Additional Material for He and Kondor (2015)

C.1 What if $c_h^P > R_K$?

Throughout the main body of the paper we have restricted our attention to the case where the equilibrium range of cash-to-capital ratio, which is $[c_l^*, c_h^*]$, is below R_K. This ensures that in the idiosyncratic stage the price $\widehat{p}_\tau = c_\tau \le R_K$ clears the market in a way that cash (capital) firms get all the cash (capital). Otherwise, suppose that $c_h^* > R_K$. Then, along the equilibrium path it is possible that $c_\tau > R_K$, and the price of capital at the idiosyncratic stage will be capped at the capital's final output R_K. As a result, cash firms sell all their capital $\frac{K_\tau}{2}$ to capital firms at a price of R_K, ending up with a total amount of cash of $\frac{C_\tau + R_K K_\tau}{2}$; while the capital firms will have K_τ units of capital but with $\frac{C_\tau - R_K K_\tau}{2}$ units of cash in their hands. This allocation is inefficient as capital firms are holding cash.

This concern is also relevant in the planner's constrained efficient allocation. Recall that overinvestment requires the planner's upper investment threshold $c_h^P > c_h^*$, hence it is quite likely that in a wide range of parameters $c_h^P > R_K$. Because the planner is facing the same information constraint, i.e., the planner cannot tell a cash firm from a capital firm, and the constrained outcome at the idiosyncratic stage is likely to be inefficient.

To illustrate that our main results hold in this case, in this Appendix we relax our parameter restriction to

$$R_K > lR_C, \text{ and } R_K < h. \tag{C.39}$$

Note that we are replacing $R_K < hR_C$ by $R_K < h$; it just says that on the margin capital is better than cash if we just consume cash (for a utility of 1).

We will fully characterize the planner's solution when $c_h^P > R_K$, which is relatively simpler than the market solution (solving for the market solution fully is much more involved). It turns out that when

$$l < \frac{R_K}{R_C} < h$$

holds, then $c_h^P > R_K$ always holds. More importantly, this Appendix shows that when $c_h^P > R_K$, then our key Proposition 6 in the main text remains valid. It is because in Proposition 6 we show the overinvestment result by establishing in the limit that $c_h^* \to h < R_K$. Then, since $c_h^P > R_K$, we know that $c_h^* < c_h^P$ in this case automatically, and the key overinvestment result holds.

C.1.1 Mechanism design approach: constrained efficient allocation at idiosyncratic stage

We first show that the mechanism design approach yields the same result as if the planner opens the trading market at the idiosyncratic stage: as discussed above, cash firms ends up with $\frac{C_\tau + R_K K_\tau}{2}$ amount of cash, while capital firms have K_τ units of capital but with $\frac{C_\tau - R_K K_\tau}{2}$ amount of cash.

Recall that cash firms cannot operate capital, but capital firms can consume cash at its reservation value of 1. Denote the allocations by $\{C_C, C_K, K_C, K_K\} \in \mathbb{R}_+^4$ where the subscript indicates the reported type. Given resource pair (C, K), the planner is maximizing

$$\max_{\{C_C, C_K, K_C, K_K\} \in \mathbb{R}_+^4} \frac{1}{2}(R_C C_C) + \frac{1}{2}(R_K K_K + C_K)$$

$$s.t. \quad C_C + C_K \le C, \tag{C.40}$$

$$K_C + K_K \le K, \tag{C.41}$$

$$C_C \ge C_K, \tag{C.42}$$

$$R_K K_K + C_K \ge R_K K_C + C_C. \tag{C.43}$$

Proposition C.1 *The solution to the above problem is*

$$K_K = K,\ K_C = 0,\ C_K = \frac{C - R_K K}{2},\ \text{and}\ C_C = \frac{C + R_K K}{2},$$

which is identical to the market solution, where the capital price is capped at $\widehat{p}_\tau = R_K$.

Proof. *We have several key observations. First, reducing K_C and increasing K_K until (C.41) binds can improve objective, relax (C.43), but still satisfies (C.41). Hence $K_C = 0$ and $K_K = K$. Second, (C.40) holds with equality. Otherwise, let $\varepsilon = (C - C_C - C_K)/2 > 0$ and raise C_C and C_K by ε, which improves objective and satisfies (C.42) and (C.43). Then we guess (C.43) binds before (C.42). Solve the problem with binding (C.42), we have $C_K = \frac{C-R_K K}{2}$ and $C_C = \frac{C+R_K K}{2}$. Since (C.42) holds with strict inequality under this solution, our claim follows.* ■

C.1.2 Property of the planner's value function $j_P(c)$

The previous subsection shows that when $c > R_K$, then the aggregate surplus at the idiosyncratic stage is

$$R_K + R_C\left(\frac{c + R_K}{2}\right) + \frac{c - R_K}{2}$$

where the second (third) term captures the cash in the cash (capital) firms. For $c \leq R_K$, the surplus is still $R_K + cR_C$. As a result, we can write the HJB equation as

$$0 = \frac{\sigma^2}{2} j_P''(c) + \xi\left(R_K + \frac{R_C + 1}{2}c + \frac{R_C - 1}{2}\min(c, R_K) - j_P(c)\right) \text{ for } c \in \left(0, c_h^P\right)$$

where the flow payoff $f(c) \equiv R_K + \frac{R_C+1}{2}c + \frac{R_C-1}{2}\min(c, R_K)$ is piecewise linear, increasing, and concave in c.

The general solution can be written as

$$j_P(c) = \begin{cases} R_K + R_C c + D_1^{below} e^{-\gamma c} + D_2^{below} e^{\gamma c} & \text{for } c \in (0, R_K) \\ \frac{R_C+1}{2}R_K + \frac{R_C+1}{2}c + D_1^{upper} e^{-\gamma c} + D_2^{upper} e^{\gamma c} & \text{for } c \in \left(R_K, c_h^P\right) \end{cases}$$

where D_1^{below}, D_2^{below}, D_1^{upper}, and D_2^{upper} are coefficients to be determined. Now we list all the boundary conditions. At $c_l^P = 0$ we have the smooth pasting condition:

$$j_P(0) = l j_P'(0). \tag{C.44}$$

At $c = R_K$ we have value matching and smooth pasting conditions on both sides

$$j_P(R_K-) = j_P(R_K+),\, j_P'(R_K-) = j_P'(R_K+) \tag{C.45}$$

At $c = c_h^P$ we have smooth pasting condition

$$j_P\left(c_h^P\right) = \left(h + c_h^P\right) j_P'\left(c_h^P\right) \tag{C.46}$$

and super contact condition

$$j_P''\left(c_h^P\right) = 0 \tag{C.47}$$

These five conditions (C.44)-(C.47) pin down five unknowns, in which four of them are coefficients for $j_P(c)$ in two intervals, and one of them is the optimal upper threshold c_h^P.

Now we show that our main result hold in this case of $c_h^P > R_K$. First, we the counterpart of Proposition

2.

