On the autocorrelation of the stock market

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December, 2018

Abstract

I introduce an index of market autocorrelation based on the prices of index options and of forward-start index options and implement it empirically at a six-month horizon. Forward-looking autocorrelation was close to zero before the subprime crisis but has been negative since late 2008, attaining a low of around −20% at the end of 2010 and remaining below −15% subsequently. I speculate that this may reflect market perceptions about the likely reaction, via quantitative easing, of policymakers to future market moves.

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Do past returns on the market forecast future returns? Is the return on the market autocorrelated? It is well-known that any asset return has zero *risk-neutral* autocorrelation (see, for example, Samuelson (1965)).[1] But *true* autocorrelation may diverge significantly from zero—a point first made by LeRoy (1973)—and fluctuate over time. It is not even clear whether one should expect positive or negative autocorrelation; indeed, both might be present simultaneously at different horizons. The former might be attributed to the influence of return-chasing investors in the investor population, as in the models of Hong and Stein (1999) and Vayanos and Woolley (2013), or to sluggish response to information; and the latter to bid-ask bounce, to overreaction as in the model of Barberis, Greenwood, Jin and Shleifer (2015), or to the response of monetary authorities to fluctuations in asset prices.

Several authors, including Fama and French (1988), Lo and MacKinlay (1988), Poterba and Summers (1988) and Moskowitz, Ooi and Pedersen (2012), have studied the properties of *realized* autocorrelation of the market return, with results that vary depending on the horizon studied and on the sample period (on the latter, see Campbell, 2018, pp. 125–7). But how can we infer the *forward-looking* autocorrelation perceived by sophisticated investors? One straightforward approach is simply to ask investors what they think, following Shiller (1987) and others. But the expectations reported in such surveys appear to be far from rational: for example, Greenwood and Shleifer (2014) argue that times when surveyed investors are bullish are associated with low, not high, subsequent returns.

I therefore take a different approach, and ask what autocorrelation must be perceived by a rational, risk-averse investor—specifically, by an unconstrained rational investor with log utility who chooses to invest his or her wealth fully in the market. It turns out to be possible to give a precise answer to this question in terms of the prices of various types of options. The fact that the autocorrelation index is computed directly from forward-looking asset prices, rather than from historical measures, is the major innovation of the paper. The price to pay is that one has to accept the log investor's perspective as being a reasonable one to adopt. Nonetheless, related approaches have proved

[1] This statement is precisely true only if interest rates are deterministic; see below.

fruitful in forecasting returns on the stock market (Martin, 2017), on individual stocks (Martin and Wagner, 2018), and on currencies (Kremens and Martin, 2018); and the approach has the obvious advantage of bringing a novel type of evidence to bear on a classic question.

The theoretical results are derived in Section 1. They show that the autocorrelation index can be calculated from the prices of European index options and of forward-start index options. The latter options are relatively exotic, but I have been able to obtain indicative price quotes from a major investment bank for a small number of days between June 2007 and December 2013. Section 2 uses these prices to calculate the autocorrelation index. Section 3 concludes.

1 Measuring autocorrelation

Today is time t; the price of the underlying asset at time t is S_t. The goal is to measure the correlation between the return on the asset over the next period, $R_{t\to t+1}$, and the return over the following period, $R_{t+1\to t+2}$. I assume that the underlying asset does not pay dividends, so $R_{t\to t+1} = S_{t+1}/S_t$. I write the rate at which money can be risklessly invested from time u to time v as $R_{f,u\to v}$; this is assumed known at time t, so $R_{f,t\to t+1}$ and $R_{f,t\to t+2}$ are one- and two-period spot rates, and $R_{f,t+1\to t+2}$ is the forward rate from $t+1$ to $t+2$.

I write $\mathbb{E}_t^*$ for the risk-neutral expectation operator whose defining property is that the time-t price of a time-$(t+2)$ payoff X_{t+2} is $\frac{1}{R_{f,t\to t+2}}\,\mathbb{E}_t^*\,X_{t+2}$; and cov_t^* and corr_t^* for the corresponding risk-neutral covariance and risk-neutral correlation operators.

