Aggregation and the relevance of some issues for others

Franz Dietrich
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Abstract

A general collective decision problem is analysed. It consists in many issues that are interconnected in two ways: by mutual constraints and by connections of relevance. The goal is to decide on the issues by respecting the mutual constraints and by aggregating in accordance with an informational constraint given by the relevance connections. Whether such aggregation is possible in a non-degenerate way depends on both types of connections and their interplay. One result, if applied to the preference aggregation problem and adopting Arrow's notion of (ir)relevance, implies Arrow's Theorem, without excluding indifferences unlike in the existing general aggregation literature.

1 Introduction

Most complex decision problems can be formalised as consisting of many binary decisions: decisions of accepting or rejecting certain propositions. For instance, establishing a preference relation R over a given set of alternatives Q consists in deciding, for each pair of alternatives $x, y \in Q$, whether or not xRy. Judging the values of different variables consists of judging, for each variable and each of its potential values, whether or not the variable takes this value. Producing a report that contains qualitative economic forecasts might involve deciding for or against many propositions: atomic ones like "inflation will increase" and compound ones like "*if* consumption will increase *and* foreign demand does *not* decrease, *then* inflation will increase" (where logical operators are italicised).

Although this separation into binary issues is usually possible, there are arguably two distinct types of interconnections – to be called *logical* connections and *relevance* connections – that can prevent us from treating the issues independently. First, the decisions on the issues may logically constrain each other; in the above examples, the preference judgments must respect conditions like transitivity, the variables might constrain each other, and the propositions in the economic report have to be logically consistent, respectively. Second – and this is the topic of this paper – some issues may be *relevant* to (the decision on) other issues. The nature and interpretation of relevance connections is context-specific. A proposition r may be relevant to another one p on the grounds that r is an (argumentative) premise of p, or that r is a causal factor bringing about p, or that r and p share some other (semantic) relation. Relevance connections are not reducible to logical connections. Two issues – say, whether traffic lights are necessary and whether the diplomatic relations to a country should be interrupted – may be considered irrelevant to each other and yet be indirectly logically related via other issues under consideration. Conversely, an issue – say that of whether country x has weapons of mass destruction – may be considered relevant to another issue – say that of whether measure y against country x is appropriate – without a (direct or indirect) logical connection in the complex decision problem considered. A more careful argument for non-reducibility is given later. Though not reducible to logical connections, the notion of relevance is arguably related to them (as reflected by the definition I will give of a relevance relation).

Now suppose that the complex decision problem is faced by a group of individuals and should be settled by aggregating the individual judgments on each proposition (issue). Many concrete aggregation models and procedures in the literature in effect account, in different ways, for both logical and relevance connections. Logical connections are represented by delimiting the set of admissible decisions, i.e., by restricting the *output* of the aggregation rule, for instance in the form of collective rationality conditions like transitivity in preference aggregation, or in the form of an overall budget constraint if different budget items are to be decided simultaneously. By contrast, relevance connections are accounted for through the way in which the decision (output of the aggregation rule) depends on the profile of individual inputs: only *relevant* information should be used. Arrow's condition of *independence of irrelevant alternatives* ("IIA") is the most famous condition excluding (arguably) irrelevant information. The *premise-based procedure* in judgment aggregation makes the decision on certain (conclusion-like) propositions dependent on people's judgments on other (premise-like) propositions considered relevant to the conclusion. The question of "what is relevant for what" may be controversial: some researchers reject Arrow's IIA condition, and the same proposition may be explained in more than one way in terms of premises.

While accounted for in concrete aggregation problems and procedures, the relevance of some issues for others has not yet been treated in general terms. In this paper, I propose to consider, in addition to logical connections, a (binary) *relevance* relation $\mathcal{R}$ between propositions (issues), and to aggregate in accordance with *independence of irrelevant propositions* ("IIP") and with a certain unanimity condition. Both of these conditions are defined with respect to $\mathcal{R}$. To allow broad applications, I leave entirely general the complex decision problem and the interpretation and formal features of the relevance relation $\mathcal{R}$: $\mathcal{R}$ might be a highly partial relation (few inter-relevances) or a close to complete relation (many inter-relevances); $\mathcal{R}$ need not be transitive or reflexive (i.e., self-relevance is not required); and $\mathcal{R}$ need not relate in a systematic way to logical connections, except that the propositions relevant to a given proposition should not logically underdetermine the latter (which might be viewed as an inherent feature of notions of relevance).

In the special case that every proposition is considered relevant just to itself (i.e., $p\mathcal{R}q \Leftrightarrow p = q$ for any propositions p, q), IIP reduces to the (restrictive) condition of *proposition-wise independence* (often simply called *independence*): here, on each proposition an isolated vote is taken, using an arbitrary voting rule but ignoring people's judgments on other propositions. A number of general results have been obtained on proposition-wise independent aggregation, in abstract aggregation models (starting with Wilson 1975) or models of logic-based judgment aggregation (starting with List and Pettit 2002). Essentially, these results establish limits to the possibility of proposition-wise independent aggregation in the presence of certain logical connections between propositions. Impossibility results with necessary conditions on logical connections are derived, for instance, by Wilson (1975), List and Pettit (2002), Pauly and van Hees (2006), Dietrich (2006), Gärdenfors (2006), Mongin (2005) and van Hees (forthcoming). Nehring and Puppe (2002, 2005, 2006) derive the first results with minimal conditions on logical connections, and Dokow and Holzman (2005) introduce minimal conditions of an algebraic kind. Other impossibility results are given, for instance, in Dietrich (forthcoming), Dietrich and List (forthcoming) and Nehring (2005). Possibilities of proposition-wise independent aggregation arise if the individual judgments fall into particular domains (List 2003) or if logical connections are modelled via subjunctive implications (Dietrich 2005).

The proposition-wise independence condition is often criticised (e.g., Chapman 2002, Mongin 2005), but has rarely been weakened in the cited general aggregation literature. The normative appeal of the condition is easily challenged by concrete examples: why, for instance, should the collective judgment on whether to introduce taxes on kerosene be independent of people's judgments on whether global warming should be prevented? The weakened independence conditions proposed in the literature are special cases of IIP: each one is obtainable from some specification of relevance $\mathcal{R}$. Let me mention the literature's two most notable independence weakenings.[1]

First, if relevance $\mathcal{R}$ is an *equivalence* relation, IIP becomes equivalent to an issue-wise independence condition with possibly *non-binary* issues (see Example 3 below). Rubinstein and Fishburn (1986) use such a non-binary issue-wise independence condition, and derive mathematically elegant representation results for decision problems and/or aggregation rules of a particular algebraic kind (see footnote 14). In judgment aggregation, Claussen and Roisland (2005) introduce a non-binary version of the discursive paradox and prove results on when it occurs; in view of typical economic decision situations, they argue for non-binary issues in judgment aggregation and use procedures satisfying a non-binary issue-wise independence condition. Also Pauly and van Hees' (2006) multi-valued logic approach can be viewed as using a non-binary independence condition. The implicit binariness assumption in the proposition-wise indepen-

[1] The first of them was originally not thought of as a weakening of proposition-wise independence, but is one under our division of the decision problem into (binary) decisions on propositions.

dence condition makes the setting inapplicable to Arrowian preference aggregation, since "Arrowian issues" are in fact ternary: there are three ways (including indifference) to rank two distinct alternatives. However, *strict* preference aggregation has binary issues, and some of the above results do indeed imply Arrow's Theorem for strict preferences.[2]

The literature's second independence relaxation – aimed at representing not non-binariness but the different status of different propositions – consists in applying proposition-wise independence only to *some* propositions, e.g. to "premises" (Dietrich 2006) or to *atomic* propositions (Mongin 2005). Mongin (2005) argues that the collective judgment on a compound proposition like $p \wedge q$ should not ignore how the individuals judge p and judge q; our relevance relation $\mathcal{R}$ would then have to satisfy $p\mathcal{R}p \wedge q$ and $q\mathcal{R}p \wedge q$.

In the first part of the paper, I introduce the model, and I consider different classes of relevance relations and corresponding types of IIP aggregation rules. The model allows us to define notions of unanimity preservation and dictatorship that generalise Arrow's notions (of weak Pareto and dictatorship) and are weaker than those used in the cited literature. In the second part, I show that if connections in terms of relevance and logic play together in certain ways, IIP and unanimity preserving aggregation rules are degenerate. Different kinds of connections lead to different forms of degenerate aggregation, ranging from semi-vetodictatorship to strong dictatorship. One of the results, if applied to the preference aggregation problem, yields Arrow's Theorem (without excluding indifferences, unlike in earlier work). In deriving conditions for impossibilities, I will sacrifice some generality for simplicity and elegance of the conditions.[3]

Many particular classes of relevance relations (some of them discussed below) are potentially of interest. I believe that much work can and should be done to investigate IIP aggregation under different classes. For instance, within suitable classes – say if relevance is transitive and relates in a systematic way to logical connections – there is a hope to obtain elegant minimal conditions for when non-degenerate aggregation is (not) possible. Further, one should explore premise-based and sequential prioritarian approaches (as explained later), and develop a distance based approach[4] compatible with IIP. The development of different types of IIP aggregation procedures should go hand in hand with developing criteria for when to consider a proposition as relevant for another (i.e., for how to define $\mathcal{R}$). This second research goal has a normative dimension. Reaching both goals would enable us to give concrete recommendations for practical group

[2]Namely those by Wilson (1975), Dietrich and List (forthcoming) and Dokow and Holzman (2005). Nehring (2003) derives an Arrow-like theorem.

[3]I make no conjecture on the nature of *minimal* conditions for impossibilities, except that they do not have a unified and structured form but the form of disjunctions of several cases. The reason is that the conditions must capture the joint and non-separable behaviour of relevance and logical connections, which is left entirely general and uncontrolled in this paper.

[4]E.g., Eckert and Pigozzi (2005) and (for fusion operators in artificial intelligence) Konieczny and Pino-Perez (2002).

decision-making.

2 Basic notions

We consider a set $N = \{1, ..., n\}$ of individuals, where $n \geq 2$, faced with a collective decision problem of a general kind.

Agenda, judgment sets. The *agenda* is an arbitrary non-empty (possibly infinite) set X of *propositions* on which a decision (acceptance or rejection) is needed. The agenda includes negated propositions: $X = \{p, \neg p : p \in X^+\}$, where X^+ is some set of non-negated propositions and "$\neg p$" stands for "not p". Notationally, double-negations cancel each other out.[5] A *judgment set* is a set $A \subseteq X$ of (accepted) propositions; it is *complete* if it contains a member of each pair $p, \neg p \in X$ ("no abstentions").

Logical interconnections. Not all judgment sets are consistent. For the agenda $X = \{a, \neg a, b, \neg b, a \wedge b, \neg(a \wedge b)\}$, the (complete) judgment set $\{a, b, \neg(a \wedge b)\}$ is inconsistent. Let $\mathcal{J}$ be a non-empty set of judgment sets A, each containing exactly one member of each pair $p, \neg p \in X$, and suppose the *consistent* judgment sets are precisely the sets in $\mathcal{J}$ and their subsets; all other judgment sets are *inconsistent.*[6] A judgment set A *entails* a proposition p (written $A \vdash p$) if $A \cup \{\neg p\}$ is inconsistent. I write $q \vdash p$ for $\{q\} \vdash p$.

