On the limits of communication in multidimensional cheap talk: A comment

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Abstract

We analyze a cheap talk game, a-la Crawford and Sobel (1982), in a multidimensional state and policy space. A feature of the multidimensional state space is that communication on one dimension often reveals information on others. We show how this feature imposes bounds on communication.

1. Introduction

In this note we analyze a cheap talk game in a multidimensional state and policy space. In this game, a sender who knows the state of the world sends a message to a receiver. Given the message and his prior beliefs about the state, the receiver chooses an action. Both the sender's preferred action and that of the receiver depend on the state of the world. The sender and the receiver have conflicting interests about which action is the appropriate one.

The analysis of cheap talk in a multidimensional environment introduces two effects. The first arises from the interaction of the multiple dimensions in the players' preferences. Even when on each dimension separately the players' conflict of interest is large, in the multidimensional environment there may be a dimension of

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compromise upon which both players agree. This may allow the sender to truthfully reveal information on this dimension (see Austen-Smith (1993), Battaglini (2002) and Chakraborty and Harbaugh (forthcoming)).

The second effect, overlooked in previous literature, involves the interaction between the different dimensions according to the prior. Generally, communication on one dimension may reveal information on others. Therefore, even when the two players have no conflict on a particular dimension, such informational spillovers may restrict their ability to communicate. We focus on this second effect and show how it limits communication in the multidimensional model.

In Section 2 we analyze a model in which a sender has lexicographic preferences. He first compares actions on the x-dimension, on which he and the receiver have a large conflict.[2] We show that there is an upper bound on the number of actions that the receiver takes in equilibrium, and hence on the level of communication. The upper bound does not depend on the sender's preferences over the y-dimension, but rather on how the state is distributed along the x and y dimensions according to the prior.

In Section 3, we extend the model to consider several senders, each receiving an imperfect signal about the state of the world. We identify necessary conditions on equilibrium strategies; fixing the preferences and the prior, we use these conditions to show that the fully revealing equilibrium proposed in Battaglini (2002) may not be robust to imperfect signals. Intuitively, as above, information spillovers impede communication when signals are imperfect.

[2] That is, if the receiver and the sender were to play a cheap-talk game restricted to the $x-$axis, there would be no information transmission in equilibrium.

In an Appendix (provided online) we extend our results to a general family of single-peaked preferences (similar to those of Crawford and Sobel (1982), but adopted to the multidimensional state space). We show that when the conflict between the sender and the receiver is large, these preferences have similar characteristics to those of the lexicographic preferences described above. We prove that communication is restricted in this environment as well. As for the lexicographic model, we identify the necessary conditions that the prior and the preferences should satisfy in order to sustain communication in equilibrium.

In a complementary paper, Chakraborty and Harbaugh (forthcoming, henceforth CH) explore the existence of equilibria with communication. In a symmetric model, with preferences and prior that are the same across the different dimensions, CH construct such equilibria for all levels of conflict. For a fixed level of conflict, they show that these equilibria are robust to small asymmetries of the preferences and the prior. In particular, the perturbed model may not satisfy the necessary conditions that we identify. Our results differ in the order of quantifiers; for fixed preferences and prior, we show that if the necessary conditions are violated, the CH equilibria do not exist for a large enough level of conflict.

Our result about the bounds on communication implies that the receiver can increase his utility if he can commit to take actions in a subset of the state space. Note that for large conflicts, this is not the case in the unidimensional model of Crawford and Sobel (1982). The result also allows us to illustrate that linking decisions in a multidimensional game may reduce the level of communication compared with separate unidimensional games, each restricted to one dimension. This is in contrast to both the papers by Battaglini (2002) and CH.

2. One sender

A receiver has to choose an action $a = (a_x, a_y)$ in $\mathbb{R}^2$. The appropriate choice of action depends on the realization of a state of the world $\theta \in \mathbb{R}^2$. The receiver initially holds a continuous and full support prior distribution F on $\mathbb{R}^2$, with density f and expectation at $\mu = (\mu_x, \mu_y)$.

A sender who is fully informed about θ (henceforth type θ), has lexicographic preferences over the receiver's actions. Let $U(a_y, \theta_y)$ represent the preferences of the sender on the y-dimension and assume that it satisfies the assumptions as in Crawford and Sobel (1982, henceforth CS). We assume that a sender of type θ prefers a to a' if $a_x > a'_x$ or if $a_x = a'_x$ and $U(a_y, \theta_y) > U(a'_y, \theta_y)$. Rather than specifying the receiver's preferences, we assume that his action is his expectation of θ, according to his posterior.

The sender observes the state θ and chooses a message in $M = \mathbb{R}^2$. Subsequently, the receiver takes his action. The strategy of the sender is a function $m : \mathbb{R}^2 \to \Delta(\mathbb{R}^2)$, and that of the receiver is his posterior expectation of θ, a function $a : \mathbb{R}^2 \to \mathbb{R}^2$. In a weak Perfect Bayesian equilibrium, $(a^*(.), m^*(.|\theta))$ satisfies:

(i) For any $\theta \in \mathbb{R}^2$, $\int_{\mathbb{R}^2} m^*(m|\theta)dm = 1$, and if $m^*(m|\theta) > 0$, m maximizes the utility of type θ over all $m' \in \mathbb{R}^2$ given $a^*(.)$,

(ii) For any $m \in \mathbb{R}^2$, if $m^*(m|\theta) > 0$ for some θ, $a^*(m) = \int_{\mathbb{R}^2} \theta \frac{m^*(m|\theta)f(\theta)}{\int_{\mathbb{R}^2} m^*(m|\theta')f(\theta')d\theta'} d\theta$.[3]

[3]Milgrom and Weber (1985) have shown that this formulation is equivalent to a distributional approach, in which the strategy of the sender is chosen ex-ante, before the sender learns θ. This

Given $(a^*(.), m^*(.|\theta))$, we say that a type θ *induces* a if there exists a message m such that $m^*(m|\theta) > 0$ and $a^*(m) = a$. An action is *induced* if there exists a type θ that induces it. Lemma 1 characterizes the equilibria of the model:

LEMMA 1 *For each induced action a' : (i) $a'_x = \mu_x$; (ii) the set of types who induce a' is connected and bordered by lines parallel to the x-axis.*

PROOF: If there is more than one induced action, they cannot differ on their x-coordinate (since all sender types would then prefer to induce the action with the highest x-coordinate). As the expectation over all actions must accord with the prior expectation, all induced actions must have an x-coordinate equal to μ_x. The separability of the preferences over the x and y dimensions and the CS assumptions imply that each set of indifferent types comprises a unique line parallel to the x-axis, from which (ii) follows.■

By Lemma 1, any communication between the sender and the receiver is restricted to the y-dimension. We proceed to characterize the bounds on such communication.

