Complementary Proofs to “Careerist Judges”

Proof of Lemma 6: (i) $q_n(n, t^*)$ increases with $t^*$:

$$
\text{sign} \frac{\partial q_n(n, t^*)}{\partial t^*} = \text{sign}(1 - q)q f(t^*)[-t^* \int_{t^*}^{1} (1 - v)f(v)dv + \int_{t^*}^{1} v f(v)dv]
$$

$$
= \text{sign} - t^* \int_{t^*}^{1} f(v)dv + t^* \int_{t^*}^{1} v f(v)dv + \int_{t^*}^{1} v f(v)dv - t^* \int_{t^*}^{1} v f(v)dv
$$

$$
= \text{sign}(1 - q)q f(t^*)[-t^* \int_{t^*}^{1} f(v)dv + \int_{t^*}^{1} v f(v)dv] > 0
$$

Similarly, I will show that $q_n(y, t^*)$ decreases with $t^*$:

$$
\text{sign} \frac{\partial q_n(s^* = y, t^*)}{\partial t^*} = \text{sign}(1 - t^*)(\int_{5}^{1} (1 - v)f(v)dv + \int_{5}^{t^*} v f(v)dv) - t^*(\int_{5}^{1} v f(v)dv + \int_{5}^{t^*} (1 - v)f(v)dv)
$$

$$
= \text{sign}(1 - q)q f(t^*)(\int_{5}^{t^*} (1 - 2t^*)f(v)dv + \int_{t^*}^{1} (1 - t^* - v)f(v)dv) < 0. \Box
$$

(ii) The proof of this is analogous to part (i). □

(iii) Follows from (i) and (ii). □

(iv) By the above claims, it is enough to show that $q_y(n, t^*) > q$, i.e., that:

$$
\frac{q(\int_{5}^{1} v f(v)dv + \int_{5}^{t^*} (1 - v)f(v)dv)}{q(\int_{5}^{1} v f(v)dv + \int_{5}^{t^*} (1 - v)f(v)dv) + (1 - q)(\int_{5}^{1} v f(v)dv + \int_{5}^{t^*} v f(v)dv)} > q
$$

which holds if

$$
\int_{5}^{1} v f(v)dv + \int_{5}^{t^*} (1 - v)f(v)dv > \int_{5}^{1} (1 - v)f(v)dv + \int_{5}^{t^*} v f(v)dv \iff \int_{5}^{1} v f(v)dv > \int_{t^*}^{1} (1 - v)f(v)dv \iff \int_{t^*}^{1} (2v - 1)f(v)dv > 0.
$$

The second part, i.e., that $q_n(s^*, t^*) > 1 - q$, has an analogous proof. □

(v) To show that $q_y(n, t^*) > \Pr(w = y|q, n, t^*)$, we need to show that:

$$
q_y(n, t^*) > \frac{q(1 - t^*)}{q(1 - t^*) + t^*(1 - q)}
$$

which, after re-arranging, is analogous to:

$$
t^*(\int_{5}^{1} v f(v)dv + \int_{5}^{t^*} (1 - v)f(v)dv) > (1 - t^*)(\int_{5}^{1} (1 - v)f(v)dv + \int_{5}^{t^*} v f(v)dv)
$$
where the last inequality is satisfied using the proof in part (iv). To show that \( q_y(y, t^*) > \Pr(w = y|q, y, t^*) \), we need to show that:

\[
q_y(y, t^*) > \frac{q t^*}{q t^* + (1 - t^*)(1 - q)} \iff \frac{\int_{t^*}^1 v f(v)dv}{\int_{t^*}^1 (1 - v)f(v)dv} > \frac{t^*}{1 - t^*}
\]

but

\[
\int_{t^*}^1 v f(v)dv > t^* \text{ and } \int_{t^*}^1 (1 - v)f(v)dv < 1 - t^*,
\]

hence the above is satisfied. We can use the analogous proof to show that \( q_n(s^*, t^*) > \Pr(w = n|q, s^*, t^*) \).

