Efficiency in Repeated Games with Local Monitoring

Francesco Nava and Michele Piccione

London School of Economics

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General Motivation

We consider environments in which each player:

1. interacts repeatedly with a subset of players
2. has to choose a common action for all his neighbors
3. is **privately informed of the players with whom he interacts**
4. observes only the actions chosen by such players
5. and cannot communicate

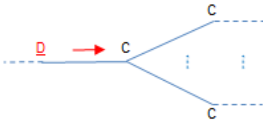
The leading examples are:

1. decentralized markets
2. local public goods games
3. ...

Complications

Characterizing Sequential Equilibria in such games is hard because:

1. Trigger strategies may not work:



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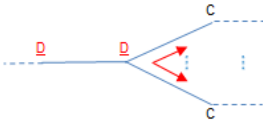

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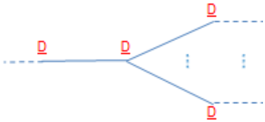

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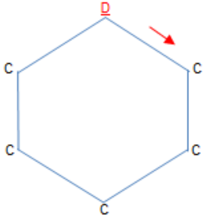

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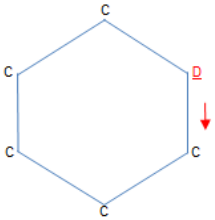

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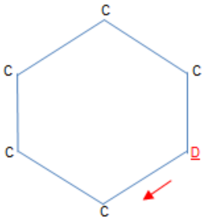

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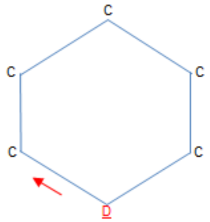

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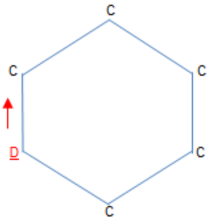

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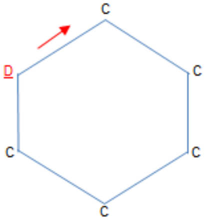

Equilibrium Properties

The analysis discusses some properties of sequential equilibria [SE]:

1. Efficiency [C]
   Cooperative behavior along the equilibrium path
2. Invariance [I]
   Behavior is independent of beliefs about the monitoring structure
3. Stability [S]
   Reversion to full cooperation in a finite time after any history

Roadmap

All results are developed for symmetric two-action games.

The main result for **patient** players shows that:

1. SE satisfying C, I and S exist.

The main results for **impatient** players instead, show that:

2. SE exist satisfying C, I and S, if monitoring is acyclic;
3. SE exist satisfying C and I, if beliefs have full support.

Classical techniques apply only to specific Prisoner's Dilemma games.
For such games we establish that SE satisfying C and I always exist.

Related Literature

The paper fits within the literature on community enforcement.

Contributions to this literature include: Ahn (1997), Ali and Miller (2009), Ben-Porath and Kaneman (1996), Deb (2009), Ellison (1994), Fainmesser (2010), Fainmesser and Goldberg (2011), Jackson et al (2010), Kandori (1992), Kinateder (2008), Lippert and Spagnolo (2008), Mihm, Toth and Lang (2009), Renault and Tomala (1998), Takahashi (2008), Vega-Redondo (2006), Wolitzky (2011), Xue (2011).

Most of these studies:

- invoke strong assumptions on the monitoring structure;
- assume that the environment is symmetric;
- restrict attention to Prisoner's Dilemmas;
- allow the strategy to depend on prior beliefs.

Wolitzky (2011) considers similar monitoring structure, but imposes more stringent assumptions about observability.

Stage Game: Information and Actions

Consider a game played by a set N of players.

An undirected graph (N, G) defines the *information network*.

Player i only observes players in his *neighborhood* N_i.

Players are **privately informed** about their neighborhood.

Beliefs regarding the information network are derived from a common prior.

Let $A_i = \{C, D\}$ denote set of actions of player i.

Players choose **a single action** for all their neighbors.

Stage Game: Payoffs

The payoff of player i is separable and satisfies:

$$v_i(a_i, a_{N_i}) = \sum_{j \in N_i} \eta_{ij} u_{ij}(a_i, a_j)$$

The payoff of i in relationship ij, $u_{ij}(a_i, a_j)$, is given by:

$i \setminus j$	C	D
C	1	$-l$
D	$1+g$	0

Assumption A1: Assume that:

1. it is efficient to cooperate, $g - l < 1$;
2. it is privately beneficial to defect when others cooperate, $g > 0$.

Repeated Game

The network is realized prior to the game and **remains constant**.

Players discount the future by $\delta \leq 1$.