Throughout, we use the notation $\tilde{x}$ to denote a random variable with a generic realization x. Let $E[\tilde{\theta}_x|\theta_y] = \int_{\mathbb{R}} \theta_x \frac{f(\theta_x,\theta_y)}{\int_{\mathbb{R}} f(\theta_z,\theta_y)d\theta_z} d\theta_x$ denote the conditional expectation of $\tilde{\theta}_x$ as a function of θ_y (this is a continuous function). Let l^μ denote the line parallel to the $y-$axis that passes through the prior. We say that $E[\tilde{\theta}_x|\theta_y]$ *crosses* l^μ *at* θ_y^* if $E[\tilde{\theta}_x|\theta_y^*] = \mu_x$ and for any $\varepsilon > 0$, there exist θ'_y, θ''_y in an $\varepsilon-$neighbourhood of θ_y^*, satisfying $E[\tilde{\theta}_x|\theta'_y] > \mu_x$ and $E[\tilde{\theta}_x|\theta''_y] < \mu_x$.

DEFINITION 1 *For a finite integer k, F satisfies the k-crossing property with respect*

implies that the functions $m^*(m|\theta)$ and $\frac{m^*(m|\theta)f(\theta)}{\int_{\mathbb{R}^2} m^*(m|\theta)f(\theta)d\theta}$ are both measurable functions of m and θ and hence the integrals in (i) and (ii) are well defined.

to the x and y dimensions if $E[\tilde{\theta}_x|\theta_y]$ *crosses* l^{μ} *at exactly k distinct values.*

The definition above, along with Lemma 1, allows us to derive our first result:

PROPOSITION 1 *Suppose that F satisfies the k-crossing property with respect to the x and y dimensions. Then any equilibrium has at most k induced actions.*

PROOF: Let L be the set of lines which are parallel to the $x-$axis, such that the $x-$coordinate of the conditional expectation above each such line is μ_x. Denote an element of L by l. By Lemma 1, any equilibrium is characterized by a subset of lines in L. However, if F satisfies the $k-$crossing property, then $|L| = k-1$. To see this, note first that there cannot be two lines l and l' in L that are between two neighboring crossings. If there exist two such lines, then the x-coordinate of the conditional expectation over the subset of $\mathbb{R}^2$ bounded by these two lines is either larger or smaller than μ_x. However, by construction, the x-coordinate of the conditional expectation both above l and above l' is μ_x, implying that it has to be so also for the state space bounded between them, a contradiction. Similar arguments show that there cannot be any line in L whose $y-$coordinate is below the $y-$coordinate of the lowest crossing, or above that of the highest crossing, and that there cannot be more than one crossing between any two neighbouring lines in L.■

A corollary to Proposition 1 is that when F satisfies the one-crossing property (for example, when θ_x and θ_y are strictly affiliated according to F), the only equilibrium is the 'babbling' equilibrium, in which the set of induced actions is a singleton.

When F satisfies the $k-$crossing property for $k > 1$, Proposition 1 does not guarantee the existence of equilibria with more than one distinct action. The prior F and the sender's utility $U(a_y, \theta_y)$ would have to satisfy additional requirements to allow for the existence of such equilibria; such requirements are analogous to the equilibrium conditions in CS, and would enable communication on the $y-$dimension (and only on the $y-$dimension).

It is important to note that the upper bound k does not depend on $U(a_y, \theta_y)$. Moreover, for any F which satisfies the k crossing property, we can find a $U(a_y, \theta_y)$ for which there is an equilibrium with exactly k different actions. Thus, the upper bound that we identify is binding.

Our result has two immediate implications. First, in the multidimensional model, the ability of the receiver to commit to particular actions, can increase communication. In the absence of commitment, the sender could transmit at most k distinct messages, all about the $y-$dimension. If, on the other hand, the receiver were able to commit to take actions of the form $(\bar{x}, y)$ for a fixed $\bar{x}$, the sender could potentially transmit more information on the $y-$dimension. The receiver may be able to implement such a scheme if he takes an irreversible action on the $x-$dimension prior to his observation of the sender's message.[4]

Second, the level of communication in a multidimensional cheap talk game may be lower than the information generated in two separate games, one along the x-dimension and the other along the y-dimension. In the two-dimensional model there are at most k distinct messages on the y-dimension. In separate unidimensional games

[4] On the other hand, in a unidimensional cheap talk game a-la CS, commitment cannot enlarge the set of equilibrium outcomes when the conflict between the sender and the receiver is large enough. For low levels of conflict, commitment can enlarge the set of equilibrium outcomes (see Dessein (2002)).

there is still no communication on the $x-$dimension but there may be equilibria with more information transmission on the $y-$dimension.[5]

Finally, in the Appendix we prove analogous results to Lemma 1 and Proposition 1 for the case of single-peaked preferences and large levels of conflict.

3. Multiple senders and noisy signals

In this section we extend the model to the case of more than one sender, and allow the senders to have imperfect signals about the state of the world. Analogously to Lemma 1, we derive conditions on equilibrium strategies and use this result to show that the equilibrium proposed in Battaglini (2002) may not be robust to imperfect signals. Battaglini (2004) shows a sense in which it is robust; the difference between the models is clarified in footnote 11.

