The Equity Risk Premium and the Riskfree Rate in an Economy with Borrowing Constraints

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Abstract

Our objective in this article is to study analytically the effect of borrowing constraints on asset returns. We explicitly characterize the equilibrium for an exchange economy with two agents who differ in their risk aversion and are prohibited from borrowing. In a representative-agent economy with CRRA preferences, the Sharpe ratio of equity returns and the riskfree rate are linked by the risk aversion parameter. We show that allowing for preference heterogeneity and imposing borrowing constraints breaks this link. We find that an economy with borrowing constraints exhibits simultaneously a relatively high Sharpe ratio of stock returns and a relatively low riskfree interest rate, compared to both representative-agent and unconstrained heterogeneous-agent economies.

JEL classification: G12, G11, D52.
Key words: Incomplete markets, portfolio choice, asset pricing, general equilibrium.

Contents

1 Introduction

A feature of standard representative agent models with constant relative risk aversion (CRRA) preferences is that the Sharpe ratio of stock returns and the risk-free rate are linked to one another. This is a major limitation. For instance, attempts to resolve the finding in Mehra and Prescott (1985) that the risk premium is too small and the risk-free rate is too high in such a model relative to the data, run into the problem that an increase in the Sharpe ratio of stock returns is associated with an increase in the risk-free rate, known as the "interest rate puzzle" (Weil 1989).

Our objective in this article is to study analytically the effect of borrowing constraints on the link between the Sharpe ratio and the risk-free rate. We do this by considering a model that is a straightforward extension of the homogeneous agent economy of Mehra and Prescott where financial markets are effectively complete. The extension is to introduce a borrowing constraint in a general equilibrium exchange economy with two agents who have CRRA preferences, and to give the borrowing constraint a meaningful role we assume that the two agents differ in their risk aversion. We characterize exactly in closed form the equilibrium of this economy. General-equilibrium economies with borrowing constraints are typically not amenable to explicit analysis and are studied using numerical simulation methods.[1] Our model is extremely tractable and amenable to rigorous theoretical analysis.

Our main result is that, unlike in a representative agent model, in an economy with borrowing constraints the Sharpe ratio of stock returns can be relatively high, while the risk-free interest rate remains relatively low. In particular, we show that the Sharpe ratio of stock returns in the constrained heterogeneous-agent economy is the same as in the representative-agent economy populated only by the more risk averse of the two agents, while the risk-free rate in the constrained heterogeneous-agent economy may be even lower than in the representative-agent economy populated by the less risk averse of the two agents. And, comparing the constrained heterogeneous-agent economy to one where agents are heterogeneous but unconstrained, we find that imposing a borrowing constraint increases the Sharpe ratio of stock returns and lowers the risk-free interest rate.

[1]A notable exception is a model of Detemple and Murthy (1997), in which explicit results can be obtained when all agents have logarithmic preferences, but differ in their beliefs about the aggregate endowment process.

Moreover, we show that the unconstrained economy with heterogeneous agents suffer from the same limitations as the representative-agent economy with CRRA preferences, namely the tight link between the Sharpe ratio of stock returns and the level of the risk-free rate (we establish this new analytical result for the unconstrained economy), which is not the case in an economy with borrowing constraints.

Borrowing constraints are an important feature of the real economy and as argued by Constantinides (2002) it is important to consider these constraints when studying the implications of asset pricing models. However, taking into account borrowing constraints is a challenging task since even in models without borrowing constraints but with heterogeneous risk aversion (Dumas 1989, Wang 1996, Chan and Kogan 2002) most of the asset-pricing results are obtained using numerical analysis.[2] In models *with* borrowing constraint, for instance Heaton and Lucas (1996) and Constantinides, Donaldson and Mehra (2002), the analysis is undertaken using numerical methods, while in Kogan and Uppal (2002) the analysis is undertaken using approximation methods that apply in the neighborhood of log utility which then limits the range of the risk aversion parameter for which the effect of borrowing constraints can be analyzed. In contrast, we characterize exactly in closed form the equilibrium in an economy with borrowing constraints.

There is another important difference between our model and the models of Heaton and Lucas (1996) and Constantinides, Donaldson and Mehra (2002), which are the two papers closest to our work. In both these models, the source of heterogeneity across agents is idiosyncratic endowment shocks and therefore the mechanism through which the borrowing constraint works is different. In Heaton and Lucas, the constraint on borrowing and a cost for trading stocks and bonds raises individual consumption variability, and hence, lowers the risk-free rate of return due to the demand for precautionary savings. Constantinides, Donaldson and Mehra model do not have trading costs; instead, they consider an overlapping generations model. In their model, the young would like to invest in equity by collateralizing future wages but are prevented from doing so because of the constraint on borrowing. On the other hand, for the middle-aged wage uncertainty has largely been resolved and so most of variation in their consumption occur from variation in

[2]Wang can solve for only some of the quantities of the model in closed form but even this is possible only for particular combinations of the number of agents and the degree of risk aversion for each of these agents.

financial wealth; thus, stock returns are highly correlated with consumption. Hence, this age cohort requires a higher rate of return for holding equity. Thus, in their model "the *deus ex machina* is the stage in the life cycle of the marginal investor."

In contrast to these two papers, in our model the source of heterogeneity is risk aversion, and therefore no additional source of risk is introduced relative to the standard representative-agent framework considered in Mehra and Prescott (1985). Moreover, because we solve for the equilibrium in closed-form, the economic forces driving the results in our paper are transparent.

Our work is also related to the paper by Basak and Cuoco (1998) who characterize the equilibrium in a model where agents differ with respect to their risk aversion and, instead of a constraint on borrowing, face a constraint on participating in the stock market. In contrast to our model where all agents face the same constraint on borrowing, in their setup the constraint is applied asymmetrically across agents; in particular, they assume that it's the less risk averse agent who is excluded from the stock market, which is counter to what one would expect.

The rest of the paper is arranged as follows. In Section 2, we describe an exchange economy with heterogeneous agents who face borrowing constraints. In Section 3, we characterize analytically the equilibrium in this economy. In Section 4, we consider the robustness of our results to more general forms of the borrowing constraint. We conclude in Section 5. Our main results are highlighted in propositions and the proofs for all the propositions are collected in the appendix.

2 A model of an exchange economy with heterogeneous agents

In this section, we study a general-equilibrium exchange (endowment) economy with multiple agents who differ in their level of risk aversion. Wang (1996) analyzes this economy for the case where there are two agents who do not face any portfolio constraints.