When seeking a measure of autocorrelation that can be computed directly from asset prices, the natural first thought is to consider *risk-neutral* autocorrelation. Unfortunately we have the following well-known result.

Result 1. *Suppose that interest rates are deterministic. Then the return on any asset has zero risk-neutral autocorrelation:* $\text{corr}_t^*(R_{t\to t+1}, R_{t+1\to t+2}) = 0$.

Proof. By the defining property of the risk-neutral expectation operator, we have $\mathbb{E}_t^*\,R_{t\to t+1} = R_{f,t\to t+1}$ and $\mathbb{E}_{t+1}^*\,R_{t+1\to t+2} = R_{f,t+1\to t+2}$. As interest rates are deterministic, the second equality implies that $\mathbb{E}_t^* R_{t+1\to t+2} = R_{f,t+1\to t+2}$ by the law of iterated expectations. So we can write

$$\begin{aligned}\operatorname{cov}_t^*(R_{t\to t+1}, R_{t+1\to t+2}) &= \mathbb{E}_t^* \left[(R_{t\to t+1} - R_{f,t\to t+1})(R_{t+1\to t+2} - R_{f,t+1\to t+2})\right] \\ &= \mathbb{E}_t^* \left[(R_{t\to t+1} - R_{f,t\to t+1})\, \mathbb{E}_{t+1}^* \left(R_{t+1\to t+2} - R_{f,t+1\to t+2}\right)\right] \\ &= 0,\end{aligned}$$

using the law of iterated expectations again for the second equality. □

Although interest rates are not deterministic, they are typically extremely stable by comparison with returns on stock indices, so Result 1 rules out the autocorrelation perceived by a risk-neutral investor as a useful measure. Moreover, it is easy to adapt the proof above to show that the risk-neutral autocorrelation of *excess* returns is zero even if interest rates are stochastic.

How, then, can we define a non-trivial measure of autocorrelation? This paper introduces an index that can be interpreted as the autocorrelation perceived by a rational, unconstrained investor with log utility whose wealth is fully invested in the market. The next result, which is also exploited by Martin (2017), provides the key to calculating this quantity.

Result 2. *Let X_T be some random variable of interest whose value becomes known at time T, and suppose that we can price a claim to $X_T R_{t\to T}$ delivered at time T. Then we can compute the expected value of X_T from the perspective of an investor with log utility whose wealth is invested in the market:*

$$\mathbb{E}_t\, X_T = \textit{time-}t \textit{ price of a claim to the time-}T \textit{ payoff } X_T R_{t\to T}. \qquad (1)$$

Proof. An investor with log utility who chooses to hold the market must perceive that the return on the market is growth-optimal. As the reciprocal of the growth-optimal return is an SDF, the right-hand side of (1) equals $\mathbb{E}_t \left[\frac{1}{R_{t\to T}} X_T R_{t\to T}\right]$, and the result follows. □

This result provides a general strategy for inferring the *true* expectation of the log investor from traded asset prices. If we can price the claim $X_T R_{t\to T}$ then we can infer the investor's expectation of X_T, even if X_T is not itself a tradable payoff. In particular, Result 2 will allow us to calculate $\mathrm{corr}_t(R_{t\to t+1}, R_{t+1\to t+2})$.

To that end, we wish to compute

$$\mathrm{cov}_t\left(R_{t\to t+1}, R_{t+1\to t+2}\right) = \mathbb{E}_t\, R_{t\to t+2} - \mathbb{E}_t\, R_{t\to t+1}\, \mathbb{E}_t\, R_{t+1\to t+2}. \tag{2}$$

By Result 2, $\mathbb{E}_t\, R_{t\to T}$ is equal to the price of a claim to the square of the return on the market, $R_{t\to T}^2$. This price can be calculated by a replication argument, as in Martin (2017), by exploiting the fact that

$$R_{t\to T}^2 = \left(\frac{S_T}{S_t}\right)^2 = \frac{2}{S_t^2}\int_0^\infty \max\left\{0, S_T - K\right\}\, dK.$$

This equation expresses the desired payoff—the squared return—as the payoff on a portfolio that holds equal quantities of calls of all strikes. Thus

$$\mathbb{E}_t\, R_{t\to T} = \frac{2}{S_t^2}\int_0^\infty \mathrm{call}_{t,T}(K)\, dK, \tag{3}$$

and setting $T = t+1$ and $T = t+2$ in this expression delivers the first two expectations on the right-hand side of (2).

