Convexity Bounds for the Stochastic Discount Factor: Theory and Evidence

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October, 2025

Background

- Asset prices are often used for assessing expectations:
  - forward rates
  - breakeven inflation
  - CDS rates
  - implied volatility
  - ...
- These are almost continuously observable
- Don't need to rely on economists' models
- And they embody the collective views of market participants
- But they may be distorted by risk: people will pay more for insurance/hedge assets that pay off in scary states of the world

Background

- Asset prices are often used for assessing expectations:
  - forward rates $\longrightarrow$ risk-neutral expected future interest rates
  - breakeven inflation $\longrightarrow$ risk-neutral expected future inflation
  - CDS rates $\longrightarrow$ risk-neutral default probabilities
  - implied volatility $\longrightarrow$ risk-neutral volatility
  - ...
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The importance of risk considerations

For any payoff X:

$$\frac{1}{R_f}\,\mathbb{E}^*\left[X\right] = \mathbb{E}[MX](= \mathrm{price}[X])$$

- Risk consideration is captured by the stochastic discount factor (SDF), M,
  - no arbitrage implies “linear pricing”
  - let $X = 1$ (discount bond): $\mathbb{E}\left[M\right] = 1/R_f$
- If there is little variation in M, risk consideration is not that important...

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- If there is little variation in M, risk consideration is not that important...
- Hansen and Jagannathan (1991): variance of the SDF

$$\text{var}\left[MR_f\right] \geq \text{SR}^2$$

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- Snow (1991): for $\theta < 0$ or $\theta > 1$, we have, for any asset return R

$$\mathbb{E}\left[(MR_f)^\theta\right] \geq \left\{\mathbb{E}\left[\left(\frac{R}{R_f}\right)^{\frac{\theta}{\theta-1}}\right]\right\}^{1-\theta}$$

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  - The θth moment of the SDF is related to the $\frac{\theta}{\theta-1}$th moments of asset returns
  - When $\theta \to 1$, the Snow bounds rely on extremely high moments of asset returns
    - hard to measure in practice
    - paradoxically, we know for sure that $\mathbb{E}[MR_f] = 1$!

What's missing?

- The Snow bounds exploit the true return distribution
  - can be estimated from historical realized return time series
- But they ignore the risk-neutral distribution completely!
  - resolving the paradox: $\mathbb{E}[M]$ is directly revealed by the riskless rate
  - the risk-neutral first moment of returns
- We can exploit both the true and the risk-neutral distributions
  - in addition to realized returns, we use information in option prices
    - known issues for option returns (eg., Jackwerth, 2000; Coval and Shumway, 2001)
    - and only a short history is available

This paper

- We show how to understand all known bounds in a simple unifying framework
- We use the framework to derive new bounds
    - that play off the true return distribution against the risk-neutral distribution
- We explain why the H-J bound should not be expected to be stable in theory
    - our empirical results show that the moments of the SDF rise exceedingly rapidly as θ rises above one, the H-J bound is perhaps ∞
- We show that certain bounds are stable and they convey nice intuitions
- For all our empirical results, the short option time series is not a problem
- But the "short" (150-year) realized return sample is a problem!

The cumulant-generating function (CGF)

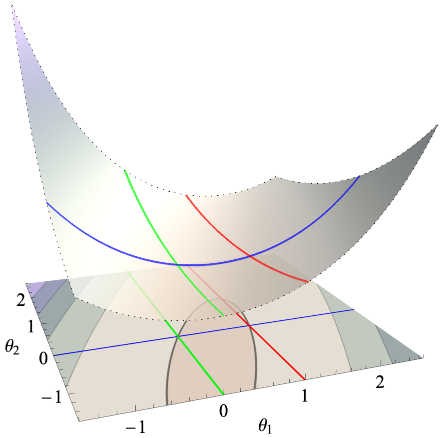
Fixing the asset return R, we consider the function

$$\boldsymbol{c}(\theta_1,\, \theta_2) = \log \mathbb{E}\left[(MR_f)^{\theta_1} \left(\frac{R}{R_f} \right)^{\theta_2} \right]$$

- $\boldsymbol{c}(0,0) = 0$: trivial
- $\boldsymbol{c}(1,0) = 0$: because SDF prices the riskless asset
- $\boldsymbol{c}(1,1) = 0$: because SDF prices the return R
- $\boldsymbol{c}(\theta_1,\theta_2)$ is a convex function

The cumulant-generating function (CGF)

- Observables from asset markets:
  - Returns: $\boldsymbol{c}(0,\, \theta_2) = \log \mathbb{E}\left[(R/R_f)^{\theta_2}\right]$
  - Options: $\boldsymbol{c}(1,\, \theta_2) = \log \mathbb{E}\left[(MR_f)\,(R/R_f)^{\theta_2}\right] = \log \mathbb{E}^*\left[(R/R_f)^{\theta_2}\right]$
    - Recall that, for any cash flow X: $\text{price}(X) = \mathbb{E}[MX] = \dfrac{1}{R_f}\,\mathbb{E}^*[X]$; here, we consider $X = (R/R_f)^{\theta_2}$
- Equilibrium models specify $\boldsymbol{c}(\theta_1,\, 0) = \log \mathbb{E}\left[(MR_f)^{\theta_1}\right]$
- For now, think of all expectations as conditional on time t information
- Rich implications: $\boldsymbol{c}(0, 1)$ "equity premium"; Cboe $\text{VIX}^2 = -2\frac{\partial \boldsymbol{c}}{\partial \theta_2}(1, 0)$; CME CVOL is such that $\log(1 + \text{CVOL}^2) = \boldsymbol{c}(1, 2)$; ...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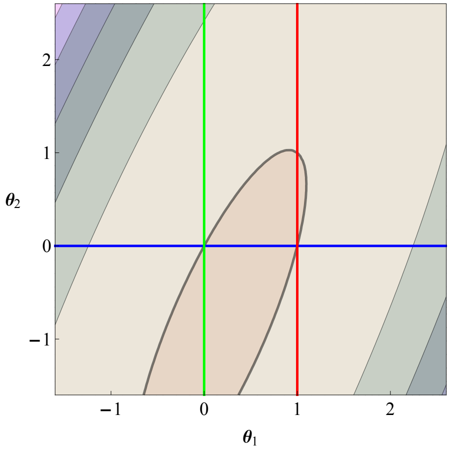
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Example 1: Lognormality (Black-Scholes)

