Consumption-Based Asset Pricing with Higher Cumulants

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Abstract

I extend the Epstein-Zin-lognormal consumption-based asset-pricing model to allow for general i.i.d. consumption growth. Information about the higher moments—equivalently, cumulants—of consumption growth is encoded in the *cumulant-generating function*. I use the framework to analyze economies with rare disasters, and argue that the importance of such disasters is a double-edged sword: parameters that govern the frequency and sizes of rare disasters are critically important for asset pricing, but extremely hard to calibrate. I show how to sidestep this issue by using observable asset prices to make inferences without having to estimate higher moments of the underlying consumption process. Extensions of the model allow consumption to diverge from dividends, and for non-i.i.d. consumption growth.

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The combination of power utility and i.i.d. lognormal consumption growth makes for a tractable benchmark model in which asset prices and expected returns can be found in closed form. This paper demonstrates that the lognormality assumption can be dropped without sacrificing tractability, thereby allowing for straightforward and flexible analysis of the possibility that, say, consumption is subject to rare disasters. There has recently been considerable interest in reviving the idea of Rietz (1988) that the presence of such disasters, or fat tails more generally, can help to explain asset pricing phenomena such as the riskless rate, equity premium and other puzzles (Barro (2006), Farhi and Gabaix (2008), Gabaix (2008), Jurek (2008)). Here, I take a different line, closer in spirit to Weitzman (2007), and argue that the importance of rare, extreme events is a double-edged sword: those model parameters which are most important for asset prices, such as disaster parameters, are also the hardest to calibrate, precisely because the disasters in question are rare.

Working under the assumptions that there is a representative agent with Epstein-Zin preferences (Epstein and Zin (1989)) and that consumption growth is i.i.d., I show in Section I that the equity premium, riskless rate, consumption-wealth ratio and mean consumption growth (the "fundamental quantities") can be simply expressed in terms of the cumulant-generating function (CGF). CGFs crop up elsewhere in the literature; one contribution of this paper is to demonstrate how neatly they dovetail with the standard consumption-based asset-pricing approach. Importantly, the framework allows for the possibility of disasters, but is agnostic about whether or not they occur. The expressions derived relate the fundamental quantities directly to the cumulants (equivalently, moments) of consumption growth. I show, for example, how the precautionary savings effect, which influences the riskless rate in a lognormal model, can be generalized in the presence of higher cumulants. By shifting the focus from moments to cumulants, I retain tractability without needing to truncate Taylor expansions (as in, say, Kraus and Litzenberger (1976)), so avoid the critique of Brockett and Kahane (1992).

In Section II, I illustrate the framework by investigating a continuous-time model featuring rare disasters, and show that the model's predictions are sensitively dependent on the calibration assumed. As a stark example, take a consumption-based model in which the representative agent has relative risk aversion equal to 4. Now add to the model a certain type of disaster that strikes, on average, once every 1,000 years, and reduces consumption by 64 per cent. (Barro (2006) documents that Germany and Greece each suffered such a fall in per capita real GDP during the Second World War.) The introduction of this disaster drives the riskless rate down by 5.9 percentage points and increases the equity premium by 3.7 per cent.[1] Very rare, very severe events exert an extraordinary influence on the benchmark model, and we do not expect to estimate their frequency and intensity directly from the data.

I document this more formally in Section II.B, where I present the results of a GMM exercise as in Hansen and Singleton (1982). I use samples consisting of 100 years of simulated data in the model economy with disasters, and show that in such a relatively short sample, GMM leads to biased and extremely inaccurate estimators of the true population parameters. In economies with still fatter tails, GMM may not be valid even asymptotically.

The remainder of the paper is devoted to finding ways around this disheartening fact. We can, for example, detect the influence of disaster events *indirectly*, by observing asset prices. I argue, therefore, that the standard approach—calibrating a particular model and trying to fit the fundamental quantities—is not the way to go. I turn things round, viewing the fundamental quantities as observables, and making inferences from them. It then becomes possible to make nonparametric statements that are robust to the details of the consumption growth process.

In this spirit, I derive, in section III, sharp and robust restrictions on preference parameters that are valid in *any* Epstein-Zin-i.i.d. model that is consistent with the observed fundamentals. The key idea is to exploit an important property of CGFs: they are always convex. The results restrict the time-preference rate, ρ, and elasticity of intertemporal substitution, ψ, to lie in a certain subset of the positive quadrant. (See Figure 7.) These parameters are of central importance for financial and macroeconomic models. The restrictions depend only on the Epstein-Zin-i.i.d. assumptions and on observed values of the fundamental quantities, and not, for example, on any assumptions about the existence, frequency or size of disasters. They are complementary to econometric or experimental estimates of ψ and ρ, and are of particular interest because there is little agreement about the value of ψ. (Campbell (2003) summarizes the conflicting evidence.) I also show how good-deal bounds (Cochrane and Saá-Requejo (2000)) can be used to provide upper bounds on risk aversion, based once again on the fundamental quantities, without calibrating a consumption process.

Section IV extends the analysis in two directions. The majority of the paper sets dividends equal to consumption or adopts the highly tractable approach, taken by Campbell

[1]The effect is smaller with Epstein-Zin preferences if the elasticity of substitution is greater than 1, but even with an elasticity of intertemporal substitution equal to 2, the riskless rate drops by 3.5 per cent.

(1986, 2003) and also advocated by Abel (1999), of modelling dividends as a power of consumption. However, several authors have argued for the importance of allowing log consumption and log dividends to be imperfectly correlated.[2] Section IV.A shows how the CGF approach can be extended to allow for this possibility, and presents a heterogeneous-agent model as a motivating application. Consistent with the argument of the rest of the paper, I find that heterogeneity is a double-edged sword. The *good* news is that heterogeneity interacts well with disasters in the sense that it can potentially give a huge boost to risk premia. The *bad* news is that heterogeneity only matters *to the extent to which it occurs at times of aggregate disaster*, so the fundamental empirical difficulty highlighted in the rest of the paper is not avoided. Again, though, it is possible to make statements that are robust to what is going on in the tails.

Finally, although presumably the framework is a good approximation to reality over time horizons long enough that the economy "looks roughly i.i.d.", it is of interest to weaken the i.i.d. assumption, and I do so in Section IV.B.

When not included in the body of the paper, proofs are in the appendix.

Related literature. Campbell and Cochrane (1999) and Bansal and Yaron (2004) modify the textbook model along different dimensions, but take care to remain in a conditionally lognormal environment. This paper explores different features, and implications, of the data, so is complementary to their work. It would, of course, be interesting to extend these papers by allowing for the possibility of jumps, but doing so would obscure the main point of this paper.

Various authors have presented analytical solutions to asset pricing models. Abel (2008) computes asset prices in economies in which agents have benchmark consumption levels, but works in a lognormal framework; and Eraker (2008) prices assets from the perspective of an Epstein-Zin representative agent, but relies on a loglinearization of the return on aggregate wealth for tractability. This approximation is likely to be particularly problematic in a disaster model in which aggregate wealth may experience severe declines. Gabaix (2009) proposes a class of reverse-engineered "linearity-generating" processes that lead to closed-form asset prices. Bonomo, Garcia, Meddahi, and Tédongap (2009) extend the framework of Garcia, Meddahi and Tédongap (2008) to provide analytical pricing formulas in a long-run risks environment with generalized disappointment aversion, although their focus is not on rare disasters. Martin (2011a, 2011b) uses CGFs

[2]For example, Cecchetti, Lam and Mark (1993), Bonomo and Garcia (1996), Campbell and Cochrane (1999), Longstaff and Piazzesi (2004), and Bansal and Yaron (2004). With the exception of Longstaff and Piazzesi (2004), consumption and dividends are not cointegrated in any of these papers.

in multi-asset models in which consumption growth is not i.i.d.

A large body of literature applies Lévy processes to derivative pricing (Carr and Madan (1998), Cont and Tankov (2004)) and portfolio choice (Cvitanić, Polimenis and Zapatero (2005), Aït-Sahalia, Cacho-Diaz and Hurd (2006)). Lustig, Van Nieuwerburgh and Verdelhan (2008) present estimates of the wealth-consumption ratio. Backus, Foresi and Telmer (2001) derive expressions relating cumulants to risk premia, though their approach is very different from that taken here.

Garcia, Luger, and Renault (2003) expand the range of assets, using options prices to obtain information about preference parameters, though they work in a conditionally log-normal framework. Backus, Chernov and Martin (2011) adopt this approach in exploring the evidence for disasters in option prices but, again, impose a particular structure on the i.i.d. dividend process. Julliard and Ghosh (2008) argue that the cross-section of asset price data is hard to square with disaster explanations of the equity premium. Consistent with the above discussion, their parameter estimates have large standard errors. They also carry out a Generalized Empirical Likelihood estimation whose results are similar to those of Section II.B.

I Asset pricing and the CGF

Define $G_t \equiv \log C_t/C_0$ and write $G \equiv G_1$. I make two assumptions.

A1 There is a representative agent with Epstein-Zin preferences, time preference rate ρ, relative risk aversion γ, and elasticity of intertemporal substitution ψ.

A2 The consumption growth, $\log C_t/C_{t-1}$, of the representative agent is (or is perceived to be) i.i.d., and the CGF of G (defined below) exists on a neighborhood of $[-\gamma, 1]$.[3]

Assumption A1 allows risk aversion γ to be disentangled from the elasticity of intertemporal substitution ψ. To keep things simple, those calculations that appear in the main text restrict to the power utility case in which ψ is constrained to equal $1/\gamma$; in this case, the representative agent maximizes

$$\mathbb{E} \sum_{t=0}^{\infty} e^{-\rho t} \frac{C_t^{1-\gamma}}{1-\gamma} \quad \text{if } \gamma \neq 1 \,, \quad \text{or} \quad \mathbb{E} \sum_{t=0}^{\infty} e^{-\rho t} \log C_t \quad \text{if } \gamma = 1 \,.$$

[3]If not, the consumption-based asset-pricing approach is invalid. This assumption implies that all cumulants, and hence all moments, of G are finite. See Billingsley (1995, Section 21).

Cogley (1990) and Barro (2009) present evidence in support of A2 in the form of variance-ratio statistics close to one, on average, across 9 (Cogley) or 19 (Barro) countries.

We need expected utility to be well defined in that

$$\mathbb{E}\sum_{t=0}^{\infty}\left|e^{-\rho t}\frac{C_t^{1-\gamma}}{1-\gamma}\right| < \infty \quad \text{if } \gamma \neq 1. \tag{1}$$

I discuss this requirement further below.

Consider an asset which pays dividend stream $\{D_t\}_{t\geq 0}$. The Euler equation relates the price of an asset this period, P_0, to the payoff next period, $P_1 + D_1$. Expectations are calculated with respect to the measure perceived by the representative agent:

$$P_0 = \mathbb{E}_0\left(e^{-\rho}\left(\frac{C_1}{C_0}\right)^{-\gamma}(D_1 + P_1)\right).$$

Iterating forward and imposing a no-bubble condition, we have the familiar equation

$$P_0 = \mathbb{E}\left(\sum_{t=1}^{\infty} e^{-\rho t}\left(\frac{C_t}{C_0}\right)^{-\gamma} D_t\right).$$

Suppose that $D_t \equiv (C_t)^\lambda$ for some constant λ. If $\lambda = 0$ then the asset is a riskless bond; if $\lambda = 1$ then it is the wealth portfolio which pays consumption as its dividend. As suggested by Campbell (1986, 2003) and Abel (1999), it is possible to view values $\lambda > 1$ as a tractable way of modelling levered claims. Writing P_0 for the price of this asset at time 0, we have

$$P_0 = \mathbb{E}\sum_{t=1}^{\infty} e^{-\rho t}\left(\frac{C_t}{C_0}\right)^{-\gamma} C_t{}^\lambda = D_0\sum_{t=1}^{\infty} e^{-\rho t}\,\mathbb{E}\,e^{(\lambda-\gamma)G_t} = D_0\sum_{t=1}^{\infty} e^{-\rho t}\left(\mathbb{E}\,e^{(\lambda-\gamma)G}\right)^t. \tag{2}$$

The last equality follows from the assumption that log consumption growth is i.i.d. To make further progress, I now introduce a pair of definitions.

Definition 1. *Given some arbitrary random variable, G, the* moment-generating function $\boldsymbol{m}(\theta)$ *and* cumulant-generating function *or* CGF $\boldsymbol{c}(\theta)$ *are defined by*

$$\begin{aligned}\boldsymbol{m}(\theta) &\equiv \mathbb{E}\exp(\theta G)\\ \boldsymbol{c}(\theta) &\equiv \log\mathbb{E}\exp(\theta G),\end{aligned}$$

for all θ for which the expectations are finite.

Here, G is an annual increment of log consumption, $G = \log C_{t+1} - \log C_t$. Notice that $\boldsymbol{c}(0) = 0$ for any growth process and that $\boldsymbol{c}(1)$ is equal to log mean gross consumption growth, so we will want $\boldsymbol{c}(1) \approx 2\%$. The CGF summarizes information about the cumulants (or, equivalently, moments) of G. We can expand $\boldsymbol{c}(\theta)$ as a power series in θ,

$$\boldsymbol{c}(\theta) = \sum_{n=1}^{\infty} \frac{\kappa_n \theta^n}{n!}, \tag{3}$$

which defines κ_n as the nth *cumulant* of log consumption growth. Some algebra shows that the first few cumulants are familiar: $\kappa_1 \equiv \mu$ is the mean, $\kappa_2 \equiv \sigma^2$ the variance, κ_3/σ^3 the skewness and κ_4/σ^4 the excess kurtosis of log consumption growth. Knowledge of the cumulants of a random variable implies knowledge of the moments, and vice versa. With this definition, (2) becomes

$$P_0 = D_0 \sum_{t=1}^{\infty} e^{-[\rho - \boldsymbol{c}(\lambda-\gamma)]t} = D_0 \cdot \frac{e^{-[\rho-\boldsymbol{c}(\lambda-\gamma)]}}{1 - e^{-[\rho-\boldsymbol{c}(\lambda-\gamma)]}}.$$

It is convenient to define the log dividend yield $d/p \equiv \log(1 + D_0/P_0)$. Then, $d/p = \rho - \boldsymbol{c}(\lambda - \gamma)$. Two special cases are of particular interest. The first is $\lambda = 0$, in which case the asset in question is the riskless bond, whose dividend yield is the riskless rate R_f. Again, it is convenient to work with the log riskless rate, $r_f = \log(1 + R_f)$. The above calculation shows that $r_f = \rho - \boldsymbol{c}(-\gamma)$. The second is $\lambda = 1$, in which case the asset pays consumption as its dividend, and can therefore be interpreted as aggregate wealth. The dividend yield is then the consumption-wealth ratio; when $\lambda = 1$, I write c/w in place of d/p. This calculation also shows that the necessary restriction on consumption growth for the expected utility to be well defined in (1) is that $\rho - \boldsymbol{c}(1 - \gamma) > 0$, or equivalently that the consumption-wealth ratio is positive.