It is more difficult to compute $\mathbb{E}_t\, R_{t+1\to t+2}$, and doing so is the main innovation of the present paper. In view of Result 2, to calculate this quantity we need the time-t price of a claim to $R_{t+1\to t+2}R_{t\to t+2}$ delivered at time $t+2$. That is, we must price a claim to $S_{t+2}^2/(S_t S_{t+1})$.

It will turn out that we can replicate this claim using *forward-start options*. A forward-start call option that is initiated at time t, for settlement at time $t+2$, has the payoff

$$\max\left\{0, S_{t+2} - K S_{t+1}/S_t\right\}$$

for some fixed K. The unusual feature of a forward-start option is that its strike price, KS_{t+1}/S_t, is not determined until the intervening time $t+1$. (The introduction of S_t, a known constant from the perspective of time t, is simply a convenient normalization.) In contrast, the strike price of a conventional option is determined at initiation. I write $\mathrm{FScall}_t(K)$ for the time-t price of the above payoff, and $\mathrm{FSput}_t(K)$ for the price of the corresponding put payoff, $\max\left\{0, KS_{t+1}/S_t - S_{t+2}\right\}$.

If we hold a portfolio consisting of $2/S_t^2\, dK$ forward-start calls for each K, the portfolio payoff is

$$\frac{2}{S_t^2}\int_0^\infty \max\left\{0, S_{t+2} - KS_{t+1}/S_t\right\}\, dK = \frac{S_{t+2}^2}{S_t S_{t+1}}.$$

Since the payoff on the portfolio of forward-start calls replicates the desired payoff, the price of the payoff $S_{t+2}^2/(S_t S_{t+1})$ is the price of the portfolio of forward-start calls, and hence

$$\mathbb{E}_t\, R_{t+1\to t+2} = \frac{2}{S_t^2}\int_0^\infty \text{FScall}_t(K)\, dK. \tag{4}$$

Before using (3) and (4) to compute the covariance $\text{cov}_t(R_{t\to t+1}, R_{t+1\to t+2})$, it will be convenient to rearrange them by replacing in-the-money calls and forward-start calls with out-of-the-money puts and forward-start puts. For vanilla options, we can do so by exploiting put-call parity, which in our context states that

$$\text{call}_{t,T}(K) - \text{put}_{t,T}(K) = S_t - \frac{K}{R_{f,t\to T}}.$$

The next result provides the corresponding relation for forward-start options.

Result 3 (Put-call parity for forward-start options)**.** *Let G_t be defined by the equation $\text{FScall}_t(G_t S_t) = \text{FSput}_t(G_t S_t)$ (so G_t is observable at time t, assuming the prices of forward-start options of all strikes are available). Then*

$$\text{FScall}_t(K) - \text{FSput}_t(K) = S_t - \frac{K}{G_t}. \tag{5}$$

If interest rates are deterministic, then G_t equals the forward (gross) interest rate for investment from time $t+1$ to $t+2$.

Proof. The time-$(t+2)$ payoff on a portfolio that is long a forward-start call and short a forward-start put, each with strike K, is $S_{t+2} - KS_{t+1}/S_t$. It follows that

$$\text{FScall}_t(K) - \text{FSput}_t(K) = S_t - \frac{1}{R_{f,t\to t+2}}\,\mathbb{E}_t^*\left[\frac{KS_{t+1}}{S_t}\right] = S_t - \lambda K,$$

where λ is the time-t price of a claim to S_{t+1}/S_t *delivered at time* $t+2$. We can pin down λ by applying the equation immediately above in the case $K = G_t S_t$ to conclude that $\lambda = 1/G_t$. This gives the result (5).