Repeated game payoffs conditional on graph G are defined as

$$U_i(\sigma_N | G) = \begin{cases} (1-\delta)\sum_{t=1}^{\infty} \delta^{t-1} v_i(a_i^t, a_{N_i}^t) & \text{if} \quad \delta < 1 \\ \Lambda_t\left((1/t)\sum_{s=1}^{t} v_i(a_i^s, a_{N_i}^s)\right) & \text{if} \quad \delta = 1 \end{cases}$$

where $\Lambda_t(\cdot)$ denotes the Banach-Mazur limit of a sequence.

Desired Properties

Definition [C]: A strategy profile is *collusive* if, along the equilibrium path, all the players always play C for any realized network G.

Definition [Π-I]: A strategy profile is a Π-*invariant equilibrium*, if it is a sequential equilibrium for any prior beliefs in Π.

Definition [Π-S]: A strategy profile satisfies Π-*stability*, if for any information network G which has positive probability for some prior belief in Π and for any history h, there exists a period T_G^h such that all the players play C in all periods greater than T_G^h.

Patient Players

Π^A be the set of priors for which posterior beliefs are well defined.

Theorem (1)

If A1 holds and $\delta = 1$, there exists a strategy satisfies C, Π^A-I, and Π^A-S.

The proof proceeds by arguing that:

- a contagion-annhilation strategy that satisfies C & Π^A-S exists;
- complying with this strategy is a best response when others do for any network G [without recourse to the one-deviation property];
- the strategy thus, satisfies Π^A-I.

A Contagion-Annihilation Strategy

The strategy employs two state variables (d_{ij}, d_{ji}) for each link ij.

Both variables (d_{ij}, d_{ji}) depend only on the history of play within the relationship, and are thus common knowledge for i and j.

The state variables are constructed so that in each relationship:

- unilateral deviations to D are punished with an extra D by the enemy;
- unilateral deviations to C are punished with an extra D by both;
- simultaneous deviations to D are not punished.

One may interpret d_{ij} as the number of D's that i and j require from i to return to the initial state.

A Contagion-Annihilation Strategy

The transition rule for (d_{ij}, d_{ji}) is defined as follows:

- in the first period $d_{ij} = d_{ji} = 0$;
- thereafter, if actions (a_i, a_j) are played:

d_{ij}	0	0	0	0	0	0	0	0	$+$	$+$	$+$	$+$
d_{ji}	0	0	0	0	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$
a_i	D	D	C	C	D	D	C	C	D	D	C	C
a_j	D	C	D	C	D	C	D	C	D	C	D	C
Δd_{ij}	0	0	1	0	0	1	0	1	-1	0	1	0
Δd_{ji}	0	1	0	0	0	2	-1	1	-1	1	0	0
Dif	0	-1	1	0	0	-1	1	0	0	-1	1	0

The interim strategy satisfies:

$$\zeta_i(h_i) = \begin{cases} C & \text{if} \quad \max_{j \in N_i} d_{ij}(h_i) = 0 \\ D & \text{if} \quad \max_{j \in N_i} d_{ij}(h_i) > 0 \end{cases}$$

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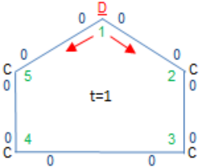

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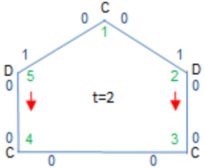

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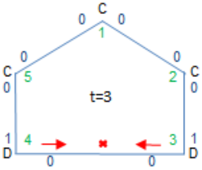

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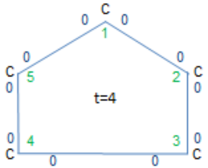

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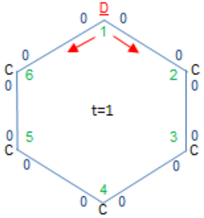

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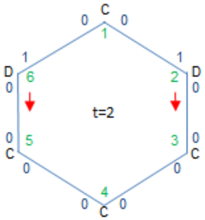

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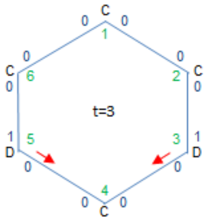

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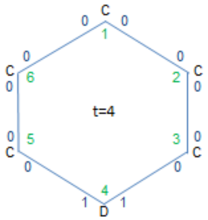

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To build some intuition, consider the following cases:

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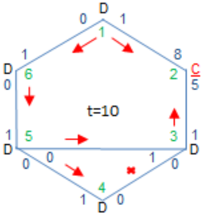
Example III: player 1 deviates to D in the first period and player 2 plays C for the first ten periods.



