On the Moments of the Stochastic Discount Factor

Ian Martin Ran Shi

February, 2026

Background

- Asset prices are often used for assessing expectations:
  - forward rates
  - breakeven inflation
  - CDS rates
  - implied volatility
  - ...

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  - forward rates $\longrightarrow$ risk-neutral expected interest rate
  - breakeven inflation $\longrightarrow$ risk-neutral expected inflation
  - CDS rates $\longrightarrow$ risk-neutral default probability
  - implied volatility $\longrightarrow$ risk-neutral volatility
  - ...

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  - ...
- These are almost continuously observable and model-free
- But they may be distorted by risk: higher WTP for insurance/hedge assets

The importance of risk considerations

- For payoff X

$$\text{price}[X] = \frac{1}{R_f}\,\mathbb{E}^*\,[X] = \mathbb{E}[MX]$$

- Risk consideration is captured by the stochastic discount factor (SDF) M
- If there is little variation in M, risk consideration is not that important...

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- Other SDF "variability" bounds: higher moments (Snow, 1991) and entropies (Bansal and Lehman, 1997; Alvarez and Jermann, 2005)
- All can be calculated from asset returns

This paper

- We show how to understand all known bounds in one unifying framework
- We use the framework to derive and estimate new bounds
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- We explain why the H-J bound is inherently unstable
  - in sample, the θth moment of SDF rises exceedingly rapidly as θ rises above one
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  - ... and behaves poorly even in population
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- Our prescription
  - certain (new) bounds are stable both in theory and in the data
  - like H-J, they are related to measures of investment opportunities (for CRRA investors)

The cumulant-generating function (CGF)

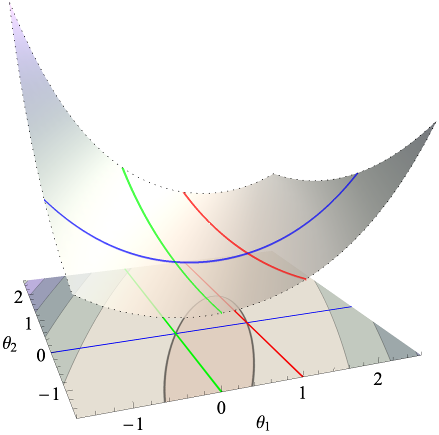
Fixing the asset return R, we consider the function

$$\boldsymbol{c}(\theta_1,\, \theta_2) = \log \mathbb{E}\left[(MR_f)^{\theta_1}\left(\frac{R}{R_f}\right)^{\theta_2}\right]$$

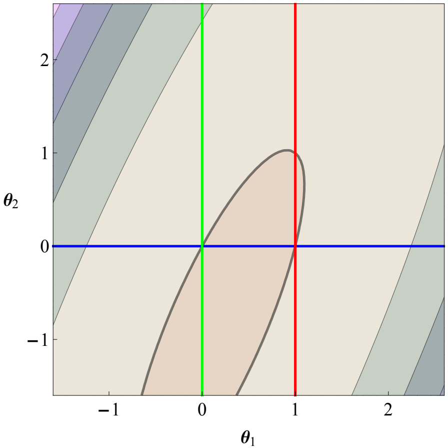
- $\boldsymbol{c}(0,0) = 0$: trivial
- $\boldsymbol{c}(1,0) = 0$: because SDF prices the riskless asset
- $\boldsymbol{c}(1,1) = 0$: because SDF prices the return R
- $\boldsymbol{c}(\theta_1,\theta_2)$ is a convex function

The cumulant-generating function (CGF)

- Observables from asset markets:
  - Returns: $\boldsymbol{c}(0,\, \theta_2) = \log \mathbb{E}\left[(R/R_f)^{\theta_2}\right]$
  - Options: $\boldsymbol{c}(1,\, \theta_2) = \log \mathbb{E}\left[(MR_f)\,(R/R_f)^{\theta_2}\right] = \log \mathbb{E}^*\left[(R/R_f)^{\theta_2}\right]$
    - Recall that, for any cash flow X: $\text{price}(X) = \mathbb{E}[MX] = \dfrac{1}{R_f}\,\mathbb{E}^*[X]$; here, we consider $X = (R/R_f)^{\theta_2}$
- Equilibrium models specify $\boldsymbol{c}(\theta_1,\, 0) = \log \mathbb{E}\left[(MR_f)^{\theta_1}\right]$
- Rich implications: $\boldsymbol{c}(0, 1)$ "equity premium"; Cboe $\text{VIX}^2 = -2\frac{\partial \boldsymbol{c}}{\partial \theta_2}(1, 0)$; CME CVOL is such that $\log(1 + \text{CVOL}^2) = \boldsymbol{c}(1, 2)$; ...