Proposition C.2 *The planner's value function under optimal policy $j_P(c)$ is strictly concave, and $j_P(c) < f(c) = R_K + \frac{R_C+1}{2}c + \frac{R_C-1}{2}\min(c, R_K)$.*

Proof. *Denote the flow payoff by*

$$f(c) = R_K + \frac{R_C+1}{2}c + \frac{R_C-1}{2}\min(c, R_K).$$

The value function $j_P(c)$ satisfies

$$0 = \frac{\sigma^2}{2}j_P''(c) + \xi(f(c) - j_P(c)) \tag{C.48}$$

with boundary conditions $j_P(0) = lj_P'(0)$, $j_P(c_h^P) = (h + c_h^P)j_P'(c_h^P)$, $j_P''(c_h^P) = 0$, and two conditions at $c = R_K$. Note that the boundary conditions imply that $j_P(c_h^P) = f(c_h^P) = \frac{R_C+1}{2}R_K + \frac{R_C+1}{2}c_h^P$ given $c_h^P > R_K$.

We show that $j_P(c)$ is concave over $[0, c_h^P]$, which implies $j_P(c) < f(c)$. First, from smooth pasting condition at c_h^P we have (recall the parameter restriction of $R_K > h$)

$$\frac{R_C+1}{2} - j_P'(c_h^P) = \frac{R_C+1}{2} - \frac{j_P(c_h^P)}{h + c_h^P} = \frac{R_C+1}{2} - \frac{\frac{R_C+1}{2}R_K + \frac{R_C+1}{2}c_h^P}{h + c_h^P} = \frac{R_C+1}{2}\frac{h - R_K}{h + c_h^P} < 0.$$

Then, taking derivative again on (C.48) and evaluate at the optimal policy point c_h^P, we have

$$j_P'''(c_h^P) = -\frac{2\xi}{\sigma^2}\left(\frac{R_C+1}{2} - j_P'(c_h^P)\right) = \frac{2\xi}{\sigma^2}\frac{R_C+1}{2}\frac{R_K - h}{h + c_h^P} > 0, \tag{C.49}$$

and as a result $j_P''(c_h^P-) < 0$. Suppose that j_P fails to be globally concave over $[0, c_h^P]$. Then there exists some point $j_P'' > 0$, and pick the largest one $\widehat{c}$ so that j_P'' is concave over $[\widehat{c}, c_h^P]$ with $j_P''(\widehat{c}) = 0$ and $j_P'''(\widehat{c}) < 0$. At $\widehat{c}$ we have $j_P(\widehat{c}) = f(c)$, $j_P'(\widehat{c}) < f'(c)$ (because $j_P(c)$ cross $f(c)$ from above), and $j_P(c)$ is strictly convex around the vicinity of $c < \widehat{c}$. Using standard argument one can show that $j_P(c)$ is strictly convex over $[0, \widehat{c}]$ (if not, pick the largest point point $\widetilde{c}$ so that $j_P''(\widetilde{c}) = 0$. But due to convexity of $j(c)$ and linearity of $f(c)$ in $[\widetilde{c}, \widehat{c}]$ we have $f(c) < j_P(c)$ strictly, contradicting with $j_P''(\widetilde{c}) = 0$). Thus $j_P(c)$ is strictly convex over $[0, \widehat{c}]$. Since $f(c)$ is concave, we have $j_P'(0) < j_P'(\widehat{c}) < f'(\widehat{c}) < f'(0) = R_C$ and $j_P(0) > f(0) = R_K$. It contradicts with the boundary condition at $c = 0$, because $j_P(0) = lj_P'(0) < lR_C < R_K$.

To sum up, $j_P(c)$ is globally concave over $[0, c_h^P]$, which also implies that $j_P(c) < f(c)$ due to (C.48). ■

We now show that when $R_K < hR_C$ holds, we have $c_h^P > R_K$.

Corollary 1 *Under parameter restriction C.39, and suppose that $R_K < hR_C$. Then $c_h^P > R_K$.*

Proof. *Suppose that $c_h^P < R_K$; then we should have the same characterization in Proposition 2. However,*

$$\frac{R_K - hR_C}{R_K - lR_C}\left(e^{c_h^P\gamma}(1 + l\gamma) - (1 - l\gamma)e^{-c_h^P\gamma}\right) - 2\gamma(c_h^P + h) = 0$$

admits no solution: if $R_K - hR_C < 0$ then even the first term is negative.[21] *Hence $c_h^P > R_K$. And, when $c \to \infty$ the marginal value of cash is 1 and the marginal value of capital is R_K. Hence when $R_K < h$, holding cash is strictly dominated by holding capital when $c \to \infty$, implying $c_h^P < \infty$.*[22] ■

[21] Note that $e^{c_h^P\gamma}(1 + l\gamma) - (1 - l\gamma)e^{-c_h^P\gamma} > e^{c_h^P\gamma} - e^{-c_h^P\gamma} > 0$.

[22] This result is also consistent with condition (C.49) which says that postponing liquidating for $c > c_h^P$ gives $j_P''(c_h^P+) > 0$.

Intuitively, when accumulated cash (relative to the capital stock) is below R_K, then the marginal value of cash is R_C. If $R_K < hR_C$, then the benefit of capital is below the cost of building capital, it is never optimal to build the capital for $c < R_K$. When $c > R_K$, the marginal value of cash is just its consumption value 1. As $R_K > h$ says the benefit of capital exceeds the cost, then the planner starts building capital when $c = c_h^P > R_K$ for sufficiently high c_h^P.

C.1.3 Welfare results

Now we have the counterpart of Proposition 3, which gives the key result about investment inefficiency. The argument is almost identical to Proposition 3 which only relies on the concavity of j_P for all policy $c \in [c_l, c_h]$ if $c_h < c_h^P$ and $c_l > 0$.

Proposition C.3 *For any $c_h < c_h^P$ and $c_l > 0$, we have*

$$\frac{\partial j_P(c; c_l, c_h)}{\partial c_l} < 0, \text{ and } \frac{\partial j_P(c; c_l, c_h)}{\partial c_h} > 0 \text{ of all } c \in [c_l, c_h].$$

Proof. *Suppose that we are given the policy pair (c_l, c_h) with $0 < c_l < c_h < c_h^P$ where c_h^P satisfies the super-contact condition $j_P''\left(c_h^P; 0, c_h^P\right) = 0$. To avoid cumbersome notation we denote the social value $j_P(c; c_l, c_h)$ given the policy pair (c_l, c_h) by $j(c; c_l, c_h)$, and denote the social value under the optimal policy $j_P\left(c; 0, c_h^P\right)$ by $j_P(c)$. We need to show that*

$$\frac{\partial j(c; c_l, c_h)}{\partial c_l} < 0 \text{ and } \frac{\partial j(c; c_l, c_h)}{\partial c_h} > 0.$$

This result further implies that for $0 < c_l^2 < c_l^1 < c_h^1 < c_h^2 < c_h^P$, we have $j\left(c; c_l^1, c_h^1\right) < j\left(c; c_l^2, c_h^2\right)$.