- If $MR_f = e^{-\frac{1}{2}\lambda^2 - \lambda Z}$ and $R/R_f = e^{\mu - \frac{1}{2}\sigma^2 + \sigma W}$, where Z and W are standard normal with correlation ρ, then

$$\boldsymbol{c}(\theta_1, \theta_2) = \mu\theta_2(1-\theta_1) + \frac{1}{2}\lambda^2\theta_1(\theta_1 - 1) + \frac{1}{2}\sigma^2\theta_2(\theta_2 - 1)$$

- The two observables

$$\begin{aligned} \boldsymbol{c}(0,\theta) &= \mu\theta + \frac{1}{2}\sigma^2\theta(\theta-1) \\ \boldsymbol{c}(1,\theta) &= \frac{1}{2}\sigma^2\theta(\theta-1) \end{aligned}$$

- We learn nothing interesting from option prices

Example 2: Jump-diffusion (Merton '76)

- If $MR_f = e^{-\frac{1}{2}\lambda^2 - \lambda Z - J_1 \omega}(1+J_1)^N$ and $R/R_f = e^{\mu - \frac{1}{2}\sigma^2 + \sigma W - J_2 \omega}(1+J_2)^N$, where N is Poisson with intensity ω and J_i are jump sizes, then

$$\begin{aligned} \boldsymbol{c}(\theta_1, \theta_2) = \mu\theta_2(1-\theta_1) + \frac{1}{2}\lambda^2\theta_1(\theta_1 - 1) + \frac{1}{2}\sigma^2\theta_2(\theta_2 - 1) + \\ + \omega\left[(1+J_1)^{\theta_1}(1+J_2)^{\theta_2} - (1+J_1\theta_1)(1+J_2\theta_2)\right] \end{aligned}$$

- The two observables

$$\begin{aligned} \boldsymbol{c}(0,\theta) &= \mu\theta + \frac{1}{2}\sigma^2\theta(\theta-1) + \omega\left[(1+J_2)^\theta - \theta J_2 - 1\right] \\ \boldsymbol{c}(1,\theta) &= \frac{1}{2}\sigma^2\theta(\theta-1) + \omega(1+J_1)\left[(1+J_2)^\theta - \theta J_2 - 1\right] \end{aligned}$$

- We need to look at option prices to detect J_1

Example 3: Parameter learning

In a pure jump (no diffusion) model, the agent update beliefs about the jump intensity ω based on the observed N (the number of realized jumps) from the prior $\omega \sim \exp(1/\overline{\omega})$

$$\begin{aligned} \boldsymbol{c}_t(\theta_1, \theta_2) = & \theta_1 \log\left(1 - \overline{\omega} J_1\right) + \theta_2 \left(\mu + \log\left(1 - \overline{\omega} J_2\right)\right) \\ & - \log\left(1 - \overline{\omega}\left[(1 + J_1)^{\theta_1}(1 + J_2)^{\theta_2} - 1\right]\right) \end{aligned}$$

If jumps are disasters:

- The true and risk-neutral moments are unbounded for very negative θ
- The positive moments of the SDF also diverge for θs greater than some critical value that is above one
- The "dismal" economy of Geweke (2001) and Weitzman (2007): every moment of the SDF may be unbounded

Example 4: Heterogeneous beliefs (Martin and Papadimitriou 2022)

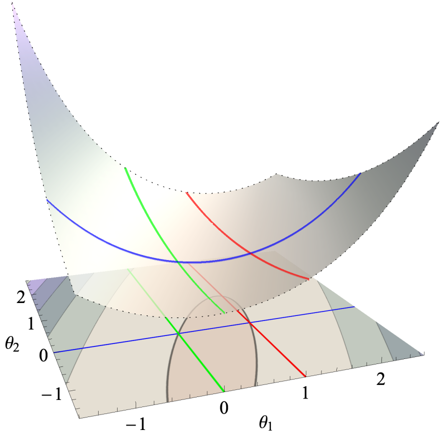
- In MP model, the two observables are both quadratic

$$\boldsymbol{c}(0,\theta) = \frac{1+\delta}{2\delta}\sigma^2\theta + \frac{1}{2}\sigma^2\theta^2$$
$$\boldsymbol{c}(1,\theta) = \frac{1+\delta}{2\delta}\sigma^2\theta(\theta-1)$$

- σ is true return volatility and $\delta > 0$ controls the amount of disagreement
- But true and risk-neutral volatility differ, so option prices contain information
- The whole surface is

$$\boldsymbol{c}(\theta_1,\theta_2) = \frac{1}{2}\left[\frac{1+\delta}{\delta}\sigma^2\left(\theta_2-\theta_1\right) + \frac{1+\delta}{1+\delta-\theta_1}\sigma^2\left(\theta_2-\theta_1\right)^2 + \log\frac{1+\delta}{1+\delta-\theta_1} - \theta_1\log\frac{1+\delta}{\delta}\right]$$

