Forecasting the Forecasts of Others: Implications for Asset Pricing[*]

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Abstract

We study the properties of rational expectation equilibria (REE) in dynamic asset pricing models with heterogeneously informed agents. We show that under mild conditions the state space of such models in REE can be infinite dimensional. This result indicates that the domain of analytically tractable dynamic models with asymmetric information is severely restricted. We also demonstrate that even though the serial correlation of returns is predominantly determined by the dynamics of stochastic equity supply, under certain circumstances asymmetric information can generate positive autocorrelation of returns.

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1   Introduction

This paper studies the properties of linear rational expectation equilibria in dynamic asset pricing models with imperfectly informed agents. Since the 1970's, the concept of rational expectation equilibrium (REE) has become central to both macroeconomics and finance, where agents' expectations are of paramount importance. There is extensive evidence that in financial markets information is distributed unevenly among different agents. As a result, prices reflect expectations of various market participants and, therefore, are essential sources of information. To extract useful information from prices, rational agents must disentangle the contribution of fundamentals from errors made by other investors. Hence, it is not only the agents' own expectations about future payoffs that matter, but also their expectations about other agents' expectations. In the terminology of Townsend [45], agents forecast the forecasts of others. Trades of each agent make the price depend on his own expectations about fundamentals, his own mistakes, and his perception of other agents' mistakes. This brings an additional layer of iterated expectations. Iterating this logic forward, prices must depend on the whole hierarchy of investors' beliefs.[1]

Iterated expectations are viewed by many economists as an important feature of financial markets and as a possible source of business cycle fluctuations. However, while dynamic analysis has become standard in both macroeconomics and finance, formal analysis of dynamic models with heterogeneous information has proven to be very difficult. The reason is that in most dynamic models successive forecasts of the forecasts of others are all different from one another. Hence, unless a recursive representation for them is available, the dimensionality of the state space of the model grows with the number of signals that the agents receive. As a result, both analytical and numerical analysis of the model becomes more complicated as the number of trading periods increases.

Although several assumptions have been identified in the literature that help preserve tractability, all of them are quite restrictive. For example, a direct way to limit the depth of higher order expectations is to assume that all private information becomes public after several periods (Albuquerque and Miao [3]; Brown and Jennings [15]; Grundy and McNichols [23]; Grundy and Kim [24]; Singleton [44]). As a result, the investors' learning problem becomes static, severely weakening the effects of

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[1]We use the terms "hierarchy of beliefs" and "hierarchy of expectations" interchangeably. They are different from what is generally referred to in game theory (Biais and Bossaerts [13]; Harsanyi [25]; Mertens and Zamir [36]). In our case, agents have common knowledge of a common prior but agents' private information is represented by infinite dimensional vectors.

asymmetric information on expectations, prices, and returns. Another way to make a model tractable is to assume that agents are hierarchically informed, i.e., they can be ranked by the amount of information they possess (Townsend [45]; Wang [49, 50]). The simplest example is the case with two classes of agents: informed and uninformed. Informed agents know the forecasting error of the uninformed agents and, therefore, do not need to infer it. Thus, the series of higher order expectations collapses. Although this information structure substantially simplifies the solution, it puts severe restrictions on the range of possible dynamics of agents' expectations and their impact on stock returns. Finally, the "forecasting the forecasts of others" problem can be avoided if payoff-relevant fundamentals are constant over time (Amador and Weill [5]; Back, Cao, and Willard [8]; Foster and Viswanathan [20]; He and Wang [26]) or if the number of signals that agents receive is not less than the number of shocks in the model.

Although the simplifying assumptions are widely used in the literature, the question as to what extent these conditions can be relaxed has not been explored. Our paper fills this gap. We consider a dynamic asset pricing model with an infinite horizon in which 1) each agent lacks some information available to others, 2) payoff-relevant fundamentals evolve stochastically over time, and 3) for each agent, the dimensionality of unobservable shocks exceeds the number of the conditioning variables. We show that this small set of conditions can make an infinite hierarchy of iterated expectations unavoidable and, therefore, an infinite number of state variables is required to describe the dynamics of the economy. The establishment of this fact is a challenging problem since even when a model does have a finite-dimensional state space, it is not obvious how to identify those state variables in which the equilibrium dynamics have a tractable form.

The identified conditions are quite intuitive. The first one guarantees that the information held by other agents is relevant to each agent's payoff. As a result, beliefs about other agents' beliefs affect each agent's demand for the risky asset. The second condition forces agents to form new sets of higher order beliefs every period. Since the fundamentals are persistent and no agent ever becomes fully informed, all agents need to incorporate the entire history of prices into their predictions. The third condition makes it impossible for agents to reconstruct unknown shocks to fundamentals (or their observational equivalents) using observable signals.

We also examine the effects of asymmetric information on asset prices. We compare two settings with identical fundamentals but different informational structures. In the first setting, the agents are fully informed, whereas in the second setting the informa-

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tion is heterogeneously distributed among them. We demonstrate that the equilibrium prices and returns are strongly affected by the allocation of information. In particular, when there are no superiorly informed agents, the diffusion of information into prices can be slow. Under certain circumstances, the resulting underreaction to new information leads to a positive autocorrelation of returns. This finding shows that momentum can be consistent with the rational expectation framework. In our model, momentum is an equilibrium outcome of agents' portfolio decisions. This distinguishes our explanation of momentum from a number of behavioral theories as well as existing rational theories, which show how momentum can appear in a partial equilibrium framework.

Our paper is related to the literature that explores the role of higher order expectations in asset pricing (Allen, Morris, and Shin [4]; Bacchetta and Wincoop [6]; Nimark [38]; Woodford [52]). In particular, Allen, Morris, and Shin [4] emphasize the failure of the law of iterated expectations for average expectations. As a consequence, under asymmetric information agents tend to underreact to private information, biasing the price towards the public signal and producing a drift in the price process. Their intuition is challenged by Banerjee, Kaniel, and Kremer [9], who argue that asymmetric information alone cannot generate price drift and suggest "differences in opinions", i.e., disagreements about higher order beliefs, as a possible source of momentum. In this paper, we show that the serial correlation of returns is determined by an interplay between two different effects. On the one hand, slow diffusion of information increases persistence of returns. On the other hand, the stochastic asset supply contributes to negative return autocorrelation (this effect is pure in the full information case). Hence, the serial correlation of returns depends on the relative strength of the two effects. In general, neither a slow diffusion of information nor a breakdown of the law of iterated expectations is sufficient to generate momentum. Nevertheless, we demonstrate that the information dispersion can have a qualitative effect on the serial correlation of returns if the effect of stochastic supply is sufficiently suppressed.

To analyze the model, we transform it from the time domain to the frequency domain. First developed by Futia [21], this technique has found many applications (Bernhardt, Seiler, and Taub [12]; Kasa [33]; Kasa, Walker, and Whiteman [34]; Walker [48]). Similar to our paper, Bernhardt, Seiler, and Taub [12] consider a dynamic model with stochastic fundamentals and heterogeneously informed agents. However, their model features a finite number of risk neutral investors who can individually affect prices and, hence, behave strategically. Thus, their work focuses on the effects of competition on information revelation, trading strategies, and price behavior. In

contrast, in our model investors are risk averse and behave competitively, as they are unable to move the price individually. Their trading behavior is determined not by a strategic revelation of information, but by risk considerations.

The rest of the paper is organized as follows. Section 2 describes the model. In Section 3, we prove that a simple version of our model has the "forecasting the forecasts of others" problem and that its dynamics cannot be described in terms of a finite number of state variables. In Section 4 we examine the impact of information dispersion on serial correlations of returns and discuss the dynamics of higher order expectations. In particular, we demonstrate that heterogeneous information accompanied by evolving fundamentals can produce momentum. Section 5 concludes. Technical details are presented in Appendices A and B.

2   Setup

This section presents our main setup, which will be used for analysis of the "forecasting the forecasts" problem and the serial correlation of returns. The setup includes most of the essential elements common to dynamic asset pricing models with asymmetric information.

The uncertainty in the model is described by a complete probability space $(\mathcal{X}, \mathcal{F}, \mu)$ equipped with a filtration $\mathcal{F}_t$. Time is discrete. There are two assets in the model: a riskless asset and a risky asset ("stock"). The riskless asset is assumed to be in infinitely elastic supply at a constant gross rate of return R. Each share of the stock pays a dividend D_t at time $t \in \mathbb{Z}$. Following Grossman and Stiglitz [22], we assume that the total amount of risky equity available to rational agents is Θ_t and attribute the time variation in Θ_t to noise trading. This assumption prevents prices from being fully revealing in an equilibrium, and thus gives a scope for asymmetric information. We assume that the processes D_t and Θ_t are stationary, adapted to the filtration $\mathcal{F}_t$, and jointly Gaussian. The normality assumption is standard for asset pricing models with asymmetric information. It ensures the linearity of conditional expectations and, therefore, makes models tractable.

There is a continuum of competitive investors who are assumed to be uniformly distributed on a unit interval $[0,1]$. All investors have an exponential utility function with the same constant risk-aversion parameter α and derive utility from their accumulated wealth in the next period. The assumption of one-period investment horizon is common in the asset pricing literature with asymmetric information (e.g., Allen,

Morris, and Shin [4], Bacchetta and Wincoop [7], Singleton [44], Walker [48]). In this case, investors have no hedging component in their demand for the risky asset. In our setup, this is especially advantageous since the calculation of the hedging demand in an economy with an infinite number of state variables presents a considerable technical challenge. Sidestepping this problem allows us to preserve tractability of the model.

