.

Efficiency in Decentralized Oligopolistic Markets*

Francesco Nava[†]

May 2013

Abstract: The paper investigates how quantity competition operates in economies in which a network describes the set of feasible trades. A model is presented in which equilibrium dictates whether an individual buys, sells, or intermediates goods. The analysis first considers small economies, and provides sufficient conditions for equilibrium existence, a characterization of prices and flows, and some negative results relating welfare to network structure. The second and central part of the analysis considers behavior in large markets, and presents necessary and sufficient conditions on the network structure for equilibria to be approximately efficient when the number of players at various locations of the economy is large.

*I am indebted to Matt Jackson and to Ted Bergstrom for their valuable comments, and to Hugo Sonnenschein, Roger Myerson, Phil Reny and Balazs Szentes for their advice. I also would like to thank the seminar participants at Harvard/MIT, UChicago, UW Madison, UCSB, Caltech, BU, UBC, VTech, LSE, UCL, Cambridge, Essex, Royal Holloway, Cardiff, Pompeo Favra, Universitat Autonoma Barcelona, Bocconi, Bank of Italy, Maastricht.

[†]London School of Economics, f.nava@lse.ac.uk.

1 Introduction

Classical models of competition rely on the anonymity of markets to explain prices and trade. In this view, exchanges in an economy take place in centralized markets and the identity of players has no effect on prices and on terms of trade. Recent models of decentralized competition depart from such a stark paradigm by considering economies in which exchanges take place in bilateral relationships. Prices and terms of trade in such economies crucially depend on the constraints imposed on the set of feasible trades and on the implied market power. Known results have analyzed economies in which the identity of buyers and sellers is exogenously determined, and in which only trades from sellers to buyers are feasible. This study aims at understanding how decentralized oligopolistic markets operate when equilibrium determines the role of players in the economy, and at presenting conditions on the structure of an economy for trade to be approximately efficient when the number of players is large.

To this end, the paper introduces a static model of trade for economies in which a network describes the set of feasible trading relationships among individuals. In the model of decentralized oligopolistic competition considered, individuals decide how much to sell to neighboring players knowing that competition at each location implies that players purchase goods at their marginal value. Changes in sales distort both the price at which players purchase goods, and the prices at which these units are sold to their neighbors, as trade distorts the marginal rate of substitution of both players involved in a transaction. Traders account for such price distortions and selfishly choose how much to sell to their neighbors in order to maximize their well-being. Equilibrium flows of goods endogenously determine whether an individual buys, sells or does both, based on preferences, production possibilities and the position held in the network. Supply chains arise endogenously in equilibrium. Intermediation and significant price dispersion are generic phenomena in small or poorly connected economies.

In economies with a small number of players, distortions inherent to any quantity competition model imply that trade is necessarily inefficient. However, when the number of players is large simple conditions can be imposed on an economy that ensure that trade is approximately efficient when the number of players is sufficiently large. To do so, for a fixed network structure the analysis considers what happens when the number of players at various locations is large. Conditions for approximate efficiency imply that intermediation must be superfluous to clear any market in which the number of players is large. If so, direct competition among large number of players eliminates resale and restores efficiency. Otherwise, intermediation would be necessary for efficiency to obtain, and intermediaries would therefore necessarily command a rent and distort trade.

The first part of the analysis develops baseline results for economies in which the number of players is small. In particular, it presents sufficient conditions for pure strategy equilibrium existence, it characterizes equilibrium prices, flows and markups, and it details some negative conclusions on welfare. A key feature of the outflow model is that resale markups are strictly positive due to the double-marginalization problem faced by players acting as intermediaries. Therefore, goods will never cycle in equilibrium, and not all linked players with different marginal rates of substitution will elect to trade. Individual would never purchase units previously sold, because a higher price would have to be paid. Moreover, individuals with lower willingness to pay might prefer not to sell their goods to players with a higher willingness to pay, as trade might increase the price paid for the units purchased.[1] Selfish behavior results in price discrimination across locations of the trading network. Intermediation is a common phenomenon that relies both on the scarcity of trading partners and on the different prices that prevail throughout the economy in equilibrium. Results on welfare first establish that trade is obviously inefficient in any economy populated by finitely many players, and then present some negative conclusions relating welfare to network structure. In particular, adding trading relationships does not necessarily improve social welfare. When new links are added, more goods may flow to low value markets since sellers may price discriminate locations in which the goods are most desired. More surprisingly, even though players have the option not to trade with any one of their neighbors, the welfare of an individual may decline when additional players belong to his neighborhood. Since trading relationships are common knowledge, whenever new links raise the demand of an individual, price discrimination by his suppliers may decrease the amount of goods sold to him and consequently his welfare.

The second part of the analysis studies behavior in economies with a large number of players. In particular, it considers economies in which players are positioned at finitely many locations connected by a network, and in which players can trade only with other players positioned either at their, or at neighboring locations. Necessary and sufficient conditions are presented for a networked economy to be approximately efficient when the number of players at every location is large. Any economy in which intermediation is required to clear markets, is inefficient independently of the size and structure of the market. Intermediaries always command a rent whenever they are needed to distribute goods, and competition among them would undermine, but not eliminate resale markups. Results establish that efficiency in such markets is equivalent the existence of trades that are both direct and efficient. Necessary and sufficient conditions for the existence of such trades are derived. Conditions are evocative of market clearing requirements for two-sided markets, and imply that the aggregate demand of

[1] This implication differs from that of the Cournot model in which any two players with different marginal rates of substitution always elect to trade.

any subset locations must be exceeded by the aggregate supply of locations in which direct trade is feasible. Equilibrium outcomes can be fully characterized even when conditions for efficiency fail. In such instances, intermediation and distortions persist even when the number of players at every location is large. Similar results are developed for economies in which only a subset of locations is large. Approximate efficiency at large locations is again equivalent to the existence of direct and efficient trades between those locations. Approximate efficiency at small locations further requires that those locations be allowed to directly trade in some large market. Results also establish that social welfare converges monotonically to efficiency as the number of players at every location grows large.

The analysis concludes by presenting an alternative quantity competition model in which individuals decide how much to buy, and in which units are sold at marginal value. Similar results hold, even though the distribution of rents differs. More rents flow to buyers, and social welfare is generally higher than when players choose how much to sell.

Literature Review: A vast and recent literature has analyzed trade in buyer-seller networks. Such models usually take the identities of buyers and the identities of sellers in an economy as exogenous characteristics of the market, and describe the set of feasible trades from sellers to buyers with a network. Papers differ mainly, because of the different models of competition analyzed. Kranton and Minehart 2001 models competition among sellers as simultaneous ascending price auctions, and studies the formation of efficient link patterns. Corominas-Bosch 2004 models trade as centralized non-cooperative bargaining game, and provides sufficient conditions on the network structure for the equilibrium of the bargaining game to coincide with the Walrasian outcome. Ilkilic 2010 discusses market power in the context of a linear-quadratic quantity competition model of two-sided markets. Lever 2010 analyses Bertrand competition between duopolists, and the relationship between network structure and welfare. Several papers on decentralized bargaining (Abreu and Manea 2012, Manea 2011, Polanski 2007, Polanski and Vega-Redondo 2013, Polanski and Winter 2010) also encompass models of trade in two-sided markets, and analyze efficiency in such markets. All of these models however, rule out intermediation by assumption, and implicitly set the identity of buyers and sellers as a primitive of the problem.

Other papers have introduced some notion of intermediation in the context of a two-sided market. Blume, Easley, Kleinberg and Tardos 2007 study buyer-seller networks in which all trades have to be mediated by price-setting middlemen. Equilibria in their model always implement an efficient allocation, in which middlemen command a positive rent if and only if they possess an essential connection in the network structure. Siedlarek 2013 allows for more general structures of intermediation in the context of a model of coalitional bargaining, and shows that efficiency again obtains when no intermediary is essential. The efficiency result however, relies partly on exogeneity of sellers and buyers and partly on the centralized nature of the bargaining protocol considered. Manea 2013 has recently developed similar results in the context of a decentralized bargaining model, and detailed frictions that might arise in such models. A related literature models competition between owners of links on a network in which individuals selfishly route flows (Chawla and Roughgarden 2007 and Acemoglu and Ozdaglar 2007). This literature usually takes a Bertrand approach to model competition, and was developed to model competition between internet providers pricing information streams.

The quantity competition model presented here differs from all of the models discussed above, as the roles of individuals in a supply chain are endogenously determined in equilibrium. Kakade, Kerns and Orthiz 2004 characterizes the competitive equilibria of a general networked market in which the roles of players are endogenous. However, price taking behavior implies that network structure cannot directly affect market power. Condorelli and Galeotti 2012 analyzes sequential trade of a single unit in a general networked market in which there is some incomplete information about the value of the good. In the model prices decrease along the supply chain as trade reveals a low value for the good. This study is closest in the spirit to current paper. But due to the complications arising from incomplete information and dynamics, the model remains stylized and does not encompass the results presented here. Recent and interesting studies in the matching literature have also addressed the problem of intermediation in decentralized markets. Prime examples in this literature are Ostrovsky 2008, and Hatfield and Kominers 2012. Although these studies are motivated by similar questions, differences in the environment and in the notion of equilibrium remain significant.

Roadmap: Section 2 analyses outflow competition. It presents the model, a characterization of equilibrium prices and flows, and results for small economies. Section 3 discusses outflow competition in large economies, and presents conditions for efficiency in large markets. Section 4 discusses inflow competition. Section 5 concludes. All proofs can be found in appendix.

2 Outflow Competition

The section begins with a description of the economy and of the outflow competition model, and proceeds with a characterization of the equilibria of the model and with results on welfare in small economies.

The Economy and Constrained Efficiency

Consider an economy with a finite set of players V, and two goods. For convenience, refer to the two goods as consumption q, and money m. Any player i in the economy can trade goods only with a subset of players $V_i \subseteq V\backslash\{i\}$, which is called the *neighborhood* of player i. Assume that $j \in V_i$ if and only if $i \in V_j$. This structure of interaction defines an undirected graph $G = (V, E)$ in which $ij \in E$ if and only if $j \in V_i$. Refer to G as the *trade network*.

Denote by q^i_j the flow of consumption good from individual i to individual j. Since trade can occur only between players that know each other, $q^i_j = 0$ whenever $ji \notin E$. For any player i, define the *total purchases* and the *total sales* of consumption good respectively as,

$$q^\circ_i = \sum_{k \in V_i} q^k_i \text{ and } q^i_\circ = \sum_{k \in V_i} q^i_k.$$

Refer to the difference between the two quantities, $q_i = q^\circ_i - q^i_\circ$, as the *net-trade* of player i, and refer to the smallest among the two, $r_i = \min\{q^\circ_i, q^i_\circ\}$, as the *resale* of player i. When an individual purchases more (fewer) units than those he sells, his resale thus, consists of all the units that he sells (buys). Bold letters are used to denote vectors of flows. In particular, $\mathbf{q}^i$ denotes the vector of consumption flows from i to his neighbors in V_i; $\mathbf{q}$ denotes the Cartesian product of all the $\mathbf{q}^i$'s; and $\mathbf{q}^{-i}$ denotes the Cartesian product of $\mathbf{q}^j$ for all $j \neq i$.

The payoff of every individual in the economy is separable in the two goods, and linear in money. The payoff derived by player i from net-trade q_i and money m is determined by the map

$$u_i(q_i) + m.$$

The net-trade of any player i is bounded below by a non-positive number $-Q_i$. Refer to Q_i as the capacity of player i. Since $Q_i > 0$ is possible, players can sell more units, than they purchase. This setup can capture both endowment and production economies. In the production interpretation of the model $-u_i(q_i)$ can be viewed as the cost of supplying $-q_i > 0$ units to the market. Non-negativity constraints on monetary holdings are neglected throughout the analysis. It is implicitly assumed that monetary endowments are sufficiently large for such constraints never to bind.[2] To discipline the problem, further invoke the following standard assumptions on payoffs.

Assumption A1 *For any player $i \in V$, u_i is three times continuously differentiable, strictly increasing and strictly concave on $[-Q_i, \infty)$.*

Social welfare of any profile of flows $\mathbf{q} \in \mathbb{R}^E_+$ is evaluated by sum of payoffs. Since the payoff of every player is quasi-linear and monetary endowments are large, any interior Pareto optimum maximizes the sum of the utilities of the non-linear good. We therefore, define a profile of

[2]The framework is evocative of Kalai, Postlewaite & Roberts 1978 which analyzes properties of core allocations in similar environments.

flows $\mathbf{q}^* \in \mathbb{R}_+^E$ to be *constrained efficient* if it solves,

$$\mathbf{q}^* \in \arg\max_{\mathbf{q}\in\mathbb{R}_+^E} \sum_{i\in V} u_i(q_i) \text{ s.t. } q_i \geq -Q_i \text{ for } \forall i \in V.$$

If the solution to this problem is interior, the constrained efficient allocation equalizes the marginal rates of substitution of any two consumers i and j that belong to the same component of the networked economy G.[3]

Outflow Competition

In the model of competition considered here, the description of the economy is common knowledge. Every player owns a trading location at which anyone of his neighbors can sell goods. Players simultaneously decide how many units of consumption to sell at each of their neighbors' trading locations. Any player i is constrained not to sell more than Q_i units of consumption. As in many quantity competition models, prices are determined at each location so that buyers pay all of their inflows at their marginal value. In particular, the price paid by player i for units sold from a neighbor j is determined by the inverse demand curve with respect to net-trade at node i,

$$p_i^j(\mathbf{q}) = p_i(q_i) = u_i'(q_i) = u_i'(q_i^{\circ} - q_{\circ}^i). \tag{1}$$

Such prices could be micro-founded in the context of a two-stage model in which suppliers first pre-commit to sales of consumption to known buyers, and then compete on prices to supply each of these buyers. Indeed, if suppliers were able to commit to outflows, and if they were to compete on prices at each local market given their outflow decisions, equation (1) would still dictate pricing, since no supplier would benefit from a unilateral deviation in price-setting game. Price reductions would not affect the quantity sold, while price increases would reduce revenues because of falling sales. This observation was first made in Kreps and Scheinkman 1983 while studying Bertrand competition with quantity pre-commitment. Their results extend immediately to the outflow framework, since no restrictions were imposed on the number of buyers.[4] Such pricing could capture behavior in markets in which local supply decisions have to be made prior to competition.

The concavity of the utility function implies that the price paid by any player i decreases when his inflows increase, increases when his outflows increase, and is not directly affected by other flows in the economy. That is, $\partial p_i(q_i)/\partial q_i^j < 0$ and $\partial p_i(q_i)/\partial q_j^i > 0$ for any neighbor $j \in V_i$. When choosing their outflows, sellers account for the distortions that their supply

[3] Any maximal connected subgraph of G is a component of G (Bollobas 1998).

[4] The proposed two-stage model would always possess Subgame Perfect equilibria in which prices and flows coincide with the Nash equilibria of the outflow competition model.

decisions might induce both on the prices they receive for each unit sold and on the price they pay for each unit bought. Thus, the *welfare* of an individual i given a profile of flows $\mathbf{q}$ is determined by the map,

$$w_i(\mathbf{q}) = u_i(q_i) + \sum_{k \in V_i} \left[p_k(q_k) q_k^i - p_i(q_i) q_i^k \right],$$

where prices are pinned down by equation (1), and where the summation denotes the trading surplus of player i. In what follows the expression *outflow equilibrium* will be used to refer to a pure strategy Nash equilibrium of the outflow competition model.