It is natural (though for the present results not necessary) to take the propositions in X to be *statements* of a formal language, and to take consistency to be standard logical consistency, as is usually assumed in the judgment aggregation literature. The formal language should be sufficiently expressive to mimic the natural language in which the real decision problem arises.[7]

A $p \in X$ is a *contradiction* if $\{p\}$ is inconsistent, and a *tautology* if $\{\neg p\}$ is inconsistent. For sets $A, B \subseteq X$, I call A *consistent with* B (and B consistent with A) if $A \cup B$ is consistent; and for $A \subseteq X$ and $p \in X$, I call A *consistent with* p (and p consistent with A) if $A \cup \{p\}$ is consistent.

Aggregation. The *(judgment) aggregation rule* is a function F that assigns to every profile $(A_1, \ldots, A_n)$ of (individual) judgment sets (in some domain of admissible profiles) a (collective) judgment set $F(A_1, \ldots, A_n) = A \subseteq X$. I will require that F satisfies the standard input and output conditions: F allows as an

[5] That is: whenever I write "$\neg q$" (where $q \in X$), I mean the other member of the pair $p, \neg p \in X$ to which q belongs; hence "$\neg\neg q$" stands for q.

[6] So $\mathcal{J}$ contains the consistent *and complete* judgment sets. Any set $\{p, \neg p\} \subseteq X$ is inconsistent. Any subset of a consistent set is consistent. Finally, $\emptyset$ is consistent, and any consistent set has a superset that is consistent and complete (hence in $\mathcal{J}$).

[7] The logic can be a classical (propositional or predicate) one, or a non-classical one such as a modal logic, as long as the logic satisfies some regularity conditions. This follows Dietrich's (forthcoming) model of judgment aggregation in general logics, which generalises List and Pettit's (2002) original model in classical propositional logic.

input all profiles $(A_1, ..., A_n)$ of complete and consistent (individual) judgment sets ("universal domain"); and F always generates a complete and consistent (collective) judgment set ("collective rationality"). In summary, F is then a function from $\mathcal{J}^n$ to $\mathcal{J}$. For instance, majority rule on $\mathcal{J}^n$, given by

$$F(A_1, ..., A_n) = \{p \in X : |\{i : p \in A_i\}| > n/2\} \text{ for all } (A_1, ..., A_n) \in \mathcal{J}^n,$$

usually violates collective rationality, i.e. takes values outside $\mathcal{J}$.

Abstract aggregation theory. One may (re)interpret the elements of X as arbitrary *attributes*, which may but need not be *propositions/judgments*, and may but need not be expressed in formal logic. Then sets $A \subseteq X$ are *attribute sets* (rather than judgment sets), the aggregation rule map profiles of individual attribute sets to a collective attribute set, and our later independence condition is an *independence of irrelevant attributes* (rather than propositions). Of course, the holders $i \in N$ of attribute sets A_i need not be human individuals.

I give two examples here; more examples follow in the next section.

Example 1: preference aggregation. For a given set of at least three "alternatives" Q, consider the agenda

$$X := \{xRy, \neg xRy : x, y \in Q\} \text{ (the \textit{preference agenda})},$$

where xRy is the proposition "x is at least as good as y". Throughout the paper, I often write xPy for $\neg yRx$. Let $\mathcal{J}$ be the set of all judgment sets $A \subseteq X$ representing a weak ordering[8], that is, for which there is a weak ordering $\succeq$ on Q such that

$$A = \{xRy \in X : x \succeq y\} \cup \{\neg xRy \in X : x \not\succeq y\}.$$

Note that there is a bijective correspondence between weak orderings on Q and judgment sets in $\mathcal{J}$; and between judgment aggregation rules $F : \mathcal{J}^n \to \mathcal{J}$ and Arrowian social welfare functions (with universal domain). The agenda X (and its consistency notion) belong to a predicate logic, as defined in Dietrich (forthcoming), drawing on List and Pettit (2004).[9]

Example 2: judging values of and constraints between variables. Suppose a group (e.g. a central bank's board or research panel) debates the values of different variables (e.g. macroeconomic variables measuring GDP, prices or consumption). Let $\mathbf{V}$ be a non-empty set of "variables". For each $V \in \mathbf{V}$ let $ran(V)$ be a non-empty set of possible "values" of V (numbers or other objects), called the *range* of V. For any variable $V \in \mathbf{V}$ and any value $v \in ran(V)$,

[8] A weak ordering on Q is a binary relation $\succeq$ on Q that is reflexive, transitive, and connected (but not necessarily anti-symmetric). Weak orderings represent fully rational preference relations, with indifferences allowed.

[9] See Dietrich and List (forthcoming) for a logic representing *strict* preference aggregation.

the group has to judge the proposition $V = v$ stating that V takes the value v.[10] These judgments should respects the (causal) constraints between variables; but, not surprisingly, the nature of these constraints is itself disputed, for instance because the group members believe in different (econometric) estimation techniques. If the variables are real-valued, some *linear* constraints like $V + 3W - U = 5$, or non-linear ones like $V^2 = W$, might be debated. Let $\mathbf{C}$ be any non-empty set of "constraints" under consideration.[11] The agenda is given by

$$X = \{V = v, \neg(V = v) : V \in \mathbf{V}, v \in ran(V)\} \cup \{c, \neg c : c \in \mathbf{C}\}.$$

A judgment set $A \subseteq X$ thus states that certain variables do (not) take certain values, and that the variables do (not) constrain each other in certain ways. To define logical connections, note first that some constraints may conflict with others (e.g., $V > W$ conflicts with $W > V$), and that some constraints may conflict with other *negated* constraints (e.g., $V \log(W) > 2$ conflicts with $\neg(V \log(W) > 0)$). Let $\mathcal{J}^*$ be some non-empty set containing for each constraint $c \in \mathbf{C}$ either c or $\neg c$ (not both); the sets in $\mathcal{J}^*$ represent consistent judgments on the constraints. Now let $\mathcal{J}$ be the set of all judgment sets $A \subseteq X$ containing exactly one member of each pair $p, \neg p \in X$ such that:

(i) each variable $V \in \mathbf{V}$ has a single value $v \in ran(V)$ with $V = v \in A$;
(ii) the family of values in (i) obeys all accepted constraints $c \in A \cap \mathbf{C}$;
(iii) the judgments on constraints are consistent: $A \cap \{c, \neg c : c \in \mathbf{C}\} \in \mathcal{J}^*$.

Note that it is consistent to hold a negated constraint $\neg c$ and yet to assign values to variables in accordance with c. Indeed, variables can stand in certain relations by coincidence, i.e. without a constraint to this effect. More precisely, a constraint states not just an *actual* relation r between variables but a *necessary* relation "necessarily r", that is (in modal logical terms) "in all possible worlds r"; the negation of this constraint is equivalent to "possibly $\neg r$", which is indeed consistent with "r", i.e. with the relation holding.

3 Independence of irrelevant propositions

The two conditions I will impose on the aggregation rule are based on a *relevance relation*, whose nature and interpretation is context-specific, as indicated earlier. Such a relevance relation is not simply reducible to logical connections (of inconsistency and entailment). For instance, one might assume that whether country x has weapons of mass destruction is relevant to whether country x should be attacked but not vice versa. This asymmetry of relevance between two (atomic) propositions need not be reflected in the logical structure $\mathcal{J}$ of the

[10] More generally, the group might consider propositions stating that V's value belongs to certain sets $S \subseteq val(V)$.

[11] A constraint might be formalised by a subset of the "joint range" $\Pi_{V \in \mathbf{V}} ran(V)$ of the family of variables $(V)_{V \in \mathbf{V}}$ (e.g. a subset of $\mathbf{R}^3$ if $\mathbf{V}$ consists of three real-valued variables), or by an expression in a logical language (see below).

agenda: $\mathcal{J}$ can be perfectly symmetric in the two propositions. This is obviously so if X contains only these two logically independent atomic propositions (and their negations). But even additional propositions that create a logical link between the two propositions need not reveal the direction of relevance. For instance, suppose the agenda X contains propositions $a, a \to b, \neg b$, where a and $a \to b$ are relevant to $\neg b$ (and plausibly also to b), but not vice versa. The asymmetry in relevance between a and $\neg b$ is not reflected in their logical interconnections, which are symmetric in a and $\neg b$ (assuming X contains just the three propositions and their negations).[12] Hence any relevance relation defined based on logical interconnections would have to declare a and $\neg b$ as relevant to each other or irrelevant to each other.[13]

So relevance must be taken on board as an additional structure. The following definition delimits what may count as a relevance relation.

Definition 1 (a) *A set $A \subseteq X$* settles *$p \in X$ if $A \vdash p$ or $A \vdash \neg p$.*
(b) *A binary relation $\mathcal{R}$ on X is a* relevance relation *(where "$r\mathcal{R}p$" means "r is relevant to p") if every $p \in X$ is settled by every consistent set A that contains either r or $\neg r$ for every proposition $r \in X$ with $r\mathcal{R}p$.*

So $\mathcal{R}$ is a relevance relation if the truth value of any $p \in X$ is determined by the truth values of the propositions relevant to p. The latter requirement seems indeed a plausible (necessary) condition on the notion of "relevance". For instance, $\mathcal{R}$ is a relevance relation if $\mathcal{R}$ is reflexive, i.e. if every proposition is self-relevant. But $\mathcal{R}$ need not be reflexive: for instance, one may assume that to a conjunction $p \wedge q$ only the conjuncts p and q are relevant, not $p \wedge q$ itself.

In the following, let $\mathcal{R}$ be an arbitrary relevance relation, and let $\mathcal{R}(p)$ denote the set $\{r \in X : r\mathcal{R}p\}$ of propositions relevant to $p \in X$. The following condition requires the collective judgment on any proposition $p \in X$ to be formed on the basis of how the individuals judge the propositions relevant to p.

Independence of Irrelevant Propositions (IIP). For all propositions $p \in X$ and all profiles $(A_1, ..., A_n)$ and $(A_1^*, ..., A_n^*)$ in the domain, if $A_i \cap \mathcal{R}(p) = A_i^* \cap \mathcal{R}(p)$ for every individual i then $p \in F(A_1, ..., A_n) \Leftrightarrow p \in F(A_1^*, ..., A_n^*)$.

This constraint on aggregation strongly depends on how relevance is specified. Roughly, the more propositions are relevant to each other, the weaker IIP is. IIP is empty if all propositions are relevant to all propositions, i.e., if

[12] That is, a truth-value assignment $(t_1, t_2, t_3) \in \{T, F\}^3$ to the propositions $a, a \to b, \neg b$ is consistent if and only if (t_3, t_2, t_1) (in which the truth-values of a and $\neg b$ are interchanged) is consistent. This is so whether $a \to b$ represents a *subjunctive* or a *material* implication. In the first case, the only inconsistent truth-value assignment is (T, T, T). In the second case, there are other inconsistent truth-value assignments (as $a \to b$ is equivalent to $\neg a \vee b$), yet without breaking the symmetry between a and $\neg b$.

[13] One might extract the intended asymmetric relevance structure from the *syntax* of the propositions in X. A syntactically founded relevance definition is plausible for some but not all agendas.