Let $\mathcal{F}_t^i \subseteq \mathcal{F}_t$ be an information set of investor i generated by some stationary signals. It is natural to assume that $\mathcal{F}_t^i$ always includes the current price of the risky asset P_t and its current dividend D_t as well as all their past realizations. In addition, investors may receive some information about noise trading (past, current, or future) or future dividends, and this information may vary across investors. Hence, each investor i at time t solves the following portfolio problem:

$$\max_{x_t^i} E[-e^{-\alpha W_{t+1}^i}|\mathcal{F}_t^i] \tag{1}$$

subject to the budget constraint

$$W_{t+1}^i = x_t^i Q_{t+1} + W_t^i R, \tag{2}$$

where $Q_{t+1} = P_{t+1} + D_{t+1} - RP_t$ is the stock's excess return at time $t+1$ and W_t^i is accumulated wealth. It is well-known that the solution to this problem is

$$x_t^i = \omega^i E[Q_{t+1}|\mathcal{F}_t^i], \tag{3}$$

where $\omega^i = (\alpha \mathrm{Var}[Q_{t+1}|\mathcal{F}_t^i])^{-1}$ is a constant due to the normality and stationarity assumptions.

The rational expectation equilibrium in the model is characterized by (i) the price process P_t, which is assumed to be adapted to the filtration $\mathcal{F}_t$ and (ii) the trading strategies x_t^i, which solve the optimization problem in Eq. (1) and in each period t satisfy the market clearing condition

$$\int_0^1 x_t^i di = \Theta_t.$$

The market clearing condition allows us to represent the equilibrium price in terms of expectations of future fundamentals D_t and Θ_t. It is convenient to define the weighted average expectation operator $\bar{E}_t[\ \cdot\ ]$, which summarizes the expectations

of individual investors:

$$\bar{E}_t[\,\cdot\,] = \Omega \int_0^1 \omega^i E[\,\cdot\,|\mathcal{F}_t^i]di, \quad \Omega = \left[\int_0^1 \omega^i di\right]^{-1}. \tag{4}$$

Using this operator, the market clearing condition can be succinctly rewritten as

$$\Omega \Theta_t = \bar{E}_t[Q_{t+1}]. \tag{5}$$

Since the equilibrium price P_t is observed by all investors, combining the definition of excess returns and Eq. (5) yields an explicit equation for the price process:

$$P_t = -\lambda \Theta_t + \beta \bar{E}_t[D_{t+1} + P_{t+1}], \tag{6}$$

where we introduce $\beta = R^{-1}$ and $\lambda = \beta \Omega$ for the ease of notation. Iterating Eq. (6) forward and imposing the standard no-bubble condition $\lim_{t\to\infty} \beta^t P_t = 0$, we obtain another representation for the equilibrium price:

$$P_t = -\lambda \Theta_t + \sum_{s=0}^{\infty} \beta^{s+1} \bar{E}_t...\bar{E}_{t+s}\left(D_{t+s+1} - \lambda\Theta_{t+s+1}\right). \tag{7}$$

Eq. (7) represents the price as a sum of iterated average expectations of future realizations of processes D_t and Θ_t. In general, the information sets of investors are different, so the law of iterated expectations for the weighted average expectation operator $\bar{E}_t[\,\cdot\,]$ does not hold. As a result, investors must forecast not only future values of D_t and Θ_t but also expectations of other agents, and, therefore, non-trivial higher order expectations arise. Eq. (7) reveals the potential source of the "forecasting the forecasts of others" problem. Indeed, the current price P_t depends on agents' future expectations, which, in turn, depend on future prices (recall that P_{t+s} is in the information set of all investors and is used for conditioning in $\bar{E}_{t+s}[\,\cdot\,]$). Hence, to find the equilibrium in the model it is necessary to solve a fixed point problem for the entire sequence of prices. As we demonstrate below, it is a very complicated problem unless the information sets $\mathcal{F}_t^i$ have a very special structure.

3 Information Dispersion and Dynamics of Prices

In this section, we rigorously prove that the information dispersion can produce the "forecasting the forecasts of others" problem, i.e., that the dynamics of the model

cannot be described in terms of a finite number of state variables. Instead of working with the very general model described in the previous section, we consider its special version. On the one hand, it allows us to keep the proof as short as possible. On the other hand, even in its reduced form the model preserves all elements which are necessary for the existence of the "forecasting the forecasts of others" problem.

First, we assume that $D_t \equiv 0$, i.e., the price of the asset is solely determined by the risk premium required by investors for liquidity provision to noise traders. The main benefit of setting dividends to zero comes from the reduction in the number of observables available to investors, and the resulting simplification of investors filtering problem. Second, we assume that the process Θ_t consists of a persistent component V_t and a transitory component $b\varepsilon_t^\Theta$:

$$\Theta_t = V_t + b\varepsilon_t^\Theta, \tag{8}$$

where b is a non-zero constant. In its turn, V_t also contains two components following AR(1) processes with the same persistence parameter a:

$$V_t = V_t^1 + V_t^2,$$

$$V_{t+1}^j = aV_t^j + \varepsilon_{t+1}^j, \quad j = 1, 2.$$

The innovations ε_t^1, ε_t^2, and ε_t^Θ are assumed to be jointly independent and normally distributed with zero mean and unit variance.

There are two types of investors labeled by $j = 1, 2$. Each type populates a subset of a unit interval with a measure of $1/2$. All investors observe the price process P_t. The informational structure of the model is determined by what investors of each type know about the persistent components V_t^1 and V_t^2.

As a benchmark, we consider a full information case in which the information sets of investors are
$$\mathcal{F}_t^1 = \mathcal{F}_t^2 = \{P_\tau, V_\tau^1, V_\tau^2 : \tau \le t\}.$$

It means that all investors observe both V_t^1 and V_t^2, i.e., have all relevant information about future fundamentals available at time t. Since the information sets of investors are identical, the law of iterated expectations holds for average expectations: $\bar{E}_t\bar{E}_{t+1}...\bar{E}_{t+s}V_{t+s+1} = E_tV_{t+s+1} = a^{s+1}V_t$. At the same time, $E_t\varepsilon_{t+s}^\Theta = 0$ for $s > 0$. Hence, the infinite sum in Eq. (7) can be computed explicitly and we obtain a simple

representation for the equilibrium price:

$$P_t = -\lambda \left(b\varepsilon_t^\Theta + \frac{1}{1 - a\beta} V_t \right). \tag{9}$$

Note that since all investors observe V_t^1 and V_t^2, the shocks ε_t^Θ can be inferred from the price. From Eq. (9) it is obvious that in the full information equilibrium the price is determined by the state variables V_t^1, V_t^2, and ε_t^Θ and the system $\{V_t^1, V_t^2, \varepsilon_t^\Theta, P_t\}$ follows a VAR process.

Next, we consider the case in which investors have heterogeneous information about V_t. In particular, we assume that in addition to the price, type j investors also observe V_t^j, but not the other component of V_t denoted by V_t^{-j}. Thus, the information sets of investors are

$$\mathcal{F}_t^1 = \{P_\tau, V_\tau^1 : \tau \leq t\}, \quad \mathcal{F}_t^2 = \{P_\tau, V_\tau^2 : \tau \leq t\}$$

and the equilibrium price is determined by the following equation:

$$P_t = -\lambda(V_t + b\varepsilon_t^\Theta) + \beta\bar{E}_t[P_{t+1}]. \tag{10}$$

In this setting, each type of investor receives information that the other type also needs but does not observe: V_t^{-j} would help type j investors to predict the future asset supply Θ_{t+s}, $s > 0$ and, therefore, the future price. The price is not fully revealing: observing the price and their own component V_t^j is insufficient for type j investors to infer the other component, V_t^{-j}. Since V_t^{-j} is related to future payoffs, the expectations of V_t^{-j} formed by type j investors affect their demand and, subsequently, the price. Therefore, while extracting information from prices, type $-j$ investors need to infer not only the missing information about V_t^j but also the expectations of type j investors about V_t^{-j}. Type j investors face a similar problem, and the infinite regress starts.

This logic may seem to be quite general, but it does not always produce an infinite number of distinct higher-order expectations. He and Wang [26] provide an example in which the higher order expectations can be reduced to first-order expectations even when investors have heterogeneous information. They consider a conceptually similar setup, but assume that V does not evolve over time and is revealed to everybody at some future moment T. In their model, each investor also tries to predict the average of investors' expectations, $\hat{V}$. However, as He and Wang [26] demonstrate, $\hat{V}$ can be written as a weighed average of V conditional on public information (history of prices) and the true value of V. Given this, type j investors' expectations of $\hat{V}$ are a weighed

average of their first-order expectations, conditional on past prices and on their past private signals. Since V is constant, the aggregation of investors' private signals also produces V. Therefore, the second (and higher) order expectations of V can also be expressed in terms of a weighted average of V conditional on past prices and the true value of V.

This logic breaks down when V_t evolves stochastically over time. In this case, investors form expectations not only about the value of V_t at a particular moment, but also about all previous realizations of V_t. New information arriving in each period changes the way investors use their past signals; they need to reestimate the whole path of their expectations. This means that investors have to track an infinite number of state variables and that the logic described above does not hold.

It is important to know whether a model has a finite or infinite number of state variables because the two cases call for different solution techniques. In the former case, the major problem is to find the appropriate state space variables. In the latter, the search for a finite set of state variables that can capture the dynamics is doomed, and the solution of such models presents a greater challenge.

Until now, the "forecasting the forecasts of others" problem was defined relatively loosely. To lay the groundwork for its rigorous treatment, we introduce the concept of Markovian dynamics.