If interest rates are deterministic, $\lambda = 1/R_{f,t+1\to t+2}$. For we can replicate the payoff S_{t+1}/S_t, paid at time $t+2$, by investing $1/R_{f,t+1\to t+2}$ in the market from time t to $t+1$ and then at the riskless rate from time $t+1$ to $t+2$. $\square$

Martin (2017) defined the volatility index $\mathrm{SVIX}^2_{t,T} = \frac{1}{T-t}\operatorname{var}^*_t\left(R_{t\to T}/R_{f,t\to T}\right)$:

$$\mathrm{SVIX}^2_{t,T} = \frac{2}{(T-t)R_{f,t\to T}S_t^2}\left[\int_0^{S_tR_{f,t\to T}} \mathrm{put}_{t,T}(K)\,dK + \int_{S_tR_{f,t\to T}}^{\infty} \mathrm{call}_{t,T}(K)\,dK\right].$$

We can define a forward volatility index FSVIX_t that is new to this paper:

$$\mathrm{FSVIX}^2_t = \frac{2}{G_tS_t^2}\left[\int_0^{S_tG_t} \mathrm{FSput}_t(K)\,dK + \int_{S_tG_t}^{\infty} \mathrm{FScall}_t(K)\,dK\right].$$

Using the put-call parity relations to substitute out calls and forward-start calls that have low strikes (i.e., are in-the-money), and then introducing these definitions, equations (3) and (4) can be rewritten as

$$\begin{aligned} \mathbb{E}_t\,R_{t\to T} &= R_{f,t\to T}\left(1+(T-t)\,\mathrm{SVIX}^2_{t,T}\right) && (6)\\ \mathbb{E}_t\,R_{t+1\to t+2} &= G_t\left(1+\mathrm{FSVIX}^2_t\right). && (7)\end{aligned}$$

These definitions lead to the following characterization.

Result 4. *The forward-looking autocovariance of returns, as perceived by the log investor, can be expressed in terms of spot and forward volatility indices as*

$$\operatorname{cov}_t(R_{t\to t+1}, R_{t+1\to t+2}) = R_{f,t\to t+2}\left(1+2\,\mathrm{SVIX}^2_{t,t+2}\right) - R_{f,t\to t+1}G_t\left(1+\mathrm{SVIX}^2_{t,t+1}\right)\left(1+\mathrm{FSVIX}^2_t\right). \tag{8}$$

This expression simplifies if interest rates are deterministic:

$$\operatorname{cov}_t(R_{t\to t+1}, R_{t+1\to t+2}) = R_{f,t\to t+2}\left(2\,\mathrm{SVIX}^2_{t,t+2} - \mathrm{SVIX}^2_{t,t+1} - \mathrm{FSVIX}^2_t - \mathrm{SVIX}^2_{t,t+1}\,\mathrm{FSVIX}^2_t\right). \tag{9}$$

Proof. Equation (8) follows on substituting equations (6) and (7) into the definition (2) of autocovariance. If interest rates are deterministic then $R_{f,t\to t+1}G_t = R_{f,t\to t+2}$ (because, as shown in Result 3, G_t is then equal to the forward rate from $t+1$ to $t+2$); equation (9) follows. $\square$

Thus for example forward-looking autocorrelation is positive if long-dated options are sufficiently expensive relative to short-dated and forward-start options.

The remaining task is to compute $\mathrm{var}_t\, R_{t\to t+1}$ and $\mathrm{var}_t\, R_{t+1\to t+2}$. As one might by now expect, the former can be computed from vanilla options and the latter from forward-start options. We have already calculated $\mathbb{E}_t\, R_{t\to t+1}$ and $\mathbb{E}_t\, R_{t+1\to t+2}$, so it only remains to find $\mathbb{E}_t\, R^2_{t\to t+1}$ and $\mathbb{E}_t\, R^2_{t+1\to t+2}$. By Result 2, the first of these is equal to the time-t price of a claim to $R^3_{t\to t+1}$ paid at time $t+1$, and since

$$\left(\frac{S_{t+1}}{S_t}\right)^3 = \frac{6}{S_t^3}\int_0^\infty K \max\left\{0, S_{t+1} - K\right\}\, dK,$$

the desired quantity is

$$\mathbb{E}_t\, R^2_{t\to t+1} = \frac{6}{S_t^3}\int_0^\infty K\, \mathrm{call}_{t,t+1}(K)\, dK. \tag{10}$$

