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Joint Commitment

Implementation without Public Randomization

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ABSTRACT. This paper investigates the effects of the ability to jointly rule out actions on the analysis of a strategic form game. If players can voluntarily sign contracts that jointly commit them to rule out some of their actions, mutually beneficial bargains can often be found. It may appear that making sure that such bargains are efficient occasionally requires players to sign random contracts, which commit them to a specific course of action based on the realization of some extraneous randomization. Indeed, the set of allocations implemented if players have a single opportunity to sign contracts that deterministically rule out some actions, is characterized and often inefficient. The analysis however shows that if players have sufficiently many opportunities in which to sign such deterministic contracts, any individually rational allocation is approximately implementable. Therefore, even in games that are not convex, the implementation of any efficient and individually rational allocation does not require any randomness in the mechanism used if players have several opportunities in which to reach an agreement.

Contents

1 Introduction

Intrinsic to the definition of game is the ability of players to fully commit to any of their available actions at any choice instance. Such definition takes action sets as exogenous elements of the problem and presumes that players can under no circumstance rule out any of the available strategies before the game takes place. If, indeed, such description is appropriate for many strategic environments, it does abstract from the institutions determining which choices are available to players. In fact, if actions sets were determined by players' commitments to rule out strategies, such abstraction would focus the analysis on the interim payoffs of the game. An alternative approach consists of, explicitly, taking into account such options to rule out strategies before the game takes place, thereby making choice sets endogenous. Actions chosen would then necessarily depend on commitments previously undertaken.

Different institutions enabling players to rule out strategies can be envisioned. Some have commitments arising from unilateral pledges, as in the case of a producer setting its capacity, others from joint agreements amongst players, as in the case of a traditional marriage. While either type of institution can enhance cooperation amongst players by eliminating tempting, but costly strategies, commitments undertaken and actions played in equilibrium may significantly differ.

The analysis characterizes the set of allocations implemented by different institutions that can be used to commit the players' course of action. The methodology adopted extends the original game by modeling explicitly how commitments arise before actions are chosen. Any extension in which all players' choices that precede the game only determine the set of feasible actions in the game is referred to as commitment game. The main conclusion of study is that if players are given sufficiently many instances in which to commit any individually rational allocation can be implemented without the need of public randomizations. In proving such result, properties of different commitment institutions will be discussed.

If commitments can arise only from unilateral pledges, implementing such payoff hull will not be possible even if player are given arbitrarily many instances in which to commit. A paper by Renou [17] characterizes the set of payoffs that can be implemented if there is a single instance in which to commit. In a separate paper I analyze what allocations can be implemented if there are multiple rounds in which to unilaterally commit and what conditions warrant the implementation of the individually rational hull.

When commitments are jointly determined, however, an extension of the game will need to specify not only a set of feasible contracts, but also the nature of the agreement amongst players necessary to enforce each of those contracts. The most cooperative class of commitment extensions requires a unanimous agreement in order to rule out any strategies. Such mechanisms allow each player to object to any commitment that modifies the structure of original game.

The characterization of the payoffs implemented by mechanisms belonging to this class shows that, if players have a single period in which to commit and if pledges can only deterministically rule out strategies, not necessarily can any efficient allocation be implemented. The concavity of the payoff hull of the original game can, indeed, lead to such negative conclusion. It may appear that in order to overcome such shortcoming in the implementation result lotteries on commitments ought to belong to the set of feasible contracts. In fact, such assumption would result in the implementation of all allocations exceeding each players worst Nash outcome, since any deviation from such a contract could be discouraged by appropriate beliefs about equilibrium play in the subgame.

Surprisingly however such random contracts may never be necessary to implement the very same payoff hull if players have several instances in which to jointly commit. In such extension at any such instance players can agree to rule out strategies, not previously eliminated.[1] The central result to the analysis of unanimous commitment games asserts that, under weak dimensionality conditions on the original game, any payoff profile that exceeds the worst Nash outcome can be approximately implemented as the number of rounds in which to commit increases. Moreover only contracts that fully commit players to a single pure strategy profile of the original game are necessary to attain such result. The claim hinges on the existence of continuation values that motivate each player to occasionally concede to unfavorable contracts at various stages, in order to support concession by the others' to contracts favoring him at later stages. Hence, time suffices in forming beliefs about sequences of concessions to deterministic contracts that implement any payoff profile that can be obtained in a single instance with random contracts. Any such sequence of beliefs can be interpreted as a description of the bargaining process amongst players trying to sign one of such deterministic contracts. An implication of such claim is that even when a single round of deterministic commitment is inefficient, not observing a random contract being signed should not, *per se*, be interpreted as a signal for inefficiency.

Even though increasing either the set of feasible contracts or the number contracting rounds implements the same outcome set, the contracts signed in either scenario do significantly differ. Indeed, increasing the number of rounds often results in players signing with positive probability contracts that would never be individually rational to them if there were a single instance in which to commit. The simplicity of a stochastic extension is counterbalanced by the considerable number of contracts required in the implementation and by the empirical counterpart of such contracts that often involve lottery-contracts being signed. A multi-round extension, though technically more involved, requires far less contracts and all of them deterministic to realize the same objective.

1.1 Related Literature

The idea that unilaterally limiting one's flexibility at the choice instance can be beneficial is deeply rooted in the game theoretic literature. Many examples of the advantages and limitations of commitment can be found in Schelling's classic [18]. A part of the literature on the topic investigates how bargaining impasses can arise form unilateral and uncertain commitments to favorable positions. Crawford [5] shows why rational players, when bargaining, may elect to commit, even when doing so leads with some positive probability to an inefficient impasse. Powell [16] shows how such conflicting commitments arise when compromise solutions are not enforceable. Ellingsen and Miettinen [8] extend results by making commitments certain introducing small costs to commit. Their analysis shows that costs reduce the multiplicity of equilibria and that equilibria in which the probability of an impasse tends to one can arise, as costs tend to zero. Hart and Moore [13] discuss the role of unilateral pledges in an environment with one sided incomplete information and bargaining. They show how a trade-off between commitment and flexibility could arise in the choice of the optimal pledge.

More recently, a literature has developed relating unilateral commitments to the endogenous timing of moves in a game. In the context of duopoly game Hamilton and Slutsky [12] show how the ability to unilaterally commit to any single action, by endogenizing the timing play, may implement the leader-follower equilibria of the game. Van Damme and Hurkens [6] provide conditions under which the equilibria of the original game are robust to unilateral commitments to single pure strategies. Renou [17] extends results further allowing all players to unilaterally commit to any subsets of their action set. For such technology, necessary and

[1] In these extensions time is immaterial. It merely defines the number of rounds left to negotiate and the set of available contracts conditional on agreements previously signed.

sufficient conditions are provided for implementation of any pure strategy profile in the commitment expansion of the original game. Therein, it is shown that commitments to single pure strategies may not suffice for the implementation of an efficient allocation, even when unilateral pledges to subsets of strategies do. Again, a trade-off between commitment and flexibility can arise, which in turns crates a rationale for the optimality of limited commitment. The results presented section **??** generalize the implications of such models. A related paper by Bade, Haeringer and Renou [1] extends results to well-behaved two players continuous games.

1.2 Roadmap

The paper proceeds as follows: section 2 presents an example motivating the analysis, section 3 sets up notation and defines properties of the commitment structures studied; section 4 characterizes the equilibria of a commitment extension and provides simplifying results; section 5 characterizes the set of payoffs implemented by a single round of unanimous commitment, gives sufficient conditions for efficiency and shows that some efficient allocations cannot be implemented by any deterministic commitment; section 6 argues that whenever agents have arbitrarily many stages in which to commit, all allocations exceeding each player's Nash threat, including the efficient ones, are implementable. Therein, it is shown how harmful concession at each commitments stage may be motivated by beliefs about favorable concessions at later stages. Section 7 reports examples motivating theory developed. Section 8 concludes and discusses related projects. In appendix (section 9) it is possible to find proofs omitted from the main text. A longer draft of the article with more results can be found on my website.

2 Example: Invasion and Deployment

Consider a scenario in which two countries decide how much to invest to conquer a land of size 1. Suppose that each country chooses how many units of its force to deploy and that sending $q \in [0, 1]$ has a total cost of $c(q) = 6q - 3q^2$. Assume that the returns on every unit deployed decrease with the total number of units deployed by the two countries for the invasion, $Q = q_1 + q_2$. Specifically, let such returns be defined by $d(Q) = 10 - 3Q$. Hence, for any possible profile of deployments, $(q_i, q_j) \in [0, 1]^2$, benefits form the invasion for country $i \neq j \in \{1, 2\}$ reduce to:

$$u_i(q_i, q_j) = d(q_i + q_j)q_i - c(q_i) = 4q_i - 3q_iq_j$$

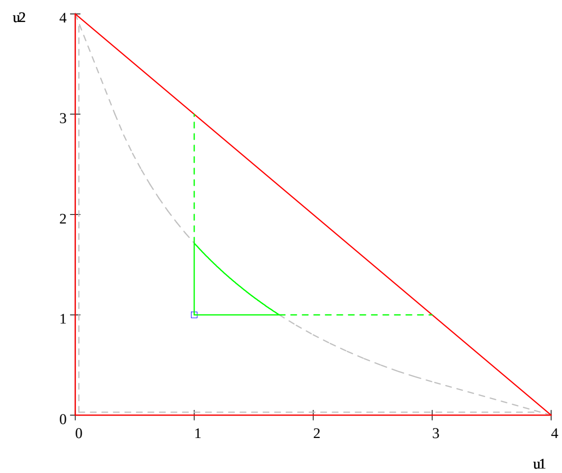
If the two countries must choose deployments independently, no course of action can ever lead to profit profiles outside:

$$\mathcal{A} = \left\{(u_1, u_2) \in \mathbb{R}^2 | u_2 \leq 16/3 + u_1 - 8\sqrt{u_1/3},\ u_1 \geq 0 \ \&\ u_2 \geq 0\right\}$$

The concavity of $\mathcal{A}$ implies that some feasible allocations can only be obtained by correlating the deployment choices of the two contenders. In fact, the set of feasible ex-ante payoff profiles, denoted by $\mathcal{U}$, consists of all payoffs in $\mathsf{co}(\mathcal{A})$. The unique Nash equilibrium for this invasion model requires both countries to invade at full force, $q_1 = q_2 = 1$ and to make unit profits. Figure 1 depicts the aforementioned payoff sets.

Indeed because any action that does not involve full deployment is strictly dominated, no player would benefit from limiting unilaterally his ability to deploy. Thus, even if players could unilaterally commit to lower deployments, they would never do so in equilibrium.

If instead both countries had the ability to limit their deployment by jointly committing to not use full force, it would be possible to increase the set of allocations implemented as a equilibria of the enlarged game to include all payoff profiles in $\mathcal{A}$ that exceed the Nash outcome of either player. For instance consider how payoff profile $(4/3, 4/3)$ could be implemented by having both countries jointly pledge to limit their deployment to $2/3$. Indeed, signing such contract would yield additional profits to each country, since any deviation at the contract signing stage would at best implement the unique Nash outcome of the original game.



1: red $\mathcal{U}$, gray $\mathcal{A}$, blue $\mathcal{N}$, green solid $\mathcal{K}_{\mathrm{U}}$

However no efficient payoff belongs to $\mathcal{A}$ and is preferred to the Nash outcome by both countries. It may appear that the implementation of such payoff profiles requires the use of a stochastic contracts that make either country the only contender with a probability in $[1/4, 3/4]$. For instance, a probability of $1/2$ would yield to both players ex-ante expected profits of 2.

The analysis presented shows that any such allocation can also be implemented without any randomness in the contracts signed, if countries have sufficiently many rounds in which offer contract on reduced deployments. Thus, having multiple stages in which to commit may defy the need for randomness in contracts. Indeed, consider how the two countries could implement with two rounds of deterministic contracting the profit allocation $(1, 2)$, that lies outside $\mathcal{A}$. With a single stage of contracting commitments not to deploy more that $(0.5512, 0.4262)$ units, respectively, would support profits $(1.5, 1)$ as an equilibrium. In fact both countries would weakly benefit by signing such an agreement. With an additional round of contract signing even payoff profile $(1, 2)$ could be supported. Indeed suppose that at the second stage contracting stage both countries randomize on three contracts that only commit the first country not to deploy, but that differ on fine prints. Fine prints must coincide as well for any such contract to be enforced. The game possesses an equilibrium in which both countries randomize three of such devices with equal probability and in which miscoordination on one of these devices leads players to sign the contract $(1.5, 1)$ at the latter round. If any deviation from such strategy is punished by the other country with the refusal to contract any further, the described strategies constitute an equilibrium of the commitment expansion of the game, since:

$$\begin{aligned} u_1(\pi_1, \pi_2) &= (1/3)0 + (2/3)1.5 \geq 1 \\ u_2(\pi_1, \pi_2) &= (1/3)4 + (2/3)1 \geq 1 \end{aligned}$$

Therefore the implementation of ex-ante payoff profiles that do not belong to the equilibrium set when there is a single occasion to commit, is possible when more occasions are given to reach an agreement. The analysis shows how with sufficiently many rounds in which to sign deterministic contracts it is possible to implement any profit profile exceeding each player's worst Nash threat, as a subgame perfect equilibrium of the enlarged contracting game.

In this model players may occasionally offer contract that are not their best interest if signed, because they believe that such behavior may favor them in the negotiations to come if no agreement is found. In the example player 1 is willing to offer contracts forcing him not to deploy, because he believes that he will offered a favorable contract at the next round if no agreement arises. This model thus provides a theory of bargaining on deterministic contracts. In the leading interpretation of the model an offer corresponds to an envoys with orders to sign a specific contract at some round.

3 Unanimous Commitment Structures

3.1 Definitions: Payoff Hulls & Equilibria

Consider a strategic form game $\{N, \{A_i, u_i\}_N\}$, where N denotes the set of players, A_i the set of actions of player $i \in N$ and $u_i : A \to \mathbb{R}$ his utility map [2] Throughout the paper, for any set X, 2^X and $\Delta(X)$ denote respectively the power set and the simplex of set X.[3] For notational convenience, for any Cartesian product $\times_{i\in N} X_i$ let $\Delta_N(X) = \times_{i\in N}\Delta(X_i)$ denote the Cartesian product of the simplices.