As preparation, we first show that $j''(c_h; c_l, c_h) < 0$ and $j''(c_l; c_l, c_h) < 0$. Because (c_l, c_h) is suboptimal, we must have $j(c; c_l, c_h) < j_P(c) \le f(c)$ (recall the above proposition). Then $0 = \frac{\sigma^2}{2} j''(c) + \xi(f(c) - j(c))$ implies that $j(c)$ is strictly concave at both ends. Second, for any policy pair (c_l, c_h) (including the market solution or the planner's solution), the smooth pasting condition (not optimality condition!) at the regulated ends implies that

$$j(c_h; c_l, c_h) - (c_h + h) j'(c_h; c_l, c_h) = 0, \tag{C.50}$$
$$j(c_l; c_l, c_h) - (c_l + l) j'(c_l; c_l, c_h) = 0. \tag{C.51}$$

Now we start proving the properties for the top policy c_h. Define $F_h(c; c_l, c_h) \equiv \frac{\partial}{\partial c_h} j(c; c_l, c_h)$, which is the marginal impact of changing the top investment policy on the social value. Differentiating the basic ODE by the policy c_h, we have $\frac{\sigma^2}{2}\frac{\partial}{\partial c_h} j''(c; c_l, c_h) - \xi \frac{\partial}{\partial c_h} j(c; c_l, c_h) = 0$, or

$$\frac{\sigma^2}{2} F_h''(c; c_l, c_h) - \xi F_h(c; c_l, c_h) = 0. \tag{C.52}$$

Moreover, take the total derivative with respect to c_h on the equality (C.50), i.e., take derivative that affects both the policy c_h and the state $c = c_h$, we have

$$\begin{aligned}
& \frac{\partial}{\partial c_h} j(c_h; c_l, c_h) + j'(c_h; c_l, c_h) = j'(c_h; c_l, c_h) + (c_h + h)\left(\frac{\partial}{\partial c_h} j'(c_h; c_l, c_h) + j''(c_h; c_l, c_h)\right) \\
\Rightarrow \quad & \frac{\partial}{\partial c_h} j(c_h; c_l, c_h) - (c_h + h)\frac{\partial}{\partial c_h} j'(c_h; c_l, c_h) = (c_h + h) j''(c_h; c_l, c_h) < 0 \\
\Rightarrow \quad & F_h(c_h; c_l, c_h) - (c_h + h) F_h'(c_h; c_l, c_h) < 0. \qquad \text{(C.53)}
\end{aligned}$$

which gives the boundary condition of $F_h(\cdot)$ at c_h. At c_l we can take total derivative with respect to c_h on *the equality (C.51), we have the boundary condition of* $F_h(\cdot)$ *at* c_l*:*

$$\frac{\partial}{\partial c_h} j(c_l; c_l, c_h) = (c_l + l)\frac{\partial}{\partial c_h} j'(c_l; c_l, c_h) \Rightarrow F_h(c_l; c_l, c_h) - (c_l + l) F_h'(c_l; c_l, c_h) = 0. \tag{C.54}$$

With the aid of these two boundary conditions, the next lemma shows that $F_h(\cdot)$ *has to be positive always. Because of the definition of* $F_h(c; c_l, c_h) \equiv \frac{\partial}{\partial c_h} j(c; c_l, c_h)$*, it implies that raising* c_h *given any state* c *and any lower policy* c_l *improves the social value. The argument for the effect of* c_l *is similar and thus omitted.*

Lemma B.1 *We have* $F_h(c) > 0$ *for* $c \in [c_l, c_h]$*.*

Proof. *We show this result in three steps.*

First, $F_h(c)$ *cannot change sign over* $[c_l, c_h]$*. Suppose that* $F_h(c_l) > 0$*; then from (C.54) we know that* $F_h'(c_l) > 0$*. Then simple argument based on ODE (C.52) implies that* $F_h(\cdot)$ *is convex and always positive. Now suppose that* $F_h(c_l) < 0$*; then the similar argument implies that* F_h *is concave and negative always. Finally, suppose that* $F_h(c_l) = 0$ *but* F_h *changes sign at some point. Without loss of generality, there must exist some point* $\widehat{c}$ *so that* $F_h'(\widehat{c}) = 0$*,* $F_h(\widehat{c}) > 0$ *and* $F_h''(\widehat{c}) < 0$*. But this contradicts with the ODE (C.52).*

Second, define $W_h(c) \equiv F_h(c) - (l + c) F_h'(c)$ *so that* $W_h'(c) = -(l + c) F_h''(c) = -\frac{2\xi(l+c)}{\sigma^2} F_h(c)$*. As a result,* $W_h'(c)$ *cannot change sign. Because we have* $W_h(c_l) = 0$*,* $W_h(c)$ *cannot change sign either.*

Third, suppose counterfactually that $F_h(c) < 0$ *so that* $W_h'(c) > 0$*. Step 2 implies that* $W_h(c) > 0$*, and* $F_h'(c_h) = \frac{h-l}{l+c}(F_h - W_h) < 0$*. But we then have*

$$W_h(c_h) = F_h(c_h) - (l + c) F_h'(c_h) = F_h(c_h) - (h + c) F_h'(c_h) + (h - l) F_h'(c_h) < 0,$$

where we have used (C.53), contradiction. Thus we have shown that $F_h(c) > 0$*.* ■

■

The next proposition naturally follows.

Proposition C.4 *When* $\gamma \to \infty$ *then in the market solution* $c_h^* \to h < R_K$*. For planner's solution, we have* $c_h^P > R_K$ *when either* $R_K - hR_C > 0$ *is sufficiently small or when* $R_K - hR_C < 0$*. It implies that firms overinvest in capital in booms in the market solution.*

C.1.4 A Model without Idiosyncratic Investment Opportunities

Suppose that at the final date, every firm with holdings (K, C) can produce $R_K K + R_C C$ units of final consumption goods. This formulation is also equivalent to the base model but with complete market, i.e., introducing some Arrow-Debrew securities contingent on firms' idiosyncratic type realization–either K or C—which obviously complete the market. Ex ante, each firm will fully hedge using these Arrow-Debrew contracts, so that each unit of capital pays off R_K units of consumption goods while each unit of cash pays off R_C units of consumption goods. This is exactly identical to the hypothetical precautionary-saving-motive model without idiosyncratic investment opportunities.

We show that the precautionary-saving-motive model is constraint efficient, a reminiscent of the first welfare theorem. To prove this result formally, denote the value functions in the complete market equilibrium by $v_{cm}(c)$ and $q_{cm}(c)$. The HJB equation for value functions become

$$0 = \underbrace{\frac{\sigma^2}{2} q_{cm}''(c)}_{\text{volatility of } dc_t} + \underbrace{\xi\left(R_C - q_{cm}(c)\right)}_{\text{final date realization}} \tag{C.55}$$

$$0 = \underbrace{\frac{\sigma^2}{2} v_{cm}''(c)}_{\text{volatility of } dc_t} + \underbrace{q_{cm}'(c)\,\sigma^2}_{\text{expected value of dividends}} + \underbrace{\xi\left(R_K - v_{cm}(c)\right)}_{\text{final date realization}} \tag{C.56}$$

In contrast, in our model the valuation equations for q and v given idiosyncratic investment opportunities are

$$0 = \underbrace{\frac{\sigma^2}{2} q''_{cm}(c)}_{\text{volatility of } dc_t} + \underbrace{\frac{\xi}{2}\left(R_C - q_{cm}(c)\right)}_{\text{becoming a cash firm}} + \underbrace{\frac{\xi}{2}\left(\frac{R_K}{c} - q_{cm}(c)\right)}_{\text{becoming a capital firm}} \quad \text{(C.57)}$$

$$0 = \underbrace{\frac{\sigma^2}{2} v''_{cm}(c)}_{\text{volatility of } dc_t} + \underbrace{q'_{cm}(c)\,\sigma^2}_{\text{expected value of dividends}} + \underbrace{\frac{\xi}{2}\left(R_C c - v_{cm}(c)\right)}_{\text{becoming a cash firm}} + \underbrace{\frac{\xi}{2}\left(R_K - v_{cm}(c)\right)}_{\text{becoming a capital firm}} \quad \text{(C.58)}$$

Take q equation as example. In (C.55), the term "final date realization" captures that with intensity ξ, the firm can use its cash holding to obtain R_C units of consumption goods. While in (C.57), this term has two components: if becoming a cash firm with intensity $\frac{\xi}{2}$, then it obtains R_C; while if becoming a capital firm with intensity $\frac{\xi}{2}$, it obtains $\frac{R_K}{\widehat{p}} = \frac{R_K}{c}$ by purchasing capital (which generates R_K) at the price of $\widehat{p} = c$.