$\mathcal{R} = X \times X$. IIP is the standard proposition-wise independence condition if each proposition is just self-relevant, i.e., $\mathcal{R}(p) = \{p\}$ for all $p \in X$. IIP is Gärdenfors' "weak" (yet still quite strong) independence if $\mathcal{R}(p) = \{p, \neg p\}$ for all $p \in X$. IIP is Dietrich's (2006) independence restricted to a subset $Y \subseteq X$ if $\mathcal{R}(p) = \{p\}$ for $p \in Y$ and $\mathcal{R}(p) = X$ for $p \in X \backslash Y$. IIP is Mongin's (2005) independence restricted to the atomic propositions (of an agenda X in a propositional language) if $\mathcal{R}(p) = \{p\}$ for p atomic and $\mathcal{R}(p) = X$ for p compound (e.g. $p = a \wedge \neg b$).

I now discuss further examples of relevance relations. In these examples, I assume that relevance does not distinguish between a proposition and its negation:

$$p\mathcal{R}q \Leftrightarrow \tilde{p}\mathcal{R}\tilde{q} \text{ for all } p, q \in X \text{ and all } \tilde{p} \in \{p, \neg p\}, \, \tilde{q} \in \{q, \neg q\}. \tag{1}$$

So $\mathcal{R}$ is given by its restriction on the set $X^+ \subseteq X$ of non-negated propositions.

Example 1 (continued). I assume throughout that for the preference agenda X relevance is defined by

$$\mathcal{R}(xRy) = \{xRy, \neg xRy, yRx, \neg yRx\} \text{ for all } xRy \in X, \tag{2}$$

i.e., it is relevant to xRy whether or not xRy and whether or not yRx. Then IIP is equivalent to Arrow's *independence of irrelevant alternatives* ("IIA"), whereas the standard proposition-wise independence condition is stronger than IIA.

Example 2 (continued). For the agenda of Example 2, one might put

$$\begin{array}{ll} \mathcal{R}(V = v) = \{V = v', \neg(V = v') : v' \in ran(V)\} & \text{for all } V = v \in X \\ \mathcal{R}(c) = \{c, \neg c\} & \text{for all constraints } c \in \mathbf{C}. \end{array} \tag{3}$$

On a modified assumption, some distinct constraints $c, c' \in \mathbf{C}$ could be declared relevant to each other, for instance if they concern the same variables.

Example 3: relevance as an equivalence relation, and non-binary issues. Examples 1 and 2 are instances of the general case where relevance is an equivalence relation, that is, $\mathcal{R}$ is reflexive (requiring self-relevance), symmetric, and transitive. Each of these three conditions is a substantial assumption on the notion of relevance. The agenda X is then partitioned into equivalence classes (of inter-relevant propositions), each one interpretable as an "issue"; and IIP is an issue-wise independence condition. An issue may be binary (of the form $\{p, \neg p\}$) or non-binary. For the preference agenda (Example 1), an issue takes the form $\{xRy, \neg xRy, yRx, \neg yRx\}$ (for options $x, y \in Q$): the issue of how to rank x relative to y. Rubinstein and Fishburn's (1986) algebraic aggregation problem can be viewed as one of the present type, where each issue consists in specifying a value in an algebraic field.[14]

Example 4: relevance as premisehood, and generalised premise-based rules. If we interpret "$r\mathcal{R}p$" as "r is a premise/reason/argument for (or against) p", IIP is the condition that the aggregation rule be *premise-based*: that the collective judgment on any proposition $p \in X$ be determined by how the individuals judge p's premises.

In principle, $\mathcal{R}$ could define an arbitrarily complex premisehood structure over a possibly complex agenda X, generalising the so-called *premise-based procedure* usually defined for simple agendas like $X^+ = \{a, b, a \wedge b\}$. For the latter agenda, this procedure decides each "premise" a and b by a (majority) vote, and decides $a \wedge b$ by logical entailment from the decisions on a and b; which satisfies IIP for the following relevance ("premisehood") relation:

$$\text{the only inter-relevances within } X^+ \text{ are } a\mathcal{R}a,\ b\mathcal{R}b,\ a\mathcal{R}a \wedge b,\ b\mathcal{R}a \wedge b. \qquad (4)$$

Call $p \in X$ a *root* proposition if p has no premise other than p (and $\neg p$). In (4), a and b are root propositions. Any root proposition $p \in X$ must be a premise to itself: otherwise p would have no premises at all, violating the definition of a relevance relation.[15] So the collective judgment on any root proposition p is (by IIP) formed solely on the basis of people's judgments on p via some voting method – majority voting if we stick close to the standard premise-based procedure – whereas decisions on non-root propositions may depend on external premises.

When interpreting $\mathcal{R}$ as a premisehood relation, additional requirements on $\mathcal{R}$ may be appropriate. Surely, symmetry should not be required (unlike in Examples 1-3). Indeed, one might require that $\mathcal{R}$ is *anti-symmetric* on X^+ (to prevent distinct propositions in X^+ from being premises to each other) or, more strongly, *acyclic* on X^+: there are no pairwise distinct $p_1, ..., p_m \in X^+$ ($m \geq 2$) with $p_1\mathcal{R}p_2\mathcal{R}p_3\mathcal{R}...\mathcal{R}p_m\mathcal{R}p_1$.[16]

Example 5: relevance as a transitive relation, and sequential priority rules. In Example 4, there may not *exist* any non-degenerate premise-based

[14] Their aggregation problem is also comparable to a variant of Example 2, in which the inter-variable constraints are exogenously imposed (rather than under decision), and all variables range over the same algebraic field $\mathcal{F}$ (e.g. $\mathbf{R}$); so the agenda is $X = \{V = v, \neg(V = v) : V \in \mathbf{V} \text{ and } v \in \mathcal{F}\}$, and the issues (equivalence classes) are the sets $\{V = v, \neg(V = v) : v \in \mathcal{F}\}$, $V \in \mathbf{V}$. Their two main theorems establish, in a different framework, correspondences between algebraic properties (like being a *hyperplane*) of the set $\mathcal{J}$ of allowed judgment sets (represented as a subset B of the $\mathcal{F}$-vector space $\mathcal{F}^{\mathbf{V}}$) and algebraic properties (like *linearity* or *additivity*) of "admissible" aggregation rules F (represented as mappings from B^n to B). Although the algebraic conditions on $\mathcal{J}$ or F exclude many applications, they apply notably to *probability aggregation* problems.

[15] Unless p is a tautology or contradiction: then even the empty set settles p.

[16] Or one might require only that the asymmetric part of $\mathcal{R}$ is acyclic.

(i.e., IIP) aggregation rule $F : \mathcal{J}^n \to \mathcal{J}$. An impossibility threat comes not only from logical interconnections between root propositions (or other propositions), but also from transitivity violations of relevance $\mathcal{R}$. To see why, let $p \in X$ and suppose the premises of p's premises – call them the "pre-premises" – are *not* premises of p. The decision on p is settled by the decisions on p's premises (as $\mathcal{R}$ is a relevance relation), which in turn depend (by IIP) on how people judge the pre-premises. This forces the decision on p to be some function f of how people judge the pre-premises. But by IIP the decision on p must be a function of how people judge p's premises (not pre-premises). So f depends on people's pre-premise judgments only indirectly: only through people's premise judgments as entailed by their pre-premise judgments – a strong restriction on f that suggests impossibility is looming.

It is debatable whether premisehood (more generally, relevance) is inherently a transitive concept. If $\mathcal{R}$ is assumed transitive – whether for conceptual reasons or just to remove one impossibility source – an interesting potential candidates for IIP aggregation arises, to be explained now. List (2004) introduced *sequential priority rules* in judgment aggregation (generalising sequential rules in standard social theory). Here the propositions of a (finite) agenda are put in a priority order $p_1, ..., p_k$ ($k = |X|$) and decided sequentially, where earlier decision logically constrain later ones. As is easily seen, such a rule is IIP if relevance is defined by $p_j \mathcal{R} p_l \Leftrightarrow j \leq l$ for all $j, l = 1, ..., k$. I now introduce similar rules relative to an arbitrary relevance relation, and I show that they are IIP provided that relevance is transitive (and well-founded). Informally, these rules decide the propositions sequentially in the order of relevance: each $p \in X$ is decided by logical entailment from previously accepted relevant propositions *except* if the latter propositions do not settle p, in which case p is decided via some local decision method (e.g. via majority voting on p). Formally, F is a *sequential priority rule* if F has universal domain and for every proposition $p \in X^+$ there is a an aggregation rule D_p for the binary agenda $\{p, \neg p\}$ (satisfying universal domain and collective rationality) such that, for all profiles $(A_1, \ldots, A_n) \in \mathcal{J}^n$,

$$F(A_1, ..., A_n) \cap \{p, \neg p\} = \begin{cases} \{\tilde{p} \in \{p, \neg p\} : F(A_1, ..., A_n) \cap & \text{if this set is} \\ \qquad\qquad (\mathcal{R}(p) \backslash \{p, \neg p\}) \vdash \tilde{p}\} & \text{non-empty} \\ D_p(A_1 \cap \{p, \neg p\}, ..., A_n \cap \{p, \neg p\}) & \text{otherwise.} \end{cases} \tag{5}$$

So the rule first decides for every root proposition $p \in X^+$ which of $p, \neg p$ to accept, based on which of $p, \neg p$ the individuals accept. Then it turns to any non-root proposition $p \in X$ to which only root proposition are relevant: if the previous decisions on the root propositions relevant to p entail p or $\neg p$, the entailed proposition(s) is (are) accepted (hence "sequential priority"); otherwise a local vote is taken on p. And so on.

Sequential priority rules can be constructed by (i) specifying the family $(D_p)_{p \in X^+}$ of ("local") aggregation rules (for instance the same rule D_p for all $p \in X^+$), and (ii) applying formula (5) recursively. This uniquely defines the sequential priority rule $F \equiv F_{(D_p)_{p \in X^+}}$ (by the transfinite recursion theorem[17]) provided that relevance $\mathcal{R}$ is a *well-founded* relation on X^+: every non-empty set $S \subseteq X^+$ has an $\mathcal{R}$-minimal element s (i.e., for no $r \in S \backslash \{s\}$, $r\mathcal{R}s$); or, more intuitively, there is no infinite sequence $p_1, p_2, ... \in X^+$ such that for all p_k the successor p_{k+1} is relevant to (and distinct from) p_k. This exclusion of "infinite relevance chains" is again a debatable assumption on the notion of relevance.

We obtain the desired IIP property by a (transfinite) inductive argument:

Proposition 1 *If relevance $\mathcal{R}$ is transitive and well-founded on X^+ and satisfies* (1)*, every sequential priority rule $F_{(D_p)_{p \in X^+}}$ satisfies IIP.*[18]

Proof. Let $\mathcal{R}$ be as specified, and let $F := F_{(D_p)_{p \in X^+}}$ be a sequential priority rule. To show IIP, I prove that all $p \in X^+$ satisfy the following: for all $(A_1, ..., A_n), (A_1^*, ..., A_n^*) \in \mathcal{J}^n$, if $A_i \cap \mathcal{R}(p) = A_i^* \cap \mathcal{R}(p)$ for all i then

$$F(A_1, ..., A_n) \cap \{p, \neg p\} = F(A_1^*, ..., A_n^*) \cap \{p, \neg p\} \neq \emptyset. \tag{6}$$

Suppose for a contradiction that the latter fails for some $p \in X^+$. By well-foundedness, there exists an $\mathcal{R}$-minimal $p \in X^+$ for which it fails. So there are $(A_1, ..., A_n), (A_1^*, ..., A_n^*) \in \mathcal{J}^n$ with $A_i \cap \mathcal{R}(p) = A_i^* \cap \mathcal{R}(p)$ for all i such that (6) is false. By p's minimality property and $\mathcal{R}$'s transitivity,

$$F(A_1, ..., A_n) \cap (\mathcal{R}(p) \backslash \{p, \neg p\}) = F(A_1^*, ..., A_n^*) \cap (\mathcal{R}(p) \backslash \{p, \neg p\}). \tag{7}$$

Let $Y := \{\tilde{p} \in \{p, \neg p\} : \text{the set (7) entails } \tilde{p}\}$.