The gross return on the λ-asset is

$$1 + R_{t+1} = \frac{D_{t+1} + P_{t+1}}{P_t} = \frac{P_{t+1}}{P_t}\left(1 + \frac{D_{t+1}}{P_{t+1}}\right) = \frac{D_{t+1}}{D_t}\left(e^{\rho-\boldsymbol{c}(\lambda-\gamma)}\right),$$

so the expected gross return is

$$1 + \mathbb{E}\, R_{t+1} = \mathbb{E}\left(\left(\frac{C_{t+1}}{C_t}\right)^{\lambda}\right) \cdot e^{\rho-\boldsymbol{c}(\lambda-\gamma)} = e^{\rho-\boldsymbol{c}(\lambda-\gamma)+\boldsymbol{c}(\lambda)}.$$

Once again, it is more convenient to work with log expected gross return, $er \equiv \log(1 + \mathbb{E}\, R_{t+1}) = \rho + \boldsymbol{c}(\lambda) - \boldsymbol{c}(\lambda - \gamma)$. Finally, I define the risk premium $rp = er - r_f$. The following result summarizes and extends the above calculations by expressing the riskless rate, dividend yield and risk premium in terms of the CGF in the Epstein-Zin case.

Result 1. *Defining* $\vartheta \equiv (1-\gamma)/(1-1/\psi)$, *we have*

$$\begin{aligned} r_f &= \rho - \boldsymbol{c}(-\gamma) + \boldsymbol{c}(1-\gamma)\,(1-1/\vartheta) && (4)\\ d/p &= \rho - \boldsymbol{c}(\lambda-\gamma) + \boldsymbol{c}(1-\gamma)(1-1/\vartheta) && (5)\\ rp &= \boldsymbol{c}(\lambda) + \boldsymbol{c}(-\gamma) - \boldsymbol{c}(\lambda-\gamma)\,. && (6) \end{aligned}$$

The Gordon growth model holds (note that $\boldsymbol{c}(\lambda) = \log \mathbb{E}(D_{t+1}/D_t)$*):*

$$d/p = rp + r_f - \boldsymbol{c}(\lambda). \tag{7}$$

The consumption-wealth ratio c/w *is given by* (5) *with* $\lambda = 1$. *As in the power utility case, these expressions are well-defined so long as* $c/w > 0$.

Equation (6) shows that the elasticity of intertemporal substitution does not affect the risk premium. It also shows that the CGF of the driving consumption process must have a significant amount of convexity over the range $[-\gamma, \lambda]$ to generate an empirically reasonable risk premium.

These expressions can be written out as power series using (3). In the power utility case, for example, equation (4) implies that

$$r_f = \rho + \kappa_1\gamma - \frac{\kappa_2}{2}\gamma^2 + \frac{\kappa_3}{3!}\gamma^3 - \frac{\kappa_4}{4!}\gamma^4 + \text{higher order terms}\,.$$

By definition of the first four cumulants, this can be rewritten as

$$r_f = \rho + \mu\gamma - \frac{1}{2}\sigma^2\gamma^2 + \frac{\text{skewness}}{3!}\sigma^3\gamma^3 - \frac{\text{excess kurtosis}}{4!}\sigma^4\gamma^4 + \text{higher order terms}\,. \tag{8}$$

If consumption growth is lognormal, skewness, excess kurtosis and all higher cumulants are zero, so this reduces to the familiar $r_f = \rho + \mu\gamma - \sigma^2\gamma^2/2$. More generally, the riskless rate is low if mean log consumption growth μ is low (an intertemporal substitution effect); if the variance of log consumption growth σ^2 is high (a precautionary savings effect); if there is negative skewness; or if there is a high degree of kurtosis. Similarly, the dividend yield is

$$\begin{aligned} d/p &= \rho + \mu(\gamma-\lambda) - \frac{1}{2}\sigma^2(\gamma-\lambda)^2 + \frac{\text{skewness}}{3!}\sigma^3(\gamma-\lambda)^3 - \\ &\quad - \frac{\text{excess kurtosis}}{4!}\sigma^4(\gamma-\lambda)^4 + \text{higher order terms}, \end{aligned}$$

and the risk premium (in either the power utility or the Epstein-Zin case) is

$$\begin{aligned} rp &= \lambda\gamma\sigma^2 + \frac{\text{skewness}}{3!}\sigma^3\left(\lambda^3 - \gamma^3 - (\lambda-\gamma)^3\right) + \\ &\quad + \frac{\text{excess kurtosis}}{4!}\sigma^4\left(\lambda^4 + \gamma^4 - (\lambda-\gamma)^4\right) + \text{higher order terms}. \end{aligned}$$

To understand what happens with Epstein-Zin preferences, it is helpful to focus on the case $\lambda = 1, \gamma > 1$. (The logic is the same if $\lambda \neq 1$; some signs are reversed if $\gamma < 1$.) The coefficients on $\kappa_n/n!$ in the power series expansions are

$$\begin{aligned}
r_f: &\quad (-1)^{n+1}\gamma^n + (-1)^n(\gamma-1)^n(1-1/\vartheta) \\
c/w: &\quad (-1)^{n+1}(\gamma-1)^n/\vartheta \\
rp: &\quad 1 + (-1)^n\gamma^n + (-1)^{n+1}(\gamma-1)^n \\
\boldsymbol{c}(1): &\quad 1
\end{aligned}$$

The Gordon growth formula (7) implies that the nth coefficient for c/w is equal to the nth coefficient for r_f, plus that for rp, minus that for (log) expected consumption growth, $\boldsymbol{c}(1)$. So it suffices to understand the comparative statics of the risk premium and of the riskless rate.

The comparative statics of the risk premium are the same in the power utility and Epstein-Zin cases. The nth coefficient is $1 + (-1)^n\gamma^n + (-1)^{n+1}(\gamma-1)^n$. The third of these terms is smaller in magnitude than the second, but has the opposite sign, so exerts an offsetting effect. Thus the nth coefficient is positive for even $n \geq 2$: the risk premium is increasing in variance and higher even cumulants. For odd $n \geq 3$, the coefficient is negative, so the risk premium is decreasing in skewness and higher odd cumulants.

The comparative statics of the riskless rate depend on both ψ and γ. With power utility, the nth coefficient in the expansion of r_f is $(-1)^{n+1}\gamma^n$. This is positive if n is odd and negative if n is even, leading to the comparative statics discussed below equation (8). There is no offsetting term in $(\gamma-1)^n$, so the riskless rate is more sensitively dependent on higher cumulants than the risk premium. In the Epstein-Zin case, we gain an extra term $(-1)^n(\gamma-1)^n(1-1/\vartheta)$. If $\psi = 1$ then $1/\vartheta = 0$ and the nth coefficient is $(-1)^{n+1}\gamma^n + (-1)^n(\gamma-1)^n$, which *does* have the offsetting term, resulting in a riskless rate that is less sensitively dependent on the higher cumulants than in the power utility case. More generally, if $1/\vartheta < 0$—if $\gamma > 1$ and $\psi > 1$—then for small n the term $(-1)^n(\gamma-1)^n(1-1/\vartheta)$ may even dominate. For large n, though, the first term always prevails, so the riskless rate depends less sensitively on high cumulants than it does in the power utility case.

We are now in a position to understand the comparative statics of the consumption-wealth ratio. With power utility, the riskless rate is the dominant influence: it is increasing in odd cumulants and decreasing in even cumulants, and the consumption-wealth ratio inherits that property. With Epstein-Zin preferences and $1 - 1/\vartheta > 0$—i.e. $\psi > 1/\gamma$, assuming $\gamma > 1$—the effects of higher cumulants on the riskless rate are muted. If $\psi = 1$, the movements of the riskless rate exactly offset the movements of the risk premium, so that the consumption-wealth ratio is constant. If $\psi > 1$, so that $1/\vartheta < 0$, then the risk premium effect is dominant; the consumption-wealth ratio is then increasing in even cumulants and decreasing in odd cumulants. Put differently, larger even cumulants lead to a *lower* wealth-consumption ratio; this is an important component of Bansal and Yaron's (2004) long-run risk model.

Equations (4)–(6), together with the Gordon growth model (7), provide another way to look at a point made by Kocherlakota (1990). In principle, given sufficient asset price and consumption data, we could determine the riskless rate, the risk premium, and CGF $\boldsymbol{c}(\cdot)$ to arbitrary accuracy. Since γ is the only preference parameter that determines the risk premium, it could be calculated from (6), given knowledge of $\boldsymbol{c}(\cdot)$. On the other hand, knowledge of the riskless rate leaves ρ and ψ indeterminate in equation (4), even given knowledge of γ and $\boldsymbol{c}(\cdot)$. So the time discount rate and elasticity of intertemporal substitution cannot be disentangled on the basis of the four fundamental quantities.

II The continuous-time case

In continuous time, the analogue of the i.i.d. growth assumption is that the log consumption path, G_t, of the representative agent follows a Lévy process. If so, $\mathbb{E}\, e^{\theta G_t} = \left(\mathbb{E}\, e^{\theta G}\right)^t$ for *arbitrary* $t \geq 0$. Given this property, we have the following result in the limit as the period length dt (which was equal to one in the discrete-time calculations) goes to zero.

Result 2 (The continuous-time case)**.** *The instantaneous riskless rate, R_f, dividend yield, D/P, and instantaneous risk premium on aggregate wealth, RP, are*

$$\begin{aligned} R_f &= \rho - \boldsymbol{c}(-\gamma) + \boldsymbol{c}(1-\gamma)\,(1 - 1/\vartheta) \\ D/P &= \rho - \boldsymbol{c}(\lambda-\gamma) + \boldsymbol{c}(1-\gamma)(1-1/\vartheta) \\ RP &= \boldsymbol{c}(\lambda) + \boldsymbol{c}(-\gamma) - \boldsymbol{c}(\lambda-\gamma). \end{aligned}$$

The Gordon growth model holds: $D/P = R_f + RP - \boldsymbol{c}(\lambda)$.

II.A A concrete example: disasters

In this section, I show how to derive a convenient continuous-time version of Barro (2006), and show that the predictions of an i.i.d. disaster model are sensitively dependent on the parameter values assumed. Suppose that log consumption follows a jump-diffusion

$$G_t = \widetilde{\mu} t + \sigma_B B_t + \sum_{i=1}^{N(t)} Y_i$$

where B_t is a Brownian motion, $N(t)$ is a Poisson counting process with parameter ω, and Y_i are i.i.d. random variables. The CGF is $\boldsymbol{c}(\theta) = \log \boldsymbol{m}(\theta)$, where

$$\boldsymbol{m}(\theta) = \mathbb{E}\, e^{\theta G_1} = e^{\widetilde{\mu}\theta} \cdot \mathbb{E}\, e^{\sigma_B \theta B_1} \cdot \mathbb{E}\, e^{\theta \sum_{i=1}^{N(1)} Y_i}.$$

Separating the expectation into two separate products is legitimate since the Poisson jumps and Y_i are independent of the Brownian component B_t. The middle term is the expectation of a lognormal random variable: $\mathbb{E}\, e^{\theta \sigma_B B_1} = e^{\sigma_B^2 \theta^2/2}$. The final term is slightly more complicated, but can be evaluated by conditioning on the number of jumps that take place before $t = 1$:

$$\begin{aligned} \mathbb{E} \exp\left\{\theta \sum_{i=1}^{N(1)} Y_i\right\} &= \sum_0^\infty \frac{e^{-\omega}\omega^n}{n!} \mathbb{E} \exp\left\{\theta \sum_1^n Y_i\right\} \\ &= \sum_0^\infty \frac{e^{-\omega}\omega^n}{n!} \left[\mathbb{E} \exp\left\{\theta Y_1\right\}\right]^n \\ &= \exp\left\{\omega \left(\boldsymbol{m}_{Y_1}(\theta) - 1\right)\right\}, \end{aligned}$$

Finally, $\boldsymbol{c}(\theta) = \widetilde{\mu}\theta + \sigma_B^2\theta^2/2 + \omega\left(\boldsymbol{m}_{Y_1}(\theta) - 1\right)$, so the cumulants $\kappa_n(G) = \boldsymbol{c}^{(n)}(0)$ are

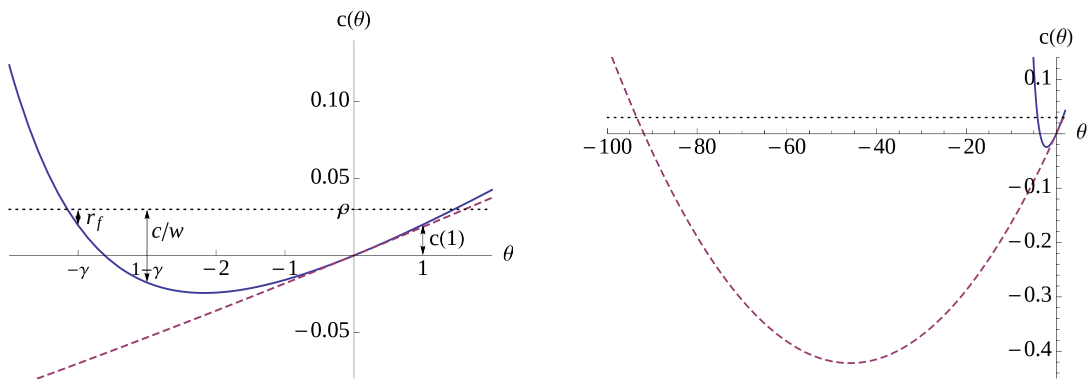
$$\kappa_n(G) = \begin{cases} \widetilde{\mu} + \omega\, \mathbb{E}\, Y & n = 1 \\ \sigma_B^2 + \omega\, \mathbb{E}\, Y^2 & n = 2 \\ \omega\, \mathbb{E}\, Y^n & n \geq 3 \end{cases}$$

Take the case in which $Y \sim N(-b, s^2)$; b is assumed to be greater than zero, so the jumps represent disasters. The CGF is then

$$\boldsymbol{c}(\theta) = \widetilde{\mu}\theta + \frac{1}{2}\sigma_B^2\theta^2 + \omega(e^{-\theta b + \frac{1}{2}\theta^2 s^2} - 1). \tag{9}$$

Figure 1a plots the CGF (9) against θ. I choose parameters according to Barro's (2006) baseline calibration—$\gamma = 4, \sigma_B = 0.02, \rho = 0.03, \widetilde{\mu} = 0.025, \omega = 0.017$—and set $b = 0.39$ and $s = 0.25$ to match the mean and variance of the distribution of jumps used in the same paper. I also plot the CGF that results in the absence of jumps ($\omega = 0$). In the latter case, I adjust the drift of consumption growth to keep mean log consumption growth constant; in the figure, this means that the two curves are tangent at the origin.