The remaining term, $\mathbb{E}_t\, R^2_{t+1\to t+2}$, is equal to the price of a claim to $R^2_{t+1\to t+2}R_{t\to t+2}$ at time $t+2$. Since

$$R^2_{t+1\to t+2}R_{t\to t+2} = \frac{S^3_{t+2}}{S_t S^2_{t+1}} = \frac{6}{S_t^3}\int_0^\infty K \max\left\{0, S_{t+2} - K S_{t+1}/S_t\right\}\, dK,$$

we have

$$\mathbb{E}_t\, R^2_{t+1\to t+2} = \frac{6}{S_t^3}\int_0^\infty K\, \mathrm{FScall}_t(K)\, dK. \tag{11}$$

Using the put-call parity relations to replace in-the-money calls with out-of-the-money puts, equations (10) and (11) become

$$\mathbb{E}_t\, R^2_{t\to t+1} - R^2_{f,t\to t+1} = \frac{6}{S_t^3}\left[\int_0^{S_t R_{f,t\to t+1}} K\, \mathrm{put}_{t,t+1}(K)\, dK + \int_{S_t R_{f,t\to t+1}}^\infty K\, \mathrm{call}_{t,t+1}(K)\, dK\right] \tag{12}$$

and

$$\mathbb{E}_t\, R^2_{t+1\to t+2} - G_t^2 = \frac{6}{S_t^3}\left[\int_0^{S_t G_t} K\, \mathrm{FSput}_t(K)\, dK + \int_{S_t G_t}^\infty K\, \mathrm{FScall}_t(K)\, dK\right]. \tag{13}$$

Equations (6), (7), (12), and (13) provide the ingredients needed to calculate the autocorrelation index

$$\mathrm{corr}_t(R_{t\to t+1}, R_{t+1\to t+2}) = \frac{\mathbb{E}_t\, R_{t\to t+2} - \mathbb{E}_t\, R_{t\to t+1}\, \mathbb{E}_t\, R_{t+1\to t+2}}{\sqrt{\mathrm{var}_t\, R_{t\to t+1}\, \mathrm{var}_t\, R_{t+1\to t+2}}}. \tag{14}$$

1.1 The autocorrelation index in homogeneous models

In many familiar theoretical models, the autocorrelation index is exactly zero. As we will see in the next section, this is counterfactual.

Forward-start call prices take a particularly simple form in the Black–Scholes (1973) model, for example. At time $t+1$, a forward-start call becomes identical to a vanilla call with strike KS_{t+1}/S_t, so by the Black–Scholes formula (with volatility σ and a continuously-compounded riskless rate of r), the forward-start call is worth

$$S_{t+1}\Phi\left(\frac{\log\frac{S_t}{K}+r+\frac{1}{2}\sigma^2}{\sigma}\right)-K\frac{S_{t+1}}{S_t}e^{-r}\Phi\left(\frac{\log\frac{S_t}{K}+r-\frac{1}{2}\sigma^2}{\sigma}\right)$$

at time $t+1$. As a claim to S_{t+1} at time $t+1$ is worth S_t at time t, the above expression implies that at time t, the forward-start call is worth the same as a one-period vanilla call: $\text{FScall}_t(K)=\text{call}_{t,t+1}(K)$. It follows by put-call parity that $\text{FSput}_t(K)=\text{put}_{t,t+1}(K)$, and hence also that $\text{FSVIX}_t^2=\text{SVIX}_{t,t+1}^2$. The autocorrelation index therefore takes a particularly simple form: as we have $\text{SVIX}_{t,T}^2=\frac{1}{T-t}(e^{\sigma^2(T-t)}-1)$,

$$\text{cov}_t(R_{t\to t+1},R_{t+1\to t+2})=e^{2r}\left(e^{2\sigma^2}-1-2(e^{\sigma^2}-1)-(e^{\sigma^2}-1)^2\right)=0.$$

That is, the autocorrelation index is zero in the Black–Scholes model. Another way to make the same point is that with constant risk aversion (through log utility) and constant volatility the risk premium is constant, so there is no room for autocorrelation to arise through the drift term. As volatility is also constant, there is no autocorrelation at all.