- Explodes when $\theta_1 \geq 1+\delta$: $(1+\delta)$th and higher moments of the SDF are unbounded



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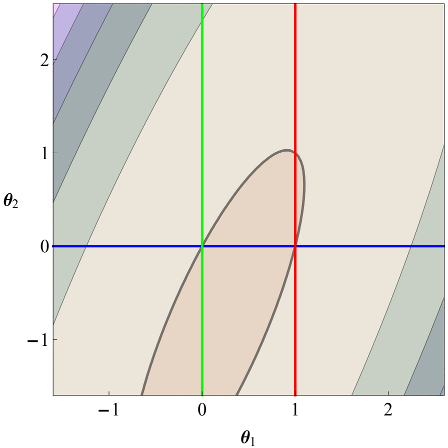

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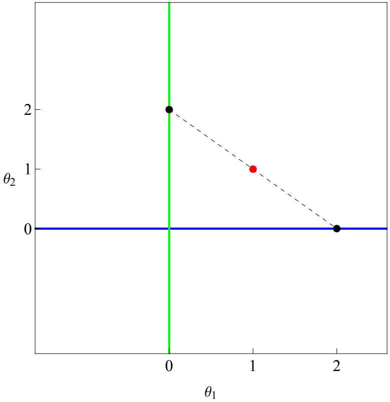
How to derive traditional results using convexity

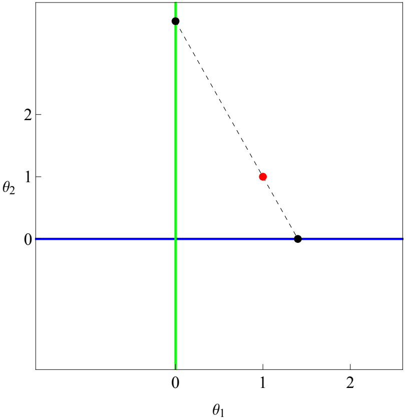


- $\frac{1}{2}\boldsymbol{c}(2,0) + \frac{1}{2}\boldsymbol{c}(0,2) \geq \boldsymbol{c}(1,1)$
- But, as $\boldsymbol{c}(1,1) = 0$, this implies

$$\mathbb{E}\left[(MR_f)^2\right] \geq \frac{1}{\mathbb{E}\left[\left(\frac{R}{R_f}\right)^2\right]}$$

- Gives the Hansen–Jagannathan bound (the minimum second moment R is such that $1/\,\mathbb{E}[(R/R_f)^2] - 1 =$ the maximal SR^2)

How to derive traditional results using convexity

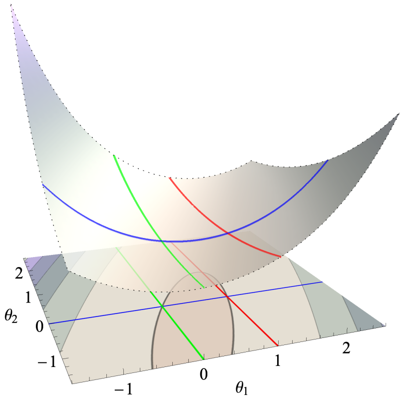


- $\frac{1}{\theta}\boldsymbol{c}(\theta, 0) + \left(1 - \frac{1}{\theta}\right)\boldsymbol{c}\left(0, \frac{\theta}{\theta-1}\right) \geq \boldsymbol{c}(1,1)$
- But, as $\boldsymbol{c}(1,1) = 0$, this implies

$$\mathbb{E}\left[(MR_f)^{\theta}\right] \geq \left\{\mathbb{E}\left[\left(\frac{R}{R_f}\right)^{\frac{\theta}{\theta-1}}\right]\right\}^{1-\theta}$$

- Gives the Snow bound

But we can do better



- Existing bounds connect the blue line (SDF moments) to the green line (historical time series) by looking through the point $(1, 1)$ where the CGF is pinned to zero
- We don't need to do this!
- We can look in any direction once we realise that the red line (risk-neutral distribution) is observable

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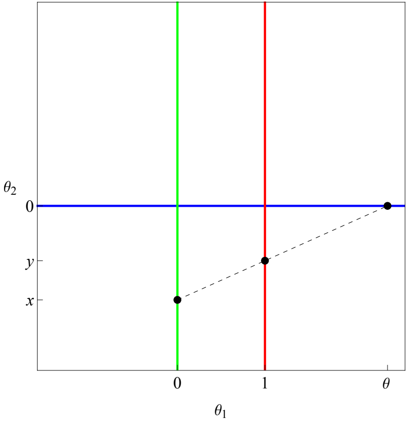

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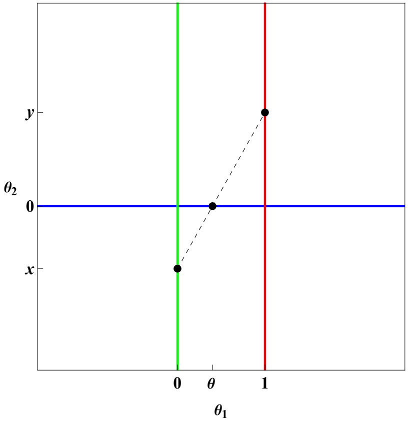

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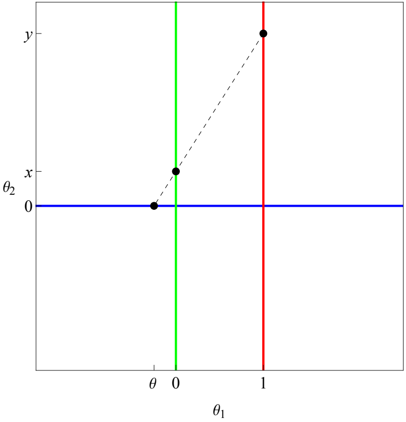