Case 1: $Y \neq \emptyset$. Then, by the first case in (5), $F(A_1, ..., A_n) \cap \{p, \neg p\} = Y$, and for the same reason $F(A_1^*, ..., A_n^*) \cap \{p, \neg p\} = Y$. This implies (6), contradicting the choice of p.

Case 2: $Y = \emptyset$. Then $p, \neg p \in \mathcal{R}(p)$: otherwise $p, \neg p \notin \mathcal{R}(p)$ by (1), implying that the set (7) contains a member of each pair $r, \neg r \in \mathcal{R}(p)$ (using p's minimality property), hence entails p or $\neg p$ (as $\mathcal{R}$ is a relevance relation), which contradicts that $Y = \emptyset$. By $p, \neg p \in \mathcal{R}(p)$, we have $A_i \cap \{p, \neg p\} = A_i^* \cap \{p, \neg p\}$ for all i. So, as for both profiles the second case in (5) applies (by $Y = \emptyset$), $F(A_1, ..., A_n) \cap \{p, \neg p\}$ and $F(A_1^*, ..., A_n^*) \cap \{p, \neg p\}$ both equal the same (non-empty) set generated by D_p. So (6) holds, contradicting the choice of p. ■

4 A unanimity condition

It would not be in the spirit of a relevance-based approach to require for *every* $p \in X$ that a unanimity for p be socially preserved: this would deny that

[17] To be precise, for every $(A_1, ..., A_n) \in \mathcal{J}^n$, $F(A_1, ..., A_n)$ is the union of the sets $f(p) := F(A_1, ..., A_n) \cap \{p, \neg p\}$, $p \in X^+$, where the function f is defined on X^+ by recursion on $\mathcal{R}$.

[18] Of course, IIP is not the only desideratum on $F_{(D_p)_{p \in X^+}}$. One might investigate for which types of logical and relevance interconnections there exist (non-degenerate) local decision methods D_p, $p \in X^+$, such that the outcomes of $F_{(D_p)_{p \in X^+}}$ are consistent (hence in $\mathcal{J}$).

disagreements on other propositions relevant to p matter. I will impose a weaker condition unanimity preservation, restricted to propositions $p \in X$ such that a unanimity for p excludes disagreements on propositions relevant to p. A special case will be Arrow's *weak Pareto principle*, which is also a restricted unanimity preservation condition.

Formally, any set $A \subseteq \{r, \neg r : r \in \mathcal{R}(p)\}$ (where $p \in X$) is called a *p-relevant judgment set*, as it represents judgments on propositions relevant to p; and $\mathcal{R}_p$ denotes the set of all p-relevant judgment sets that are consistent with p and complete (i.e. contain for each $r \in \mathcal{R}(p)$ either r or $\neg r$). Each $A \in \mathcal{R}_p$ actually *entails* p, as $\mathcal{R}$ is a relevance relation. The sets in $\mathcal{R}_p$ represent those truth-value assignments to the propositions relevant to p for which p rather than $\neg p$ follows. If $\mathcal{R}(p) = \{p\}$ (only p is relevant to p), $\mathcal{R}_p$ contains only $\{p\}$ (or is empty if p is a contradiction). But usually $\mathcal{R}_p$ is more complex.[19]

In the following, let $\mathcal{P}$ be a fixed set of ("*privileged*") propositions $p \in X$ for which $|\mathcal{R}_p| = 1$. If all individuals hold $p \in \mathcal{P}$, they cannot (rationally) disagree on propositions relevant to p, as they must all hold the single set in $\mathcal{R}_p$; and so we may require this unanimity to be preserved socially.

Unanimity Preservation. For every profile $(A_1, ..., A_n)$ in the domain and every privileged proposition $p \in \mathcal{P}$, if $p \in A_i$ for all individuals i then $p \in F(A_1, ..., A_n)$.

I assume throughout that for the preference agenda $\mathcal{P}$ is defined as

$$\mathcal{P} := \{\neg xRy : x, y \in Q, x \neq y\} = \{yPx : x, y \in Q, x \neq y\}, \quad (8)$$

the set of strict ranking propositions; this makes unanimity preservation equivalent to the weak Pareto principle.

If each $p \in X$ has $\mathcal{R}(p) = \{p\}$ (and is not a contradiction), then we may define $\mathcal{P} = X$, so that unanimity preservation applies globally. In the earlier example of $X^+ = \{a, b, a \wedge b\}$ with relevance satisfying (4), $\mathcal{P}$ may contain $a \wedge b$ (as $\mathcal{R}_{a \wedge b}$ contains just $\{a, b\}$) but not $\neg(a \wedge b)$ (as $\mathcal{R}_{\neg(a \wedge b)} = \{\{\neg a, b\}, \{a, \neg b\}, \{\neg a, \neg b\}\}$). So a unanimity for $\neg(a \wedge b)$ can be overruled.

The default specification of $\mathcal{P}$, applicable unless $\mathcal{P}$ is explicitly defined otherwise, is

$$\mathcal{P} = \{p \in X : |\mathcal{R}_p| = 1\}.$$

This maximal choice of $\mathcal{P}$ is often natural (though not necessary for our results). For the preference agenda, (8) is the default specification.[20]

[19] For all xRy in the preference agenda (for $x \neq y$ and relevance given by (2)), $\mathcal{R}_{xRy}$ contains $\{xRy, yRx\}$ ("x and y are indifferent") and $\{xRy, \neg yRx\}$ ("x is better than y"); and $\mathcal{R}_{\neg xRy}$ contains just $\{\neg xRy, yRx\}$. If $b \in X$ and $\mathcal{R}(b) = \{a, a \to b, bb\}$ (where a and b are distinct atomic propositions and $\to$ is material or subjunctive implication), $\mathcal{R}_b$ contains $\{a, a \to b, b\}$, $\{\neg a, a \to b, b\}$ and $\{\neg a, a \to b, \neg b\}$ but not $\{a, a \to b, \neg b\}$. In general, for any $p \in X$, $\mathcal{R}_p$ contains the restrictions of full judgments sets $A \in \mathcal{J}$ containing p to the propositions that are relevant to p or negated relevant to p: $\mathcal{R}_p = \{A \cap \{r, \neg r : r \in \mathcal{R}(p)\} : A \in \mathcal{J} \text{ with } p \in A\}$.

[20] Except that (8) excludes the tautologies $xRx \in X$ from $\mathcal{P}$, but this makes no difference as any aggregation rule $F : \mathcal{J}^n \to \mathcal{J}$ preserves unanimities for tautologies.

5 Semi-vetodictatorship and semi-dictatorship

From now on, we consider an IIP and unanimity preserving aggregation rule $F : \mathcal{J}^n \to \mathcal{J}$, relative to some arbitrary fixed relevance relation $\mathcal{R}$. I show that if the interconnections between propositions in terms of relevance and logic play together in certain ways, F is forced to be of some degenerate form: a (semi-)dictatorship or (semi-)vetodictatorship.

First, how should these degenerate rules be defined? Our relevance-based framework allows us to generalise the standard social-choice-theoretic definitions. Recall that an (Arrowian) "dictator" is an individual who can socially enforce his *strict* preferences between options, but not necessarily his *indifferences.* Similarly, a "vetodictator" can prevent ("veto") any *strict* preference, but not necessarily any indifference. Put in our terminology, a dictator (vetodictator) can enforce (veto) any *privileged* proposition of the preference agenda (see (8)). The following definitions generalise this to arbitrary agendas.

Definition 2 *An individual i is*

(a) *a* dictator *(respectively,* semi-dictator*) if, for every privileged proposition $p \in \mathcal{P}$, $p \in F(A_1, ..., A_n)$ for all $(A_1, ..., A_n) \in \mathcal{J}^n$ such that $p \in A_i$ (respectively, such that $p \in A_i$ and $p \notin A_j$, $j \neq i$);*

(b) *a* vetodictator *(respectively,* semi-vetodictator*) if, for every privileged proposition $p \in \mathcal{P}$, i has a* veto *(respectively,* semi-veto*) on p, i.e. a judgment set $A_i \in \mathcal{J}$ not containing p such that $p \notin F(A_1, ..., A_n)$ for all $A_j \in \mathcal{J}$, $j \neq i$ (respectively, for all $A_j \in \mathcal{J}$, $j \neq i$, containing p).*

In the standard models without a relevance relation, *conditional entailments* between propositions (first used by Nehring and Puppe 2002/2005) have proven useful to understand agendas. Roughly, $p \in X$ conditionally entails $q \in X$ if p together with other propositions in X entails q (where the "other" propositions should not by themselves entail q or $\neg p$). I will not use conditional entailments here, as they reflect only *logical* connections between propositions. Rather, I now define *constrained entailments*, which reflect both logical and relevance connections. It will turn out that certain paths of constrained entailments lead to degenerate aggregation.[21]

Definition 3 *For propositions $p, q \in X$, if $\{p\} \cup Y \vdash q$ for a set $Y \subseteq \mathcal{P}$ consistent with every set in $\mathcal{R}_p$ and every set in $\mathcal{R}_{\neg q}$, I say that p* constrained

[21] Nehring and Puppe (2002/2005) use paths of *conditional* entailment to define their *total blocked* agendas. Total blockedness (and Dietrich and List's (forthcoming) *pathconnectedness* condition) can be obtained from the present path conditions by a particular specification of relevance (see footnote 25). For totally blocked agendas, Nehring and Puppe obtain strong dictatorship by imposing that F satisfies proposition-wise independence, an unrestricted unanimity condition, and a monotonicity condition. Dokow and Holzman (2005) show that monotonicity can be replaced by an algebraic agenda condition. I impose relevance-based conditions on F, and obtain less than strong dictatorship.

entails q (in virtue of Y), *and write* $p \vdash_* q$ *or* $p \vdash_Y q$.[22]

The amount of constrained entailments is crucial for whether impossibilities arise. Trivially, every *unconditional* entailment is also a constrained entailment (in virtue of $Y = \emptyset$). Intuitively, the more inter-relevances there are between propositions, the larger the sets $\mathcal{R}_p$, $p \in X$, tend to be and also the smaller the set of privileged propositions $\mathcal{P}$ tends to be; so the stronger the demands on Y in constrained entailments, and hence the fewer constrained entailments, and the more room for possibility.