2
1
θ_2
0
−1
−1
0
θ_1
1
2



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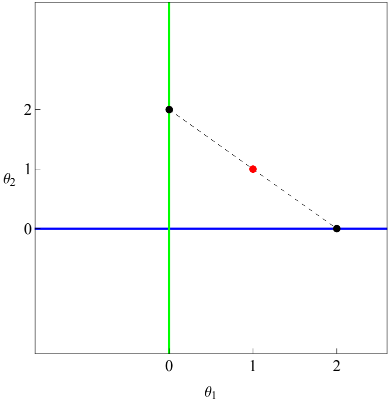
How to derive traditional results using convexity

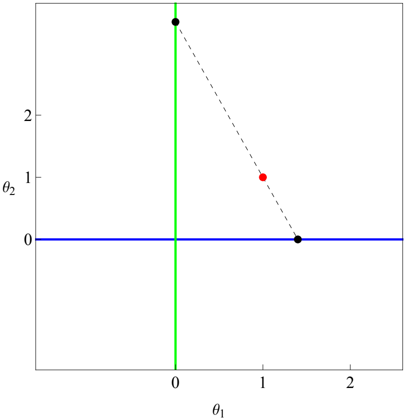


- $\frac{1}{2}\boldsymbol{c}(2,0) + \frac{1}{2}\boldsymbol{c}(0,2) \geq \boldsymbol{c}(1,1)$
- But, as $\boldsymbol{c}(1,1) = 0$, this implies

$$\mathbb{E}\left[(MR_f)^2\right] \geq \frac{1}{\mathbb{E}\left[\left(\frac{R}{R_f}\right)^2\right]}$$

- Gives the Hansen–Jagannathan bound (the minimum second moment R is such that $1/\,\mathbb{E}[(R/R_f)^2] - 1 =$ the maximal SR^2)

How to derive traditional results using convexity



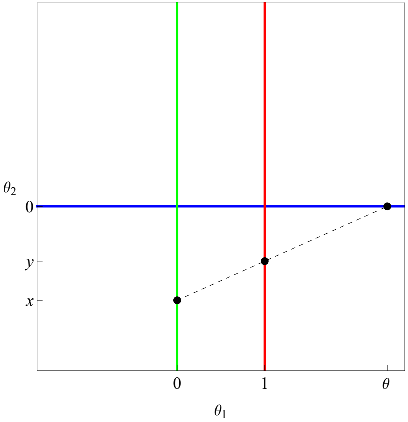
- $\frac{1}{\theta}\boldsymbol{c}(\theta,0)+\left(1-\frac{1}{\theta}\right)\boldsymbol{c}\left(0,\frac{\theta}{\theta-1}\right)\geq\boldsymbol{c}(1,1)$
- But, as $\boldsymbol{c}(1,1)=0$, this implies

$$\mathbb{E}\left[(MR_f)^{\theta}\right]\geq\left\{\mathbb{E}\left[\left(\frac{R}{R_f}\right)^{\frac{\theta}{\theta-1}}\right]\right\}^{1-\theta}$$

- Gives the Snow (1991) bound
- Paradoxically, when $\theta\to 1$, RHS R^{∞}
- ... hard (if not impossible) to measure

  while we know $\mathbb{E}[MR_f]\equiv 1$!

But we can do better



- Existing bounds connect the blue line (SDF moments) to the green line (historical time series) by looking through the point $(1, 1)$ where the CGF is pinned to zero
- We don't need to do this!
- We can look in any direction once we realise that the red line (risk-neutral distribution) is observable

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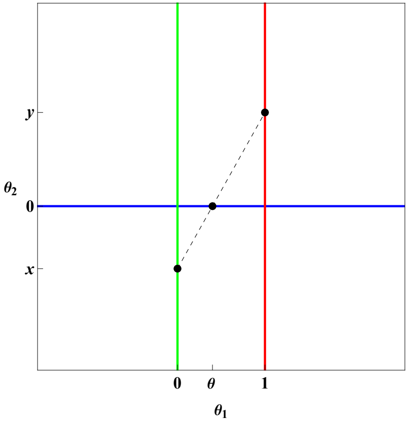