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Result (New moment bounds)

For $\theta < 0$ or $\theta > 1$, we have

$$\mathbb{E}\left[(MR_f)^\theta\right] \geq \sup_{y\in\mathbb{R}} \left\{\mathbb{E}^*\left[\left(\frac{R}{R_f}\right)^y\right]\right\}^\theta \left\{\mathbb{E}\left[\left(\frac{R}{R_f}\right)^{\frac{\theta}{\theta-1}y}\right]\right\}^{1-\theta}$$

- Inequality is reversed for $\theta \in (0,1)$
- When $y = 1$, reduces to the Snow (1991) bound
- When $y = 1$ and $\theta = 2$, equivalent to the Hansen–Jagannathan bound
- This result lets us derive nontrivial bounds even for a single fixed R (but we could also optimize across R)

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$$\boldsymbol{c}(\theta, 0) \geq \sup_{y \in \mathbb{R}} \; \theta \, \boldsymbol{c}(1, y) + (1 - \theta) \, \boldsymbol{c}\left(0, \frac{\theta}{\theta - 1} y\right)$$

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Summary so far

- We have derived new bounds on the moments of the SDF that play the true and risk-neutral distributions off against one another
- Couldn't we just do this by plugging returns on option strategies into, say, the Hansen–Jagannathan bound (Liu, 2021)?
    - problem: we only observe option prices over a short time series
    - this problem is particularly severe for option strategies: highly skewed and fat-tailed
    - our approach allows us to sidestep this issue

Conditional and unconditional CGFs

- Our bounds apply conditionally or unconditionally
- Conditional risk-neutral CGF $\boldsymbol{c}_t(1,\theta)$ is easy to measure

$$\begin{aligned}\boldsymbol{c}_t\left(1,\theta\right) &= \log \mathbb{E}_t^*\left[\left(R_{t+1}/R_{f,t+1}\right)^\theta\right] \\ &= \log\left\{1+\theta(\theta-1)\Bigg[\underbrace{\int_0^1 K^{\theta-2}\,\mathrm{put}_t(KR_{f,t+1})\,\mathrm{d}K+\int_1^\infty K^{\theta-2}\,\mathrm{call}_t(KR_{f,t+1})\,\mathrm{d}K}_{\text{the price of a portfolio of options with different strikes } K}\Bigg]\right\},\end{aligned}$$

- But conditional true distribution of R is hard to measure, so we work unconditionally
- As we measure conditional risk-neutral distribution perfectly, most statistical uncertainty is associated with the true distribution, not the risk-neutral distribution

Conditional and unconditional CGFs

- Unconditional true CGF of R is

$$\boldsymbol{c}(0,\theta) = \log \mathbb{E}\left[\left(R_{t+1}/R_{f,t+1} \right)^{\theta} \right]$$

- Assuming stationarity and ergodicity, we approximate, for large T,

$$\boldsymbol{c}(0,\theta) = \log \frac{1}{T} \sum_{t=0}^{T-1} \left(R_{t+1}/R_{f,t+1} \right)^{\theta}$$

Conditional and unconditional CGFs

- Unconditional counterpart of risk-neutral CGF is

$$
\begin{aligned}
\boldsymbol{c}(1,\theta) &= \log \mathbb{E}\left[M_{t+1}R_{f,t+1}\left(R_{t+1}/R_{f,t+1}\right)^{\theta}\right] \\
&= \log \mathbb{E}\left(\mathbb{E}_t\left[M_{t+1}R_{f,t+1}\left(R_{t+1}/R_{f,t+1}\right)^{\theta}\right]\right) \\
&= \log \mathbb{E}\left(\underbrace{\mathbb{E}_t^*\left[\left(R_{t+1}/R_{f,t+1}\right)^{\theta}\right]}_{\exp\{\boldsymbol{c}_t(1,\theta)\}}\right)
\end{aligned}
$$

- So we approximate, for large T, (replacing $\mathbb{E}$ with $\mathbb{E}_T = 1/T\sum_t$)

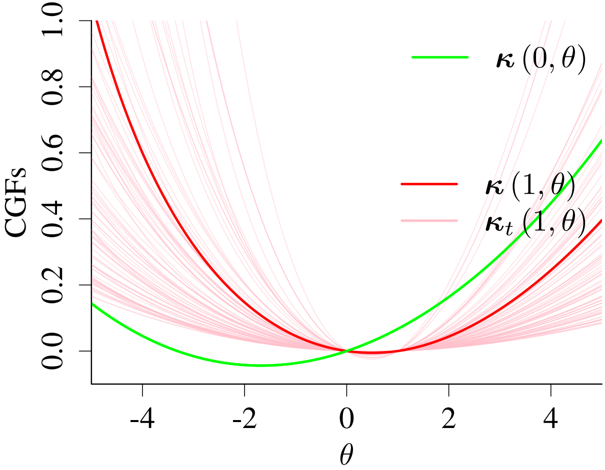
$$
\boldsymbol{c}(1,\theta) = \log \frac{1}{T}\sum_{t=1}^{T} \exp\left\{\boldsymbol{c}_t(1,\theta)\right\}
$$

Data

- Market returns:
  - Global Financial Data: 1871–1926
  - CRSP: 1926–2022

  Baseline sample is monthly from 1872 to 2022
- Jordà–Schularick–Taylor Macrohistory Database (1872–2020): annual returns from Jordà, Knoll, Kuvshinov, Schularick, and Taylor (2019)
- S&P 500 index options from OptionMetrics