In particular, consider first the case of two senders. Suppose that each sender $i \in \{1,2\}$ has a signal $\tilde{s}^i$ about $\theta \in \mathbb{R}^2$. Assume that the realizations of $\tilde{s}^i$ are in $\mathbb{R}^2$, and that given $\theta \in \mathbb{R}^2$, they are distributed according to $F(s^1, s^2|\theta)$, with density $f(s^1, s^2|\theta)$ and marginal densities $f(s^1|\theta)$ and $f(s^2|\theta)$. For a given θ, denote by $\tilde{s}^i(\theta)$ the random variable generated by $f(s^i|\theta)$. We will sometimes assume that signals are imperfect in the following sense:

Assumption A1 *$f(s^1, s^2|\theta)$ has full support for any $\theta \in \mathbb{R}^2$.*

The message space of each sender is $\mathbb{R}^2$ and we assume that both senders transmit their messages simultaneously. Thus, in contrast to the analysis with one sender, in

[5] The two implications above are best illustrated when $U(a_y, \theta_y)$ enables full communication in a game restricted to the $y-$dimension.

the current model each sender may view the receiver's actions as a lottery $\tilde{a} = (\tilde{a}_x, \tilde{a}_y)$, as the actions might depend on the information transmitted to the receiver by the other sender. Assume that sender 1 prefers the lottery $\tilde{a}$ to $\tilde{a}'$ if $E(\tilde{a}_x) > E(\tilde{a}'_x)$, and if $E(\tilde{a}_x) = E(\tilde{a}'_x)$ then his preferred lottery is decided according to some preferences on the $y-$dimension (the specifics of these preferences are not important for our next result). Sender 2 has the analogous reverse preferences, deciding first according to the $y-$dimension.[6] All other assumptions are as in Section 2.

Given his type s^i, sender i's strategy, $m^i(.|s^i)$, is a density function over $\mathbb{R}^2$.[7] Given a state θ, the density of sender i's messages is $h^i(m|\theta) = \int\limits_{\mathbb{R}^2} m^i(m|s^i)f(s^i|\theta)ds^i$. The measure of a message m sent by sender i is therefore $h^i(m) = \int_{\mathbb{R}^2} h^i(m|\theta)f(\theta)d\theta$.

The next result imposes restrictions on equilibrium strategies when signals are imperfect. Denote by $\hat{M}^i = \bigcup\limits_{s^i \in \mathbb{R}^2} Supp\{m^i(.|s^i)\}$ the set of all messages that are sent in equilibrium by sender i. A set of messages $\bar{M}^i \subset \hat{M}^i$ has a zero measure if $\int_{m \in \bar{M}^i} h^i(m)dm = 0$.

Lemma 2 *Suppose A1 is satisfied. In any equilibrium: (i)* $\forall \hat{m}^1 \in \hat{M}^1$*, up to a measure zero of messages,* $E(\tilde{\theta}_x|\hat{m}^1) = \mu_x$*; (ii)* $\forall \hat{m}^2 \in \hat{M}^2$*, up to a measure zero of messages,* $E(\tilde{\theta}_y|\hat{m}^2) = \mu_y$.

Proof: Denote the conditional density function of sender 1 over the messages of sender 2 by $g(.|s^1)$ and the receiver's action on the $x-$dimension given the equilibrium

[6] In the Appendix (Remark 1) we discuss how these assumptions about the preferences accord with single-peaked preferences and high levels of conflict.

[7] This follows by the same reasons enlisted in footnote 3; accordingly, the measures presented below are well defined by the equilibrium conditions.

messages $\hat{m}^1$ and $\hat{m}^2$, by $E(\tilde{\theta}_x|\hat{m}^1, \hat{m}^2)$.

To insure that sender 1 does not deviate from his equilibrium strategy, any equilibrium must satisfy the following condition, for any $s^1 \in \mathbb{R}^2$, any m in the support of the strategy of s^1, and any $\hat{m}^1 \in \hat{M}^1$:

$$\int_{\hat{M}^2} E(\tilde{\theta}_x|m, m^2)g(m^2|s^1)dm^2 \geq \int_{\hat{M}^2} E(\tilde{\theta}_x|\hat{m}^1, m^2)g(m^2|s^1)dm^2 \tag{1}$$

Integrating the left-hand-side of (1) over the strategy of sender 1, sustains the inequality:

$$\int_{\hat{M}^1}\int_{\hat{M}^2} E(\tilde{\theta}_x|m, m^2)g(m^2|s^1)m^1(m|s^1)dm^2 dm \geq \int_{\hat{M}^2} E(\tilde{\theta}_x|\hat{m}^1, m^2)g(m^2|s^1)dm^2 \tag{2}$$

We can integrate both sides of (2) over s^1 to get:

$$\mu_x \geq E(\tilde{\theta}_x|\hat{m}_1) \text{ for any } \hat{m}^1 \in \hat{M}^1, \tag{3}$$

Note that the integration over the right-hand-side of (2) follows from Assumption A1 that guarantees that every couple of messages in $\hat{M}^1 \times \hat{M}^2$ is on the equilibrium path.

Finally, note that by the law of iterated expectations,

$$\int_{\hat{M}^1} E(\tilde{\theta}_x|\hat{m}_1)h^1(\hat{m}_1)d\hat{m}_1 = \mu_x,$$

which together with (3), implies condition (i). Similarly we can derive condition (ii) for sender 2.■

Given Lemma 2, we analyze the robustness of the equilibrium proposed in Battaglini (2002) to imperfect signals.[8] To this end, we assume that the preferences of sender

[8]On the robustness of these equilibria to assumptions about the state space, see Ambrus and Takahashi (2006).