More generally, let us say that a model is *homogeneous* if interest rates are constant and call prices have the property that $\text{call}_{t,T}(K)=Kg(S_t/K,T-t)$ for some function g. Many of the leading option-pricing models have this property, including the Black–Scholes (1973) model, Merton's (1976) jump-diffusion model, the variance-gamma model of Madan, Carr and Chang (1998), and the Heston (1993) model among others; the Dupire (1994) local volatility framework is an example of a setting in which the homogeneity property does *not* hold.

In a homogeneous model, the relationship between forward-start options and vanilla European options is trivial, as in the Black–Scholes example above. For, a forward-start call with strike K, initiated at time t for final settlement at time $t+2$, has the payoff $\max\{0, S_{t+2} - KS_{t+1}/S_t\}$ at time $t+2$. From the perspective of time $t+1$, this is equivalent to the payoff on a vanilla call with strike KS_{t+1}/S_t. By the homogeneity assumption, at time $t+1$ the vanilla call (and hence also the forward-start call) is worth

$$\frac{KS_{t+1}}{S_t} g\left(\frac{S_{t+1}}{KS_{t+1}/S_t}, 1\right) = \frac{KS_{t+1}}{S_t} g\left(S_t/K, 1\right).$$

The forward-start call is therefore worth $Kg(S_t/K, 1)$ at time t. In other words, by the homogeneity property, $\mathrm{FScall}_t(K) = \mathrm{call}_{t,t+1}(K)$. It is now straightforward to show, using the put-call parity relations for vanilla and forward-start options, that $\mathrm{FSput}_t(K) = \mathrm{put}_{t,t+1}(K)$ by put-call parity. As a result, $\mathrm{FSVIX}_t^2 = \mathrm{SVIX}_{t,t+1}^2$.

2 Empirical results

I obtained indicative price quotes for 6-month and 12-month vanilla call and put options on the S&P 500 index, together with 6-month-6-month-forward-start options, from a major investment bank. All prices were supplied for a range of dates—June 15, 2007; June 20, 2008; November 21, 2008; February 20, 2009; December 17, 2010; July 15, 2011; December 20, 2012; and December 20, 2013—and for at-the-money, 5% and 10% out-of-the-money strikes for puts and for calls, together with the level of S&P 500 spot and the bank's internally marked 6-month and 12-month interest rate. I also obtained daily updated prices of vanilla European call and put options on the S&P 500 index from OptionMetrics in order to plot the daily time-series shown in Figure 2.

I calculate the autocorrelation index (14) using expressions (6), (7), (12), and (13), interpolating linearly between option prices inside the range of observed strikes. To extrapolate option prices where necessary, I assume a flat volatility smile outside the range of observed strikes, following the approach of Carr and Wu (2009): in other words, for out-of-the-money puts with moneyness below the lowest observed strike, I use the Black–Scholes implied volatility

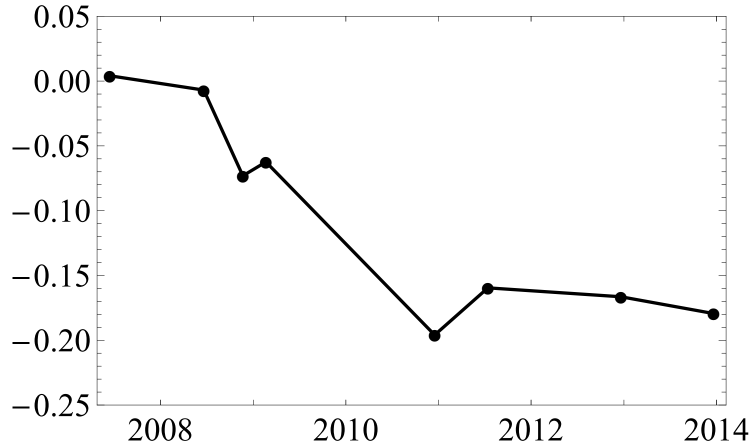

Figure 1: The autocorrelation of the S&P 500, $\mathrm{corr}_t(R_{t \to t+6\mathrm{mo}}, R_{t+6\mathrm{mo} \to t+12\mathrm{mo}})$.

at the lowest observed strike price, and for out-of-the-money calls I use the Black–Scholes implied volatility at the highest observed strike price. Lastly, I set G_t equal to the forward rate from $t+1$ to $t+2$.