Zooming out on Figure 1a, we obtain Figure 1b, which further illustrates the equity premium and riskless rate puzzles. With jumps, the CGF is visible at the right-hand side of the figure; the CGF explodes so quickly as θ declines that it is only visible for θ greater than about -5. The jump-free lognormal CGF has incredibly low curvature. For a realistic riskless rate and equity premium, the model requires a risk aversion above 80.



(a) r_f, c/w, and $\boldsymbol{c}(1)$ can be read off the CGF (b) Zooming out

Figure 1: Left: The CGF with (solid) and without (dashed) jumps. The figure assumes that $\rho = 0.03$ and $\gamma = 4$. Right: Zooming out. Without jumps, we need enormously high γ to avoid the riskless rate and equity premium puzzles.

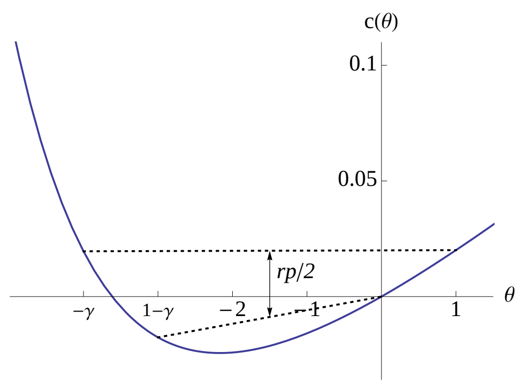


Figure 2: The risk premium. The figure assumes that $\gamma = 4$.

The riskless rate, consumption-wealth ratio and mean consumption growth can be read directly off the graph, as indicated by the arrows in Figure 1a. The risk premium can be calculated from these via the Gordon growth formula, or read directly off the graph, as in Figure 2, by drawing a line from $(-\gamma, \boldsymbol{c}(-\gamma))$ to $(1, \boldsymbol{c}(1))$ and another from $(1-\gamma, \boldsymbol{c}(1-\gamma))$ to $(0, 0)$. The midpoint of the first line lies above the midpoint of the second by convexity of the CGF. The risk premium is twice the distance from one midpoint to the other.

The standard lognormal model predicts a counterfactually high riskless rate: in Figure 1a, this is reflected in the fact that the no-jumps CGF lies well below ρ for reasonable values of θ. Similarly, the standard lognormal model predicts a counterfactually low equity premium: the no-jump CGF is practically linear over the range $[-\gamma, 1]$. Conversely, the disaster CGF has a shape which allows it to match observed fundamentals closely.

Table I shows how changes in the calibration of the distribution of disasters affect

	ω	b	s	R_f	C/W	RP	R_f^*	C/W^*	RP^*
Baseline case	**0.017**	**0.39**	**0.25**	**1.0**	**4.8**	**5.7**	**-0.9**	**2.8**	**5.7**
High ω	0.022			-2.4	3.1	7.4	-2.5	3.0	7.4
Low ω	0.012			4.5	6.4	4.1	0.7	2.6	4.1
High b		0.44		-1.9	3.6	7.5	-2.6	2.9	7.5
Low b		0.34		3.5	5.8	4.4	0.4	2.7	4.4
High s			0.30	-2.2	3.8	8.1	-3.1	2.9	8.1
Low s			0.20	3.2	5.5	4.2	0.5	2.7	4.2

Table I: The impact of different assumptions about the distribution of disasters. $\widetilde{\mu} = 0.025$, $\sigma = 0.02$. Unasterisked group assumes power utility, $\rho = 0.03$, $\gamma = 4$. Asterisked group assumes Epstein-Zin preferences, $\rho = 0.03$, $\gamma = 4$, $\psi = 1.5$.

the fundamental quantities. I consider the power utility case with $\rho = 0.03$ and $\gamma = 4$, and the Epstein-Zin case with the same time-preference rate and risk aversion but higher elasticity of intertemporal substitution, $\psi = 1.5$. The model's predictions are sensitively dependent on the parameter values. Small changes in any of ω, b or s have large effects on the equity premium (and, with power utility, on the riskless rate; this effect is muted in the Epstein-Zin case). Given that these parameters are hard to estimate—disasters happen very rarely—this is problematic.

n		R_f	C/W	RP	R_f^*	C/W^*	RP^*
1	deterministic	10.35	8.51	0.00	4.22	2.39	0.00
2	lognormal	7.11	6.69	1.62	3.01	2.59	1.62
3		4.71	5.68	2.97	1.73	2.70	2.97
4		3.04	5.15	4.12	0.65	2.76	4.12
5		2.04	4.91	4.88	−0.08	2.79	4.88
∞	true model	1.04	4.76	5.73	−0.92	2.80	5.73

Table II: The impact of approximating the disaster model by truncating at the nth cumulant. Unasterisked group assumes power utility; asterisked group assumes Epstein-Zin preferences. All parameters as in baseline case of Table I.

Table II investigates the consequences of truncating the CGF at the nth cumulant. When $n = 2$, this is equivalent to making a lognormality assumption, as noted above. With $n = 3$, it can be thought of as an approximation which accounts for the influence of skewness; $n = 4$ also allows for kurtosis. As is clear from the table, even calculations based on fourth- or fifth-order approximations do not fully capture the impact of disasters.

II.B A GMM exercise

What would estimates of ρ and γ look like if the baseline disaster model were a literal description of reality? This section carries out a GMM exercise using the baseline calibration. I simulate 100 years of annual consumption data, back out asset prices and returns using the above results, and estimate the parameters ρ and γ from the sample analogues of the moment conditions

$$\mathbb{E}\left[e^{-\rho}\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}(R_{t+1}-R_{f,t+1})\right]=0 \quad \text{and} \quad \mathbb{E}\left[e^{-\rho}\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}(1+R_{f,t+1})\right]=1. \tag{10}$$

These equations apply in the power utility case; the conclusions of this section also apply in the Epstein-Zin case if ρ is replaced by $\widetilde{\rho} \equiv \rho + (1-1/\vartheta)\cdot \boldsymbol{c}(1-\gamma)$. The moment conditions (10) identify ρ and γ in the model; see the Appendix for a proof.

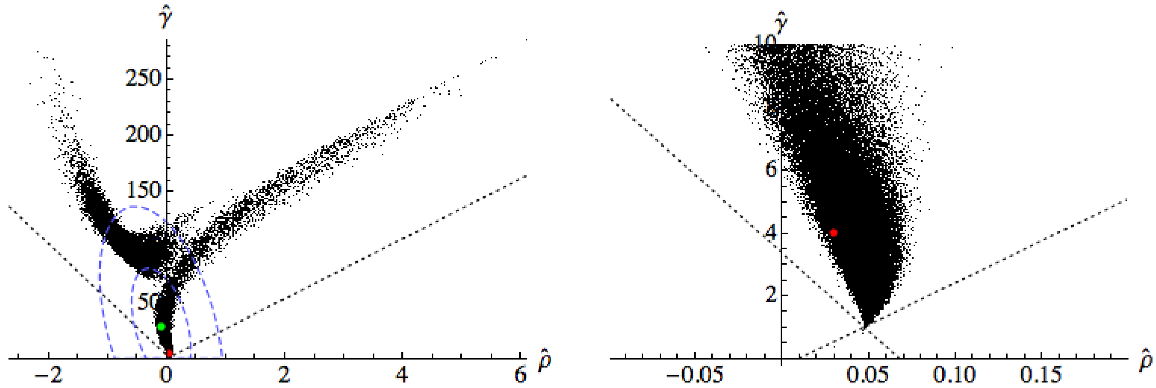
Explicitly, the estimates $\widehat{\rho}$ and $\widehat{\gamma}$ are computed by solving

$$\sum_{t=1}^{T}\left(\frac{C_{t+1}}{C_t}\right)^{-\widehat{\gamma}}(R_{t+1}-R_{f,t+1})=0$$

for $\widehat{\gamma}$, and using the result to determine $\widehat{\rho}$ from

$$\frac{1}{T}\sum_{t=1}^{T}e^{-\widehat{\rho}}\left(\frac{C_{t+1}}{C_t}\right)^{-\widehat{\gamma}}(1+R_{f,t+1})=1.$$

I show in the supplementary appendix that the standard regularity conditions hold in this calibration, so that GMM estimates are consistent and asymptotically Normal. It turns out, though, that 100 years is not enough data for these asymptotic results to apply even approximately. Figure 3 shows what happens when the GMM procedure is repeated 100,000 times. Each black dot represents an estimate $(\widehat{\rho},\widehat{\gamma})$ generated from 100 years of annual data simulated in the baseline disaster calibration; for clarity, the left panel excludes 7 estimates for which $\widehat{\gamma}$ lies above 300 (with a maximum of 416). In each of the 100,000 histories, I also compute the mean realized equity premium and mean consumption growth. Across the 100,000 histories, the sample means of these variables line up perfectly with their population counterparts, at 5.7% and 2.0% respectively, and both have a standard deviation of 0.5%.

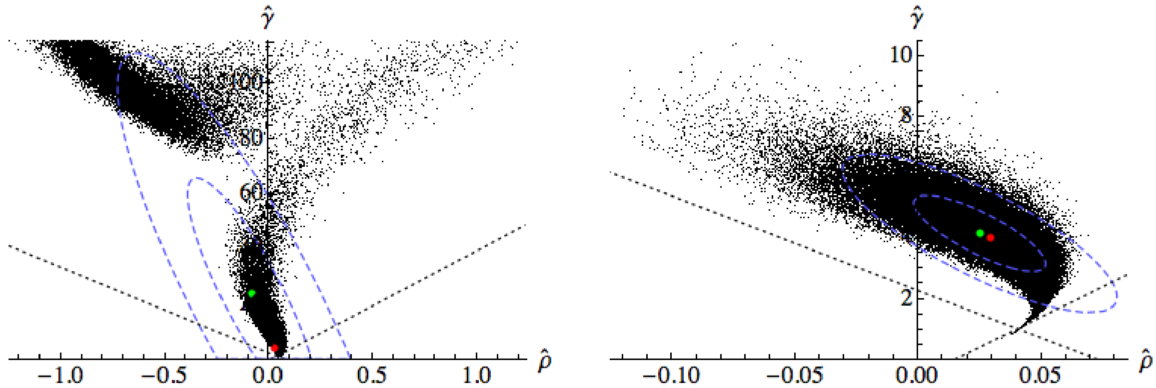


(a) GMM estimates of (ρ, γ) (b) Zooming in

Figure 3: Each of the 100,000 small black dots represents a GMM estimate of (ρ, γ) from 100 years of simulated annual data in the disaster model. The ellipses indicate confidence regions in which 50% (inner ellipse) and 95% (outer ellipse) of the mass of the distribution of $(\widehat{\rho}, \widehat{\gamma})$ would lie if $(\widehat{\rho}, \widehat{\gamma})$ were Normally distributed. Dotted lines in each panel indicate the model-free parameter restrictions derived in Section III: just 10 of the estimates $(\widehat{\rho}, \widehat{\gamma})$ lie in the lower admissible region, visible in the right-hand panel, in which $\gamma < 1$.

The mean estimate $(\overline{\rho}, \overline{\gamma}) = (-0.086, 29.5)$, generated by averaging the resulting 100,000 estimates $(\widehat{\rho}, \widehat{\gamma})$, is marked with a green dot. (For comparison, Kocherlakota (1996) carries out the same exactly identified GMM exercise in real-world data. In my notation, he estimates $\widehat{\rho} = -0.077$ and $\widehat{\gamma} = 17.95$.) The true value $(\rho, \gamma) = (0.03, 4)$ is marked with a red dot. The standard deviation of the estimates $\widehat{\rho}$ is 0.426; the standard deviation of the estimates $\widehat{\gamma}$ is 43.5; and the correlation between $\widehat{\rho}$ and $\widehat{\gamma}$ is -0.42. The dashed ellipses show confidence regions within which 50% (small ellipse) and 95% (large ellipse) of the sample points would lie if the data were Normally distributed. Larger estimates of risk aversion tend to be associated with smaller estimates of the time preference rate: the procedure is struggling to match the data by increasing risk aversion, at the cost of having to assuming extreme patience in order to match the riskless rate. Finally, dotted lines indicate model-free parameter restrictions that will be derived in the next section.