1 (sender 2) on the y-dimension ($x-$dimension) allow for full communication on this dimension, if he and the receiver were to play a cheap talk game restricted to this dimension. Battaglini (2002) considers the case in which the senders have perfect information about θ and constructs a fully revealing equilibrium (henceforth, FRE).[9] In this equilibrium, sender 1 truthfully reveals θ_y (and only θ_y) and sender 2 truthfully reveals θ_x (and only θ_x).

In what follows, we fix the prior $F(\theta)$ and the preferences of the senders, and perturb their signals. Our notion of robustness demands that the strategies in the game with imperfect signals converge to the strategies of the FRE (when signals converge to be perfect). As the exact values of messages are not important, we will focus on the informational content of the strategies.

Specifically, consider the messages sent by sender i given θ, the random variable $\tilde{m}^i(\theta)$, with density $h^i(m|\theta)$. Consider further the random variables, $E[\tilde{\theta}|\tilde{m}^1(\theta)]$ and $E[\tilde{\theta}|\tilde{m}^2(\theta)]$, i.e., the expectations about $\tilde{\theta}$ upon observing the messages generated by the strategies of sender 1 and sender 2 respectively. When signals are perfect, given any state θ, the strategies in the FRE induce the random variables $\tilde{m}^i_{FRE}(\theta)$. Note that $E[\tilde{\theta}|\tilde{m}^1_{FRE}(\theta)] = E[\tilde{\theta}|\theta_y]$ and $E[\tilde{\theta}|\tilde{m}^2_{FRE}(\theta)] = E[\tilde{\theta}|\theta_x]$.

Definition 2 *The FRE is robust to imperfect signals, if: (i) There exists a sequence of signals,* $\{\tilde{s}^1_n, \tilde{s}^2_n\}_{n=1}^{\infty}$*, satisfying A1 for any* $n \geq 1$*, such that for any* $\theta \in \mathbb{R}^2$*,* $\{(\tilde{s}^1_n(\theta), \tilde{s}^2_n(\theta))\}_{n=1}^{\infty}$ *converges in probability to* (θ, θ)*;*[10] *(ii) There exists a cor-*

[9] In the case of one sender, the assumption that the sender *perfectly* observes the state of the world, as in Section 2, is without loss of generality. With more than one sender this is not the case, as will be illustrated by Proposition 2.

[10] Our requirement that the signals converge to θ is without loss of generality. Signals that converge

responding sequence of equilibria inducing for any θ a sequence of random variables $\{(E[\tilde{\theta}|\tilde{m}_n^1(\theta)], E[\tilde{\theta}|\tilde{m}_n^2(\theta)])\}_{n=1}^{\infty}$ that converges in distribution to $(E[\tilde{\theta}|\theta_y], E[\tilde{\theta}|\theta_x])$.

PROPOSITION 2 *If according to the prior $F(\theta)$, $E[\tilde{\theta}_x|\theta_y]$ is not constant in θ_y or $E[\tilde{\theta}_y|\theta_x]$ is not constant in θ_x, then the FRE is not robust to imperfect signals.*[11]

PROOF: Assume without loss of generality that $E[\tilde{\theta}_x|\theta_y]$ is not constant in θ_y. Consider the FRE and assume that it is robust to imperfect signals. Fix $\theta \in \mathbb{R}^2$. For any n, consider the random variable $E[\tilde{\theta}_x|\tilde{m}_n^1(\theta)]$. For any $n \geq 1$, the signals $\tilde{s}_n^1$ and $\tilde{s}_n^2$ satisfy A1 and hence (3) holds. Thus, for any realization m of $\tilde{m}_n^1(\theta)$ we have $E[\tilde{\theta}_x|m] \leq \mu_x$. This implies that $E[E[\tilde{\theta}_x|\tilde{m}_n^1(\theta)]] \leq \mu_x$ for all n. As $E[\tilde{\theta}|\tilde{m}_n^1(\theta)] \xrightarrow{D} E[\tilde{\theta}|\theta_y]$ this implies that $E[E[\tilde{\theta}_x|\tilde{m}_n^1(\theta)]] \rightarrow E[\tilde{\theta}_x|\theta_y]$. We therefore have, for any θ, that $E[\tilde{\theta}|\theta_y] \leq \mu_x$. By the continuity of $F(.)$ and by the law of iterated expectations, this implies $E[\tilde{\theta}_x|\theta_y] = \mu_x$ for any θ. This is a contradiction, as $E[\tilde{\theta}_x|\theta_y]$ is not constant in θ_y.■

Note that our results extend to the case of more than two senders. Suppose that there is a set of senders J with $|J| > 2$. First, by Assumption A1, any profile of messages $(m^1, ..., m^{|J|}) \in \prod_{i \in J} \hat{M}^i$ is on the equilibrium path and so the proof of Lemma 2 follows as above. Second, in the FRE, at least two senders must use strategies which are fully revealing about a certain dimension of the state. We can then repeat the

to some injective function of θ may be modified to converge to θ while maintaining their informational content.

[11]For an example illustrating that the FRE may be robust to imperfect signals, see Battaglini (2004), who assumes that $F(\theta)$ is the (improper) uniform distribution on $\mathbb{R}^2$. This assumption implies that any two dimensions are independent according to the prior, and in our context, that θ_x and θ_y are independently distributed.

proof of Proposition 2 focusing on these two senders.