Figure 1 shows the 6-monthly autocorrelation of the S&P 500 index—that is, $\mathrm{corr}_t(R_{t \to t+6\mathrm{mo}}, R_{t+6\mathrm{mo} \to t+12\mathrm{mo}})$—on a sample of dates. Prior to the subprime crisis, autocorrelation was close to zero (at 0.004 on June 15, 2007 and -0.007 on June 20, 2008). It declined sharply following the subprime crisis, with a minimum of -0.20 on December 17, 2010. By the time of my final data point, on December 20, 2013, the autocorrelation was -0.18.

As shown by Hobson and Neuberger (2012), the prices of forward-start options are not tightly constrained by the prices of ostensibly closely related vanilla options. This fact is precisely what makes them interesting; nonetheless, it should be emphasized that they are exotic derivatives, with all the caveats that entails—most notably, that the foward-start option market is not nearly as liquid as the vanilla option market. Moreover the magnitudes of the autocorrelation index calculated above may, at first sight, seem unreasonably large: in particular, it might seem that strategies designed to exploit reversals should earn Sharpe ratios that are too good to be true. In order to assess this possibility, the next result shows how to use vanilla option prices to calculate the maximum attainable Sharpe ratio perceived by the log investor.

Result 5. *The maximal Sharpe ratio over the period from t to $t+n$, as perceived by the log investor, satisfies*

$$\textit{max Sharpe ratio} \leq R_{f,t\to t+n}\sqrt{\int_0^\infty \frac{2S_t}{K^3}\Omega_{t,t+n}(K)\,dK}\,, \tag{15}$$

where $\Omega_{t,t+n}(K)$ is the time t price of an out-of-the-money European option with strike K expiring at time $t+n$:

$$\Omega_{t,t+n}(K) = \begin{cases} \text{put}_{t,t+n}(K) & \textit{if } K \leq S_t R_{f,t\to t+n} \\ \\ \text{call}_{t,t+n}(K) & \textit{if } K > S_t R_{f,t\to t+n} \end{cases}$$

Proof. Using the result of Hansen and Jagannathan (1991) and the fact that $M_{t\to t+n} = 1/R_{t\to t+n}$, we have

$$\begin{aligned} \text{max Sharpe ratio} &\leq R_{f,t\to t+n}\sqrt{\text{var}_t \frac{1}{R_{t\to t+n}}} \\ &= \sqrt{R^2_{f,t\to t+n}\,\mathbb{E}_t \frac{1}{R^2_{t\to t+n}} - 1}\,. \end{aligned} \tag{16}$$

By the result of Breeden and Litzenberger (1978), as rewritten by Carr and Madan (1998), the time t price of a claim to $f(S_{t+n})$ paid at time $t+n$ is

$$\mathbb{E}_t \frac{f(S_{t+n})}{R_{t\to t+n}} = \frac{f(S_t R_{f,t\to t+n})}{R_{f,t\to t+n}} + \int_0^\infty f''(K)\Omega_{t,t+n}(K)\,dK.$$

Setting $f(K) = S_t/K$, this implies that

$$\mathbb{E}_t \frac{1}{R^2_{t\to t+n}} = \frac{1}{R^2_{f,t\to t+n}} + \int_0^\infty \frac{2S_t}{K^3}\Omega_{t,t+n}(K)\,dK. \tag{17}$$

The result follows on substituting (17) into (16). □

Figure 2 plots the time series of the right-hand side of inequality (15) of Result 5, which provides an upper bound on the maximal Sharpe ratio. The dates on which the autocorrelation index is calculated in Figure 1 are marked with crosses. While the maximum attainable Sharpe ratio (as perceived by the