The figures demonstrate three features of GMM estimation in the disaster calibration. First, there is extraordinary dispersion in the estimates. Estimates of γ extend up to more than 400. Estimates of ρ, the time discount rate, range between about -2 and 6: in "time-discount-factor" terms, writing $\beta = e^{-\widehat{\rho}}$, these correspond to $\beta = 7.39$ and $\beta = 0.0025$, respectively. Second, $\widehat{\rho}$ is a downward-biased estimator of ρ and $\widehat{\gamma}$ is an upwardbiased estimator of γ. Third, the estimates $(\widehat{\rho}, \widehat{\gamma})$ are far from Normally distributed when using 100 years of data: the confidence ellipses utterly fail to capture the shape of the distribution. (Contrast Figure 3a with Figure 4b.) A formal test of Normality is superfluous, since amongst the 100,000 sample points there are several estimates $\widehat{\rho}$ that lie more than 10 standard deviations above the mean $\overline{\rho}$ (i.e. are larger than about 4.2). The probability of a 10-sigma event under the Normal distribution is about 10^{-23}, so even one such occurrence in 100,000 samples would be sufficient to reject the hypothesis of Normality at the 0.00001% level.



(a) GMM estimates of (ρ, γ), after trimming

(b) GMM estimates of (ρ, γ) in a lognormal world

Figure 4: Left: The message of the previous figure is unaltered if the most extreme 5% of realizations $(\widehat{\rho}, \widehat{\gamma})$ is discarded. Right: GMM estimates of (ρ, γ) from 100 years of simulated annual data in a lognormal model.

These conclusions are not driven by extreme outliers: Figure 4a shows the result of trimming the data by discarding the largest 5% of pairs ordered by $\widehat{\gamma}$. The GMM estimates are still biased—$(\overline{\rho}, \overline{\gamma}) = (-0.076, 23.6)$—and the confidence ellipses are still large.

As a sanity check, and as a contrast with the disaster calibration, Figure 4b shows what happens in a lognormal world. It reports the results of carrying out the same exercise in a calibration without disasters ($\omega = 0$), with σ_B adjusted upwards so that the risk premium remains constant. As one would expect, GMM provides an accurate estimate of the underlying population parameters, and the sample distribution is approximately Normal.

With more extreme specifications of disaster sizes, even worse behavior is possible (Kocherlakota (1997)). If the random variables inside the expectations in (10) do not have finite second moment, the GMM estimators are not even asymptotically Normal. In the case of the moment condition for the riskless bond, we require $\boldsymbol{c}(-2\gamma)$ to be finite;

in the case of the consumption claim, we require $\boldsymbol{c}(2-2\gamma)$ to be finite. If the former is finite, the latter is too, so in summary the GMM approach is only valid if $\boldsymbol{c}(-2\gamma)$ is finite. This is *not* assured by the assumptions that ensure finite utility, or a finite consol price. Below, I construct an example for which the GMM approach fails not only in finite samples, but even asymptotically: Figure 6b plots the CGF of (an extreme case of) such a distribution.

III Restrictions on preference parameters

I now assume that the riskless rate, consumption-wealth ratio, and risk premium (and hence expected consumption growth, via the Gordon growth model (7)) are *observable* and take the values given in Table III.[4] The observables provide us with information about the shape of the CGF; this section shows how to use this information to derive restrictions on the preference parameters that must hold in *any* Epstein-Zin/i.i.d. model, no matter what is going on in the tails. The restrictions are of particular interest when there are small sample biases, as in the previous section, and they continue to be valid when GMM breaks down entirely, as discussed at the end of the previous section.

riskless rate	r_f	0.02
risk premium	rp	0.06
consumption-wealth ratio	c/w	0.06

Table III: Assumed values of the observables.

For example, $r_f = \rho - \boldsymbol{c}(-\gamma)$ in the power utility case, so observation of the riskless rate tells us something about ρ and something about the value taken by the CGF at $-\gamma$. Similarly, observation of the consumption-wealth ratio tells us something about ρ and something about the value taken by the CGF at $1-\gamma$. Next, $\boldsymbol{c}(1) = \log \mathbb{E}(C_1/C_0)$ is pinned down by the Gordon growth formula (7), and $\boldsymbol{c}(0) = 0$ by definition. How, though, can we get control on the enormous range of possible consumption processes and CGFs? One approach is to exploit

Fact 1. *CGFs are convex.*

[4] As noted in the above GMM exercise, the equity premium can be accurately estimated, even in a world with rare disasters: in the baseline calibration, the equity premium conditional on no disasters is very close to the unconditional equity premium.

Proof. Since $\boldsymbol{c}(\theta) = \log \boldsymbol{m}(\theta)$, we have

$$\begin{aligned} \boldsymbol{c}''(\theta) &= \frac{\boldsymbol{m}(\theta) \cdot \boldsymbol{m}''(\theta) - \boldsymbol{m}'(\theta)^2}{\boldsymbol{m}(\theta)^2} \\ &= \frac{\mathbb{E}\, e^{\theta G}\, \mathbb{E}\, G^2 e^{\theta G} - \left(\mathbb{E}\, G e^{\theta G}\right)^2}{\boldsymbol{m}(\theta)^2} \,. \end{aligned}$$

The numerator of this expression is positive by a version of the Cauchy-Schwartz inequality which states that $\mathbb{E}\, X^2 \cdot \mathbb{E}\, Y^2 \geq \mathbb{E}(|XY|)^2$ for any random variables X and Y. In this case, we need to set $X = e^{\theta G/2}$ and $Y = G e^{\theta G/2}$. (See Billingsley (1995), for further discussion of this and other properties of CGFs.) □

This fact can be used to derive sharp preference parameter bounds based on observables.

Result 3. *In the power utility case, we have*

$$r_f - c/w \leq \frac{c/w - \rho}{\gamma - 1} \leq rp + r_f - c/w. \tag{11}$$

In the Epstein-Zin case, we have

$$r_f - c/w \leq \frac{c/w - \rho}{1/\psi - 1} \leq rp + r_f - c/w. \tag{12}$$

Moreover, these bounds are sharp: for any $\varepsilon > 0$ there are distributions of consumption growth, and parameter choices for ρ, γ and ψ, that are consistent with the observables and satisfy $\frac{c/w-\rho}{1/\psi-1} < r_f - c/w + \varepsilon$; and other distributions of consumption growth and parameter choices that are consistent with the observables and satisfy $\frac{c/w-\rho}{1/\psi-1} > rp + r_f - c/w - \varepsilon$.

Proof. The result is best understood geometrically. Look at Figure 5, which plots a generic CGF: it is convex and passes through the origin. We know from Result 2 that asset prices provide information about some key points on the CGF—$\boldsymbol{c}(-\gamma)$, $\boldsymbol{c}(1-\gamma)$, and $\boldsymbol{c}(1)$—and therefore about the gradients of the dashed lines marked X, Y, and Z.

Now, because $\boldsymbol{c}(\theta)$ is convex, the gradients of X, Y, and Z, in that order, are increasing:

$$\frac{\boldsymbol{c}(1-\gamma) - \boldsymbol{c}(-\gamma)}{(1-\gamma) - (-\gamma)} \leq \frac{\boldsymbol{c}(0) - \boldsymbol{c}(1-\gamma)}{0 - (1-\gamma)} \leq \frac{\boldsymbol{c}(1) - \boldsymbol{c}(0)}{1 - 0}.$$

These inequalities can be rearranged to give

$$\frac{\boldsymbol{c}(-\gamma)}{-\gamma} \leq \frac{\boldsymbol{c}(1-\gamma)}{1-\gamma} \leq \boldsymbol{c}(1). \tag{13}$$

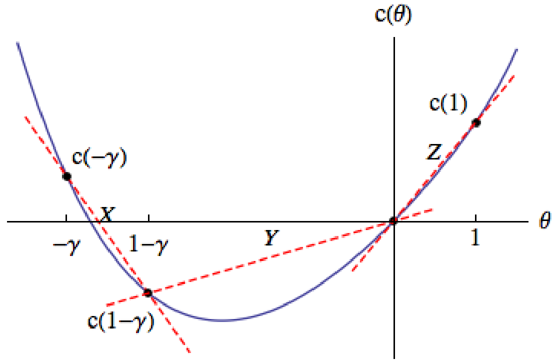


Figure 5: Convexity of the CGF implies that line X has the smallest slope and line Z the largest.

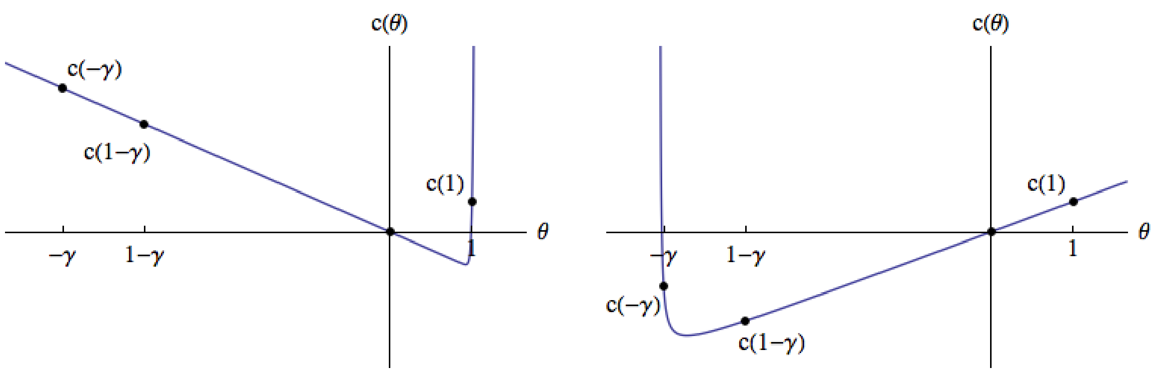
But from equation (5) we have, in the Epstein-Zin case,

$$\frac{c/w - \rho}{1/\psi - 1} = \frac{\boldsymbol{c}(1-\gamma)}{1-\gamma}. \tag{14}$$

Putting (13) and (14) together, we have

$$\frac{\boldsymbol{c}(-\gamma)}{-\gamma} \leq \frac{c/w - \rho}{1/\psi - 1} \leq \boldsymbol{c}(1).$$

The result (12) follows on rearranging the left-hand inequality using (4) and (5), and substituting out $\boldsymbol{c}(1)$ using the Gordon growth model; and (11) is a special case of (12).



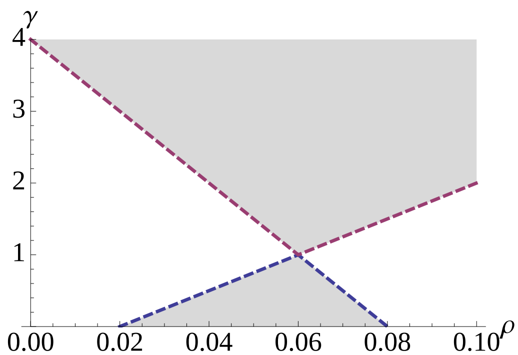
(a) $\frac{c/w-\rho}{1/\psi-1} \approx r_f - c/w$

(b) $\frac{c/w-\rho}{1/\psi-1} \approx rp + r_f - c/w$

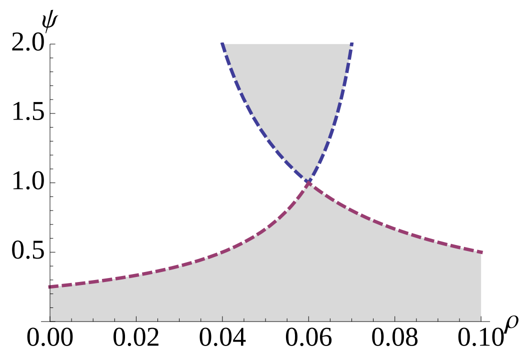
Figure 6: CGFs of two possible distributions of consumption growth that are consistent with the observables and illustrate why the bounds in Result 3 cannot be improved.

The detailed proof that the bounds are sharp is in the Appendix, but the main idea can be understood by comparing the generic CGF depicted in Figure 5 to the two CGFs shown in Figure 6, which indicate how the two extremes can be approximately attained. The Appendix demonstrates that there are distributions of consumption growth whose CGFs look like those in Figure 6—i.e., have the properties that (i) they are almost linear on the intervals $[-\gamma, 0]$ in the case represented by Figure 6a, or on $[1-\gamma, 1]$ in the case represented by Figure 6b; and (ii) they are consistent with the observables, which pins down $\boldsymbol{c}(1)$ and the gap $\boldsymbol{c}(1-\gamma) - \boldsymbol{c}(-\gamma)$. $\square$

The intuition is that as ψ approaches one, the consumption-wealth ratio approaches ρ. Therefore, if the consumption-wealth ratio is to be far from ρ, ψ must be far from one. Result 3 turns this qualitative statement into a quantitative one, without making assumptions about what is going on in the tails. Using the values $rp = 6\%, r_f = 2\%, c/w = 6\%$, we have the restriction that $-0.04 \leq (0.06 - \rho)/(1/\psi - 1) \leq 0.02$. The shaded areas in Figure 7 illustrate where the parameters must lie. If $\psi > 1$, then ρ is constrained to lie between 0.02 and 0.08; if also $\psi < 2$, then ρ must lie between 0.04 and 0.07. If $\gamma = 1$ in the power utility case, or if $\psi = 1$ in the Epstein-Zin case, then ρ is exactly identified by the consumption-wealth ratio.



(a) Power utility case: γ and ρ



(b) Epstein-Zin case: ψ and ρ

Figure 7: Parameter restrictions for i.i.d. models with $rp = 6\%$, $r_f = 2\%$ and $c/w = 6\%$.