Define set of ex-ante and ex-post feasible payoff profiles, respectively, by:

$$\mathcal{U} = \{u(\mu) \in \mathbb{R}^N \,|\mu \in \Delta(A)\} \quad \& \quad \mathcal{A} = \{u(a) \in \mathbb{R}^N \,|a \in A\}$$

Also, let $\mathcal{I} = \{u(\sigma) \in \mathbb{R}^N \,|\sigma \in \Delta_N(A)\}$ denote the set of ex-ante feasible payoffs if players are bound not correlate their strategies. By construction any payoff profile that belongs to Nash equilibrium payoff hull $\mathcal{N}$ also belongs to the independent hull, $\mathcal{N} \subseteq \mathcal{I}$. The Nash equilibrium strategy set is denoted by $\mathcal{E} \subseteq \Delta_N$.[4] The set of payoff profiles belonging to the weak Pareto frontier of the game is denoted by $\mathcal{P}$.[5]

Finally, let *Nash rational* payoff hull consists of all ex-ante feasible payoffs that exceed the worst Nash equilibrium outcome of each player. That is:

$$\underline{\mathcal{N}} = \left\{ u \in \mathcal{U} | u_i \geq \min_{u' \in \mathcal{N}} u'_i,\, \forall i \in N \right\} \supseteq \mathcal{N}$$

The notation $\underline{u}^i$ is used recurrently to denote a payoff profile that yields worst Nash outcome to player i. That is: $\underline{u}^i \in \arg\min_{u\in\mathcal{N}} u_i$. The term *Nash threat* is used throughout to refer to such payoff profile.

3.2 Joint Commitment

3.2.1 Deterministic Commitments

This section describes how commitment expansions of a strategic form game are modeled throughout the paper. First, the framework is introduced, then properties of such expansions are discussed.

[2] I adopt the following common conventions: $A = \times_{i\in N} A_i$, $A_{-j} = \times_{i\in N\setminus j} A_i$ & $A_J = \times_{i\in J} A_i$ for $\forall J \in 2^N$. Similar conventions are adopted for elements.

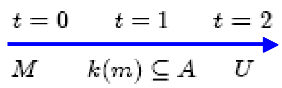
[3] For X a set, $\Delta(X) = \left\{\mu \in \mathbb{R}_+^X \,|\, \sum_X \mu(x) = 1\right\}$.

[4] Specifically $\mathcal{E} = \{\sigma \in \Delta_N(A)\,|u_i(a_i, \sigma_{-i}) - u_i(\sigma)) \leq 0 \text{ for } \forall a_i \in A_i \;\&\; i \in N\}$ and $\mathcal{N} = \{u(\sigma) \in \mathcal{U}\,|\sigma \in \mathcal{E}\}$.

[5] Specifically, define $\mathcal{P} = \{u \in \mathcal{U}\,|\nexists u' \in \mathcal{U} : u' - u > \mathbf{0}\}$.

In the context of the paper, the term commitment is used to refer to a voluntary and fully enforceable pledge to rule out actions before the choice stage. Therefore, a commitment rules out, but does not necessarily rule in strategies. In this setup, each player's pledge may depend on the pledges made by others, but not on the actions chosen by others.[6]

For any strategic form game a commitment structure, $\{M_i, k_i\}_{i \in N}$, consists of a collection of message spaces, M_i, and of a collection of functions mapping message profiles into non-empty action sets: $k_i : M \rightarrow 2^{A_i} \backslash \emptyset$. To ease the analysis, only consider commitment structures with finite messaging spaces. For a given commitment structure, the extension of the game $\{N, \{M_i, k_i, A_i, u_i\}_N\}$ consists of a two stage simultaneous move game. At first, players choose which messages to send, then commitments are realized based on messages sent and actions are chosen within the, consequently, restricted strategy sets. Assume that all player at the choice stage know all messages sent at the commitment stage. *Perfect monitoring* of the messages makes the extended game one of complete information and enhances the ability to punish deviating players.



A commitment structure is *voluntary*: if for any player $i \in N$ and there exists message $m_i \in M_i$ such that $k_i(m) = A_i$ independently of messages sent by the other players $m_{-i} \in M_{-i}$. In any such structure, if an player ever commits, he must do so voluntarily, because he could have prevented any commitment on his part by sending the no commitment message, m_i. The analysis focuses on voluntary structures, because they entail an inalienable right for players to control actions and restraints that affect their own choice set in the original game.

A commitment structure is *with veto* whenever for any agent $i \in N$, there exists message $m_i \in M_i$ such that $k(m) = A$ independently of the profile of messages of the others $m_{-i} \in M_{-i}$.[7] Such a commitment structure requires all to agree for anyone to commit.

A commitment structure is said to be *independent* (or unilateral) if it is voluntary and if for any player $i \in N$ and message profile $m \in M$:

$$k_i(m) = k_i(m_i, m'_{-i}) \text{ for } \forall m'_{-i} \in M_{-i}$$

Such a mechanisms are without veto, since no player can ever block unilateral commitments undertaken by others. Indeed, independence implies that $k_i : M_i \rightarrow 2^{A_i} \backslash \emptyset$ for any $i \in N$. A particular type of independent commitment structure is discussed in detail by Renou in [17].

A commitment structure is said to be *unanimous* if it is with veto and if $k_i(m) \neq A_i$ for some $m \in M$ and $i \in N$, implies that, for $\forall j, l \in N$:

$$k_l(m'_j, m_{-j}) = A_l \text{ for } \forall m'_j \in M_j \backslash m_j$$

Also unanimous structures require an agreement amongst all players to enforce any given pledge, since all players have the ability to block any commitment. For convenience, define a unanimous commitment structure to be in *canonical form* if $M_i = M_*$ for any $i \in N$ and that $k_i(m) \neq A_i$ for some $i \in N$ implies that $m = m_i^N \in M$. The only restriction upon simple unanimity is that all players have a common messaging space. Pledges in a canonical form unanimous supergame can, thus, arise only on the main diagonal of the game. Focusing

[6] This assumption is invoked in order to guarantee existence of equilibria at the action stage. In fact, if such assumption were not invoked, it could be that the extent of each player's commitment be unknown to him, when choosing which action to play [if others randomize]. This would undermine assumptions necessary that grant existence.

[7] Any commitment structure with veto is voluntary, but the converse is not generally true.

on such subclass simplifies the analysis, while posing no limitation on implementation of any particular allocation. In a two player setup this would account to:

$1\backslash 2$	m_1	m_2	m_3	...
m_1	$k(m_1^2)$	A	A	...
m_2	A	$k(m_2^2)$	A	...
m_3	A	A	$k(m_3^2)$	A
...	...	...	A	...

This paper will discuss in detail the set of allocations implemented by the class of unanimous commitment structures. It will also be shown that allowing also agreements that commit only those who are willing to commit, cannot reduce the set of allocations implemented.[8] The set feasible commitments structures on action space A is denoted by:

$$\mathrm{K} = \left\{(k, M)\,\middle|k : M \to \times_{i\in N}[2^{A_i}\backslash\emptyset]\right\}$$

Similarly let U denote the sets of unanimous structures. That is, $\mathrm{U} = \{(k, M) \in \mathrm{K}|k \text{ unanimous}\}$. In such definitions the dependence on the original action space is omitted for notational ease.

3.2.2 Stochastic Commitments

All the mechanisms described up to this subsection were deterministic. Because the analysis discusses when focusing on deterministic devices entails no loss, consider a richer class of mechanisms that includes random commitment devices. In such scenario, players can agree sign contracts that randomly enforce restrictions on their set of available choices. Again, as a benchmark, assume that after the lottery is performed all information about messages sent and commitments realized is released to all agents.

Formally, a stochastic commitment structure consists of a map from messages profiles to lotteries on the set of feasible commitments. That is $k : M \to \Delta(\times_{i\in N}[2^{A_i}\backslash\emptyset])$. Any joint distribution on action profiles is an example of such a lottery. Denote the set of stochastic structures by:

$$\mathrm{S} = \left\{(k, M)\,\middle|k : M \to \Delta(\times_{i\in N}[2^{A_i}\backslash\emptyset])\right\}$$

Equilibrium outcomes of a stochastic commitment structure cannot necessarily be implemented by any deterministic commitment device. Indeed, mechanisms in this class presume that players have the ability to condition how to rule out their actions upon the outcome of some randomization device. Such delegation ability may allow players to sustain in equilibrium distributions of commitment profiles that would not have be self-enforcing for any given deterministic device.

Notation $k(A'|m)$ will be used to denote the probability that players are committed to profile $A' \subseteq A$ given that they signed up to lottery $k(m)$. As, usual messaging spaces will be assumed finite. A stochastic commitment structure is said to be independent if $k(m) = \prod_{i\in N} k_i(m_i)$ for any message profile $m \in M$ and for $k_i : M_i \to \Delta(2^{A_i}\backslash\emptyset)$ for any player $i \in N$.

[8] A commitment structure is *direct* whenever $M_j = k(M) \subseteq A$, for any agent $j \in N$.
An intermediate form of commitment may require agreement only by those committed. Indeed a direct commitment structure is said to be *consensual*, if it is direct and if for any agent $i \in N$ it satisfies:

$$k_i(m) = \begin{cases} \mathsf{proj}_i m_i & \text{if} \quad m_i = m_j \text{ for } \forall j \in N(m_i) \\ \Delta_i & \text{if} \quad \text{otw} \end{cases}$$

for $N(m_i) = \left\{j \in N : \mathsf{proj}_j m_i \neq A_j\right\}$. Consensual commitment structure is voluntary, but without veto.

3.3 Envoy Interpretation of the Model

To interpret such mechanism consider an environment in which all players have an envoy they may send to offer a contracts. When the envoys meet they will sign a contract only if all offer corresponds exactly to what they have been told to sign. In this context contracts may differ even when they lead to the same restrictions on future behavior, since $k(m) = k(m')$ for $m \neq m'$ is possible. Such differences may be interpreted as the formulation of the contract or the location where signing will take place. The contracts offered by the envoys if signed require all players to rule out some courses of action, either randomly or deterministically.

4 Subgame Perfect Equilibria of a Commitment Game

All propositions and remarks in the following sections should be completed with the, systematically, omitted quantifier: "For any complete information strategic form game...". Also, all sets, should be taken to depend on the original game. First equilibrium behavior in the committed subgames is discussed, then the commitment supergame is analyzed.

4.1 Equilibria of the Committed Subgame

Consider a subgame in which actions are restricted to a non-empty subset $A'_i \subseteq A_i$ for any player $i \in N$. The perfect monitoring assumption requires any player to know all commitments made by all others. Let $\langle u, A' \rangle = \left\{N, \{A'_i, u_i\}_N\right\}$ denote such subgame. The set of Nash equilibrium strategies and payoffs for such a subgame are, respectively, defined by:

$$\begin{aligned} \mathcal{E}\left\langle u, A'\right\rangle &= \left\{\sigma \in \Delta_N(A') \left| u_i(a_i, \sigma_{-i}) - u_i(\sigma) \leq 0 \text{ for } \forall a_i \in A'_i \ \& \ i \in N \right.\right\} \\ \mathcal{N}\left\langle u, A'\right\rangle &= \left\{u(\sigma) \in \mathcal{U} \left| \sigma \in \mathcal{E}\left\langle u, A'\right\rangle \right.\right\} \end{aligned}$$

Since any such subgame is a strategic form game Nash equilibria exist. More interestingly, for any two commitment subgames that can be ranked by inclusion, if an equilibrium strategy of the larger subgame is not ruled by the restrictions in the smaller subgame, then that strategy profile must, also, be an equilibrium of the smaller subgame. Formally:

Remark 1 *For any $A' \subseteq A'' \subseteq A$, if $\sigma \in \mathcal{E}\langle u, A''\rangle$ and $\sigma \in \Delta(A')$, then $\sigma \in \mathcal{E}\langle u, A'\rangle$.*

In fact, whenever an equilibrium strategy of the remains available, conditional on the other players behaving according to it, it is, still, optimal for any agent to follow such strategy, because the set of possible deviations from it is smaller than in the larger subgame. Therefore, any equilibrium of the original game remains an equilibrium of any subgame in which the corresponding strategy is not ruled out by the choice of commitments.

4.2 Equilibria of the Commitment Supergame

4.2.1 Deterministic Commitment

For any commitment structure $(k, M) \in \mathrm{K}$, denoted only by k in this section, define the set of feasible maps from message profiles to Nash continuation payoffs of the committed subgame by:

$$V(k) = \left\{v \in \mathcal{U}^M | v(m) \in \mathcal{N}\langle u, k(m)\rangle \text{ for } \forall m \in M\right\}$$

Implicit in such definition of continuation values is the ability of agents to condition their course of action not only upon the commitments realized, but also upon messages sent by other players at the contracting stage. Therefore, equilibrium action profiles may differ in the same restricted action domain $k(m) = k(m')$, whenever such commitments originate from different message profiles, $m \neq m'$. This is a direct consequence of the perfect monitoring assumption discussed and invoked previously. Weaker monitoring structures limiting the punishment ability of the players could reduce the set of implementable payoffs.