Definition. Let X_t be a multivariate stationary adapted random process. We say that X_t *admits Markovian dynamics* if there exists a collection of $n < \infty$ adapted random processes $\bar{Y}_t = \{Y_t^i\}$, $i = 1 \ldots n$, such that the joint process $(X_t, \bar{Y}_t)$ is a Markov process:

$$Prob\left(X_t \leq x, \bar{Y}_t \leq y | X_\tau, \bar{Y}_\tau : \ \tau \leq t - 1\right) = Prob\left(X_t \leq x, \bar{Y}_t \leq y | X_{t-1}, \bar{Y}_{t-1}\right).$$

While it is obvious that any Markov process admits Markovian dynamics, the opposite is not true. The following example helps to clarify the difference.

Let ε_t, $t \in \mathbb{Z}$ be *i.i.d.* standard normal random variables and consider an MA(1) process X_t such that $X_t = \varepsilon_t - \theta\varepsilon_{t-1}$ where θ is a constant. X_t is not a Markov process, or even an n-Markov process: $Prob\left(X_t | X_\tau : \ \tau \leq t - 1\right) \neq Prob\left(X_t | X_{t-1}, \ldots, X_{t-n}\right)$ for any n. However, X_t can be easily extended to a Markov process by augmenting it with ε_t.

Applying the concept of Markovian dynamics to our model with heterogeneous information, we get the main result of the paper.

Theorem 1 *Suppose that type j investors' information set is given by $\mathcal{F}_t^j = \{P_\tau, V_\tau^j : \tau \leq t\}$, $j = 1, 2$. Then, in the linear equilibrium the system $\{V^1, V^2, \varepsilon^\Theta, P\}$ does not admit Markovian dynamics.*

Proof. See Appendix A.

The main idea of the proof is to use the following result from the theory of stationary Gaussian processes: if a multivariate process admits Markovian dynamics, then it can be described by a set of rational functions in the frequency domain. We start the proof assuming that the equilibrium admits Markovian dynamics, so the price function in the frequency domain is rational. The proof then consists of showing that if we work only with rational functions, it is impossible to simultaneously satisfy the market clearing condition and solve the filtering problem of each agent. This contradiction proves that the equilibrium does not admit Markovian dynamics.

To the best of our knowledge, we are the first who rigorously demonstrate that the infinite regress problem does indeed arise naturally in a rational expectation equilibrium of dynamic models with heterogeneous information. Ironically, Townsend [45] who inspired the study of the infinite regress problem and coined the term "forecasting the forecasts of others", actually developed a model without the infinite regress problem (Kasa [33]; Pearlman and Sargent [39]; Sargent [43]). This illustrates how challenging it could be even to diagnose the presence of the "forecasting the forecasts of others" problem in a particular setting. Theorem 1 suggests a set of typical conditions for its existence. If each agent lacks some information available to other agents, fundamentals evolve stochastically over time, and dimensionality of unknown shocks exceeds that of conditioning variables then an infinite hierarchy of iterated expectations arises. Almost all of these conditions are also necessary. For example, as demonstrated by He and Wang [26], if the fundamentals are constant, then it is possible to avoid the "forecasting the forecasts of others" problem.

The infinite regress problem can be viewed as a rational justification for technical analysis. To a statistician who has only public information, the dynamics of prices might appear to be quite simple. However, for an investor who has additional private information the joint dynamics of prices and his private signals can become very complicated. We show that even with a very simple specification of the fundamentals, asymmetric information makes equilibrium dynamics highly non-trivial. To be as efficient as possible, agents use the entire history of prices in their predictions. Moreover, as Theorem 1 demonstrates, investors cannot summarize the available information with a finite number of state variables. In other words, investors' strategies become path-

dependant, often in a convoluted way. This suggests that in financial markets where the asymmetry of information is common, the entire history of prices may be very important to investors and have a profound effect on their actions.

A model with an infinite number of state variables calls for an appropriate solution technique. In Appendix B we describe one of the ways to solve the model. Since an analytical solution is not feasible, we construct a numerical approximation to the equilibrium. This approach allows us to evaluate the effects of various information distributions on prices and returns and explore the dynamics of higher order expectations.

4 Higher order expectations and autocorrelations of returns

In this section we explore the link between asymmetric information and autocorrelation of returns. A non-zero autocorrelation implies the predictability of next period returns by current returns and the existence of potentially profitable trading opportunities. In the asset pricing literature, positive serial correlations are often associated with momentum whereas negative serial correlations imply a reversal in stock returns.[2]

Consider again the general model presented in Section 2. We first observe that the autocorrelation structure of returns in rational expectation equilibria is determined to a large extent by the assumed process for the stochastic supply of equity Θ_t. The market clearing condition in Eq. (5) implies that the unconditional autocovariance of excess returns Q_t is solely determined by the contemporaneous covariance between excess returns and the supply of equity:

$$\text{Cov}(Q_{t+1}, Q_t) = \Omega \text{Cov}(Q_t, \Theta_t). \tag{11}$$

Our derivation of Eq. (11) explicitly assumes that investors are myopic and have no hedging demand. However, Eq. (11) would also hold in a more general setting. If investors had a CARA utility function over the long horizon, any hedging demand resulting solely from the asymmetric information would be a linear function of investors' forecast errors. Since by definition, the forecast errors are orthogonal to the public information set including past prices and dividends, the covariance of the hedging demand with past returns would be zero and Eq. (11) would still hold. For example,

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[2]The empirical literature on momentum is very large. See Jegadeesh and Titman [30] for the initial contribution and Jegadeesh and Titman [31] for a review.

Eq. (11) is valid in the setups of He and Wang [26] and Wang [49].

Thus to get further insights, we have to make some specific assumptions about Θ_t. The next theorem characterizes the autocorrelation of returns when the stochastic supply of equity follows an AR(1) process.

Theorem 2 *Suppose Θ_t follows an AR(1) process: $\Theta_t = a_\Theta \Theta_{t-1} + b\varepsilon_t^\Theta$. If $0 \leq a_\Theta < 1/R$ and all current and past dividends as well as all investors' exogenous signals about dividends are uncorrelated with Θ_t, then excess returns are negatively autocorrelated: $Cov(Q_{t+1}, Q_t) < 0$. Conversely, if $1/R < a_\Theta < 1$ then $Cov(Q_{t+1}, Q_t) > 0$.*

Proof. First, we demonstrate that if the stated conditions hold, then $\text{Cov}(\Theta_t, P_t) \neq 0$. Indeed, suppose the opposite is true, i.e., $\text{Cov}(\Theta_t, P_t) = 0$. Since Θ_t is an AR(1) process,

$$\text{Cov}\left(\Theta_t, P_s\right) = a_\Theta^{t-s} \text{Cov}\left(\Theta_s, P_s\right) = 0, \quad s = t, t-1, \ldots \tag{12}$$

Hence, Θ_t is uncorrelated with past prices. Using the representation for the price from Eq. (7)

$$P_t = -\lambda \Theta_t + \sum_{s=0}^{\infty} \beta^{s+1} \bar{E}_t ... \bar{E}_{t+s} \left(D_{t+s+1} - \lambda \Theta_{t+s+1}\right)$$

and computing the covariance of both sides with Θ_t we get

$$\text{Cov}\left(\lambda \Theta_t, \sum_{s=0}^{\infty} \beta^{s+1} \bar{E}_t ... \bar{E}_{t+s} \left(D_{t+s+1} - \lambda \Theta_{t+s+1}\right)\right) > 0. \tag{13}$$

However, all expectations in Eq. (13) are functions of past prices and agents' exogenous signals, which are orthogonal to Θ_t. Thus,

$$\text{Cov}\left(\lambda \Theta_t, \sum_{s=0}^{\infty} \beta^{s+1} \bar{E}_t ... \bar{E}_{t+s} \left(D_{t+s+1} - \lambda \Theta_{t+s+1}\right)\right) = 0. \tag{14}$$

We arrive at a contradiction implying that $\text{Cov}(\Theta_t, P_t) \neq 0$. A direct computation shows that in the case of full information $\text{Cov}(\Theta_t, P_t) < 0$. Hence, in any information dispersion setup, which can be continuously transformed into the full information setup by changing precisions of signals, prices are negatively correlated with Θ_t. Combining this result and Eq. (12) we get

$$\text{Cov}\left(Q_{t+1}, Q_s\right) = \Omega a_\Theta^{t-s}(1 - a_\Theta R)\text{Cov}(\Theta_t, P_t), \quad s = t, t-1, \ldots . \tag{15}$$

Hence, the autocorrelations of returns are negative when Θ_t follows an AR(1) process with $0 \leq a_\Theta \leq 1/R$. ∎

It is interesting to note that Theorem 2 seemingly contradicts the findings of Brown and Jennings [15] who obtain a positive autocorrelation of returns in a rational expectation equilibrium with persistent noise trading. The two results can be reconciled by realizing that while our model is stationary and has an infinite horizon, the model of Brown and Jennings (1989) has only two periods. In their case, several artificial features such as deterministic changes in conditional variance of returns or the impact of boundary conditions can have substantial effect on autocorrelations of returns. In a stationary model, we eliminate these obscuring effects and show that asymmetric information alone cannot generate momentum when the demand from noise traders follows an AR(1) process.

The impossibility of generating momentum with AR(1) traders emphasizes that slow diffusion of information or breakdown of the law of iterated expectations are not sufficient conditions for momentum. In our model, the autocorrelation of returns is determined by the interplay of two different effects. On the one hand, the downward sloping demand curve of rational investors together with the mean reverting stochastic supply produce a negative autocorrelation of returns (this effect is easiest to see in the full information case). On the other hand, since investors are heterogeneously informed, the diffusion of information into prices may be slow, and this increases the autocorrelation of returns. Hence, the statistical properties of returns depend on which effect is stronger. When the demand from noise traders follows an AR(1) process, Eq. (15) shows that the latter effect is always stronger.