For the preference agenda X (Example 1), many constrained entailments (hence impossibilities) arise. For instance, $xRy \vdash_{\{yPz\}} xPz$ (if x, y, z are pairwise distinct), as yPz is in $\mathcal{P}$ and is consistent with all sets in $\mathcal{R}_{xRy}$ and all sets in $\mathcal{R}_{\neg xPz} = \mathcal{R}_{zRy}$ (see footnote 19). By contrast, *no* non-trivial constrained entailments arise in our example $X^+ = \{a, b, a \wedge b\}$ with relevance given by (4) and (1); for instance, it is not the case that $a \vdash_{\{\neg(a\wedge b)\}} \neg b$ since $\neg(a \wedge b) \notin \mathcal{P}$; and it is not the case that $a \vdash_{\{b\}} a \wedge b$, as $\{b\}$ is inconsistent with $\{a, \neg b\} \in \mathcal{R}_{\neg(a\wedge b)}$. As a result, our impossibilities will not apply to this agenda – and cannot, as the premise-based procedure for odd n (see Example 4) satisfies all conditions.

To obtain impossibility results, richness in constrained entailments is not sufficient. At least one constrained entailment $p \vdash_* q$ must hold in a "truly" constrained sense. By this I mean more than that p does not *un*conditionally entail q, i.e. more than that p is consistent with $\neg q$: I mean that every p-relevant judgment set $A \in \mathcal{R}_p$ is consistent with every $\neg q$-relevant judgment set $B \in \mathcal{R}_{\neg q}$.

Definition 4 *For propositions* $p, q \in X$, p truly constrained entails q *if* $p \vdash_* q$ *and moreover every* $A \in \mathcal{R}_p$ *is consistent with every* $B \in \mathcal{R}_{\neg q}$.

For instance, if relevance is an equivalence relation (as in Example 3) that partitions X into *pairwise logically independent* subagendas[23] (as for the preference agenda) then all constrained entailments across equivalence classes are truly constrained. Also, $p \vdash_* q$ is truly constrained if $p \nvdash q$ and moreover $\mathcal{R}(p) = \{p\}$ and $\mathcal{R}(\neg q) = \{\neg q\}$ (e.g. p might be "basic premises" in Example 4 or 5).

Our impossibility results rest on path conditions of the following kind.

Definition 5 (a) *For propositions* $p, q \in X$, *if* X *contains propositions* $p_1, ..., p_m$ ($m \geq 2$) *with* $p = p_1 \vdash_* p_2 \vdash_* ... \vdash_* p_m = q$, *I write* $p \vdash\vdash q$; *if moreover one of these constrained entailments is truly constrained, I write* $p \vdash\vdash_{\text{true}} q$.

[22] Many alternative notions of constrained entailment turn out to be non-suitable: they do not preserve interesting properties along paths of constrained entailments. The present definition is the weakest one to preserve semi-winning coalitions. The requirement that $Y \subseteq \mathcal{P}$ allows one to apply unanimity preservation. In view of different results to those derived here, it might be fruitful to impose additional requirements on Y, e.g. that Y be consistent also with sets in $\mathcal{R}_{\neg p}$ and/or in $\mathcal{R}_q$.

[23] That is, if X_1, X_2 are distinct subagendas, $A \cup B$ is consistent for all consistent $A \subseteq X_1, B \subseteq X_2$.

(b) *A set* $Z \subseteq X$ *is* pathlinked (in X) *if* $p \vdash\vdash q$ *for all* $p, q \in Z$, *and* truly pathlinked (in X) *if moreover* $p \vdash\vdash_{\text{true}} q$ *for some (hence all)* $p, q \in Z$.

While pathlinkedness forces to a (limited) form of *neutral* aggregation (see Lemma 3), true pathlinkedness forces to the following degenerate aggregation rules.

Theorem 1 *If the set* $\mathcal{P}$ *of privileged propositions is inconsistent and truly pathlinked, there is a semi-vetodictator.*

Theorem 2 *If the set* $\{p, \neg p : p \in \mathcal{P}\}$ *of privileged or negated privileged propositions is truly pathlinked, there is a semi-dictator.*

In the present (and all later) theorems, the qualification "truly" can be dropped if relevance is restricted to taking a form for which pathlinkedness (of the set in question) implies true pathlinkedness, for instance if $\mathcal{R}$ is restricted to being an equivalence relation that partitions X into logically independent subagendas.[24]

Under the conditions of Theorems 1 and 2, there may be more than one semi-(veto)dictator, and moreover there need not exist any (veto)dictator.[25]

Many applications are imaginable. The preference agenda (Example 1) is discussed later. If in Example 4 or 5 we define $\mathcal{P}$ as the set of root propositions, and if these root propositions are interconnected in the sense of Theorem 1 (2), then some individual is semi-(veto)decisive on all "fundamental issues"; and hence, premise-based or sequential prioritarian aggregation rules take a degenerate form (at least with respect to the local decision methods D_p for root propositions $p \in X$). Let me discuss Example 2 in more detail.

Example 2 (continued) For many instances of this aggregation problem (of judging values of and constraints between variables), the conditions of Theorems 1 and 2 hold, so that semi-(veto)dictatorships are the only solutions. To make

[24] The argument for the latter is as follows. By Lemma 1, all constrained entailments *within* any of the subagendas are unconditional. This implies that the pathlinked set in question contains propositions linked by a path containing a constrained entailment *across* subagendas. The latter is truly conditional by the earlier remark.

[25] Suppose $\mathcal{R}(p) = p$ for all $p \in X$ (only *self*-relevance allowed), $\mathcal{P} = X$ (all propositions privileged), and $|X| < \infty$. Then constrained entailment reduces to simple conditional entailment, and hence pathlinkedness of X reduces to Nehring and Puppe's (2002/2005) *total blockedness* condition whereby there is a path of *conditional* entailments between any $p, q \in X$. Dokow and Holzman (2005) show that any *parity rule*, defined on $\mathcal{J}^n$ by $F(A_1, ..., A_n) = \{p \in X : |\{i \in M : p \in A_i\}| \text{ is odd}\}$ where $M \subseteq N$ has odd-sized subgroup, takes values in $\mathcal{J}$ for certain agendas X that are totally blocked (hence pathlinked, in fact *truly* pathlinked) and satisfy an algebraic condition. Such a parity rule is also IIP and unanimity preserving, and hence provides the required counterexample because every $i \in M$ a semi-dictator and a semi-vetodictator, but not a dictator and not a vetodictator (unless $|M| = 1$).

this point, let relevance be again given by (3), and let the privileged propositions be given by

$$\mathcal{P} = \{V = v : V \in \mathbf{V} \& v \in ran(V)\} \cup \{c, \neg c : c \in \mathbf{C}\}. \tag{9}$$

Also, let $\mathbf{V}$ contain more than one variable (to make it interesting), and assume[26]

$$\{\neg c : c \in \mathbf{C}\} \notin \mathcal{J}^*; \tag{10}$$

First, consider Theorem 1. Obviously, $\mathcal{P}$ is inconsistent. Often, $\mathcal{P}$ is also truly pathlinked. The latter could be shown by establishing that

(a) $\mathcal{P}_1 := \{V = v : V \in \mathbf{V} \& v \in ran(V)\}$ is truly pathlinked, and

(b) for all $c \in \mathbf{C}$ there are $p, q, r, s \in \mathcal{P}_1$ with $c \vdash_* p$, $q \vdash_* c$, $\neg c \vdash_* r$, $s \vdash_* \neg c$.

Part (a) might even hold in the sense of, for every $V = v, V' = v' \in \mathcal{P}_1$ with $V \neq V'$, a truly constrained entailment $V = v \vdash_* V' = v' \in \mathcal{P}_1$ (rather than an indirect path $V = v \vdash\vdash V' = v'$); indeed, there might be a set of constraints $C \subseteq \mathbf{C}$ and a set of value assignments $D \subseteq \mathcal{P}_1$ such that $V = v \vdash_{C \cup D} V' = v'$ (hence, under the constraints in C, the set of value assignments $\{V = v\} \cup D$ implies that $V' = v'$).

Part (b) might hold for the following reasons. Consider a constraint $c \in \mathbf{C}$. Plausibly, $V = v \vdash_D \neg c$ for some $V = v \in \mathcal{P}_1$ and $D \subseteq \mathcal{P}_1$; here, $\{V = v\} \cup D$ is a set of value assignments violating the constraint c. It is also plausible that $c \vdash_D V = v$ for some $V = v \in \mathcal{P}_1$ and some $D \subseteq \mathcal{P}_1$; here, the value assignments in D imply, under the constraint c, that $V = v$. Moreover, we may have $V = v \vdash_D c$ for some $V = v \in \mathcal{P}_1$ and $D \subseteq \mathcal{P}_1$: this is so if the set of value assignments $\{V = v\} \cup D$ violates all constraints in C except c, hence entails c by (10). Finally, we may have $\neg c \vdash_{\tilde{C} \cup D} V = v$ for some $V = v \in \mathcal{P}_1$ and $D \subseteq \mathcal{P}_1$, and some set $\tilde{C}$ of negated constraints; indeed, suppose $\{\neg c\} \cup \tilde{C}$ contains the negations of all except of one constraint in $\mathbf{C}$, hence entails the remaining constraint by (10); then, under the remaining constraint, the value assignments in D might imply that $V = v$.

Now consider Theorem 2. The special form (9) of $\mathcal{P}$ in fact implies that the conditions of Theorem 2 hold whenever those of Theorem 1 hold (hence in many cases, as argued above). Specifically, let $\mathcal{P}$ be truly pathlinked. To prove that also $\{p, \neg p : p \in \mathcal{P}\}$ is truly pathlinked, it suffices to show that, for all $V = v \in \mathcal{P}$, there is a $p \in \mathcal{P}$ with $\neg(V = v) \vdash\vdash p$ and $p \vdash\vdash \neg(V = v)$. Consider any $V = v$, and choose any $p \in \mathbf{C}$ $(\subseteq \mathcal{P})$. As $\mathcal{P}$ is pathlinked and by (9) contains $\neg p$ and $V = v$, we have $\neg p \vdash\vdash V = v$ and $V = v \vdash\vdash \neg p$; hence (using Lemma 4 below) $\neg(V = v) \vdash\vdash p$ and $p \vdash\vdash \neg(V = v)$, as desired.

I now derive lemmas that will help both prove the theorems and understand constrained entailment. I first give a sufficient condition for a constrained entailment to reduce to an unconditional one.

[26] Condition (10) requires that *at least one* constraint between variables holds, i.e. that the variables are not totally independent from each other. This assumption can be natural when the question is not *whether* but only *how* the variables affect each other, as it is the case for macroeconomic variables.

Lemma 1 *For all $p, q \in X$ with $\mathcal{R}(p) \subseteq \mathcal{R}(\neg q)$ or $\mathcal{R}(\neg q) \subseteq \mathcal{R}(p)$, $p \vdash_* q$ if and only if $p \vdash q$.*

Proof. Let p, q be as specified. Obviously, $p \vdash q$ implies $p \vdash_\emptyset q$. Suppose for a contradiction that $p \vdash_* q$, say $p \vdash_Y q$, but $p \nvdash q$. Then $\{p, \neg q\}$ is consistent. So there is an $B \in \mathcal{J}$ containing p and $\neg q$. Then

- the set $B \cap \{r, \neg r : r\mathcal{R}p\}$ is consistent with p, so is in $\mathcal{R}_p$;
- the set $B \cap \{r, \neg r : r\mathcal{R}\neg q\}$ is consistent with $\neg q$, so is in $\mathcal{R}_{\neg q}$.