Outflow Equilibrium *Flows* $\mathbf{q} \in R_+^E$ *constitute an outflow equilibrium, if for any* $i \in V$,

$$\mathbf{q}^i \in \arg\max_{\mathbf{c}^i \in \mathbb{R}_+^{V_i}} w_i(\mathbf{c}^i, \mathbf{q}^{-i}) \text{ s.t. } c_\circ^i \leq Q_i.$$

The outflow constraint $q_\circ^i \leq Q_i$ requires total sales not exceed capacity. This restriction is only imposed to guarantee that a player's action set does not depend on the supply decisions of other players', but does not affect results presented in the remainder of the analysis.[5]

Outflow Equilibrium Existence

The next part of the analysis presents sufficient conditions for outflow equilibrium existence, and a first characterization of equilibrium flows of consumption. As in numerous other imperfect competition studies, bounds are imposed on the slope and the curvature of every demand function to guarantee that the payoff of every player remains well-behaved. Denote the elasticity of the inverse demand curve of player i with respect to quantity by $\eta_i(q) = -(Q_i + q)u_i''(q)/u_i'(q)$. Also, denote player i's *total cost* of supplying outflows and player i's *revenue* from suppling units to market $j \in V_i$ respectively by

$$\begin{aligned} C_i(\mathbf{q}^i, \mathbf{q}^{-i}) &= -u_i(q_i) + u_i'(q_i) q_i^\circ, \\ R_j^i(q_j^i, \mathbf{q}^{-i}) &= u_j'(q_j) q_j^i. \end{aligned}$$

The total cost of supplying outflows is determined by the adding the cost of forgone net-trades $-u_i(q_i)$ to the expenditure on inflows $u_i'(q_i)q_i^\circ$. The welfare of an individual can thus, be expressed as the sum of the revenues that he makes in each market in which he sells, minus the total cost of supplying such outflows.

Consider the following constraints on the elasticity of the slope of the inverse demand curve with respect to quantity.

[5] All of the results presented (including those on existence) would also hold under the weaker outflow constraint $q_i \geq -Q_i$. However, the model would cease to be a game as the set feasible actions would be determined in equilibrium by means of a fixed point argument.

Assumption A2 *For any player $i \in V$, the utility u_i satisfies at least one of the following two conditions for any $q > -Q_i$:*
[B1] $-(Q_i + q)u_i'''(q)/u_i''(q) \in [-1, 2]$;
[B2] $-(Q_i + q)u_i'''(q)/u_i''(q) \in [-\eta_i(q)/V_i, 2\eta_i(q)]$.

The next result establishes that any one of two constraints guarantees the existence of an outflow equilibrium, and characterizes the conditions identifying any outflow equilibrium.

Proposition 1 *If A1 and A2 hold:*
(a) an outflow equilibrium exists;
(b) any outflow equilibrium $\mathbf{q}$ is a solution $(\mathbf{q}, \boldsymbol{\mu}) \in \mathbb{R}_+^E \times \mathbb{R}_+^V$ to the complementarity problem

$$f_j^i(\mathbf{q}, \boldsymbol{\mu})q_j^i = 0 \quad \text{and} \quad f_j^i(\mathbf{q}, \boldsymbol{\mu}) = u_i'(q_i) - u_j'(q_j) - u_j''(q_j)q_j^i - u_i''(q_i)q_i^\circ + \mu_i \geq 0 \quad \text{for} \quad ij \in E;$$
$$f_i(\mathbf{q}, \boldsymbol{\mu})\mu_i = 0 \quad \text{and} \quad f_i(\mathbf{q}, \boldsymbol{\mu}) = Q_i - q_\circ^i \geq 0 \quad \text{for} \quad i \in V.$$

The first part of the proposition follows, because the set of feasible outflows of every player is non-empty, convex and compact, and because either of the two conditions guarantees that best reply maps are continuous and single-valued (and that Brouwer's fixed point theorem thus applies). Condition B1 implies that best reply maps are single-valued, as it requires revenues to be concave and total costs to be convex in any market. Condition B2 instead, requires total costs to be convex and revenues to be concave only when revenues increase in a market. In turn this suffices to establish that best reply maps are single-valued, as the payoff of every player is proven to be concave when increasing. Any combination of the bounds in the two conditions would also grant existence, as the lowerbounds discipline only total costs, while the upperbounds only revenues. It can be readily verified that common families of preferences meet the proposed conditions for outflow equilibrium existence.

Remark 2 *An outflow equilibrium exists if one of the following two conditions holds:*

$$\text{(a)} \quad u_i(q) = \beta_i(Q_i + q)^{\alpha_i} \quad \text{for} \quad \alpha_i \in (0, 1) \quad \beta_i \in \mathbb{R}_{++} \quad \text{any } i \in V;$$
$$\text{(b)} \quad u_i(q) = -\beta_i e^{-\alpha_i(Q_i + q)} \quad \text{for} \quad \alpha_i \in \mathbb{R}_{++} \quad \beta_i \in \mathbb{R}_{++} \quad \text{any } i \in V.$$

The second part of proposition 1 characterizes outflow equilibria as a solution to the system of best responses (where μ_i denotes the multiplier on the capacity constraint of player $i \in V$). When the outflow constraint $q_\circ^i \leq Q_i$ does not bind, the optimality for a positive outflow $q_j^i > 0$ requires that,

$$p_j(q_j) - p_i(q_i) = -p_j'(q_j)q_j^i - p_i'(q_i)q_i^\circ.$$

If so, the markup on the flow q_j^i (i.e. the difference between price received and the marginal cost of forgone consumption) is completely determined by two wedges: one distorts of the price received from player j, while the other distorts the price paid on all inflows purchased.

The first wedge is due to the fact that i is a Cournot supplier of j, while the second wedge is due to the fact i is a monopsonistic buyer at his location. Pricing behavior in the outflow model favors suppliers as the demand curve is used to clear each local market. Section 4 explores the consequences of the alternative setup in which sellers own the trading location and buyers can pre-commit to inflows.

Four Player Examples

Before the formal discussion of outflow equilibrium properties, consider a simple economy with four players, labeled $\{a, b, c, d\}$. Interpret Q_i as the initial endowment of player i, and let $Q_a = 5$, $Q_b = 2$, and $Q_c = Q_d = 1/2$. Let the preferences for consumption of player i satisfy $u_i(q) = (Q_i + q)^{1/2}$. If no trade takes place, social welfare is worth 5.06. Constrained efficiency in this economy requires all consumers to split the consumption good equally whenever the trade network is connected. Social welfare at this allocation is maximal and equal to 5.66. Equal sharing however, is not an outflow equilibrium even when all trades are feasible. When the trade network is complete, in the unique outflow equilibrium player a sells to all of his neighbors, and player b resells some of the goods purchased from a to c and d. Players c and d do not trade with each other since they are identical and in a symmetric position.[6] Equilibrium flows do not equalize marginal rates of substitutions. The price paid by consumers c and d for each unit of consumption purchased is 0.41. This price exceeds the price charged by consumer a to b on the units traded, 0.34. Even though a has the option not to sell to his competitor, b, he prefers to do so, because it is profitable, and because it is impossible to prevent b from supplying the final consumers, c and d. Thus, player b is able to impose a 21% markup on all the units that he resells. Equilibrium flows for this economy are reported in first network of figure 1. Consumption, prices and welfare can be found in the first matrix of table 1. In equilibrium, consumers a and b curtail their supply to c and d in order to maximize their gains from trade. The allocation is inefficient and social welfare is equal to 5.61.

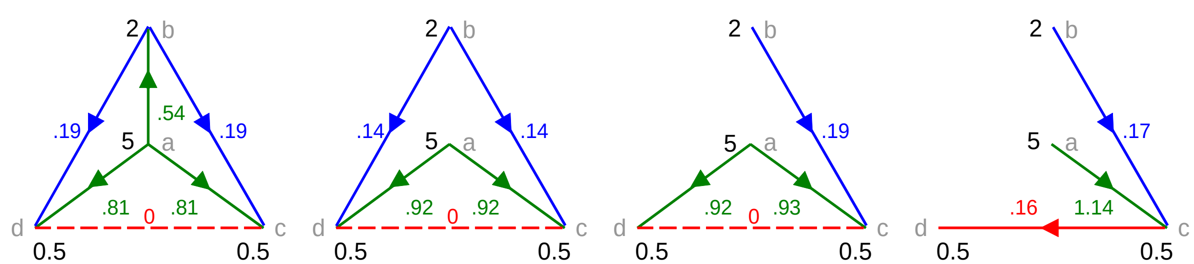

FIGURE 1: Each network depicts one of the four examples: on the vertices are endowments and identities and on the edges are equilibrium flows and their direction.

[6] If the link cd were removed from the trade network, equilibrium flows and prices would not be affected, since no trade takes place between c and d.

It could be conjectured that severing the link between players a and b would favor a by giving him the opportunity to commit not to sell to b. However, this is not the case. When the link ab is severed, consumption of every player, except b, increases. The final consumers, c and d, purchase more goods at a lower price and are better off. But consumers a and b are worse off. The equilibrium remains inefficient, and social welfare decreases further to 5.59. A unique price is paid for every unit of consumption purchased by c and d, namely 0.40. This price coincides with the Cournot equilibrium price for the economy without a network. The equilibrium of this economy is described in the second network of figure 1 and in the second matrix of table 1.

	p	x	w		p	x	w		p	x	w		p	x	w
a	-	2.84	2.53	a	-	3.16	2.51	a	-	3.15	2.53	a	-	3.86	2.41
b	0.34	2.16	1.44	b	-	1.72	1.42	b	-	1.81	1.42	b	-	1.83	1.42
c	0.41	1.50	0.82	c	0.40	1.56	0.83	c	0.39	1.62	0.83	c	0.39	1.65	0.87
d	0.41	1.50	0.82	d	0.40	1.56	0.83	d	0.42	1.42	0.80	d	0.62	0.66	0.71
+	-	8.00	5.61	+	-	8.00	5.59	+	-	8.00	5.58	+	-	8.00	5.41

TABLE 1: Each matrix reports equilibrium prices paid, consumption $x = Q + q$, and welfare for each player and society in one of the four economies.

If link between players b and d is further removed from the trade network, consumer d remains with only a and c as potential suppliers, while consumer c can still purchase from both a and b. In equilibrium a and b still supply all of their neighbors. But, even though consumer c ends up with more consumption good than d, he opts not to resell to d. Indeed, player c prefers to forgo the revenues he could make, since selling to player d would increase the price he pays on all the units purchased. In the outflow model, linked players with different marginal rates of substitution occasionally prefer not to trade, because a commitment not to resell can significantly reduce the price paid on all the units purchased. The equilibrium of this economy is described in the third network of figure 1 and in the third matrix of table 1. Since player c has two suppliers, while player d has only one that is active, player c pays a lower price for consumption than d. Player a sells more units in the competitive market than in the one in which he is a monopolist. Social welfare decreases further to 5.58. Finally consider the economy in which all individuals can only trade with c. In such a market players a and b sell to c, who with some of the units purchased supplies d. The equilibrium for this economy is characterized in the fourth network of figure 1 and in the fourth matrix of table 1. Player c's markup on the units sold to d is of 58%. Resale takes place despite such a high markup. Sales from player c to player d are constrained by the effects that such a trade has on the price paid by player c to his suppliers. Social welfare drops significantly to 5.41.

Consumers a and d are worse off than in the previous environment, while consumer c is better off since any trade with d has to be mediated by him.

Outflow Equilibrium Properties

This subsection presents two results about outflow equilibria in economies with a finite number of players. The first result mainly addresses the properties of equilibrium flows and pricing, while the latter presents several negative conclusions on welfare.

In the outflow model consumption flows from players with low marginal value to players with high marginal value, as assumption A1 implies that $q_i^j > 0$ only if $u_j'(q_j) > u_i'(q_i)$. The worst possible use of the goods owned is therefore consumption and not trade. Hence, no buyer would be willing to pay more than this value for the last unit purchased. This observation also implies that consumption flows only in one direction on every link and that at most $|E|/2$ flows are positive in any equilibrium. Individuals sell, or resell, goods to their neighbors only if the gains from trade can compensate them both for the monopsony price distortion on inflows and for the Cournot distortion on outflows. A positive difference in marginal rates of substitution is necessary, but not sufficient for trade to take place among pairs of linked individuals. Small differences in marginal rates of substitution may not suffice for trade to take place, as the monopsony distortion curtails trades between players with similar marginal values. Equilibrium retail markups are always strictly positive, as $q_i^\circ \geq r_i > 0$ implies that $p_j(q_j) > p_i(q_i)$ even when q_j^i is small. Resale however, remains common phenomenon that arises both because of the limited number of trading relationships that can be used to transfer goods, and because of the sellers' incentives to price discriminate neighboring buyers. The latter motive explains why even a fully connected economy may display equilibrium resale. The next proposition summarizes several useful properties of outflow equilibria. For convenience refer to an individual as a *source* if he does not buy consumption, and refer to an individual as a *sink* if he does not sell consumption.

Proposition 3 *If A1 and A2 hold, in any outflow equilibrium* $\mathbf{q}$*:*
(a) $q_j^i > 0$ *implies* $p_j(q_j) > p_i(q_i)$*, and the converse may not hold;*
(b) goods do not cycle and prices strictly increase along any supply chain;
(c) players with marginal utility lower/higher than their neighbors are sources/sinks;
(d) if unconstrained, sources sell to all their neighbors with strictly higher marginal utility;
(e) if $i, j \in V_k$ *and* $p_j(q_j) > p_i(q_i)$*, then* i *buys from* k *only if* j *buys from* k*.*

Part (a) follows from the previous discussion, and (b) is an immediate consequence of goods being resold at strictly positive markups. In fact, because the marginal utility of consumption strictly increases along a supply chain, it can never be that an individual buys some of the units he previously sold. Since goods do not cycle, flows of goods move from sources to sinks. Flows however, can have more than one source and/or sink in equilibrium. Part (c) establishes that individuals with lower marginal utility than all their neighbors are sources and that individuals with higher marginal utility than all their neighbors are sinks. In fact, individuals with lower marginal utility than their neighbors would never buy, because only players with lower marginal utility could supply them. Similarly, individuals with higher marginal utility than their neighbors would never sell. Part (d) shows that, if unconstrained, sources must sell to every neighbor with higher marginal utility. A positive difference in marginal rates of substitution is not only necessary, but also sufficient for trade to take place, because sources have no inflows, and because outflow price distortions vanish with outflows. Part (e) finally, establishes that if two players have a neighbor in common, that neighbor sells to the low marginal utility player only if he sells to the high marginal utility player. Results contained in the web-appendix show that in economies in which all individuals can trade with each other, the ranking of marginal utilities coincides with the ranking of supply costs.[7]

The next result exploits some of the properties of outflow equilibria to derive several negative conclusions on welfare. Given the quasi-linear payoff structure, it possible to evaluate *social welfare* by summing the welfare of each player in the economy,

$$\sum_{i\in V} w_i(\mathbf{q}).$$

Results first establish that inefficiencies are common feature to the outflow equilibrium model, and then show that adding links might have unexpected consequences on individual and social welfare in equilibrium.

Proposition 4 *If A1 holds, the following conclusions also hold for $i, j \in V$:*
(a) a constrained efficient equilibrium $\mathbf{q}$ exists if and only if $\mathbf{q} = 0$ is constrained efficient;
(b) equilibrium social welfare in a network (V, E) can be higher than in a network $(V, E \cup ij)$;
(c) equilibrium welfare of i in a network (V, E) can be higher than in a network $(V, E \cup ij)$.