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In this appendix we maintain all the assumptions as in the model considered in Section 2, but for the sender's preferences, which we assume to be represented by:

$$V(a,\theta) = U(b_x + \theta_x - a_x, b_y + \theta_y - a_y),$$

where the function U is strictly decreasing in $|b_x + \theta_x - a_x|$ and in $|b_y + \theta_y - a_y|$ and attains its unique maximum at $(0,0)$. We interpret the vector $b = (b_x, b_y) \in \mathbb{R}^2$ as the conflict between the sender and the receiver. We assume that at least for some $i \in \{x,y\}$, $b_i \neq 0$. Without loss of generality, we assume that $b_y \neq 0$. Let $\beta \in \mathbb{R}$ be defined by $b_x = \beta b_y$ and let $b = ||(b_x, b_y)||$. Fixing β, we consider sequences of vectors, $\{(b_x^n, b_y^n)\}_{n=1}^{\infty}$, for which $b_x^n = \beta b_y^n$ for any $n \in \{1,2,...,\infty\}$, and $b \to \infty$. Finally, assume that U is twice-differentiable and that the limits below are well-defined:[12]

$$\alpha^* \equiv \lim_{x\to\infty, y\to\infty, \frac{x}{y}\to\beta} \left(-\frac{U_x(x,y)}{U_y(x,y)}\right) \text{ and } \gamma^* \equiv \lim_{x\to\infty, y\to\infty, \frac{x}{y}\to\beta} \left(-\frac{U_{xx}(x,y) + \alpha^* U_{xy}(x,y)}{\alpha^* U_{yy}(x,y) + U_{xy}(x,y)}\right)$$

We say that action a is *induced in* $\Theta \subseteq \mathbb{R}^2$ if there exists $\theta \in \Theta$ that induces it. When a is induced in $\mathbb{R}^2$, we simply say that a is *induced*.

LEMMA 3 *For any compact and convex subset $\Theta \subseteq \mathbb{R}^2$, for any $\varepsilon > 0$, $\exists \bar{b} < \infty$ such that for any two actions a and a' in Θ, that are induced in Θ, if $b > \bar{b}$, then: (i) $|\frac{a_y - a'_y}{a_x - a'_x} - \alpha^*| < \varepsilon$; (ii) for any θ and θ' in Θ who are indifferent between the two actions, $|\frac{\theta_y - \theta'_y}{\theta_x - \theta'_x} - \gamma^*| < \varepsilon$.*

PROOF OF LEMMA 3: (i) As a and a' are induced in Θ, by continuity and the convexity of Θ, there must be a type $\theta \in \Theta$ that is indifferent between the two

[12] All our assumptions are satisfied, for example, by the family of utility functions $V(\left(\sum_{i\in\{x,y\}} \lambda_i |a_i - (b_i + \theta_i)|^p\right)^{\frac{1}{p}})$ where $V(.)$ is strictly decreasing and $1 < p < \infty$.

actions. Consider then the indifference curve of θ that goes through a. The slope of the indifference curve is

$$\frac{da_y}{da_x} = -\frac{U_x(\theta_x + b_x - a_x, \theta_y + b_y - a_y)}{U_y(\theta_x + b_x - a_x, \theta_y + b_y - a_y)}.$$

Consider sequences of vectors $\{(b_x^n, b_y^n)\}_{n=1}^{\infty}$ for which $b_x^n = \beta b_y^n$ for a fixed β and for any $n \in \{1, 2, ..., \infty\}$, and let $b \to \infty$. By the compactness of Θ, $\frac{\theta_x + b_x - a_x}{\theta_y + b_y - a_y} \underset{b\to\infty}{\longrightarrow} \beta$ and thus the limit of $\frac{da_y}{da_x}$ does not depend neither on θ nor on a. This convergence is uniform w.r.t. θ and a, hence (i) follows.

(ii) Consider some type θ who is indifferent between a and a' :

$$U(\theta_x + b_x - a_x, \theta_y + b_y - a_y) = U(\theta_x + b_x - a'_x, \theta_y + b_y - a'_y)$$

Total differentiation with respect to θ, along with the mean value theorem, implies (assuming without loss of generality that $a_x \neq a'_x$):

$$\frac{d\theta_y}{d\theta_x} = -\frac{U_{xx}(\theta_x + b_x - \hat{a}_x, \theta_y + b_y - \hat{a}_y) + U_{yx}(\theta_x + b_x - \hat{a}_x, \theta_y + b_y - \hat{a}_y)\frac{(a'_y - a_y)}{(a'_x - a_x)}}{U_{yy}(\theta_x + b_x - \check{a}_x, \theta_y + b_y - \check{a}_y)\frac{(a'_y - a_y)}{(a'_x - a_x)} + U_{yx}(\theta_x + b_x - \check{a}_x, \theta_y + b_y - \check{a}_y)}$$

for $\hat{a}$ and $\check{a}$ which are between a and a'. As above, $\frac{\theta_x + b_x - \hat{a}_x}{\theta_y + b_y - \hat{a}_y} \underset{b\to\infty}{\longrightarrow} \beta$ and $\frac{\theta_x + b_x - \check{a}_x}{\theta_y + b_y - \check{a}_y} \underset{b\to\infty}{\longrightarrow} \beta$. By (i) above, and by the compactness of Θ, we have that $\frac{d\theta_y}{d\theta_x} \underset{b\to\infty}{\longrightarrow} \gamma^*$ uniformly (w.r.t. θ and a). By continuity, the curve of indifferent types is connected. By compactness and the uniform convergence, (ii) follows.■

We now prove that communication is bounded also when the sender's preferences are single-peaked. To this end, we modify some of the definitions introduced in Section 2. First, denote the set of induced actions by A^* and let $\tilde{a}^*$ be the receiver's equilibrium choice of action.

Definition 3 *For a finite k, an equilibrium has k actions up to ε if*

$$k = \min_{\substack{A' \subseteq A^* \\ |A'| < \infty}} \{|A'|;\ \Pr(\tilde{a}^* \in A') > 1 - \varepsilon\}.$$

Second, we can span the state space according to the $\alpha-$dimension, that parallel to lines with slope α^*, with a generic coordinate θ_α, and the $\gamma-$dimension, parallel to lines with slope γ^*, with a generic coordinate θ_γ. We can then define the conditional expectations $E(\tilde{\theta}_\gamma|\theta_\alpha)$ in the two-dimensional space spanned by the γ and α dimensions, and extend the definition of the k-crossing property analogously.