For any feasible profile of continuation payoffs, v, denote the simultaneous move commitment supergame by: $\langle v, M\rangle = \{N, \{M_i, v_i\}_N\}$. Let $\mathcal{K}(k)$ denote the set of subgame perfect equilibrium payoffs of a game extended by means of commitment structure k. Such set comprises all Nash payoff profiles of any the feasible game $\langle v, M\rangle$:

$$\mathcal{K}(k) = \cup_{v\in V(k)} \mathcal{N}\langle v, M\rangle$$

Behavioral strategy profiles of a commitment game are denoted by (π, σ) and satisfy $\pi \in \Delta_N(M)$ and $\sigma(m) \in k(m)$, for any $m \in M$. Occasionally, a continuation maps in $V(k)$ is denoted by $u(\cdot|\sigma) = u(\sigma(\cdot))$, for $\sigma(m) \in \mathcal{E}\langle u, k(m)\rangle$ the behavioral strategy following a message profile $m \in M$.[9]

4.2.2 Stochastic Commitment

The definition of stochastic commitment structure had any profile of messages $m \in M$ correspond to a lottery $k(m)$ on set of feasible action profiles $\times_{i\in N}[2^{A_i}\backslash\emptyset]$. Therefore, for any $k \in \mathrm{S}$, the set of feasible continuation payoffs would be defined by:[10]

$$V(k) = \left\{v \in \mathcal{U}^M | v(m) = w(k(m)|m) \text{ for } w(A'|m) \in \mathcal{N}\left\langle u, A'\right\rangle \text{ for } \forall A' \subseteq A, \text{ for } \forall m \in M\right\}$$

In this context, the definition of equilibrium payoff hull for the commitment supergame, $\mathcal{K}(k)$, remains unchanged with respect to the deterministic case once the appropriate definition of continuation values is established.

5 Implementation with a Single Round of Commitment

This section characterizes payoffs implemented when a unanimous agreement is required to commit. If deterministic, such contracts generally lead to efficiency gains, but not necessarily Pareto efficiency. Indeed, in such contracting environment random pledges may be needed to implement any efficient allocation.

First, properties of unanimous structures are investigated, then the set payoffs implemented by the entire class U are characterized and, finally, efficiency is discussed. All proofs of the claims to follow only make use of unanimous structures that are in canonical form.

5.1 Deterministic Commitment

The initial remarks outline elementary properties of the payoff sets that can be implemented by unanimous commitment structures. First, notice that any payoff that exceeds the worst Nash threat for all players and that belongs to equilibrium set of some committed subgame can be supported as an equilibrium of an appropriately chosen commitment structure. That is, for $\mathcal{K}_{\mathrm{U}} = \cup_{k\in \mathrm{U}} \mathcal{K}(k)$ and $\mathcal{V} = \cup_{A' \in \times_{i\in N}[2^{A_i}\backslash\emptyset]} \mathcal{N}\langle u, A'\rangle$:

[9] Thus, for any behavioral strategy (π, σ):

$$u_i(\pi|\sigma) = \sum_M \pi(m) u_i(m|\sigma) = \sum_M \pi(m) \sum_A \sigma(a|m) u_i(a) = \sum_A u_i(a) \sum_M \sigma(a|m)\pi(m)$$

[10] For $w(k(m)|m) = \sum_{A'\subseteq A} k(A'|m) w(A'|m)$.

Remark 2 *For any $u \in \mathcal{V} \cap \underline{\mathcal{N}}$ there exists a unanimous commitment structure, $k \in \mathrm{U}$, such that $u \in \mathcal{K}(k)$. Hence, $\mathcal{K}_{\mathrm{U}} \supseteq \mathcal{V} \cap \underline{\mathcal{N}}$.*

Because any player would have a weak incentive to commit to such a payoff, whenever all the others agreed to it, if threatened by his worst equilibrium outcome. As a consequence, all Nash equilibria of the original game belong to the set of unanimous commitment payoffs since any Nash strategy yields a payoff in $\underline{\mathcal{N}}$. That is, $\mathcal{K}_{\mathrm{U}} \supseteq \mathcal{V} \cap \underline{\mathcal{N}} \supseteq \mathcal{N}$. Also, observe that unanimous pledges never implement a payoff outside the Nash rational hull, because some player would have an incentive to veto such pledge:

Remark 3 *For any unanimous commitment structure, $k \in \mathrm{U}$, and any $u \in \mathcal{K}(k)$ it must be that $u \in \underline{\mathcal{N}}$. Hence, $\underline{\mathcal{N}} \supseteq \mathcal{K}_{\mathrm{U}}$.*

The third observation points out that an allocation implemented in pure strategies by unanimous pledges if and only if it belongs to the hull $\mathcal{V} \cap \underline{\mathcal{N}}$.

Remark 4 *There exists $k \in \mathrm{U}$ and $v \in V(k)$ such that $\pi(m) = 1$ for some $m \in M$ & $v(\pi) = u \in \mathcal{K}(k)$ if and only if $u \in \mathcal{V} \cap \underline{\mathcal{N}}$.*

The last and more useful remark states that enlarging the messaging space of a given unanimous commitment structure cannot reduce the set of payoffs that can be attained as subgame perfect equilibria of a unanimous commitment extension. More precisely:

Remark 5 *For $(M, k), (\overline{M}, \overline{k}) \in \mathrm{U}$, if $\overline{M} \supseteq M$ and if $k(m) = \overline{k}(m)$ for any $m \in M$, then $\mathcal{K}(\overline{k}) \supseteq \mathcal{K}(k)$.*

This observation relies on the cooperation that unanimity requires for players to commit. The remark, significantly, simplifies the analysis of any unanimous structure.

Before deriving tight bounds on the set of implementable allocation, notice that perfect monitoring can be exploited by players to correlate their strategies. Indeed, any payoff in the convex hull of the Nash equilibria of the original game can be approximately implemented by some unanimous commitment structure. Formally, denoting by $\mathsf{co}(\cdot)$ the convex hull operator, the following holds true:

Proposition 6 *For any $u \in \mathsf{co}(\mathcal{N})$ and any $\varepsilon > 0$ there exists a $k \in \mathrm{U}$ and a $\widehat{u} \in \mathcal{K}(k)$ such that $\|u - \widehat{u}\| < \varepsilon$. Moreover, it is possible to find one such k that never restricts any player's strategy.*

This proposition asserts that $\mathsf{co}(\mathcal{N}) \subseteq \mathsf{cl}(\mathcal{K}_{\mathrm{U}})$, for $\mathsf{cl}(\cdot)$ the closure of a set. Therefore, a unanimous commitment mechanism can be found that mimics any direct communication. Its is possible to do so, because the perfect monitoring assumption allows players to condition their actions upon the messages previously sent and publicly observed. The proposition, also, asserts that any such payoff can be implemented by a structure in which commitments never materialize in equilibrium. As discussed later, this result makes the analysis of structures in which players have several rounds in which to commit more tractable.

The next proposition provides an upper-bound on the equilibrium hull for games that possess a unique Nash equilibrium outcome.

Proposition 7 *If $|\mathcal{N}| = 1$, then $\mathcal{K}_{\mathrm{U}} \subseteq \overline{\mathcal{V} \cap \underline{\mathcal{N}}} \cap \mathsf{co}(\mathcal{V} \cap \underline{\mathcal{N}})$.*

For $\overline{\mathcal{V}\cap\underline{\mathcal{N}}} = \{u \in \underline{\mathcal{N}} | \exists u' \in \mathcal{V}\cap\underline{\mathcal{N}} : u' \geq u\}$ denoting the set of payoffs exceeding the Nash threat, but dominated by some element in $\mathcal{V}\cap\underline{\mathcal{N}}$. This proposition identifies an upper-bound for the payoffs attainable without communication. The constructed bound is contained in the convex hull of pure Nash equilibria of the commitment extension that is $\mathsf{co}(\mathcal{V}\cap\underline{\mathcal{N}})$. Indeed, the smallest convex set containing the Nash payoff set will, by (1), be $\mathsf{co}(\mathcal{V}\cap\underline{\mathcal{N}})$. Such upper-bound in (2) is not, in general, convex. The most interesting consequence of part (2) of the claim is that all allocations that are efficient within the commitment payoff set, $\mathcal{K}_{\mathrm{U}}$, always, belong to the set $\mathcal{V}\cap\underline{\mathcal{N}}$. Hence, for games with a unique equilibrium payoff, when no communication is permitted, the pure strategy equilibria of the messaging stage suffice in implementing all efficient allocations within $\mathcal{K}_{\mathrm{U}}$. Another consequence of the claim is that, in such games, unanimous commitments do not, in general, lead to Pareto efficiency. Examples, in section 7, show that not all payoffs in $\overline{\mathcal{V}\cap\underline{\mathcal{N}}}$ are necessarily implemented and that any tighter characterization depends on the specific incentive structure of the original game.

The next proposition provides an upper-bound on the set of commitment equilibria for the general case. Let $\mathcal{U}_{\mathcal{V}} = \mathsf{cc}(\mathcal{V}, \mathsf{co}(\mathcal{N}))$, for $\mathsf{cc}(\cdot,\cdot)$ the convex combination operator[11] and $\underline{\mathcal{N}}_i = \{u \in \mathcal{U} | u_i \geq \min_{\mathcal{N}} u_i\}$, then it must be that:

Proposition 8 *$\mathcal{K}_{\mathrm{U}} \subseteq \cap_{i\in N}\mathsf{co}(\underline{\mathcal{N}}_i \cap \mathcal{U}_{\mathcal{V}})$.*

Notice that such upper-bound set is convex, because any intersection of convex sets is itself convex. Also, such bound is weakly bigger than any bound in the preceding propositions, since:

$$\mathsf{co}(\mathcal{V}\cap\underline{\mathcal{N}}) \subseteq \mathsf{co}(\mathcal{U}_{\mathcal{V}}\cap\underline{\mathcal{N}}) \subseteq \cap_{i\in N}\mathsf{co}(\mathcal{U}_{\mathcal{V}}\cap\underline{\mathcal{N}}_i)$$

Though it may, still, be that $[\cap_{i\in N}\mathsf{co}(\mathcal{U}_{\mathcal{V}}\cap\underline{\mathcal{N}}_i)] \cap \mathcal{P} = \emptyset$, in which case efficiency could never attained. Therefore, this bound provides a necessary, but not sufficient, condition to test efficiency, both with and without communication. The proof of this claim depends almost entirely on individual rationality and does not make use of all equilibrium implications. Indeed, the fact that any player must be indifferent amongst all messages sent with positive probability was never used in the proof.[12] This observation can potentially tighten the result. Currently, the paper refrains from doing so, because gains derived from such tightening are limited, while characterization of the bounding set becomes more complex.

5.1.1 Efficiency

Having the ability to jointly commit may often leads to mutually beneficial agreements amongst players. Indeed any allocation improving on the worst Nash outcome of each player will be implementable, since $\mathcal{A}\cap\underline{\mathcal{N}} \subseteq \mathcal{K}_{\mathrm{U}}$. If the game were convex and compact any efficient allocation would therefore be implementable with a single round of contracting since $\mathcal{P} \subseteq \mathcal{A}$.

Efficient allocations, however, are not necessarily implementable whenever the game is not convex. Indeed, it is possible to show that whenever no efficient payoff belongs to the set $\mathcal{V}\cap\underline{\mathcal{N}} \subseteq \mathcal{I}\cap\underline{\mathcal{N}}$, then no efficient allocation will ever belong to the commitment equilibrium payoff hull. Specifically:

Proposition 9 *If $\mathcal{V}\cap\underline{\mathcal{N}}\cap\mathcal{P} = \emptyset$, then $\mathcal{K}_{\mathrm{U}}\cap\mathcal{P} = \emptyset$.*

[11] For any two sets, X & Y, denote $\mathsf{cc}(X,Y) = \{\alpha x + (1-\alpha)y | x \in X,\ y \in Y\ \&\ \alpha \in [0,1]\}$.

[12] For any two messages sent with positive probability $m', m'' \in \{m|\pi_i(m) > 0\}$ by player $i \in N$:

$$u_i(\pi|\sigma) = u_i(m', \pi_{-i}|\sigma) = u_i(m'', \pi_{-i}|\sigma)$$

Indeed, this is the case for the non-convex prisoner's dilemma that will be discussed in the examples section. But, even in games meeting such assumption, hostile to coordination, unanimous pledges generally achieve efficiency gains on the equilibrium set of the original game. The next sections show how either the ability to enforce random contracts or the ability to commit at several rounds can always lead to efficiency.

5.2 Stochastic Commitment

When such agreements can be used agents are allowed to jointly delegate their choice of commitment to a third party who is able to forcefully impose the choice of commitment conditional on the realization of some lottery. Such powerful technology helps agents to discipline their incentives problems ultimately leading to Pareto efficiency. In fact, because any payoff in $\mathcal{A}$ may attained as a payoff of a subgame for some fully restrictive profile of commitments, it must be that any payoff in $\mathcal{U}$ be attained by some lottery on such commitment profiles. Therefore any rationalizable allocation will be implementable, because all players would agree on any one of them if threatened by their worst possible outcome. Consequently, the following holds true, for $\mathcal{K}_{\mathrm{U}}^{\mathrm{S}} = \cup_{k \in \mathrm{U} \cap \mathrm{S}} \mathcal{K}(k)$:

Proposition 10 $\mathcal{K}_{\mathrm{U}}^{\mathrm{S}} = \underline{\mathcal{N}}$.

Even though it may appear that random contracts are necessary to implement the entire Nash rational hull, the rest of analysis shows that this is not he case. Indeed, if players were given several rounds in which to sign pure deterministic contracts, there would be no need for randomness in contracts in order to implement the aforementioned payoff hull. Though outcomes in the two scenarios may coincide, contracts signed in any given equilibrium could significantly differ. In fact, increasing the number of rounds will make up for reducing the number of contracts.

5.3 Envoy Interpretation

Again consider the envoy interpretation of the model. If restrictions on the feasible set of contracts require envoys make only deterministic offers, efficient agreements may never be signed. Indeed, to little time would be given to the envoys in order to reach such mutually beneficial agreement given the limited set of contracts. If instead also random contracts can be offered and enforced, any allocation exceeding the worst Nash threat corresponds to one such contract and can be implemented.

6 Implementation with Multiple Rounds of Commitment

Up to this point players were given a single instance in which to reach an agreement upon which contract to sign. This section relaxes this assumption by giving to the players several occasions in which to rule out some of their actions. It is shown that under weak dimensionality conditions on the original game relaxing such assumption guarantees that each Nash rational allocation can be implemented by some commitment extension. The mechanisms used at each round will have players unanimously agree on which actions to rule out and will thus entail no public randomization. The multiple rounds are exploited by the players to make concessions that enable them to sustain mutually beneficial agreements.