Comparing (C.55)-(C.56) to the planner's solution in Section 3.1.1, we see that

$$j_P(c) = v_{cm}(c) + c q_{cm}(c).$$

Denote the new endogenous (dis)investment boundaries by c_h^{cm} and c_l^{cm}. Because we knew that the constrained efficient solution features $c_l^{cm} = 0$, we need to be careful in the associated boundary conditions at the lower bound boundary. At upper boundary, we have

$$\frac{v\left(c_h^{cm}\right)}{q\left(c_h^{cm}\right)} = h,\ v'\left(c_h^{cm}\right) = 0, \text{ and } q'\left(c_h^{cm}\right) = 0. \quad \text{(C.59)}$$

At the lower boundary, taking into account of possibility that $c_l^* = 0$ binds at zero, we have the complementary-slack condition:

$$\frac{v\left(c_l^{cm}\right)}{q\left(c_l^{cm}\right)} = l,\ v'\left(c_l^{cm}\right) = 0, \text{ and } q'\left(c_l^{cm}\right) \leq 0 \text{ with strict inequality if } c_l^{cm} = 0. \quad \text{(C.60)}$$

The first condition $\frac{v(c_l^{**})}{q(c_l^{**})} = l$ have to hold because in equilibrium only a fraction of firms are liquidating their capital at c_l^{cm}, who must be indifferent between selling or liquidating their capital. The intuition behind $v'\left(c_l^{cm}\right) = 0$ and $q'\left(c_l^{cm} = 0\right) < 0$ is as follows. Note that c in the functions $v(\cdot)$ and $q(\cdot)$ capture the aggregate cash liquidity. Loosely speaking, if the aggregate liquidity is strictly negative say $-\varepsilon$, then the value of cash is higher (relative to $q(c = 0)$) because cash can be used to reduce the amount of capital that needs to be liquidated. In contrast, given $c = -\varepsilon$ or 0, the optimal policy with regard to existing capital is unchanged (think about those capitals that end up not to be liquidated), which explains $v'\left(c_l^{cm} = 0\right) = 0$.

The following proposition formally shows that $c_l^{cm} = 0$ and $c_h^{cm} = c_h^P$, which coincide with the planner's solution.

Proposition C.5 *In the complete market economy, there is an equilibrium for any set of parameters where*

1. *firms do not consume before the aggregate stage,*
2. *each firm in each state $c \in [0, c_h^{cm}]$ is indifferent in the composition of her portfolio,*
3. *each firm holding capital use every positive cash shock to build capital if and only if $c = c_h^{cm}$ and finance the negative cash shocks by liquidating the capital if and only if $c = 0$,*
4. *the value of holding a unit of cash and the value of holding a unit of capital are described by*

$$q_{cm}(c) = R_C + e^{-c\gamma} L_1 + e^{c\gamma} L_2, \quad \text{(C.61)}$$

$$v_{cm}(c) = R_K + e^{-c\gamma}\left(D_1^P - c L_1\right) + e^{c\gamma}\left(D_2^P - c L_2\right). \quad \text{(C.62)}$$

where constants L_1, L_2, D_1^P, D_2^P and c_h^{cm} are determined by boundary conditions from (C.59) and (C.60):

$$\frac{v_{cm}\left(c_h^{cm}\right)}{q_{cm}\left(c_h^{cm}\right)} = h,\ \frac{v_{cm}\left(0\right)}{q_{cm}\left(0\right)} = l,\ v_{cm}'\left(c_h^{cm}\right) = q_{cm}'\left(c_h^{cm}\right) = v_{cm}'\left(0\right) = 0. \tag{C.63}$$

In this equilibrium, $v_{cm}(c)$ is increasing in c, $q_{cm}(c)$ is decreasing in c. Hence $p_{cm}(c) \equiv \frac{v_{cm}(c)}{q_{cm}(c)}$ is increasing in c, implying the optimality of $c_l^{cm} = 0$,

5. *finally, we have $c_h^{cm} = c_h^P$, $D_1^P = D_1$ and $D_2^P = D_2$ given in the planner's solution, so that $j_P(c) = v_{cm}(c) + cq_{cm}(c)$ for all $c \in \left[0, c_h^P\right]$.*

Proof. *Given (C.61)-(C.62), the boundary conditions in (C.63) are*

$$\begin{aligned}
\frac{R_K + R_C c_h^{cm} + e^{c_h^{cm}\gamma} D_2^P + e^{-c_h^{cm}\gamma} D_1^P}{R_C + e^{-c_h^{cm}\gamma} L_1 + e^{c_h^{cm}\gamma} L_2} &= h + c_h^{cm}; && \text{(C.64)}\\
\frac{R_K + D_2^P + D_1^P}{R_C + L_1 + L_2} &= l; && \text{(C.65)}\\
\gamma e^{c_h^{cm}\gamma}\left(D_2^P - c_h^{cm} L_2\right) - L_2 e^{c_h^{cm}\gamma} - \gamma e^{-c_h^{cm}\gamma}\left(D_1^P - c_h^{cm} L_1\right) - L_1 e^{-c_h^{cm}\gamma} &= 0; && \text{(C.66)}\\
-\gamma e^{-c_h^{cm}\gamma} L_1 + \gamma e^{c_h^{cm}\gamma} L_2 &= 0; && \text{(C.67)}\\
\gamma D_2^P - L_2 - \gamma D_1^P - L_1 &= 0. && \text{(C.68)}
\end{aligned}$$

Adding c_h^{cm} times (C.67) to (C.66) gives

$$e^{c_h^{cm}\gamma}\left(\gamma D_2^P - L_2\right) - e^{-c_h^{cm}\gamma}\left(\gamma D_1^P + L_1\right) = 0.$$

Together with (C.68), this implies

$$\gamma D_2^P = L_2,\ \text{and}\ -L_1 = \gamma D_1^P. \tag{C.69}$$

Substituting this into (C.67) gives

$$e^{-c_h^{cm}\gamma} D_1^P + e^{c_h^{cm}\gamma} D_2^P = 0. \tag{C.70}$$

Also, as $L_1 + L_2 = \gamma L_3 - \gamma L_4$, (C.65) implies that

$$R_K + D_2^P + D_1^P = l\left(R_C + \gamma D_2^P - \gamma D_1^P\right) \tag{C.71}$$

and by (C.69), (C.64) is equivalent to

$$R_K + R_C c_h^{cm} + e^{c_h^{cm}\gamma} D_2^P + e^{-c_h^{cm}\gamma} D_1^P = \left(h + c_h^{cm}\right)\left(R_C - \gamma D_1^P e^{-c_h^{cm}\gamma} + \gamma D_2^P e^{c_h^{cm}\gamma}\right). \tag{C.72}$$