When trade is necessary to achieve efficiency, any outflow equilibrium of an economy with a finite number of players is necessarily inefficient, as price distortions curtail trade in any local market. Social welfare moreover, can decline when new trading relationships are added to

[7] Complete networks thus, guarantee that low marginal utility players sell more. However, without further discipline on preferences it is impossible to guarantee that players with low marginal utilities also buy less from their neighbors. Section 2 of the web-appendix provides sufficient conditions on preferences for this to be the case.

the network.[8] A new trading link can further distort the allocation of consumption, as profit maximization by sellers may reallocate consumption from high value to low value buyers. The equilibrium payoff of a player can also, decline when new trading partner is added to his neighborhood. If a new trading partner increases the marginal value of consumption of player i (due to the option value of reselling units to the new partner), players selling units to i might curtail their supply to i in order to extract some of the surplus generated in the new relationship. Occasionally this phenomenon is so pronounced that in any outflow equilibrium player i's payoff declines when a new trade partner is added to his neighborhood.

These simple observations are proven in appendix: the first result formally, while the latter two by introducing two prototypical examples. All of the conclusions on welfare rely on the market power frictions implicit in any quantity competition model.[9] The main aim of the remainder of the analysis is to provide conditions on the network structure for such frictions to vanish when the number of players grows large.

Comments on Outflow Competition and Market Power: In the model presented nodes on a network were interpreted as separate local markets. Competitors used their access to different locations to price discriminate their customers. As discrimination within a local market was ruled out by linear pricing, discriminating across markets was welfare maximizing for suppliers. Preferences and access to markets jointly determined prices, welfare and market power. Goods were exchanged at local prices that differed from the competitive equilibrium price. Resale at positive markups was common even in well connected economies, and was driven by the arbitrage opportunities that the different prices in the economy offered to traders. The monopsony wedges were the main force limiting resale in the model, as the cost of supplying units was shown to increase along any supply chain. Although explicit characterization of market power remains desirable, non-linearities in complementarity problem characterizing equilibria implied that such results could not be derived for general functional forms.

3 Large Markets and Efficiency

This section analyzes outflow equilibrium behavior in economies with a large number of players. The main aim of the section is to provide necessary and sufficient conditions on the

[8] A negative relationship between social welfare and network density obtains in numerous other studies on networks, and is often referred to as Braess's Paradox in the context of this literature.

[9] Despite the negative conclusions obtained proposition 4, it would be interesting to argue that a link aways exists that, if added to the trade network, does not decrease either social or individual welfare. If so, the complete network would be both welfare maximizing and pairwise stable. The proof of such a conjecture however, remains an open question.

network topology to ensure that outflow equilibrium trades become efficient when the number of players in the economy is large. To impose some discipline on the network structure as the number of players diverges to infinity, the analysis introduces the notion of community structure of a trade network. A community is going to be defined as a complete subgraph in which all players share the same neighbors. Any trade network is going to be represented by a corresponding network among communities. Our analysis fixes the topology among communities and analyzes outflow equilibrium behavior when communities are large. The first preliminary result presents necessary and sufficient conditions on the network structure for the existence of trades among communities which are efficient and direct (without resale). Such conditions are an adaptation of Hall's marriage theorem to our more complex environment.[10] The analysis proceeds to show that these conditions are necessary and sufficient for the existence of an efficient outflow equilibrium when the number of players in every community is large. When all communities have the same magnitude, conditions imply that aggregate resale vanishes, and that efficiency attains in every community. When communities have different magnitudes, efficiency in the largest communities again requires the absence of intermediation between those communities. Smaller communities however, may remain inefficient unless the largest communities can mediate all of their trades. The section concludes with some examples and by discussing the relationship between social welfare and market size, and the existence of symmetric equilibria. Throughout the section and without loss of generality, restrict attention to economies in which the trade network G is connected, and in which trade is necessary to attain efficiency.

Community Structure, Efficiency and Market Clearing

The notion of community structure of a given trade network is now introduced. A subset of players $C \subseteq V$ is said to be a *community* in (V, E), if $V_i \cup \{i\} = V_j \cup \{j\} \supseteq C$ for any two players $i, j \in C$, and if $V_i \cup \{i\} \neq V_k \cup \{k\}$ for any player $k \in V \backslash C$. A community differs from clique in that all players need to share the same neighbors to belong to the same community.[11] Denote by $\mathbf{C}$ the set of communities of a network (V, E). The set $\mathbf{C}$ partitions the vertices of the original trade network into disjoint subsets of players. The *community structure* of a network (V, E) consists of network $(\mathbf{C}, \mathbf{E})$ with communities as vertices $\mathbf{C}$ and with edges between any two communities $C, K \in \mathbf{C}$ defined so that $CK \in \mathbf{E}$ if $ij \in E$ for some $i \in C$ and $j \in K$. The definition of community further implies that if two communities are linked then all of their players can trade with each other. The remainder of the analysis presents results in terms of the community structure of a trade network. The approach is

[10] Hall's marriage theorem provides conditions on a bipartite graph for the existence of a match that clears the short side of the market (Bollobas 1998).

[11] A clique is a maximal complete subgraph.

without loss of generality since there is a one-to-one mapping between community structure and network structure.

The definition of efficiency deliberately abstracts from the trade network, and sets as a benchmark the surplus that can be generated in an economy in which all trades can be executed.

Efficiency: *A profile of net-trades* $\bar{\mathbf{q}} \in R_+^V$ *is efficient for an economy* $\{V, \mathbf{Q}, \mathbf{u}\}$ *if it solves*

$$\bar{\mathbf{q}} \in \arg\max_{\mathbf{q} \in \mathbb{R}^V} \sum_{i \in V} u_i(q_i) \ \text{ s.t. } \ q_i \geq -Q_i \ \text{ for } \forall i \in V.$$

Assumption A1 trivially implies that a such an allocation exist, and that it would attain as a competitive equilibrium in the corresponding centralized market. Let $D = \{i \in V | \bar{q}_i > 0\}$ denote the set of players who would demand consumption at the efficient profile $\bar{\mathbf{q}}$, and let $S = \{i \in V | \bar{q}_i < 0\}$ denote the set of players who would supply consumption at the efficient profile $\bar{\mathbf{q}}$. Refer to players in D as *buyers,* and to players in S as *sellers*. For any community $C \in \mathbf{C}$, denote by $q_C^+ = \sum_{i \in C \cap D} \bar{q}_i$ the aggregate demand of the community, and denote by $q_C^- = \sum_{i \in C \cap S} \bar{q}_i$ the aggregate supply of the community. The efficient net-trade of the community can thus, be defined by the difference between these two quantities, $\bar{q}_C = q_C^+ + q_C^-$.[12] For any subset of communities $T \subseteq \mathbf{C}$, let W_T denote the set of communities that that can trade with at least one community in T. That is

$$W_T = T \cup \{K \in \mathbf{C} \mid CK \in \mathbf{E} \ \text{ for some } \ C \in T\}.$$

Define the *excess-supply* and the *excess-demand* faced by a subset of communities $T \subseteq \mathbf{C}$ respectively by

$$\begin{aligned} \sigma(T, \bar{\mathbf{q}}) &= -\sum_{C \in W_T} q_C^- - \sum_{C \in T} q_C^+ \text{ and} \\ \delta(T, \bar{\mathbf{q}}) &= \sum_{C \in W_T} q_C^+ + \sum_{C \in T} q_C^-. \end{aligned}$$

The excess-supply of a group of communities T is defined by the difference between the aggregate supply of communities who can directly sell to communities in T and the aggregate demand of communities in T. The excess-demand of a group of communities is similarly defined by the difference between the aggregate demand of communities who can purchase directly from communities in T and the aggregate supply of communities in T.[13] Given these definitions, it is possible to introduce an important condition on an economy that will play

[12] The aggregate supply of any community is always expressed as a non-positive number in the context of the model.

[13] The definitions of excess-demand and excess-supply depend on the network structure and on the notion of efficiency, but not on any element of the outflow competition setup.

an important role in the remainder of the analysis.

Condition MC: *Economy* $\{\mathbf{C}, \mathbf{E}, \mathbf{Q}, \mathbf{u}\}$ *satisfies MC if* $\sigma(T, \bar{\mathbf{q}}) \geq 0$ *for any* $T \subseteq \mathbf{C}$.

The condition requires any group of communities to face a non-negative excess-supply from communities to which they are linked to. A simple economy satisfying condition MC is one in which every seller is linked every buyer. If so, MC holds trivially as the aggregate excess supply equals zero by market clearing, $\sigma(\mathbf{C}, \bar{\mathbf{q}}) = 0$.[14] The next result is an adaptation of Hall's marriage theorem to our environment. It establishes that condition MC is both necessary and sufficient for the existence of direct flows of consumption from sellers to buyers that support an efficient allocation $\bar{\mathbf{q}}$. The proposition also establishes that condition MC is equivalent to requiring any group of export communities to face a non-negative excess demand.

Proposition 5 *For any economy* $\{\mathbf{C}, \mathbf{E}, \mathbf{Q}, \mathbf{u}\}$ *the following three statements are equivalent:*
(a) the economy satisfies MC;
(b) $\delta(T, \bar{\mathbf{q}}) \geq 0$ *for any* $T \subseteq \mathbf{C}$*;*
(c) there exists $\mathbf{q} \in \mathbb{R}_+^E$ *such that:*

$$\bar{q}_i = \sum_{j \in S \cap V_i} q_i^j \quad \text{for } \forall i \in D, \qquad (i)$$
$$\bar{q}_i = -\sum_{j \in D \cap V_i} q_j^i \quad \text{for } \forall i \in S. \qquad (ii)$$

The third statement in proposition amounts to the existence of consumption flows that clear markets in environments in which intermediation is impossible. Thus, the result implies that efficient and direct flows of consumption exist if and only if any subset of communities can have its efficient net-trade met by those communities to which they are linked to.[15] As in the marriage theorem, the more surprising part of the result is that condition MC is sufficient for the existence of direct and efficient flows of consumption, since necessity obtains trivially. The existence of direct and efficient consumption flows will be essential to the analysis of behavior in large markets, as resale would always distort in the proposed quantity competition model.

Large Economies and Outflow Competition Efficiency

[14] As condition MC hinges on the definitions of D and S and in turn on the definition of efficiency, some knowledge on preferences and technologies would required to test whether condition MC holds whenever the network is not complete.

[15] In the context of Hall's marriage theorem, condition MC would simplify to having any group of players on one side of the market linked to a group of players on the other side of the market which has at least its size. Also in that context the definition of MC would rely both on the network structure and on the notion of efficiency (as a match between players on two sides of the market has value, whereas one between players on the same side has none).

The next results provide sufficient conditions on the community structure for the existence of an efficient outflow equilibrium when the number of players in some communities is large. Taking limits as communities become large given fixed the community structure is a convenient approach to modelling large markets, as it affects the extent of the competition in each community without changing the overall topology among communities. If communities were interpreted as countries and the community structure as the network of the trade agreements among countries, our analysis would aim at providing conditions on the network of trade agreements to ensure that trade is efficient when countries are large.

The analysis begins by considering large markets in which communities can differ in size, but remain comparable in their magnitude. To do so, for any possible community structure $(\mathbf{C}, \mathbf{E})$ define a convenient sequence of increasing economies, in which players grow at the same rate in every community.

Replica: $\{\mathbf{C}^z, \mathbf{E}^z, \mathbf{Q}^z, \mathbf{u}^z\}$ *is a z-replica economy of* $\{\mathbf{C}, \mathbf{E}, \mathbf{Q}, \mathbf{u}\}$ *if for* $z \in N_+$*:*
[R1] $C^z = \{i.s | i \in C \;\; \& \;\; s \in \{1, ..., z\}\}$ for any $C \in \mathbf{C}$;
[R2] $\mathbf{C}^z = \{C^z | C \in \mathbf{C}\} \;\; \& \;\; \mathbf{E}^z = \{C^z K^z | CK \in \mathbf{E}\}$;
[R3] $Q^z_{i.s} = Q_i \;\; \& \;\; u^z_{i.s} = u_i$ for any $i \in V \;\; \& \;\; s \in \{1, ..., z\}$.

The first condition states that in a z-replica each community consists of z copies of the players in the original community. The second establishes that community structure is not affected by replication. The third and final condition clarifies that all copies of a player have the same capacity and preferences. While increasing competition within each community, the notion of replica preserves the community structure in an economy and the composition of players within each community (as the same players belong to every community in larger numbers). A convenient feature of such a balanced replication process is that the efficient net-trades of players do not change as an economy grows large. Therefore, buyers (sellers) in an economy remain buyers (sellers) in anyone of its replicas.

The notion of replica is introduced for sake of tractability, since any large market with communities of comparable magnitude, populated by finitely many types of players, can be well approximated by a replica economy. A sequence of replica economies $\{\mathbf{C}^z, \mathbf{E}^z, \mathbf{Q}^z, \mathbf{u}^z\}_{z=1}^{\infty}$ is said to *converge to efficiency*, if there exists a sequence of outflow equilibria in which net-trades converge pointwise to the efficient net-trades for every player in every community. The first trivial observation establishes that trade can converge to efficiency only if there exists a limit outflow equilibrium in which no player resells significant amounts of consumption.

Proposition 6 *When A1 holds, if a sequence of replica economies converges to efficiency then there exists an equilibrium in which no individual resells consumption in the limit.*

In the outflow model intermediaries command a rent (that distorts trade) whenever they are necessary to distribute goods. Players may become price takes as sellers (when each local market becomes more competitive), but never as buyers since they retain the monopsony power when purchasing goods at their individual location. Thus, the wedge on inflow prices cannot disappear whenever resale persists.

The next result is central to the analysis of large markets, and shows that condition MC is both necessary and sufficient for outflow equilibrium net-trades to converge to efficiency when all communities have comparable magnitudes.

Proposition 7 *When A1 holds, a sequence of replica economies* $\{\mathbf{C}^z, \mathbf{E}^z, \mathbf{Q}^z, \mathbf{u}^z\}_{z=1}^{\infty}$ *converges to efficiency if and only if MC holds in the economy* $\{\mathbf{C}, \mathbf{E}, \mathbf{Q}, \mathbf{u}\}$*.*

Whenever intermediation is superfluous, competition among large numbers of sellers at each local market eliminates rents on all trades. If so, outflow equilibrium net-trades converge to efficiency, and a unique price reigns in the limiting economy. Large economies failing condition MC however, cannot converge to efficiency.[16] In such scenarios, a subset of players would have to resell a non-negligible amount of consumption for the efficient allocation to obtain, and such players would necessarily extract a rent whenever they are required to mediate trade. The result thus views anonymous centralized Walrasian markets as approximations of a non-anonymous decentralized markets in which a large number of buyers and sellers can directly trade with each other (as would be the case in an economy in which every community with a non-negligible aggregate demand was able trade with every other community with a non-negligible aggregate supply).

The next part of the analysis modifies the definition of replica to consider large economies in which communities differ in magnitude. In particular, unbalanced replication processes are introduced in which a subset of communities grows at a common rate, while the remaining communities remain small. Because convergence to efficiency would always be determined by the communities with the largest magnitude, the approach is almost without loss.

Unbalanced Replica: $\{\mathbf{C}^z, \mathbf{E}^z, \mathbf{Q}^z, \mathbf{u}^z\}$ *is a* $\hat{\mathbf{C}}$*-unbalanced* z*-replica economy of* $\{\mathbf{C}, \mathbf{E}, \mathbf{Q}, \mathbf{u}\}$ *if for* $\hat{\mathbf{C}} \subset \mathbf{C}$ *and* $z \in N_+$*:*

[U1] $C^z = \{i.s | i \in C \;\; \& \;\; s \in \{1, ..., z\}\}$ for $C \in \hat{\mathbf{C}}$ and $C^z = C$ otherwise;

[U2] $\mathbf{C}^z = \{C^z | C \in \mathbf{C}\} \;\; \& \;\; \mathbf{E}^z = \{C^z K^z | CK \in \mathbf{E}\}$;

[U3] $Q^z_{i.s} = Q_i \;\; \& \;\; u^z_{i.s} = u_i$ for any $i \in V \;\; \& \;\; s \in \{\emptyset, 1, ..., z\}$.