(vi) Note that \( q_y(y, \frac{1}{2}) > q_n(y, \frac{1}{2}) \), and by the above claims this holds for all \( t^* > \frac{1}{2} \) and \( s^* = y \). On the other hand, \( q_y(n, t^* \to 1) \to q \) and \( q_n(n, t^* \to 1) \to 1 \). Since \( q_n(n, t^*) \) increases with \( t^* \) and \( q_y(n, t^*) \) decreases with \( t^* \), there exists a unique \( \tilde{t}(q) \) such that \( q_y(n, \tilde{t}(q)) = q_n(n, \tilde{t}(q)) \) and for all \( t^* < (>) \tilde{t}(q) \), \( q_y(n, t^*) > (\leq) q_n(n, t^*) \). With the uniform distribution, i.e., \( f(v) = 2 \), then \( q_y(n, q) = \frac{2 - q}{2 - q} \geq \frac{1 + q}{1 + 2q} = q_n(n, q) \) for all \( q \geq .5 \) which implies that \( \tilde{t}(q) > q \).

(vii) To see this, recall that:

\[
\tilde{p}(y) = q_y^2 + (1 - q_y) \Pr(w = y|q, s, t),
\]

hence:

\[
\frac{\partial \tilde{p}(y)}{\partial t} |_{s^* = n} = (2q_y - \Pr(w = y|q, n, t)) \frac{\partial q_y}{\partial t} + (1 - q_y) \frac{\partial \Pr(w = y|q, n, t)}{\partial t}
\]

but when \( s^* = n \), \( \frac{\partial q_y}{\partial t} < 0 \). Also, \( \frac{\partial \Pr(w = y|q, n, t)}{\partial t} \) and \( 2q_y - \Pr(w = y|q, s, t) > 0 \) by the above. Similarly,

\[
\frac{\partial \tilde{p}(n)}{\partial t} = (2q_n - \Pr(w = n|q, n, t)) \frac{\partial q_n}{\partial t} + (1 - q_n) \frac{\partial \Pr(w = n|q, n, t)}{\partial t} > 0.
\]

An analogous analysis holds for the derivatives w.r.t. \( q \).

This completes the proof of Lemma 6.

**Proof of Proposition 1, Step 2 (Uniqueness).** I show a sufficient condition for uniqueness, i.e., that at the equilibrium value of \( t^* \), whenever \( \frac{\partial}{\partial t} \frac{q_y(n, t) - \theta \beta(n, t)}{q_n(n, t) + \theta \beta(n, t)} < 0 \), then:

\[
\left| \frac{\partial}{\partial t} \Pr(w = y|q, n, t) \right| > \left| \frac{\partial}{\partial t} \frac{q_y(n, t) - \theta \beta(n, t)}{q_n(n, t) + \theta \beta(n, t)} \right|
\]

Consider first \( \frac{\Pr(w = y|q, n, t)}{\Pr(w = n|q, n, t)} = \frac{q(1 - t)}{t(1 - q)} \). Then:

\[
\left| \frac{\partial}{\partial t} \Pr(w = y|q, n, t) \right| = \frac{q}{(1 - q)t^2}
\]
Now consider
\[
\frac{\partial}{\partial t} q_y(n, t) - \theta \beta(n, t) = \frac{1}{q_n + \theta \beta} \left( -1 + \theta \frac{\partial \beta}{\partial q_y} + \theta \frac{\partial \beta}{\partial q_y} q_y + \theta \beta \right)
\]
\[
= \frac{\partial q_y}{\partial n} \left( (1 + \theta \frac{\partial \beta}{\partial q_n} (q_y - \theta \beta) + \theta \frac{\partial \beta}{\partial q_n} q_n + \theta \beta) \right)
\]
but since
\[
\frac{\partial \beta}{\partial q_y} > 0, \quad \frac{\partial \beta}{\partial q_n} < 0, \quad \frac{\partial q_n}{\partial t} > 0 \text{ and } \frac{\partial q_y}{\partial t} < 0,
\]
it is enough to show that
\[
\frac{q}{(1 - q)t^2} > \frac{1}{q_n} \left[ -\frac{\partial q_y}{\partial t} + \frac{\partial q_n}{\partial t} q_y - \partial \beta \right]
\]
Plugging in the equilibrium condition and the expressions for the derivatives, the right-hand-side becomes:
\[
\frac{1}{q_n} \left[ \frac{2q_y(1 - q_y)}{(2 - t)t} + \frac{q}{(1 + t)} \frac{2q_n(1 - q_n)}{t(1 - q)} \right]
\]
Let \( q_x \in \{q_y, q_n \} \) such that \( q_x(1 - q_x) = \max \{q_y(1 - q_y), q_n(1 - q_n)\} \). It is therefore sufficient to prove that:
\[
\frac{q}{t} > \frac{2q_x(1 - q_x)}{q_n} \frac{(1 + t)(1 - q) + q(2 - t)}{(2 - t)(1 + t)}
\]
But the above equation holds both when \( q_x = q_n \) and when \( q_x = q_y \).

**Proof of Proposition 1, Step 4 (\( \frac{\partial \nu(e, \theta)}{\partial \nu} > 0 \)).**

By total differentiation of the equilibrium condition:
\[
\frac{\partial t}{\partial q} \bigg|_{t=\nu} = \frac{\partial}{\partial q} \frac{q_y(n, t) - \theta \beta(n, t)}{q_y(n, t) + \theta \beta(n, t)} - \frac{\partial}{\partial q} \frac{\Pr(w = y | q, n, t)}{\Pr(w = n | q, n, t)} \bigg|_{t=\nu}
\]
I show that when \( \Pr(w = y | q, n, t) = \frac{q_y(n, t) - \theta \beta(n, t)}{q_y(n, t) + \theta \beta(n, t)} \frac{\Pr(w = y | q, n, t)}{\Pr(w = n | q, n, t)} \bigg|_{t=\nu} \)
\[
\frac{\partial \Pr(w = y | q, n, t)}{\partial q} > \frac{\partial q_y(n, t) - \theta(n, t)}{\partial q} \frac{q_n(n, t) + \theta(n, t)}{\partial q}
\]
which, along with step 2 of the Proposition, implies that \( \frac{\partial t}{\partial q} \bigg|_{t=\nu} > 0 \). As in step 2, it is enough to show the inequality for \( \theta = 0 \), i.e., we have to show that:
\[
\frac{(1 - t)}{t(1 - q)^2} > \frac{1}{q_n} \frac{\partial q_y}{\partial q} - \frac{\partial q_n q(1 - t)}{\partial q} t(1 - q)
\]
\[
\frac{(1 - t)}{t(1 - q)^2} > \frac{1}{q_n} \frac{\partial q_y(1 - q)}{q(1 - q)} + \frac{q_n(1 - q_n)(1 - t)}{t(1 - q)^2}
\]
but since \( q_y > q_n \) in equilibrium, and for all \( q \) and \( t \), \( q_y > 1 - q_n \), it is sufficient to show that:

\[
\frac{(1-t)}{t(1-q)^2} > \frac{q_n(1-q_n)}{q_n} \frac{1}{q(1-q)} + \frac{(1-t)}{t(1-q)^2},
\]

\[
\frac{q(1-t)}{t(1-q) + q(1-t)} > \frac{q(1-t)}{(1-q)(1+t) + q(1-t)}
\]

which holds for all \( t, q \in [0.5, 1] \). This implies that \( \frac{\partial u}{\partial v} |_{t=v} > 0 \). 