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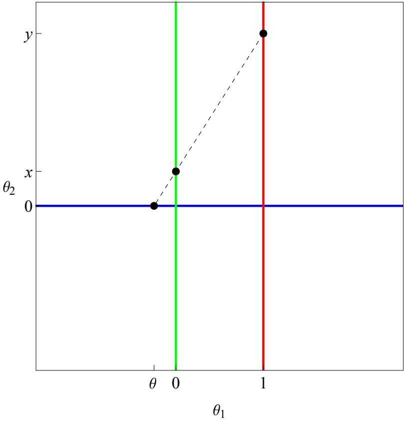

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Result (New moment bounds)

For $\theta < 0$ or $\theta > 1$, we have

$$\mathbb{E}\left[(MR_f)^\theta\right] \geq \sup_{y\in\mathbb{R}} \left\{\mathbb{E}^*\left[\left(\frac{R}{R_f}\right)^y\right]\right\}^\theta \left\{\mathbb{E}\left[\left(\frac{R}{R_f}\right)^{\frac{\theta}{\theta-1}y}\right]\right\}^{1-\theta}$$

- Inequality is reversed for $\theta \in (0, 1)$
- When $y = 1$, reduces to the Snow (1991) bound
- When $y = 1$ and $\theta = 2$, equivalent to the Hansen–Jagannathan bound
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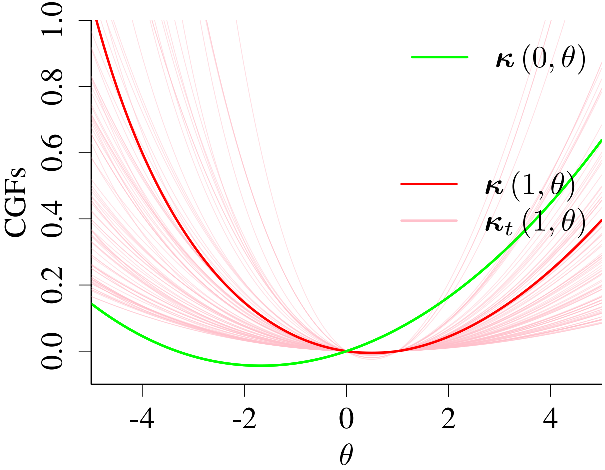
The two observable slices of the CGF in the data



Two special cases

Recall the object:

$$\theta\, \boldsymbol{c}(1, y) + (1-\theta)\, \boldsymbol{c}\left(0, \frac{\theta}{\theta - 1} y\right)$$

Consider two special cases:

- $\theta = 2$: the lower bound for $\log \mathbb{E}\left[(MR_f)^2\right]$ equals

$$\sup_y\ 2\boldsymbol{c}(1,\, y) - \boldsymbol{c}(1,\, 2y)$$

  i.e., the difference between two convex functions—badly behaved!

- $\theta = 1/2$: the upper bound for $\log \mathbb{E}\left[\sqrt{MR_f}\right]$

$$\inf_y\ \frac{1}{2}\boldsymbol{c}(1,\, y) + \frac{1}{2}\boldsymbol{c}\,(0,\, -y),$$

  i.e., the sum of two convex functions—well behaved

Instability of volatility bounds

There is no good reason to expect the population volatility of the SDF to be well behaved

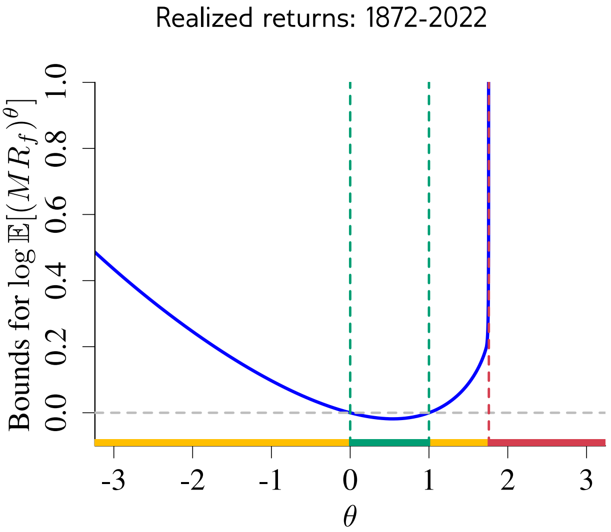
To make things worse, we have the following (simplified) results in the sample

Result

Fix $\theta > 1$. *If the most extreme put strike (of a put with positive bid price) in the dataset is lower than the lowest observed return in sample, then the lower bound can be made arbitrarily large.*