2.1 The aggregate endowment process

The infinite-horizon exchange economy has an aggregate endowment, D_t, that evolves according to

$$dD_t = \mu\, D_t\, dt + \sigma\, D_t\, dZ_t,$$

where μ and σ are constant parameters. We assume that the growth rate of the endowment is positive, $\mu - \sigma^2/2 > 0$. Without much loss of generality we also assume that $D_0 = 1$.

2.2 Financial assets

We assume that there are two assets available for trading in the economy. The first asset is a short-term risk-free bond, available in zero net supply, which pays the interest rate r_t that will be determined in equilibrium. The second asset is a stock that is a claim on the aggregate endowment. The price of the stock is denoted by S_t. The cumulative stock return process is given by

$$\frac{dS_t + D_t dt}{S_t} = \mu_{St} dt + \sigma_{St} dZ_t, \tag{1}$$

with μ_{St} and σ_{St} to be determined in equilibrium.

2.3 Preferences

There are two competitive agents in the economy. The utility function of both agents is time-separable and is given by

$$\mathrm{E}_0 \left[\int_0^\infty e^{-\rho t} \frac{1}{1-\gamma} \left(C_{\gamma,t}^{1-\gamma} - 1 \right) \, dt \right],$$

where ρ is the constant subjective time discount rate, and C_t is the flow of consumption. The agent's relative risk aversion equals γ, and for agents with unit risk aversion ($\gamma = 1$), the utility function is logarithmic:

$$\mathrm{E}_0 \left[\int_0^\infty e^{-\rho t} \ln C_{1,t} \, dt \right].$$

We assume that the first agent has risk aversion greater than one, while the second agent has unit risk aversion. Most of our results can be easily generalized to an arbitrary combination of risk aversion coefficients[3].

[3] See Appendix B.

2.4 Individual endowments

We assume that both agents are initially endowed with shares of the stock. We will let $\omega_{\alpha,0}$, $\alpha \in \{1, \gamma\}$, denote the initial share of the aggregate endowment owned by the agent with relative risk aversion equal to α.

2.5 The constraint on borrowing

We consider a leverage constraint that restricts the proportion of individual wealth that can be invested in the risky asset. The base case of our model assumes that borrowing is prohibited. We establish our analytical results under this assumption. As an extension, we analyze numerically a more general case, when the proportion of individual wealth invested in the risky asset is bounded from above, $\pi \leq \overline{\pi} > 1$.

2.6 The competitive equilibrium

The equilibrium in this economy is defined by the stock price process, P_t, the interest rate process r_t, and the portfolio and consumption policies, such that (i) given the price processes for financial assets, the consumption and portfolio choices are optimal for the agents, (ii) the goods market and the markets for the stock and the bond clear.

3 The equilibrium and asset prices

In this section, we characterize an equilibrium in the economy described above. We compare the equilibrium in this economy with homogeneous representative-agent economies. We conclude by comparing the equilibrium in the economy with constraints to the one that is unconstrained.

3.1 Equilibrium in the economy with borrowing constraints

We look for an equilibrium in which the policy of the less risk averse agent is affected by the borrowing constraint, while the more risk averse agent is effectively unconstrained. Clearly, one could construct other equilibria by lowering the risk-free rate relative to the values that we identify. The equilibrium we identify has an intuitive appeal, since it can be also interpreted as approximating an economy in which a small amount of borrowing is allowed. In such an economy, while portfolio holdings of both agents would consist almost entirely of the risky asset, the more risk averse agent would be unconstrained. Our numerical results in Section 4 further illustrate this point.

The following proposition characterizes equilibrium prices and allocations in the constrained economy. To simplify notation, we let $R_t = \int_0^t r_s ds$ denote the cumulative return on the risk-free asset and define $\bar{R}_t = R_t - (\rho + \mu - \gamma\sigma^2)t$. The short-term interest rate can then be recovered from the process R_t by differentiation.

Proposition 1 *Let* $\rho > \max[(1-\gamma)\mu + \frac{\gamma(\gamma-1)}{2}\sigma^2, 0]$. *There exists a competitive equilibrium in which*

(i) The consumption processes of the two agents are given by

$$C_{\gamma,t} = (1-A)\exp\left(\frac{\bar{R}_t + (1-\gamma)(\mu - \gamma\sigma^2/2)t}{\gamma}\right) D_t, \tag{2}$$

$$C_{1,t} = A\exp(\bar{R}_t) D_t, \tag{3}$$

where the constant $A = C_{1,0}/D_0 \in [0,1]$ *and the deterministic process* $\bar{R}_t$ *are determined as a unique solution of the following system of equations:*

$$(1-A)\exp\left(\frac{\bar{R}_t + (1-\gamma)(\mu - \gamma\sigma^2/2)t}{\gamma}\right) + A\exp(\bar{R}_t) = 1, \tag{4}$$

$$A - \rho(1-\omega_{\gamma,0})\int_0^\infty \exp(-\bar{R}_t - \rho t)\, dt = 0. \tag{5}$$

(ii) The instantaneous Sharpe ratio of stock returns equals

$$\frac{\mu_{St} - r_t}{\sigma_{St}} = \gamma\sigma; \tag{6}$$

(iii) The instantaneous volatility of stock returns equals

$$\sigma_{St} = \sigma; \tag{7}$$

(iv) The risk-free interest rate process r_t is deterministic and is given by

$$r_t = \frac{d\bar{R}_t}{dt} + (\rho + \mu - \gamma\sigma^2).$$

Proposition 1 states that in equilibrium the moments of asset returns are deterministic. Moreover, the instantaneous Sharpe ratio and volatility of stock returns are constant. The reason for why the moments of returns are not affected by shocks to the aggregate endowment, which is the only source of uncertainty in this economy, is very intuitive. Since the growth rate of the aggregate endowment process is independent of its past history, the moments of asset returns may depend only on the distribution of wealth in the economy. Because the agents cannot borrow, in equilibrium they both invest all of their wealth in the stock, and therefore their wealth processes are instantaneously perfectly correlated. Thus, the cross-sectional wealth distribution in the economy evolves in a locally predictable manner. Moreover, since both agents have CRRA preferences, their consumption policies (consumption rate as a fraction of individual wealth) depend only on the contemporaneous investment opportunity set in the economy, that is, on the wealth distribution. Thus, we conclude that the instantaneously riskless rate of change of the wealth distribution in the economy must be a function of the wealth distribution itself, implying that the latter evolves deterministically over time, and hence all moments of asset returns are also deterministic functions of time.