The weakness of the information effect may not be specific to the case with the AR(1) equity supply. A distinctive property of the AR(1) process is that today's expectations of future realizations decline monotonically (in absolute terms) with the horizon. As a result, the supply of equity and prices tend to revert. Thus, it is reasonable to expect that for all processes with $a_s \geqslant a_{s+1}$, where $a_s = E(\varepsilon_{t-s}^\Theta \Theta_t)$ the effect of heterogeneous information is weaker than the effect of mean reverting equity supply.

We would like to emphasize that our conclusion crucially depends on the specification of equity supply as an exogenous process. If instead we assume that it is endogenously determined (e.g., as a function of past returns), the autocorrelation of returns may become positive. To illustrate this point, suppose that the time variation in equity supply is produced by feedback traders (in the terminology of De Long,

Shleifer, Summers, and Waldmann [17]), whose demand for the stock is proportional to the last period return. Since Eq. (11) still holds, we immediately obtain that in this case, the first-order autocorrelation of returns is positive.

Although the asymmetry of information cannot change the sign of return autocorrelations when Θ_t follows an $AR(1)$ process, the situation is different if the exogenous equity supply follows a more general process. As an example, we consider the case when Θ_t is an $AR(2)$ process defined as $\Theta_t = (a_{1\Theta} + a_{2\Theta})\Theta_{t-1} - a_{1\Theta}a_{2\Theta}\Theta_{t-2} + b\varepsilon_t^\Theta$. To demonstrate the effect of asymmetric information on the dynamics of returns, we choose the parameters $a_{1\Theta}$ and $a_{2\Theta}$ so that the serial correlation of returns is negative in the full information case. Specifically, we set $a_{1\Theta} = 0.55$ and $a_{2\Theta} = 0.8$. The innovations ε_t^Θ are i.i.d. with the standard normal distribution and $b = 0.2$. Also, we assume that the dividend D_t consists of three components: $D_t = D_t^1 + D_t^2 + b_D\varepsilon_t^D$. The first two components follow $AR(1)$ processes:

$$D_t^i = aD_{t-1}^i + b_i\varepsilon_t^i, \qquad i = 1, 2.$$

The shocks ε_t^1, ε_t^2, and ε_t^D are also i.i.d. with the standard normal distribution. In the numerical example we set $a = 0.8$, $b_1 = b_2 = 0.7$, and $b_D = 0.9$. For simplicity, the risk-free rate is set to be zero, so $R = 1$.[3] As in Section 3, we assume that there are two types of investors. Type i investors perfectly observe D_t^i, so $\mathcal{F}_t^i = \{P_s, D_s^i\}_{-\infty}^t$. This model contains all elements required for the existence of the "forecasting the forecasts of others" problem and it is very likely that it does not admit a Markovian dynamics. Hence, the statistical properties of returns can be explored only numerically. The detailed description of our computations is relegated to Appendix B.

Table I reports serial correlations of returns in models with the full information and heterogeneous information. For the chosen set of parameters, returns are negatively autocorrelated when investors are fully informed. However, the autocorrelation is positive for the same fundamentals in the heterogeneous information case. It means that if stochastic supply follows an $AR(2)$ process, momentum can exist in the rational expectation equilibrium. Intuitively, in this case the negative autocorrelation of returns produced by mean reversion of the equity supply is relatively weak, and the information dispersion effect is strong enough to change the sign of the autocorrelation from negative to positive. However, the magnitudes of autocorrelations are rather small,

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[3]Although our results are obtained for this specific combination of parameters, we also examined alternative models with other parameters and found that our qualitative conclusions are not driven by this particular choice.

	Full Info	Heterogeneous Info
$\mathrm{Corr}(Q_{t+1}, Q_t)$	-0.0002	0.0001
$\mathrm{Std}(\Delta_t)$	0	1.6574
$\mathrm{Corr}(\Delta_t, \Delta_{t-1})$	—	0.6217
$\mathrm{Corr}(\Theta_t, \Delta_t)$	—	-0.3930
$\mathrm{Corr}(\varepsilon_t^D, \Delta_t)$	—	0.6121

Table I: Statistics of returns Q_t and the informational component of the price Δ_t in the full information and heterogeneous information equilibria.

and our setup falls short of explaining momentum quantitatively.

To better understand the impact of asymmetric information on dynamics of returns, it is convenient to decompose the price from Eq. (7) into a component that is determined solely by the fundamentals and a correction term Δ_t produced by heterogenous information. Denoting the expectation operator with respect to full information at time t as E_t, we obtain the following representation for the price:

$$P_t = \underbrace{-\lambda\Theta_t + \sum_{s=0}^{\infty} \beta^{s+1} E_t(D_{t+s+1} - \lambda\Theta_{t+s+1})}_{\text{full information component}} + \Delta_t, \tag{16}$$

where

$$\Delta_t = \sum_{s=0}^{\infty} \beta^{s+1}(\bar{E}_t...\bar{E}_{t+s} - E_t)(D_{t+s+1} - \lambda\Theta_{t+s+1}). \tag{17}$$

The term Δ_t represents the effect of asymmetric information. If investors are fully informed, the chain of iterated average expectations collapses and the information term disappears: $\Delta_t = 0$.[4] In the heterogeneous information setup, due to the violation of the law of iterated expectations for average expectations $\bar{E}_t$, all higher order expectations are different. Therefore, Δ_t is non-trivial and has two effects. On the one hand, it produces a gap between the price of the firm and its fundamental value. On the other hand, it affects the dynamics and statistical properties of prices and returns.

To illustrate the impact of Δ_t, we compute its volatility and autocorrelation as well as its correlations with Θ_t and ε_t^D. The results are presented in Table I. First, note that in the heterogeneous information case the standard deviation of Δ_t is quite high and represents a substantial part of the price variation. Hence, higher order expectations play an important role in the price formation. Second, Table I shows that Δ_t
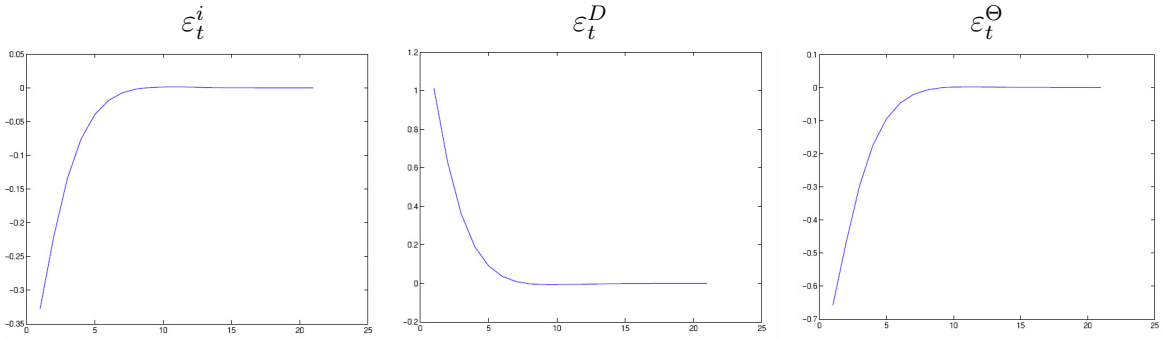
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[4]In the full information solution, the constant λ is also different.

Figure 1: Impulse response of Δ_t to the shocks ε_t^i, ε_t^D, and ε_t^Θ.

is highly persistent under heterogeneous information: its autocorrelation coefficient is 0.62. Intuitively, when investors are heterogeneously informed, they all make forecasting errors. The errors made by one type of investors depend not only on fundamentals but also on the errors made by the other type. In the absence of fully informed arbitrageurs, errors are persistent since it takes several periods for investors to realize that they made mistakes and to correct them.

This intuition is also illustrated by the impulse response functions of Δ_t presented in Figure 1. Indeed, Δ_t has high negative loadings on ε_t^i and ε_t^Θ and positive loadings on ε_t^D. When there is a shock to one of the persistent components D_t^i, the investors who observe D_t^i increase their demand and drive the price upward. Investors who cannot observe D_t^i partially attribute the price increase to a negative shock to the aggregate asset supply Θ_t and underreact relative to the full information case. Hence, the price is lower and Δ_t is negative. However, observing subsequent realizations of prices and dividends, uninformed investors infer that is was a shock to D_t^i and also increase their demand, driving the price up towards its full information level. A similar logic explains negative loadings of Δ_t on ε_t^Θ.

A positive reaction of Δ_t to ε_t^D is also quite natural. When ε_t^D is positive, investors observe a higher dividend. Unable to disentangle the persistent components D_t^1 and D_t^2 from ε_t^D, they partially attribute the increase in dividends to positive ε_t^1 or ε_t^2. As a result, investors value the asset more and drive the price up. As time passes and new realizations of dividends become available, they learn their mistake and Δ_t decreases.

The described relations between Δ_t and the fundamental shocks also explain the correlations reported in Table I. Indeed, Δ_t is negatively correlated with Θ_t and positively with ε_t^D. Given that ε_t^D are i.i.d., the latter correlation is a direct consequence of the positive impulse response of Δ_t to ε_t^D. The former correlation is less straightforward, since Θ_t is a persistent process. However, if learning is relatively fast, then the

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sign of the correlation $\text{Corr}(\Theta_t, \Delta_t)$ is mostly determined by the correlation between Δ_t and ε_t^Θ, which is negative.