- The worst monthly return is -29% (September 1931)
- Put options with strikes more than 29% out of the money have positive *bid* prices on 73% of days in our sample

For all θs: the convexity bounds for the SDF moments



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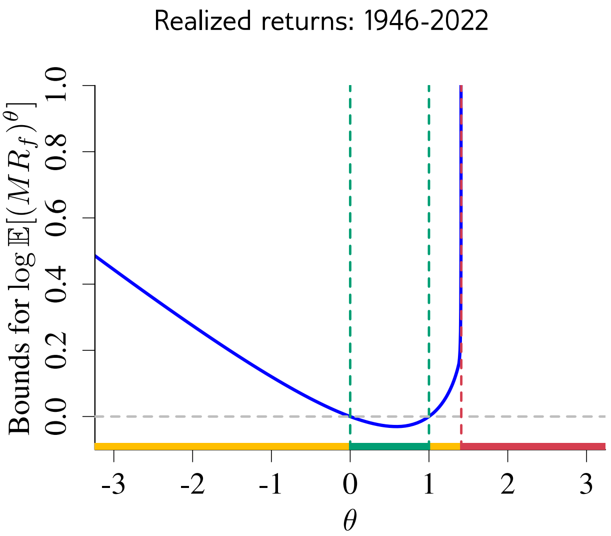

The singularity is near

sample	est.	bootstrap CI
1872-2022	1.72	$(1.52, 2.05)$
1946-2022	1.38	$(1.20, 1.60)$
1996-2022	1.44	$(1.23, 1.78)$

- The singularity mostly appears before two
- Can emerge naturally in models with parameter learning and disagreements

The good news is...

Result

The moment bounds are well-behaved when $\theta \in (0, 1)$, in the sense that the minimization problem over y on the right-hand side of the inequality has a unique interior minimum.

- The variance of the SDF is appealing because it relates the Sharpe ratio
    - a measure of the attractiveness of investment opportunities
    - ... from the perspective of a one-period investor with quadratic utility

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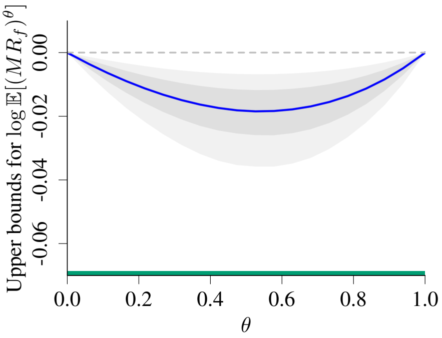
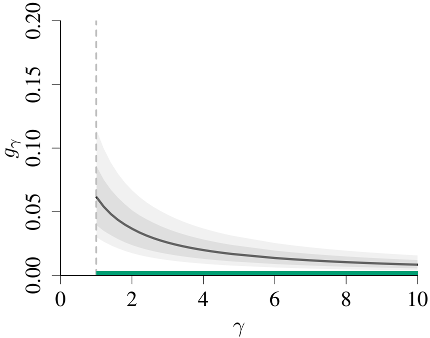
Do these intermediate moments of the SDF have similar properties?

- To do so, we adopt the perspective of a one-period CRRA-γ investor
- The attractiveness of investment opportunities can be quantified using the willingness-to-pay (WTP)—the fraction of initial wealth—to be allowed to trade risky assets, namely g_γ

An economic interpretation for the well-behaved SDF moments

Realized returns: 1872–2022





Bounds for the SDF: $\theta \in (0, 1)$

WTP: $\gamma = \frac{1}{1-\theta} > 1$

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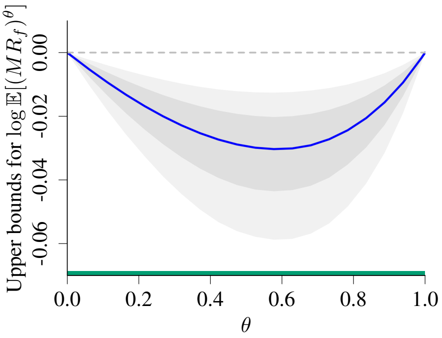
Realized returns: 1946–2022



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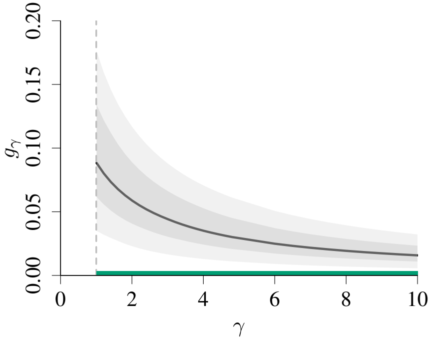