The fact that the cross-sectional wealth distribution evolves deterministically has another important implication. The consumption policies of both agents (consumption as a share of individual wealth) then must be deterministic as well, therefore the volatility of consumption growth of each agent coincides with the volatility of the growth rate of aggregate endowment. The standard CCAPM relation then implies that the maximum Sharpe ratio is given by the product of the volatility of aggregate endowment growth and the relative risk aversion coefficient of the effectively unconstrained agent, i.e., of the more risk averse agent. Because there is only one source of risk in this economy, the aggregate stock returns are instantaneously perfectly correlated with shocks to the aggregate endowment and therefore the Sharpe ratio of stock returns coincides with the maximum achievable Sharpe ratio, thus the Sharpe ratio of stock returns is effectively set by the more risk averse of the two agents.

3.2 Comparison with representative agent economies

Having characterized the competitive equilibrium, we are now in a position to identify the impact of heterogeneity on the properties of asset returns. We compare our heterogeneous economy to a representative agent economy populated by identical agents with a relative risk aversion of $\gamma^\star$. By the same logic as above, we look for an equilibrium which is supposed to approximate an economy in which a small amount of borrowing is allowed, that is, we are looking for an equilibrium in which the representative agent is unconstrained. The solution to this problem is well-known. The moments of asset returns in this economy are given by

$$\sigma_S = \sigma, \qquad \frac{\mu_S - r}{\sigma_S} = \gamma^\star \sigma, \qquad r = \rho + \gamma^\star \mu - \frac{\gamma^\star(1+\gamma^\star)}{2}\sigma^2. \tag{8}$$

Both the Sharpe ratio of stock returns and the risk-free rate depend on the same preference parameter. This gives rise to the well-known interest rate puzzle (Weil 1989): in a representative agent model with CRRA preferences, realistic values of the Sharpe ratio of stock returns are associated with unrealistically high levels of the risk-free rate.

The economy with borrowing constraints has properties that are markedly different from those in the representative-agent economy. Comparing the results in Proposition 1 with equation (8), we see that the Sharpe ratio of stock returns in the constrained heterogeneous-agent economy is the same as in the representative-agent economy populated only by the second, more risk averse, of the two agents, i.e., the economy with $\gamma^\star = \gamma$. At the same time, the risk-free rate in the constrained heterogeneous economy is lower than the corresponding value suggested by (8), which we will henceforth denote by $r(\gamma^\star)$. The following proposition summarizes the properties of the risk-free rate.

Proposition 2 *The risk-free interest rate in the economy with the borrowing constraint is a monotonically decreasing function of time. At time* 0*, the initial value of the interest rate is given by*

$$r_0 = z[r(1) + (1-\gamma)\sigma^2] + (1-z)r(\gamma), \tag{9}$$

where $r(\gamma^\star)$ *denotes the risk-free rate in a representative-agent economy with risk aversion equal to* $\gamma^\star$*, as given in (8), and* $z = \gamma A/[1 + (\gamma - 1)A] \in [0,1]$*, where* $A = C_{1,0}/D_0$ *is the time-zero* *consumption share of the log-utility agent (see Proposition 1). The initial value of the interest rate is a convex combination of* $r(\gamma)$ *and* $r(1) + (1-\gamma)\sigma^2$ *and the weight,* z*, is a decreasing function of the wealth distribution* $\omega_{\gamma,0}$*. In the long run, as time approaches infinity,*

(i) If $\mu - \gamma\sigma^2/2 > 0$*, then* $r(\gamma) > r(1)$ *and* $\lim_{t\to\infty} r_t = \rho + \mu - \gamma\sigma^2 = r(1) + (1-\gamma)\sigma^2$*;*

(ii) If $\mu - \gamma\sigma^2/2 < 0$*, then* $r(\gamma) < r(1)$ *and* $\lim_{t\to\infty} r_t = \rho + \gamma\mu - \gamma(1+\gamma)\sigma^2/2 = r(\gamma)$*;*

(iii) If $\mu - \gamma\sigma^2/2 = 0$*, then* $r_t = r(\gamma) = r(1)$*.*

Case (i) of Proposition 2 is the one in which the "interest rate puzzle" can arise in a representative agent economy: a relatively high Sharpe ratio in the economy with $\gamma^\star = \gamma$ is also associated with a relatively high risk-free interest rate, i.e. $r(\gamma) > r(1)$. It is also the case that is relevant for empirical analysis, since most reasonable parameter choices would satisfy the condition $\mu - \gamma\sigma^2/2 > 0$, which says that the risk-adjusted growth rate of the economy is positive. The proposition shows that in this case, in the heterogeneous economy, the risk-free rate is always lower than in the representative-agent economy with risk aversion equal to γ, i.e., $r_t < r(\gamma)$. Moreover, for sufficiently large values of t, or equivalently for an initial wealth distribution with enough wealth controlled by the log-agent, the risk-free rate in the heterogeneous economy is even lower than in the log-agent economy, that is, $r_t < r(1)$. Thus, in contrast to the representative agent economies, in our heterogeneous economy the risk-free rate is almost entirely divorced from the Sharpe ratio of stock returns. In fact, in an economy with only a small fraction of wealth controlled by the more risk averse type of agents, the Sharpe ratio of stock returns is the same as in a homogeneous economy with risk aversion of γ, while the risk-free rate is even *lower* than in a homogeneous economy with risk aversion of one.

The intuition underlying the result in Proposition 2 is the following. When most of the wealth in the economy is controlled by the log investor, the level of expected stock returns is close to that in a homogeneous economy with a log-utility representative agent, that is, $\rho+\mu$. This is because the consumption rate of the log-utility agent is a constant fraction of his/her wealth, given by the time preference parameter ρ (e.g., Merton, 1969). Market clearing requires that the wealth of the log agent is approximately equal to the stock price, while his/her consumption approximately equals the aggregate endowment, from which the result on the price level and expected stock returns follows immediately. However, following Proposition 1, we argued that it is quite intuitive why the Sharpe ratio of stock returns is determined by risk aversion of the effectively unconstrained, more risk averse investor. Thus, the presence of the borrowing constraint drives a wedge between the risk-free rate and the Sharpe ratio of stock returns.

3.3 Comparison with an economy without borrowing constraints

To further isolate the effect of the borrowing constraint, we consider the benchmark economy where agents are heterogeneous and there is no constraint on borrowing. This is precisely the setting studied by Wang (1996). We assume that the agent's preferences in the unconstrained economy are identical to those in the constrained economy. Unfortunately, the asset prices in the unconstrained economy cannot be computed in closed form, which limits the scope of our analysis. Nevertheless, some comparative results can be established.

Proposition 3 *The instantaneous Sharpe ratio of stock returns in the unconstrained economy falls between σ and $\gamma\sigma$ and hence is lower or equal to that in the economy with borrowing constraints, regardless of the cross-sectional distribution of wealth in each of the economies.*

Proposition 3 establishes that imposing the borrowing constraint raises the Sharpe ratio of stock returns. In the unconstrained economy, there is no well-defined marginal investor, risk aversion of both agents affects the Sharpe ratio of stock returns. As we argued above, imposing a borrowing constraint effectively makes the logarithmic agent infra-marginal and the Sharpe ratio of stock returns is now set by the more risk averse agents. Thus, it is not surprising that in the constrained economy the Sharpe ratio is higher than in its unconstrained counterpart.