The only difference with respect to the notion of replica lies in the first condition which states that only communities in $\hat{\mathbf{C}}$ are large. In contrast to the definition of replica, efficient

[16] A convenient feature of the proposed balanced replication process is that condition MC can imposed directly on the unreplicated economy rather than on the entire sequence of replicas.

net-trades may now change as an economy gets replicated, since the composition of players in the economy might be affected by the replication process. In the limit economy, most of the surplus will originate in the large communities. Let $\hat{V} = \cup_{C \in \hat{\mathbf{C}}} C$ denote the set of players located in one of the large communities. The definition of replica now implies that the set of players in located in small communities is given by $V \backslash \hat{V}$ in any of of its $\hat{\mathbf{C}}$-unbalanced z-replicas. For any $\hat{\mathbf{C}}$-unbalanced replica economy $\{\mathbf{C}^z, \mathbf{E}^z, \mathbf{Q}^z, \mathbf{u}^z\}$, let $\bar{\mathbf{q}}^z$ denote the efficient net-trades in the economy, and let $\bar{\mathbf{q}}^\infty = \lim_{z \to \infty} \bar{\mathbf{q}}^z$. To guarantee that the efficient net-trades remain bounded when an unbalanced replica grows large, an additional assumption on preferences will be imposed.

Assumption A3 *For any player $i \in V$, the utility u_i satisfies $\lim_{q \to \infty} u_i'(q) = 0$.*

A sequence of $\hat{\mathbf{C}}$-unbalanced replica economies $\{\mathbf{C}^z, \mathbf{E}^z, \mathbf{Q}^z, \mathbf{u}^z\}_{z=1}^\infty$ is said to *converge to approximate efficiency*, if there exists a sequence of outflow equilibria in which net-trades converge pointwise to the efficient net-trades for every player in any community belonging to $\hat{\mathbf{C}}$. The notion of approximate efficiency is introduced as the surplus of any small community is negligible compared to aggregate surplus when an economy is large. For any economy $\{\mathbf{C}, \mathbf{E}, \mathbf{Q}, \mathbf{u}\}$ and any subset of communities $\hat{\mathbf{C}}$, consider the economy $\{\hat{\mathbf{C}}, \hat{\mathbf{E}}, \mathbf{Q}, \mathbf{u}\}$ obtained by deleting communities that do not belong to $\hat{\mathbf{C}}$,

$$\hat{\mathbf{E}} = \left\{ CK | C, K \in \hat{\mathbf{C}} \quad \cap \quad CK \in \mathbf{E} \right\}.$$

The next result generalizes proposition 7 and establishes why the existence of efficient and direct flows of consumption in the economy $\{\hat{\mathbf{C}}, \hat{\mathbf{E}}, \mathbf{Q}, \mathbf{u}\}$ is essential for a sequence $\hat{\mathbf{C}}$-unbalanced replicas to converge to approximate efficiency.

Proposition 8 *When A1 and A3 hold, a sequence of $\hat{\mathbf{C}}$-unbalanced replicas $\{\mathbf{C}^z, \mathbf{E}^z, \mathbf{Q}^z, \mathbf{u}^z\}_{z=1}^\infty$ converges to approximate efficiency if and only if MC holds in the economy $\{\hat{\mathbf{C}}, \hat{\mathbf{E}}, \mathbf{Q}, \mathbf{u}\}$.*

Whenever intermediation is superfluous in the largest communities, equilibria in which flows converge to approximate efficiency exist. In these equilibria, all goods are traded at a unique price in every large community. No distortion affects pricing, as a large number of sellers compete to supply any group of buyers belonging to $\hat{\mathbf{C}}$.[17] Prices however, may differ in small communities as market power and resale rents can still distort pricing at such locations. The result again views anonymous centralized Walrasian markets as approximations of behavior in large communities of non-anonymous decentralized markets in which a large number of buyers and sellers can directly trade with each other.

[17] A convenient feature of the proposed unbalanced replication process is that condition MC can imposed directly on the subnetwork of large communities rather than on the entire sequence of replicas.

The final result on large markets imposes further assumptions on the community structure to guarantee that net-trades converge to efficiency in every community of an unbalanced replica. In particular consider the following additional requirement.

Condition IC: *A sequence of* $\hat{\mathbf{C}}$*-unbalanced replicas* $\{\mathbf{C}^z, \mathbf{E}^z, \mathbf{Q}^z, \mathbf{u}^z\}_{z=1}^{\infty}$ *satisfies IC if*
(a) for any $i \in V \backslash \hat{V}$ *such that* $\bar{q}_i^{\infty} < 0$*, there exist* $j \in \hat{V} \cap V_i$*;*
(b) for any $i \in V \backslash \hat{V}$ *such that* $\bar{q}_i^{\infty} > 0$*, there exist* $j \in \hat{V} \cap V_i$ *such that* $\bar{q}_j^{\infty} < 0$*.*

The two conditions together imply that any small community requiring trade must be linked to larger one. The second condition further implies that a large number of sellers competes to supply any buyer living in a small community. Condition IC may hold even when the economy $\{\mathbf{C}, \mathbf{E}, \mathbf{Q}, \mathbf{u}\}$ violates MC. The final result establishes that, whenever IC holds, net-trades converge to efficiency even in small communities. The result obtains, as IC implies that players in small communities have access to a large pool of buyers or sellers to meet their efficient net-trades.

Proposition 9 *When A1 and A3 hold, a sequence of* $\hat{\mathbf{C}}$*-unbalanced replicas* $\{\mathbf{C}^z, \mathbf{E}^z, \mathbf{Q}^z, \mathbf{u}^z\}_{z=1}^{\infty}$ *converges to efficiency if and only if MC holds in the economy* $\{\hat{\mathbf{C}}, \hat{\mathbf{E}}, \mathbf{Q}, \mathbf{u}\}$ *and IC holds for the sequence of* $\hat{\mathbf{C}}$*-unbalanced replicas.*

Condition IC guarantees that large communities can directly clear every local market and attain the efficient net-trades. If so, intermediation and distortions vanish even in small markets as competition from the larger communities disciplines prices by reducing rents on every trade. Convergence to efficiency would not obtain in smaller markets, if IC were to fail, as price distortions would necessarily curtail trade in some of the small and poorly connected communities.

Other studies on two-sided networks and matching have exploited variants of condition MC to clear markets and achieve efficiency in decentralized markets. All of these studies however, had to rule out intermediation to establish the necessity of such conditions. Within the outflow competition framework, MC was proven to be necessary and sufficient for convergence to efficiency even in environments in which resale was feasible. This observation differs from most other studies exploiting Hall theorem type arguments in which the necessity of MC to clear markets is regularly built in the trading environment. Although the analysis relied on particular unbalanced replication processes, results would extend to more general replication processes in which all communities grow at possibly heterogeneous rates. Convergence to approximate efficiency in those environments would still rely on MC holding among the largest communities of the limit economy. Conditions for convergence to efficiency however, would slightly differ in those environments, as communities that are neither large nor small may occasionally mediate trade between larger and smaller communities without creating frictions. The main aim of this part of the analysis was to provide simple conditions for trading behavior in large decentralized markets to emulate behavior in large centralized markets. Sequences of replicas were only invoked here, as a parsimonious method to represent large decentralized markets in which every location is populated by finitely many types of traders.

Examples: Large Economies and Replication

Before proceeding to the final results, consider three examples of replica economy. In each example the economy consists of three communities C_a, C_b, C_c. The preferences of all the individuals in community C_i satisfy $u_i(q) = (Q_i + q)^{1/2}$ for $i \in \{a, b, c\}$, where $Q_a = 2$, $Q_b = 1$, and $Q_c = 0$. In this example it is convenient to interpret Q_i and $Q_i + q_i$ respectively as the endowment and the equilibrium consumption of a player in community i. Community C_a is populated only by sellers, community C_c is only by buyers, while players in community C_b are neither sellers nor buyers. The first two examples present two replica economies that differ only in their community structure, while the third one differs in the replication process. Begin by considering the economy in which all the three communities are connected, and form a grand community (depicted in the left plot of figure 2). The economy trivially satisfies condition MC, and thus converges to efficiency if replicated sufficiently many times. The equilibrium consumption of every player in the economy converges to one. Consumption in the sellers' community decreases monotonically, while consumption in the buyers' community increases monotonically. The price paid by players in community C_b converges from below to the competitive equilibrium price, 1/2. The price paid by buyers in the import community C_c instead, monotonically decreases to the same value. Equilibrium resale by players in community C_b vanishes, and such players do not trade in the limit economy. Per-capita social welfare increases monotonically as the economy grows large. The left plots in figures 3 and 4 depict consumption and prices in the unique equilibrium of the proposed sequence of replicas.

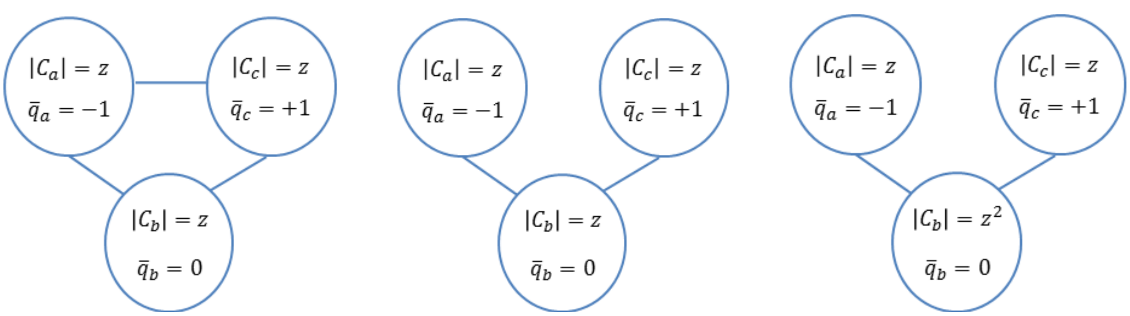

FIGURE 2: Community structure. Communities appear as linked circles.

Next consider the same economy, but suppose that sellers in community C_a cannot trade directly with buyers in community C_c (depicted in the central plot of figure 2). If so, players in community C_b act as middlemen buying from sellers in community C_a to supply buyers in community C_c. The economy cannot satisfy condition MC, since no direct trade between sellers and buyers is feasible. Thus, no sequence of outflow equilibria of its replicas can ever converge to efficiency. Outflow equilibrium consumption in the three communities does not converge. In the limit economy, consumption by players in communities C_a and C_b exceeds that of any player in community C_c. The price paid by middlemen in community C_b first grows and then declines converging to a value below the competitive equilibrium price. The price paid by buyers in community C_c instead, monotonically decreases, but always remains significantly above the competitive price. The limit markup made by middlemen is approximately 30%. Per-capita social welfare increases monotonically as the economy grows large, but remains inefficient in the limit economy. The central plots in figures 3 and 4 depict consumption and prices in the unique equilibrium of the proposed sequence of replicas. When all communities grow at the same rate, the outflow model recognizes that the second community structure cannot attain efficiency while mimicking an anonymous Walrasian market, as players in community C_b must act as intermediaries while transferring a non-negligible amount of consumption from sellers in C_a to buyers in C_c.

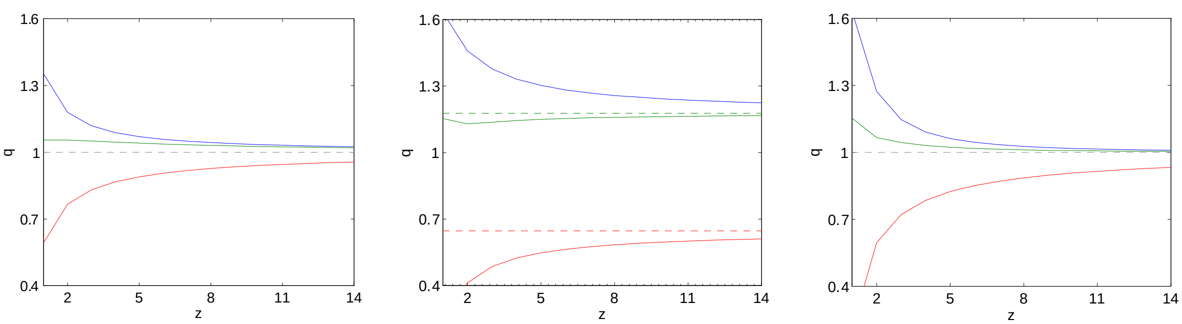

FIGURE 3: In each plot on the vertical axis consumption on the horizontal the replica.

The final example considers an unbalanced replica of the second community structure, in which players in community C_b grow at a faster rate than other players in the economy (depicted in the right plot of figure 2). In particular, the z^{th} element of the unbalanced replica considered here possesses z players in communities C_a and C_c, and z^2 in community C_b. Any economy in the sequence still violates condition MC, as no direct trade is feasible between the sellers' community and buyers' community. However, the unique symmetric equilibrium of this sequence of unbalanced replicas also converges to efficiency. Approximate efficiency obtains, as only one community is large in the limit and thus MC among large communities trivially holds. Efficiency also obtains, since large communities can clear the aggregate demand and the aggregate supply of any smaller community when the market is sufficiently large. The example highlights why results on unbalanced replicas also apply to replication processes in which communities grow at different rates, and why intermediation between communities has to take place in the largest communities. Consumption and prices for such a sequence of economies are plotted in the right plots in figures 3 and 4.

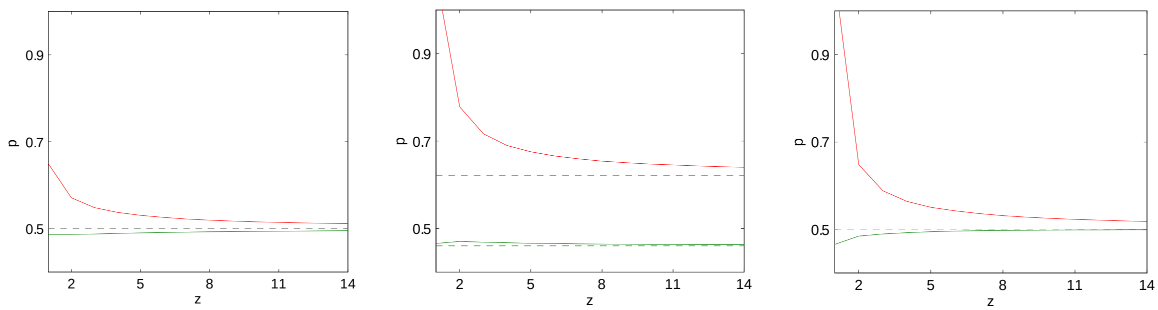

FIGURE 4: In each plot on the vertical axis prices on the horizontal the replica.

Concluding Results on Balanced Replication

The analysis concludes with two minor results on balanced replication. The first result presents sufficient condition for symmetric equilibrium existence in large balanced replicas, while the latter relates welfare to the size of the market. For any sequence of symmetric equilibria of a replica, let $\breve{q}^i_j = \lim_{z\to\infty} z q^{i.t}_{j.s}(z)$ denote the amount of goods sold in the limit economy from an individual of type i to all individuals of type j. In any symmetric equilibrium of the limit economy optimality of flows requires,

$$\breve{q}^i_j(u'_j(\breve{q}_j) - u'_i(\breve{q}_i) + u''_i(\breve{q}_i)\breve{q}^\circ_i - \mu_i) = 0,$$

where μ_i denotes the non-negative multiplier on the outflow constraint, $\breve{q}^i_\circ \leq Q_i$. Outflow price distortions vanish in a symmetric equilibrium, because in the limit economy infinitely many individuals compete to supply each neighbor. The price distortions on inflows instead, persist for those individuals who resell consumption in the limit economy. However, since the outflow wedges were the complicating factor in the proof of existence, stronger results can be stated for the limit economy.