A commitment structure with $t+1$ rounds consists of a sequence of maps $\{k_0, ..., k_t\} = k^t$ that at any instance $s \in \{0, ..., t\}$ and at information stage $m^{s-1} \in M^{s-1}$ for any player $i \in N$ satisfies:

$$k_{i,s}(m^{s-1}) : M_s \to 2^{k_{i,s-1}(m^{s-1})} \backslash \emptyset$$

Such maps provide each player for given choice set $k_{s-1}(m^{s-1})$ and for any message profile sent m_s a new choice set for the latter round commitment round $k_s(m^s)$. By construction any action that is ever ruled out can never be ruled in again. The convention used requires that $k_{-1}(m^{-1}) = A$.

Simple multi-round commitment structures may have players only commit to actions in their choice sets :

$$k_s(m^s) \in A \cup \{A\}$$

In such scenario if the commitment maps are unanimous at each rounds whenever an agreement is found on a pure strategy profile of the original game, pledges end since all players are committed to a single action. If instead no agreement is found, $k_s(m^s) = A$, players can again try to commit at the next round, $s-1$.

Denote the class of commitment mechanisms with $t+1$ rounds that are unanimous at each round by U^t. Equilibrium continuation values are then defined for $s < t$ by:

$$\begin{aligned} V_s(m^{s-1}|k^t) &= \left\{ v \in \mathcal{U}^{M_s} | v(m_s) \in \mathcal{N} \left\langle \overline{v}, M_{s+1} \right\rangle \text{ for } \overline{v} \in V_{s+1}(m^s|k^t) \text{ and } \forall m_s \in M_s \right\} \\ V_t(m^{t-1}|k^t) &= \left\{ v \in \mathcal{U}^{M_t} | v(m_t) \in \mathcal{N} \left\langle u, k_t(m^t) \right\rangle \text{ for } \forall m_t \in M_t \right\} \end{aligned}$$

Also, let $\mathcal{K}(k^t) = \cup_{v \in V_0(k^t)} \mathcal{N} \langle v, M_0 \rangle$ denote the equilibrium payoff hull of commitment structure k^t and let $\mathcal{K}^t_{\mathrm{U}} = \cup_{k^t \in \mathrm{U}^t} \mathcal{K}(k^t)$. Before proceeding to the main implementation result, a four propositions on the evolution of the equilibrium correspondence are presented. The first states that increasing the message spaces does not reduce the equilibrium payoff hull if the commitment structure is unanimous at each round. The generalization of remark 5 thus becomes:

Proposition 11 *For $\left(M^t, k^t\right), (\overline{M}^t, \overline{k}^t) \in \mathrm{U}^t$, if $\overline{M}^t \supseteq M^t$ and if $k_s(m^s) = \overline{k}_s(m^s)$ for any $m^s \in M^s$ and $s \le t$, then $\mathcal{K}(\overline{k}^t) \supseteq \mathcal{K}(k^t)$.*

The second claims that the set of equilibrium payoff hull weakly increases with the number of rounds and always belongs to the Nash rational hull.

Proposition 12 *$\underline{\mathcal{N}} \supseteq \mathcal{K}^{t+1}_{\mathrm{U}} \supseteq \mathcal{K}^t_{\mathrm{U}} \supseteq \mathcal{V} \cap \underline{\mathcal{N}}$, for any $t \ge 0$.*

The third states that players can still exploit the commitment extension in order to communicate at each round.

Proposition 13 *$\mathsf{cl}(\mathcal{K}^{t+1}_{\mathrm{U}}) \supseteq \mathsf{co}(\mathcal{K}^t_{\mathrm{U}})$, for any $t \ge 0$.*

The last claim provides a technique to enlarge the set of implementable payoffs at each round of commitment. On this proposition hinges much of the proof of implementation result.

Proposition 14 *For $u_t \in \mathcal{K}^t_{\mathrm{U}}$, $u \in \mathcal{A}$ and $q \in \mathbb{N}_+$, if $(1/q^{N-1})u + (1 - 1/q^{N-1})u_t \in \underline{\mathcal{N}}$ then:*

$$(1/q^{N-1})u + (1 - 1/q^{N-1})u_t \in \mathcal{K}^{t+1}_{\mathrm{U}}$$

The final and central result provides conditions for which the use of multiple round unanimous structures suffices for the implementation of the Nash rational payoff hull. Contracts required for such claim are very simple and involve at each layer only commitments to pure strategies of the original game. For $\underline{N} = \{i \in N | \exists u \in \underline{\mathcal{N}} \text{ s.t. } u_i > \underline{u}_i^i\}$, $\mathsf{aff}(\cdot)$ the affine hull operator and $\mathsf{vert}(\cdot)$ the vertices of a polytope, the statement of the claim is as follows:[13]

Theorem 15 *If* $\mathsf{vert}(\mathcal{U} \cap \mathsf{aff}(\underline{\mathcal{N}})) \subseteq \mathcal{V}$ *and if there exist* $u \in \mathsf{co}(\mathcal{V} \cap \underline{\mathcal{N}})$, $u' \in \mathcal{V}$, $\alpha \in [0,1]$ *and* $j \in \underline{N}$ *such that:*

$$\begin{aligned} \alpha u_i' + (1-\alpha)u_i &> \underline{u}_i^i \quad \textit{for} \quad \forall i \in \underline{N} \backslash j \\ \alpha u_i' + (1-\alpha)u_i &\geq \underline{u}_i^i \quad \textit{for} \quad \forall i \in N \backslash (\underline{N} \backslash j) \end{aligned}$$

then for $C = \lim_{t\to\infty} \mathsf{cl}(\mathcal{K}_{\mathrm{U}}^t)$, *it must be that* $\mathsf{cl}(C) = \underline{\mathcal{N}}$.

The first is a mere dimensionality assumption. It guarantees that the Nash rational payoff hull can be spanned elements in $\mathcal{V}$, which includes all payoffs in $\mathcal{A}$. Such assumption is without loss when there are only two players, because $\mathcal{U} \cap \mathsf{aff}(\underline{\mathcal{N}})$ is either $\mathcal{U}$ or $\mathcal{N}$ or a facet of $\mathcal{U}$. When there are more than two players, even though the assumption holds in great generality, cases in which it does not can be constructed. The second assumption, instead, guarantees that it be possible to implement at some round a payoff in the relative interior of the Nash rational hull. This assumption is needed for the implementation result, because it makes sure that at some layer there be equilibria in which all relevant players, $\underline{N}$, receive a payoff above the Nash threat. This condition is in the spirit similar to the multiplicity assumption required to implement payoffs in a finitely repeated game in [3]. An example in which the latter assumption fails in a two player game will be reported in the next section.

In the proof of this result, all binding contracts signed at any stage by the players with strictly positive probability entail a full commitment to a strategy profile in A that can never be renegotiated. Thus, empirical counterpart of contracting simply results in a pure commitment contract being signed sometime before the game plays out.

In such a contracting environments joint randomizations on action profiles are realized by voluntary probabilistic concessions to unfavorable contracts at earlier stages in order to encourage coordination on favorable contracts at latter stages if an agreement does not materialize. In fact, even contracts that yield payoffs lower than the Nash threat for some player may occasionally be observed, because of beliefs about future concessions. Thus, the theorem, to the extent of its assumptions, makes the point that even in non-convex games there is non need to observe stochastic contracts for an efficient allocation to be implemented. The model implicitly generates a non-cooperative theory for how players bargaining upon deterministic contracts can implement efficient outcomes.

The generality of the second assumption on the original game depends on the class games to which the original game belongs. The implementation is easier for continuous compact games. The web-appendix discusses such matters in detail. The next section presents examples that clarify the dynamics underlying the proof of the implementation theorem.

6.1 Envoy Interpretation

In the envoy interpretation of the model a commitment structure with several rounds corresponds to players having the ability to repeatedly send envoys offering contracts to be signed.

[13] For $x, y \in \mathbb{R}^n$ let $L(x,y) = \{(1-\lambda)x + \lambda y | \lambda \in \mathbb{R}\}$, the *line*. Then, a set H is a *flat* if $x, y \in H$ implies $L(x,y) \subseteq H$. The *affine hull* of X, denoted $\mathsf{aff}(X)$, consists of the intersection of all flats that contain X.

If X is a convex polytope, then $x \in \mathsf{vert}(X)$ if $y, z \in X$, $\lambda \in (0,1)$ and $x = (1-\lambda)z + \lambda y$ implies $x = y = z$.

Even when only deterministic contracts can be enforced, the possibility of sending the envoys at several occasions allows players to reach efficient agreements. Indeed, players may now engage in occasional concessions, because they believe that such concessions will be rewarded if no agreement is reached and the concession is observed. Such bargaining on which contracts to sign may result in an efficient without any randomization on the commitments to be undertaken. The UN security council may be taken as an empirical counterpart of such model in that offers are made repeatedly by envoys and unanimity is required for offers to be enforced.

7 Two Players Examples

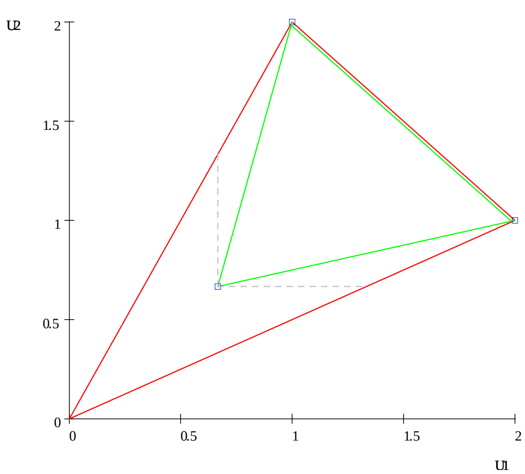
For graphical clarity, this section focuses on two player games. The first subsection presents examples in which the implementation result holds. Such examples should clarify how several rounds of commitment can generate mutually beneficial concessions amongst players. The second subsection instead provides an example in which the implementation theorem fails.

7.1 When the Implementation Result Holds

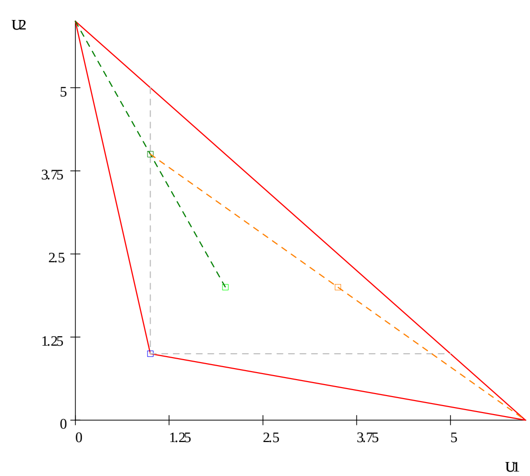
A typical example of a game which requires no pledges to be made in equilibrium for efficiency to be attained with a single round of commitment is the battle of the sexes:

$1\backslash 2$	n	c
n	$2, 1$	$0, 0$
c	$0, 0$	$1, 2$

Indeed, no restriction is needed to implement any efficient allocation because, by proposition 6, any payoff in the convex hull of Nash equilibria of the original game can be arbitrarily closely approximated by an equilibrium of a unanimous structure, $\mathsf{co}(\mathcal{N}) \subseteq \mathsf{cl}(\mathcal{K}^0_{\mathrm{U}})$. In any such equilibrium, the commitment structure is used as a direct communication device. Also, $\mathsf{cl}(\mathcal{K}^1_{\mathrm{U}}) = \underline{\mathcal{N}}$, since two stages may be needed to approximately implement some payoffs in $\underline{\mathcal{N}} \backslash \mathsf{co}(\mathcal{N})$. The commitment equilibrium set is depicted in figure 2A.



2A: In red $\mathcal{U}$, in blue $\mathcal{N}$, in gray $\underline{\mathcal{N}}$ & in green a subset of $\mathcal{K}^0_{\mathrm{U}}$



2B: In blue $\mathcal{N}$, light green, dark green, orange an element of $\mathcal{K}^0_{\mathrm{U}}$, $\mathcal{K}^1_{\mathrm{U}}$, $\mathcal{K}^2_{\mathrm{U}}$

The second example of this section requires infinitely many contracting stages to implement any efficient Nash rational payoff, while the third and final only requires two stages. Consider a prisoners' dilemma type game, in which the hull of feasible payoffs, $\mathcal{U}$, cannot be spanned by independent randomizations, $\mathcal{I} \subsetneq \mathcal{U}$. Indeed, consider the game:

$1\backslash 2$	n	c
n	$1,1$	$6,0$
c	$0,6$	$2,2$

In such context, any efficient outcome corresponds to a specific randomization on action profiles (n,c) and (c,n). The payoff hulls for this game is depicted in figure 2B. In every committed subgame of this game each player does not cooperate unless he has committed to do so, because cooperation remains strictly dominated. The set $\mathcal{V}\cap\underline{\mathcal{N}}$ for this game consists of the Nash equilibrium $(1,1)$ and of the allocation $(2,2)$ obtained in the subgame in which both players commit to cooperation. In fact these are the only two possible pure strategy equilibria of the commitment extension. No deterministic single stage commitment, if accepted, can ever yield to any player a payoff grater than 2. Indeed, by proposition 7 it follows that:

$$\mathcal{K}_{\text{U}} \subseteq \mathsf{co}(\mathcal{V}\cap\underline{\mathcal{N}}) = \mathsf{co}((1,1),(2,2))$$

This game, however, satisfies the assumptions of the implementation theorem with multiple rounds of commitment. Indeed, if players can repeatedly commit efficiency gains can be attained with each additional stage of contracting. Potential gains, though, will diminish as the number of stages increases. In fact, at any stage t it is possible to implement payoffs, $u_t \in \mathcal{K}^t_{\text{U}}$, that are no further than $1/2^{t-1}$ utils from the closest efficient payoff, $\|u_t,\mathcal{P}\|_+ \leq 1/2^{t-1}$. The following sequence of games $\{\langle v_s, M_s\rangle\}_{s=0}^{\infty}$ for continuation payoffs $v_s \in V(k^s)$ of a canonical form unanimous structure can be used to verify the validity of the claim:

$t=1$	m	m'	...
m	$0,6$	$2,2$	...
m'	$2,2$	$0,6$	...
...	....	...	...