Overall, when investors are heterogeneously informed, the information may penetrate into prices quite slowly. As a result, the sign of the serial correlation of returns may change, and returns can be positively autocorrelated in the equilibrium, an outcome that points towards a rational explanation for momentum. This mechanism is distinct from behavioral stories, which attribute momentum to under-reaction or delayed over-reaction caused by cognitive biases (Barberis, Shleifer, and Vishny [10]; Daniel, Hirshleifer, and Subrahmanyam [16]; Hong and Stein [27]). It is also quite different from existing rational explanations, which are based on partial equilibrium models (Berk, Green, and Naik [11]; Johnson [32]; Sagi and Seasholes [42]).

5 Concluding remarks

In this paper, we develop a dynamic asset pricing model with heterogeneous information and study the structure of its rational expectation equilibrium. The model demonstrates the mechanics of the "forecasting the forecasts of others" problem and shows that an infinite hierarchy of higher order expectations naturally arises in such a setting. Moreover, we give a formal proof that under very mild conditions it is impossible to describe the dynamics of the economy using a finite number of state variables.

Due to the complexity of the problem, we make several simplifying assumptions. However, it is reasonable to believe that the intuition behind our conclusions is also valid in more realistic models. First, while our model considers only one firm, it would be interesting to analyze a similar setup with multiple firms. Such an analysis would extend existing static models (Admati [1]; Easley and O'Hara [19]; Hughes, Liu, and Liu [28]) and grant new insights into the effect of information distribution on cross-correlations of prices and returns.[5] Next, in this paper we consider myopic investors with no hedging demand. This significantly simplifies the analysis; otherwise we would have to solve a dynamic program with an infinite dimensional state space. The impact of hedging could be non-trivial and requires further investigation.

In our model, agents are exogenously endowed with their information; they can neither buy new information, nor release their own if they find this exchange profitable. It might be interesting to relax this assumption and introduce a market for information.

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[5]See Biais, Bossaerts, and Spatt [14] and Watanabe [51] for some results in the dynamic setting.

Such extensions have been studied in a static setting by Admati and Pfleiderer [2], Verrecchia [46], and others, but the dynamic properties of the market for information have not been thoroughly explored[6].

Although our analysis pertains mostly to asset pricing, it may provide new insights into various aspects of the " forecasting the forecasts of others" problem and iterated expectations in general. The intuition behind our results is also relevant to other fields. For example, higher order expectations naturally arise in several different macroeconomic settings (Lorenzoni [35]; Rondina [41]; Woodford [52]), in the analysis of exchange rate dynamics (Bacchetta and Wincoop [6]), in models of industrial organization where firms have to extract information about unknown cost structure of their competitors (Vives [47]). Adapting our approach to the analysis of higher order expectations in these fields may be a fruitful direction for future research.

Appendix A

Proof of Theorem 1

The main idea of the proof is to transform the analysis from the time domain into the frequency domain. Therefore, before proceeding with the proof, we introduce some notations and state several relevant results from the theory of stochastic processes.

Time and Frequency Domains

Consider an infinite sequence of three-dimensional random vectors $\varepsilon_t = (\varepsilon_t^1, \varepsilon_t^2, \varepsilon_t^\Theta)$, $t \in \mathbb{Z}$. The random variables ε_t^1, ε_t^2, ε_t^Θ are independent, normal, and have zero mean and unit variance. We say that the process X_t is a stationary regular Gaussian process adapted to the filtration generated by ε_t if it can be written as

$$X_t = \sum_{k=0}^{\infty} x_k \varepsilon'_{t-k},$$

where the three-dimensional constant coefficients x_k are square summable:

$$\sum_{k=0}^{\infty} x_k x'_k < \infty.$$

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[6]See Naik [37] for analysis of a monopolistic information market in a dynamic framework.

The set of such processes is a pre-Hilbert space H with an inner product of vectors

$$X_t = \sum_{k=0}^{\infty} x_k \varepsilon'_{t-k} \quad \text{and} \quad Y_t = \sum_{k=0}^{\infty} y_k \varepsilon'_{t-k}$$

defined as

$$E[X_t Y_t] = \sum_{k=0}^{\infty} x_k y'_k.$$

Thus, two processes are orthogonal if they are uncorrelated. The infinite sequence of coefficients x_k, $k = 0, 1, \ldots$ can be viewed as a representation of the process in the time domain.

Instead of working with an infinite number of coefficients, it is often convenient to introduce the frequency domain representation. We will say that a three-dimensional function $X(z)$ is a z-representation of the process X_t if

$$X(z) = \sum_{k=0}^{\infty} x_k z^k.$$

The components of $X(z)$ are well-defined analytical functions in the unit disk $D_0 = \{z : |z| < 1\}$ in the complex plane $\mathbb{C}$. The corresponding inner product takes the following form:

$$E[X_t Y_t] = \frac{1}{2\pi i} \oint X(z) Y(z^{-1})' \frac{dz}{z}, \tag{A1}$$

where the integral is taken in the complex plane along the contour $|z| = 1$. Eq. (A1) creates a bridge between the correlation structure of random variables and the location of singularities of their z-representation functions.

Let L be a shift operator defined as $L\varepsilon_t = \varepsilon_{t-1}$. Using its z-representation we can write the process X_t as:

$$X_t = X(L)\varepsilon'_t. \tag{A2}$$

From (A2) it is clear that $zX(z)$ corresponds to the lagged process X_{t-1}.

Markovian Dynamics in the Frequency Domain

The concept of Markovian dynamics introduced in Section 3 can be concisely defined in the frequency domain representation. We rely on the following result from the theory of stationary Gaussian processes (see Doob [18] for the original results and Ibragimov and Rozanov [29] for a textbook treatment).

Theorem 3 *Let X_t be a regular stationary Gaussian process with discrete time defined on a complete probability space $(\Omega, \mathcal{F}, \mu)$. The process X_t admits Markovian dynamics with a finite number of Gaussian state variables if and only if its z-representation is a rational function of z.*

The following examples serve to illustrate the theorem. They show that several well-known processes admitting Markovian dynamics have indeed rational z-representation.

Example 1. Let X_t be i.i.d. over time: $X_t = \varepsilon_t$. In this case, $x_0 = 1$ and $x_k = 0$, for $k = 1, 2....$. Thus $X(z) \equiv 1$.

Example 2. Let X_t be an AR(1) process: $X_t = aX_{t-1} + \varepsilon_t$. In this case, $x_k = a^k$, for $k = 0, 1, ...$. Thus $X(z) = (1 - az)^{-1}$.

Example 3. Let X_t be an MA(1) process: $X_t = \varepsilon_t - \theta\varepsilon_{t-1}$. In this case, $x_0 = 1$, $x_1 = -\theta$, $x_k = 0$, for $k = 2, 3, ...$. Thus $X(z) = 1 - \theta z$.

Main Part of the Proof

We showed in Section 3 that the equilibrium price function satisfies the following equation:

$$P_t = -\lambda(V_t + b\varepsilon_t^\Theta) + \beta\bar{E}_t[P_{t+1}], \tag{A3}$$

where $0 < \beta < 1$. Recall that $\lambda = \beta\Omega$, where

$$\Omega^{-1} = \frac{1}{2\alpha}\left(\frac{1}{\text{Var}[Q_{t+1}|\mathcal{F}_t^1]} + \frac{1}{\text{Var}[Q_{t+1}|\mathcal{F}_t^2]}\right).$$

Hence, λ is endogenous, i.e., it is determined in the equilibrium along with the process for P_t. However, to prove that the equilibrium does not admit Markovian dynamics it is sufficient to prove that Eq. (A3) has no solution that admits Markovian dynamics for any arbitrary $\lambda \neq 0$ treated as an exogenous constant. This is the path we follow below.

Because we focus on a stationary linear equilibrium, we assume that the price P_t is a stationary regular adapted Gaussian process represented as

$$P_t = \sum_{k=0}^{\infty} f_k \varepsilon_{t-k}^1 + \sum_{k=0}^{\infty} f_k \varepsilon_{t-k}^2 + b\sum_{k=0}^{\infty} f_k^\Theta \varepsilon_{t-k}^\Theta, \tag{A4}$$

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or in the frequency domain:

$$P_t = f(L)\varepsilon_t^1 + f(L)\varepsilon_t^2 + bf^{\Theta}(L)\varepsilon_t^{\Theta}, \quad f(z) = \sum_{k=0}^{\infty} f_k z^k, \quad f_{\Theta}(z) = \sum_{k=0}^{\infty} f_k^{\Theta} z^k. \tag{A5}$$

We prove Theorem 1 in several steps. First, we use the frequency domain representation of the equilibrium price and reformulate the equilibrium conditions in terms of the functions $f(z)$ and $f_{\Theta}(z)$. Second, we assume that the system of processes $\{V^1, V^2, \varepsilon^{\Theta}, P\}$ admits Markovian dynamics. In this case, according to Theorem 3 $f(z)$ and $f_{\Theta}(z)$ must be rational. Finally, we demonstrate that rational $f(z)$ and $f_{\Theta}(z)$ cannot satisfy the equilibrium conditions. This contradiction proves Theorem 1.