Proposition 10 *If A1 holds, the following three results follow:*
(a) if $u'''_i \geq 0$, a symmetric outflow equilibrium exists in the limit economy;
(b) if MC holds, an efficient outflow equilibrium exists in the limit economy;
(c) if $V_i \supseteq S$ for any $i \in D$, a unique outflow equilibrium exists in the limit economy.

In either case, revenues in each local market are concave in the limit economy. Proposition 10 shows that, when (a) holds, the costs of supplying outflows are convex in the limit by the assumption on the third derivative. Cases (b) and (c) instead hold, as condition MC implies the existence of an efficient outflow equilibrium in the limit economy. The stronger conditions on the market structure imposed in (c) further imply that all equilibria converge to efficiency when all sellers and buyers can directly trade.

The final result establishes why the proposed notion of balanced replica implies that larger markets are more efficient. In particular, the proposition establishes that per-capita social welfare increases monotonically as an economy gets replicated, whenever a unique symmetric equilibrium exists. Intuitively, the definition of balanced replica yields the result as competition increases uniformly at every location of the trade network.

Proposition 11 *If A1 holds and if any z-replica possesses a unique symmetric equilibrium, then per-capita social welfare increases every time the economy is replicated.*

Even economies failing MC become more competitive (though not perfectly competitive) as the number of players grows large. The proof of the proposition exploits the definition of balanced replica to establish a link between social welfare and network structure by studying how changes in the number of player at each location would affect the Jacobian matrix of the complementarity problem characterizing the symmetric equilibria of a replica. Although the result offers limited testable implications as it relies on the balanced nature of the proposed replication process, it establishes an interesting property of such a process.

The aim of the section was to present conditions under which competition in large oligopolistic decentralized markets could mimic perfect competition. The results showed that trade had to be either direct, or mediated by a very large number of intermediaries, for this to be the case. Economies, in which trade among buyers (or sellers) was necessary, would instead, never approximate perfect competition and efficiency as distortions would inevitably affect pricing in such markets.

4 Inflow Competition

This section outlines a similar quantity competition model, and compares it to the outflow model. The web-appendix presents a more detailed discussion and examples. For sake of brevity, the analysis assumes that players are constrained in the amount of units that they can purchase. Alternative specifications in which the supply side is constrained would yield similar results.

Inflow Competition: In the inflow competition model, every player owns a trading location at which anyone of his neighbors can buy goods. Rather than deciding on how many units to sell, players simultaneously decide how many units of consumption to buy at each of their neighboring locations. Any player i is constrained not to buy more than Q_i units of consumption. Prices are determined at each location so that all units are sold at their marginal cost. The price paid by player i for units sold from a neighbor j is determined by the inverse supply curve with respect to net-trade at node j,

$$p_i^j(\mathbf{q}) = p^j(q_j) = u_j'(q_j) = u_j'(q_j^\circ - q_\circ^j).$$

Buyers expect such prices when choosing their demands. Players sell all of their outflows at unique price which coincides with the marginal value of the last unit supplied. Assumption A1 again implies that $\partial p^j(q_j)/\partial q_i^j > 0$ and $\partial p^j(q_j)/\partial q_j^i < 0$ for any $i \in V_j$. The price earned by player j decreases when his inflows increase, and increases when his outflows increase. Again an argument à la Kreps and Scheinkman could be used to show that if individuals were to commit to their inflows, price competition among buyers would lead to such prices in each local market.[18] For $X_i = \left\{ \mathbf{q}_i \in \mathbb{R}_+^{V_i} \middle| \, q_i^\circ \leq Q_i \right\}$, the problem of an individual $i \in V$ thus, reduces to

$$\max_{\mathbf{q}_i \in X_i} u_i(q_i) + \textstyle\sum_{j \in V_i} [p^i(q_i) q_j^i - p^j(q_j) q_i^j].$$

If $q_i^j > 0$ and $q_i^\circ < Q_i$, optimality of the flow q_i^j requires that

$$p^i(q_i) - p^j(q_j) = -\frac{\partial p^j(q_j)}{\partial q_j} q_i^j - \frac{\partial p^i(q_i)}{\partial q_i} q_\circ^i = 0.$$

If so, the markup on the flow q_i^j (i.e. the difference between the marginal value and the price paid) is completely determined by two wedges: namely monopoly price distortion on all units sold, and the Cournot distortion on the units purchased from seller j. Optimality in the inflow model differs from the outflow model, as a different distortions affect equilibrium pricing. Whereas suppliers were able to commit to their sales in the outflow model, buyers are able to commit to their purchases in the inflow model. The ability to commit to trade flows benefits the players executing trades by allowing them to appropriate more gains from trade. Thus, an inflow economy is in general more efficient than an outflow economy, as more units flow to individuals with higher marginal values. The expression *inflow equilibrium* is used to refer to a pure strategy Nash equilibrium of the inflow competition model.

Result Survey: Almost all of the results developed in the context of the outflow model also,

[18] Individuals offering a lower price would be worse off since part of their demand would not be not met. Individual offering a higher price would be worse off since their demand could be met at a lower price.

apply to the inflow model. Sufficient conditions for inflow equilibrium existence differ slightly form conditions imposed in the outflow model, and are reported in the web-appendix. In the inflow model sellers supply all their customers at a single price. Buyers however, purchase goods from different suppliers at different prices. It is in their best interest to do so, because price distortions would increase their expenditure, if they were to concentrate their demand in a single market. As in the outflow model resale is a common feature of equilibrium behavior, but linked individuals with different marginal rates of substitution do not necessarily trade. Sufficient conditions for trade to take place between pairs of linked individuals require gains from trade to exceed the outflow price distortion of the buyer. Examples reported in the web-appendix establish that adding links can still reduce social welfare or the welfare of one of the two individuals being connected. Results on large markets would not be affected by the change in the pricing paradigm. Again, economies in which intermediation would not vanish would never become attain efficiency, whereas economies satisfying the condition MC would.

5 Conclusions

When does a centralized Walrasian market well approximate behavior in a decentralized oligopolistic market? Providing a simple answer to this question was the main aim of the analysis. A tractable model of oligopolistic competition in networked markets was introduced. Distinguishing features of the model were option to resell goods and the endogenous identity of buyers and sellers in the economy. Obviously, Walrasian competition would never well approximate competition in markets of finite size as market power would always distort trade. However, Walrasian markets were shown to well approximate behavior large decentralized markets in which sufficiently many sellers could directly compete to supply any group of buyers, and in which no specific individual or group of individuals was required to intermediate goods in the economy. In such scenarios efficiency would obtain and all trades at large locations would be executed directly without recourse to intermediation. If so, intermediation would persist only to clear markets at locations of negligible size.

Strong assumptions on trade costs between individuals were implicit in both of the quantity competition models presented. Trade was assumed to be costless between linked individuals, but extremely costly between any other pair of traders. Such restrictions however, were only imposed for sake of clarity. In fact, the model could be easily generalized by assuming the network to be complete and trade costs to be heterogeneous between pairs of players. Similar results would hold. Assumptions also, required the marginal utility of consumption to be positive for any player in the economy. However, the proposed framework would well approximate environments in which those who resell units do not want to consume, if the marginal utility of consumption of such players were chosen to be sufficiently low. Other limitations of the analysis were the omission of an explicit network formation model, and consequent impossibility of migration. Indeed, it would be interesting to know if migration would always lead to efficient community structures. However, this question lies beyond the scope of this manuscript.

References

[1] Abreu D., Manea M., "Bargaining and Efficiency in Networks", Journal of Economic Theory 147/1, 2012.

[2] Acemoglu D., Ozdaglar A., "Competition in Parallel Serial-Networks", IEEE Journal on Selected Areas in Communication, 25/6, 2007.

[3] Ahuja R., Magnanti T., Orlin J., "Network Flows", Prentice Hall, 1993.

[4] Blume L., Easley D., Kleinberg J., Tardos E., "Trading Networks with Price-Setting Agents", Electronic Commerce, Proceedings of the 8th ACM conference, 2007.

[5] Bollobas B., "Modern Graph Theory", Springer, 1998.

[6] Chawla S., Roughgarden T., "Bertrand Competition in Networks", SAGT, 2008.

[7] Condorelli D., Galeotti A., "Bilateral Trading in Networks", Working Paper, 2012.

[8] Corominas-Bosch M., "Bargaining in a Network of Buyers and Sellers", Journal of Economic Theory 115/1, 2004.

[9] Cournot A., "Recherches sur les Principes Mathematiques de la Théorie des Richesses", 1838, Translation Macmillan, 1897.

[10] Goyal S., Joshi S., "Networks of Collaboration in Oligopoly", Games and Economic Behavior 43, 2003.

[11] Hatfield J., Kominers S., "Matching in Networks with Bilateral Contracts", American Economic Journal: Microeconomics, 4/1, 2012.

[12] Ilkilic R., "Cournot Competition on a Network of Markets and Firms", Working Paper, 2010.

[13] Jackson M., "Social and Economic Networks", Princeton University Press, 2008.

[14] Kalai E., Postlewaite A., Roberts J., "Barriers to Trade and Disadvantageous Middlemen: Nonmonotonicity of the Core", Journal of Economic Theory 19/1, 1978.

[15] Kakade S., Kearns M., Ortiz L., "Graphical Economics", Lecture Notes in Computer Science 3120, 2004.

[16] Karamardian D., "The Complementarity Problem", Mathematical Programming 2, 1972.

[17] Kolstad C., Mathiesen L., "Necessary and Sufficient Conditions for Uniqueness of a Cournot Equilibrium", Review of Economic Studies 54/4, 1987.

[18] Kranton R., Minehart D., "Competition in Buyer-Seller Networks", American Economic Review 91/3, 2001.

[19] Kreps D., Scheinkman J., "Quantity Pre-commitment and Bertrand Competition Yield Cournot Outcomes", Bell Journal of Economics 14/2, 1983.

[20] Lever C., "Price Competition on a Buyer-Seller Network", Working Paper, 2010.

[21] Manea M., "Bargaining on Stationary Networks", American Economics Review 101/5, 2011.

[22] Manea M., "Intermediation in Networks", Working Paper, 2013.

[23] Polanski A., "Bilateral bargaining in networks", Journal of Economic Theory 134, 2007.

[24] Polanski A., Vega-Redondo F., "Markets, Bargaining and Networks with Heterogeneous Agents", Working Paper, 2013.

[25] Polanski A., Winter E., "Endogenous Two-Sided Markets with Repeated Transactions", BE Journal of Theoretical Economics 10, 2010.

[26] Ostrovsky M., "Stability in Supply Chain Networks", American Economic Review 98/3, 2008.

[27] Siedlarek J., "Exchange with Intermediation in Networks", Working Paper, 2013.

6 Appendix

Outflow Equilibrium Existence

Proof of Proposition 1. Begin by establishing (a). For every player $i \in V$, the set of feasible outflows $X_i = \left\{\mathbf{q}^i \in \mathbb{R}^{V_i} | q^i_\circ \leq Q_i \quad \cap \quad q^i_j \geq 0\right\}$ is clearly non-empty, compact, convex. Sufficient conditions for the best reply maps to be single-valued require: that for every player i revenues from the sales to each neighbor $j \in V_i$ be concave in q^i_j; that his costs of supplying units be convex in outflows $\mathbf{q}^i$; and that one of the two conditions be strict. Revenues are concave in each market, if for any $ij \in E$

$$\partial^2 R^i_j(q^i_j, \mathbf{q}^{-i})/(\partial q^i_j)^2 = 2u''_j(q_j) + q^i_j u'''_j(q_j) \leq 0. \tag{E1}$$

Since q_i is a linear function of every outflow q^i_j and since outflows affect costs only though consumption q_i, costs $C_i(\mathbf{q}^i, \mathbf{q}^{-i})$ are a convex in the vector $\mathbf{q}^i$ whenever $C_i(\mathbf{q}^i, \mathbf{q}^{-i})$ is convex in q_i

$$\partial^2 C_i(\mathbf{q}^i, \mathbf{q}^{-i})/(\partial q_i)^2 = -u''_i(q_i) + q^\circ_i u'''_i(q_i) \geq 0. \tag{E2}$$

Assumptions A1 and B1 imply that E1 and E2 hold, with at least one of the two holding strictly. In particular, since by feasibility $Q_j + q_j \geq q^\circ_j \geq q^i_j$, A1 and the upperbound in B1 imply that revenues are concave since

$$2u''_j(q_j) + q^i_j u'''_j(q_j) \leq (2u''_j(q_j) + (Q_j + q_j)u'''_j(q_j))\mathbb{I}(u'''_j(q_j) > 0) \leq 0,$$

where $\mathbb{I}(\cdot)$ denotes the indicator function.[19] Moreover, since $Q_i + q_i \geq q^\circ_i$, A1 and the lowerbound in B1 imply that costs are convex since

$$-u''_i(q_i) + q^\circ_i u'''_i(q_i) \geq (-u''_i(q_i) + (Q_i + q_i)u'''_i(q_i))\mathbb{I}(u'''_i(q_i) < 0) \geq 0.$$

Since both indicator maps cannot hold at once either revenues are strictly concave, or costs are strictly convex. Thus, A1 and B1 imply that payoffs are strictly concave and continuous for each player. Strict concavity of payoffs and the compactness and convexity of choice set X_i require the best-response correspondences to be single-valued. Continuity of the payoffs implies (by Berge's theorem of the maximum) that best responses are continuous. Thus, the existence of outflow equilibrium is guaranteed by Brouwer's fixed point theorem.

Next observe that sufficient conditions for best reply maps to be single-valued do not need to discipline payoffs when the revenues from selling units are decreasing. In fact, such

[19] In particular $\mathbb{I}(A) = \begin{cases} 1 & \text{if} \quad A \text{ is true} \\ 0 & \text{if} \quad A \text{ is false} \end{cases}$.

outflows could never be a best reply for the player selling the units, as marginal costs are positive by assumption A1. Thus, to grant existence it suffices to show A1 and B2 imply that E1 and E2 hold whenever revenues increase. The rest of the argument shows that A1 and B2 imply that the revenue of player i from sales to every neighbor $j \in V_i$ is concave in q^i_j and that his costs of supplying units are convex in $\mathbf{q}^i$, whenever the revenue from selling units to i increases. Revenues in market i increase, if $u'_i(q_i) + q^j_i u''_i(q_i) \geq 0$. If so,

$$V_i u'_i(q_i) + q^{\circ}_i u''_i(q_i) \geq 0.$$

where the implication holds by summing over all neighbors j. Thus, A1 and the lowerbound in B2 imply that costs are convex when revenues increase, since

$$-u''_i(q_i) + q^{\circ}_i u'''_i(q_i) \geq V_i u'_i(q_i) \left(-\frac{u''_i(q_i)}{u'_i(q_i)V_i} - \frac{u'''_i(q_i)}{u''_i(q_i)} \right) \mathbb{I}(u'''_i(q_i) < 0) \geq 0.$$

Similarly, observe that A1 and the upperbound in B2 imply that revenues are concave when revenues increase, since

$$2u''_j(q_j) + q^i_j u'''_j(q_j) \leq -u'_i(q_i) \left(\frac{u'''_i(q_i)}{u''_i(q_i)} - \frac{2u''_i(q_i)}{u'_i(q_i)} \right) \mathbb{I}(u'''_j(q_j) > 0) \leq 0.$$

As one of the two conditions on revenues and costs holds strictly, assumptionsA1 and B2 imply that the payoff of each player is strictly concave and continuous whenever increasing. The strict concavity of payoffs and the compactness and convexity of the choice set correspondence imply that the best-responses are single-valued. Again Brouwer's fixed point theorem applies and implies existence. Also observe that any combination of the two assumptions B1 and B2 would similarly grant existence.