Then, we observe that the system (C.70)-(C.72) is equivalent with the following system for the planner's problem with $D_2^P = D_2$, $D_1^P = D_1$ and $c_h^{cm} = c_h^P$:

$$\begin{aligned}
R_K + D_1 + D_2 &= l\left(R_C - \gamma D_1 + \gamma D_2\right),\\
R_K + R_C c_h^P + D_1 e^{-\gamma c_h^P} + D_2 e^{\gamma c_h^P} &= \left(h + c_h^P\right)\left(R_C - \gamma D_1 e^{-\gamma c_h^P} + \gamma D_2 e^{\gamma c_h^P}\right),\\
D_1 e^{-\gamma c_h^P} + D_2 e^{\gamma c_h^P} &= 0.
\end{aligned}$$

with $D_1 = -\frac{(R_K - lR_C)e^{2\gamma c_h^P}}{(1+l\gamma)e^{2\gamma c_h^P} - (1-l\gamma)}$, $D_2 = \frac{R_K - lR_C}{(1+l\gamma)e^{2\gamma c_h^P} - (1-l\gamma)}$.

Finally, to show that price is monotonically increasing in this economy we show that $v_{cm}'(c) > 0$ and

$q'_{cm}(c) < 0$ *for every* $c \in (0, c_h^{cm})$. *It is easy to check that*

$$\begin{aligned} q'_{cm}(c) &= -\gamma e^{-c\gamma} L_1 + \gamma e^{c\gamma} L_2 = \gamma^2 e^{-c\gamma} D_1 + \gamma^2 e^{c\gamma} D_2 = \\ &= \gamma^2 (R_K - l R_C) e^{c\gamma} \frac{1 - e^{2\gamma\left(c_h^P - c\right)}}{e^{2\gamma c_h^P} + l\gamma\left(e^{2\gamma c_h^P} - 1\right) + 1} < 0. \end{aligned}$$

This also verifies the complementarity-slackness condition in (C.60). And,

$$\begin{aligned} v'_{cm}(c) &= \gamma e^{c\gamma} (D_2 - c L_2) - L_2 e^{c\gamma} - \gamma e^{-c\gamma} (D_1 - c L_1) - L_1 e^{-c\gamma} \\ &= -c\gamma^2 D_2 e^{c\gamma} - c\gamma^2 D_1 e^{-c\gamma} = c\gamma^2 (R_K - l R_C) e^{c\gamma} \frac{e^{2\gamma\left(c_h^P - c\right)} - 1}{e^{2\gamma c_h^P} + l\gamma\left(e^{2\gamma c_h^P} - 1\right) + 1} \geq 0. \end{aligned}$$

Q.E.D. ■

C.2 The role of Cobb-Douglas technology in Section 5.1

In Section 5.1 we introduce a Cobb-Douglas technology which combines cash and capital to produce some final goods; and recall in the base model this technology is not needed. This note explains why we introducing this technology.

It is a tradition in the pecuniary externality literature to think about distortionary tax scheme as small transfer which results in some first-order incentive/welfare implications. But, because the linear production technology used in the main model may well push individual firms to take some cornered solution (as we find in our numerical solution), firms are insensitive to marginal tax transfers. In other words, the linear technology implies a cornered solution on the lower side, and as a result distortionary tax schemes which only affect incentives slightly have a hard time to induce some real effect.

We introduce the Cobb-Douglas technology $\phi K^\alpha C^{1-\alpha}$ in the aggregate stage in order to break the corner solution for the disinvestment threshold. Cobb-Douglas technology naturally implies a higher marginal value of cash when the aggregate cash level is lower. This can be seen here:

$$0 = q'(c)\sigma^2 + v'(c)\frac{\xi}{2}\left(-p(c) + \frac{c^2}{p(c)}\right) + \frac{\sigma^2}{2}v''(c) + \xi\left(\frac{R_C p(c) + R_K}{2} - v(c)\right) + \underbrace{\phi\alpha c^{1-\alpha}}_{\text{extra mv of capital}} \tag{C.73}$$

$$0 = q'(c)\frac{\xi}{2}\left(-p(c) + \frac{c^2}{p(c)}\right) + \frac{\sigma^2}{2}q''(c) + \xi\left(\frac{1}{2}\left(R_C + \frac{R_K}{p(c)}\right) - q(c)\right) + \underbrace{\phi(1-\alpha)c^{-\alpha}}_{\text{ex mv of cash}} \tag{C.74}$$

relative to (12), there is an extra flow $\phi(1-\alpha)c^{-\alpha}$ in the cash value equation which increases when c drops. In fact, because it features the Inada condition so that the marginal value of cash goes to infinity when $C_t = 0$, it automatically guarantees that the equilibrium disinvestment threshold take an interior solution with a zero-order first condition (rather than be cornered at $c = 0$ with non-zero first-order condition).

As a result, with Cobb-Douglas technology, in the market solution firms are taking interior solutions on both the upper investment and lower disinvestment thresholds, both with zero first-order conditions. Because policy interventions are in the form of small distortionary tax schemes, this helps us illustrate the pecuniary externality with small policy interventions, an exercise that we are performing in Section 5.

C.3 The model with collateralized borrowing and proof of Proposition 9

As a preparation we first analyze the model with collateralized borrowing; we then give the proof of Proposition 9 in Section 5.2.

We highlight three parameters with superscript b which have special roles in characterizing the equilibrium. In the economy with collateralized borrowing, we use R_K^b to denote the productivity of capital; and, per unit of capital firms get l^b units of cash when liquidating while need to invest h^b units of cash when investing.

We conjecture that firms always max out their borrowing capacity (and verify later this holds in equilibrium). In the idiosyncratic stage after the heterogenous technology shocks hit, there are $K_\tau/2$ units of capital to be sold. On the cash side, in addition to $C_\tau/2$ units of the cash, collateralized borrowing implies that there are bK_τ units of extra cash in aggregate from external creditors. Hence the equilibrium capital price is $\widehat{p}_\tau = \frac{bK_\tau + C_\tau/2}{K_\tau/2} = 2b + c_\tau$. For capital firms being willing to purchase capital, we require

$$R_K^b > 2b + c_\tau, \tag{C.75}$$

which, as in the base model, can hold in equilibrium because c_τ will be bounded endogenously.

For assets that can be used as collateral, it is useful to note that $\widehat{p}_\tau - b$ is the "effective" capital price. This is because each unit of capital can be used to borrow b units of cash, and firms with their own cash of $\widehat{p}_\tau - b = b + c_\tau$ can buy a unit of capital. When $c_\tau = -b$ which is lower bound of the aggregate net cash holding, the effective price of capital drops to zero.