Intuitively, one could also conjecture that imposing the borrowing constraint lowers the risk-free interest rate. This is because imposing the constraint reduces the demand for borrowing on behalf of the less risk averse investors, so for the bond market to clear, the more risk averse investors should not be willing to lend, and therefore the risk-free rate must fall. This argument is heuristic, since it ignores the general-equilibrium effects that the borrowing constraint has on the dynamic properties of stock returns and the risk-free rate. Nevertheless, this intuition is appealing and is formalized in Proposition 4 below.

Because the wealth distribution in the unconstrained economy cannot be derived explicitly, it is difficult to compare interest rates in the constrained and the unconstrained economies while controlling for the wealth distribution. In the following proposition, we take a different approach, by assuming that the consumption distribution in the two economies is identical and comparing the corresponding interest rates. This is not a standard comparative statics experiment, since the consumption distribution in the two economies being same is not equivalent to the wealth distribution being the same. However, together with Proposition 3 this establishes the following important result. For an unconstrained economy with any wealth distribution one can always find a wealth distribution in a constrained economy with the same preferences to simultaneously achieve a lower value of the risk-free rate and a higher value of the Sharpe ratio of stock returns.

Proposition 4 *Given the same cross-sectional distribution of consumption in the constrained and the constrained economies, the risk-free interest rate in the constrained economy is lower than or equal to that in the unconstrained economy.*

Propositions 3 and 4 show that, *holding the agents' preferences fixed*, imposing a borrowing constraint increases the Sharpe ratio of stock returns and lowers the risk-free interest rate. One could, however, argue that since the distribution of risk-aversion coefficients is not directly observable, one would often treat it as a free parameter in calibration, and therefore an unconstrained economy could potentially have properties similar to a constrained economy, albeit with a different choice of risk aversion parameters. The following proposition demonstrates that this is the case. In fact, an unconstrained heterogeneous economy exhibits a tradeoff between the Sharpe ratio of stock returns and the risk-free rate that is very similar to the one in representative-agent economies with CRRA preferences. In the latter case, the Sharpe ratio of stock returns, denoted by $SR(\gamma)$, and the risk-free rate are related by

$$r(\gamma) = \rho + \frac{\mu}{\sigma} SR(\gamma) - \frac{1+\gamma^{-1}}{2} SR(\gamma)^2.$$

For realistic choices of model parameters, a high Sharpe ratio of returns implies a relatively high risk-free rate. As the following proposition shows, the situation is not very different in an unconstrained heterogeneous economy.

Proposition 5 *Let SR_t^{unc} denote the instantaneous Sharpe ratio of stock returns in an unconstrained heterogeneous-agent economy. Then the risk-free interest rate and the Sharpe ratio of stock returns satisfy*

$$r_t^{unc} > \rho + \frac{\mu}{\sigma} SR_t^{unc} - \left(SR_t^{unc}\right)^2 . \tag{10}$$

The inequality (10) does not explicitly depend on the preference parameter γ or the distribution of wealth in a heterogeneous economy, that is, it applies for any wealth distribution within a particular economy and also across various economies, differing in agents' risk aversion. Figure 1 below illustrates the implication of Proposition 5. Note that, as the wealth distribution shifts from the log-utility agent to the more risk averse agent (as ω_γ increases), both the interest rate and the Sharpe ratio rise in the unconstrained economy. However, to achieve a high value of the Sharpe ratio, the risk-free rate must be unrealistically high. On the other hand, a constrained economy with the same parameter values can generate a high Sharpe ratio while the risk-free rate remains relatively low.

Finally, we find that the borrowing constraint does not have an obvious systematic effect on the volatility of stock returns. In the constrained economy, the instantaneous return volatility equals the volatility of the endowment process σ, while in the unconstrained economy it can be either higher or lower, depending on the choice of model parameters.

4 Numerical analysis: More general borrowing constraints

In this section, we analyze a more general case, when the proportion of individual wealth invested in the risky asset is bounded from above, $\pi \leq \overline{\pi} > 1$. An explicit solution in this case is not available and we have to resolve to numerical simulations. Our objective is to illustrate that the explicit solution in the economy without borrowing is qualitatively similar to the behavior of an economy with a relatively tight restriction on leverage.

We consider an economy in which the moments of the aggregate endowment growth are given by $\mu = 0.018$ and $\sigma = 0.033$ (these correspond to the unconditional moments of the century-long U.S. aggregate consumption series). We set the subjective time discount rate to $\rho = 0.02$. We assume that the more risk averse agent in the economy has the relative risk aversion parameter $\gamma = 10$. We set $\overline{\pi} = 1.05$, i.e., the agents cannot borrow more than 5% of their individual wealth. We solve for equilibrium prices and strategies using the same iterative procedure as in Kogan and Uppal (2002).

Our results are shown in Figure 1. In the region where the borrowing constraint is binding, which corresponds approximately to $\omega_\gamma > 0.05$, the Sharpe ratio of stock returns is close to the value in the economy without borrowing, which is the same as in the representative-agent economy with risk aversion of γ. The risk-free rate is monotonically increasing in ω_γ, as in the economy analyzed above. Note that if most of wealth in the economy is controlled by the less risk-averse, log-utility, agent, the interest rate can be lower than in a representative-agent log-utility economy, as predicted by our analytical solution above (the results of Proposition 2, obtained in the limit of large values of time t are equivalent to the limit of the wealth distribution ω_γ approaching zero, since the wealth distribution in the economy without borrowing is a deterministic monotone function of time and $\lim_{t\to\infty} \omega_{\gamma,t} = 0$).