To the extent that $D_t = C_t^\lambda$ is a reasonable approximation of leverage, we can say even more. For, we observe the consumption-wealth ratio c/w, wealth risk premium rp_w *and* the dividend yield on the market d/p and market risk premium rp_m (that is, observe (4)–(6), together with the expressions that result on substituting in $\lambda = 1$). The following relationships hold:

$$\begin{aligned} c/w - d/p &= \boldsymbol{c}(\lambda - \gamma) - \boldsymbol{c}(1-\gamma) \\ \vartheta(\rho - c/w) + c/w - d/p &= \boldsymbol{c}(\lambda - \gamma) \\ rp_m + r_f - d/p &= \boldsymbol{c}(\lambda) \end{aligned}$$

By convexity, we have

$$\frac{\boldsymbol{c}(\lambda-\gamma)-\boldsymbol{c}(1-\gamma)}{\lambda-1} \leq \frac{\boldsymbol{c}(\lambda-\gamma)}{\lambda-\gamma} \leq \frac{\boldsymbol{c}(\lambda)}{\lambda}\,.$$

Substituting in, we have joint bounds on ρ, γ and ψ:

$$\frac{c/w-d/p}{\lambda-1} \leq \frac{\vartheta(\rho-c/w)+c/w-d/p}{\lambda-\gamma} \leq \frac{rp_m+r_f-d/p}{\lambda}\,.$$

III.A Hansen-Jagannathan and good-deal bounds

Hansen and Jagannathan (1991) derived a bound that relates the standard deviation and mean of the stochastic discount factor, M, to the Sharpe ratio on an arbitrary asset, SR:

$$SR \leq \frac{\sigma(M)}{\mathbb{E}\,M}. \tag{15}$$

In the Epstein-Zin-i.i.d. setting, the right-hand side of (15) becomes

$$\begin{aligned}\frac{\sigma(M)}{\mathbb{E}\,M} &= \sqrt{\frac{\mathbb{E}\,M^2}{(\mathbb{E}\,M)^2}-1} \\ &= \sqrt{e^{\boldsymbol{c}(-2\gamma)-2\boldsymbol{c}(-\gamma)}-1}.\end{aligned} \tag{16}$$

Combining (15) and (16), we obtain the Hansen-Jagannathan bound in CGF notation:

$$\log\left(1+SR^2\right) \leq \boldsymbol{c}(-2\gamma)-2\boldsymbol{c}(-\gamma)\,. \tag{17}$$

Cochrane and Saá-Requejo (2000) observe that inequality (15) suggests a natural way to restrict asset-pricing models. Suppose $\sigma(M)/\mathbb{E}\,M \leq h$; then (15) implies that the maximal Sharpe ratio is less than h. In CGF notation, the good-deal bound is written

$$\boldsymbol{c}(-2\gamma)-2\boldsymbol{c}(-\gamma) \leq \log\left(1+h^2\right). \tag{18}$$

Suppose, for example, that we wish to impose the restriction that Sharpe ratios above 1 are too good a deal to be available. Then the good-deal bound is $\boldsymbol{c}(-2\gamma)-2\boldsymbol{c}(-\gamma) \leq \log 2$. This expression can be evaluated under particular parametric assumptions about the consumption process. In the case in which consumption growth is lognormal, with volatility of log consumption equal to σ, it supplies an upper bound on risk aversion: $\gamma \leq \sqrt{\log 2}/\sigma$ (which is about 42 if $\sigma = 0.02$). However, this upper bound is rather weak, and in any case the postulated consumption process is inconsistent with observed features of asset markets such as the high equity premium and low riskless rate. Alternatively, one might model the consumption process as subject to disasters in the sense of Section II.A. In this case, the good-deal bound implies tighter restrictions on γ, but these restrictions are sensitively dependent on the disaster parameters.

In order to progress from (18) to a bound on γ and ρ which does not require parametrization of the consumption process, we want to relate $\boldsymbol{c}(-2\gamma) - 2\boldsymbol{c}(-\gamma)$ to quantities which can be directly observed. For example, the Hansen-Jagannathan bound (17) improves on a conclusion which follows from the convexity of the CGF, namely, that

$$0 \leq \boldsymbol{c}(-2\gamma) - 2\boldsymbol{c}(-\gamma) . \tag{19}$$

This trivial inequality follows by considering the value of the CGF at the three points $\boldsymbol{c}(0)$, $\boldsymbol{c}(-\gamma)$, and $\boldsymbol{c}(-2\gamma)$. Convexity implies that the average slope of the CGF is more negative (or less positive) between -2γ and $-\gamma$ than it is between $-\gamma$ and 0:

$$\frac{\boldsymbol{c}(-\gamma) - \boldsymbol{c}(-2\gamma)}{\gamma} \leq \frac{\boldsymbol{c}(0) - \boldsymbol{c}(-\gamma)}{\gamma}.$$

Equation (19) follows immediately, given that $\boldsymbol{c}(0) = 0$. Combining (18) and (19), we obtain the (underwhelming!) result that $0 \leq \log(1 + h^2)$.

However, we can sharpen (19) by comparing the slope of the CGF between -2γ and $-\gamma$ to the slope between $-\gamma$ and $1 - \gamma$ (rather than between $-\gamma$ and 0):

$$\frac{\boldsymbol{c}(-\gamma) - \boldsymbol{c}(-2\gamma)}{\gamma} \leq \frac{\boldsymbol{c}(1-\gamma) - \boldsymbol{c}(-\gamma)}{1}.$$

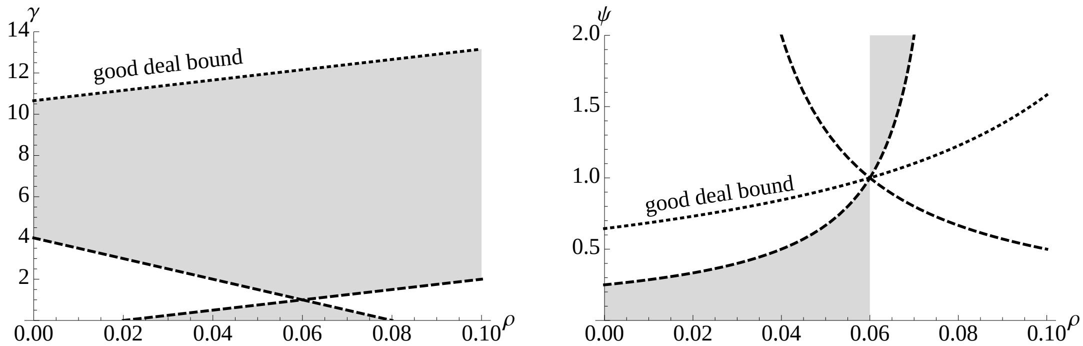
This implies, by Result 1, that $\boldsymbol{c}(-2\gamma) - 2\boldsymbol{c}(-\gamma) \geq (\gamma - 1)(c/w - r_f) + \vartheta(c/w - \rho)$, and hence

Result 4. *If the maximal Sharpe ratio is less than or equal to h, then we must have*

$$(\gamma - 1)(c/w - r_f) + \vartheta(c/w - \rho) \leq \log\left(1 + h^2\right) . \tag{20}$$

The important feature of this result is that by exploiting the observable consumption-wealth ratio and riskless rate, we do not need to take a stand on what is going on in the tails.

Figure 8 reproduces the bounds of Figure 7, adding in the good deal bound (20) with $h = 0.75$, i.e. ruling out Sharpe ratios above 0.75. Shaded areas indicate admissible parameter values. In the right-hand panel, admissible values lie below the line marked 'good deal bound' when $\psi < 1$ and above it when $\psi > 1$. There are also admissible values of ρ and ψ not visible in the figure, with ρ large and ψ close to zero. The figure assumes



(a) Power utility case: γ and ρ

(b) Epstein-Zin case: ψ and ρ

Figure 8: Shaded areas indicate admissible parameter values for i.i.d. models with $rp = 6\%$, $r_f = 2\%$, $c/w = 6\%$, and a maximal Sharpe ratio of 0.75.

$\gamma = 4$, but the admissible regions are unaltered for any value of γ between 0 and 8.5. The line marked 'good deal bound' steepens as γ increases, while continuing to pass through the point $(0.06, 1)$; once γ rises above 8.5, the good deal bound starts to impose tighter constraints on ψ and ρ.

IV Extensions

IV.A Log consumption not proportional to log dividends

Thus far we have operated under the assumption that log dividends are proportional to log consumption. This section considers more general scenarios in which consumption and dividends may differ, with a motivating application to a heterogeneous-agent economy. Again, the goal will be to make statements that are independent of particular assumptions about tail behavior.

To introduce some notation, suppose that a power utility agent with consumption C_t is pricing an asset paying dividends D_t, so that the asset's price satisfies

$$P_0 = D_0 \cdot \mathbb{E} \int_{t=0}^{\infty} e^{-\rho t} \left(\frac{C_t}{C_0} \right)^{-\gamma} \cdot \frac{D_t}{D_0} \, dt \, .$$

Defining the bivariate CGF $\boldsymbol{c}(\theta_1, \theta_2) \equiv \log \mathbb{E}\, e^{\theta_1 \log(C_{t+1}/C_t) + \theta_2 \log(D_{t+1}/D_t)}$, it is almost immediate, following the same logic as before, that $D_0/P_0 = \rho - \boldsymbol{c}(-\gamma, 1)$. The next result, whose proof is straightforward, collects this fact together with the riskless rate and risk premium.

Result 5. *If consumption growth* $G_t \equiv \log C_t/C_0$ *and dividend growth* $H_t \equiv \log D_t/D_0$ *are distinct, potentially correlated, Lévy processes, then*

$$\begin{aligned} D_0/P_0 &= \rho - \boldsymbol{c}(-\gamma, 1) \\ R_f &= \rho - \boldsymbol{c}(-\gamma, 0) \\ RP &= \boldsymbol{c}(0,1) + \boldsymbol{c}(-\gamma, 0) - \boldsymbol{c}(-\gamma, 1). \end{aligned}$$

Constantinides and Duffie (1996) have shown that accounting for heterogeneity may contribute to an understanding of the equity premium puzzle. On the other hand, Grossman and Shiller (1982) have shown that if agents' consumption processes follow diffusions, risk premia are unaffected by heterogeneity. The tension between these two results will be resolved by showing that heterogeneity matters to the extent that it is present *at times of aggregate jumps*. The presence of jumps lends a discrete-time flavor to the model, and as a result it lies closer on the spectrum to Constantinides-Duffie than to Grossman-Shiller.

I assume that agents suffer idiosyncratic shocks to consumption, perhaps because agents have labor income risk which is uninsurable for moral hazard reasons.[5] All agents have power utility. Agent i's log consumption process is given by

$$\log \frac{C_{i,t}}{C_{i,0}} = \underbrace{\mu t + \sigma B_t + \sum_{j=1}^{N_t} Y_j}_{\text{common to all agents}} + \underbrace{\sigma_1 B_{it} - \frac{1}{2}\sigma_1^2 t}_{\text{type (i)}} + \underbrace{\sum_{j=1}^{N_{i,t}} X_{i,j}}_{\text{type (ii)}} + \underbrace{\sum_{j=1}^{N_t} Y_{i,j}}_{\text{type (iii)}} \qquad (21)$$

Here B_t is a Brownian motion, and Y_j are the i.i.d. sizes of jumps, which occur at times dictated by the Poisson process N_t, with arrival rate ω; these shocks are common to all agents. I assume that the jumps are bad—or at least, not good—news on average, so $\mathbb{E}\, e^{Y_j} \leq 1$. There are also three types of idiosyncratic shocks: (i) an idiosyncratic Brownian motion component, $B_{i,t}$; (ii) jumps whose size $X_{i,k}$ and timing (determined by the Poisson process $N_{i,t}$) are idiosyncratic; and (iii) jumps of idiosyncratic size $Y_{i,k}$, whose timing coincides with aggregate disasters, to capture the fact that disasters do not have the same impact on all agents. I assume that $X_{i,j}$ and $Y_{k,l}$ are i.i.d. across i, j, k and l, and $N_{i,t}$ are Poisson processes, independent across i, with arrival rate ω_2. Finally, σ_1 and ω_2 are constant across all agents i.

Aggregate quantities are computed by summing over agents i; I assume that a law of large numbers holds so that this process is equivalent to taking an expectation over i. With

[5] Storesletten, Telmer and Yaron (2004) show that idiosyncratic shocks are highly persistent, and large, with a standard deviation of about 0.25.

this assumption, together with the normalization that for all i and k, $\mathbb{E}\, e^{X_{i,k}} = \mathbb{E}\, e^{Y_{i,k}} = 1$, (21) implies that aggregate consumption evolves according to $\log \frac{C_t}{C_0} = \mu t + \sigma B_t + \sum_{j=1}^{N_t} Y_j$.

The upshot is that all agents attach the same value to the "equity" claim to aggregate consumption, so as in Constantinides and Duffie (1996) there is a no-trade equilibrium in which agent i consumes $C_{i,t}$ at time t. The Euler equation holds for each agent i, so the price of equity, P, must satisfy

$$P = \mathbb{E} \int_0^\infty e^{-\rho t} \left(\frac{C_{i,t}}{C_{i,0}} \right)^{-\gamma} \cdot C_t \, dt\,. \tag{22}$$

Result 5 now applies with $G_t = \log C_{it}/C_{i0}$ and $H_t = \log C_t/C_0$. Defining $\boldsymbol{m}_D(\theta) \equiv \mathbb{E}\, e^{\theta Y_j}$, $\boldsymbol{m}_2(\theta) \equiv \mathbb{E}\, e^{\theta X_{i,k}}$, and $\boldsymbol{m}_3(\theta) \equiv \mathbb{E}\, e^{\theta Y_{i,j}}$, the CGF of (G_1, H_1) is $\boldsymbol{c}(\theta_1, \theta_2) = \mu\,(\theta_1 + \theta_2) + \frac{1}{2}\sigma^2\,(\theta_1 + \theta_2)^2 + \frac{1}{2}\sigma_1^2\theta_1(\theta_1 - 1) + \omega_2\,[\boldsymbol{m}_2(\theta_1) - 1] + \omega\,[\boldsymbol{m}_D\,(\theta_1 + \theta_2)\,\boldsymbol{m}_3\,(\theta_1) - 1]$. The dividend yield, riskless rate, and risk premium follow from Result 5.