The two observable slices of the CGF in the data



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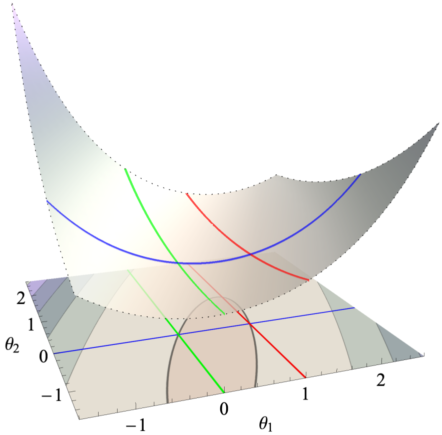
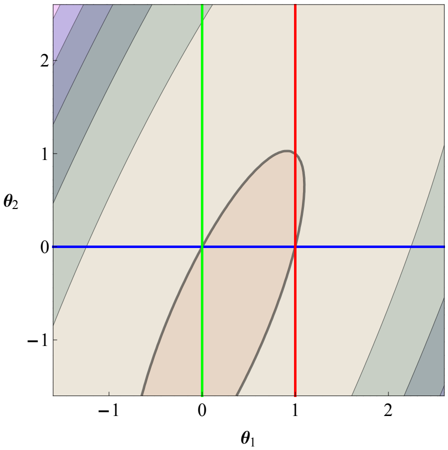




Two special cases

Recall the object:

$$\theta\, \boldsymbol{c}(1, y) + (1-\theta)\, \boldsymbol{c}\left(0, \frac{\theta}{\theta - 1} y\right)$$

Consider two special cases:

- $\theta = 2$: the lower bound for $\log \mathbb{E}\left[(MR_f)^2\right]$ equals

$$\sup_y \; 2\boldsymbol{c}(1,\, y) - \boldsymbol{c}(1,\, 2y)$$

  i.e., the difference between two convex functions—badly behaved!

- $\theta = 1/2$: the upper bound for $\log \mathbb{E}\left[\sqrt{MR_f}\right]$

$$\inf_y \; \frac{1}{2}\boldsymbol{c}(1,\, y) + \frac{1}{2}\boldsymbol{c}\,(0,\, -y),$$

  i.e., the sum of two convex functions—well behaved

Instability of volatility bounds

There is no good reason to expect the population volatility of the SDF to be well behaved

- It explains the familiar evidence that the Sharpe ratio of option strategies is unstable (Jackwerth 2000; Coval and Shumway 2001; Goetzmann, Ingersoll, Spiegel and Welch 2007; Bondarenko 2003; Jones 2006; Driessen and Maenhout 2007; Santa-Clara and Saretto 2009; Broadie, Chernov and Johannes 2009; ...)

To make things worse, we have the following (simplified) results in the sample

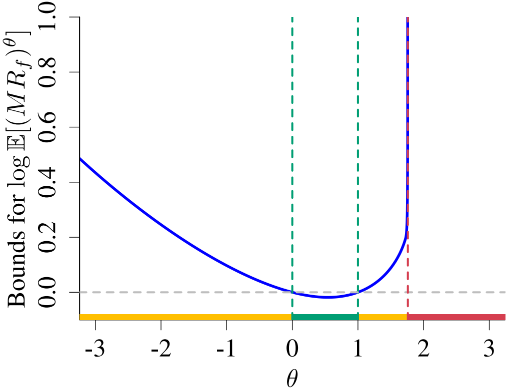
Result

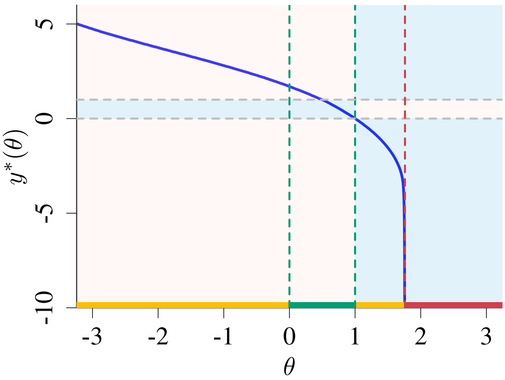
Fix $\theta > 1$. If the most extreme put strike (of a put with positive bid price) in the dataset is lower than the lowest observed return in sample, then the lower bound can be made arbitrarily large.

- The worst monthly return is -29% (September 1931)
- Put options with strikes more than 29% out of the money have positive *bid* prices on 73% of days in our sample

The convexity bounds for the SDF moments

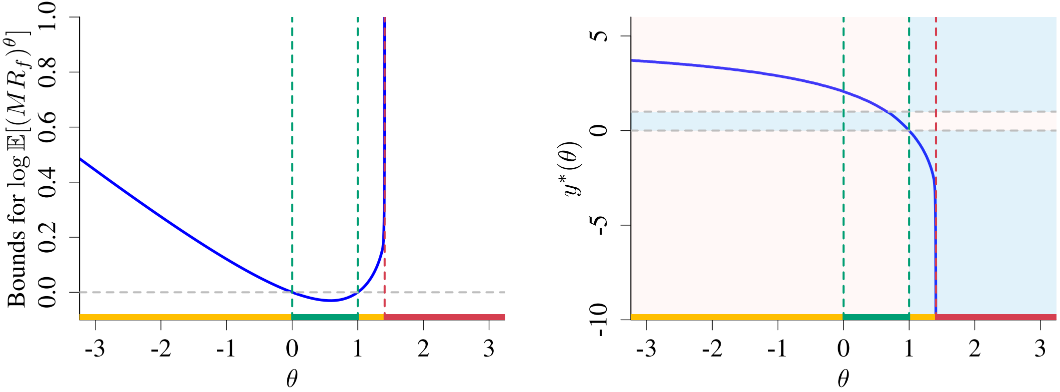
Realized returns: 1872-2022





The convexity bounds for the SDF moments

Realized returns: 1946-2022



The singularity is near

Over the one-month horizon

sample	est.	bootstrap CI	$\mathbb{E}$-bootstrap CI	$\mathbb{E}^*$-bootstrap CI
1872-2022	1.72	$(1.52, 2.05)$	$(1.50, 1.97)$	$(1.62, 1.87)$
1946-2022	1.38	$(1.20, 1.60)$	$(1.20, 1.56)$	$(1.34, 1.45)$
1996-2022	1.44	$(1.23, 1.78)$	$(1.23, 1.77)$	$(1.39, 1.52)$