One of these two sets is a superset of the other one, as $\mathcal{R}(p) \subseteq \mathcal{R}(\neg q)$ or $\mathcal{R}(\neg q) \subseteq \mathcal{R}(p)$; call this superset A. As $p \vdash_Y q$, $A \cup Y$ is consistent. So, as $A \vdash p$ and $A \vdash \neg q$, $\{p, \neg q\} \cup Y$ is consistent. So $\{p\} \cup Y \nvdash q$, a contradiction. ■

The next fact helps in choosing the set Y in a constrained entailment.

Lemma 2 *For all $p, q \in X$, if $p \vdash_* q$ then $p \vdash_Y q$ for some set Y containing no proposition relevant to p or to $\neg q$.*

Proof. Let $p, q \in X$, and assume $p \vdash_* q$, say $p \vdash_Y q$. The proof is done by showing that $p \vdash_{Y \backslash (\mathcal{R}(p) \cup \mathcal{R}(\neg q))} q$. Suppose for a contradiction that not $p \vdash_{Y \backslash (\mathcal{R}(p) \cup \mathcal{R}(\neg q))} q$. Then

(*) $\{p, \neg q\} \cup Y \backslash (\mathcal{R}(p) \cup \mathcal{R}(\neg q))$ is consistent.

I show that

(**) $p \vdash p'$ for all $p' \in Y \cap \mathcal{R}(p)$ and $\neg q \vdash q'$ for all $q' \in Y \cap \mathcal{R}(\neg q)$,

which together with (*) implies that $\{p, \neg q\} \cup Y$ is consistent, a contradiction since $p \vdash_Y q$. Suppose for a contradiction that $p' \in Y \cap \mathcal{R}(p)$ but $p \nvdash p'$. Then there is a $B \in \mathcal{J}$ containing p and $\neg p'$. The set $A := B \cap \{r, \neg r : r\mathcal{R}p\}$ does not entail $\neg p$, hence entails p (as $\mathcal{R}$ is a relevance relation), i.e. $A \in \mathcal{R}_p$. So $A \cup Y$ is consistent (as $p \vdash_Y q$), a contradiction since $A \cup Y$ contains both p' and $\neg p'$. For analogous reasons, for all $q' \in Y \cap X^l$ it cannot be that $\neg q \nvdash q'$. ■

Now I introduce a notion of (semi-)decisive coalitions and show that semi-decisiveness is preserved along paths of constrained entailments.

Definition 6 *A possibly empty coalition $C \subseteq N$ is* decisive *(respectively,* semi-decisive*) for $p \in X$ if its members have judgment sets $A_i \in \mathcal{J}$, $i \in C$, containing p, such that $p \in F(A_1, ..., A_n)$ for all $A_i \in \mathcal{J}$, $i \in N \backslash C$ (respectively, for all $A_i \in \mathcal{J}$, $i \in N \backslash C$, not containing p).*

While a decisive coalition for p can (by appropriate judgment sets) always socially enforce p, a semi-decisive coalition can do so provided all other individuals reject p. Let $\mathcal{W}(p)$ and $\mathcal{C}(p)$ be the sets of decisive and semi-decisive coalitions for $p \in X$, respectively.

Lemma 3 *For all $p, q \in X$, if $p \vdash_* q$ then $\mathcal{C}(p) \subseteq \mathcal{C}(q)$. In particular, if $Z \subseteq X$ is pathlinked, all $p \in Z$ have the same semi-decisive coalitions.*[27]

[27] While constrained entailment preserves semi-decisiveness, it does not preserve decisiveness.

Proof. Suppose $p, q \in X$, and $p \vdash_* q$, say $p \vdash_Y q$, where by Lemma 2 w.l.o.g. $Y \cap \mathcal{R}(p) = Y \cap \mathcal{R}(\neg q) = \emptyset$. Let $C \in \mathcal{C}(p)$. So there are sets $A_i \in \mathcal{J}$, $i \in C$, containing p, such that $p \in F(A_1, ..., A_n)$ for all $A_i \in \mathcal{J}$, $i \in N \backslash C$, containing $\neg p$. By Y's consistency with every $A \in \mathcal{R}_p$, it is possible to change each A_i, $i \in C$, into a set (still in $\mathcal{J}$) that contains every $y \in Y$ and has the same intersection with $\mathcal{R}(p)$ as A_i; this change preserves the required properties, i.e. it preserves that $p \in A_i$ for all $i \in C$ (as $\mathcal{R}$ is a relevance relation), and preserves that $p \in F(A_1, ..., A_n)$ for all $A_i \in \mathcal{J}$, $i \in N \backslash C$, containing $\neg p$ (by $Y \cap \mathcal{R}(p) = \emptyset$ and IIP). So we may assume w.l.o.g. that $Y \subseteq A_i$ for all $i \in C$. Hence, by $\{p\} \cup Y \vdash q$, all A_i, $i \in C$, contain q.

To establish that $C \in \mathcal{C}(q)$, I consider any sets $A_i \in \mathcal{J}$, $i \in N \backslash C$, all containing $\neg q$, and I show that $q \in F(A_1, ..., A_n)$. We may assume w.l.o.g. that $Y \subseteq A_i$ for all $i \in N \backslash C$, by an argument like the one above (using that Y is consistent with all $A \in \mathcal{R}_{\neg q}$, $\mathcal{R}$ is a relevance relation, $Y \cap \mathcal{R}(\neg q) = \emptyset$, and IIP). As $\{\neg q\} \cup Y \vdash \neg p$, all A_i, $i \in N \backslash C$, contain $\neg p$. Hence $p \in F(A_1, ..., A_n)$. Moreover, $Y \subseteq F(A_1, ..., A_n)$ by $Y \subseteq \mathcal{P}$. So, as $\{p\} \cup Y \vdash q$, $q \in F(A_1, ..., A_n)$, as desired. ■

I now prove the two theorems, after stating a last (obvious) lemma.

Lemma 4 (contraposition) *For all $p, q \in X$ and all $Y \subseteq \mathcal{P}$, $p \vdash_Y q$ if and only if $\neg q \vdash_Y \neg p$.*

Proof of Theorem 1. Let $\mathcal{P}$ be inconsistent and truly pathlinked. I first prepare the proof by establishing three simple claims.

Claim 1. (i) The set $\mathcal{C}(p)$ is the same for all $p \in \mathcal{P}$; call it $\mathcal{C}_0$. (ii) The set $\mathcal{C}(\neg p)$ is the same for all $p \in \mathcal{P}$.

Part (i) follows from Lemma 3. Part (ii) follows from it too because, by Lemma 4, $\{\neg p : p \in \mathcal{P}\}$ is like $\mathcal{P}$ pathlinked, q.e.d.

Claim 2. $\emptyset \notin \mathcal{C}_0$ and $N \in \mathcal{C}_0$.

By unanimity preservation, $N \in \mathcal{C}_0$. Suppose for a contradiction that $\emptyset \in \mathcal{C}_0$. Consider any judgment set $A \in \mathcal{J}$. Then $F(A, ..., A)$ contains all $p \in \mathcal{P}$, by $N \in \mathcal{C}_0$ if $p \in A$, and by $\emptyset \in \mathcal{C}_0$ if $p \notin A$. Hence $F(A, ..., A)$ is inconsistent, a contradiction, q.e.d.

By Claim 2, there is a minimal coalition C in $\mathcal{C}_0$ (with respect to inclusion), and $C \neq \emptyset$. By $C \neq \emptyset$, there is a $j \in C$. Write $C_{-j} := C \backslash \{j\}$. As $\mathcal{P}$ is truly pathlinked, there exist $p \in \mathcal{P}$ and $r, s \in X$ such that $p \vdash\vdash r$, $r \vdash_* s$ truly, and $s \vdash\vdash p$.

Claim 3. $\mathcal{C}(r) = \mathcal{C}(s) = \mathcal{C}_0$; hence $C \in \mathcal{C}(r)$ and $C_{-j} \notin \mathcal{C}(s)$.

By Lemma 3, $\mathcal{C}(p) \subseteq \mathcal{C}(r) \subseteq \mathcal{C}(s) \subseteq \mathcal{C}(p)$. So $\mathcal{C}(r) = \mathcal{C}(s) = \mathcal{C}(p) = \mathcal{C}_0$, q.e.d.

Now let Y be such that $r \vdash_Y s$, where by Lemma 2 w.l.o.g. $Y \cap \mathcal{R}(r) = Y \cap \mathcal{R}(\neg s) = \emptyset$. By $C \in \mathcal{C}(r)$, there are judgment sets $A_i \in \mathcal{J}$, $i \in C$, containing r, such that $r \in F(A_1, ..., A_n)$ for all $A_i \in \mathcal{J}$, $i \in N \backslash C$, not containing r. I assume w.l.o.g. that

$$\text{for all } i \in C_{-j},\ Y \subseteq A_i, \text{ hence (by } \{r\} \cup Y \vdash s) \ s \in A_i, \tag{11}$$

which I may do by an argument like that in the proof of Lemma 3 (using that Y is consistent with all $A \in \mathcal{R}_r$, $\mathcal{R}$ is a relevance relation, $Y \cap \mathcal{R}(r) = \emptyset$, and IIP). By (11) and as $C_{-j} \notin \mathcal{C}(s)$ (see Claim 3), there are sets $B_i \in \mathcal{J}$, $i \in N \backslash C_{-j}$, containing $\neg s$, such that, writing $B_i := A_i$ for all $i \in C_{-j}$,

$$\neg s \in F(B_1, ..., B_n). \tag{12}$$

I may w.l.o.g. modify the sets B_i, $i \in N \backslash C_{-j}$, into new sets in $\mathcal{J}$ as long as their intersections with $\mathcal{R}(\neg s)$ stays the same (because the new sets then still contain $\neg s$ as $\mathcal{R}$ is a relevance relation, and still satisfy (12) by IIP). First, I modify the set B_i for $i = j$: as $r \vdash_* s$ truly, $B_j \cap \{t, \neg t : t \in \mathcal{R}(\neg s)\}$ $(\in \mathcal{R}_{\neg s})$ is consistent with any $A \in \mathcal{R}_r$, hence with $A_j \cap \{t, \neg t : t \in \mathcal{R}(r)\}$, so that I may assume that $A_j \cap \{t, \neg t : t \in \mathcal{R}(r)\} \subseteq B_j$; which implies that

$$B_i \cap \mathcal{R}(r) = A_i \cap \mathcal{R}(r) \text{ for all } i \in C. \tag{13}$$

Second, I modify the sets B_i, $i \in N \backslash C$: I assume (using that $Y \cap \mathcal{R}(\neg s) = \emptyset$ and Y's consistency with every $A \in \mathcal{R}_{\neg s}$) that

$$\text{for all } i \in N \backslash C,\ Y \subseteq B_i, \text{ hence (as } \{\neg s\} \cup Y \vdash \neg r) \ \neg r \in B_i. \tag{14}$$

The definition of the sets A_i, $i \in C$, and (14) imply, via (13) and IIP, that

$$r \in F(B_1, ..., B_n). \tag{15}$$

By (12), (15), and the inconsistency of $\{r, \neg s\} \cup Y$, the set Y is not a subset of $F(B_1, ..., B_n)$. So there is a $y \in Y$ with $y \notin F(B_1, ..., B_n)$. We have $\{j\} \in \mathcal{C}(\neg y\}$ for the following two reasons.