$t=2$	m	m'	...
m	$6,0$	$1,4$	...
m'	$1,4$	$6,0$	...
...	....	...	...

$t=3$	m	m'	...
m	$6,0$	$3.5,2$	...
m'	$3.5,2$	$6,0$	...
...	....	...	...

$t=4$	m	m'	...
m	$0,6$	$4.75,1$	...
m'	$4.75,1$	$0,6$	...
...	....	...	...

$t=5$	m	m'	...
m	$0,6$	$2.375,3.5$	...
m'	$2.375,3.5$	$0,6$	...
...	....	...	...

...

These mechanisms are unanimous and pure because $(6,0),(0,6),(2,2) \in \mathcal{A} \subseteq \mathcal{V}$ and because for $u_t = v_{t+1}(m,m') \in \mathcal{K}(k^t)$, for any $t>0$.[14] The sequence $u_t = v_{t+1}(m,m')$ of equilibrium payoffs possesses the desired property, $\|u_t,\mathcal{P}\|_+ \leq 1/2^{t-1}$. The same technique may be used to produce: $u_6 = (1.1875, 4.75)$, $u_7 = (3.59375, 2.375)$ and so on. A similar argument together with remark 11 shows that even the symmetric payoff profiles, $\overline{u}_t = (u_{2,t}, u_{1,t})$, can belong to the equilibrium set of a multi-round commitment structure. Therefore by remark 13, the convex hull of such payoff profiles belongs to the closure of equilibrium payoff hull of the preceding commitment stage. That is, $\mathsf{co}((1,1),u_t,\overline{u}_t) \subset \mathsf{cl}(\mathcal{K}(k^{t+1}))$. Equilibrium payoff sets can, similarly, be expanded in any direction, not only the efficient one, so long as there are

[14] If any player randomizes uniformly on described messages, the other is indifferent amongst all such messages.

payoffs in that direction belonging to $\underline{\mathcal{N}}$. For instance, k^7 so that:

$t=7$	m	...	...	m'	...
m	$0,6$	$1.1875,4.75$	...	$1.1875,4.75$	...
...	$1.1875,4.75$	$0,6$	...	...	...
...	....	...	$0,6$	$1.1875,4.75$	...
m'	$1.1875,4.75$	...	$1.1875,4.75$	$0,6$	...
...	...	...	...	...	...

For an appropriate number of such messages, call it $n \in \mathbb{N}$, the following allocation can, also, be implemented: $([1.1875 - 1.1875/n], [4.75 + 1.25/n]) \in \mathcal{K}(k^7)$. Indeed, for $n = 7$, payoff profile $(1.018, 4.929) \in \mathcal{K}(k^7)$.

The last two observations applied iteratively, guarantee approximate implementation of any payoff profile belonging to the Nash rational hull in a finite number of stages. Efficiency gains at each stage are driven by the fear of the other player abandoning any further negotiation at all latter stages, if no probabilistic concession to unfavorable contracts is ever made. Therefore, contracts that yield to some player a payoff below his min-max value could be observed with positive probability in equilibrium. Such phenomenon cannot be avoided, since all efficient payoff profiles of the game involve randomizations on action profiles that are dominated for one of the two players.

It is trivial to verify that had the agents the ability to stochastically delegate their choice of commitment, any Nash rational allocation could be implemented with a single stage commitment extension. Specifically, any individually rational efficient allocation could be implemented by a stochastic commitment: $p(6,0) \oplus (1-p)(0,6)$ for $p \in [1/6, 5/6]$.

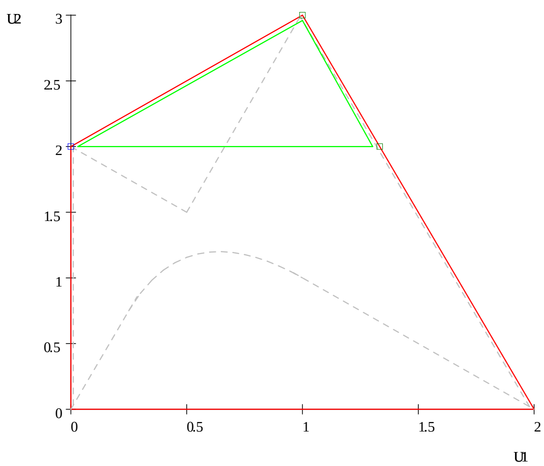
The next example, depicted in figure 3A, consists of a game in which the set $\mathcal{I} \cap \underline{\mathcal{N}}$ is not connected and the implementation of the Nash rational hull occurs in a finite number of stages.

$1\backslash 2$	n	c
n	$0,2$	$2,0$
c	$0,0$	$1,3$

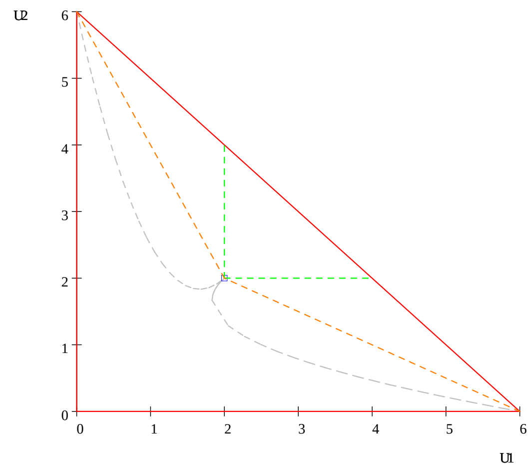
The unique Nash equilibrium of this game has both players not cooperating and receiving a payoff of $(0, 2)$. Additionally, a joint commitment to cooperate is mutually beneficial and therefore $(1, 3) \in \mathcal{V} \cap \underline{\mathcal{N}} \subseteq \mathcal{K}^0_{\mathrm{U}}$. With an additional round of contracting however it is possible to implement the efficient allocation most favorable to player one, $(4/3, 2)$. Indeed, such allocation would be obtained as an equilibrium of a game in which both players coordinate on whether to sign a contract favorable to player one or to proceed to negotiations at the latter round. As in the game below:

$1\backslash 2$	m	...	m'	...
m	$2,0$	$1,3$	$1,3$	$0,2$
...	$1,3$	$2,0$	$1,3$	$0,2$
m'	$1,3$	$1,3$	$2,0$	$0,2$
...	$0,2$	$0,2$	$0,2$	...

Thus, since $(0, 2), (1, 3), (4/3, 2)$ all belong $\mathcal{K}^1_{\mathrm{U}}$ and because the convex hull of such allocations is the Nash rational hull it is possible to conclude that $\mathsf{cl}(\mathcal{K}^2_{\mathrm{U}}) = \underline{\mathcal{N}}$.



3A: in red $\mathcal{U}$, in gray $\mathcal{I}$, in blue $\mathcal{N}$ & in green $\underline{\mathcal{N}}$



3B: in red $\mathcal{U}$, in gray $\mathcal{I}$, in blue $\mathcal{K}_{\text{U}}$ & in green $\underline{\mathcal{N}}$

7.2 When the Implementation Result Fails

In the final example no deterministic commitment can ever enlarge the set of equilibrium payoffs independently of the number of contracting rounds permitted. Indeed, consider the game:

$1\backslash 2$	l	c	r
t	$2,2$	$1,1$	$1,1$
m	$1,1$	$0,0$	$6,0$
d	$1,1$	$0,6$	$0,0$

Such game possesses a unique Nash equilibrium outcome, in which the first agent plays t and the second l. Uniqueness can be shown by iterative elimination of strictly dominated strategies. This equilibrium lies on the boundary of the independent utility hull. Additionally, $\mathcal{N} = \mathcal{V} \cap \underline{\mathcal{N}}$ because any payoff $u \in \mathcal{V} \subseteq \mathcal{I}$ satisfies the following inequality for $i \neq j \in \{1, 2\}$:

$$u_i \leq u_j + 1.5(2 - u_j)^2$$

In this game, the upper-bound on the commitment equilibrium payoff set, derived in proposition 7, requires: $\mathcal{K}_{\text{U}} \subseteq \mathsf{co}(\mathcal{V} \cap \underline{\mathcal{N}}) = \mathcal{N}$. Therefore, a single round of commitment does not enlarge the set of equilibrium payoffs in such environment.

Increasing the number of rounds does not help, since any commitment vetoed with at the last round would also be vetoed at earlier rounds given that any disagreement can only lead to the unique Nash outcome. Indeed, because no player can ever stand to lose, there is no room for beneficial concessions when bargaining upon which deterministic contract to sign.

8 Conclusions

The analysis discussed how the availability of commitment devices can lead to the implementation of allocations that are not self-enforcing in the original game. It was shown that endowing players with the ability to unanimously pledge to rule out strategies did not necessarily lead to full cooperation. Indeed, even though efficiency gains were often obtained, without further assumptions on the original game, there would be no guarantee of ever implementing an efficient payoff profile. In non-convex games, conditions were found preventing any efficient allocation from ever being implemented through a single stage of contracting. It was also observed that perfect monitoring of pledges provided players with a communication channel and that the use of such channel did not require any pledge to be enforced.

The main contribution in the analysis of such devices was the approximate implementation of the Nash rational hull in extensions with multiple rounds of commitment. Indeed increasing the number of commitment rounds was shown to enlarge the set of implementable payoffs, because at any instance occasional concessions to unfavorable pledges could be exchanged with beliefs in a more favorable treatment at later instances, were an agreement not to be found. Therefore, in such cooperative commitment environment, bargaining on contracts can, for appropriately chosen beliefs, implement any allocation that a stochastic device would implement. When players made use of the multiple rounds of contracting, the model could be interpreted as generating a non-cooperative theory of bargaining on contracts involving at each round threats to dismiss any further negotiation and occasional concessions to unfavorable outcomes to sustain cooperation.

Thus, the empirical counterpart of such bargaining would merely result in the observation of some pure strategy contract being signed by all sometime before the game takes place. The commitments undertaken in this process and thus observed, cannot necessarily be implemented with a single round of contracting, since further concession may be required to support them in equilibrium.

The unanimity assumption does significantly increase the set of implementable payoffs, when compared to environment in which commitment are to be taken unilaterally, because such technology enhanced cooperation. A consensual technology, requiring agreement just amongst those committed, even though more flexible and, arguably, more empirically relevant, does not lead to efficiency gains on the unanimous case. The analysis of such devices is forthcoming.

Quite different dynamics arise if the original game is one with incomplete information. In such case, pledges do not only discipline incentives, but they also disclose information about the types of agents playing the game. In such scenario even unanimous pledges may not suffice in implementing efficient profiles.

References

[1] Bade S., Haeringer G. & Renou L., “Bilateral Commitment”, The Fondazione Eni Enrico Mattei Note di Lavoro Series, 2006.

[2] Bertsekas D. P., “Nonlinear Programming”, Athena Scientific, 2nd Edition, 1999.

[3] Benoit J. & Krishna V., “Finitely Repeated Games”, *Econometrica*, 1985, **53**, 905-922.

[4] Border K. C., “Fixed Point Theorems with Applications to Economics and Game Theory”, Cambridge University Press, 1985.

[5] Crawford V., “A Theory of Disagreement in Bargaining”, *Econometrica*, 1982, **50**, 607-637.

[6] van Damme E. & Hurkens S., “Commitment Robust Equilibria and Endogenous Timing”, *Games & Economic Behavior*, 1996, **15**, 290-311.

[7] Dasgupta P. & Maskin E., “The Existence of Equilibrium in Discontinuous Economic Games I: Theory”, *Review of Economic Studies*, 1986, **53**, 1-26.

[8] Ellingsen T. & Miettinen T., "Disagreement and Authority", Working Paper, 2007.

[9] Forges F., "An Approach to Communication Equilibrium", *Econometrica*, 1986, **54**, 1375-1385.

[10] Geanakoplos J., "Nash and Walras equilibrium via Brouwer", *Economic Theory*, 2003, **21**, 585-603.

[11] Grunbaum B., "Convex Polytopes", Springer, 2003.

[12] Hamilton J. H. & Slutsky S. M., "Endogenous Timing in Duopoly Games: Stackelberg or Cournot Equilibria", *Games & Economic Behavior*, 1990, **2**, 29-46.

[13] Hart O. & Moore J., "Agreeing Now to Agree Later: Contracts that Rule Out but do not Rule In", *American Law & Economics Association Annual Meeting*, 2005, **37**.

[14] Myerson R. B., "Game Theory – Analysis of Conflict", Harvard University Press, 1991.

[15] Nau R. & McCardle K., "Coherent Behavior in Noncooperative Games", *Journal of Economic Theory*, 1990, **50**, 424-444.

[16] Powell R. L., "War as a Commitment Problem", *Internatinal Organization*, 2006, **60**, 169-203.

[17] Renou L., "Commitment Games", *Working Paper*, 2006.

[18] Schelling T., "The Strategy of Conflict", Harvard University Press, 1960.

[19] Valentine F. A., "Convex Sets", McGraw Hill Series in Higher Mathematics, 1964.

9 Appendix

Recall that $\underline{u}^i \in \arg\min_{u \in \mathcal{N}} u_i$. Let $\sigma_N(a) = \prod_{i \in N} \sigma_i(a_i)$ denote the joint probability of action profile $a \in A$ when the players choose strategies $\sigma_i \in \Delta_i$.