- The singularity mostly emerges before two
- The majority of estimation uncertainty comes from the realized returns

The singularity is near

Over the one-year horizon

sample	est.	bootstrap CI	$\mathbb{E}$-bootstrap CI	$\mathbb{E}^*$-bootstrap CI
1872-2022	1.67	$(1.27, 2.50)$	$(1.27, 2.05)$	$(1.60, 2.10)$
1946-2022	1.28	$(1.15, 1.50)$	$(1.14, 1.42)$	$(1.23, 1.38)$
1996-2022	1.36	$(1.14, 1.92)$	$(1.13, 1.73)$	$(1.32, 1.52)$
JKKST annual	1.36	$(1.23, 1.56)$	$(1.21, 1.52)$	$(1.32, 1.51)$

- The singularity mostly emerges before two
- The majority of estimation uncertainty comes from the realized returns
- Calendar year returns drop severe mid-year crashes (-65%, June 1931 to June 1932)
- Less observations, narrower CI: sensitivity to extreme market crashes

So what can we do?

Result

The moment bounds are well-behaved when $\theta \in (0, 1)$, in the sense that the minimization problem over y on the right-hand side of the inequality has a unique interior minimum.

- The variance of the SDF is appealing because it relates the Sharpe ratio
  - a measure of the attractiveness of investment opportunities
  - ... from the perspective of a one-period investor with quadratic utility

Do these intermediate moments of the SDF hold similar properties?

- To do so, we adopt the perspective of a one-period CRRA-γ investor
- The attractiveness of investment opportunities can be quantified using the willingness-to-pay (WTP)—the fraction of initial wealth—to be allowed to trade risky assets, namely g_γ

An WTP-based interpretation

Define

$$B(\theta) = \inf_{y \in \mathbb{R}} \ \theta \boldsymbol{c}(1, y) + (1-\theta)\boldsymbol{c}\left(0, \frac{\theta}{\theta - 1} y\right) \quad \text{when } \theta \in (0,1).$$

- Our previous results say $\boldsymbol{c}(\theta, 0) = \log \mathbb{E}[(MR_f)^\theta] \le B(\theta)$

We also have the following result

Result

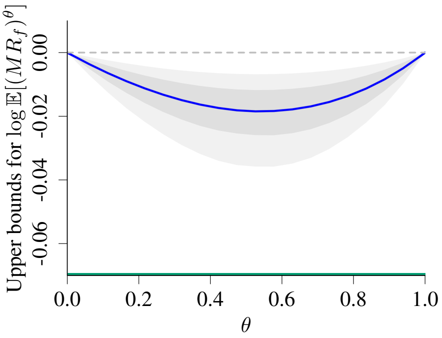
The lower bounds are informative about the attractiveness of the investment opportunity set:

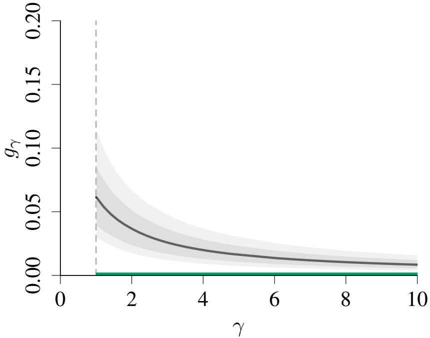
$$g_\gamma \ge \left| \frac{B(\theta)}{\theta} \right| \quad \textit{for } \theta = 1 - \frac{1}{\gamma},\ \gamma > 1$$

- For $\gamma \in (0,1)$, i.e., $\theta < 0$, similar result ($B(\theta) = \sup_y \ldots$)
- A related result for $\gamma = 1$ (log investors, more on this later)

An economic interpretation for the well-behaved SDF moments

Realized return sample: 1872–2022



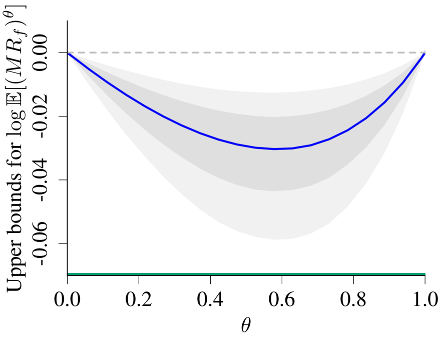


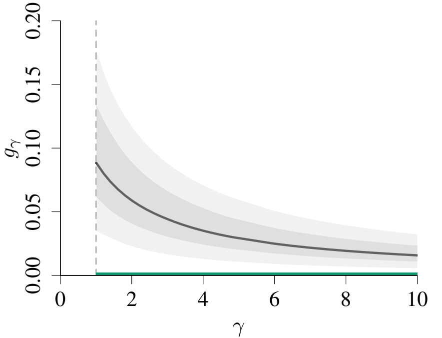
Convexity bounds for the SDF: $\theta \in (0, 1)$