To prove (b) finally observe that part (a) implies that solutions can be found by Kuhn-Tucker first order conditions. Therefore, the system of first order conditions can be expressed as the complementarity problem stated in (b), as the optimality of a flow q^i_j implies that

$$\begin{aligned}
&q^i_j = 0 \;\text{ if }\; u'_i(q_i) - u''_i(q_i)q^{\circ}_i - u'_j(q_j) + \mu_i > 0 \;\text{ and,}\\
&q^i_j > 0 \;\text{ s.t. }\; u'_i(q_i) - u''_i(q_i)q^{\circ}_i - u'_j(q_j) - u''_j(q_j)q^i_j + \mu_i = 0 \;\text{ otherwise,}\\
&\text{for }\; \mu_i \geq 0,\; Q_i - q^i_{\circ} \geq 0,\; \text{and }\; \mu_i(Q_i - q^i_{\circ}) = 0.
\end{aligned}$$

■

Basic Properties of the Outflow Model

Proof of Proposition 3. (a) First order necessary conditions immediately establish the result since $q_i^j > 0$ implies

$$u_j'(q_j) - u_i'(q_i) = \mu_i - u_j''(q_j)q_j^i - u_i''(q_i)q_i^\circ > 0,$$

where μ_i denotes the non-negative multiplier on the outflow constraint, $q_\circ^i \leq Q_i$, and where the latter terms are positive by assumption A1. Moreover, the converse clearly fails due to the positive price distortions inherent to the model.

(b) Let $T(q) = \{ij \in E | q_j^i > 0\}$ be the set of active trading links. If $ij \in T(\mathbf{q})$ by first order optimality $u_i'(q_i) < u_j'(q_j)$. Thus if a cycle $c = \{ij, jk, ..., li\} \in T(\mathbf{q})$, a contradiction arises since $u_i'(q_i) < u_j'(q_j) < u_k'(q_k) < ... < u_l'(q_l) < u_i'(q_i)$.

(c) If for $i \in V$ and for any $j \in V_i$ equilibrium dictates that $u_i'(q_i) \leq u_j'(q_j)$, then i cannot buy from any neighbor, since $u_i'(q_i) > u_j'(q_j)$ is necessary for $q_i^j > 0$. Similarly if $u_i'(q_i) \geq u_j'(q_j)$ for any $j \in V_i$, player i cannot be selling to any neighbor, since $u_i'(q_i) < u_j'(q_j)$ is necessary for $q_j^i > 0$.

(d) By part (c) if i is a source $q_i^\circ = 0$. Which in turn implies that, if A1 holds, player i sells to any $j \in V_i$ with $u_i'(q_i) < u_j'(q_j)$, provided that $q_\circ^i < Q_i$, since there exists $q_j^i > 0$ for which

$$-u_i'(q_i) + u_j'(q_j) + q_j^i u_j''(q_j) = 0.$$

(e) If A1 holds, optimality of the trade from k to i requires

$$u_k'(q_k) - u_k''(q_k) \textstyle\sum_{l \in V_k} q_k^l + \mu_k = u_i'(q_i) + q_i^k u_i''(q_i) < u_i'(q_i) < u_j'(q_j).$$

Which is both necessary and sufficient for a trade from k to j to occur. ■

Proof of Proposition 4. (a) A profile of flows $\mathbf{q}$ is constrained efficient if for any feasible flow q_j^i,

$$\begin{aligned} [u_j'(q_j) - u_i'(q_i) - \lambda_i]q_j^i &= 0 \\ [q_i + Q_i]\lambda_i &= 0, \end{aligned} \tag{2}$$

where λ_i denotes the multiplier of the capacity constraint of player i. Optimality conditions for an outflow equilibrium instead, require that for any feasible flow q_j^i,

$$\begin{aligned} [u_j'(q_j) - u_i'(q_i) - \mu_i + u_j''(q_j)q_j^i + u_i''(q_i)q_i^\circ]q_j^i &= 0 \\ [q_i + Q_i]\mu_i &= 0, \end{aligned} \tag{3}$$

where μ_i denotes the multiplier of the capacity constraint. If $\mathbf{q} = \mathbf{0}$ is an outflow equilibrium, it must satisfy conditions 3. If so, however, $\mathbf{q} = \mathbf{0}$ immediately satisfies conditions 2 for constrained efficiency, by setting $\lambda_i = \mu_i$. Moreover, if $\mathbf{q} = \mathbf{0}$ is constrained efficient, it must satisfy conditions 2. But if so, $\mathbf{q} = \mathbf{0}$ immediately satisfies conditions 3 for an outflow equilibrium. What remains to be proven is that if $\mathbf{q} \neq \mathbf{0}$ is constrained efficient, then $\mathbf{q}$ cannot be an outflow equilibrium. But this observation is trivial, as conditions 3 and 2 can coincide only $\mathbf{q} = \mathbf{0}$ when utility is concave by assumption A1.

(b) Consider a market with three consumers $\{a, b, c\}$. Let Q_i denote the consumption endowment of player i and let $Q_a = 2$, $Q_b = 1$, and $Q_c = 0$. Let preferences of any player satisfy $u_i(q) = (Q_i + q)^{1/2}$. Increasing the set of trading relationships from $\{ac\}$ to $\{ac, ab\}$ reduces social welfare. In fact, if only players a and c were allowed to trade, player a would sell 0.4 units to c at a price of 0.8, and social welfare would stand at 2.9. If instead, consumer a were allowed to trade with b as well as c, he would elect to supply both neighbors: b with 0.2 units and c with 0.36 units at different prices. Individual a's price discrimination of players b and c would decrease sales to c. Player a would opt to curtail his supply to c in order to extract higher marginal rents from c, since he would be able to recoup the loss in revenue by selling to b. Social welfare would thus decline to 2.89. Flows, prices and quantities for the two economies are reported in figure 2 (left and center) and table 2 (left and center).

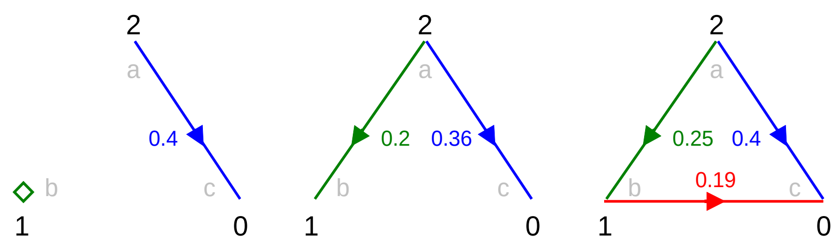

FIGURE 2: On the vertices endowments and identities and on the edges flows.

E1	p	x	w
a	-	1.60	1.58
b	-	1.00	1.00
c	0.79	0.40	0.32
+	-	3.00	2.90

E2	p	x	w
a	-	1.44	1.59
b	0.46	1.20	1.00
c	0.83	0.36	0.30
+	-	3.00	2.89

E3	p	x	w
a	-	1.35	1.54
b	0.49	1.06	1.03
c	0.65	0.59	0.39
+	-	3.00	2.96

TABLE 2: Prices, consumption & welfare: left $\{ac\}$, center $\{ac, ab\}$, right $\{ac, ab, bc\}$.

(c) Consider an economy: with four players $\{a, b, c, d\}$; with initial endowments $\{2.97, 1, 0, 0.03\}$; and in which the preferences of any player satisfy $u_i(q) = (Q_i + q)^{1/2}$. When the set of feasible trades increases from $\{ad, bc\}$ to $\{ad, bc, dc\}$, player d's welfare decreases. If only trades in $\{ad, bc\}$ are feasible in the unique equilibrium of this economy players a and b supply their respective customers as monopolies.

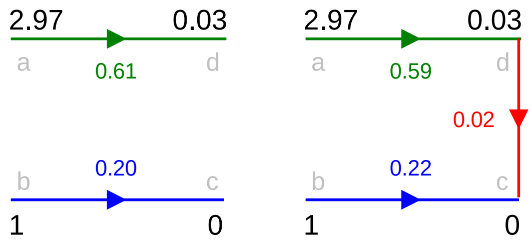

FIGURE 3: On the vertices endowments and identities and on the edges flows.

E1	p	q	w
a	-	2.36	1.92
b	-	0.80	1.12
c	1.12	0.20	0.22
d	0.63	0.64	0.42
+	-	4.00	3.68

E2	p	q	w
a	-	2.38	1.93
b	-	0.78	1.11
c	1.02	0.24	0.24
d	0.65	0.60	0.41
+	-	4.00	3.69

TABLE 3: Prices, consumption & welfare: left $\{ad, bc\}$, right $\{ad, dc, bc\}$.

But when the link cd is added to the network, player d competes with b to supply c. In the unique equilibrium consumer d is worse off than when he cannot sell to c, since his payoff decreases from 0.42 to 0.41. Even though player d chooses to supply c, having the option to sell affects the quantity sold to him from a, and thus reduces his welfare. All gains from trade on the newly created link are either kept by c or transferred to a. Player a being the monopoly supplier of d is able to extract more rents, because he faces a steeper demand schedule whenever d has the option to resell. Flows, allocations and prices for the two economies are reported in table 3 and in figure 3. ■

Large Markets, Efficiency, and the Competitive Equilibrium

Proof of Proposition 5. The first step of the argument shows that (a) and (b) are equivalent. If $\sigma(H, \bar{\mathbf{q}}) \geq 0$ for $\forall H \subseteq \mathbf{C}$, it must be that for any $T \subseteq \mathbf{C}$

$$\begin{aligned}\delta(T, \bar{\mathbf{q}}) &= \delta(T, \bar{\mathbf{q}}) - \delta(\mathbf{C}, \bar{\mathbf{q}}) = -\textstyle\sum_{C \in \mathbf{C} \backslash W_T} q_C^+ - \sum_{C \in \mathbf{C} \backslash T} q_C^- = \\ &= \sigma(\mathbf{C} \backslash W_T, \bar{\mathbf{q}}) - \textstyle\sum_{C \in (\mathbf{C} \backslash T) \backslash W_{\mathbf{C} \backslash W_T}} q_C^- \geq 0,\end{aligned}$$

where the first equality holds since the efficient net trades add to zero $\delta(\mathbf{C}, \bar{\mathbf{q}}) = 0$, and where the second equality holds since $\mathbf{C} \backslash T \supseteq W_{\mathbf{C} \backslash W_T}$ for $\forall T \subseteq \mathbf{C}$. Similarly if $\delta(H, \bar{\mathbf{q}}) \geq 0$

for $\forall H \subseteq \mathbf{C}$, then for any $T \subseteq \mathbf{C}$

$$\sigma(T, \bar{\mathbf{q}}) = \delta(\mathbf{C} \backslash W_T, \bar{\mathbf{q}}) + \sum_{C \in (\mathbf{C} \backslash T) \backslash W_{\mathbf{C} \backslash W_T}} q_C^+ \geq 0.$$

To prove the final step it is convenient to map condition MC to the original network structure. Let $D_i = D \cap V_i$, $S_i = S \cap V_i$, and $S_H = \cup_{i \in H} S_i$. First establish that MC holds if and only if for any $H \subseteq D$

$$\tilde{\sigma}(H, \bar{\mathbf{q}}) = -\sum_{j \in S_H} \bar{q}_j - \sum_{i \in H} \bar{q}_i \geq 0.$$

For any $H \subseteq D$, let $T(H) = \{C \in \mathbf{C} \mid C \cap H \neq \emptyset\}$. If MC holds, observe that for any $H \subseteq D$

$$\tilde{\sigma}(H, \bar{\mathbf{q}}) \geq -\sum_{C \in W_{T(H)}} \sum_{i \in C \cap S} \bar{q}_i - \sum_{C \in T(H)} \sum_{i \in C \cap D} \bar{q}_i = \sigma(T(H), \bar{\mathbf{q}}) \geq 0,$$

where the first inequality holds as $T(H)$ may include more buyers than H. Similarly, for any $T \subseteq \mathbf{C}$, let $H(T) = \{i \in D \mid i \in C \text{ for } C \in T\}$. If $\tilde{\sigma}(H, \bar{\mathbf{q}}) \geq 0$ for any $H \subseteq D$, MC holds as for any $T \subseteq \mathbf{C}$,

$$\sigma(T, \bar{\mathbf{q}}) = -\sum_{C \in W_T} \sum_{i \in C \cap S} \bar{q}_i - \sum_{i \in H(T)} \bar{q}_i = \tilde{\sigma}(H(T), \bar{\mathbf{q}}) \geq 0,$$

where the first inequality holds as $i \in S_{H(T)}$ if and only if $i \in C \cap S$ for some $C \in W_T$.

The next step exploits the previous simplification to establish that (c) implies (a) by contradiction. Observe that whenever (ii) holds for any $H \subset D$

$$-\sum_{i \in S_H} \bar{q}_i = \sum_{i \in S_H} \sum_{j \in D_i} q_j^i \geq \sum_{i \in S_H} \sum_{j \in H \cap D_i} q_j^i = \sum_{j \in H} \sum_{i \in S_j} q_j^i,$$

where first equality holds by (ii) and the last holds since any pair of players i, j that satisfies $j \in H$ and $i \in S_j$, also satisfies $i \in S_H$ and $j \in H \cap D_i$. If MC were violated, $\tilde{\sigma}(H, \bar{\mathbf{q}}) < 0$ for some $H \subset D$. But if so, the previous observation would imply that (i) would also be violated for some player $j \in H$ since

$$-\sum_{j \in H} \bar{q}_j + \sum_{j \in H} \sum_{i \in S_j} q_j^i \leq -\sum_{j \in H} \bar{q}_j - \sum_{i \in S_H} \bar{q}_i = \tilde{\sigma}(H, \bar{\mathbf{q}}) < 0.$$

The final step proves that $\tilde{\sigma}(H, \bar{\mathbf{q}}) \geq 0$ for any $H \subseteq D$ implies (c) by induction on D. First establish that the result holds for $|D| = 1$. Let i denote the only buyer in the economy. By assumption we have that $\tilde{\sigma}(D, \bar{\mathbf{q}}) \geq 0$, which in turn implies that $\tilde{\sigma}(D, \bar{\mathbf{q}}) = 0$ as supply cannot exceed aggregate demand by construction. Thus flows satisfying both (i) and (ii) can be found by setting $q_i^j = -\bar{q}_j$ for any $j \in S = S_i$.

Next suppose that $\tilde{\sigma}(H, \bar{\mathbf{q}}) \geq 0$ for any $H \subseteq D$ is sufficient whenever $|D| \leq m-1$ to prove that it is sufficient for $|D| = m$. Initially assume that $H \subset D$ exists such that $\tilde{\sigma}(H, \bar{\mathbf{q}}) = 0$.