A naive econometrician who uses aggregate consumption in calculations of these fundamentals is wrongly dropping the is in (22), i.e. replacing G_t in the CGF by H_t, or equivalently using the function $\boldsymbol{c}(0, \theta_1 + \theta_2)$ in place of $\boldsymbol{c}(\theta_1, \theta_2)$. The discrepancies between the true values and incorrect predictions based on aggregate quantities (denoted by bars) are

$$\begin{aligned} D/P - \overline{D/P} &= -\sigma_1^2\gamma(\gamma+1)/2 - \omega_2\,[\boldsymbol{m}_2(-\gamma) - 1] - \omega\,\boldsymbol{m}_D(1-\gamma)\,[\boldsymbol{m}_3(-\gamma) - 1] && (23)\\ R_f - \overline{R_f} &= -\sigma_1^2\gamma(\gamma+1)/2 - \omega_2\,[\boldsymbol{m}_2(-\gamma) - 1] - \omega\,\boldsymbol{m}_D(-\gamma)\,[\boldsymbol{m}_3(-\gamma) - 1] && (24)\\ RP - \overline{RP} &= \omega\,[\boldsymbol{m}_D(-\gamma) - \boldsymbol{m}_D(1-\gamma)]\,[\boldsymbol{m}_3(-\gamma) - 1]\,. && (25) \end{aligned}$$

All three types of heterogeneity influence the dividend yield and riskless rate. But if there are no aggregate disasters—$\omega = 0$—then heterogeneity has no effect on the risk premium, as in Grossman and Shiller (1982). Even if there are disasters, heterogeneity of type (i) or type (ii) has no effect on risk premia, as in Krueger and Lustig (2010). Using these expressions, we can sign the effects of heterogeneity no matter what the distribution of jumps.

Result 6 (Robust implications of heterogeneity)**.** *Heterogeneity drives down the dividend yield and riskless rate, and increases the risk premium:* $D/P \leq \overline{D/P}$*;* $R_f \leq \overline{R_f}$*; and* $RP \geq \overline{RP}$*.*

Campbell (2003) and Cochrane (2008) have argued, based on calculations in lognormal or approximately lognormal models, that heterogeneity is unlikely to be *quantitatively* relevant for the equity premium puzzle. The good news is that (25) shows that this conclusion is altered—heterogeneity may be quantitatively important—in the presence of jumps. Suppose that aggregate and idiosyncratic type (iii) jumps in log consumption are Normally distributed, $Y_j \sim N(-b, s^2)$ and $Y_{i,k} \sim N(-s_i^2/2, s_i^2)$. Then $\boldsymbol{m}_D(\theta) = e^{-b\theta + s^2\theta^2/2}$ and $\boldsymbol{m}_3(\theta) = e^{s_i^2\theta(\theta-1)/2}$. From (25), this increases the equity premium by

$$\omega\left(e^{b\gamma+s^2\gamma^2/2} - e^{b(\gamma-1)+s^2(\gamma-1)^2/2}\right)\left(e^{s_i^2\gamma(\gamma+1)/2} - 1\right).$$

This increases extremely rapidly in γ: relative to the baseline calibration, $s_i = 0.25$ causes the equity premium to increase by an *extra* 528bp if $\gamma = 4$, and by 1982bp if $\gamma = 5$.

But here too, disasters are a double-edged sword. The *bad* news is that heterogeneity only matters for risk premia to the extent that it is present at times of aggregate disaster. This is frustrating from the point of view of empirical work, since it means that the considerable amount of information about heterogeneity in "normal" times that is available in individual-level datasets is irrelevant for risk premia. Thus the difficulties in identifying parameters discussed in previous sections extend to the heterogeneous agent case. This may explain why several authors have failed to find empirical evidence that heterogeneity matters. For example, Heaton and Lucas (1996) and Lettau (2002) do not consider the relevance of disasters. Cogley (2002) does, but truncates at the third moment; moreover his data only run from 1980, so do not contain any examples of disasters in Barro's (2006) sense.

IV.B Non-i.i.d. consumption growth

An extensive literature has documented that time variation in valuations contributes a sizeable proportion of equity volatility. To capture this feature while still allowing for arbitrarily distributed consumption growth, I now drop the assumption that consumption growth is i.i.d. Valuations are then time-varying, so that we have nontrivial valuation shocks Q_{t+1}:

$$1 + \frac{W_{t+1}}{C_{t+1}} = \left(1 + \frac{W_t}{C_t}\right) Q_{t+1}. \tag{26}$$

In the i.i.d. case, valuation ratios were constant and $Q_{t+1} \equiv 1$. I now assume only that $\mathbb{E}_t\, Q_{t+1} = 1$. The joint behavior of consumption growth and valuation ratios can be summarized by the conditional CGF $\boldsymbol{c}_t(\theta_1, \theta_2) = \log \mathbb{E}_t \exp\{\theta_1 \log(C_{t+1}/C_t) + \theta_2 \log Q_{t+1}\}$.

An Epstein-Zin investor's Euler equation implies that

$$\frac{W_t}{C_t} = \mathbb{E}_t\left[e^{-\rho\vartheta}\left(\frac{C_{t+1}}{C_t}\right)^{-\vartheta/\psi}(1 + R_{m,t+1})^{\vartheta-1}\left(\frac{C_{t+1} + W_{t+1}}{C_t}\right)\right]. \tag{27}$$

Using (26), we have

$$1 + R_{m,t+1} = \left(1 + \frac{C_t}{W_t}\right) \frac{C_{t+1}}{C_t} Q_{t+1}\,, \tag{28}$$

so the expected return moves around with the valuation ratio C_t/W_t. We also have

$$\frac{C_{t+1} + W_{t+1}}{C_t} = \frac{W_t}{C_t} \left(1 + \frac{C_t}{W_t}\right) \frac{C_{t+1}}{C_t} Q_{t+1}\,. \tag{29}$$

Substituting (28) and (29) into (27) and rearranging, we find that

$$1 + \frac{C_t}{W_t} = \exp\left\{\rho - \boldsymbol{c}_t(1-\gamma, \vartheta)/\vartheta\right\}\,.$$

Using this together with (28), $\mathbb{E}_t(1 + R_{m,t+1}) = \exp\{\rho + \boldsymbol{c}_t(1,1) - \boldsymbol{c}_t(1-\gamma,\vartheta)/\vartheta\}$. Similarly, the price of a one-period bond, B_t, is

$$B_t = \mathbb{E}_t\left[e^{-\rho\vartheta}\left(\frac{C_{t+1}}{C_t}\right)^{-\vartheta/\psi}(1 + R_{m,t+1})^{\vartheta-1}\right] = \exp\left\{-\rho + (1/\vartheta - 1)\boldsymbol{c}_t(1-\gamma,\vartheta) + \boldsymbol{c}_t(-\gamma, \vartheta-1)\right\}.$$

Now define, in line with previous notation, the consumption-wealth ratio $c/w_t \equiv \log(1 + C_t/W_t)$, riskless rate $r_{f,t} \equiv -\log B_t$, and risk premium $rp_t \equiv \log \mathbb{E}_t(1 + R_{m,t+1}) - r_{f,t}$; we have

$$\begin{aligned} c/w_t &= \rho - \boldsymbol{c}_t(1-\gamma,\vartheta)/\vartheta \\ r_{f,t} &= \rho + (1 - 1/\vartheta)\boldsymbol{c}_t(1-\gamma,\vartheta) - \boldsymbol{c}_t(-\gamma,\vartheta-1) \\ rp_t &= \boldsymbol{c}_t(1,1) + \boldsymbol{c}_t(-\gamma,\vartheta-1) - \boldsymbol{c}_t(1-\gamma,\vartheta). \end{aligned}$$

As before, the fundamentals provide information about the values taken at various points by the CGF, and as before, the CGF is a convex function. Now, however, because the points $(1,1)$, $(1-\gamma,\vartheta)$, and $(-\gamma,\vartheta-1)$ do not lie on a line, there are no direct constraints imposed by convexity in the general case.[6] However, I show in the appendix that the hypothesis that Q_{t+1} and C_{t+1}/C_t are independent of one another cannot be rejected in the data. If we assume that they are indeed independent then the analysis simplifies nicely,[7] and the previous results can be extended—perfectly, in the power utility case—to the case of non-i.i.d. consumption growth.

[6]Except in the limiting case $\psi = \infty$; then, the points *do* lie on a line and without any further assumptions the convexity argument goes through. Equation (12) holds, and hence $r_{f,t} \le \rho \le r_{f,t} + rp_t$.

[7]Even if aggregate consumption growth is independent of Q_{t+1}, the consumption growth of marginal investors may be an important determinant of valuation ratios. In the supplementary appendix, I illustrate this possibility with an equilibrium model and show how the methodology of the paper can be adapted to it.

Result 7. *With power utility, Results 3 and 4 continue to hold, with fundamentals replaced by conditional fundamentals. With Epstein-Zin preferences, the right-hand inequality in* (12) *holds in the region* $\psi \geq 1/\gamma$*, and the left-hand inequality holds in the region* $\psi \leq \min\{1/\gamma, 2/(1+\gamma)\}$*; and Result 4 holds if* $\vartheta < 0$ *or* $\vartheta > 2$*.*

V Conclusion

As pointed out by Rietz (1988), Barro (2006) and Weitzman (2007), the tails of the distribution of consumption growth exert an enormous influence on asset prices. This paper takes an agnostic approach regarding the existence and importance of disasters, and introduces a framework that handles general i.i.d. consumption growth processes. The framework leads to simple and intuitive formulas that shed light on the impact of the higher cumulants of consumption growth on the riskless rate, consumption-wealth ratio and risk premium.

The framework is flexible: in a heterogeneous-agent example, I show that consumption heterogeneity gives a huge extra kick to the disaster logic if it is present at times of disaster, and I show how the parameter bounds derived under the i.i.d. assumption can be extended to the case with non-i.i.d. consumption growth and time-varying valuation ratios, under an independence assumption that cannot be rejected in the data.

Disasters are a double-edged sword. They exert an enormous influence on asset prices, and hence provide a potential explanation for the equity premium puzzle (good); but the predictions of disaster models are sensitively dependent on the assumptions made about the hard-to-estimate parameters governing the size and frequency of disasters (bad). A central theme of this paper is that we can sidestep this discouraging observation by exploiting a simple but powerful fact: CGFs are always convex. This convexity imposes constraints that hold no matter what is going on in the tails—even in economies in which the distribution of disasters is so severe that GMM is invalid even asymptotically.

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A Proof of Result 1

The Epstein-Zin first-order condition leads to the pricing formula

$$P = \mathbb{E} \sum_{1}^{\infty} e^{-\rho\vartheta t} \left(\frac{C_t}{C_0} \right)^{-\vartheta/\psi} (1 + R_{m,0\to t})^{\vartheta - 1} (C_t)^{\lambda} ,$$

where $\vartheta = (1-\gamma)/(1-1/\psi)$ and $R_{m,0\to t}$ is the cumulative return on the wealth portfolio from period 0 to period t. I assume that $\psi \neq 1$ for convenience. Now,

$$1 + R_{m,s-1\to s} = \frac{C_s + W_s}{W_{s-1}} = \frac{C_s}{C_{s-1}} \left(\frac{C_{s-1}}{W_{s-1}} + \frac{W_s}{C_s} \frac{C_{s-1}}{W_{s-1}} \right) = \frac{C_s}{C_{s-1}} e^{\nu} ,$$

where the last equality follows by making the assumption—which will subsequently be verified—that the consumption-wealth ratio is constant. I have defined $1 + C/W \equiv e^{\nu}$. It follows that $1 + R_{m,0\to t} = (C_t/C_0)e^{\nu t}$, and hence that

$$P = (C_0)^{\lambda} \cdot \mathbb{E} \sum_{1}^{\infty} e^{-\rho\vartheta t} \left(\frac{C_t}{C_0} \right)^{\lambda - \vartheta/\psi} \left(\frac{C_t}{C_0} \right)^{\vartheta - 1} e^{\nu(\vartheta-1)t} = \frac{(C_0)^{\lambda}}{e^{\rho\vartheta + \nu(1-\vartheta) - \boldsymbol{c}(\lambda-\gamma)} - 1},$$

so long as $\rho\vartheta + \nu(1-\vartheta) - \boldsymbol{c}(\lambda - \gamma) > 0$. Finally, we have $D/P = e^{\rho\vartheta + \nu(1-\vartheta) - \boldsymbol{c}(\lambda-\gamma)} - 1$.

Defining d/p as usual, $d/p = \rho\vartheta + \nu(1-\vartheta) - \boldsymbol{c}(\lambda - \gamma)$. The assumption imposed above is therefore that $d/p > 0$; this is the same requirement as in the power utility case. Setting $\lambda = 1$, we get an expression for $c/w \equiv \nu$ which can be solved for ν, giving $\nu = \rho - \boldsymbol{c}(1-\gamma)/\vartheta$: a constant, as assumed. Substituting back, we have $d/p = \rho - \boldsymbol{c}(\lambda - \gamma) + \boldsymbol{c}(1-\gamma)(1-1/\vartheta)$ and hence $r_f = \rho - \boldsymbol{c}(-\gamma) + \boldsymbol{c}(1-\gamma)(1-1/\vartheta)$. Since the price-dividend ratio is constant, $1 + R_{t+1} = (D_{t+1}/D_t)(1 + D_t/P_t)$, so $er = d/p + \log \mathbb{E}(D_{t+1}/D_t) = \rho\vartheta + \nu(1-\vartheta) - \boldsymbol{c}(\lambda-\gamma) + \boldsymbol{c}(\lambda)$, and hence the risk premium $rp = er - r_f = \boldsymbol{c}(\lambda) + \boldsymbol{c}(-\gamma) - \boldsymbol{c}(\lambda - \gamma)$.