Remark 1. *For any $A' \subseteq A'' \subseteq A$, if $\sigma \in \mathcal{E}\langle u, A''\rangle$ and $\sigma \in \Delta_N(A')$, then $\sigma \in \mathcal{E}\langle u, A'\rangle$.*

Proof. Indeed, for $\forall i \in N$, if $\sigma_i \in \arg\max_{\sigma_i' \in A_i''} u_i(\sigma_i', \sigma_{-i})$ it must be that $\sigma_i \in \arg\max_{\sigma_i' \in A_i'} u_i(\sigma_i', \sigma_{-i})$, since $A_i' \subseteq A_i''$. ■

9.1 Implementation with a Single Round

Remark 2. *For any $u \in \mathcal{V} \cap \underline{\mathcal{N}}$ there exists a unanimous commitment structure $k \in \mathrm{U}$ such that $u \in \mathcal{K}(k)$. Hence, $\mathcal{K}_{\mathrm{U}} \supseteq \mathcal{V} \cap \underline{\mathcal{N}}$.*

Proof. Consider $\overline{u} = u(\overline{\sigma}) \in \mathcal{V} \cap \underline{\mathcal{N}}$. Since $\overline{u} \in \mathcal{V}$, there exist $A_i' \subseteq A_i$ such that $\overline{u} \in \mathcal{N}\langle u, A'\rangle$. Then consider a canonical form unanimous commitment structure with messaging spaces $M_* = \{1, 2\}$ and such that

$$k(m) = \begin{cases} A' & \text{if} \quad m = (1, ..., 1) \\ A & \text{if} \quad \text{otw} \end{cases}$$

Such structure implements $\overline{u}$ as a pure strategy Nash equilibrium, so long as $\overline{u} \in \underline{\mathcal{N}}$. In fact, the following subgame perfect strategy implements the desired payoff $\overline{u}$ for any $m \in M$ and $i \in N$:

$$\begin{aligned} \pi_i(1) &= 1 \\ \sigma_i(m) &= \begin{cases} \overline{\sigma}_i & \text{if } \ m_j = 1 \text{ for } \forall j \in N \\ \in \arg\min_{\sigma' \in \mathcal{E}} u_k(\sigma') & \text{if } \ m_j = 1 \text{ for } \forall j \in N \backslash k \\ \in \mathcal{E} & \text{otherwise} \end{cases} \end{aligned}$$

Proving the desired result. ■

REMARK 3. *For any unanimous commitment structure $k \in \mathrm{U}$ and any $u \in \mathcal{K}(k)$ it must be that $u \in \underline{\mathcal{N}}$. Hence, $\underline{\mathcal{N}} \supseteq \mathcal{K}_{\mathrm{U}}$.*

Proof. Suppose, by contradiction that $\overline{u} \in \mathcal{K}(k)$ for some $k \in \mathrm{U}$ and that $\overline{u} \notin \underline{\mathcal{N}}$. By assumption $\exists i \in N$ such that $\underline{u}_i^i > \overline{u}_i$. But if this were the case, agent i would profit by vetoing any commitment. Indeed, denoting player i's veto message by message $\overline{m}_i \in M_i$, it must be that for any $v \in V(k)$ and $\pi_{-i} \in \times_{j \in N \backslash i} \Delta(M_j)$:[15]

$$v_i(\overline{m}_i, \pi_{-i}) \geq \underline{u}_i^i > \overline{u}_i$$

Because when not committed at the action stage subgame perfection requires $v(\overline{m}_i, m_{-i}) \in \mathsf{co}(\mathcal{N})$ which would contradict $\overline{u} \in \mathcal{K}(k)$. ■

REMARK 4. *There exists $k \in \mathrm{U}$ and $v \in V(k)$ such that $\pi(m) = 1$ for some $m \in M$ & $v(\pi) = u \in \mathcal{K}(k)$ if and only if $u \in \mathcal{V} \cap \underline{\mathcal{N}}$.*

Proof. In the proof of remark 2 it was shown that if $u \in \mathcal{V} \cap \underline{\mathcal{N}}$, then u can be implemented by $k \in \mathrm{U}$ *and* $v \in V(k)$ *such that* $\pi(m) = 1$ *for some* $m \in M$. Therefore only the converse needs to be shown. If $k \in \mathrm{U}$ *and* $v \in V(k)$ *such that* $\pi(m) = 1$ *for some* $m \in M$ *and* $v(\pi) = u \in \mathcal{K}(k)$, then:

$$v(\pi) = v(m) \in \mathcal{N} \langle u, k(m) \rangle \subseteq \mathcal{V}$$

Since by remark 3 $v(\pi) \in \underline{\mathcal{N}}$, claim holds true. ■

REMARK 5. *For $(M, k), (\overline{M}, \overline{k}) \in \mathrm{U}$, if $\overline{M} \supseteq M$ and if $k(m) = \overline{k}(m)$ for any $m \in M$, then* $\mathcal{K}(\overline{k}) \supseteq \mathcal{K}(k)$.

Proof. To show this first note that if $u \in \mathcal{K}(k)$ then there exists $v \in V(k)$ such that $u = v(\pi)$ and $v(\pi) \in \mathcal{N} \langle v, M \rangle$. Given such a map v construct a different continuation value $\overline{v}$ on $\mathcal{U}^{\overline{M}}$ such that:

$$\overline{v}(m) = \begin{cases} v(m) & \text{if } \ m \in M \\ \underline{u}^i & \text{if } \ m_i \notin M_i \ \ \& \ \ m_{-i} \in M_{-i} \\ ? & \text{if } \ \text{otherwise} \end{cases}$$

The so defined map $\overline{v}$ must belong to $V(\overline{k})$ since $\overline{v}(m) \in \mathcal{N} \langle u, k(m) \rangle$ for any $m \in M$ and $\overline{v}(m) \in \mathcal{N}$ when $m_i \notin M_i$ and $m_{-i} \in M_{-i}$ for some $i \in N$. Additionally, it must be that $\overline{v}(\pi) = v(\pi)$ and $\overline{v}(\pi) \in \mathcal{N} \langle \overline{v}, \overline{M} \rangle$, because all Nash requirements hold for any $i \in N$:

$$\overline{v}_i(\pi) \geq \overline{v}_i(m_i, \pi_{-i}) \ \text{ for } \forall m_i \in \overline{M}_i$$

Thus, $u \in \mathcal{K}(\overline{k})$ since, for so chosen continuation values, no benefit can ever be derived from deviating to one of the additional strategies. ■

[15] Recall that $\overline{m}_i$ is such that $k(\overline{m}_i, m_{-i}) = A$ for any $m_{-i} \in M_{-i}$.

Proposition 6. *For any* $u \in \mathsf{co}(\mathcal{N})$ *and any* $\varepsilon > 0$ *there exists a* $k \in \mathrm{U}$ *and a* $\widehat{u} \in \mathcal{K}(k)$ *such that* $\|u - \widehat{u}\| < \varepsilon$. Moreover, it is possible to find one such k that never restricts any player's strategy.

Proof. By Caratheodory's theorem (reported below) it is known that, for any $u \in \mathsf{co}(\mathcal{N})$, there exists a $\mathcal{N}' \subseteq \mathcal{N}$ with $|\mathcal{N}'| \leq N + 1$ such that one may express:

$$u = \textstyle\sum_{\mathcal{N}'} u'\theta(u')$$

For some probability measure θ on a subset of Nash equilibrium strategy profiles, $\mathcal{N}'$. The prove intends to construct a unanimous commitment structure approximating any such payoff as a Nash equilibrium.

For any canonical form unanimous commitment structure (M_*, k) consider the set $\Theta_{M_*} = \left\{t \in \mathbb{N}^{|\mathcal{N}'|} | \sum_{\mathcal{N}'} t(u')/\, |M_*| = 1\right\}$ of probability distributions on the Nash outcomes. Let θ_{M_*} denote the closest element (in the sup-norm) to θ in this set:

$$\theta_{M_*} = \arg\min_{t \in \Theta_{M_*}} \|t - \theta\|_+$$

For any order on the Nash payoffs, define:

$$u[\theta_{M_*}] \equiv [\underbrace{u', ..., u'}_{|M_*|\,\theta_{M_*}(u')\text{-times},}, ..., \underbrace{u'', ..., u''}_{.....}, ..., \underbrace{u''', ..., u'''}_{|M_*|\,\theta_{M_*}(u''')\text{-times}}]$$

Then, construct a canonical commitment structure, k, in which continuation values $u(\cdot|\sigma) \in V(k)$ can satisfy:

$$\begin{aligned} u(m|\sigma) &= u(m_{1,2}, m'_{-1,2}|\sigma) \quad \text{for any } m \in M \text{ and } m'_{-1,2} \in M_{-1,2} \\ u(\cdot, m_{-i}|\sigma) &= \mathsf{per}[u[\theta_{M_*}]] \quad \text{for any } m_{-i} \in M_{-i} \text{ and } i \in \{1, 2\} \end{aligned}$$

where $\mathsf{per}[\cdot]$ denotes of the permutation operator. Such structure obtains, for instance, by labeling $M_* = \{1, 2, ..., |M_*|\}$ and requiring payoff profiles to be cyclic permutations, denote $\mathsf{cp}[\cdot]$, of the same payoff vector $\mathsf{per}[u[\theta_{M_*}]]$. That is for any $m_1 = l \in M_*$:

$$u(l, \cdot, m_{-1,2}|\sigma) = \mathsf{cp}^l[u(1, \cdot, m_{-1,2}|\sigma)]$$

For clarification, the table below depicts for $N = 2$ the commitment supergame associated to such mechanism. Label elements in the Nash subset by: $\mathcal{N}' = \left\{u^1, ..., u^{|\mathcal{N}'|}\right\}$.

$1\backslash 2$	1	2	...	$\theta_{M_*}(1)$	$\theta_{M_*}(1)+1$	...	$\lvert M_*\rvert - \theta_{M_*}(\mathcal{N}')+1$	...	$\lvert M_*\rvert$
1	u^1	u^1	...	u^1	u^2	...	$u^{\lvert\mathcal{N}'\rvert}$	...	$u^{\lvert\mathcal{N}'\rvert}$
2	$u^{\lvert M_\theta\rvert}$	u^1	...	...	u^1	...	...	$u^{\lvert\mathcal{N}'\rvert}$	$u^{\lvert\mathcal{N}'\rvert}$
...	...	$u^{\lvert M_\theta\rvert}$	...	...	...	...	...	...	$u^{\lvert\mathcal{N}'\rvert}$
$\theta_{M_*}(1)$	...	...	$u^{\lvert M_\theta\rvert}$	u^1	...	...	...	...	...
...	...	...	...	...	...	...	...	...	...
$\lvert M_*\rvert$	u^1	...	u^1	u^2	...	...	...	$u^{\lvert\mathcal{N}'\rvert}$	u^1

It needs to be shown that such a canonical commitment structure exists, is compatible with unanimity and possesses a Nash equilibrium with payoffs:

$$u_{M_*} = \textstyle\sum_{\mathcal{N}'} u'\theta_{M_*}(u')$$

For σ defined by $u(m|\sigma) = u(\sigma(m))$, the unanimity requirements are always met by a commitment structure that satisfies $k(m) = A$ for any $m \in M$, since for any $m_* \in M_*$:

$$\sigma(m) \in \mathcal{E}\langle u, A\rangle$$

Thus, there is no need to restrict any players strategy to implement such payoffs.

To show that there is a NE of the commitment supergame with the desired payoff, notice that any player $i \in N\backslash\{1,2\}$ is indifferent amongst all of his messages game, because by construction his choice does not affect payoffs. Also, remark that any player $i \in \{1,2\}$ is indifferent between all of his messages when the remaining players randomize uniformly on their messaging spaces, $\pi_l(m_l) = 1/|M_*|$ for any $m_l \in M_*$ & $l \in N\backslash i$. In fact, for $j \neq i \in \{1,2\}$ and $m_i \in M_*$:

$$\begin{aligned} u_i(m_i, \pi_{-i}|\sigma) &= \textstyle\sum_{M_{-i}} u_i(m|\sigma)\pi_{-i}(m_{-i}) = \sum_{M_{-i}} u_i(m|\sigma)/|M_*|^{N-1} = \\ &= |M_*|^{N-2}\left[\textstyle\sum_{m_j \in M_*} u_i(m|\sigma)\right]/|M_*|^{N-1} = \\ &= \left[\textstyle\sum_{\mathcal{N}'} u_i' |M_*|\,\theta_{M_*}(u')\right]/|M_*| = \sum_{\mathcal{N}'} u_i'\theta_{M_*}(u') \end{aligned}$$

Hence, for anyone to randomize uniformly in the supergame would constitute a subgame perfect equilibrium of the commitment expansion of the original game yielding the desired payoff. Consequently, $u_{M_*} \in \mathcal{K}(k)$ for the above described unanimous commitment structure.

It still needs to be shown that for $\forall\varepsilon > 0$ there $\exists\delta < \infty$ such that $\|u - u_{M_*}\| \leq \varepsilon$, whenever $|M_*| > \delta$. Because of continuity of u with respect to θ, it suffices to show that $\|\theta - \theta_{M_*}\|_+ \leq \zeta(\varepsilon)$. This obtains, for instance, by letting $|M_*| > \delta = 2/\zeta(\varepsilon)$. By way of contradiction suppose that $|M_*| > 2/\zeta(\varepsilon)$ & $\|\theta - \theta_{M_*}\|_+ > \zeta(\varepsilon)$. For the moment, also, assume that $\arg\max_{u'} |\theta(u') - \theta_{M_*}(u')|$ is unique. By hypothesis there exist a u' such that $|\theta(u') - \theta_{M_*}(u')| > \zeta(\varepsilon)$. Because both measures integrate to unity, if $\theta(u') > \theta_{M_*}(u')$, there also exists u'' such that $\theta(u'') < \theta_{M_*}(u'')$. But then, it is possible to construct $\theta'_{M_*} \in \Theta_{M_*}$ that is closer to θ, than θ'_{M_*}. Indeed, let:

$$\theta'_{M_*}(u) = \begin{cases} \theta_{M_*}(u) + 1/|M_*| & \text{if} \quad u = u' \\ \theta_{M_*}(u) - 1/|M_*| & \text{if} \quad u = u'' \\ \theta_{M_*}(u) & \text{if} \quad \text{otherwise} \end{cases}$$

In fact, $|\theta(u') - \theta_{M_*}(u')| > \left|\theta(u') - \theta'_{M_*}(u')\right|$ since $\zeta(\varepsilon) > 2/|M_*| > 1/|M_*|$. Also:

$$\left|\theta(u'') - \theta'_{M_*}(u'')\right| < \max\left\{1/M_*, \left|\theta(u'') - \theta'_{M_*}(u'')\right|\right\} < \left|\theta(u') - \theta_{M_*}(u')\right|$$

But this would contradict $\theta_{M_*} \in \arg\min_{t \in \Theta_{M_*}} \|t - \theta\|_+$, since $\|\theta - \theta_{M_*}\|_+ > \left\|\theta - \theta'_{M_*}\right\|_+$ and $\theta_{M_*} \in \Theta_{M_*}$. If the $\arg\max_{u'} |\theta(u') - \theta_{M_*}(u')|$ is not unique applying the described procedure to one of the maxima produces a vector with equal sup-norm distance from θ, but with one less component attaining such maximal distance. Iterating on the procedure until the maximizer is unique provides the result. ■

In the proof of the above result it was asserted that any point in the convex hull of a the Nash equilibria could be represented by a randomization on no more than $N+1$ elements of Nash equilibrium payoff set. Such result is a straightforward application of Caratheodory's theorem, reported below. The proof of this theorem can be found in [2].