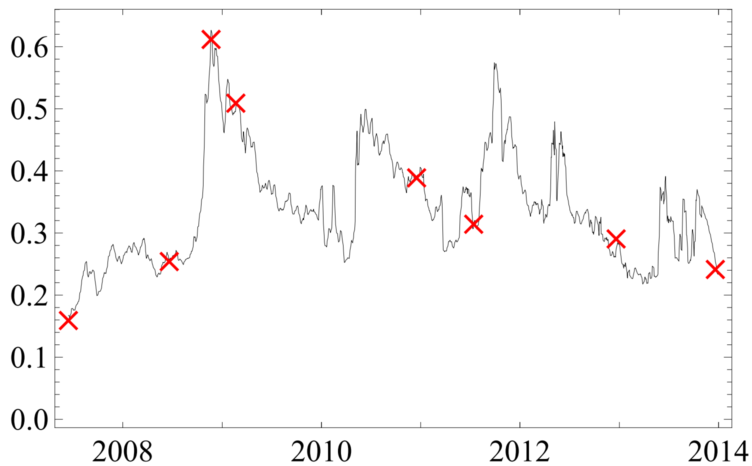

Figure 2: The maximal Sharpe ratio. Crosses indicate the dates on which the autocorrelation index is computed in Figure 1.

log investor) spiked in late 2008, it was not implausibly high. Thus, although there are several potential ways reversal strategies might be implemented in practice, none of them has an unreasonably high Sharpe ratio from the perspective of the log investor.

3 Conclusion

This paper has introduced a new index of autocorrelation and constructed it at the six-month horizon using indicative prices obtained from a major investment bank on various days between mid 2007 and late 2013. Prior to the crisis, implied autocorrelation was close to zero. Following the crisis, autocorrelation has turned sharply negative, dropping to -0.2 in December 2010 and remaining below -0.15 subsequently. At first sight it might seem that autocorrelations so far from zero should imply the existence of trading strategies that are too good to be true; I show, however, that although the maximum Sharpe ratio implied by the framework does spike during the subprime crisis, it is not unreasonably high at any stage of the sample period.

The negative implied autocorrelation is consistent with the recent model of Barberis, Greenwood, Jin and Shleifer (2015) in which rational investors interact with investors whose expectations about future returns are based (irrationally) on extrapolating from past returns.

Another possibility is that the sharply negative autocorrelation in the years following the crisis may reflect market participants' beliefs about the behavior of policymakers. The FOMC statement of September 21, 2010 contains the following paragraph,[2] which heralded a second round of quantitative easing (QE2):

> The Committee will continue to monitor the economic outlook and financial developments and is prepared to provide additional accommodation if needed to support the economic recovery and to return inflation, over time, to levels consistent with its mandate.

It would not be unreasonable to conclude from this that policy would be more expansive conditional on further declines in the market and relatively more contractionary conditional on further rises, and hence to anticipate a decline in market autocorrelation. Indeed, Cieslak, Morse and Vissing-Jorgensen (2018) argue that the behavior of stock returns over the "FOMC cycle" is consistent with this view (though they focus on shorter horizons and emphasize the importance of timing within the cycle). Consistent with this interpretation, the low point of the autocorrelation measure occurs in December, 2010.

As the autocorrelation index depends only on asset prices, it has the great advantage of being computable, in principle, in real time. The central novel feature of the index is that it is based on the prices of forward-start index options. A further contribution of the paper, therefore, is to point out that such options—variants of the more familiar "plain vanilla" European call and put options—have a natural economic application. It is sometimes tempting, when confronted with a cliquet, a lookback, a Napoleon, Himalayan, Bermudan, Asian, best-of, worst-of, or rainbow option, or with any other member

[2]The precise phrasing of the paragraph was discussed extensively during the meeting: see pages 78, 98, 101, 113, and 124–6 of the Transcript of the Federal Open Market Committee Meeting on September 21, 2010 (available at https://www.federalreserve.gov/monetarypolicy/files/FOMC20100921meeting.pdf). One consequence of the discussion was that the phrase "as needed" was replaced with "if needed," which was felt to emphasize the conditionality of any potential Fed action more clearly.

of the bewildering menagerie of exotic derivatives, to conclude that such contracts play no more significant a role than to transfer resources between groups of quants. Precisely because there is an element of truth in this caricature, financial economists have a role to play in pointing out when some seemingly obscure derivative contract is in fact of economic interest.

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