Step 1. It is convenient to start with the filtering problem of each agent. Eq. (A3) implies that each type i investor must find the best estimate of P_{t+1} given his information set $\mathcal{F}_t^i = \{V_s^i, P_s\}_{-\infty}^t$. Since some components of P_t are known to investor i, the information set $\mathcal{F}_t^i = \{V_s^i, P_s\}_{-\infty}^t$ is identical to $\mathcal{F}_t^i = \{V_s^i, Z_s^i\}_{-\infty}^t$, where

$$Z_t^i = f(L)\varepsilon_t^{-i} + bf_{\Theta}(L)\varepsilon_t^{\Theta}. \tag{A6}$$

The filtering problem is equivalent to finding a projector G such that

$$E[Z_{t+1}^i | \mathcal{F}_t^i] = G(L)Z_t^i. \tag{A7}$$

By definition, $Z_{t+1}^i - G(L)Z_t^i$ is orthogonal to all Z_s^i, $s \leq t$:

$$E[\left(Z_{t+1}^i - G(L)Z_t^i\right) Z_s^i] = 0. \tag{A8}$$

Calculating expectations we get

$$E[Z_{t+1}^i Z_s^i] = \frac{1}{2\pi i} \oint \left\{ \frac{1}{z^2} f(z) \frac{1}{z^{t-s}} f(z^{-1}) + b^2 \frac{1}{z^2} f_{\Theta}(z) \frac{1}{z^{t-s}} f_{\Theta}(z^{-1}) \right\} dz,$$

$$E[G(L)Z_t^i Z_s^i] = \frac{1}{2\pi i} \oint \left\{ \frac{1}{z} G(z) f(z) \frac{1}{z^{t-s}} f(z^{-1}) + b^2 \frac{1}{z} G(z) f_{\Theta}(z) \frac{1}{z^{t-s}} f_{\Theta}(z^{-1}) \right\} dz.$$

After collecting all terms, the orthogonality condition (A8) takes the form

$$\frac{1}{2\pi i} \oint \frac{1}{z^k} U(z) dz = 0, \quad k = 1, 2, \ldots, \tag{A9}$$

where the function $U(z)$ is

$$U(z) = \left(\frac{1}{z} - G(z)\right)\left(f(z)f(z^{-1}) + b^2 f_\Theta(z)f_\Theta(z^{-1})\right). \tag{A10}$$

Using Eq. (A3), we get a set of equations for $f(z)$ and $f_\Theta(z)$:

$$-\frac{2\lambda}{1-az} - 2f(z) + \beta\frac{f(z) - f(0)}{z} + \beta G(z)f(z) = 0, \tag{A11}$$

$$(1 - \beta G(z))f_\Theta(z) + \lambda = 0. \tag{A12}$$

For further convenience we introduce the function

$$g(z) = \frac{1}{\lambda}(\beta G(z) - 1). \tag{A13}$$

Using this definition, Eqs. (A11), (A12), and (A10) can be rewritten as

$$f(z) = \frac{\beta f(0) - z(a\beta f(0) - 2\lambda)}{(1-az)(\beta - z + \lambda z g(z))}, \tag{A14}$$

$$f_\Theta(z) = \frac{1}{g(z)}, \tag{A15}$$

$$U(z) = \frac{\beta - z - \lambda z g(z)}{\beta z}\left(f(z)f(z^{-1}) + b^2 f_\Theta(z)f_\Theta(z^{-1})\right). \tag{A16}$$

Step 2. Assume that the system of processes $\{V^1, V^2, \varepsilon^\Theta, P\}$ admits Markovian dynamics. Then, the projection of P_{t+1} on $\mathcal{F}_t^i$ also admits Markovian dynamics. By Theorem 3, its z-representation $G(z)$ must be rational. Hence, from Eqs. (A13), (A14), and (A15), the functions $g(z)$, $f(z)$, and $f_\Theta(z)$ should be rational as well. Given the rationality of $G(z)$, $f(z)$, and $f_\Theta(z)$, Eq. (A9) implies that $U(z)$ is analytic in $\overline{\mathbb{C}}$ except for a finite number of poles within the unit circle $D_0 = \{z : |z| < 1\}$ and $U(\infty) = 0$. To prove that our model has non-Markovian dynamics, it is sufficient to show that there exist no rational functions $f(z)$, $f_\Theta(z)$, $g(z)$ and $U(z)$ such that 1) $f(z)$, $f_\Theta(z)$, and $g(z)$ are analytic inside the unit circle and $U(z)$ is analytic outside the unit circle; 2) $U(\infty) = 0$; 3) Eqs. (A14), (A15), and (A16) hold.

Step 3. Inspired by the structure of Eqs. (A14) and (A16), we introduce the

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function $H(z)$ such that

$$g(z) = (\beta - z + \lambda z g(z))H(z)/\beta \qquad \text{(A17)}$$

Remarkably, all the equilibrium functions $f(z)$, $f_\Theta(z)$, $g(z)$, and $U(z)$ can easily be rewritten in terms of the function $H(z)$ and all analyticity conditions on the equilibrium functions can be translated into conditions on the function $H(z)$. Thus, the question of the existence of the equilibrium with Markovian dynamics reduces to the question of the existence of a function $H(z)$ satisfying a set of conditions.

From Eq. (A17) the function $g(z)$ is

$$g(z) = \frac{(\beta - z)H(z)}{\beta - \lambda z H(z)}. \qquad \text{(A18)}$$

Recall that β is the implicit discount factor in Eq. (A3) satisfying $0 < \beta < 1$. Since $f_\Theta(z) = 1/g(z)$, we have

$$f_\Theta(z) = \frac{\beta - \lambda z H(z)}{(\beta - z)H(z)}. \qquad \text{(A19)}$$

Substitution of Eq. (A18) into Eq. (A14) gives

$$f(z) = \frac{\beta f(0) - z(a\beta f(0) - 2\lambda)}{\beta(1 - az)(\beta - z)}(\beta - \lambda z H(z)). \qquad \text{(A20)}$$

If $a\beta f(0) - 2\lambda \neq 0$ we can define z_1 as

$$z_1 = \frac{\beta f(0)}{a\beta f(0) - 2\lambda}$$

and rewrite the function $f(z)$ in the following form:

$$f(z) = (a\beta f(0) - 2\lambda)\frac{z_1 - z}{\beta(1 - az)(\beta - z)}(\beta - \lambda z H(z)). \qquad \text{(A21)}$$

The structure of the function $U(z)$ depends on whether $a\beta f(0) - 2\lambda \neq 0$ (z_1 is finite) or $a\beta f(0) - 2\lambda = 0$ ($z_1 = \infty$). In the first case, using Eqs. (A15), (A18), and (A21)

$U(z)$ can be rewritten as

$$U(z) = \left(\frac{1}{zH(z)} - \frac{2\lambda}{\beta} \right)$$
$$\left[\frac{(a\beta f(0) - 2\lambda)^2 (z - z_1)(z^{-1} - z_1)}{\beta^2 (1 - az)(1 - az^{-1})} H(z)H(z^{-1}) + b^2 \right] \frac{1}{g(z^{-1})}. \quad (A22)$$

Since $g(z)$ does not have poles in D_0 (and, consequently, $g(z^{-1})$ does not have poles in D_∞), the analyticity of $U(z)$ in D_∞ implies the analyticity of $U^g(z) = U(z)g(z^{-1})$ in D_∞. Hence,

$$U^g(z) = \left(\frac{1}{zH(z)} - \frac{2\lambda}{\beta} \right) \left[\frac{(a\beta f(0) - 2\lambda)^2 (z - z_1)(z^{-1} - z_1)}{\beta^2 (1 - az)(1 - az^{-1})} H(z)H(z^{-1}) + b^2 \right] \quad (A23)$$

must be analytical in D_∞. In a special case $a\beta f(0) - 2\lambda = 0$ (i.e., $z_1 = \infty$), the function $U^g(z)$ takes a simpler form:

$$U^g(z) = \left(\frac{1}{zH(z)} - \frac{2\lambda}{\beta} \right) \left[\frac{4\lambda^2}{a^2\beta^2 (1 - az)(1 - az^{-1})} H(z)H(z^{-1}) + b^2 \right]. \quad (A24)$$

First, consider the general case with $a\beta f(0) - 2\lambda \neq 0$ (finite z_1). The following lemma describes the properties of $H(z)$.

Lemma 1 *The function $H(z)$ can satisfy the equilibrium conditions only if*

1. *$H(z)$ is rational;*

2. *$H(\beta) = \frac{1}{\lambda}$;*

3. *$H(z)$ may have poles only at z_1 or z_1^{-1}, where $z_1 \neq 0$, and the order of poles cannot exceed 1;*

4. *$H(z)$ may have zeros only at a^{-1} and $z_1 \in D_\infty$, and the order of zeros cannot exceed 1.*

Proof.

Statement 1 follows from the assumed rationality of $f_\Theta(z)$ and Eq. (A19).

Since $f_\Theta(z)$ must be analytic in D_0, the pole $z = \beta$ in Eq. (A19) must cancel, and this implies $1 - \lambda H(\beta) = 0$. This is Statement 2.

Also, the analyticity of $f_\Theta(z)$ implies that $H(z) \neq 0$ for $z \in D_0$. For the rest of the statements we need the conditions on functions $f(z)$ and $U(z)$.

By assumption, $f(z)$ is analytic in D_0. The pole β in Eq. (A21) disappears since $1 - \lambda H(\beta) = 0$ (Statement 2), but $H(z)$ may have first-order poles at $z = 0$ and $z = z_1$ (if $z_1 \neq 0$) or a second-order pole at $z = 0$ (if $z_1 = 0$), which potentially may not violate the analyticity of $f(z)$. An existence of a pole at $z = 0$ of any order would mean that the function $g(z)$ has a pole at $z = 0$ (this follows from Eq. (A18)) in contradiction to its analyticity in D_0. Hence, in D_0 the function $H(z)$ may have only the pole $z = z_1 \neq 0$, and the order of this pole cannot exceed 1. In D_∞ the possible poles are determined by the analyticity of $U(z)$. Indeed, a pole in $H(z)$ must be canceled by a zero in

$$\frac{(z - z_1)(z^{-1} - z_1)}{(1 - az)(1 - az^{-1})} H(z^{-1}).$$

Clearly, it can be 1) either z_1 or z_1^{-1} depending on whether z_1 is in D_∞ or D_0, or 2) a zero of $H(z^{-1})$. The second option contradicts $H(z) \neq 0$ for $z \in D_0$ and we arrive at Statement 3.