B Identification in Section II.B

I show in the supplementary appendix that the regularity conditions for the GMM estimators to be consistent and asymptotically Normal hold in this calibration. It remains to check that the parameters are identified, i.e. that the population moment conditions (10) are satisfied only at the true parameter vector (ρ, γ). The proof that they are is another application of convexity of the CGF; in particular, *it does not depend on any details of the specific calibration*. The moment conditions can be rewritten as

$$\widehat{\rho} = \log \mathbb{E}\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\widehat{\gamma}} (1 + R_{i,t+1})\right] \tag{30}$$

for the two assets $i = 1, 2$. I write $\widehat{\rho}$ and $\widehat{\gamma}$ rather than ρ and γ to emphasize that $\widehat{\rho}$ and $\widehat{\gamma}$ are to be estimated from sample analogues of (30). In general we might have any two assets with different values of λ_i; if $i = 1$ refers to the consumption claim and $i = 2$ to the riskless asset, then $\lambda_1 = 1$ and $\lambda_2 = 0$.

With just one asset, ρ and γ are not identified: we have a locus of points $(\widehat{\rho}, \widehat{\gamma})$ consistent with (30), which can be thought of as a function $\widehat{\rho}(\widehat{\gamma})$. With two assets, we have two such functions, $\widehat{\rho}_i(\widehat{\gamma})$; the parameters are identified if the two functions cross only at $\widehat{\gamma} = \gamma$, i.e. we must check that $\widehat{\gamma} = \gamma$ is the unique solution of $\widehat{\rho}_1(\widehat{\gamma}) = \widehat{\rho}_2(\widehat{\gamma})$.

We have $\widehat{\rho}_1(\widehat{\gamma}) = \widehat{\rho}_2(\widehat{\gamma})$ if and only if $\log \mathbb{E}\left(\frac{C_{t+1}}{C_t}\right)^{-\widehat{\gamma}}(1 + R_{1,t+1}) = \log \mathbb{E}\left(\frac{C_{t+1}}{C_t}\right)^{-\widehat{\gamma}}(1 + R_{2,t+1})$. The model generates constant valuation ratios, so returns move in lockstep with consumption growth, via $1 + R_{i,t+1} = (C_{t+1}/C_t)^{\lambda_i} e^{\rho - \boldsymbol{c}(\lambda_i - \gamma)}$. Substituting this in, we seek to solve

$$-\boldsymbol{c}(\lambda_1 - \gamma) + \boldsymbol{c}(\lambda_1 - \widehat{\gamma}) = -\boldsymbol{c}(\lambda_2 - \gamma) + \boldsymbol{c}(\lambda_2 - \widehat{\gamma}).$$

Clearly, $\widehat{\gamma} = \gamma$ is a solution, so it remains to show that it is the *unique* solution. We can rewrite the above equation as

$$\boldsymbol{c}(\lambda_1 - \widehat{\gamma}) - \boldsymbol{c}(\lambda_2 - \widehat{\gamma}) = K, \tag{31}$$

where $K = \boldsymbol{c}(\lambda_1 - \gamma) - \boldsymbol{c}(\lambda_2 - \gamma)$ may be positive, negative or zero, depending on the particular values of λ_1 and λ_2, γ, and the particular distribution in question.

Without loss of generality, assume that $\lambda_1 < \lambda_2$. I will show that $\boldsymbol{c}(\lambda_1 - \widehat{\gamma}) - \boldsymbol{c}(\lambda_2 - \widehat{\gamma})$ is monotone increasing in $\widehat{\gamma}$. This will establish that the unique solution of equation (31) occurs at $\widehat{\gamma} = \gamma$, as required. To establish monotonicity, pick an arbitrary $\gamma_1 > \gamma_2$. We must show that

$$\boldsymbol{c}(\underbrace{\lambda_1 - \gamma_1}_{A}) - \boldsymbol{c}(\underbrace{\lambda_2 - \gamma_1}_{A+D}) > \boldsymbol{c}(\underbrace{\lambda_1 - \gamma_2}_{B}) - \boldsymbol{c}(\underbrace{\lambda_2 - \gamma_2}_{B+D}).$$

I have simplified the notation by introducing A, B, and D, where $A < B$ and $D > 0$. So, I must show that $\boldsymbol{c}(A) - \boldsymbol{c}(A + D) > \boldsymbol{c}(B) - \boldsymbol{c}(B + D)$, or equivalently,

$$\boldsymbol{c}(B + D) - \boldsymbol{c}(B) > \boldsymbol{c}(A + D) - \boldsymbol{c}(A)$$

for arbitrary $A < B$ and $D > 0$. This holds by strict convexity of $\boldsymbol{c}(\cdot)$.

C Result 3 is sharp

Given knowledge of c/w, r_f, and rp, could there be a better bound on ρ and ψ than those supplied by Result 3? The answer is no. For fixed c/w, r_f, and rp, there are possible distributions for consumption growth, and values of ρ, γ and ψ, that come arbitrarily close to making the inequality $\frac{c/w-\rho}{1/\psi-1} \geq r_f - c/w$ hold with equality. (That is, for any $\varepsilon > 0$ we can have $\frac{c/w-\rho}{1/\psi-1} \leq r_f - c/w + \varepsilon$.) Figure 6a illustrates. Similarly for the other inequality $\frac{c/w-\rho}{1/\psi-1} \leq rp + r_f - c/w$, as illustrated by Figure 6b.

The strategy will be to exhibit possible distributions and preference parameters under which the bounds are arbitrarily nearly attained. We must, of course, respect the observables, $c/w = \rho - \boldsymbol{c}(1-\gamma)/\vartheta$, $rp = \boldsymbol{c}(1) + \boldsymbol{c}(-\gamma) - \boldsymbol{c}(1-\gamma)$, and r_f (which jointly determine $\boldsymbol{c}(1)$, by the Gordon growth formula (7)). Equivalently, we must restrict to considering CGFs that match $\boldsymbol{c}(1)$ and $\boldsymbol{c}(1-\gamma) - \boldsymbol{c}(-\gamma)$; subsequently, ρ can be set to match c/w.

Consider first the left-hand inequality in (12), $\frac{c/w-\rho}{1/\psi-1} \geq r_f - c/w$. The proof of Result 3 shows that this bound would hold with equality if the line X had the same slope as the line Y: that is, if the CGF were a straight line between $-\gamma$ and 0. This is approximately the case in Figure 6a; it cannot *literally* be true, since if the CGF is a straight line over any interval then consumption growth must be deterministic. Nonetheless, we can make the CGF arbitrarily close to a straight line over this interval, while lining up with $\boldsymbol{c}(1)$ and with $\boldsymbol{c}(1-\gamma) - \boldsymbol{c}(-\gamma)$ as required for consistency with the observables.

Suppose G takes just two values: $\boldsymbol{c}(1) + \log a$ with probability q, and $\boldsymbol{c}(1) + \log b$ with probability $1 - q$. Here a is greater than 1 and b is between 0 and 1. To ensure that the distribution respects the mean of consumption growth, i.e. that $\mathbb{E}\, e^G = e^{\boldsymbol{c}(1)}$, we set $b = (1 - aq)/(1 - q)$; this requires that $aq < 1$. Write $\kappa(\theta)$ for the CGF of G. (I do so to avoid confusion between $\kappa(1)$, which for general values of a, b, q is arbitrary, and the observable $\boldsymbol{c}(1)$; of course, my choice of b ensures that $\kappa(1) = \boldsymbol{c}(1)$.) We have

$$\kappa(\theta) = \boldsymbol{c}(1)\theta + \log\left[qa^\theta + (1-q)\left(\frac{1-aq}{1-q}\right)^\theta\right].$$

To generate the extreme example, we will want to have q very close to zero, a very large, and b close to 1. Rearranging the above equation, to understand what happens when $q \approx 0$,

$$\kappa(\theta) = \boldsymbol{c}(1)\theta + \theta\log\left(\frac{1-aq}{1-q}\right) + \underbrace{\log(1-q) + \log\left[1 + \frac{q}{1-q}a^\theta\left(\frac{1-q}{1-aq}\right)^\theta\right]}_{\text{these converge to zero as } q \text{ tends to zero}}.$$

As q tends to zero, the last two terms converge to zero *uniformly* in $a > 1$ and $\theta < 0$: for any $\varepsilon > 0$, we can pick q sufficiently small that $\left|\kappa(\theta) - \boldsymbol{c}(1)\theta - \theta\log\left(\frac{1-aq}{1-q}\right)\right| < \varepsilon$. Given this small q, we choose a so that the (approximate) slope of $\kappa(\theta)$, for $\theta < 0$, matches up with the observable $\boldsymbol{c}(1-\gamma) - \boldsymbol{c}(-\gamma)$:

$$\boldsymbol{c}(1) + \log\left(\frac{1-aq}{1-q}\right) = \boldsymbol{c}(1-\gamma) - \boldsymbol{c}(-\gamma) = \boldsymbol{c}(1) - rp\,, \quad \text{or equivalently,} \quad a = \frac{1-(1-q)e^{-rp}}{q}.$$

The left-hand inequality is then tight up to an arbitrarily small error, as in Figure 6a.

In the other direction, we seek a distribution whose CGF is almost linear between $1-\gamma$ and 1, while holding fixed $\boldsymbol{c}(1)$ and $\boldsymbol{c}(1-\gamma) - \boldsymbol{c}(-\gamma)$, as illustrated in Figure 6b. Suppose that $G = \boldsymbol{c}(1) + E$, where with probability q, E is distributed on $(-\infty, 0)$ with pdf $f_1(x) = \lambda_1 e^{\lambda_1 x}$ for some $\lambda_1 > \gamma$, and with probability $1-q$, E is distributed on $(0, \infty)$ with pdf $f_2(x) = \lambda_2 e^{-\lambda_2 x}$ for some $\lambda_2 > 1$. To match means, we need $\mathbb{E}\, e^G = e^{\boldsymbol{c}(1)}$, or equivalently $q = (1+\lambda_1)/(\lambda_1 + \lambda_2)$. Imposing this, the CGF of G is

$$\kappa(\theta) = \boldsymbol{c}(1)\theta + \log\left[1 + \frac{\theta(\theta-1)}{(\lambda_1+\theta)(\lambda_2-\theta)}\right].$$

Ultimately we will have λ_1 close to γ and λ_2 very large; thus q will be close to zero.

For $\theta \in [1-\gamma, 1]$, it is immediate, since $\lambda_1 > \gamma$, that $\left|\frac{\theta(\theta-1)}{\lambda_1+\theta}\right|$ is uniformly bounded. Thus for large λ_2, $\kappa(\theta) = \boldsymbol{c}(1)\theta + O(\lambda_2^{-1})$: given arbitrarily small $\varepsilon > 0$, we can pick a large λ_2 such that $|\kappa(\theta) - \boldsymbol{c}(1)\theta| < \varepsilon$ for all $\theta \in [1-\gamma, 1]$, i.e. the CGF is almost linear between $1-\gamma$ and 1.

Given this λ_2, it remains only to choose $\lambda_1 > \gamma$ so that $\kappa(1) + \kappa(-\gamma) - \kappa(1-\gamma) = rp$:

$$\log\left[1 + \frac{\gamma(\gamma+1)}{(\lambda_1-\gamma)(\lambda_2+\gamma)}\right] - \log\left[1 + \frac{\gamma(\gamma-1)}{(\lambda_1+1-\gamma)(\lambda_2+\gamma-1)}\right] = rp\,. \qquad (32)$$

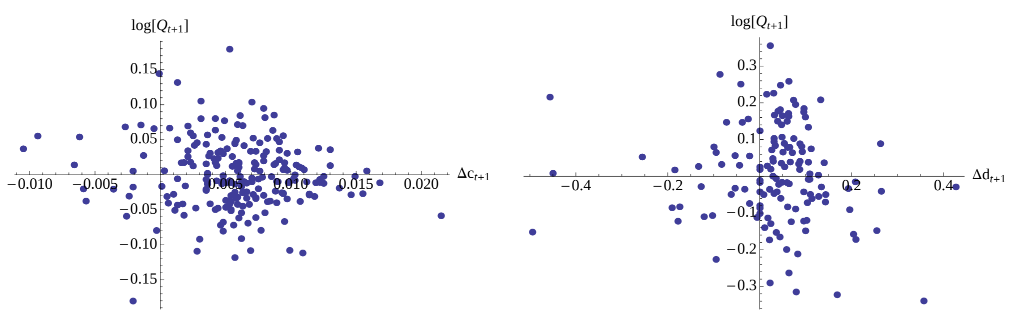
For arbitrary fixed λ_2, the left-hand side of this equation is continuous in λ_1; it tends to zero as λ_1 tends to infinity, and to ∞ as λ_1 tends to γ. So by the intermediate value theorem, there must be a value of λ_1 such that (32) holds. (In fact, for large λ_2, we will have $\lambda_1 \approx \gamma + \frac{\gamma(\gamma+1)}{(e^{rp}-1)\lambda_2}$.)

D Extensions

Dividends not equal to consumption

Proof of Result 6. Define $\boldsymbol{c}_j(\theta) = \log \boldsymbol{m}_j(\theta)$, $j = 2, 3$, and $\boldsymbol{c}_D(\theta) = \log \boldsymbol{m}_D(\theta)$. Given the assumptions made above, $\boldsymbol{c}_j(1) = 0$ and $\boldsymbol{c}_D(1) \leq 0$. By convexity, $[\boldsymbol{c}_j(0) - \boldsymbol{c}_j(-\gamma)]/\gamma < \boldsymbol{c}_j(1) - \boldsymbol{c}_j(0) = 0$. Equivalently, $\boldsymbol{c}_j(-\gamma) > 0$, and hence $\boldsymbol{m}_j(-\gamma) > 1$ for $j = 2, 3$. Similarly, $\boldsymbol{c}_D(1-\gamma) - \boldsymbol{c}_D(-\gamma) < \boldsymbol{c}_D(1) - \boldsymbol{c}_D(0) \leq 0$, so $\boldsymbol{c}_D(-\gamma) > \boldsymbol{c}_D(1-\gamma)$; it follows that $\boldsymbol{m}_D(-\gamma) > \boldsymbol{m}_D(1-\gamma)$. Finally, it is clear that $\boldsymbol{m}_D(\theta) > 0$ for all θ. The result follows from (23)–(25). □

The non-i.i.d. case



(a) Data from Lustig et al (2008) (b) Data from Robert Shiller's website

Figure 9: Changes in valuation ratios plotted against growth in cashflows.