Proposition 3 *Suppose that F satisfies the k-crossing property with respect to the γ and α dimensions. Then for any $\varepsilon > 0$ there exists a $\bar{b} < \infty$ such that for all $b > \bar{b}$, any equilibrium has at most k actions up to ε.*

Proof of Proposition 3: We start with some definitions and notation. Recall that equilibrium is defined by a pair of strategies, $(a^*(.), m^*(.|\theta))$. Let $M^* = \bigcup_{\theta \in \mathbb{R}^2} Supp\{m^*(.|\theta)\}$. For the purpose of the proof, we can restrict attention to equilibria in which $m = a^*(m)$ for any $m \in M^*$. This implies that $M^* = A^*$.[13]

The sender's strategy induces a measure on $\mathbb{R}^2 \times \mathbb{R}^2$, $g(\theta, m) = m^*(m|\theta)f(\theta)$. For any $A \subset A^*$ in the Borel $\sigma-$algebra of $\mathbb{R}^2$, the density of θ conditional on an action in A being induced is $g(\theta|A) = \int_A \frac{g(\theta,m)}{\int_{\mathbb{R}^2}\int_A g(\theta',m')d\theta' dm'} dm$. For any $A \subset A^*$ in the Borel $\sigma-$algebra of $\mathbb{R}^2$, the probability measure of the set of types that induce actions in A is $G(A) = \int\limits_{\mathbb{R}^2}\int\limits_A g(\theta, m) dm d\theta$. The set of all types who induce some action a is termed the *support set* of action a and is denoted by $S(a)$.

[13] M^* is in the Borel $\sigma-$algebra of $\mathbb{R}^2$ as $m^*(m|\theta)$ is measurable in m and θ. Thus, also A^* is in the Borel $\sigma-$algebra of $\mathbb{R}^2$.

We will use two geometrical constructions in the Proof of Proposition 3. First, S^η is the compact subset of $\mathbb{R}^2$, bounded by a square constructed symmetrically around the prior expectation and parallel to the x and y dimensions, that has a measure of $1-\eta$ according to F. For any η and any two parallel lines l' and l'' let $S^\eta_{l',l''}$ be the subset of S^η that is between l' and l''.

Second, when we span the space according to the $\alpha-$dimension and the $\gamma-$dimension, we denote by μ_γ (μ_α) the γ-coordinate (α-coordinate) of the prior expectation. Finally, let L denote the set of lines with slope γ^*, such that the $\gamma-$coordinate of the conditional expectation of the subset of $\mathbb{R}^2$ on each side of the line equals μ_γ. We denote an element of L by l.

The proof of Proposition 3 follows three steps. Step 1 below, will allow us to translate results on compact sets to non-compact ones when b is large. Step 2 focuses on compact subsets and invokes Lemma 3 to show how, when b is large, the support sets of equilibrium actions are characterized by lines in L. In Step 3 we show that when F satisfies the k-crossing property with respect to the γ and α dimensions, then the number of lines in L is $k-1$. Finally, we combine the three steps to conclude that for high enough b there will be at most k actions up to ε in equilibrium.

Step 1 *For any $\delta > 0$ there exists an $\eta' > 0$, such that for any $\eta < \eta'$, in any equilibrium, there is at most a probability measure δ of types in S^η who induce actions in $(S^\eta)^c$.*[14]

[14] As A^*, S^η, and $(S^\eta)^c$ are in the Borel $\sigma-$algebra of $\mathbb{R}^2$, also their intersections are in the Borel $\sigma-$algebra of $\mathbb{R}^2$. By the measurability of $m^*(.|\theta)$, this implies that the statement in Step 1 is well-defined in all equilibria.

Proof of Step 1: In what follows, we set "east" to be the upward direction on the $x-$axis. Consider the subset of $\mathbb{R}^2$ to the east of S^η which is bounded above and below by the extensions of the north and south sides of S^η. Denote this set by E^η and let $E^{\eta*} = A^* \cap E^\eta$.

Assume, by way of contradiction, that the probability measure of the set of types in S^η who support actions in $E^{\eta*}$ is $\delta/4$, for some $\delta > 0$. Let $g(\theta_x|E^{\eta*})$ be the marginal distribution of $g(\theta|E^{\eta*})$ on the $x-$axis.

We first focus on the case in which the set of types inducing actions in $E^{\eta*}$ is comprised of: (i) in S^η, the set of measure $\delta/4$ under $F(.)$ that is to the east-most in S^η; (ii) in $(S^\eta)^c$, those to the east-most in $(S^\eta)^c$. Let $\bar{g}(\theta_x|E^{\eta*})$ denote the relevant marginal distribution for this specification.

The $x-$coordinate of the conditional expectation over the union of the support sets of all actions in $E^{\eta*}$ is $\int_{S^\eta} \theta_x \bar{g}(\theta_x|E^{\eta*})d\theta_x + \int_{(S^\eta)^c} \theta_x \bar{g}(\theta_x|E^{\eta*})d\theta_x$. As $\bar{g}(\theta, m)$ is a proper density function, we have that $\int_{(S^\eta)^c} \theta_x \bar{g}(\theta_x|E^{\eta*})d\theta_x \underset{\eta\to 0}{\to} 0$ and $\int_{S^\eta} \theta_x \bar{g}(\theta_x|E^{\eta*})d\theta_x \underset{\eta\to 0}{\to} \bar{k}_\delta$ for some finite $\bar{k}_\delta$. As we focused on the case in which the support sets of actions in $E^{\eta*}$ are to the east-most of both S^η and $(S^\eta)^c$, then in any equilibrium, $\limsup_{\eta\to 0}\{\int_{S^\eta} \theta_x g(\theta_x|E^{\eta*})d\theta_x + \int_{(S^\eta)^c} \theta_x g(\theta_x|E^{\eta*})d\theta_x\} \leq \bar{k}_\delta$.