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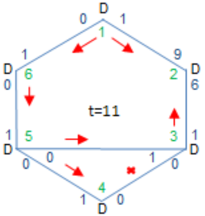

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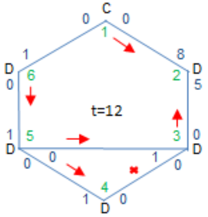

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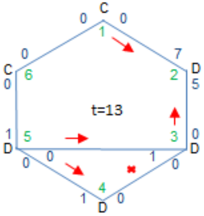

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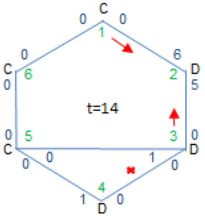

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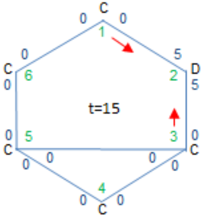

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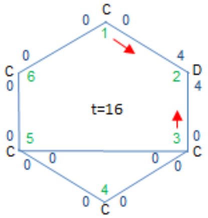

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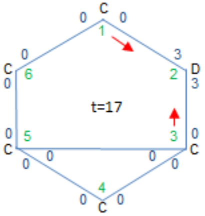

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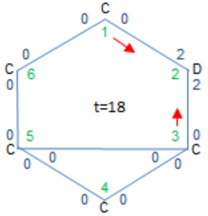

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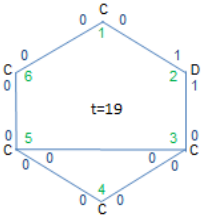


Preliminary Definitions

Define the *excess defection* on link ij as:

$$e_{ij} = d_{ij} - d_{ji}$$

Fix G, and for any $h \in H$ and any path $\pi = (j_1, .., , j_m)$ define:

$$E_\pi = \sum_{k=1}^{m-1} e_{j_k j_{k+1}}$$

Let P_{if} be the set of paths with initial node i and terminal node f

Let $S(h)$ denote the set of *sources* of punishments in the network:

$$S(h) = \{i \in N : E_\pi(h) \leq 0 \text{ for any } \pi \in P_{if}, \text{ for any } f \in N\}$$

Preliminary Result

Lemma (1)

Consider a network G. For any $h \in H$ and any $a \in A_N$:

(1) If $\pi \in P_{if}$

a_i	D	D	C	C
a_f	D	C	D	C
$\Delta E_\pi(h, a)$	0	-1	1	0

(2) If $\varkappa \in P_{ii}$

$$E_\varkappa(h) = 0$$

(3) If $\pi, \pi' \in P_{if}$

$$E_\pi(h) = E_{\pi'}(h)$$

(4) $S(h)$ is non-empty.

Prove it!

Intuition Theorem 1

The proof first establishes Π^A-S by showing that the set $S(h)$ always expands in equilibrium:

(A) $S(h^t) \subseteq S(h^{t+1})$
(B) $S(h^t) \subset S(h^{t+k})$ for some $k > 0$ if $S(h^t) \subset N$

Since Π^A-S holds and $\delta = 1$, the payoff in every relationship converges to 1.

To prove Π^A-I, we show that no player can deviate to play infinitely more D's than his opponents, as any deviation leading to the play of (D, C) in a relationship is met by a punishment of (C, D) in same relationship.

Prove it!

Robustness & Comments

Theorem 1 is robust with respect to:

- uncertainty in the number of players;
- heterogeneity in payoffs if A1 holds in all relationships;
- uncertainty in payoffs as long as A1 is met in all realizations.

Arbitrary patience is required, since histories exist for which ζ_i is not IC:

- for any history such that $(d_{ij}, d_{ji}) = (0, M)$ for any large M;
- if $\delta < 1$ and $\eta_{ij} > 0$, one can find M such that ζ_i is not IC for i.

However, (d_{ij}, d_{ji}) must grow unbounded to prevent D's from cycling.

Impatient Players

This section circumvents the problem of defections growing unbounded by restricting the class of admissible priors.

As before, the proposed equilibrium strategy ξ_i:

- relies on two state variables (d_{ij}, d_{ji}) for each relationship ij
- requires a player i to defect iff at least one of his d_{ij} is positive

The transition rule differs and depends on the sign of the parameter l.

Changes take place mainly off the equilibrium path and imply that d_{ij} is bounded by 2 for any history.