WTP to trade risky assets: $\gamma > 1$

An economic interpretation for the well-behaved SDF moments

Realized return sample: 1946–2022





Convexity bounds for the SDF: $\theta \in (0, 1)$

WTP to trade risky assets: $\gamma > 1$

In a lognormal world: $\boldsymbol{c}(\theta, 0) = 1/2\lambda^2\theta(\theta - 1)$: $\boldsymbol{c}(1/2,\, 0) = \lambda^2/8 \approx 3\%$

Entropy as measures of variability

- Alternative measures of variability: $\mathbb{E}\, f(MR_f) - f(\mathbb{E}\, MR_f)$ for convex function f
- When $f(x) = x^2$, variance
  - Variance and, more generally, θth moment bound with $\theta \notin (0,1)$ are unstable
- When $f(x) = x \log x$, we obtain Theil's first entropy measure (Stutzer 1995): the gradient of $\boldsymbol{c}(\theta, 0)$ at $\theta = 1$

$$L_1(MR_f) = \mathbb{E}^* \log(MR_f)$$

- When $f(x) = -\log x$, we obtain Theil's second entropy measure (Bansal and Lehmann 1997, Alvarez and Jermann 2005): the absolute value of the gradient of $\boldsymbol{c}(\theta, 0)$ at $\theta = 0$

$$L_2(MR_f) = -\mathbb{E} \log(MR_f)$$

- Convexity of the CGF also supplies bounds on these quantities, and they behave well

Result (Entropy bounds)

$$L_1(MR_f) \geq \sup_{y\in\mathbb{R}} \; y\,\mathbb{E}^* \log \frac{R}{R_f} - \underbrace{\log \mathbb{E}\left[\left(\frac{R}{R_f}\right)^y\right]}_{\boldsymbol{c}(0,y)}$$

$$L_2(MR_f) \geq \sup_{y\in\mathbb{R}} \; y\,\mathbb{E} \log \frac{R}{R_f} - \underbrace{\log \mathbb{E}^*\left[\left(\frac{R}{R_f}\right)^y\right]}_{\boldsymbol{c}(1,y)}$$

- RHSs: difference between a linear and a convex function, so well behaved
- First bound is completely new (impossible to derive without the risk-neutral CGFs)
- Second bound generalizes Alvarez–Jermann
  - For $y = 1$: $L_2(MR_f) \geq \mathbb{E}[\log R - \log R_f]$
  - Expected log returns: attractiveness of investment opportunities for log investors
  - Our lower bound is the smallest possible WTP to trade risky assets for log investors

Result (Entropy bounds)

$$L_1(MR_f) \geq \sup_{y\in\mathbb{R}} \; y\,\mathbb{E}^* \log \frac{R}{R_f} - \underbrace{\log \mathbb{E}\left[\left(\frac{R}{R_f}\right)^y\right]}_{\boldsymbol{c}(0,y)}$$

$$L_2(MR_f) \geq \sup_{y\in\mathbb{R}} \; y\,\mathbb{E} \log \frac{R}{R_f} - \underbrace{\log \mathbb{E}^*\left[\left(\frac{R}{R_f}\right)^y\right]}_{\boldsymbol{c}(1,y)}$$

- Under lognormality, both LHSs are the same, and equal to $\frac{1}{2}\operatorname{var}\log(MR_f)$
- Deviating from lognormality: let κ_n be the nth cumulant of $\log(MR_f)$

$$L^{(1)}(MR_f) - L^{(2)}(MR_f) = \sum_{n=3}^{\infty} \frac{n-2}{n!}\kappa_n = \frac{\kappa_3}{6} + \frac{\kappa_4}{12} + \frac{\kappa_5}{40} + \frac{\kappa_6}{180} + \cdots,$$

Result (Entropy bounds)

$$L_1(MR_f) \geq \sup_{y \in \mathbb{R}} \; y \, \mathbb{E}^* \log \frac{R}{R_f} - \underbrace{\log \mathbb{E}\left[\left(\frac{R}{R_f}\right)^y\right]}_{\boldsymbol{c}(0,y)}$$

$$L_2(MR_f) \geq \sup_{y \in \mathbb{R}} \; y \, \mathbb{E} \log \frac{R}{R_f} - \underbrace{\log \mathbb{E}^*\left[\left(\frac{R}{R_f}\right)^y\right]}_{\boldsymbol{c}(1,y)}$$

- The optimizing values y^* have particularly nice interpretations (more on this later)

Bounds for the first entropy measure

sample	est.	bootstrap CI	$\mathbb{E}$-bootstrap CI	$\mathbb{E}^*$-bootstrap CI
Panel A: one month				
1872-2022	0.088	$(0.033, 0.175)$	$(0.033, 0.177)$	$(0.084, 0.093)$
1946-2022	0.173	$(0.070, 0.344)$	$(0.071, 0.348)$	$(0.165, 0.182)$
1996-2022	0.157	$(0.016, 0.445)$	$(0.016, 0.439)$	$(0.149, 0.165)$
Panel B: one year				
1872-2022	0.070	$(0.026, 0.151)$	$(0.027, 0.150)$	$(0.062, 0.079)$
1946-2022	0.143	$(0.060, 0.301)$	$(0.058, 0.313)$	$(0.129, 0.158)$
1996-2022	0.122	$(0.013, 0.476)$	$(0.013, 0.478)$	$(0.110, 0.133)$
JKKST annual	0.078	$(0.028, 0.178)$	$(0.028, 0.179)$	$(0.068, 0.088)$

- The majority of estimation uncertainty still comes from the realized returns
- Less observations, wider CI