Thus, there exists an η' such that for all $\eta < \eta'$, the conditional expectation over the support sets of actions in $E^{\eta*}$ is actually not in $E^{\eta*}$. This is a contradiction, as the conditional expectation over the support sets must equal the expectation over the actions, where the latter are in E^η. Finally, the same exercise can be applied to other subsets analogous to E^η in $(S^\eta)^c$.$\square$

STEP 2 *For any $\zeta > 0$ there exists an $\bar{\eta} > 0$ such that for all $\eta < \bar{\eta}$, there exists a $\bar{b}(\eta) < \infty$, such that for all $b > \bar{b}(\eta)$, for any $a \in S^{\eta}$ that is induced in S^{η}, there exist $l, l' \in L$ such that $|G(a) - F(S^{\eta}_{l,l'})| < \zeta$.*

Proof of step 2: Take some $\hat{a} \in A^* \cap S^{\eta}$ that is induced in S^{η}. Denote by $\bar{l}$ a line parallel to the $\gamma-$axis that intersects the closure of $S(\hat{a})$ in S^{η} and separates S^{η} so that $S(\hat{a}) \cap S^{\eta}$ is "below" it and similarly let $\underline{l}$ be a line parallel to the $\gamma-$axis that intersects the closure of $S(\hat{a})$ and separates S^{η} so that $S(\hat{a}) \cap S^{\eta}$ is "above" it. Let $\hat{l}$ be an element of $\{\bar{l}, \underline{l}\}$.

The proof proceeds as follows. In claim 1 we show that any line $\hat{l} \in \{\bar{l}, \underline{l}\}$ separates S^{η} in the sense that the measure of types above (below) it who support actions below (above) it is 'small'. We use this result in claim 2 where we prove that for high b the support set of $\hat{a}$ nearly equals the set of all types in S^{η} that are below $\bar{l}$ and above $\underline{l}$. In claim 3 we show that the lines $\bar{l}$ and $\underline{l}$ must converge to lines in L.

Claim 1: *For any $\delta > 0$, $\exists \eta'$ such that for all $\eta < \eta'$, there exists a $b'(\eta) < \infty$ such that for all $b > b'(\eta)$, the probability measure of all types above (below) $\hat{l}$ and in S^{η} that support actions below (above) $\hat{l}$ and in S^{η}, is bounded by δ.*

Proof of Claim 1: We focus on proving that the probability measure of all types above $\bar{l}$ and in S^{η} that support actions below $\bar{l}$ and in S^{η}, is bounded by δ, as the proof for the other cases is analogous. Let $A' \subset A^*$ be the set of actions below $\bar{l}$ and in S^{η} that are supported by some types above $\bar{l}$ and in S^{η} (note that $\hat{a} \notin A'$). Let $\hat{\theta}$ be in the intersection of $\bar{l}$ and the closure of $S(\hat{a}) \cap S^{\eta}$. A curve of types in S^{η} who are indifferent between $\hat{a}$ and some $a' \in A'$ must separate $\hat{\theta}$ from the types who induce

a', as $\hat{\theta}$ weakly prefers $\hat{a}$. Such a curve cannot therefore be strictly below $\bar{l}$.

In cases in which all types below each such curve prefer a' to $\hat{a}$, Lemma 3 guarantees that the measure of the set of types above $\bar{l}$ in S^{η} that prefer a' to $\hat{a}$ converges to zero (uniformly) as $b \to \infty$.

In what follows we focus therefore on cases in which all types below each such curve prefer $\hat{a}$ to a'. Suppose by way of contradiction that the measure of the set of types above $\bar{l}$ in S^{η} who induce actions in A' is bounded from below by a strictly positive number. In the cases we now focus on, Lemma 3 guarantees that the measure of the set of types below $\bar{l}$ and in S^{η} that prefer a' to $\hat{a}$ converges to zero (uniformly) as $b \to \infty$. From this it follows that the set of types in $\mathbb{R}^2$ who induce actions in A' converges, as $\eta \to 0$ and $b \to \infty$, to the set of types above $\bar{l}$ and in S^{η} who induce actions in A'.

As we assume that the measure of this set is strictly positive, this implies that the measure of this set of types under $F(.)$ is strictly positive. As $F(.)$ is continuous, this set of types must have a strictly positive width (the width of a set is defined as the infimum of the distance between any two parallel lines containing the set and if this is not possible, as infinite). As the set has a strictly positive width and is above $\bar{l}$, its conditional expectation under $g(\theta|A')$ is both above $\bar{l}$ and bounded away from it. This, however, is a contradiction as the conditional expectation over actions in A', which must be below $\bar{l}$, must accord with the conditional expectation over the set of types in $\mathbb{R}^2$ who induce actions in A'.$\square$

Claim 2: *For any $\zeta > 0$ there exists an $\bar{\eta} > 0$ such that for all $\eta < \bar{\eta}$, there exists* *a* $\bar{b}(\eta) < \infty$, *such that for all* $b > \bar{b}(\eta)$, $|G(\hat{a}) - F(S^{\eta}_{\bar{l},\underline{l}})| < \zeta$.

Proof of Claim 2: We will show that nearly all types in $S^{\eta}_{\bar{l},\underline{l}}$ support $\hat{a}$, and that nearly all types that support $\hat{a}$ are in $S^{\eta}_{\bar{l},\underline{l}}$. For the first observation, we have to consider types in $S^{\eta}_{\bar{l},\underline{l}}$ who induce actions either in $S^{\eta}_{\bar{l},\underline{l}}$ (but not $\hat{a}$), or in $S^{\eta} \backslash S^{\eta}_{\bar{l},\underline{l}}$, or in $(S^{\eta})^c$. For the second observation, as by construction $(S(\hat{a}) \cap S^{\eta}) \subseteq S^{\eta}_{\bar{l},\underline{l}}$, we only have to consider types in $(S^{\eta})^c$ who support $\hat{a}$.