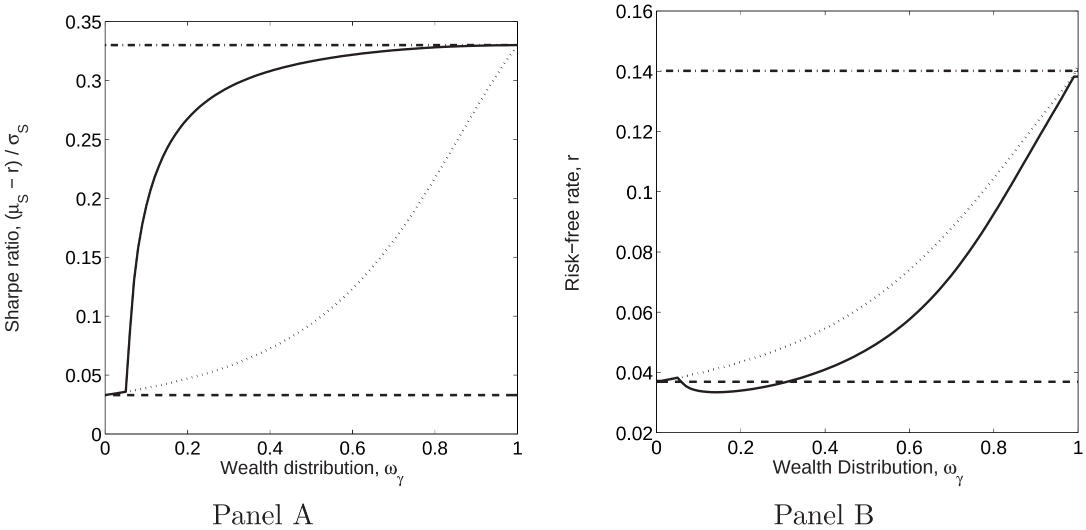

Figure 1: Effect of borrowing constraint on Sharpe ratio and risk-free rate

Panel (a) plots the instantaneous Sharpe ratio of stock returns in the constrained economy (solid line) and in the unconstrained economy (dotted line) as a function of the wealth distribution, ω_γ. Panel (b) gives the corresponding plots of the risk-free interest rate. The following parameter values are used: $\mu = 0.018$, $\sigma = 0.033$, $\rho = 0.02$. The more risk averse agent in the economy has $\gamma = 10$. The constraint on borrowing is given by $\pi \leq 1.05$, i.e., the agents cannot borrow more than 5% of their individual wealth. The dashed and dashed-dotted lines correspond to the representative-agent economies with risk aversion of $\gamma = 1$ and $\gamma = 10$ respectively.

5 Conclusion

In this article, we study a general equilibrium exchange economy with multiple agents who differ in their degree of risk aversion and face borrowing constraints. We show that, unlike in a representative agent model, in an economy with borrowing constraints the Sharpe ratio of stock returns can be relatively high, while the risk-free interest rate remains relatively low. In particular, the Sharpe ratio of stock returns in the constrained heterogeneous-agent economy is the same as in the representative-agent economy populated only by the more risk averse of the two agents, while the risk-free rate in the constrained heterogeneous-agent economy may be even lower than in the representative-agent economy populated by the less risk averse of the two agents. And, comparing the constrained heterogeneous-agent economy to one where agents are heterogeneous but unconstrained, we find that imposing a borrowing constraint increases the Sharpe ratio of stock returns and lowers the risk-free interest rate. Moreover, we show that the heterogeneous-agent unconstrained economies suffer from the same limitations as the representative-agent economies with CRRA preferences, namely the tight link between the Sharpe ratio of stock returns and the level of the risk-free rate, which is not the case in economies with borrowing constraints.

6 Appendix A: Proofs and technical results

Proof of Proposition 1

We first examine the decision problem of individual agents, subject to the market prices given in parts (ii–iv) of the Proposition. We then show that markets clear as long as the system of equations (4,5) has a solution. Finally, we prove that such a solution exists and is unique.

Individual agents' consumption/portfolio choice

Since the first, more risk averse, agent is unconstrained in equilibrium, his problem can be formulated in an equivalent static form (see Cox and Huang, 1989)

$$\max_{C_{\gamma,t}} E_0\left[\int_0^\infty e^{-\rho t}\frac{C_{\gamma,t}^{1-\gamma}}{1-\gamma}dt\right], \tag{A1}$$

subject to the budget constraint

$$E_0\left[\int_0^\infty e^{-R_t}\xi_t C_{\gamma,t}dt\right] = \omega_\gamma E_0\left[\int_0^\infty e^{-R_t}\xi_t D_t dt\right] = \omega_{\gamma,0}S_0. \tag{A2}$$

where ξ_t is the density of the equivalent martingale measure (EMM density). Given (ii) ξ_t takes the following form

$$\xi_t = e^{-\frac{1}{2}\gamma^2\sigma^2 t-\gamma\sigma W_t}. \tag{A3}$$

The optimal consumption of the first agent then satisfies

$$e^{-\rho t}C_{\gamma,t}^{-\gamma} = \lambda_1 e^{-R_t}\xi_t, \tag{A4}$$

where λ_1 is the Lagrange multiplier on his budget constraint. Thus,

$$C_{\gamma,t} = \lambda_1^{-\frac{1}{\gamma}} e^{\frac{R_t}{\gamma}} e^{\left(\frac{-\rho-\gamma\mu+\gamma(1+\gamma)\sigma^2/2}{\gamma}\right)t} D_t. \tag{A5}$$

To solve the problem of the log-utility agent, we use the technique developed in Cvitanic and Karatzas (1992) for portfolio optimization with constraints. Specifically, we introduce a fictitious market in which the diffusion component of stock returns is the same as in the original market, but the EMM density is now given by

$$\tilde{\xi}_t = e^{-\frac{1}{2}\sigma^2 t - \sigma W_t} \tag{A6}$$

and the interest rate by

$$\tilde{r}_t = r_t - (1-\gamma)\,\sigma^2. \tag{A7}$$

It is easy to check that the expected stock return in the fictitious market is the same as in the original market. If it turns out that the optimal portfolio strategy for the agent in the fictitious market satisfies the original constraints, this strategy would also be optimal in the original market (see Cvitanic and Karatzas, 1992).

The log-utility agent's problem in fictitious market is

$$\max_{C_{1,t}} E_0 \left[\int_0^\infty e^{-\rho t} \ln C_{1,t} ds \right], \tag{A8}$$

subject to

$$E_0 \left[\int_0^\infty e^{-R_t + (1-\gamma)\sigma^2 t} \tilde{\xi}_t C_{1,t} dt \right] = (1 - \omega_{\gamma,0}) S_0. \tag{A9}$$

The optimality condition takes form

$$e^{-\rho t} C_{1,t}^{-1} = \lambda_2 e^{-R_t + (1-\gamma)\sigma^2 t} \tilde{\xi}_t, \tag{A10}$$

and therefore

$$C_{1,t} = \lambda_2^{-1} e^{R_t} e^{(-\rho - \mu + \gamma\sigma^2)t} D_t. \tag{A11}$$