For capital firms with capital K_τ and cash C_τ, they will use their cash holding C_τ, together with credit $K_\tau b$, to purchase capital from the market at the effective capital price $b + c_\tau$. The final consumption goods, net of borrowing payment, is

$$\underbrace{K_\tau \left(R_K^b - b\right)}_{\text{own capital holdings}} + \underbrace{\frac{C_\tau + K_\tau b}{b + c_\tau}\left(R_K^b - b\right)}_{\text{purchasing capital from market}} = K_\tau \underbrace{\frac{(2b + c_\tau)\left(R_K^b - b\right)}{b + c_\tau}}_{\text{marginal payoff of capital}} + C_\tau \underbrace{\frac{R_K^b - b}{b + c_\tau}}_{\text{marginal payoff of cash}} \tag{C.76}$$

Note, under condition (C.75), it is optimal to exhaust the borrowing capacity (at a marginal cost of 1) to puchase the capital from the market (at a marginal benefit of $\frac{R_K^b - b}{b + c_\tau}$). For cash firms, their payoff is

$$\underbrace{C_\tau R_C}_{\text{net cash holdings}} + \underbrace{K_\tau (2b + c_\tau) R_C}_{\text{selling capital to the market}}. \tag{C.77}$$

Hence, the marginal payoff for capital is $K_\tau (2b + c_\tau)$, while for cash it is R_C.

Now we move on to aggregate stage, and denote the value of capital and cash by $v^b(c)$ and $q^b(c)$, respectively. The similar structure for the market equilibrium as in the base model prevails, i.e. firms build (dismantle) capital when the aggregate cash-to-capital ratio c_t reaches an endogenous upper (lower) threshold c_h^{*b} (c_l^{*b}). In the inaction region $c^b \in \left(c_l^{b*}, c_h^{*b}\right)$, we have the same evolution of state variable $dc_t = \sigma dZ_t$, and the values of capital and cash satisfy

$$\begin{aligned}
0 &= q^{b\prime}(c)\,\sigma^2 + \frac{\sigma^2}{2} v^{b\prime\prime}(c) - \xi v^b(c) + \frac{\xi}{2}\left[\frac{(2b + c)\left(R_K^b - b\right)}{b + c} + R_C (c + 2b)\right], \\
0 &= \frac{\sigma^2}{2} q^{b\prime\prime}(c) - \xi q^b(c) + \frac{\xi}{2}\left[R_C + \frac{R_K^b - b}{b + c}\right].
\end{aligned}$$

Here, we have used the marginal payoffs of capital and cash for either capital or cash firms given in (C.76) and (C.77). The boundary conditions are the same as the base model:

$$q^{b\prime}\left(c_l^{*b}\right) = v^{b\prime}\left(c_l^{*b}\right) = 0,\ v^{b\prime}\left(c_l^{*b}\right) = l^b q^b\left(c_l^{*b}\right), \tag{C.78}$$

$$q^{b\prime}\left(c_h^{*b}\right) = v^{b\prime}\left(c_h^{*b}\right) = 0,\ v^{b}\left(c_h^{*b}\right) = h^b q^b\left(c_h^{*b}\right). \tag{C.79}$$

We now show that there is a simple relationship between the economy with and without borrowing both in the planner's case and in the decentralized case. define three "translated" parameters as

$$R_K = R_K^b - b,\ h = h^b - b, \text{ and } l = l^b - b, \tag{C.80}$$

and consider the no-borrowing economy with the above three parameters. For the market equilibrium, denote the resulting capital and cash value functions $v(c)$ and $q(c)$ respectively, with equilibrium thresholds (c_l^*, c_h^*). Analogously, denote $j(\cdot)$ and c_h^P as the social planner's solution. We have the following proposition.

Proposition C.6 *Consider the economy with borrowing. For the social planner, we have $j^b(c) = j(c+b)$, $c_l^{P,b} = -b$ and $c_h^{P,b} = c_h^P - b$. For the market equilibrium the capital and cash value functions are given by*

$$v^b(c) = v(c+b) + bq(c+b), \text{ and } q^b(c) = q(c+b). \tag{C.81}$$

Hence the capital price is $p^b(c) = p(c+b) + b$, and the investment and disinvestment thresholds are given by $c_l^{b*} = c_l^* - b$ and $c_h^{b*} = c_h^* - b$.

Proof. *Recall that in the base model without borrowing, if $R = R^b - b$, our value functions satisfy*

$$0 = \frac{\sigma^2}{2} q''(c) + \frac{\xi}{2}(R_C - q(c)) + \frac{\xi}{2}\left(\frac{R_K^b - b}{c} - q(c)\right), \tag{C.82}$$

$$0 = q'(c)\sigma^2 + \frac{\sigma^2}{2} v''(c) + \frac{\xi}{2}(R_c c - v(c)) + \frac{\xi}{2}\left(R^b - b - v(c)\right). \tag{C.83}$$

We only show q^b. If $q^b(c) = q(c+b)$, we have

$$0 = \frac{\sigma^2}{2} q^{b\prime\prime}(c) + \frac{\xi}{2}\left(R_C - q^b(c)\right) + \frac{\xi}{2}\left(\frac{R_K^b - b}{c+b} - q^b(c)\right)$$

$$\Leftrightarrow \quad 0 = \frac{\sigma^2}{2} q''(c+b) + \frac{\xi}{2}(R_C - q(c+b)) + \frac{\xi}{2}\left(\frac{R_K^b - b}{c+b} - q(c+b)\right)$$

which holds always as we can view $c+b$ as c in (C.82). Similarly we can show the result for $v^b(c)$. The investment and disinvestment thresholds and the social planner's solution are obvious given this result. ■

C.3.1 Proof of Proposition 9

For simplicity we set $\gamma = 1$. Our results rely on two lemmas. The first lemma gives the market solution.

Lemma B.2 *For any $a > 1$, $c_l = x, c_h = ax$ is an equilibrium in the limit $x \to 0$, if*

$$\begin{aligned} \frac{3(a-1) - (a+1)\ln a}{\ln a} x &= l, \\ \frac{3(a-1) - (a+1)\ln(a)}{\ln(a)} x + o(x) &= h. \end{aligned}$$

Proof. *One can show $\lim_{x\to 0} x f_l(x, ax) = \frac{\ln(a)}{4(a-1)}, \lim_{\varepsilon \to 0} g_l(x, ax) = 1 - \frac{a\ln(a)}{a-1}$. Thus, we have*

$$\lim_{x\to 0} \frac{p_l(x; x, ax)}{x} = \lim_{x\to 0} \frac{R_K + \frac{xR_C}{2} + \frac{R_C}{2} m(x, ax) + \frac{R_K}{2}(g_l(x, ax) - x f_l(x, ax))}{x\frac{R_C}{2} + R_K x f_l(x, ax)}$$

$$= \lim_{x\to 0} \frac{R_K + \frac{R_K}{2}\left(1 - \frac{a\ln(a)}{a-1} - \frac{\ln(a^4)}{4(a-1)}\right)}{x\frac{R_C}{2} + R_K \frac{\ln(a^4)}{4(a-1)}} = \frac{3(a-1) - (a+1)\ln a}{\ln a}.$$

Hence if $l = \frac{3(a-1)-(a+1)\ln a}{\ln a} x$ *then* $x = L(ax)$ *in the limit. Similarly*

$$\lim_{x\to 0} axf_h(x,ax) = \frac{a\ln(a)}{\gamma(a-1)}, \lim_{x\to 0} g_h(x,ax) = 1 - \frac{\ln(a)}{a-1}.$$