Consider the subgraph (V', E') with vertices $V' = S' \cup D'$ with $D' = H$ and $S' = S_H$, and with edges restricted to $E' = E \cap \{ij | i \in S' \cap j \in D'\}$. This subgraph satisfies $\tilde{\sigma}(K, \bar{\mathbf{q}}) \geq 0$ for any $K \subseteq D'$ trivially, since no condition was altered. Thus, since $|H| < m$, by the induction assumption it is possible to find flows $\mathbf{q} \in \mathbb{R}_+^{E'}$ such that (i) and (ii) hold in the subgraph,

$$\begin{array}{rcll} \bar{q}_j & = & \sum_{i \in S_j} q_j^i & \text{for } \forall j \in H, \\ \bar{q}_i & = & -\sum_{j \in D_i \cap H} q_j^i & \text{for } \forall i \in S_H. \end{array}$$

To conclude the proof it suffices to show that given such flows the remaining players of the original graph still satisfy MC. Define by $\hat{\mathbf{q}} \in \mathbb{R}^V$ as the efficient net-trades $\bar{\mathbf{q}}$ shifted by such flows $\mathbf{q}$. That is for any $i \in V$, let

$$\hat{q}_i = \bar{q}_i - q_i^\circ + q_\circ^i.$$

Consider the subgraph (V'', E'') with vertices $V'' = S'' \cup D''$ with $D'' = D \backslash H$ and $S'' = S \backslash S_H$, and with edges restricted to $E'' = E \cap \{ij | i \in S'' \cap j \in D''\}$. For any $K \subset D \backslash H$ it must be that

$$\begin{array}{rcl} \tilde{\sigma}''(K, \hat{\mathbf{q}}) & = & \tilde{\sigma}''(K, \hat{\mathbf{q}}) + \tilde{\sigma}''(H, \hat{\mathbf{q}}) + \sum_{i \in S_H \cap S_K} \hat{q}_i = \\ & = & \tilde{\sigma}(K, \bar{\mathbf{q}}) - \sum_{i \in S_K} \sum_{j \in D_i \cap H} q_j^i + \tilde{\sigma}(H, \bar{\mathbf{q}}) + \sum_{i \in S_H \cap S_K} (\bar{q}_i + \sum_{j \in D_i \cap H} q_j^i) = \\ & = & \tilde{\sigma}(K, \bar{\mathbf{q}}) + \tilde{\sigma}(H, \bar{\mathbf{q}}) + \sum_{i \in S_H \cap S_K} \bar{q}_i = \tilde{\sigma}(K \cup H, \bar{\mathbf{q}}) \geq 0, \end{array}$$

since $\tilde{\sigma}''(H, \hat{\mathbf{q}}) = \tilde{\sigma}(H, \bar{\mathbf{q}}) = 0$ and $\sum_{i \in S_H \cap S_K} \hat{q}_i = 0$. Which in turn implies by induction that flows $\mathbf{q}'' \in \mathbb{R}_+^{E''}$ exist that satisfy condition (i) and (ii), since $|D \backslash H| < m$.

Finally if $\tilde{\sigma}(L, \bar{\mathbf{q}}) > 0$ for any $L \subset D$, consider $H \in \arg\min_{L \subset D} \tilde{\sigma}(L, \bar{\mathbf{q}})$ and choose any profile of flows $\dot{\mathbf{q}}$ from S_H to $D \backslash H$ such that

$$\sum_{j \in D \backslash H} \sum_{i \in S_H \cap S_j} \dot{q}_j^i = \tilde{\sigma}(H, \bar{\mathbf{q}}).$$

Let $\ddot{\mathbf{q}} \in \mathbb{R}^V$ denote the efficient net-trades $\bar{\mathbf{q}}$ adjusted for such flows $\dot{\mathbf{q}}$. After such transfers, $\tilde{\sigma}(H, \ddot{\mathbf{q}}) = 0$ and $\tilde{\sigma}(L, \ddot{\mathbf{q}}) \geq 0$ for any $L \subseteq D$, since

$$\tilde{\sigma}(L, \ddot{\mathbf{q}}) \geq \tilde{\sigma}(L, \bar{\mathbf{q}}) - \tilde{\sigma}(H, \bar{\mathbf{q}}) \geq 0.$$

Thus, the $\ddot{\mathbf{q}}$ economy satisfies all the conditions required in the previous step of the proof and MC is sufficient. ■

Proof of Proposition 6. Let $(V(z), E(z))$ denote the trade network associated to community structure $(\mathbf{C}^z, \mathbf{E}^z)$. For convenience, occasionally denote $u_i(q_i(z))$ by $u_i(z)$. Whenever the equilibrium of the replicas converges to efficiency, it must be that $\lim_{z\to\infty}(u_j'(z) - u_i'(z)) = 0$ for any two players $i, j \in V(z)$ for which $\lim_{z\to\infty} q_i(z) > -Q_i$ and $\lim_{z\to\infty} q_j(z) > -Q_j$. Suppose by contradiction that some player $i \in V(z)$ resells units in the limit economy,

$$\lim_{z\to\infty} r_i(z) = \lim_{z\to\infty} \min\left\{q_i^\circ(z), q_\circ^i(z)\right\} > 0.$$

If so, first order optimality for flows from i to his neighbors $j \in V_i(z)$ require that, when $q_j^i(z) > 0$,

$$\lim_{z\to\infty}(u_j'(z) - u_i'(z)) = \lim_{z\to\infty} \left(\mu_i(z) - u_i''(z)q_i^\circ(z) - u_j''(z)q_j^i(z)\right) > 0,$$

where $\mu_i(z)$ denotes the non-negative multiplier on the outflow constraint, $q_\circ^i(z) \leq Q_i$. Which contradicts the assumption that the economy becomes competitive. ■

Proof of Proposition 7. First the necessity of MC is proven. MC holds in an economy $\{\mathbf{C}, \mathbf{E}, \mathbf{Q}, \mathbf{u}\}$ if and only if MC holds in any of its replicas $\{\mathbf{C}^z, \mathbf{E}^z, \mathbf{Q}^z, \mathbf{u}^z\}_{z=1}^{\infty}$. If MC were not hold, by proposition 5, no direct flows would exist from seller to buyers that support the efficient net-trades in the original economy. Resale among players would thus be necessary for the efficient net-trades to be an outcome of such economy. Recall that $(V(z), E(z))$ denotes the trade network associated to community structure $(\mathbf{C}^z, \mathbf{E}^z)$. Define the *minimal resale* in the z-replica economy as

$$r(z) = \min_{\mathbf{q}\in\mathbb{R}_+^{E(z)}} \max_{i\in V(z)} r_i(z) \text{ s.t. (i) and (ii) on page 17.}$$

MC fails if and only if $r(1) > 0$. The definition of replica implies $r(1) = r(z)$, because minimizing the maximum resale requires all players of the same type to buy and sell the identical amounts. This is the case, since the average flows across any two player types in a replica (that is $\sum_{s=1}^{z} \sum_{t=1}^{z} q_{j.t}^{i.s}/z^2$) define flows in the original economy in which resale exceeds $r(1)$ (by efficiency), which in turn implies that $r(1) \leq r(z)$, as the average resale of a player of type i has to exceed the maximal resale of a player of type i. Thus, if $r(1) > 0$, any profile of flows leading to the efficient allocation requires at least one player to resell a positive amount of goods in the limit economy. But, by proposition 6 no such outcome can be an efficient limiting outflow equilibrium since no resale would take place in such a limiting equilibrium.

The next part of the proof establishes that MC is sufficient for the existence of an efficient symmetric outflow equilibrium in the limit economy. First observe that the solution of the complementarity problem defining the symmetric equilibrium flows is lower hemi-continuous in the replica counter z, as each optimality condition defining the problem is continuous and differentiable in $1/z$, (see problem CP in the proof of proposition 11). Therefore, consider flows in the original economy $\mathbf{q} \in \mathbb{R}_+^E$ satisfying (i) and (ii) in proposition 5. Such flow exist because MC holds. If so, $q_i = \bar{q}_i$ for any player $i \in V$. Define the sequence of flows $\mathbf{q}(z) \in \mathbb{R}_+^{E(z)}$ as follows: $q_{j.t}^{i.s}(z) = q_j^i/z$ for any $(i.s)(j.t) \in E(z)$, and $q_{j.t}^{i.s}(z) = 0$ otherwise. Such flows are direct and satisfy conditions (i) and (ii) in the z-replica. Moreover, $\lim_{z\to\infty} q_{j.t}^{i.s}(z) = 0$. Thus, $\lim_{z\to\infty} \mathbf{q}(z)$ satisfies all the outflow equilibrium requirements in the limit economy. In particular, if such flows were chosen by others, no player would be able to profitably affect the prices of the goods sold in the limit, as deviations on his behalf could only reduce prices since $\lim_{z\to\infty} q_{j.t}^{i.s}(z) = 0$. As gains from deviating from $\mathbf{q}(z)$ decrease along the sequence of replicas and vanish in the limit, the limit of $\mathbf{q}(z)$ is efficient, and belongs to the limit of the symmetric outflow equilibrium correspondence. Lower hemi-continuity of the equilibrium correspondence then guarantees the existence of a selection of equilibrium correspondence that converges to such a limit point.[20] ■

Proof of Proposition 8. The proof of the result emulates that of proposition 7. Let $\left(\hat{V}, \hat{E}\right)$ denote the trade network associated to community structure $(\hat{\mathbf{C}}, \hat{\mathbf{E}})$. Let $\hat{V}(z) = \cup_{C\in\hat{\mathbf{C}}} C^z$ denote the set of players located in a large community in the z-replica. To establish the necessity of MC holding in the economy $\{\hat{\mathbf{C}}, \hat{\mathbf{E}}, \mathbf{Q}, \mathbf{u}\}$ observe that if MC were not hold, by proposition 5 no direct flows would exist from seller to buyers that support the efficient net-trades in the economy $\{\hat{\mathbf{C}}, \hat{\mathbf{E}}, \mathbf{Q}, \mathbf{u}\}$. If so, resale among players in $\hat{V}$ would required for the efficient net-trades to be an outcome of the economy. Let $\hat{\mathbf{q}} \in \mathbb{R}^{\hat{V}}$ denote the profile of efficient net-trades in the economy $\{\hat{\mathbf{C}}, \hat{\mathbf{E}}, \mathbf{Q}, \mathbf{u}\}$, and let $\bar{\mathbf{q}}^z \in \mathbb{R}^{V(z)}$ denote the profile of efficient net-trades in the economy $\{\mathbf{C}^z, \mathbf{E}^z, \mathbf{Q}, \mathbf{u}\}$. Such profiles exist by assumption A1. Observe that $\lim_{z\to\infty} \bar{q}_{i.s}^z = \hat{q}_i$ for any $i \in \hat{V}$ and any $s \in \{1, ..., z\}$, as net-trades of players in $V\backslash\hat{V} = V(z)\backslash\hat{V}(z)$ become negligible when the economy $\{\mathbf{C}^z, \mathbf{E}^z, \mathbf{Q}, \mathbf{u}\}$ is large. This is the case since the efficient allocation of consumption is independent of $\sum_{i\in V\backslash\hat{V}} \bar{q}_i^z$ when z is sufficiently large given that: assumption A3 implies that $\lim_{z\to\infty} \bar{q}_i^z < \infty$ for any $i \in V\backslash\hat{V}$; the constraint on sales implies $\lim_{z\to\infty} \bar{q}_i^z \geq -Q_i$; and the set $V(z)\backslash\hat{V}(z)$ contains a finite number of players. Moreover, since the total number of units sold by players in $V(z)\backslash\hat{V}(z)$ cannot exceed $\sum_{i\in V\backslash\hat{V}} Q_i$, resale among players in $\hat{V}(z)$ is always required to achieve the efficient net-trades when condition MC fails in the economy $\{\hat{\mathbf{C}}, \hat{\mathbf{E}}, \mathbf{Q}, \mathbf{u}\}$. If so, the argument developed in proposition 7 applies, and establishes that any outflow equilibrium must be inefficient since at least a player in $\hat{V}(z)$ resells non-negligible amount of consumption in any given profile of flows that gives rise to the efficient net-trades.

[20] A direct proof of sufficiency is possible, but more involved.

The next part of the argument establishes why MC is sufficient for convergence to approximate efficiency. Observe that the solution of the complementarity problem defining the symmetric equilibrium flows is lower hemi-continuous in the replica counter z, as each optimality condition defining the problem is continuous and differentiable in z. In particular, optimality of a flow $q_{j.t}^{i.s}(z) = q_j^i > 0$ in a symmetric equilibrium of a $\hat{\mathbf{C}}$-unbalanced z-replica requires that

$$u_i'(q_i) - u_j'(q_j) - u_j''(q_j)q_j^i - u_i''(q_i)q_i^\circ + \mu_i = 0,$$

where μ_i denotes the multiplier on the capacity constraint (satisfying $\mu_i(Q_i - q_\circ^i) = 0$) and where

$$\begin{aligned} q_i &= z\sum_{k\in V_i\cap\hat{V}}\left(q_i^k - q_k^i\right) + \sum_{k\in V_i\setminus\hat{V}}\left(q_i^k - q_k^i\right), \\ q_i^\circ &= z\sum_{k\in V_i\cap\hat{V}} q_i^k + \sum_{k\in V_i\setminus\hat{V}} q_i^k. \end{aligned}$$

This establishes the lower hemi-continuity in z of the complementarity problem defining the symmetric outflow equilibria, as each optimality condition defining the problem is continuous and differentiable in z.

Now construct candidate flows that converge to approximate efficiency. For the economy $\{\hat{\mathbf{C}}, \hat{\mathbf{E}}, \mathbf{Q}, \mathbf{u}\}$, let $\tilde{\mathbf{q}} \in \mathbb{R}_+^{\hat{E}}$ denote a profile of flows satisfying (i) and (ii) in proposition 5. Such flows exist because MC holds, and satisfy $\sum_{j\in\hat{V}_i}(\tilde{q}_i^j - \tilde{q}_j^i) = \hat{q}_i$ for any player $i \in \hat{V}$. Consider the sequence of flows $\mathbf{q}(z) \in \mathbb{R}_+^{E(z)}$ obtained by setting $q_{j.t}^{i.s}(z) = \tilde{q}_j^i/z$ for any $(i.s),(j.t) \in \hat{V}(z)$, while setting the remaining flows $q_j^i(z)$ according to their respective symmetric equilibrium optimality conditions (defined above). Observe that by construction for any player in $i \in \hat{V}(z)$,

$$q_i(z) = \hat{q}_i + \sum_{k\in V_i\setminus\hat{V}}\left(q_i^k(z) - q_k^i(z)\right).$$

Consider any player $k \in V_i(z)\setminus\hat{V}(z)$, observe that $i \in V_k(z)\cap\hat{V}(z)$, if so $\lim_{z\to\infty} q_i^k(z) = 0$ or else the capacity constraint of player k would be violated. Similarly, $\lim_{z\to\infty} q_k^i(z) = 0$, or else $q_k(z)$ would diverge to infinity, which is impossible as player $i \in V_k(z)\cap\hat{V}(z)$ would not be choosing his flow to k optimally because $q_k^i(z) > 0$ is equivalent to $u_i'(q_i(z)) < u_k'(q_k(z))$ (which cannot hold in the limit as A1 implies $u_i'(q_i(z)) > 0$, while A3 that $\lim_{z\to\infty} u_k'(q_k(z)) = 0$). This establishes that flows $\mathbf{q}(z)$ converge to approximate efficiency as $\lim_{z\to\infty} q_i(z) = \hat{q}_i = \bar{q}_i^\infty$.

The proof concludes by establishing that flows $\mathbf{q}(z)$ must be arbitrarily close to symmetric equilibrium flows in the limit as z diverges. To verify that flows $q_{j.t}^{i.s}(z) = \tilde{q}_j^i/z$ for any $(i.s),(j.t) \in \hat{V}(z)$ are arbitrarily close to equilibrium flows as z diverges, observe that the conjectured flows satisfy $\lim_{z\to\infty} q_{j.t}^{i.s}(z) = 0$. Therefore, $\lim_{z\to\infty}\mathbf{q}(z)$ satisfies all the outflow equilibrium requirements for trades on links $ij \in \hat{E}$ in the limit economy. In particular, if such flows were chosen by others, no player would be able to profitably affect the prices of the goods sold in the limit, as deviations on his behalf could only reduce prices since $\lim_{z\to\infty} q_{j.t}^{i.s}(z) = 0$. Since gains from deviating from $\mathbf{q}(z)$ decrease along the sequence of unbalanced replicas and vanish in the limit, the limit of $\mathbf{q}(z)$ is approximately efficient and belongs to the limit of the symmetric outflow equilibrium correspondence. Lower hemi-continuity of the equilibrium correspondence then guarantees the existence of a selection of equilibrium correspondence that converges to such a limit point. ■

Proof of Proposition 9. Necessity of IC is immediate from the proof of proposition 8 and the following considerations. If IC were violated, convergence to efficiency in small communities would require either trade across and within small communities, or trade between buyers in small communities and buyers in large communities. Either scenario would necessarily result in distortions. In the first scenario distortions would be a trivial consequence of proposition 3, while the second distortions would appear as inflow price distortions would never vanish for players purchasing units in the limit economy.