Market clearing conditions

Let us define

$$\bar{R}_t = R_t - (\rho + \mu - \gamma\sigma^2)t. \tag{A12}$$

Then, equations (A5) and (A11) take form

$$C_{\gamma,t} = \lambda_1^{-\frac{1}{\gamma}} e^{\left(\frac{\bar{R}_t + (1-\gamma)(\mu - \gamma\sigma^2/2)t}{\gamma}\right)} D_t, \tag{A13}$$

$$C_{1,t} = \lambda_2^{-1} e^{\bar{R}_t} D_t. \tag{A14}$$

The market clearing condition in the consumption market is then given by

$$\lambda_1^{-\frac{1}{\gamma}} e^{\left(\frac{\bar{R}_t + (1-\gamma)(\mu - \gamma\sigma^2/2)t}{\gamma}\right)} + \lambda_2^{-1} e^{\bar{R}_t} = 1, \tag{A15}$$

which should hold for every $t \in [0, \infty)$. The condition (A15) at time $t = 0$ implies that $\lambda_2^{-1} = 1 - \lambda_1^{-\frac{1}{\gamma}} = C_{1,0}/D_0$. Let us denote $C_{1,0}/D_0$ as A and express (A15) as

$$(1 - A) \exp\left(\frac{\bar{R}_t + (1-\gamma)(\mu - \gamma\sigma^2/2)t}{\gamma}\right) + A \exp(\bar{R}_t) = 1. \tag{A16}$$

Consider now the budget constraint of the log-utility agent:

$$(1 - \omega_{\gamma,0}) S_0 = E_0 \left[\int_0^\infty e^{-R_t + (1-\gamma)\sigma^2 t} \tilde{\xi}_t C_{1,t} dt\right] = A \int_0^\infty e^{-\rho t} dt = \frac{A}{\rho}. \tag{A17}$$

Since

$$S_0 = E_0 \left[\int_0^\infty e^{-R_t} \xi_t D_t dt\right] = \int_0^\infty e^{-R_t} e^{(\mu - \gamma\sigma^2)t} dt = \int_0^\infty e^{-\bar{R}_t - \rho t} dt, \tag{A18}$$

equation (A17) is equivalent to

$$A = \rho(1 - \omega_{\gamma,0}) \int_0^\infty e^{-\bar{R}_t - \rho t} dt. \tag{A19}$$

As long as the budget constraint of the log-utility agent is satisfied, so is the budget constraint of the non-log agent, which follows from equations (A13), (A16), and (A18). Finally, note also that according to (A18), the ratio of the stock price to the aggregate endowment is a deterministic function of time, and hence the instantaneous volatility of stock returns equals σ. Similarly, the volatility of wealth of the log-utility agent (computed using the EMM density $\tilde{\xi}_t$) is equal to σ. Hence, the log-utility agent invests all of his/her wealth in the stock market, and therefore, the no-borrowing constraint is satisfied. The same is true for the non-log agent. Thus, we conclude that the equilibrium postulated in Proposition 1 exists as long as the system of two equations (A16, A19) (equations (4) and (5) in Proposition 1) has a solution.

Existence and uniqueness of solution to equations (4) and (5)

Differentiating (4) with respect to A we have

$$(\partial_A \bar{R}_t)\frac{1-(1-\gamma)A\exp(\bar{R}_t)}{\gamma} = \exp\left(\frac{\bar{R}_t+(1-\gamma)(\mu-\gamma\sigma^2/2)t}{\gamma}\right) - \exp(\bar{R}_t). \tag{A20}$$

Consider two cases:

Case: $\mu - \gamma\sigma^2/2 > 0$

Equation (4) implies that in this case $\bar{R}_t \geq 0$. From equation (A20) then it follows that $\partial_A \bar{R}_t < 0$. Let us define a mapping

$$I(A) = \rho(1-\omega_{\gamma,0})\int_0^\infty e^{-\bar{R}_t-\rho t}dt. \tag{A21}$$

Differentiating (A21) with respect to A we have

$$I'(A) = -\rho(1-\omega_{\gamma,0})\int_0^\infty \partial_A \bar{R}_t e^{-\bar{R}_t-\rho t}dt \geq 0. \tag{A22}$$

From (4) we have

$$0 < I(0) = (1-\omega_{\gamma,0})\rho/[\rho-(1-\gamma)(\mu-\gamma\sigma^2/2)] < 1$$

$$0 < I(1) = (1-\omega_{\gamma,0}) < 1.$$

Therefore, by the Brouwer Fixed-Point theorem, the system of equations (4), (5) has a solution. To show that the solution is unique, we compute the second derivative of $I(A)$:

$$I''(A) = \rho(1-\omega_{\gamma,0})\int_0^\infty [(\partial_A \bar{R}_t)^2 - \partial_{AA}\bar{R}_t]e^{-\bar{R}_t-\rho t}dt. \tag{A23}$$

Differentiating (A20) with respect to A we have

$$\partial_{AA}\bar{R}_t\left(1-(1-\gamma)A\exp(\bar{R}_t)\right) - (1-\gamma)\partial_A\bar{R}_t(1+A\partial_A\bar{R}_t)\exp(\bar{R}_t) =$$

$$= \partial_A\bar{R}_t\left(\exp\left(\frac{\bar{R}_t+(1-\gamma)(\mu-\gamma\sigma^2/2)t}{\gamma}\right) - \exp(\bar{R}_t)\right) + (1-\gamma)\partial_A\bar{R}_t\exp(\bar{R}_t).$$

Therefore,

$$\left[(\partial_A \bar{R})^2_{A,t} - \partial_{AA}\bar{R}_t\right](1-(1-\gamma)A\exp(\bar{R}_t)) =$$

$$\partial_A \bar{R}_t(\gamma-1)\frac{\left[1+(2\gamma-1)Ae^{\bar{R}_t}\right]\exp\left(\frac{\bar{R}_t+(1-\gamma)(\mu-\gamma\sigma^2/2)t}{\gamma}\right)+\exp(\bar{R}_t)\left(1-A\exp(\bar{R}_t)\right)}{1-(1-\gamma)A\exp(\bar{R}_t)}.$$

Since $\partial_A \bar{R}_t < 0$ and $\gamma > 1$, we conclude that $I''(A) < 0$ and the uniqueness of the solution follows.

Case: $\mu - \gamma\sigma^2/2 \leq 0$

Equation (4) implies that in this case $\bar{R}_t \leq 0$. From equation (A20) then it follows that $\partial_A \bar{R}_t > 0$ and, therefore, $I'(A) \leq 0$. This implies that the equation $I(A) = A$ has a unique solution.