- B_j contains $\neg y$; otherwise $y \in B_i$ for all $i \in N$, so that $y \in F(B_1, ..., B_n)$ by $y \in \mathcal{P}$.
- Consider any sets $C_i \in \mathcal{J}$, $i \neq j$, not containing $\neg y$, i.e. containing y. I show that $\neg y \in A := F(C_1, ..., C_{j-1}, B_j, C_{j+1}, ..., C_n)$. For all $i \neq j$, $C_i \cap \{t, \neg t : t \in \mathcal{R}(y)\}$ is consistent with y, hence is in $\mathcal{R}_y$; as $\mathcal{R}_y$ for analogous reasons also contains $B_i \cap \{t, \neg t : t \in \mathcal{R}(y)\}$ and as $|\mathcal{R}_y| = 1$ (by $y \in \mathcal{P}$), it follows that $C_i \cap \mathcal{R}(y) = B_i \cap \mathcal{R}(y)$. Hence, by $y \notin F(B_1, ..., B_n)$ and IIP, $y \notin A$. So $\neg y \in A$, as desired.

By $\{j\} \in \mathcal{C}(\neg y)$ and Claim 1, $\{j\} \in \mathcal{C}(\neg q)$ for all $q \in \mathcal{P}$. So j is a semi-vetodictator. ■

Proof of Theorem 2. Let $\{p, \neg p : p \in \mathcal{P}\}$ be truly pathlinked. I will reduce the proof to that of Theorem 1. I start again with two simple claims.

Claim 1. The set $\mathcal{C}(q)$ is the same for all $q \in \{p, \neg p : p \in \mathcal{P}\}$; call it $\mathcal{C}_0$.

This follows immediately from Lemma 3, q.e.d.

Claim 2. $\emptyset \notin \mathcal{C}_0$ and $N \in \mathcal{C}_0$.

By unanimity-preservation, $N \in \mathcal{C}(p)$ for all $p \in \mathcal{P}$; hence $N \in \mathcal{C}_0$. This implies, for all $p \in \mathcal{P}$, that $\emptyset \notin \mathcal{C}(\neg p)$; hence $\emptyset \notin \mathcal{C}_0$, q.e.d.

Now by an analogous argument to that in the proof of Theorem 1, but based this time on the present Claims 1 and 2 rather than on the two first claims in Theorem 1's proof, one can show that there exists an individual j such that $\{j\} \in \mathcal{C}(\neg q)$ for all $q \in \mathcal{P}$. So, by the present Claim 1 (which is stronger than the first claim in Theorem 1's proof),

$$\{j\} \in \mathcal{C}(q) \text{ for all } q \in \mathcal{P}. \tag{16}$$

So j is a semi-dictator, for the following reason. Let $q \in \mathcal{P}$ and let $(A_1, ..., A_n) \in \mathcal{J}^n$ be such that $q \in A_j$ and $q \notin A_i$, $i \neq j$. By (16) there is a set $B_j \in \mathcal{J}$ containing q such that $q \in F(B_1, ..., B_n)$ for all $B_i \in \mathcal{J}$, $i \neq j$, not containing q. Since $|\mathcal{R}_q| = 1$ (by $q \in \mathcal{P}$), and since $\mathcal{R}_q$ contains $A_j \cap \{t, \neg t : t \in \mathcal{R}(q)\}$ and $B_j \cap \{t, \neg t : t \in \mathcal{R}(q)\}$, the last two sets are identical. So $A_j \cap \mathcal{R}(q) = B_j \cap \mathcal{R}(q)$. Hence, using IIP and the definition of B_j, $q \in F(A_1, ..., A_n)$, as desired. ■

6 Dictatorship and strong dictatorship

In fact, the semi-dictator of Theorem 2 is in many cases (including preference aggregation) a dictator, and in some cases even a strong dictator in the sense of the following definition that generalises the classical notion of strong dictatorship in social choice theory.

Definition 7 *An individual i is a* strong dictator *if $F(A_1, ..., A_n) = A_i$ for all $(A_1, ..., A_n) \in \mathcal{J}^n$.*

So a strong dictator imposes his judgments on all rather than just privileged propositions. I will give simple criteria for obtaining (weak or strong) dictatorship, in terms of the following irreversibility property.

Definition 8 *For $p, q \in X$, p* irreversibly *constrained entails q if $p \vdash_Y q$ for a set Y for which $\{q\} \cup Y \nvdash p$.*

So a constrained entailment $p \vdash_* q$ is irreversible if the constrained entailment is not a "constrained equivalence", i.e. if p and q do not conditionally entail each other (for at least one choice of Y). If X is the preference agenda, *all* constrained entailments between (distinct) propositions are irreversible. For instance, $xRy \vdash_* xRz$ is irreversible (for distinct options x, y, z), since $xRy \vdash_{\{yPz\}} xRz$, where $\{xRz, yPz\} \nvdash xRy$.

By the next result, the semi-dictatorship of Theorem 2 becomes a dictatorship if we only slightly strengthen the pathlinkedness condition: in *at least one* path, *at least one* constrained entailment should be irreversible.

Definition 9 (a) *For propositions* $p, q \in X$*, I write* $p \vdash\vdash_{\text{irrev}} q$ *if* X *contains propositions* $p_1, ..., p_m$ $(m \geq 2)$ *with* $p = p_1 \vdash_* p_2 \vdash_* ... \vdash_* p_m = q$*, where at least one of these constrained entailments is irreversible.*

(b) *A pathlinked set* $Z \subseteq X$ *is* irreversibly pathlinked (in X) *if* $p \vdash\vdash_{\text{irrev}} q$ *for some (hence all)* $p, q \in Z$*.*

Theorem 3 *If the set* $\{p, \neg p : p \in \mathcal{P}\}$ *of privileged or negated privileged propositions is truly and irreversibly pathlinked, some individual is a dictator.*

As an application, I obtain the full Arrow theorem by proving that, if X is the preference agenda, $\{p, \neg p : p \in \mathcal{P}\}$ is truly and irreversibly pathlinked.[28]

Corollary 1 *(Arrow's Theorem) For the preference agenda, some individual is a dictator.*

Proof. Let X be the preference agenda. I show that (i) $\mathcal{P}$ is pathlinked, and (ii) there are $r, s \in \mathcal{P}$ with true and irreversible constrained entailments $r \vdash_* \neg s \vdash_* r$. Then, by (i) and Lemma 4, $\{\neg p : p \in \mathcal{P}\}$ is (like $\mathcal{P}$) pathlinked, which together with (ii) implies that $\{p, \neg p : p \in \mathcal{P}\}$ is truly and irreversibly pathlinked, as desired.

(ii): For any pairwise distinct options $x, y, z \in Q$, we have $xPy \vdash_{\{yPz\}} xRz$ $(= \neg zRx)$, and $xRz \vdash_{\{zPy\}} xPy$, in each case truly and irreversibly.

(i): Consider any $xPy, x'Py' \in \mathcal{P}$. I show that $xPy \vdash\vdash x'Py'$. The paths to be constructed depend on whether $x \in \{x', y'\}$ and whether $y \in \{x', y'\}$. As $x \neq y$ and $x' \neq y'$, the following list of cases is exhaustive. Case $x \neq x', y' \& y \neq x', y'$: $xPy \vdash_{\{x'Px, yPy'\}} x'Py'$. Case $y = y' \& x \neq x', y'$: $xPy \vdash_{\{x'Px\}} x'Py = x'Py'$. Case $y = x' \& x \neq x', y'$: $xPy \vdash_{\{yPy'\}} xPy' \vdash_{\{x'Px\}} x'Py'$. Case $x = x' \& y \neq y', x'$: $xPy \vdash_{\{yPy'\}} xPy'$. Case $x = y' \& y \neq x', y'$: $xPy \vdash_{\{x'Px\}} x'Py \vdash_{\{yPx\}} x'Px$. Case $x = x' \& y = y'$: $xPy \vdash_\emptyset xPy$. Case $x = y' \& y = x'$: taking any $z \in Q \backslash \{x, y\}$, $xPy \vdash_{\{yPz\}} xPz \vdash_{\{yPx\}} yPz \vdash_{\{zPx\}} yPx$. ■

The proof of Theorem 3 uses two further lemmas. For any set $\mathcal{S}$ of coalitions $C \subseteq N$, I define $\overline{\mathcal{S}} := \{C^* \subseteq N : C \subseteq C^* \text{ for some } C \in \mathcal{S}\}$.

Lemma 5 *For all* $p, q \in X$*,*

(a) $p \vdash_* q$ *irreversibly if and only if* $\neg q \vdash_* \neg p$ *irreversibly;*

(b) *if* $p \vdash_* q$ *irreversibly then* $\overline{\mathcal{C}(p)} \subseteq \mathcal{C}(q)$*.*

Proof. Let $p, q \in X$. Part (a) follows from Lemma 4 and the fact that, for all $Y \subseteq \mathcal{P}$, $\{q\} \cup Y \nvdash p$ if and only if $\{\neg p\} \cup Y \nvdash \neg q$.

[28] This property of $\{p, \neg p : p \in \mathcal{P}\}$ strengthens Nehring's (2003) finding that the preference agenda is totally blocked, which gave him already a weaker version of Arrow's theorem. The tedious part of our proof (showing that $\mathcal{P}$ is pathlinked) is analogous to Nehring's proof and also to proofs on the *strict* preference agenda by Dietrich and List (forthcoming) and Dokow and Holzman (2005).

Regarding (b), suppose $p \vdash_* q$ irreversibly, say $p \vdash_Y q$ with $\{q\} \cup Y \nvdash p$. We can assume w.l.o.g. that $Y \cap \mathcal{R}(p) = Y \cap \mathcal{R}(\neg q) = \emptyset$, since otherwise we could replace Y by $Y^* := Y \backslash (\mathcal{R}(p) \cup \mathcal{R}(\neg q))$, for which still $p \vdash_{Y^*} q$ (by the proof of Lemma 2) and $\{q\} \cup Y^* \nvdash p$. To show $\overline{\mathcal{C}(p)} \subseteq \mathcal{C}(q)$, consider any $C^* \in \overline{\mathcal{C}(p)}$. So there is a $C \in \mathcal{C}(p)$ with $C \subseteq C^*$. Hence there are $A_i \in \mathcal{J}$, $i \in C$, containing p, such that $p \in F(A_1, ..., A_n)$ for all $A_i \in \mathcal{J}$, $i \in N \backslash C$, containing $\neg p$. Like in earlier proofs, I may suppose w.l.o.g. that, for all $i \in C$, $Y \subseteq A_i$ (using that Y is consistent with all $A \in \mathcal{R}_p$, $\mathcal{R}$ is a relevance relation, IIP, and $Y \cap \mathcal{R}(p) = \emptyset$); hence, by $\{p\} \cup Y \vdash q$, $q \in A_i$ for all $i \in C$. Further, as $\{\neg p, q\} \cup Y$ is consistent (by $\{q\} \cup Y \nvdash p$), there are sets $A_i \in \mathcal{J}$, $i \in C^* \backslash C$, such that $\{\neg p, q\} \cup Y \subseteq A_i$ for all $i \in C^* \backslash C$.