Thus, for

$$\begin{aligned}
\lim_{x\to 0}\frac{p_h(x;x,ax)}{x} &= \lim_{x\to 0}\frac{R_K + \frac{axR_C}{2} + \frac{R_C}{2}m(x,ax) + \frac{R_K}{2}(g_h(x,ax) - axf_h(x,ax))}{x\frac{R_C}{2} + R_K x f_h(x,ax)} = \\
&= \lim_{x\to 0}\frac{R_K + \frac{axR_C}{2} + \frac{R_C}{2}m(x,ax) + \frac{R_K}{2}\left(1 - \frac{\ln(a)}{a-1} - \frac{a\ln(a)}{a-1}\right)}{x\frac{R_C}{2} + R_K x f_h(x,ax)} \\
&= \frac{1 + \frac{1}{2}(g_h(x,ax) - axf_h(x,ax))}{\frac{1}{2}xf_h(x,ax)} = \frac{3(a-1) - (a+1)\ln(a)}{\ln(a)}
\end{aligned}$$

Hence if

$$h = \frac{3(a-1) - (a+1)\ln(a)}{\ln(a)}x + o(x)$$

Then $x = H(ax)$ *in the limit. QED.* ■

Because $l(k) = O(\varepsilon^k)$, $h(k) = O(\varepsilon^k) + \varepsilon$ and $k < 1$, $\varepsilon = o(O(\varepsilon^k))$, which implies the above lemma applies. Hence the market solution has

$$c_l^* = O(\varepsilon^k), \; c_h^* = aO(\varepsilon^k)$$

where $a > 1$ is the solution to $3(a-1) = a\ln a$. The next lemma gives the planner's solution.

Lemma B.3 *For* x *being sufficiently small, suppose that*

$$l = x, \text{ and } h = x + O\left(x^{3(1+\alpha)}\right)$$

where the constant α *can be either positive or negative. Then the social planner's solution satisfies*

$$c_h^P \propto \begin{cases} x^{1+\alpha} & \text{if } \alpha < 0 \\ x & \text{if } \alpha = 0 \\ x^{1+1.5\alpha} & \text{if } \alpha > 0 \end{cases}.$$

Proof. *Without loss of generality we fix* $\gamma = 1$. *The planner's solution* c_h^P *satisfies*

$$\left(R_K - \left(x + O\left(x^{3(1+\alpha)}\right)\right)R_C\right)\left[e^{c_h^P}(1+x) - (1-x)e^{-c_h^P}\right] = 2(R_K - xR_C)\left(c_h^P + x + O\left(x^{3(1+\alpha)}\right)\right) \tag{C.84}$$

It is easy to show $c_h^P \to 0$ as $x \to 0$. Then Taylor expansion implies that

$$\begin{aligned}
& e^{c_h^P}(1+x)-(1-x)e^{-c_h^P} \\
= & \left(1+c_h^P+\frac{1}{2}\left(c_h^P\right)^2+\frac{1}{6}\left(c_h^P\right)^3+o\left(\left(c_h^P\right)^3\right)\right)(1+x) \\
& -(1-x)\left(1-c_h^P+\frac{1}{2}\left(c_h^P\right)^2-\frac{1}{6}\left(c_h^P\right)^3+o\left(\left(c_h^P\right)^3\right)\right) \\
= & 2x+2c_h^P+x\left(c_h^P\right)^2+\frac{1}{3}\left(\left(c_h^P\right)^3\right)+o\left(\left(c_h^P\right)^3\right)
\end{aligned}$$

hence the RHS of (C.84) is

$$\begin{aligned}
& \left(R_K-\left(x+O\left(x^{3(1+\alpha)}\right)\right)R_C\right)\left[2x+2c_h^P+x\left(c_h^P\right)^2+\frac{1}{3}\left(\left(c_h^P\right)^3\right)+o\left(\left(c_h^P\right)^3\right)\right] \\
= & 2R_Kx+2R_Kc_h^P+R_Kx\left(c_h^P\right)^2+\frac{R_K}{3}\left(c_h^P\right)^3-2\gamma R_C\left(x+O\left(x^{3(1+\alpha)}\right)\right)x-2R_Cc_h^P\left(x+O\left(x^{3(1+\alpha)}\right)\right) \\
& -R_C\left(x+O\left(x^{3(1+\alpha)}\right)\right)x\left(c_h^P\right)^2-\frac{R_C}{3}\left(c_h^P\right)^3\left(x+O\left(x^{3(1+\alpha)}\right)\right)+o\left(\left(c_h^P\right)^3\right);
\end{aligned}$$

the LHS of (C.84) is

$$2Rc_h^P+2R_Kx+2R_KO\left(x^{3(1+\alpha)}\right)-2R_Cxc_h^P-2R_Cx\left(x+O\left(x^{3(1+\alpha)}\right)\right);$$

and (C.84) therefore is

$$\begin{aligned}
& R_Kx\left(c_h^P\right)^2+\frac{R_K}{3}\left(c_h^P\right)^3-2R_Cc_h^PO\left(x^{3(1+\alpha)}\right)-R_C\left(x+O\left(x^{3(1+\alpha)}\right)\right)x\left(c_h^P\right)^2 \\
= & \frac{R_C}{3}\left(c_h^P\right)^3\left(x+O\left(x^{3(1+\alpha)}\right)\right)+o\left(\left(c_h^P\right)^3\right)+2R_KO\left(x^{3(1+\alpha)}\right).
\end{aligned}$$

Now, we write the above equation as

$$\begin{aligned}
& \underbrace{R_Kx\left(c_h^P\right)^2}_{(a)}+\underbrace{\frac{R_K}{3}\left(c_h^P\right)^3}_{(b)}-\underbrace{2R_KO\left(x^{3(1+\alpha)}\right)}_{(c)} \\
= & \underbrace{-x\left(c_h^P\right)^2\left[-R_C\left(x+O\left(x^{3(1+\alpha)}\right)\right)-\frac{R_C}{3}c_h^P\right]}_{(1)}+\underbrace{c_h^PO\left(x^{3(1+\alpha)}\right)R_C\left[2+\frac{1}{3}\left(c_h^P\right)^2\right]}_{(2)}-\underbrace{o\left(\left(c_h^P\right)^3\right)}_{(3)}
\end{aligned}$$

Here, term (1) on LHS is dominated by (a) on RHS, term (2) on LHS is dominated by (c) on RHS, and the term (3) on LHS is dominated by (b) on RHS. As a result, it must be that c_h^P is determined by LHS=0 when x is sufficiently small. We have the following three cases to consider.

1. *If $\alpha > 0$, we conjecture that term (b) is at a higher order so it is negligible. Thus c_h^P is determined by $x\left(c_h^P\right)^2 R_K - 2R_KO\left(x^{3(1+\alpha)}\right) = 0$, or $c_h^P = \sqrt{2}\left(O\left(x^{3(1+\alpha)}\right)/x\right)^{\frac{1}{2}} = \sqrt{2}O\left(x^{1+1.5\alpha}\right)$. This also implies that $\left(c_h^P\right)^3 = O\left(x^{3+4.5\alpha}\right)$ which is indeed at a higher order than term (c).*

2. *If $\alpha = 0$, then we LHS=0 implies that*

$$2R_KO\left(\varepsilon^3\right)-\frac{R_K}{3}\left(c_h^P\right)^3=R_Kx\left(c_h^P\right)^2$$

which implies that $c_h^P = O(x)$.