Next part of the proof establishes why MC and IC are sufficient for convergence to efficiency. Let $\mathbb{I}(\cdot)$ denote the indicator function. As in the proof of proposition 8 for the economy $\{\hat{\mathbf{C}}, \hat{\mathbf{E}}, \mathbf{Q}, \mathbf{u}\}$, consider flows $\tilde{\mathbf{q}} \in \mathbb{R}_+^{\hat{E}}$ satisfying (i) and (ii) in proposition 5. Such flow exist because MC holds, and satisfy $\sum_{j\in\hat{V}_i}(\tilde{q}_i^j - \tilde{q}_j^i) = \hat{q}_i$ for any player $i \in \hat{V}$. Consider the sequence of flows $\mathbf{q}(z) \in \mathbb{R}_+^{E(z)}$ obtained by setting for any $ij \in E(z)$

$$q_{j.t}^{i.s}(z) = \begin{cases} \frac{\tilde{q}_j^i}{z} & \text{if } \ (i.s),(j.t) \in \hat{V}(z) \\ -\frac{\bar{q}_i^\infty \mathbb{I}(\bar{q}_i^\infty<0)}{z\left|V_i\cap\hat{V}\right|} & \text{if } \ i.s \in V(z)\backslash\hat{V}(z) \text{ and } \ j.t \in \hat{V}(z) \\ \frac{\bar{q}_j^\infty \mathbb{I}(\bar{q}_i^\infty>0)}{z\left|V_j\cap\hat{V}\cap S\right|} & \text{if } \ j.t \in V(z)\backslash\hat{V}(z) \text{ and } \ i.s \in \hat{V}(z) \end{cases}$$

while setting the remaining flows $q_{j.t}^{i.s}(z)$ to zero. The proposed flows converge to efficiency since for any player in $i \in \hat{V}(z)$ by construction it must be that

$$\lim_{z\to\infty} q_i(z) = \hat{q}_i + \lim_{z\to\infty} \textstyle\sum_{k\in V_i\backslash\hat{V}} \left(q_i^k(z) - q_k^i(z)\right) = \bar{q}_i^\infty.$$

Similarly for players $i \in V(z)\backslash\hat{V}(z)$ we have that by construction $q_i(z) = \bar{q}_i^\infty$. The proposed flows $\mathbf{q}(z)$ must be arbitrarily close to symmetric equilibrium flows in the limit as z diverges. This is the case because conjectured flows $\mathbf{q}(z)$ satisfy $\lim_{z\to\infty} q_{j.t}^{i.s}(z) = 0$. Again $\lim_{z\to\infty}\mathbf{q}(z)$ satisfies all the outflow equilibrium requirements for trades on all links in the limit economy. If such flows were chosen by others, no player would be able to profitably affect the prices of the goods sold in the limit, as deviations on his behalf could only reduce prices by $\lim_{z\to\infty} q_{j.t}^{i.s}(z) = 0$, and because no player resells a non-negligible amount of consumption. As gains from

deviating from $\mathbf{q}(z)$ again decrease along the sequence of replicas and vanish in the limit, the limit of $\mathbf{q}(z)$ is approximately efficient and belongs to the limit of the symmetric equilibrium correspondence. Lower hemi-continuity of the equilibrium correspondence then guarantees the existence of a selection of equilibrium correspondence that converges to such a limit point.
■

Proof of Proposition 10. (a) Since in any symmetric equilibrium of the limiting economy the outflow wedges vanish, revenues in each market are concave. Since the third derivative is positive, costs of supplying units are convex. Thus existence of a symmetric equilibrium in the limit economy follows as in proposition 1.

(b) This is a consequence of vanishing price distortions in any efficient limiting economy (which requires concave revenues and convex costs) and of proposition 7 (which shows that MC implies that a limiting outcomes can be efficient).

(c) Let $i.s^* \in \arg\min_{j.t \in V(z)} u'_{j.t}(q_{j.t}(z))$. If $\bar{\mathbf{q}} \neq \mathbf{0}$, for any sequence of outflow equilibrium flows $\mathbf{q}(z) \in \mathbb{R}_+^{E(z)}$ it must be that $i.s^* \in S(z)$, because such a player does not purchase consumption by part (b) of proposition 3, and because by definition of competitive equilibrium $0 \geq q_{i.s^*}(z) \geq \bar{q}_i$. By contradiction suppose that there exists a sequence of outflow equilibria that does not converge to efficiency. If so, the set of players linked to $i.s^*$ and with marginal utility strictly higher than $i.s^*$ diverges, since $V_j(z) \supseteq S(z)$ for $\forall j \in D(z)$ implies $V_j(z) \supseteq D(z)$ for $\forall j \in S(z)$, and since $\lim_{z\to\infty} |D(z)| = \lim_{z\to\infty} z\,|D| = \infty$. This immediately yields a contradiction if $\lim_{z\to\infty} (q_{i.s^*}(z)) > -Q_i$, because $Q_i < \infty$ and because by part (c) of proposition 3 player $i.s^*$ would sell a strictly positive amount of consumption in the limit to all his neighbors with strictly higher marginal utility.

If, instead, $\lim_{z\to\infty} (q_{i.s^*}(z)) = -Q_i$, let $V_+(z) = \{k \in V(z) | q_k(z) > -Q_k\}$ and let $i.s_* \in \arg\min_{j.t \in V_+(z)} u'_{j.t}(q_{j.t}(z))$. First notice that: $\lim_{z\to\infty} |V_+(z)| = \infty$, since $\lim_{z\to\infty} z \sum_{i\in V} Q_i = \infty$ and since $u'' < 0$. Thus no player in $V \backslash V_+(z)$ sells to $i.s_*$ for z large enough, since a large and diverging number players have strictly higher marginal utility than $i.s_*$, if $\bar{\mathbf{q}} \neq \mathbf{0}$. Hence, in the limit $i.s_*$ does not buy. If $i.s_* \in S(z)$ for z large, assuming that the sequence of outflow equilibria does not converge efficiency again yields a contradiction. In fact, part (c) of proposition 3 would imply that player $i.s_*$ sells a strictly positive amount of goods in the limit to all his neighbors with strictly higher marginal utility which is impossible since $\lim_{z\to\infty} (q_{i.s_*}(z)) > -Q_k$, since $Q_i < \infty$, and because $i.s_*$ has infinitely many neighbors with higher marginal utility in the limit economy. A contradiction arises even if $i.s_* \in D(z)$ for z large and if the sequence of outflow equilibria does not converge to efficiency. In particular if $i.s_* \in D(z)$ for z large enough, it must be that $\bar{q}_i > 0 \geq \lim_{z\to\infty} q_{i.s_*}(z)$, since $i.s_*$ only sells for z large enough. Moreover, by definition of $i.s_*$ it must be that, for any $j.t \in V_+(z)$,

$$u'_{i.s_*}(\bar{q}_i) < u'_{i.s_*}(q_{i.s_*}(z)) \leq u'_{j.t}(q_{j.t}(z)),$$

which in turn by concavity implies that $\bar{q}_j > q_{j.t}(z)$ for any $j.t \in V_+(z)$. Also, notice that $\bar{q}_j \geq q_{j.t}(z) = 0$ for any $j.t \in V \backslash V_+(z)$. Hence, provided that $\bar{\mathbf{q}} \neq \mathbf{0}$, contradiction arises, since

$$0 = \sum_{j.t \in V(z)} \bar{q}_{j.t} > \sum_{j.t \in V(z)} q_{j.t}(z).$$

Thus the limit outflow equilibrium must be efficient. ■

Proof of Proposition 11. Define the total quantity sold from an individual of type i to all individuals of type j in the unique symmetric equilibrium of a z-replica by $\mathring{q}^i_j = z q^i_j(z)$. The inequalities defining the symmetric equilibrium of a z-replica (a complementarity problem) can be written in terms of such quantities as follows

$$\begin{aligned} f^i_j(\mathring{\mathbf{q}}, \boldsymbol{\mu}|z) &= u'_i(\mathring{q}_i) - u'_j(\mathring{q}_j) - u''_j(\mathring{q}_j)(\mathring{q}^i_j/z) - u''_i(\mathring{q}_i) \textstyle\sum_{k \in V_i} \mathring{q}^k_i + \mu_i \geq 0, \\ f_i(\mathring{\mathbf{q}}, \boldsymbol{\mu}|z) &= Q_i - \mathring{q}^i_\circ \geq 0, \end{aligned} \qquad \text{(CP)}$$

where $f^i_j \mathring{q}^i_j = 0$ and $f_i \mu_i = 0$. Let $\mathbf{f}(\mathring{\mathbf{q}}, \boldsymbol{\mu}|z)$ denote such complementarity problem. Define the set active constraints at the equilibrium of the z-replica by

$$T(\mathring{\mathbf{q}}, \boldsymbol{\mu}|z) = \left\{ ij \in E | \mathring{q}^i_j > 0 \right\} \cup \left\{ i \in V | \mu_i > 0 \right\}.$$

Let $\mathbf{f}_T(\mathring{\mathbf{q}}, \boldsymbol{\mu}|z)$ denote the complementarity problem obtained by restricting attention to the active constraints. By assumption any replica economy possesses a unique equilibrium and conditions for existence are met. Thus, the Jacobian of the problem must be positive definite at the unique solution (Kolstad and Mathiensen 1987),

$$J_T(\mathring{\mathbf{q}}, \boldsymbol{\mu}|z) = \nabla \mathbf{f}_T(\mathring{\mathbf{q}}, \boldsymbol{\mu}|z) > 0,$$

where only the principal minor of Jacobian associated the active constraints has to be considered. The implicit function theorem further implies that at the unique equilibrium of such problem

$$\frac{\partial \mathbf{f}_T}{\partial \mathbf{q}} \frac{\partial \mathbf{q}}{\partial z} + \frac{\partial \mathbf{f}_T}{\partial \boldsymbol{\mu}} \frac{\partial \boldsymbol{\mu}}{\partial z} + \frac{\partial \mathbf{f}_T}{\partial z} = 0 \quad \Rightarrow \quad \frac{\partial (\mathbf{q}, \boldsymbol{\mu})_T}{\partial z} = -J_T(\mathring{\mathbf{q}}, \boldsymbol{\mu}|z)^{-1} \frac{\partial \mathbf{f}_T}{\partial z},$$

where the definition of the complementarity problem requires that

$$\frac{\partial f^i_j(\mathring{\mathbf{q}}, \boldsymbol{\mu}|z)}{\partial z} = \frac{u''_j(\mathring{q}_j) \mathring{q}^i_j}{z^2} \quad \text{and} \quad \frac{\partial f_i(\mathring{\mathbf{q}}, \boldsymbol{\mu}|z)}{\partial z} = 0.$$

Let $L(\mathring{\mathbf{q}}|z) = \{ij{\in}E|\mathring{q}^i_j > 0\}$ denote the set of active flows. Define: $J_L(\mathring{\mathbf{q}}, \boldsymbol{\mu}|z)^{-1}$ to be the leading minor of $J_T(\mathring{\mathbf{q}}, \boldsymbol{\mu}|z)^{-1}$ associated with indexes in $L(\mathring{\mathbf{q}}|z)$; $\mathbf{x} = \{u''_j(\mathring{q}_j)\mathring{q}^i_j\}_{ij\in L}$; and $\mathbf{Z} = \{z^{ij}_{kl}\}_{ij,kl\in L}$ as follows

$$z^{ij}_{kl} = \begin{cases} 1/z & \text{if} \quad ij = kl \\ 1 & \text{if} \quad j = k \ \cap \mathring{q}^i_j > 0 \\ 0 & \text{if} \quad \text{otherwise} \end{cases}.$$

For such notation, one gets that

$$\begin{aligned} \mathbf{Zx} &= \left\{u''_j(\mathring{q}_j)(\mathring{q}^i_j/z) + u''_i(\mathring{q}_i)\textstyle\sum_{k\in V_i} \mathring{q}^k_i\right\}_{ij\in L}, \\ \frac{\partial \mathbf{q}}{\partial z} &= -(1/z^2)J_L(\mathring{\mathbf{q}}, \boldsymbol{\mu}|z)^{-1}\mathbf{x}, \end{aligned}$$

where the second equality obtains as the the replica counter never appears in an outflow constraint. The matrix $\mathbf{Z}$ is positive definite, because for an appropriate order of links it is lower triangular, and because all elements on the main diagonal are positive.[21] Differentiating per-capita social welfare with respect to z one gets that

$$\begin{aligned} \frac{\partial W(\mathring{\mathbf{q}})}{\partial z} &= \frac{\partial}{\partial z}\left(\frac{1}{V}\textstyle\sum_{i\in V} u_i(\mathring{q}_i)\right) = \frac{1}{V}\textstyle\sum_{ij\in E}(u'_j(\mathring{q}_j) - u'_i(\mathring{q}_i))(\partial q^i_j/\partial z) = \\ &= -\frac{1}{V}\textstyle\sum_{ij\in E}\left(u''_j(\mathring{q}_j)(\mathring{q}^i_j/z) + u''_i(\mathring{q}_i)\sum_{k\in V_i}\mathring{q}^k_i\right)(\partial q^i_j/\partial z) = \\ &= -\frac{1}{V}\mathbf{x}'\mathbf{Z}'\frac{\partial \mathbf{q}}{\partial z} = \frac{1}{Vz^2}\mathbf{x}'\mathbf{Z}'J_L(\mathring{\mathbf{q}}, \boldsymbol{\mu}|z)^{-1}\mathbf{x} \geq 0. \end{aligned}$$

The last expression is positive since it is a bilinear form and because both $\mathbf{Z}'$ and $J_L(\mathring{\mathbf{q}}, \boldsymbol{\mu}|z)$ are positive definite. In fact, because both are positive definite, consider the positive definite square root H of $J_L(\mathring{\mathbf{q}}, \boldsymbol{\mu}|z)^{-1}$ (i.e. $J_L(\mathring{\mathbf{q}}, \boldsymbol{\mu}|z)HH = I$). Then $\mathbf{Z}'J_L(\mathring{\mathbf{q}}, \boldsymbol{\mu}|z)^{-1} = H^{-1}(H\mathbf{Z}'H)H$. Therefore $\mathbf{Z}'J_L(\mathring{\mathbf{q}}, \boldsymbol{\mu}|z)^{-1}$ and $H\mathbf{Z}'H$ have the same eigenvalues. Since $H\mathbf{Z}'H = H'\mathbf{Z}'H$, such matrix is positive definite and thus has only non-negative eigenvalues. The third equality uses the observation that $\partial q^i_j/\partial z \neq 0$ implies that the first order condition must hold with equality. In fact, if $\partial q^i_j/\partial z < 0$, then $\mathring{q}^i_\circ < Q_i$ clearly holds and $\mathring{q}^i_j > 0$ or else $\mathring{q}^i_j$ could not decrease in equilibrium. If $\partial q^i_j/\partial z > 0$ instead, then $\mathring{q}^i_j > 0$ clearly holds and $\mathring{q}^i_\circ < Q_i$ or else $\mathring{q}^i_j$ could not increase in equilibrium. ■

[21] The matrix can be arranged in a triangular fashion for any profile of equilibrium flows, because goods do not cycle in the economy.