Proof of Proposition 2

Differentiating equation (4) with respect to t at $t = 0$ proves (9). To show that z is a decreasing function of the $\omega_{\gamma,0}$, it is enough to prove that A is a decreasing function of $\omega_{\gamma,0}$, since z is monotonically increasing in A. $A = I(A, \omega_{\gamma,0})$ holds for every $\omega_{\gamma,0} \in [0,1]$. Differentiating this equality in $\omega_{\gamma,0}$, we find that

$$[1 - \partial_A I(A, \omega_{\gamma,0})]\partial_{\omega_{\gamma,0}} A = \partial_{\omega_{\gamma,0}} I < 0.$$

At the fixed point of the mapping $I(A)$, it must be that $\partial I(A, \omega_{\gamma,0}) < 1$, and hence $\partial_{\omega_{\gamma,0}} A < 0$.

To establish the asymptotic properties of the risk-free rate, we examine (4) as t approaches infinity.

(i) Case $\mu - \gamma\sigma^2/2 > 0$: $\lim_{t\to\infty} \bar{R}_t = -\ln A$, and therefore, $\lim_{t\to\infty} r_t = \rho + \mu - \gamma\sigma^2$.

(ii) Case $\mu - \gamma\sigma^2/2 < 0$: $\lim_{t\to\infty} \bar{R}_t + (1-\gamma)(\mu - \gamma\sigma^2/2)t = const$, and therefore, $\lim_{t\to\infty} r_t = \rho + \gamma\mu - \gamma(1+\gamma)\sigma^2/2$.

(iii) Case $\mu - \gamma\sigma^2/2 = 0$: $\bar{R}_t = 0$ and $r_t = \rho + \mu - \gamma\sigma^2$.

Proof of Propositions 3–5

First, we establish some properties of the unconstrained economy. The equilibrium allocation of consumption in such economy is Pareto-optimal and can be recovered as a solution of the central planner's problem (see Wang, 1996)

$$\max_{C_\gamma + C_1 = D} \frac{C_\gamma^{1-\gamma}}{1-\gamma} + \lambda \ln C_1 \tag{A24}$$

for a suitable choice of the utility weight λ. Let $u(D;\lambda)$ denote the solution of (A24), which can be interpreted as a utility function of the representative agent (social planner). Using the optimality conditions, it is easy to show that (A24) implies

$$\partial_D C_\gamma = \frac{C_\gamma}{D-(1-\gamma)C_1} \tag{A25}$$

$$\partial_{DD} C_\gamma = \frac{(1-\gamma)\gamma C_\gamma C_1}{(D-(1-\gamma)C_1)^3}, \tag{A26}$$

and therefore,

$$\partial_D u(D;\lambda) = \lambda \frac{1}{C_1}, \tag{A27}$$

and

$$\partial_{DD} u(D;\lambda) = -\lambda \frac{1}{C_1} \frac{\gamma}{D-(1-\gamma)C_1}. \tag{A28}$$

According to the consumption CAPM, the instantaneous Sharpe ratio of stock returns is given by

$$SR_0^{unc} = \sigma \frac{-D_0 \partial_{DD} u(D_t;\lambda)}{\partial_D u(D_t;\lambda)} = \sigma \frac{\gamma}{(1-A)+\gamma A} \in [\sigma, \gamma\sigma], \tag{A29}$$

where $A = C_{1,0}/D_0$ denotes the consumption share of the log-utility agent at time zero, as in the constrained economy characterized in Proposition 1. This proves Proposition 3.

The risk-free rate in the unconstrained economy can be computed using the derived utility function of the representative agent. Specifically,

$$r_t^{unc} = \rho - \frac{D_t \partial_{DD} u(D_t;\lambda)}{\partial_D u(D_t;\lambda)}\mu - \frac{1}{2}\frac{D_t^2 \partial_{DDD} u(D_t;\lambda)}{\partial_D u(D_t;\lambda)}\sigma^2. \tag{A30}$$

It then follows that

$$r_0^{unc} = \rho + \frac{\gamma}{1+(\gamma-1)A}\left[\mu - \frac{1+\gamma-\frac{\gamma(1-\gamma)A}{1+(\gamma-1)A}}{(1+(\gamma-1)A)}\frac{\sigma^2}{2}\right]. \tag{A31}$$

Now compare r_0^{unc} to the interest rate in the constrained economy with the same initial distribution of consumption, as given by Proposition 1. We find that

$$r_0^{unc} - r_0 = \frac{\gamma A \sigma^2}{2(1-A+\gamma A)^3}\left[(\gamma-1)^3 A^2 + A(\gamma^3+\gamma^2-5\gamma+3) + 2(\gamma-1)\right].$$

It is easy to see that since $A \in [0,1]$ and $\gamma \geq 1$,

$$r_0^{unc} - r_0 \geq 0.$$

This proves Proposition 4. The result of Proposition 5 follows from (A29) and (A31).

7 Appendix B: General case

In this section we consider a general case of two agents with CRRA utility functions.

$$U_t^1(c_1) = E_t\left[\int_t^\infty e^{-\rho(s-t)}\frac{c_{\gamma_1,s}^{1-\gamma_1}}{1-\gamma_1}ds\right] \text{ and } U_t^2(c_2) = E_t\left[\int_t^\infty e^{-\rho(s-t)}\frac{c_{\gamma_2,s}^{1-\gamma_2}}{1-\gamma_2}ds\right]. \quad \text{(B1)}$$

Without loss of generality we assume that $\gamma_1 > \gamma_2$. We prove the following analog of Proposition 1:

Proposition 6 *Let* $\rho > \max[(1-\gamma_1)\mu + \frac{\gamma_1(\gamma_1-1)}{2}\sigma^2, 0]$. *There exists a competitive equilibrium in which*

(i) The consumption processes of the two agents are given by

$$C_{\gamma_1,t} = (1-A)\exp\left(\frac{\bar{R}_t + (\gamma_2-\gamma_1)(\mu + (1-\gamma_1-\gamma_2)\sigma^2/2)t}{\gamma_1}\right)D_t, \quad \text{(B2)}$$

$$C_{\gamma_2,t} = A\exp(\bar{R}_t)D_t, \quad \text{(B3)}$$

where the constant $A = C_{1,0}/D_0 \in [0,1]$ *and the deterministic process* $\bar{R}_t$ *are determined as a solution of the following system of equations:*

$$(1-A)\exp\left(\frac{\bar{R}_t + (\gamma_2-\gamma_1)(\mu + (1-\gamma_1-\gamma_2)\sigma^2/2)t}{\gamma_1}\right) + A\exp\left(\frac{\bar{R}_t}{\gamma_2}\right) = 1, \quad \text{(B4)}$$

$$A\int_0^\infty e^{-\frac{\gamma_2-1}{\gamma_2}\bar{R}_t - \psi t}dt - (1-\omega_{\gamma_1,0})\int_0^\infty e^{-\bar{R}_t-\psi t}dt = 0, \quad \text{(B5)}$$

where

$$\psi = \rho + (\gamma_2 - 1)(\mu - \gamma_2\sigma^2/2). \quad \text{(B6)}$$

(ii) The instantaneous Sharpe ratio of stock returns equals

$$\frac{\mu_{St} - r_t}{\sigma_{St}} = \gamma_1\sigma; \quad \text{(B7)}$$

(iii) The instantaneous volatility of stock returns equals

$$\sigma_{St} = \sigma; \quad \text{(B8)}$$

(iv) The risk-free interest rate process r_t is deterministic and is given by

$$r_t = \frac{d\bar{R}_t}{dt} + (\rho + \gamma_2\mu - (\gamma_1 + \gamma_2(\gamma_2 - 1)/2)\sigma^2).$$