Caratheodory Theorem *If $X \subset \mathbb{R}^n$, then $\forall x \in \mathsf{co}(X)$, there $\exists X' \subseteq X$ and a probability measure μ on X' such that $|X'| \leq n+1$ & $x = \sum_{X'} x'\mu(x')$.*

Proposition 7. *If $|\mathcal{N}| = 1$, then $\mathcal{K}_{\mathrm{U}} \subseteq \overline{\mathcal{V} \cap \underline{\mathcal{N}}} \cap \mathsf{co}(\mathcal{V} \cap \underline{\mathcal{N}})$.*

Proof. Let $\underline{u}$ denote the unique Nash payoff profile. For any $k \in \mathrm{U}$ and $m \in M$ such that $k(m) \neq A$, note that if $\pi(m) > 0$ in some equilibrium of the commitment supergame, it must be that $u(m|\sigma) \in \mathcal{V} \cap \underline{\mathcal{N}}$. Indeed, $u(m|\sigma) \in \mathcal{V}$ by subgame perfection and $u(m|\sigma) \in \underline{\mathcal{N}}$ because were this not the case there would exist an agent $j \in N$ for which $u_j(m|\sigma) < \underline{u}_j$. But such player would then benefit from vetoing any commitment rather than sending message m_j. Therefore $u(\pi|\sigma) \in \mathcal{K}_{\mathrm{U}}$ and $\pi(m) > 0$ implies $u(m|\sigma) \in \mathcal{V} \cap \underline{\mathcal{N}}$. Hence the inclusion $\mathcal{K}_{\mathrm{U}} \subseteq \mathsf{co}(\mathcal{V} \cap \underline{\mathcal{N}})$ holds.

Also, note that if $u = v(\pi) \in \mathcal{K}_{\mathrm{U}}$ for some $v \in V(k)$, since $v(m) \in \mathcal{V} \cap \underline{\mathcal{N}}$ for any $m \in M$ such that $\pi(m) > 0$, it must be that for any message profile m such that $k(m) \neq A$ and any player $i \in N$:

$$v_i(\pi) = \pi_{-i}(m_{-i})v_i(m) + (1 - \pi_{-i}(m_{-i}))\underline{u}_i \leq v_i(m)$$

which proves the inclusion $\mathcal{K}_{\mathrm{U}} \subseteq \overline{\mathcal{V} \cap \underline{\mathcal{N}}}$. ■

The next proposition derives an upper-bound for the general case. Such bound can be tightened since not all equilibrium restrictions are taken into account.

Proposition 8. $\mathcal{K}_{\mathrm{U}} \subseteq \cap_{i \in N} \mathsf{co}(\mathcal{U}_{\mathcal{N}} \cap \underline{\mathcal{N}}_i)$.

Proof. First, note that if for any $k \in \mathrm{U}$ and $v \in V(k)$ the equilibrium is in pure strategies it must belong to the desired set, since any such equilibrium payoff belongs to $\mathcal{V} \cap \underline{\mathcal{N}}$ by remark 4 and because:

$$\mathcal{V} \cap \underline{\mathcal{N}} \subseteq \mathsf{co}(\mathcal{I} \cap \underline{\mathcal{N}}) \subseteq \mathsf{co}(\mathcal{U}_{\mathcal{N}} \cap \underline{\mathcal{N}}) \subseteq \cap_{i \in N} \mathsf{co}(\mathcal{U}_{\mathcal{N}} \cap \underline{\mathcal{N}}_i)$$

For any $k \in \mathrm{U}$ and $v \in V(k)$, any mixed equilibrium payoff profile can be written as $v(\pi) = \sum_{M_i} \pi(m_i) v(m_i, \pi_{-i})$. By unanimity for any $m_i \in M_i$ there exist at most one profile $m_{-i} \in M_{-i}$ such that $k(m) \neq A$. Therefore for any m_i such that $\pi_i(m_i) > 0$:

$$v(m_i, \pi_{-i}) = \pi_{-i}(m_{-i})v(m) + (1 - \pi_{-i}(m_{-i}))u'(m_i) \in \mathcal{U}_{\mathcal{N}}$$

for $u'(m_i) = \sum_{\overline{m}_{-i} \in M_{-i} \setminus m_{-i}} \pi_{-i}(\overline{m}_{-i})/(1 - \pi_{-i}(m_{-i}))] v(m_i, \overline{m}_{-i}) \in \mathsf{co}(\mathcal{N})$. Since any player's choice of commitment is optimal, it must also be that $v(m_i, \pi_{-i}) \in \underline{\mathcal{N}}_i$, because otherwise agent i could benefit by deviating to the veto message. Collecting the last few observations, implies that $v(\pi) \in \mathsf{co}(\mathcal{U}_{\mathcal{N}} \cap \underline{\mathcal{N}}_i)$, for any $i \in N$. Therefore, it must be that $v(\pi) \in \mathcal{K}(k)$ implies $v(\pi) \in \cap_{i \in N} \mathsf{co}(\mathcal{U}_{\mathcal{N}} \cap \underline{\mathcal{N}}_i)$. ■

Proposition 9. *If* $\mathcal{V} \cap \underline{\mathcal{N}} \cap \mathcal{P} = \emptyset$*, then* $\mathcal{K}_{\mathrm{U}} \cap \mathcal{P} = \emptyset$*.*

Proof. The stated assumption, immediately, implies that $\mathcal{N} \cap \mathcal{P} = \emptyset$. It needs to be shown that for any game satisfying this assumption and for any continuation payoff $v \in V(k)$ generated by some unanimous structure $k \in \mathrm{U}$, it must be that:

$$\mathcal{N} \langle v, M_* \rangle \cap \mathcal{P} = \emptyset$$

Indeed any payoff of the supergame $\langle v, M_* \rangle$ that does not lead to a commitment must be inefficient, since $\mathcal{N} \cap \mathcal{P} = \emptyset$. Therefore any mixed equilibrium of such supergame is inefficient, since it must put positive probability on the no commitment outcome. Also, any pure strategy equilibrium of the supergame $\langle v, M_* \rangle$ must be inefficient since it belongs to $\mathcal{V} \cap \underline{\mathcal{N}}$. ■

Lemma 10. $\mathcal{K}^{\mathrm{S}}_{\mathrm{U}} = \underline{\mathcal{N}}$.

Proof. Consider $u \in \underline{\mathcal{N}}$. Such payoff profile can be represented as $u = u(\mu)$, for $\mu \in \Delta(A)$. Then consider the canonical unanimous commitment structure (M_*, k) satisfying $M_* = \{1, 2\}$ and:

$$k(m) = \begin{cases} \mu & \text{if} \quad m = (1, ..., 1) \\ A & \text{if} \quad \text{otherwise} \end{cases}$$

This commitment structure is well defined since $\mu \in \Delta(\times_{i\in N} 2^{A_i}\backslash\emptyset)$. If the contract is ever implemented no options will be left to any player at the action stage. Thus, one must only check that signing the contract more beneficial than vetoing it for any player. But choosing continuation values $v \in V(k)$ so that $v(m_i = 2, 1, ..., 1) = \underline{u}^i$ proves the claim, since for any $i \in N$:

$$v_i(1, ..., 1) = u_i(\mu) \geq \underline{u}_i^i$$

Again no payoff below the worst Nash threat would ever be accepted because of the voluntary nature of the mechanism. ■

9.2 Multiple Rounds of Commitment

For convenience denote the veto message by m_v. That is, $k_t(m_v, m_{-i,t}|A') = A'$, for any $m_{-i,t} \in M_{-i,t}$ and $i \in N$. First a result that will be used repeatedly in the proves that follow is established:

Remark 16 *If $k_0(m_0) = A$ and $v \in V_0(k^t)$ then $v(m_0) \in \mathcal{K}_{\mathrm{U}}^{t-1}$.*

Proof. By the second assumption $v(m_0) \in \mathcal{N}\langle \overline{v}, M_1\rangle$ for $\overline{v} \in V_1(m_0|k^t)$. Then define $\overline{M}_s(\cdot) = M_{s+1}(\cdot, m_0)$ and $\overline{k}_s(\cdot) = k_{s+1}(\cdot, m_0)$ for $s \in \{0, ..., t-1\}$. Because $k_0(m_0) = A$ it must be that $\overline{k}^{t-1} \in \mathrm{U}^{t-1}$. Additionally $V_1(m_0|k^t) = V_0(\overline{k}^t)$. Therefore $v(m_0) \in \mathcal{N}\left\langle \overline{v}, \overline{M}_0\right\rangle \subseteq \mathcal{K}_{\mathrm{U}}^{t-1}$. ■

PROPOSITION 11. *For $\left(M^t, k^t\right), (\overline{M}^t, \overline{k}^t) \in \mathrm{U}^t$, if $\overline{M}^t \supseteq M^t$ and if $k_s(m^s) = \overline{k}_s(m^s)$ for any $m^s \in M_s$ and $s \leq t$, then $\mathcal{K}(\overline{k}^t) \supseteq \mathcal{K}(k^t)$.*

Proof. Let (σ, π^t) be an equilibrium strategy of k^t supported by continuation values v^t, so that $v_t(\pi_t) \in \mathcal{K}(k^t)$. Then consider choosing continuation values for the larger commitment structure $\overline{k}^t$ so that at each information stage $m^{s-1} \in M^{s-1}$, for any $s \leq t$:

$$\overline{v}_s(m_s|m^{s-1}) = \begin{cases} v_s(m_s|m^{s-1}) & \text{if} \quad m_s \in M_s \\ \underline{u}^i & \text{if} \quad m_{i,s} \in \overline{M}_{i,s}\backslash M_{i,s} \;\; \& \;\; m_{-i,s} \in M_{-i,s} \\ ... & \text{if} \quad \text{otherwise} \end{cases}$$

Such continuation values would be supported strategy $(\overline{\sigma}, \overline{\pi}^t) = (\sigma, \pi^t)$ as an equilibrium of $\overline{k}^t$. Therefore, $\overline{v}_t(\overline{\pi}_t) = v_t(\pi_t) \in \mathcal{K}(k^t)$. ■

COROLLARY 12. *$\underline{\mathcal{N}} \supseteq \mathcal{K}_{\mathrm{U}}^{t+1} \supseteq \mathcal{K}_{\mathrm{U}}^t \supseteq \mathcal{V} \cap \underline{\mathcal{N}}$, for any $t \geq 0$.*

Proof. The last inclusion is a direct consequence of remark 2. To prove the first inclusion notice that playing according to strategy $\pi_{i,s}(m_v|m^{s-1}) = 1$ at each information stage $m^{s+1} \in M^{s+1}$ for any $s \leq t$ leaves all players uncommitted. Hence such strategy yields a payoff in $\mathcal{N}$ independently of the choices made by others. Therefore no payoff outside $\underline{\mathcal{N}}$ could ever be implemented since at least one player would possess a profitable deviation.

It must still be shown that if $u \in \mathcal{K}_{\mathrm{U}}^t$ then $u \in \mathcal{K}_{\mathrm{U}}^{t+1}$. Let $u \in \mathcal{K}(k^t)$ and consider a commitment structure $\overline{k}^{t+1} \in \mathrm{U}^{t+1}$ such that $\overline{k}^{s+1}(\overline{m}^{s+1}) = k^s(m^s)$ if $\overline{m}_{r+1} = m_r$ for any $r \in \{0, ..., s\}$ and if $\overline{k}^0(\overline{m}^0) = A$. Because the two commitment structures correspond in any subgame after the first round if no commitment arises it must be that $u \in \mathcal{N}\left\langle v_1(\overline{m}^0), \overline{M}_1\right\rangle$ for some $v_1(\overline{m}^0) \in V_1(\overline{m}^0|\overline{k}^{t+1})$. Indeed, in the subgame following message $\overline{m}^0$ the equilibrium strategy for k^t remains an equilibrium. It remains to show that there exists $v_0 \in V_0(\overline{k}^{t+1})$ so that $u \in \mathcal{N}\left\langle v_0, \overline{M}_0\right\rangle$. Consider the following candidate for v_0:

$$v_0(m_v, m_{-i,0}) = u \;\; \text{for any } m_{-i.0} \in M_{-i,0} \;\&\; i \in N$$

Because unanimous mechanisms are with veto, the message $m_{i,t} = m_v$ can be sent by any agent $i \in N$ at the 1st round. Additionally the continuation value requirements are met for $\overline{m}^0 = m_v^N$, since $u \in \mathcal{N}\left\langle v_1(\overline{m}^0), \overline{M}_1\right\rangle$.