3. *If* $\alpha < 0$, *we conjecture that term (a) is at a higher order so it is negligible. Then* c_h^P *is determined by*

$$\frac{R_K}{3}\left(\left(c_h^P\right)^3\right) - 2R_K O\left(x^{3(1+\alpha)}\right) = 0$$

which implies that $c_h^P = \sqrt[3]{6}\,(x)^{1+\alpha}$.

■

Recall that $l(k) = O\left(\varepsilon^k\right)$, $h(k) = O\left(\varepsilon^k\right) + \varepsilon$; applying the above lemma, we know that

$$c_h^P(k) \propto \begin{cases} O\left(\varepsilon^{1/3}\right) & \text{if } k < 1/3 \\ O\left(\varepsilon^k\right) & \text{if } k = 1/3 \\ O\left(\varepsilon^{(1-k)/2}\right) & \text{if } k > 1/3 \end{cases}$$

Because $c_l^*(k) = O\left(\varepsilon^k\right)$ and $c_h^*(k) = aO\left(\varepsilon^k\right)$, for $k > 1/3$ we have $c_h^P(k) > c_h^*(k)$, i.e. overinvestment in booms.

C.4 An alternative equilibrium

In the main text we showed that an equilibrium exist when $h - l$ is sufficiently small; it is possible that the type of equilibrium presented in the main text does not exist. In this subsection we provide some insights on the type of equilibrium that arises instead.

While the system (12)-(13), (7)-(9) always have a solution, for some parameters this solution implies that for a c sufficiently close to c_l^*, the price is below the threshold l. This obviously cannot be an equilibrium—firms would dismantle whenever the price drops below the liquidation benefit l. For that set of parameters we can construct the equilibrium as follows. There exists a $c_x \in (c_l^*, c_h^*)$, so that for every $c \in [c_l^*, c_x]$ we have $p(c) = \frac{v(c)}{q(c)} = l$, and an endogenous fraction of capital are dismantled at every instant. That is, in this range the price is constant in c and firms dismantle an increasing fraction of their capital as c drops further from c_x. The following proposition describes this equilibrium.

Proposition C.7 *Suppose that there is a* $c_h^* < R_K$, $c_x \in (l, c_h^*)$, q_0, A_1, A_2, A_3, A_4 *solving (12)-(13), (31)*

$$\frac{\xi}{2\sigma^2}\left(R_C + \frac{R_K}{c_x}\right)(l - c_x) = q'(c_x)$$

$$l\frac{\xi}{2\sigma^2}\left(R_C + \frac{R_K}{c_x}\right)(l - c_x) = v'(c_x)$$

$$\frac{v(c_x)}{q(c_x)} = l, \frac{v(c_h^*)}{q(c_h^*)} = h, v'(c_h^*) = q'(c_h^*) = 0.$$

Then there exists a market equilibrium with partial liquidation where

1. *firms do not consume before the final date,*
2. *each firm in each state* $c \in [l, c_h^*]$ *is indifferent in the composition of its asset holdings,*
3. *firms do not build or dismantle capital when* $c \in (c_x, c_h^*)$ *and, in aggregate, firms spend every positive cash shock to build capital iff* $c = c_h^*$ *and cover the negative cash shocks by liquidating a fraction of capital iff* $c \in [l, c_x]$. *When* $c = l$, *firms finance every negative cash shock by liquidating capital.*

4. *the value of cash and the value of capital are given by* $q(c)$ *and* $v(c)$ *are the same as in the base model for* $c \in (c_x, c_h^*)$*; for* $c \in (l, c_x)$ *they are*

$$q(c) = q_0 + \frac{\xi}{2\sigma^2}\left[(R_C l - R_K)(c-l) - \frac{R_C}{2}\left(c^2 - l^2\right) + lR_K(\ln c - \ln l)\right], v(c) = lq(c)$$

and the price in the aggregate stage is $p = l$ *when* $c \in [l, c_x]$.

5. *In the idiosyncratic stage, each cash firm sells all its capital to the cash firms who are not hit by the shock of the price* $\hat{p}_\tau = c$.

Proof. Under the conditions of the Proposition, firms start to disinvest whenever $p(c) = l$. Given the liquidation rate $y(c)\,dt = -dK/K$, then its impact on the aggregate cash-to-capital ratio c is

$$x(c)\,dt \equiv \frac{dC}{K} - \frac{C}{K}\frac{dK}{K} = -\frac{ldK}{K} - \frac{C}{K}\frac{dK}{K} = (l+c)\,y(c)\,dt,$$

so c evolves as $dc = x(c)\,dt + \sigma dZ_t$. We must have $v(c) = lq(c)$ as firms are always indifferent in liquidating the capital, and v and q satisfies:

$$\begin{aligned} 0 &= x(c)\,q'(c) + \frac{\sigma^2}{2}q''(c) + \frac{\xi}{2}\left(R_C + \frac{R_K}{c}\right) - \xi q(c) \\ 0 &= x(c)\,v'(c) + q'(c)\,\sigma^2 + \frac{\sigma^2}{2}v''(c) + \frac{\xi}{2}(R_C c + R_K) - \xi v(c) \end{aligned}$$

Using $v(c) = lq(c)$, we obtain

$$\begin{aligned} 0 &= x(c)\,lq'(c) + \frac{\sigma^2}{2}lq''(c) + \frac{\xi l}{2}\left(R_C + \frac{R_K}{c}\right) - \xi lq(c) \\ 0 &= x(c)\,lq'(c) + q'(c)\,\sigma^2 + \frac{\sigma^2}{2}lq''(c) + \frac{\xi}{2}(R_C c + R_K) - \xi lq(c) \end{aligned}$$

Eliminating identical terms, we get

$$q'(c) = \frac{\xi}{2\sigma^2}\left(R_C + \frac{R_K}{c}\right)(l-c) = 0.$$

As $q'(c_l) = 0$ has to hold, $c_l = l$. The closed-form solution is

$$q(c) = q_0 + \frac{\xi}{2\sigma^2}\left[(R_C l - R_K)(c-l) - \frac{R_C}{2}\left(c^2 - l^2\right) + lR_K(\ln c - \ln l)\right]$$

And, we have $q''(c) = -\frac{\xi}{2\sigma^2}\left(R_C + \frac{lR_K}{c^2}\right) < 0$. We know that of $c \in [l, c_x]$ we have $v(c) = lq(c)$ which allows us to back out the endogenous drift of c:

$$x(c) = \frac{-\frac{\sigma^2}{2}q''(c) - \frac{\xi}{2}\left(R_C + \frac{R_K}{c}\right) + \xi q(c)}{q'(c)},$$

and thus the endogenous liquidation rate $y(c) = \frac{x(c)}{l+c}$. For $c > c_x$ we have the ODE as usual. We then search of the c_x, c_h pair that satisfies the conditions of the proposition. ■

Plotting v, q and p give very similar graphs to Figure 2 with the main difference that at the range $c \in [l, c_x]$ the price is flat at the level l. In the same range $q(c)$ is decreasing implying that $v(c) = lq(c)$ is also decreasing.