Consider a type in $S^{\eta}_{\bar{l},\underline{l}}$ that induces some $a' \in S^{\eta}_{\bar{l},\underline{l}}$ for $a' \neq \hat{a}$. The curve of types who are indifferent between $\hat{a}$ and a' must cross either $\underline{l}$ or $\bar{l}$. By Lemma 3, for any such a', any such curve converges in S^{η} (uniformly w.r.t. a) to either $\underline{l}$ or $\bar{l}$ when $b \to \infty$. In other words, for any η there exists a $b(\eta) < \infty$, such that for all $b > b(\eta)$, the measure of those who induce actions in $S^{\eta}_{\bar{l},\underline{l}}$ other than $\hat{a}$ is bounded by $\zeta/4$.

By claim 1, for any $\zeta > 0$, $\exists \eta'$ such that for all $\eta < \eta'$, there exists $\bar{b}(\eta) = \max\{b'(\eta), b(\eta)\}$ such that for all $b > \bar{b}(\eta)$, the measure of types who are in $S^{\eta}_{\bar{l},\underline{l}}$ and support actions in $S^{\eta} \backslash S^{\eta}_{\bar{l},\underline{l}}$ is less than $\zeta/4$. By Step 1, we can choose $\bar{\eta} < \eta'$ such that there is at most a probability measure $\zeta/4$ of types in $S^{\eta}_{\bar{l},\underline{l}}$ that induce actions in $(S^{\eta})^c$. Finally, we can choose $\bar{\eta} < \zeta/4$ so that there is at most a probability measure $\zeta/4$ of types in $(S^{\eta})^c$.$\Box$

In the following claim, let the distance between two sets A and B, be defined as $d(A, B) = \sup_{a \in A}\{\inf_{b \in B} ||a - b||\}$.

Claim 3: *For any* $\zeta > 0$ *there exists an* $\bar{\eta} > 0$ *such that for all* $\eta < \bar{\eta}$, *there exists a* $\bar{b}(\eta) < \infty$, *such that for all* $b > \bar{b}(\eta)$, $d(\bar{l} \cap S^{\eta}, l \cap S^{\eta}) < \zeta$ *and* $d(\underline{l} \cap S^{\eta}, l' \cap S^{\eta}) < \zeta$ *for some* l *and* l' *in* L.

Proof of Claim 3: Focus on the subset of S^{η} above or below $\bar{l}$ that has the larger measure (say it is the subset below $\bar{l}$). For this subset, consider the set of all actions induced in S^{η} and contained in this subset. By Lemma 3, the $\gamma-$coordinate of the expectation over these actions must converge to μ_{γ} as $b \to \infty$. But the expectation over these actions must also equal the conditional expectations over the union of their support sets. According to Claim 1 and Step 1, this union coincides with the set of all types below $\bar{l}$ (which is measurable under F) up to a probability measure of 4ζ (either those in S^{η} and above $\bar{l}$ that support actions in S^{η} and below $\bar{l}$ and vice versa, or those in S^{η} supporting actions in $(S^{\eta})^c$ and vice versa). Thus, by choosing a small enough η and accordingly a large enough b, the $\gamma-$coordinate of the conditional expectation over the set of types below $\bar{l}$ converges to μ_{γ}. By the continuity of F, this implies that $\bar{l}$ converges (uniformly) to some $l \in L$ in S^{η}. The same argument applies to show that $\underline{l}$ converges to some $l' \in L$.□

Claims 2 and 3, and the compactness of S^{η} which implies uniform convergence with respect to a and θ, prove the statement in Step 2.□

STEP 3 *If F satisfies the $k-$crossing property with respect to the γ and α* dimensions, *then* $|L| = k - 1$.

Proof of step 3: The proof follows that of Proposition 1 in the text.□

We can now combine the three steps above to show that for a high enough b, any equilibrium has at most k actions up to ε. By Step 1, for any ε, there exists η' such that for all $\eta \leq \eta'$ the probability measure of types who support actions in $(S^{\eta})^c$ is at most $\varepsilon/3$. By Step 2, for any ε, there exists an $\bar{\eta}$ and a $\bar{b}(\varepsilon, \bar{\eta})$ such that for all $\eta < \bar{\eta}$

and $b > \bar{b}(\varepsilon, \eta)$, the set of types that is in between any two neighbouring lines in L must belong, but for a measure of $\varepsilon/3k$, to a support set of only one action. By Step 3, there are at most k such sets and each has a strictly positive measure. Therefore, for any ε, we can choose $\eta < \min\{\eta', \bar{\eta}, \varepsilon/3\}$, and hence there exists a $\bar{b}(\varepsilon, \eta)$ such that for all $b > \bar{b}(\varepsilon, \eta)$, in any equilibrium, the probability measure of those who support the $k' \leq k$ actions in S^{η} is at least $1 - \varepsilon$.■

Remark 1 In this appendix we have extended our results for the model with one sender and lexicographic preferences, to the case of single-peaked preferences. In Section 3 in the text we consider the case of multiple senders (with lexicographic preferences) and noisy signals. We have assumed that on the dimension of conflict, a sender compares between lotteries according to their expectations. Intuitively, this will arise when a sender is risk neutral on this dimension.

One can also extend our analysis in Section 3 to the case of single-peaked preferences. We can then prove an analogous result to Lemma 3 about how a sender compares between lotteries when b is large. Again, this will depend on the risk preferences of the sender on the dimension of conflict. In particular, one can show that whenever $\lim_{x\to\infty, y\to\infty, \frac{x}{y}\to\beta} \sqrt{U_x^2 + U_y^2}$ exists, then the single-peaked preferences converge to be lexicographic where the dimension that is orthogonal to α^* becomes the dimension that takes precedence. If $\lim_{x\to\infty, y\to\infty, \frac{x}{y}\to\beta} \sqrt{U_x^2 + U_y^2}$ is finite (and non zero), then a sender is risk neutral on this dimension and compares two lotteries $\tilde{a}$ and $\tilde{a}'$ according to the expected action of the receiver on this dimension, as we assume in Section 3. If $\lim_{x\to\infty, y\to\infty, \frac{x}{y}\to\beta} \sqrt{U_x^2 + U_y^2} = \infty$, then the sender becomes infinitely risk averse on this dimension.