Figure 9 illustrates the relationship between $\log Q_{t+1}$ and Δc_{t+1} in two data sets. Figure 9a shows data compiled for the period 1952.II to 2006.IV by Lustig, Van Nieuwerburgh, and Verdelhan (2008). (I thank them for making their data available to me.) The correlation between $\log Q_{t+1}$ and $\log(C_{t+1}/C_t)$ in their data is close to zero, at -0.10. Figure 9b shows the corresponding results using Robert Shiller's annual series P and D from 1871 to 2008, available on his website. Based on these figures, it seems reasonable to entertain the possibility that Q_{t+1} is independent of C_{t+1}/C_t. Lack of correlation does not imply independence, of course, so after the proof of Result 7 (below) I report several tests of the null hypothesis that Q_{t+1} and C_{t+1}/C_t are independent. The hypothesis cannot be rejected at the 5% level on either dataset, using any of the tests.

Proof of Result 7. (I drop subscripts t.) The independence assumption implies that $\boldsymbol{c}_t(\theta_1,\theta_2)$ can be decomposed as $\boldsymbol{c}_1(\theta_1)+\boldsymbol{c}_2(\theta_2)$, where $\boldsymbol{c}_1(\theta_1)=\log \mathbb{E}_t\, e^{\theta_1 \log(C_{t+1}/C_t)}$ and $\boldsymbol{c}_2(\theta_2)= \log \mathbb{E}_t\, e^{\theta_2 \log Q_{t+1}}$. Since $\mathbb{E}_t\, Q_{t+1}=1$, $\boldsymbol{c}_2(1)=0$. Then we have

$$\vartheta(\rho - c/w) = \boldsymbol{c}_1(1-\gamma)+\boldsymbol{c}_2(\vartheta) \tag{33}$$

$$c/w - r_f + \vartheta(\rho - c/w) = \boldsymbol{c}_1(-\gamma)+\boldsymbol{c}_2(\vartheta-1) \tag{34}$$

$$r_f + rp - c/w = \boldsymbol{c}_1(1) \tag{35}$$

In the power utility case, $\vartheta=1$, so the $\boldsymbol{c}_2(\cdot)$ terms in (33)–(35) drop out since $\boldsymbol{c}_2(0)=\boldsymbol{c}_2(1)=0$. The proof then goes through exactly as in the proof of Result 3.

It remains to consider the Epstein-Zin case. Given (33)–(35), the upper bound holds iff

$$\frac{\boldsymbol{c}_1(1-\gamma)}{1-\gamma}+\frac{\boldsymbol{c}_2(\vartheta)}{1-\gamma} \le \boldsymbol{c}_1(1)\,.$$

Now, $\boldsymbol{c}_1(1-\gamma)/(1-\gamma) \le \boldsymbol{c}_1(1)$ by the standard convexity argument. So the upper bound will hold if $\boldsymbol{c}_2(\vartheta)/(1-\gamma)\le 0$. Thus we need *either* $\boldsymbol{c}_2(\vartheta)\ge 0$ and $\gamma>1$ *or* $\boldsymbol{c}_2(\vartheta)\le 0$ and $\gamma<1$. But the convexity of $\boldsymbol{c}_2(\cdot)$ means that it is easy to characterize precisely when $\boldsymbol{c}_2(\cdot)$ is positive or negative: since $\boldsymbol{c}_2(0)=\boldsymbol{c}_2(1)=0$, and $\boldsymbol{c}_2(\cdot)$ is convex, $\boldsymbol{c}_2(x)\le 0$ iff $x\in[0,1]$. So the upper bound holds if $\gamma>1$ and $\vartheta\notin(0,1)$, or if $\gamma<1$ and $\vartheta\in[0,1]$.

The lower bound holds iff

$$\frac{\boldsymbol{c}_1(1-\gamma)}{1-\gamma}+\frac{\boldsymbol{c}_2(\vartheta)}{1-\gamma} \ge \boldsymbol{c}_1(1-\gamma)-\boldsymbol{c}_1(-\gamma)+\boldsymbol{c}_2(\vartheta)-\boldsymbol{c}_2(\vartheta-1)\,.$$

Now, $\boldsymbol{c}_1(1-\gamma)-\boldsymbol{c}_1(-\gamma)\le \boldsymbol{c}_1(1-\gamma)/(1-\gamma)$ by the standard convexity argument. So the lower bound holds if $\boldsymbol{c}_2(\vartheta)-\boldsymbol{c}_2(\vartheta-1)\le \boldsymbol{c}_2(\vartheta)/(1-\gamma)$; equivalently, if

$$\frac{\gamma}{\gamma-1}\boldsymbol{c}_2(\vartheta)\le \boldsymbol{c}_2(\vartheta-1).$$

This condition clearly holds if the left-hand side is negative and the right-hand side is positive, and fails if the converse is true. Given the discussion above about when $\boldsymbol{c}_2(x)$ is positive or negative, it holds if $\gamma>1$ and $\vartheta\in(0,1)$, or if $\gamma<1$ and $\vartheta\notin[0,2]$.

These conditions are equivalent to those given in the statement of the result.

Finally, we must show that the good deal bound generalizes under the given conditions. The analogue of (18) is $\boldsymbol{c}_1(-2\gamma) - 2\boldsymbol{c}_1(-\gamma) + \boldsymbol{c}_2(2(\vartheta-1)) - 2\boldsymbol{c}_2(\vartheta-1) \leq \log(1+h^2)$. In the power utility case, $\vartheta = 1$ and the $\boldsymbol{c}_2(\cdot)$ terms drop out; the proof is then as before.

It remains to deal with the cases $\vartheta < 0$ and $\vartheta > 2$. I will show that in either case,

$$\boldsymbol{c}_2(2(\vartheta-1)) - 2\boldsymbol{c}_2(\vartheta-1) \geq (\gamma-1)\boldsymbol{c}_2(\vartheta-1) - \gamma\boldsymbol{c}_2(\vartheta). \tag{36}$$

Combining this with $\boldsymbol{c}_1(-2\gamma) - 2\boldsymbol{c}_1(-\gamma) \geq (\gamma-1)\boldsymbol{c}_1(-\gamma) - \gamma\boldsymbol{c}_1(1-\gamma)$, which holds as before, and using (33) and (34), we have

$$\boldsymbol{c}_1(-2\gamma) - 2\boldsymbol{c}_1(-\gamma) + \boldsymbol{c}_2(2(\vartheta-1)) - 2\boldsymbol{c}_2(\vartheta-1) \geq (\gamma-1)\boldsymbol{c}_1(-\gamma) - \gamma\boldsymbol{c}_1(1-\gamma) + (\gamma-1)\boldsymbol{c}_2(\vartheta-1) - \gamma\boldsymbol{c}_2(\vartheta)$$

So the good-deal bound $\sigma(M)/\mathbb{E}\,M \leq h$ implies, as required, that $(\gamma-1)(c/w - r_f) + \vartheta(c/w-\rho) \leq \log(1+h^2)$. It only remains to establish (36). This is equivalent to showing that $\boldsymbol{c}_2(2(\vartheta-1)) - \boldsymbol{c}_2(\vartheta-1) \geq \gamma\left(\boldsymbol{c}_2(\vartheta-1) - \boldsymbol{c}_2(\vartheta)\right)$. Suppose first that $\vartheta < 0$. Then we know by convexity that

$$\frac{\boldsymbol{c}_2(\vartheta-1) - \boldsymbol{c}_2(2(\vartheta-1))}{1-\vartheta} \leq \boldsymbol{c}_2(\vartheta) - \boldsymbol{c}_2(\vartheta-1)\,.$$

It follows that $\boldsymbol{c}_2(2(\vartheta-1)) - \boldsymbol{c}_2(\vartheta-1) \geq (1-\vartheta)(\boldsymbol{c}_2(\vartheta-1) - \boldsymbol{c}_2(\vartheta))$. We're done so long as $1-\vartheta \geq \gamma$. This requires that $1-\gamma \geq \vartheta$, which is equivalent to $\vartheta/\psi \leq 0$; and this holds. Alternatively, suppose $\vartheta > 2$. Then $\boldsymbol{c}_2(2(\vartheta-1)) > \boldsymbol{c}_2(\vartheta-1) > 0$ and $\boldsymbol{c}_2(\vartheta-1) - \boldsymbol{c}_2(\vartheta) < 0$ so trivially the required result holds. $\square$

Testing the hypothesis that Q_{t+1} and C_{t+1}/C_t are independent

Given a set of bivariate random vectors $(X_1, Y_1), \ldots, (X_N, Y_N)$, we can form the rank vectors R_N and S_N of the samples $X_1, \ldots, X_N$ and $Y_1, \ldots, Y_N$ respectively. So, for example, the kth element of R_N, denoted R_{Nk}, would equal 3 if X_k were the third-smallest member of $\{X_1, \ldots, X_N\}$.

Throughout this section, the null hypothesis of interest is that the random variables X_i and Y_i are independent for each i. (This is slightly more convenient notation; below I will set $X_i = C_i/C_{i-1}$ and $Y_i = Q_i$.) I consider four different tests of the null: Kendall's test, based on the signs of appropriate products of differences; Spearman's test, based on a rank correlation coefficient; a test that detects a broader class of alternatives to independence, due to Hoeffding (1948) and subsequently adapted for large samples by Blum, Kiefer and Rosenblatt (BKR, 1961); and a test due to Hong (1996). The first three tests assume that the bivariate random vectors (X_i, Y_i) are independent over time. If this assumption is violated, then the tests are likely to over-reject the null hypothesis of independence between X_i and Y_i within pairs. The fourth test, due to Hong, explicitly accounts for the effects of non-independence over time, and is based on a kernel-smoothed sum of squared cross-correlations between the residuals from autoregressions of X_t on $X_{t-1}, \ldots, X_{t-p}$, and of Y_t on $Y_{t-1}, \ldots, Y_{t-p}$.

Kendall's test is based on $N(N-1)/2$ paired sign statistics $Q((X_i, Y_i), (X_j, Y_j))$, where for $1 \leq i < j \leq N$,

$$Q((a,b),(c,d)) = \begin{cases} 1 & \text{if } (d-b)(c-a) > 0 \\ -1 & \text{if } (d-b)(c-a) < 0 \end{cases}.$$

The test statistic is

$$K = \frac{\sum_{i=1}^{n-1} \sum_{j=i+1}^{N} Q((X_i, Y_i), (X_j, Y_j))}{[N(N-1)(2N+5)/18]^{1/2}},$$

which is asymptotically standard Normal under the null hypothesis.

Spearman's rank correlation coefficient, κ_N, is the classical (sample) correlation coefficient applied to the vectors R_N and S_N:

$$\kappa_N = \frac{\sum_{i=1}^{N} (R_{Ni} - \overline{R_N})(S_{Ni} - \overline{S_N})}{\left[\sum_{i=1}^{N} (R_{Ni} - \overline{R_N})^2 \sum_{i=1}^{N} (S_{Ni} - \overline{S_N})^2\right]^{1/2}}.$$

Under the null, $\sqrt{N}\kappa_N$ is asymptotically standard Normal.

Construction of the BKR test statistic is more involved; full details, together with discussion of Kendall's test and Spearman's rank correlation coefficient, can be found in Hollander and Wolfe (1999, chapter 8). The BKR test statistic is B_N; we reject the null if $\frac{1}{2}\pi^4 N \cdot B_N$ is larger than critical values given in BKR (1961) and Hollander and Wolfe (1999, Table A.33).

Data set	K	p-value	κ_N	$\kappa_N\sqrt{N}$	p-value	$\frac{1}{2}\pi^4 N \cdot B_N$	p-value
Lustig et al.	-1.581	0.114	-0.0877	-1.295	0.195	2.790	0.0534
Shiller	-1.262	0.207	-0.0799	-0.935	0.350	1.722	0.212

Table IV: Testing for independence of Q_{t+1} and C_{t+1}/C_t using Kendall's test, Spearman's test, and the BKR test.

Finally, I compute Hong's (1996) test statistic Q_N^* for a variety of lag lengths p and smoothing parameters M, using the quadratic-spectral kernel

$$k(x) = \begin{cases} \frac{25}{12\pi^2 x^2}\left[\frac{\sin(6\pi x/5)}{6\pi x/5} - \cos(6\pi x/5)\right] & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}.$$

The test statistic has an asymptotically standard Normal distribution under the null.

	$M=1$	$M=2$	$M=3$	$M=5$	$M=10$
$p=1$	1.26	1.70	1.82	1.56	1.26
$p=2$	1.31	1.59	1.62	1.31	0.92
$p=3$	1.15	1.26	1.15	0.64	0.18
$p=5$	0.89	1.22	1.27	0.97	0.54
$p=10$	0.93	1.46	1.67	1.50	1.25

Table V: Values taken by Hong's (1996) test statistic Q_N^*, which is asymptotically distributed $N(0,1)$ under the null, in the Lustig et al. data.

	$M=1$	$M=2$	$M=3$	$M=5$	$M=10$
$p=1$	0.06	-0.23	-0.29	-0.32	-0.69
$p=2$	0.34	-0.17	-0.21	-0.17	-0.55
$p=3$	0.32	-0.20	-0.24	-0.17	-0.53
$p=5$	0.64	0.05	-0.16	-0.18	-0.44
$p=10$	-0.26	-0.66	-0.57	0.17	0.37

Table VI: Values taken by Hong's (1996) test statistic Q_N^*, which is asymptotically distributed $N(0,1)$ under the null, in the Shiller data.

As is clear from Tables IV, V and VI, the null hypothesis of independence is not rejected in either dataset by any of the four tests.