Since $H(z) \neq 0$ for $z \in D_0$, $H(z)$ may have zeros in D_∞ only. However, a zero of $H(z)$ may produce a pole in $U(z)$ (the term $b^2/(zH(z))$ in Eq. (A23)) which would contradict its analyticity in D_∞. This happens unless the pole is canceled by a pole in another term in Eq. (A23). Such a pole can be either a^{-1} or a pole of $H(z^{-1})$. Since for $z^{-1} \in D_0$ the function $H(z^{-1})$ may have a first-order pole in z_1 only, Statement 4 follows. ∎

Lemma 1 implies that if $z_1 = 0$ (i.e., $f(0) = 0$), then the function $H(z)$ must have a linear form $H(z) = A(z - a^{-1})^k$, where A is a non-zero constant and k is either 0 or 1. In the latter case, Eq. (A23) and the condition $U^g(\infty) = 0$ yields the following equation for A:

$$\frac{4\lambda^2 A^2}{a^2 \beta^2} + b^2 = 0. \tag{A25}$$

Obviously, it cannot be satisfied. If $H(z)$ is a constant ($k = 0$) then the condition $U^g(\infty) = 0$ reduces to $b^2 = 0$ which is also violated. Thus, we have showed that if $z_1 = 0$ then the equilibrium does not admit Markovian dynamics.

Consider now the case of $z_1 \neq 0$. According to Lemma 1, if $z_1 \neq 0$ the function $H(z)$ must have the following representation:

$$H(z) = \frac{A(z - z_1)^{k_1}(z - a^{-1})^{k_2}}{(z - z_1)^{p_1}(z - z_1^{-1})^{p_2}},$$

where k_1, k_2, p_1, and p_2 can be either 0 or 1, $k_1 \neq 1$ if $p_1 = 1$, and A is a constant. Let the number of poles in $H(z)$ be P and the number of zeros be K: $K = k_1 + k_2$,

$P = p_1 + p_2$. The condition $U^g(\infty) = 0$ imposes some restrictions on K and P, summarized in Lemma 2.

Lemma 2 *If $z_1 \neq 0$, then either $P = K$ or $P = K + 1$.*

Proof. Since $z_1 \neq 0$, we have

$$\frac{(z - z_1)(z^{-1} - z_1)}{(1 - az)(1 - az^{-1})} \longrightarrow \frac{z_1}{a} \neq 0 \quad \text{as} \quad z \longrightarrow \infty.$$

If $P < K$ we have $H(z) \to \infty$ as $z \to \infty$. Because $H(0) = const \neq 0$ (Lemma 1) the function $U^g(z)$ from Eq. (A23) is not finite at infinity: $U^g(z) \to \infty$. If $P > K + 1$ then $zH(z) \to 0$ as $z \to \infty$ and again $U^g(z) \to \infty$. $\blacksquare$

The asymptotic condition $U^g(\infty) = 0$ also imposes restrictions on the constant A. If $z_1 \neq 0$ and $P = K$, then the condition $U^g(\infty) = 0$ implies that

$$f(0)^2 \frac{z_1^{k_1 - p_1 + p_2 - 1}}{a^{k_2 + 1}} A^2 + b^2 = 0. \tag{A26}$$

If $P = K + 1$ the condition $U^g(\infty) = 0$ yields

$$\frac{1}{A} - \frac{2\lambda}{\beta} = 0. \tag{A27}$$

Note that the constant A is also fixed by Lemma 1, which states that $H(\beta) = 1/\lambda$. Thus,

$$A = \frac{1}{\lambda} \frac{(\beta - z_1)^{p_1} (\beta - z_1^{-1})^{p_2}}{(\beta - z_1)^{k_1} (\beta - a^{-1})^{k_2}}. \tag{A28}$$

Combining the implications of Lemmas 1 and 2, it is convenient to summarize all possible cases in one table:

	poles			
zeros	$\varnothing$	z_1	z_1^{-1}	z_1, z_1^{-1}
$\varnothing$	(1)	(2)	(3)	X
z_1	X	X	(4)	X
a^{-1}	X	(5)	(6)	(7)
z_1, a^{-1}	X	X	X	X

Note that according to Lemma 1 in all cases involving z_1 we assume that $z_1 \neq 0$. In the rest of the proof, we show that none of the seven cases can realize.

Case 1: $k_1 = k_2 = p_1 = p_2 = 0$, i.e., $H(z) = A$.

In this case, Eqs. (A26) and (A28) reduce to

$$\frac{f(0)^2}{az_1}A^2 + b^2 = 0, \qquad A = \frac{1}{\lambda}. \tag{A29}$$

The analyticity of the function $U^g(z)$ in D_∞ is achieved only if the pole $z = a^{-1}$ cancels out. This can happen for three reasons:

1. $z_1 = a$;

2. $z_1 = a^{-1}$;

3. $-\frac{\lambda}{\beta} + \frac{a}{2A} = 0$.

If $z_1 = a$, Eq. (A29) reduces to

$$\frac{f(0)^2}{a^2}A^2 + b^2 = 0,$$

which obviously does not have solutions. Similarly, if $z_1 = a^{-1}$, Eq. (A29) yields

$$f(0)^2 A^2 + b^2 = 0,$$

which also cannot be satisfied. In the third case, the exogenous parameters of the model should satisfy a very specific condition

$$a\beta = 2, \tag{A30}$$

which cannot be fulfilled since $0 < a < 1$, $0 < \beta < 1$.

Case 2: $k_1 = k_2 = p_2 = 0$, $p_1 = 1$, i.e., $H(z) = A/(z - z_1)$.

In this case, Eqs. (A27) and (A28) reduce to

$$A = \frac{\beta}{2\lambda}, \qquad A = \frac{\beta - z_1}{\lambda}.$$

Combining these equations we get $z_1 = \beta/2$. The pole $z = a^{-1}$ in the function $U^g(z)$ disappears only if

$$-\frac{\lambda}{\beta} + \frac{1 - az_1}{2A} = 0.$$

Substituting for A and z_1 in this equation yields $a\beta = 0$, which cannot be satisfied.

Case 3: $k_1 = k_2 = p_1 = 0$, $p_2 = 1$, i.e., $H(z) = A/(z - z_1^{-1})$.

Similar to case 2, Eqs. (A27) and (A28) reduce to

$$A = \frac{\beta}{2\lambda}, \qquad A = \frac{\beta - z_1^{-1}}{\lambda}.$$

Combining these equations we get $z_1^{-1} = \beta/2$. The pole $z = a^{-1}$ in the function $U^g(z)$ disappears only if

$$-\frac{\lambda}{\beta} + \frac{1 - \frac{a}{z_1}}{2A} = 0.$$

After simple algebraic manipulations we again get $a\beta = 0$, which cannot be satisfied.

Case 4: $k_2 = p_1 = 0$, $k_1 = p_2 = 1$, i.e., $H(z) = A(z - z_1)/(z - z_1^{-1})$.

Since $H(z)$ cannot have zeros in D_0 (Lemma 1), $z_1 \in D_\infty$. However, in this case a pole of $U^g(z)$ in D_∞ arises, and it can be canceled only if $z_1 = a^{-1}$. Eq. (A26) reduces to

$$f(0)^2 \frac{A^2}{a^2} + b^2 = 0 \tag{A31}$$

and cannot be satisfied.

Case 5: $k_1 = p_2 = 0$, $k_2 = p_1 = 1$, i.e., $H(z) = A(z - a^{-1})/(z - z_1)$, $z_1 \neq a^{-1}$.

In this case Eq. (A26) reduces to

$$\frac{f(0)^2}{a^2 z_1^2} A^2 + b^2 = 0$$

which cannot be satisfied.

Case 6: $k_1 = p_1 = 0$, $k_2 = p_2 = 1$, i.e., $H(z) = A(z - a^{-1})/(z - z_1^{-1})$, $z_1 \neq a$.

Similar to case 5, Eq. (A26) reduces to

$$\frac{f(0)^2}{a^2} A^2 + b^2 = 0 \tag{A32}$$

which cannot be satisfied.

Case 7: $k_1 = 0$, $k_2 = p_1 = p_2 = 1$, i.e., $H(z) = A(z - a^{-1})/((z - z_1)(z - z_1^{-1}))$, $z_1 \neq a, z_1 \neq a^{-1}$.

Note that in this case the function $U^g(z)$ always has a pole in D_∞ (z_1 or z_1^{-1}), which does not cancel out. This violates the analyticity of $U^g(z)$ in D_∞ and precludes case 7.

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Finally, consider a special case in which $a\beta f(0) - 2\lambda = 0$ (it can be understood as $z_1 = \infty$). Lemma 3 below is an analog of Lemma 1; it describes the conditions on the function $H(z)$.

Lemma 3 *The function $H(z)$ can satisfy the equilibrium conditions only if*

1. *$H(z)$ is rational;*

2. *$H(\beta) = \frac{1}{\lambda}$;*

3. *$H(z)$ is analytic in the whole complex plane $\mathbb{C}$;*

4. *$H(z)$ may have a zero at a^{-1} only, and the order of this zero cannot exceed 1.*

Proof.