Bounds for the second entropy measure

	A–J measure ($y = 1$)					
sample	est.	bootstrap CI	est.	bootstrap CI	$\mathbb{E}$-bootstrap CI	$\mathbb{E}^*$-bootstrap CI
(a) one-month horizon						
1872-2022	0.052	$(0.023, 0.080)$	0.062	$(0.023, 0.122)$	$(0.023, 0.120)$	$(0.060, 0.065)$
1946-2022	0.066	$(0.036, 0.100)$	0.089	$(0.038, 0.174)$	$(0.038, 0.174)$	$(0.084, 0.094)$
1996-2022	0.067	$(0.006, 0.125)$	0.091	$(0.009, 0.262)$	$(0.009, 0.258)$	$(0.087, 0.097)$
(b) one-year horizon						
1872-2022	0.049	$(0.023, 0.072)$	0.057	$(0.023, 0.108)$	$(0.023, 0.105)$	$(0.053, 0.063)$
1946-2022	0.063	$(0.034, 0.093)$	0.084	$(0.035, 0.173)$	$(0.035, 0.164)$	$(0.076, 0.096)$
1996-2022	0.067	$(0.007, 0.118)$	0.091	$(0.010, 0.273)$	$(0.010, 0.265)$	$(0.083, 0.104)$
JKKST annual	0.045	$(0.018, 0.076)$	0.050	$(0.018, 0.117)$	$(0.018, 0.114)$	$(0.048, 0.055)$

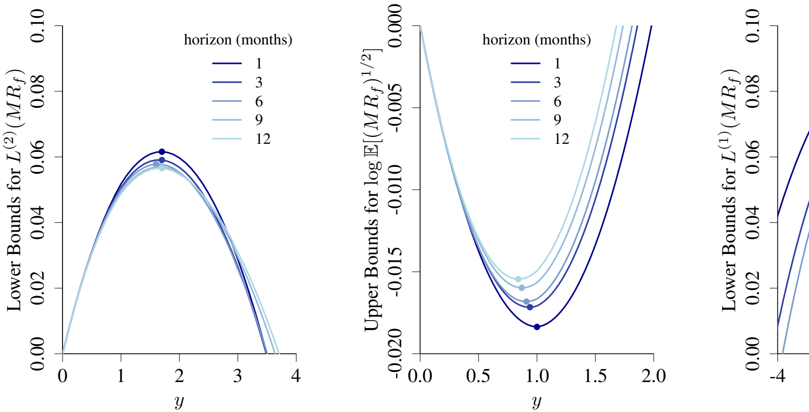
- $L^{(2)} < L^{(1)}$: the SDF tends to be positively skewed, heavy-tailed, etc.

Bounds for the intermediate moment $\theta = 1/2$

sample	est.	bootstrap CI	$\mathbb{E}$-bootstrap CI	$\mathbb{E}^*$-bootstrap CI
(a) one-month horizon				
1872-2022	-0.018	$(-0.036, -0.007)$	$(-0.035, -0.007)$	$(-0.018, -0.018)$
1946-2022	-0.029	$(-0.057, -0.012)$	$(-0.057, -0.012)$	$(-0.030, -0.029)$
1996-2022	-0.029	$(-0.081, -0.003)$	$(-0.081, -0.003)$	$(-0.029, -0.029)$
(b) one-year horizon				
1872-2022	-0.015	$(-0.029, -0.006)$	$(-0.029, -0.006)$	$(-0.016, -0.015)$
1946-2022	-0.026	$(-0.052, -0.011)$	$(-0.051, -0.011)$	$(-0.026, -0.025)$
1996-2022	-0.025	$(-0.084, -0.003)$	$(-0.083, -0.003)$	$(-0.026, -0.025)$
JKKST annual	-0.015	$(-0.033, -0.005)$	$(-0.033, -0.005)$	$(-0.015, -0.015)$

- Similar patterns for all the intermediate moments when $\theta \in (0, 1)$

Multiple horizons



- The optimizing values y^* in the entropy bounds: measures of the market risk aversion

Merton–Samuelson redux

- We can interpret the optimizing values of y in the entropy bounds when $M \propto R^{-\gamma}$
- The SDF of a myopic power utility investor holding the S&P 500
- If R/R_f is lognormal, the Merton–Samuelson (1969) calculation yields:

$$\gamma = \frac{\mu}{\sigma^2} \approx 2$$

- Under our framework, without any distributional assumptions:
  - y^* optimizing $L_1(MR_f)$: $-\gamma$
  - y^* optimizing $L_2(MR_f)$: γ

Implied market risk aversion

	first entropy measure $L^{(1)}$		second entropy measure $L^{(2)}$	
horizon in months	estimate	bootstrap CI	estimate	bootstrap CI
1	2.36	$(1.42, 3.38)$	1.69	$(1.00, 2.48)$
2	2.21	$(1.41, 3.11)$	1.67	$(1.05, 2.40)$
3	2.14	$(1.38, 3.08)$	1.65	$(1.02, 2.36)$
4	2.17	$(1.43, 3.10)$	1.63	$(1.09, 2.27)$
5	2.15	$(1.40, 3.14)$	1.62	$(1.03, 2.35)$
6	2.07	$(1.34, 3.12)$	1.63	$(1.03, 2.39)$
9	1.84	$(1.06, 2.95)$	1.66	$(1.00, 2.50)$
12	1.74	$(0.96, 2.99)$	1.66	$(0.99, 2.65)$

Conclusion

- A unifying framework to understand SDF bounds
  - Instead of hunting across the cross-section of assets, we optimally extract information from the time series of, and option prices on, a fixed asset
- Even for the S&P 500, option prices point to very high Sharpe ratios, and to very high, perhaps even infinite, SDF volatility
  - Rethinking the mean-variance framework?
- The problem isn't the short options time series
- The problem is that 150 years of market return realizations are not enough
- Higher moments of M are similarly fragile
- But the intermediate moments and entropy measures have good properties
  - They are closely related to measures of market risk aversion and of the attractiveness of investment opportunities