Given the so constructed continuation values, for any agent $i \in N$ it would be optimal to veto the 1st layer when all the other players are vetoing themselves, since:

$$v_{i,0}(m_v^N) = v_{i,0}(m_{i,0}, m_v^{N-1}) \text{ for any } m_{i,0} \in M_{i,0}$$

Therefore such strategies would constitute a Nash equilibrium of the 1st stage game, implying that $u \in \mathcal{K}(\overline{k}^{t+1})$. ■

Proposition 13. $\mathsf{cl}(\mathcal{K}_{\mathrm{U}}^{t+1}) \supseteq \mathsf{co}(\mathcal{K}_{\mathrm{U}}^{t})$, *for any* $t \geq 0$.

Proof. This claim is a straightforward application of propositions 6 and 12. Because applying the technique provided in that proof it is possible to, approximately, implement as an equilibrium of the $(t+1)^{\text{st}}$ stage of commitment any payoff in the convex hull of the Nash equilibria of the t^{th} stage. That is, for any $u \in \mathsf{co}(\mathcal{K}(k^t))$ for some $k^t \in \mathrm{U}^t$ it is possible to construct $k^{t+1} \in \mathrm{U}^t$ such that $u \in \mathsf{cl}(\mathcal{K}(k^t))$. Such direct communication is accomplished just as in proposition 6. ■

Proposition 14. *For* $u_t \in \mathcal{K}_{\mathrm{U}}^t$, $u \in \mathcal{A}$ *and* $q \in \mathbb{N}_+$, *if* $(1/q^{N-1})u + (1 - 1/q^{N-1})u_t \in \underline{\mathcal{N}}$ *then:*

$$(1/q^{N-1})u + (1 - 1/q^{N-1})u_t \in \mathcal{K}_{\mathrm{U}}^{t+1}$$

Proof. Under the stated assumptions let $u = u(a)$ for $a \in A$. Then, consider a canonical form commitment structure (M_{*t+1}, k_{t+1}) such that $k_{t+1}(m^N) = a$ for any $m \in M_{*0}(a) \subset M_{*0}$ and such that $|M_{*0}(a)| \geq q$. Then, the following continuation values are feasible, $v_0 \in V_0(k^{t+1})$, for $k^{t+1} \in \mathrm{U}^{t+1}$:

$$v_0(m) = \begin{cases} u & \text{if} \quad m_i = m_j \in M_{*0}(a) \text{ for any } i,j \in N \\ u_t & \text{if} \quad m \in M_0(a) \ \ \& \ \ m_i \neq m_j \text{ for some } i,j \in N \\ \underline{u}^i & \text{if} \quad m_i \in M_{*0} \backslash M_{*0}(a) \ \ \& \ \ m_{-i} \in M_{-i,0}(a) \\ ... & \text{if} \quad \text{otherwise} \end{cases}$$

Such continuation values are feasible since $u, u_t, \underline{u}^i \in \mathcal{K}_{\mathrm{U}}^t$. For so chosen continuation values for all players to randomize uniformly on $M_{*t+1}(a)$ provides an allocation with the desired payoff, $(1/q^{N-1})u + (1 - 1/q^{N-1})u_t$. Such strategy is part of an equilibrium because any deviation from it by any player would give him is Nash threat value and would, therefore, not be profitable by $(1/q^{N-1})u + (1 - 1/q^{N-1})u_t \in \underline{\mathcal{N}}$. ■

Theorem 15. *If* $\mathsf{vert}(\mathcal{U} \cap \mathsf{aff}\,(\underline{\mathcal{N}})) \subseteq \mathcal{V}$ *and if there exist* $u \in \mathsf{co}\,(\mathcal{V} \cap \underline{\mathcal{N}})$, $u' \in \mathcal{V}$, $\alpha \in [0,1]$ *and* $j \in \underline{N}$ *such that:*

$$\begin{aligned} \alpha u_i' + (1-\alpha)u_i &> \underline{u}_i^i \quad \textit{for} \quad \forall i \in \underline{N} \backslash j \\ \alpha u_i' + (1-\alpha)u_i &\geq \underline{u}_i^i \quad \textit{for} \quad \forall i \in N \backslash (\underline{N} \backslash j) \end{aligned}$$

then for $C = \lim_{t\to\infty} \mathsf{cl}(\mathcal{K}_{\mathrm{U}}^t)$, *it must be that* $\mathsf{cl}(C) = \underline{\mathcal{N}}$.

Proof. First it is shown that under the stated assumptions it is possible to construct $u_3 \in \mathcal{K}_{\mathrm{U}}^3$ such that $u_{3,i} > \underline{u}_i^i$ for $\forall i \in \underline{N}$. Then, it is argued that, whenever such a utility profile exists, it is possible to approximately implement any allocation in the Nash rational set. If $\underline{\mathcal{N}} = \mathcal{N}$, there is nothing to prove so assume $\underline{N} > 0$.

Recall that by proposition 12 $\mathcal{V} \cap \underline{\mathcal{N}} \subseteq \mathcal{K}_{\mathrm{U}}^0$. Also note that by proposition 13 it must be that $\mathsf{co}\,(\mathcal{V} \cap \underline{\mathcal{N}}) \subseteq \mathcal{K}_{\mathrm{U}}^1$. If there exist values $u_1 \in \mathsf{co}\,(\mathcal{V} \cap \underline{\mathcal{N}})$, $u' \in \mathcal{V}$ and $\alpha \in [0,1]$

for which $\alpha u' + (1-\alpha)u_1$ satisfies the inequalities in the assumptions of the theorem, then such inequalities must hold also for any payoff profile $(1/q^{N-1})u' + (1-1/q^{N-1})u_1$ for which $(1/q^{N-1}) \leq \alpha$ and $q \in \mathbb{N}_+ \cup \{\infty\}$, since u_1 belongs to $\underline{\mathcal{N}}$ which is convex. For any such value of q it is possible to implement as a subgame perfect equilibrium of the contracting extension the payoff profile:

$$u_2 = (1/q^{N-1})u' + (1-1/q^{N-1})u_1 \in \mathcal{K}_{\mathrm{U}}^2$$

By having players randomize uniformly on q contracts that commit all players to action sets $A' \subseteq A$ that satisfy $u' \in \mathcal{N}\langle u, A'\rangle$. Indeed so long as any deviation from such strategy is punished with the Nash threat and if miscoordination leads to $u_1 \in \mathcal{K}_{\mathrm{U}}^1$ being implemented, no player would ever have an incentive to deviate from such strategy conditional on believing that the others are playing according to it.

Therefore, there exists $u_2 \in \mathcal{K}_{\mathrm{U}}^2$ such that $u_{2,i} > \underline{u}_i^i$ for $\forall i \in \underline{N}\backslash j$ for some $j \in \underline{N}$. It must be shown that there exists an $\gamma \in (0,1)$ and $u'' \in \mathcal{V}$ such that $\beta u''_i + (1-\beta)u_{2,i} > \underline{u}_i^i$ for $\forall i \in \underline{N}$ and $\beta \in (0,\gamma]$.

For this to be true it would suffice to find $u'' \in \mathcal{V}$ such that $u''_j > \underline{u}_j^j$ and $u''_i = \underline{u}_i^i$ for $\forall i \in N\backslash\underline{N}$. Because $\mathsf{vert}(\mathcal{U} \cap \mathsf{aff}\,(\underline{\mathcal{N}})) \subset \mathcal{V}$, such u exists. In fact, note that $\mathcal{U} \cap \mathsf{aff}\,(\underline{\mathcal{N}}) = \mathsf{co}(\mathsf{vert}(\mathcal{U} \cap \mathsf{aff}\,(\underline{\mathcal{N}}))) \supseteq \underline{\mathcal{N}}$ implies that for any $u \in \mathcal{U} \cap \mathsf{aff}\,(\underline{\mathcal{N}})$ it must be that $u_i = \underline{u}_i^i$ for $\forall i \in N\backslash\underline{N}$ and that for $\forall j \in \underline{N}$ there exist $u'' \in \mathsf{vert}(\mathcal{U} \cap \mathsf{aff}\,(\underline{\mathcal{N}}))$ such that $u''_j > \underline{u}_j^j$, because the vertices could not otherwise span $\mathcal{U} \cap \mathsf{aff}\,(\underline{\mathcal{N}})$. Thus, the desired payoff profile $u'' \in \mathsf{vert}(\mathcal{U} \cap \mathsf{aff}\,(\underline{\mathcal{N}}))$ exists.

Again, for $q \in \mathbb{N}_+$, such that $1/q^{N-1} \leq \gamma$, it is possible to support as an of the contracting extension payoff profile:

$$u_3 = (1/q^{N-1})u'' + (1-1/q^{N-1})u_2 \in \mathcal{K}_{\mathrm{U}}^3$$

By having players randomize uniformly on q contracts that commit them to action sets $A'' \subseteq A$ that satisfy $u'' \in \mathcal{N}\langle u, A''\rangle$.

Now, a weakly increasing sequence of sets $\{C_t\}_{t\geq 3}$ such that $C_t \subseteq \mathcal{K}_{\mathrm{U}}^t$ is constructed. It will then be shown that $\mathsf{cl}(\lim_{t\to\infty} C_t) = \underline{\mathcal{N}}$. Let $V = \mathsf{vert}(\mathcal{U} \cap \mathsf{aff}\,(\underline{\mathcal{N}})) \subseteq \mathcal{V}$, consider $u_3 \in \mathcal{K}_{\mathrm{U}}^3$ such that $u_{3,i} > \underline{u}_i^i$ for $\forall i \in \underline{N}$ and define $C_3 = \{u_3\}$. Then for any $u_t \in C_t$ and $v \in V$ define:

$$q(v, u_t) = \inf_{q \in \mathbb{N}_+} q \quad \text{s.t.} \quad (1/q^{N-1})v_i + (1-1/q^{N-1})u_{i,t} \geq \underline{u}_i^i, \text{ for } \forall i \in \underline{N}$$
$$U_t = \left\{(1/q(v,u_t)^{N-1})v + (1-1/q(v,u_t)^{N-1})u_t | u_t \in C_t \;\;\&\;\; v \in V\right\}$$

For such objects, the recursion equation defining the weakly increasing sequence of equilibrium sets can be stated as:

$$C_{t+1} = \begin{cases} C_t \cup U_t & \text{if} \quad t \geq 2 \text{ even} \\ \mathsf{co}(C_t) & \text{if} \quad t \geq 2 \text{ odd} \end{cases}$$

Any payoff profile $u_{t+1} \in C_{t+1}$ also belongs to the equilibrium set $\mathcal{K}_{\mathrm{U}}^{t+1}$, if $C_t \subseteq \mathcal{K}_{\mathrm{U}}^t$. This follows by corollary 13 if t is odd and if t is even, either because $C_t \subseteq \mathcal{K}_{\mathrm{U}}^t \subseteq \mathcal{K}_{\mathrm{U}}^{t+1}$ or because for any payoff in U_t there is an equilibrium of some commitment structure that yields that payoff. Namely, for $(1/q(v,u_t)^{N-1})v + (1-1/q(v,u_t)^{N-1})u_t$, the equilibrium would entail all players uniformly randomize on $q(v,u_t)$ contracts that commit them to action sets $A(v) \subseteq A$ such that $v \in \mathcal{N}\langle u, A(v)\rangle$ if signed by all.[16] In such equilibrium continuation values following coordination on such a contract would lead to the payoff v, following miscoordination on such contracts to the payoff profile u_t, while any deviation of an agent $i \in N$ to a different contract would punished with his Nash threat, $\underline{u}^i$. Specifically,...

[16] If $v \in \mathcal{N}\langle u, A(v)\rangle$ for some $A(v) \subseteq A$, if suffices to have the initial commitment structure satisfy $k_0(m^N) = A(v)$ for any $m \in M_{*0}(v) \subset M_{*0}$.

Since the sequence $\{C_t\}_{t\geq 3}$ is weakly increasing and since at any stage it belongs to the Nash rational payoff set $\underline{\mathcal{N}}$, such sequence must converge. Let $C = \lim_{t\to\infty} C_t$. It is immediate that C is convex, because all odd steps convexify. Then, suppose by way of contradiction that $\mathsf{cl}(C) \subsetneq \underline{\mathcal{N}}$. Since both sets are convex there exists $u_* \in \partial\mathsf{cl}(C)\backslash\partial\underline{\mathcal{N}}$. For any payoff $u \in \underline{\mathcal{N}}\backslash\partial\underline{\mathcal{N}}$ define:

$$U(u) = \left\{(1/q(v,u)^{N-1})v + (1 - 1/q(v,u)^{N-1})u | v \in V\right\}$$

and note that $\mathsf{co}(U(u)) \supseteq \underline{\mathcal{N}} \cap B_e(u)$, for some ball of radius ε around u.[17] Such observation immediately implies a contradiction, if a payoff $u_* \in \partial\mathsf{cl}(C_*)\backslash\partial\underline{\mathcal{N}}$ can implemented, $u_* \in C_t$, through a finite number of layers t, since $\mathcal{K}_{\mathrm{U}}^{t+2} \supseteq \mathsf{co}(U(u_*)) \supseteq \underline{\mathcal{N}} \cap B_e(u_*)$ would imply that $u_* \notin \partial\mathsf{cl}(C)$. Any payoff $u_t \in C_t$ chosen to be arbitrarily close arbitrarily close to u_*, will have $\mathsf{co}(U(u_t))$ arbitrarily close to $\mathsf{co}(U(u_*))$, by continuity. Therefore, since $\mathsf{co}(U(u_*))\backslash C \neq \emptyset$ by $u_* \in \partial\mathsf{cl}(C)$, for any sequence of equilibrium payoffs $\{u_t\}$ converging to u_*, it will be the case that:

$$\lim_{t\to\infty} \mathsf{co}(U(u_t))\backslash C = \mathsf{co}(U(u_*))\backslash C \neq \emptyset$$

Implying that for any $\eta > 0$ there exists some number of layers T such that whenever $t > T$ and for some norm $d(\cdot)$:

$$d(\mathsf{co}(U(u_t)), \mathsf{co}(U(u_*))) < \eta$$

But this would lead to a contradiction since it would imply that for that for t big enough $\mathsf{co}(U(u_t))\backslash C \neq \emptyset$, because otherwise the norm could not converge. ■

[17] Where the radius ε can potentially depend on u.