Similar to Lemma 1, Statements 1 and 2 follow from the required conditions on $f_\Theta(z)$ and $g(z)$. Also, the analyticity of $f_\Theta(z)$ implies that $H(z) \neq 0$ for $z \in D_0$. Next, Eq. (A20) can be rewritten as

$$f(z) = \frac{2\lambda}{a\beta(1-az)(\beta-z)}(\beta - \lambda z H(z)). \tag{A33}$$

By assumption, $f(z)$ is analytic in D_0. The pole z_0 in Eq. (A33) disappears since $1 - \lambda H(\beta) = 0$ (Statement 2), but $H(z)$ may have a first-order pole at $z = 0$ which potentially may not violate the analyticity of $f(z)$. However, an existence of a pole at $z = 0$ of any order would mean that the function $g(z)$ has a pole at $z = 0$ (this follows from Eq. (A18)) in contradiction to its analyticity in D_0. Hence, the function $H(z)$ is analytic in D_0. In D_∞ the possible poles are determined by the analyticity of $U(z)$. Indeed, from Eq. (A24) a pole in $H(z)$ must be canceled by a zero in

$$\frac{1}{(1-az)(1-az^{-1})}H(z^{-1}).$$

Clearly, it can be only a zero of $H(z^{-1})$. However, such zeros don't exist since $H(z) \neq 0$ for $z \in D_0$. Thus, we arrive at Statement 3.

Since $H(z) \neq 0$ for $z \in D_0$, $H(z)$ may have zeros only in D_∞. However, a zero of $H(z)$ may produce a pole in $U(z)$ (the term $b^2/(zH(z))$ in Eq. (A24)) which would contradict its analyticity in D_∞. This happens unless the pole is canceled by a pole in another term in Eq. (A24). Such a pole can be either a^{-1} or a pole of $H(z^{-1})$. Since for $z^{-1} \in D_0$ the function $H(z^{-1})$ is analytic, Statement 4 follows. ∎

Taken together, the statements of Lemma 3 effectively imply that either $H(z) = A$ or $H(z) = A(1 - az)$, where A is a non-zero constant. However, in the first case Eq. (A24) yields $U^g(\infty) = -2\lambda b^2/\beta$. This is a contradiction to the condition $U^g(\infty) = 0$. In the second case, the condition $U^g(\infty) = 0$ implies

$$\frac{4\lambda^2}{a^2\beta^2}A^2 + b^2 = 0,$$

which also cannot be satisfied. This completes the proof of the theorem.

Appendix B

In this appendix, we describe the numerical method of computing approximate solutions to the version of our model presented in Section 4. To construct this approximation, we assume that all information is revealed to all investors after k periods, so the information set of investor i is $\mathcal{F}_t^i = \{P_\tau, D_\tau^i : \tau \leq t; \; D_\tau^{-i}, \Theta_\tau : \tau \leq t - k\}$. As before, $-i$ denotes an investor type other than type i. When Θ_t follows an AR(2) process, it is convenient to characterize the state of the economy Ψ_t in terms of the current values of D_t^1, D_t^2, Θ_t, Θ_{t-1}, and by their k lags:

$$\Psi_t = (\psi_t, \psi_{t-1}, \ldots, \psi_{t-k})', \quad \text{where} \quad \psi_\tau = (D_\tau^1, D_\tau^2, \Theta_\tau, \Theta_{\tau-1})'.$$

The demand of type i investors is

$$X_t^i = \omega_i E[Q_{t+1}|\mathcal{F}_t^i] = \omega_i(aD_t^i - RP_t + E[aD_t^{-i} + P_{t+1}|\mathcal{F}_t^i]).$$

We look for the equilibrium price process as a linear function of state variables: $P_t = P\Psi_t$, where P is a $(1 \times 4(k+1))$ constant matrix. In the matrix form, the dynamics of ψ_t are:

$$\psi_{t+1} = a_\psi \psi_t + \varepsilon_{t+1}^\psi,$$

where

$$a_\psi = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & -(a_{1\Theta} + a_{2\Theta}) & a_{1\Theta}a_{2\Theta} \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \varepsilon_t^\psi = \begin{pmatrix} \varepsilon_t^1 \\ \varepsilon_t^2 \\ \varepsilon_t^\Theta \\ 0 \end{pmatrix}, \quad \text{Var}(\varepsilon_t^\psi) = \begin{pmatrix} b_V^2 & 0 & 0 & 0 \\ 0 & b_V^2 & 0 & 0 \\ 0 & 0 & b^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Consequently, the dynamics of Ψ_t can be described as:

$$\Psi_{t+1} = A_\Psi \Psi_t + B_\Psi \varepsilon_{t+1}^\psi, \quad \text{where} \quad A_\Psi = \begin{pmatrix} a_\psi & 0 & \dots & 0 & 0 \\ I_4 & 0 & \dots & 0 & 0 \\ 0 & I_4 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & I_4 & 0 \end{pmatrix}, \quad B_\Psi = \begin{pmatrix} I_4 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Here I_4 is a four-dimensional unit matrix. In terms of the state variables Ψ_t the demand can be rewritten as

$$X_t^i = \omega_i(aD_t^i - RP_t + E[aD_t^{-i} + PA_\Psi \Psi_t | \mathcal{F}_t^i]).$$

Introducing $(1 \times 4(k+1))$ constant matrices $D^1 = (1,0,0,\dots,0)$, $D^2 = (0,1,0,\dots,0)$ and $D = (1,1,0,\dots,0)$, we get

$$X_t^i = -\omega_i RP_t + \omega_i(aD + PA_\Psi)E[\Psi_t | \mathcal{F}_t^i].$$

Thus, we have to calculate $E[\Psi_t | \mathcal{F}_t^i]$. Denoting the observations of agent i at time t as $y_t^i = (P_t, D_t^i)'$, we can gather all his relevant observations into one vector $Y_t^i = (y_t^i, y_{t-1}^i, \dots, y_{t-k+1}, \psi_{t-k})$. It is also convenient to introduce a set of $\tilde{P}_\tau$, $\tau = t-k+1 \dots t$ in order to separate the informative part of the price:

$$\tilde{P}_t = P_t,$$

$$\tilde{P}_{t-1} = P_{t-1} - P^k \psi_{t-k-1},$$

$$\dots$$

$$\tilde{P}_{t-k+1} = P_{t-k+1} - P^2 \psi_{t-k-1} - \dots - P^k \psi_{t-2k}.$$

Now we can put all observations in a matrix form:

$$Y_t^i = H^i \Psi_t, \quad \text{where} \quad H^i = \begin{pmatrix} h^i \\ h^i J \\ h^i J^2 \\ \vdots \\ h^i J^k \\ O_{3\times 3k} \quad I_3 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & I_4 & 0 & \dots & 0 \\ 0 & 0 & I_4 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & I_4 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad h^i = \begin{pmatrix} P \\ V^i \end{pmatrix}$$

We use the following well-known fact: if (Ψ, Y) are jointly normal with zero mean, then

$$E[\Psi|Y] = \beta'Y, \quad \text{where} \quad \beta = \text{Var}(Y)^{-1}E(Y\Psi'),$$

$$\text{Var}[\Psi|Y] = \text{Var}(\Psi) - E(Y\Psi')'\text{Var}(Y)^{-1}E(Y\Psi').$$

In our particular case we have:

$$\text{Var}(Y_t^i) = H^i\text{Var}(\Psi_t)H^{i'}.$$

$$E(Y^i\Psi') = H^i\text{Var}(\Psi_t)$$

From the dynamic equation for Ψ_t we find that

$$\text{Var}(\Psi_t) = A_\Psi\text{Var}(\Psi_t)A_\Psi' + B_\Psi\text{Var}(\varepsilon_t^\psi)B_\Psi'.$$

Iteration of this equation yields

$$\text{Var}(\Psi_t) = \sum_{l=0}^{\infty} A_\Psi^l B_\Psi\text{Var}(\varepsilon_t^\psi)B_\Psi'A_\Psi^{l\,'}.$$

Thus, the demand of agent i is

$$X_t^i = -\omega_i RP_t + \omega_i(aD + PA_\Psi)\text{Var}(\Psi_t)H^{i'}(H^i\text{Var}(\Psi_t)H^{i'})^{-1}H^i\Psi_t.$$

Imposing the market clearing condition and taking into account that there is an equal number of agents of each type we get

$$-RP_t + \frac{1}{2}(aD + PA_\Psi)\text{Var}(\Psi_t)H^{1'}(H^1\text{Var}(\Psi_t)H^{1'})^{-1}H^1\Psi_t$$
$$+ \frac{1}{2}(aD + PA_\Psi)\text{Var}(\Psi_t)H^{2'}(H^2\text{Var}(\Psi_t)H^{2'})^{-1}H^2\Psi_t = \Omega\Theta\Psi_t,$$

where $\Theta = (0,0,1,0,0,\ldots,0)$ and Ω is defined in Section 2. Rearranging the terms we get:

$$P_t = \frac{1}{2}\beta(aD + PA_\Psi)\text{Var}(\Psi_t)\times$$
$$\times \left[H^{1'}(H^1\text{Var}(\Psi_t)H^{1'})^{-1}H^1 + H^{2'}(H^2\text{Var}(\Psi_t)H^{2'})^{-1}H^2\right]\Psi_t - \lambda\Theta\Psi_t,$$

where $\lambda = \Omega/R$. Comparing this equation with the price representation $P_t = P\Psi_t$, we get a matrix equation:

$$P = \frac{1}{2}\beta(aD + PA_\Psi)\mathrm{Var}(\Psi_t)\times$$
$$\times \left(H^{1\prime}(H^1\mathrm{Var}(\Psi_t)H^{1\prime})^{-1}H^1 + H^{2\prime}(H^2\mathrm{Var}(\Psi_t)H^{2\prime})^{-1}H^2\right) - \lambda\Theta. \quad \text{(B1)}$$

As a result, when all information is revealed after k lags the equilibrium condition transforms into a system of non-linear equations (B1) that determines P. A numerical solution to these equations gives us an approximation to the REE with heterogeneous information.

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