**Proof of Proposition 2, Step 2** \((t^c(q) < \hat{t} \text{ for all } q)\):

Let \( s^c = n \) and \( t = t^c \). I will show that there is a unique \( \hat{t} < 1 \) satisfying \( \tau_{\hat{t}}(y, y, \sigma^c) = \tau_t(n, y, \sigma^c) \), and that for all \( t > \hat{t} \), \( \tau(y, y, \sigma^c) < \tau(n, y, \sigma^c) \). This implies that an equilibrium with \( t > \hat{t} \) cannot exist, since then the expected utility from ruling \( n \), an average over \( \tau(n, y, \sigma^c) \) and \( \tau(n, n, \sigma^c) \) where \( \tau(n, n, \sigma^c) > \tau(n, y, \sigma^c) \) is higher than the expected utility from ruling \( y \), an average over \( \tau(y, n, \sigma^c) \) and \( \tau(y, y, \sigma^c) \) where \( \tau(y, y, \sigma^c) > \tau(y, n, \sigma^c) \).

The expression for \( \tau(y, y, \sigma^c)|_{s^c=n} \) is

\[
\tau(y, y, \sigma^c)|_{s^c=n} = \frac{\int_{1/5}^{1/5} t^c f(t) dt + \int_{1/5}^{t^c} t(1-t) f(t) dt}{\int_{1/5}^{1/5} t f(t) dt + \int_{1/5}^{t^c} (1-t) f(t) dt}
\]

Taking the derivative of \( \tau(y, y, \sigma^c)|_{s^c=n} \) w.r.t \( t^c \), it is

\[
\frac{d\tau(y, y, \sigma^c)}{dt^c}|_{s^c=n} = \frac{(1-t^c) f(t^c)(t^c - \tau(y, y, \sigma^c))}{(\int_{1/5}^{1/5} t f(t) dt + \int_{1/5}^{t^c} (1-t) f(t) dt)^2}
\]

Therefore, \( \tau(y, y, \sigma^c) \) is a monotonously decreasing function as long as \( t^c < \tau(y, y, \sigma^c) \) and a monotonously increasing function when \( t^c > \tau(y, y, \sigma^c) \). When \( t^c \to 0.5 \), \( t^c < \tau(y, y, \sigma^c) \) and when \( t^c \to 1 \), \( \tau(y, y, \sigma^c) < t^c \). Therefore, there exists \( t' \) such that \( t' = \tau_t(y, y, \sigma^c) \). Moreover, \( t' \) is unique, since when \( t^c > \tau(y, y, \sigma^c) \), \( \frac{dr(y, y, \sigma^c)}{dt^c} < 1 \). Thus, for all \( t^c < (<) t' \), \( t^c < (<) \tau(y, y, \sigma^c) \) and \( \frac{dr(y, y, \sigma^c)}{dt^c} < (<) 0 \).

On the other hand, \( \tau(n, y, \sigma^c) \) is an average over \( t \) for \( t > t^c \) and thus increases with \( t^c \) for all \( t^c > 0.5 \). Also, since only values of \( t > t^c \) are included in the computation of \( \tau(n, y, \sigma^c) \), then \( \tau(n, y, \sigma^c) > t^c \) for all \( t^c \). By the above, when \( t^c \to 1 \), \( \tau(n, y, \sigma^c) > t^c > \tau(y, y, \sigma^c) \). When \( t^c = 0.5 \), by Lemma 3, \( \tau(y, y, \sigma^c) = \tau(n, n, \sigma^c) > \tau(n, y, \sigma^c) \). Then, there must exist some \( \hat{t} \in (0.5, 1) \) satisfying \( \tau_{\hat{t}}(y, y, \sigma^c) = \tau_{\bar{t}}(n, y, \sigma^c) \). Moreover, it must be that \( \hat{t} < t' \) because for all \( t^c > t' \), \( \tau(n, y, \sigma^c) > t^c > \tau(y, y, \sigma^c) \). Because \( \tau(y, y, \sigma^c) \) decreases monotonously for \( t^c < \hat{t} \) and \( \tau(n, y, \sigma^c) \) increases monotonously in \( t^c \); \( \hat{t} \) is unique.