I have to show that $q \in F(A_1, ..., A_n)$ for all $A_i \in \mathcal{J}$, $i \in N \backslash C^*$, containing $\neg q$. Consider such sets A_i, $i \in N \backslash C^*$. Again, we may assume w.l.o.g. that for all $i \in N \backslash C^*$, $Y \subseteq A_i$ (as Y is consistent with all $A \in \mathcal{R}_{\neg q}$, $\mathcal{R}$ is a relevance relation, IIP, and $Y \cap \mathcal{R}(\neg q) = \emptyset$), which by $\{\neg q\} \cup Y \vdash \neg p$ implies that $\neg p \in A_i$ for all $i \in N \backslash C^*$. In summary then,

$$A_i \supseteq \begin{cases} \{p, q\} \cup Y & \text{for all } i \in C \\ \{\neg p, q\} \cup Y & \text{for all } i \in C^* \backslash C \\ \{\neg p, \neg q\} \cup Y & \text{for all } i \in N \backslash C^*. \end{cases}$$

So $p \in F(A_1, ..., A_n)$ (by the choice of the sets A_i, $i \in C$) and $Y \subseteq F(A_1, ..., A_n)$ (by $Y \subseteq \mathcal{P}$). Hence, as $\{p\} \cup Y \vdash q$, $q \in F(A_1, ..., A_n)$, as desired. ■

In the following characterisation of decisive coalitions it is crucial that $p \in \mathcal{P}$.

Lemma 6 *If* $p \in \mathcal{P}$, $\mathcal{W}(p) = \{C \subseteq N$ *: all coalitions* $C^* \supseteq C$ *are in* $\mathcal{C}(p)\}$.

Proof. Let $p \in \mathcal{P}$ and $C \subseteq N$. If $C \in \mathcal{W}(p)$ then clearly all coalitions $C^* \supseteq C$ are in $\mathcal{C}(p)$. Conversely, suppose all coalitions $C^* \supseteq C$ are in $\mathcal{C}(p)$. As $C \in \mathcal{C}(p)$, there are sets A_i, $i \in C$, containing p, such that $p \in F(A_1, ..., A_n)$ for all sets A_i, $i \in N \backslash C$, not containing p. To show that $C \in \mathcal{W}(p)$, consider any sets A_i, $i \in N \backslash C$ (containing or not containing p); I show that $p \in F(A_1, ..., A_n)$. Let $C^* := C \cup \{i \in N \backslash C : p \in A_i\}$. By $C \subseteq C^*$, $C^* \in \mathcal{C}(p)$. So there are sets B_i, $i \in C^*$, containing p, such that $p \in F(B_1, ..., B_n)$ for all sets B_i, $i \in N \backslash C^*$, not containing p. By $|\mathcal{R}_p| = 1$, $A_i \cap \mathcal{R}(p) = B_i \cap \mathcal{R}(p)$ for all $i \in C^*$. So, by IIP and the definition of the sets B_i, $i \in C^*$, and since $p \notin A_i$ for all $i \in N \backslash C^*$, $p \in F(A_1, ..., A_n)$, as desired. ■

Proof of Theorem 3. Let $\{p, \neg p : p \in \mathcal{P}\}$ be truly and irreversibly pathlinked. By Theorem 2, there is a semi-dictator i. I show that i is a dictator.

Claim. For all $q \in \{p, \neg p : p \in \mathcal{P}\}$, $\mathcal{C}(q)$ contains all coalitions containing i.

Consider any $q \in \{q, \neg q : q \in \mathcal{P}\}$ and any coalition $C \subseteq N$ containing i. By true pathlinkedness there exist $p \in \mathcal{P}$ and $r, s \in X$ such that $p \vdash\vdash r \vdash_* s \vdash\vdash q$, where $r \vdash_* s$ is a truly constrained entailment. By $\{i\} \in \mathcal{C}(p)$ and Lemma 3,

$\{i\} \in \mathcal{C}(r)$. So, by Lemma 5(b), $C \in \mathcal{C}(s)$. Hence, by Lemma 3, $C \in \mathcal{C}(q)$, q.e.d.

By this claim and Lemma 6, $\{i\} \in \mathcal{W}(p)$ for all $p \in \mathcal{P}$. This implies that i is a dictator, by an argument similar to the one that completed the proof of Theorem 2. ■

Finally, for what agendas do we even obtain *strong* dictatorship? Surely not for the preference agenda, as it is well-known that Arrow's conditions only imply weak dictatorship.[29]

Trivially, if *all* propositions are privileged, every dictatorship is strong:

Corollary 2 *If $\mathcal{P} = X$, and X is truly and irreversibly pathlinked, some individual is a strong dictator.*

But the assumption $\mathcal{P} = X$ removes nearly all generality: unanimity preservation becomes the standard unrestricted unanimity principle, and the relevance relation becomes essentially equivalent to one that declares as relevant for a proposition $p \in X$ only p itself (by $|\mathcal{R}_p| = 1$). However, strong dictatorship follows under a much less restrictive condition than $\mathcal{P} = X$. Call $p \in X$ *logically equivalent to* $A \subseteq X$ if A entails p and p entails all $q \in A$ (i.e., intuitively, if p is equivalent to the conjunction of all $q \in A$). For instance, $a \wedge b$ is equivalent to $\{a, b\}$ (where $a, b, a \wedge b \in X$).

Corollary 3 *If $\{p, \neg p : p \in \mathcal{P}\}$ is truly and irreversibly pathlinked and each proposition in X is logically equivalent to a set of negated privileged propositions $A \subseteq \{\neg p : p \in \mathcal{P}\}$, some individual is a strong dictator.*

Proof. Let the assumptions hold. By Theorem 3, there is a dictator i. To show that i is a strong dictator, I consider any $(A_1, ..., A_n) \in \mathcal{J}^n$, and I show that $A_i = F(A_1, ..., A_n)$. Obviously, it suffices to show that $F(A_1, ..., A_n) \subseteq A_i$. Suppose $q \in F(A_1, ..., A_n)$. By assumption, q is logically equivalent to some $A \subseteq \{\neg p : p \in \mathcal{P}\}$. For all $\neg p \in A$, we have $\neg p \in F(A_1, ..., A_n)$ (by $q \vdash \neg p$), hence $p \notin F(A_1, ..., A_n)$, and so $p \notin A_i$ (as $p \in \mathcal{P}$ and i is a dictator), implying that $\neg p \in A_i$. This shows that $A \subseteq A_i$. So $q \in A_i$ (since $A \vdash q$), as desired. ■

The preference agenda X, which has *not strongly* dictatorial solutions, indeed violates the extra condition in Corollary 3: some propositions in X (namely precisely the privileged propositions xPy) are not logically equivalent to any set of negated privileged propositions xRy.

Example 2 (continued) As argued earlier, $\{p, \neg p : p \in \mathcal{P}\}$ is truly pathlinked in many instances of this aggregation problem. The other conditions in Corollary 3 also often hold, so that strong dictatorship follows. The reasons are simple.

[29] Lexicographic dictatorships satisfy all conditions but are only weak dictatorships.

First, X is often rich in irreversible entailments. For instance, if X contains $V = 3$ and $W = 3$ and the constraint $W > V$, then $V = 3 \vdash_* \neg(W = 3)$ irreversibly, since $V = 3 \vdash_{\{W>V\}} \neg(W = 3)$ but $\{\neg(W = 3), W > V\} \nvdash V = 3$; or, if X contains a constraint c that is strictly stronger than another constraint $c' \in \mathbf{C}$, then $c \vdash_* c$ irreversibly, since $c \vdash_\emptyset c'$ but not $c' \nvdash c$.

Second, if $\mathcal{P}$ is again given by (9), each proposition $q \in X$ is indeed logically equivalent to a set of negated privileged propositions $A \subseteq \{\neg p : p \in \mathcal{P}\}$: if q is in $\{c, \neg c : c \in \mathbf{C}\}$ or has the form $\neg(V = v)$, one may simply take $A = \{q\}$; otherwise q has the form $V = v$, and one should take $A = \{\neg(V = v') : v' \in ran(V)\backslash\{v\}\}$.

7 Conclusion

The above impossibility findings might be interpreted as showing how relevance $\mathcal{R}$ should *not* be specified. To enable non-degenerate aggregation, $\mathcal{R}$ must display sufficiently many inter-relevances. But such a richness in inter-relevances may imply that collective decisions have to be made in a "wholistic" manner: many semantically unrelated decisions must be bundled and decided simultaneously. Two propositions, say one on traffic regulations and one on diplomatic relations with Argentina, have to be treated simultaneously if the relevance relation (specified sufficiently richly to enable non-degenerate aggregation rules) displays some possibly indirect link between the two.[30] Large and semantically disparate decision problems are a hard challenge in practice.

8 References

Chapman, B. (2002) Rational Aggregation, Polit. Philos. Econ. 1(3): 337-354

Claussen, C. A., Roisland, O. (2005) Collective economic decisions and the discursive paradox, working paper, Central Bank of Norway Research Division

Dietrich, F. (2006) Judgment aggregation: (im)possibility theorems, *Journal of Economic Theory* 126(1): 286-298

Dietrich, F. (forthcoming) A generalised model of judgment aggregation, *Social Choice and Welfare*

Dietrich, F. (2005) The possibility of judgment aggregation under subjunctive implications, working paper

Dietrich, F., List, C. (forthcoming) Arrow's theorem in judgment aggregation, *Social Choice and Welfare*

[30] That is, if the two propositions are related in terms of the transitive, symmetric and reflexive closure of relevance $\mathcal{R}$. This closure partitions the totality of propositions (questions) into equivalence classes of irreducible decision problems. If there is a single equivalence class, *all* decisions (including those on traffic regulations and on diplomatic relations) have to be treated simultaneously.

Dokow, E., Holzman, R. (2005) Aggregation of binary evaluations, working paper, Technion Israel Institute of Technology

Eckert, D., Pigozzi, G. (2005) Belief merging, judgment aggregation and some links with social choice theory, in *Belief Change in Rational Agents*, J. Delgrande et al. (eds.)

Gärdenfors, P. (2006) An Arrow-like theorem for voting with logical consequences, *Economics and Philosophy* 22(2): 181-190

Konieczny, S., Pino-Perez, R. (2002) Merging information under constraints: a logical framework, *Journal of Logic and Computation* 12(5): 773-808

List, C. (2004) A model of path-dependence in decisions over multiple propositions, *American Political Science Review* 98(3): 495-513

List, C., Pettit, P. (2002) Aggregating sets of judgments: an impossibility result. *Economics and Philosophy* 18: 89-110

List, C., Pettit, P. (2004) Aggregating sets of judgments: two impossibility results compared. *Synthese* 140(1-2): 207-235

Mongin, P. (2005) Factoring out the impossibility of logical aggregation, working paper, CNRS, Paris

Nehring, K. (2003) Arrow's theorem as a corollary. *Economics Letters* 80(3): 379-382

Nehring, K., Puppe, C. (2002) Strategy-proof social choice on single-peaked domains: possibility, impossibility and the space between, working paper, University of California at Davies

Nehring, K., Puppe, C.(2005) The structure of strategy-proof social choice, part II: non-dictatorship, anonymity and neutrality, working paper, University of Karlsruhe

Nehring, K., Puppe, C. (2006) Consistent judgment aggregation: the truth-functional Case, working paper, University of Karlsruhe

Pauly, M., van Hees, M. (2006) Logical constraints on judgment aggregation, *Journal of Philosophical Logic* 35: 569-585

Rubinstein, A., Fishburn, P. (1986) Algebraic aggregation theory, *Journal of Economic Theory* 38: 63-77

van Hees, M. (forthcoming) The limits of epistemic democracy, *Social Choice and Welfare*

Wilson, R. (1975) On the Theory of Aggregation, *Journal of Economic Theory* 10: 89-99