Proof. Following the same line of arguments as before we find that the first agent's solution to the unconstrained problem with EMM density

$$\xi_t = e^{-\frac{1}{2}\gamma_1^2\sigma^2 t - \gamma_1\sigma W_t} \tag{B9}$$

is

$$C_{\gamma_1,t} = \lambda_1^{-\frac{1}{\gamma_1}} e^{\frac{R_t}{\gamma_1}} e^{\left(\frac{-\rho-\gamma_1\mu+\gamma_1(1+\gamma_1)\sigma^2/2}{\gamma_1}\right)t} D_t. \tag{B10}$$

where λ_1 is the Lagrange multiplier on his budget constraint. To solve the problem of the less risk-averse second agent, we use the technique developed in Cvitanic and Karatzas (1992) for portfolio optimization with constraints. Specifically, we introduce a fictitious market in which the diffusion component of stock returns is the same as in the original market, but the EMM density is now given by

$$\xi_t = e^{-\frac{1}{2}\gamma_2^2\sigma^2 t - \gamma_2\sigma W_t} \tag{B11}$$

and the interest rate by

$$\tilde{r}_t = r_t - (\gamma_2 - \gamma_1)\,\sigma^2. \tag{B12}$$

Thus, the second agent solves

$$\max_{C_{\gamma_2,t}} E_0 \left[\int_0^\infty e^{-\rho s} \frac{c_{\gamma_2,s}^{1-\gamma_2}}{1-\gamma_2} ds \right], \tag{B13}$$

subject to

$$E_0 \left[\int_0^\infty e^{-R_t + (\gamma_2-\gamma_1)\sigma^2 t} \tilde{\xi}_t C_{\gamma_2,t} dt \right] = (1 - \omega_{\gamma_1,0}) S_0. \tag{B14}$$

As a result, his consumption is given by

$$C_{\gamma_2,t} = \lambda_2^{-\frac{1}{\gamma_2}} e^{\frac{R_t}{\gamma_2}} e^{\left(\frac{-\rho-\gamma_2\mu+(\gamma_1+\gamma_2(\gamma_2-1)/2)\sigma^2}{\gamma_2}\right)t} D_t. \tag{B15}$$

Market clearing conditions

Let us define

$$\bar{R}_t = R_t - (\rho + \gamma_2\mu - (\gamma_1 + \gamma_2(\gamma_2 - 1)/2)\sigma^2)t. \tag{B16}$$

The market clearing condition in the consumption market is then given by

$$(1 - A)\exp\left(\frac{\bar{R}_t + (\gamma_2 - \gamma_1)(\mu + (1 - \gamma_1 - \gamma_2)\sigma^2/2)t}{\gamma_1}\right) + A\exp\left(\frac{\bar{R}_t}{\gamma_2}\right) = 1, \tag{B17}$$

where, as before, $A = C_{1,0}/D_0 = \lambda_2^{-\frac{1}{\gamma_2}} = 1 - \lambda_1^{-\frac{1}{\gamma_1}} = C_{1,0}/D_0$. Consider now the budget constraint of the less risk-averse agent:

$$(1 - \omega_{\gamma,0})S_0 = E_0\left[\int_0^\infty e^{-R_t + (\gamma_2 - \gamma_1)\sigma^2 t}\tilde{\xi}_t C_{\gamma_2,t}dt\right] = A\int_0^\infty e^{-\frac{\rho}{\gamma_2}t}e^{\frac{\gamma_2 - 1}{\gamma_2}\left[-R_t + (\frac{\gamma_2}{2} - \gamma_1)\sigma^2 t\right]}. \tag{B18}$$

Since

$$S_0 = E_0\left[\int_0^\infty e^{-R_t}\xi_t D_t dt\right] = \int_0^\infty e^{-R_t}e^{(\mu - \gamma_1\sigma^2)t}dt = \int_0^\infty e^{-\bar{R}_t - \left(\rho + (\gamma_2 - 1)(\mu - \gamma_2\sigma^2/2)\right)t}dt, \tag{B19}$$

equation (B18) is equivalent to

$$A\int_0^\infty e^{-\frac{\rho}{\gamma_2}t}e^{\frac{\gamma_2 - 1}{\gamma_2}\left[-R_t + (\frac{\gamma_2}{2} - \gamma_1)\sigma^2 t\right]} = (1 - \omega_{\gamma,0})\int_0^\infty e^{-\bar{R}_t - \left(\rho + (\gamma_2 - 1)(\mu - \gamma_2\sigma^2/2)\right)t}dt, \tag{B20}$$

or

$$A\int_0^\infty e^{-\frac{\gamma_2 - 1}{\gamma_2}\bar{R}_t - \psi t}dt = (1 - \omega_{\gamma_1,0})\int_0^\infty e^{-\bar{R}_t - \psi t}dt. \tag{B21}$$

As long as the budget constraint of the second agent is satisfied, so is the budget constraint of the first agent, which follows from equations (B17) and (B19). Finally, note also that according to (B19), the ratio of the stock price to the aggregate endowment is a deterministic function of time, and hence the instantaneous volatility of stock returns equals σ. Similarly, the volatility of wealth of the second agent (computed using the EMM density $\tilde{\xi}_t$) is equal to σ. Hence, the second agent invests all of his wealth in the stock market, and therefore, the no-borrowing constraint is satisfied. The same is true for the first agent. Thus, we conclude that the equilibrium postulated in Proposition 1 exists as long as the system of two equations (B17, B21) has a solution.

Existence of solution to equations (B17) and (B21)

Let us define a mapping

$$I(A) = A \int_0^\infty e^{-\frac{\gamma_2 - 1}{\gamma_2} \bar{R}_t - \psi t} dt - (1 - \omega_{\gamma_1,0}) \int_0^\infty e^{-\bar{R}_t - \psi t} dt. \tag{B22}$$

From (B17) and (B22) we have

$$I(0) < 0 < I(1).$$

Therefore, the system of equations (B17), (B21) has a solution.

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