**Proof of Proposition 4, Part (ii).** We have to show that \( t^f(q) < t^c(q) \). \( t^f(q) \) solves:
\[ \Pr(w = y|q, n, t^f(q)) \Gamma_y = \Pr(w = n|q, n, t^f(q)) \Gamma_n + \Gamma \]  

where \( \Gamma_y = \tau(y, y, \sigma^f) - \tau(y, n, \sigma^f) \), \( \Gamma_n = \tau(n, n, \sigma^f) - \tau(n, y, \sigma^f) \) and \( \Gamma = \tau(n, y, \sigma^f) - \tau(y, n, \sigma^f) \). I will show that at \( t^f(q) \),

\[ \tilde{p}(y) \Gamma_y > \tilde{p}(n) \Gamma_n + \Gamma \]  

for \( \tilde{p}(d) = (1 - q_d) \Pr(w = d|q, n, t^f(q)) + q_d^2 \), which implies that at \( t^f(q) \), the utility from \( y \) is higher than the utility from \( n \) if appeals are allowed, meaning that the equilibrium solution \( t^c(q) \) must admit \( t^c(q) > t^f(q) \).

Plugging (1) into (2), we have to show that:

\[
\begin{align*}
q_y(q_y - \Pr(w = y|q, n, t^f(q))) \Gamma_y &> q_n(q_n - \Pr(w = n|q, n, t^f(q))) \Gamma_n \\
\Gamma_y &> q_n(q_n - \Pr(w = n|q, n, t^f(q))) \\
\Gamma_n &> q_y(q_y - \Pr(w = y|q, n, t^f(q)))
\end{align*}
\]

However, I now show that for all values of \( t \), \( \frac{\Gamma_y}{\Gamma_n} > 1 \), whereas for all values of \( t \), \( \frac{q_n(q_n - \Pr(w = n|q, n, t^f(q)))}{q_y(q_y - \Pr(w = y|q, n, t^f(q)))} < 1 \), which implies the desired result.

To see that \( \frac{\Gamma_y}{\Gamma_n} > 1 \), I calculate the reputations for a cutoff point \( t \) from each action and state of the world:

\[
\Gamma_y = \tau(y, y, \sigma) - \tau(y, n, \sigma)
\]

\[
= \frac{\int_0^1 2v^2 dv + \int_0^t 2v(1-v) dv}{\int_0^1 2v dv + \int_0^t 2(1-v) dv + \int_0^t 2vdv} - \frac{\int_0^1 2v(1-v) dv + \int_0^t 2v^2 dv}{\int_0^1 2v dv + \int_0^t 2(1-v) dv + \int_0^t 2vdv} = \frac{1}{t^2} - \frac{1}{3}t^3 - \frac{1}{6}
\]

and similarly

\[
\Gamma_n = \tau(n, n, \sigma) - \tau(n, y, \sigma) = -\frac{t + t^2 + \frac{1}{2} - \frac{1}{3}t^3}{(1 + t)(1 - t)^2}
\]

and therefore

\[
\Gamma_y > \Gamma_n \iff \frac{1 - 4t^4 + 4t^3 - t^2 + 2t - 1}{6(t + 1)t^2(2 - t)} > 0
\]

which holds for all \( t > \frac{1}{2} \).

To see that \( \frac{q_n(q_n - \Pr(w = n|q, n, t^f(q)))}{q_y(q_y - \Pr(w = y|q, n, t^f(q)))} < 1 \) for all \( t \), I simplify the expression and find that this holds iff:

\[
\frac{(1 - q)(1 + t)(1 - t)}{t^2(1 - 2q)^2 + 1 + 2t(1 - 2q)} < \frac{q(2 - t)t}{t^2(1 - 2q)^2 + 4q^2 + 4qt(1 - 2q)}
\]

which holds for all \( t \geq \frac{1}{2} \) and \( q \geq \frac{1}{2} \). This completes the proof.\[\blacksquare\]