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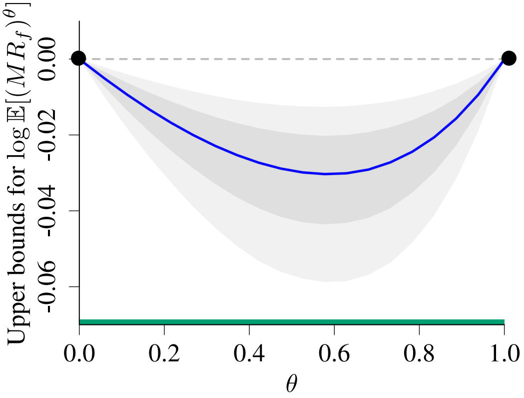

Entropy as measures of variability

- Alternative measures of variability: $\mathbb{E}\, f(MR_f) - f(\mathbb{E}\, MR_f)$ for convex function f
- When $f(x) = x^2$, variance
- When $f(x) = x \log x$ (Stutzer 1995): the gradient at $\theta = 1$

$$L_1(MR_f) = \mathbb{E}^* \log(MR_f)$$

- When $f(x) = -\log x$ (Bansal and Lehmann 1997; Alvarez and Jermann 2005): the absolute value of the gradient at $\theta = 0$

$$L_2(MR_f) = -\,\mathbb{E} \log(MR_f)$$

- Convexity of the CGF also supplies bounds on these quantities, and they behave well

Result (Entropy bounds)

$$L_1(MR_f) \geq \sup_{y \in \mathbb{R}} \; y\, \mathbb{E}^* \log \frac{R}{R_f} - \underbrace{\log \mathbb{E}\left[\left(\frac{R}{R_f}\right)^y\right]}_{\boldsymbol{c}(0,y)}$$

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- RHSs: difference between a linear and a convex function, so well behaved
- First bound is completely new (impossible to derive without the risk-neutral CGFs)
- Second bound generalizes Bansal–Lehmann/Alvarez–Jermann
  - For $y = 1$: $L_2(MR_f) \geq \mathbb{E}[\log R - \log R_f]$
  - Expected log returns: attractiveness of investment opportunities for log investors
  - Our lower bound is the smallest possible WTP to trade risky assets for log investors

Result (Entropy bounds)

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- The optimizing values y^* have particularly nice interpretations

Merton–Samuelson redux

- We can interpret the optimizing values of y in the entropy bounds when $M \propto R^{-\gamma}$
- The SDF of a myopic power utility investor holding the S&P 500
- If R/R_f is lognormal, the Merton–Samuelson (1969) calculation yields:

$$\gamma = \frac{\mu}{\sigma^2} \approx 2$$

- Under our framework, without any distributional assumptions:
  - y^* optimizing $L_1(MR_f)$: $-\gamma$
  - y^* optimizing $L_2(MR_f)$: γ

Implied risk aversion

	first entropy measure $L^{(1)}$		second entropy measure $L^{(2)}$	
horizon in months	estimate	bootstrap CI	estimate	bootstrap CI
1	2.36	$(1.42, 3.38)$	1.69	$(1.00, 2.48)$
2	2.21	$(1.41, 3.11)$	1.67	$(1.05, 2.40)$
3	2.14	$(1.38, 3.08)$	1.65	$(1.02, 2.36)$
4	2.17	$(1.43, 3.10)$	1.63	$(1.09, 2.27)$
5	2.15	$(1.40, 3.14)$	1.62	$(1.03, 2.35)$
6	2.07	$(1.34, 3.12)$	1.63	$(1.03, 2.39)$
9	1.84	$(1.06, 2.95)$	1.66	$(1.00, 2.50)$
12	1.74	$(0.96, 2.99)$	1.66	$(0.99, 2.65)$

Conclusion

- A unifying framework to understand existing SDF bounds and derive new ones

- For the US market (S&P 500): very high, perhaps infinite, SDF vol. and Sharpe ratios
  - this can appear in any sample
  - ...which is an intrinsic problem even at the population level
  - the foundation of the mean-variance paradigm is questionable
  - higher moments of the SDF are similarly fragile

- But the intermediate moments and entropy measures have good properties
  - directly translate to measures of market risk aversion and investment opportunities
  - should be the focus of performance measures and model diagnostics