If $I > 0$, the transitions satisfy:

d_{ij}	0	0	0	0	0	0	0	0	$+$	$+$	$+$	$+$
d_{ji}	0	0	0	0	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$
a_i	D	D	C	C	D	D	C	C	D	D	C	C
a_j	D	C	D	C	D	C	D	C	D	C	D	C
Δd_{ij}	0	0	2	0	0	d_{ji}	0	d_{ji}	-1	0	0	0
Δd_{ji}	0	2	0	0	0	0	-1	0	-1	0	0	0

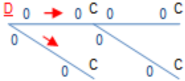
If $I < 0$, the transitions satisfy:

d_{ij}	0	0	0	0	0	0	0	0	$+$	$+$	$+$	$+$
d_{ji}	0	0	0	0	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$
a_i	D	D	C	C	D	D	C	C	D	D	C	C
a_j	D	C	D	C	D	C	D	C	D	C	D	C
Δd_{ij}	0	0	1	0	0	0	0	2	-1	2-d_{ij}	2-d_{ij}	2-d_{ij}
Δd_{ji}	0	1	0	0	-1	-1	-1	2-d_{ji}	-1	2-d_{ji}	2-d_{ji}	2-d_{ji}

If $I = 0$, choose either transition.

To build some intuition, consider the following cases:

Example I: player 1 deviates to D in the first two periods and $l \geq 0$.



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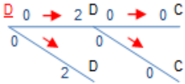

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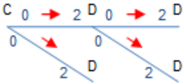

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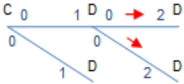

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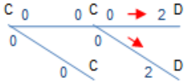

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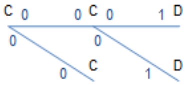

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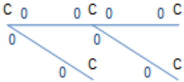


Example II

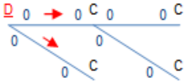
Example III

Example IV

To build some intuition, consider the following cases:

Example I

Example II: player 1 deviates to D in the first period, player 2 deviates to C in the second period and $l \geq 0$.



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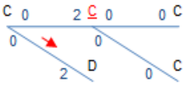

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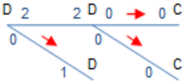

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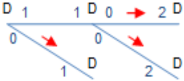


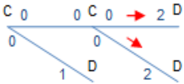
Example III

Example IV

To build some intuition, consider the following cases:

Example I

Example II: player 1 deviates to D in the first period, player 2 deviates to C in the second period and $I \geq 0$.



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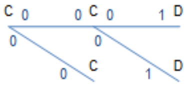

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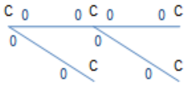


Example III

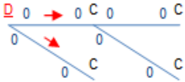
Example IV

To build some intuition, consider the following cases:

Example I

Example II

Example III: player 1 deviates to D in the first two periods and $l \leq 0$.



Example IV

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Example I

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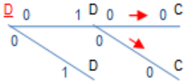
Example III: player 1 deviates to D in the first two periods and $l \leq 0$.



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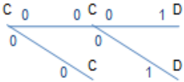

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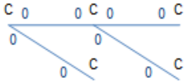

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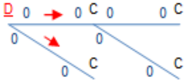
Example I

Example II

Example III

Example IV: player 1 deviates to D in the first period, player 2 deviates to C in the second period and $l \leq 0$.



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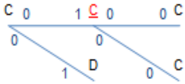

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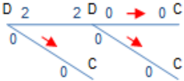

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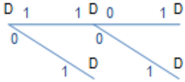

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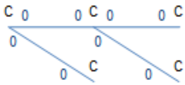


Preliminary Result

The next result shows that player i never expects his neighbors to play D due to past play in relationships to which he does not belong, if:

1. all deviations have occurred in player i's neighborhood;
2. no two neighbors of player i are linked by a path.

For a history h and a network G let $\mathcal{D}(G, h)$ denote the set of players who deviated in the past.

Lemma (3)

Consider a network G, a player $i \in N$, and a history $h \in H$ such that:

(i) $\mathcal{D}(G, h) \subseteq N_i \cup \{i\}$;

(ii) *If $j \in \mathcal{D}(G, h)$, link ij is a bridge in G.*

Then, $d_{jk}(h) = 0$ for any $j \in N_i$ and $k \in N_j \backslash \{i\}$. *Prove it!*

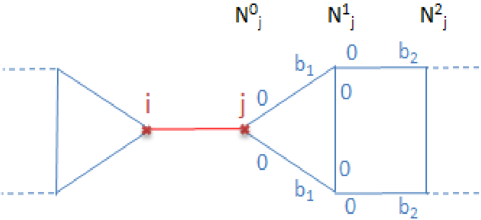
Intuition Lemma 3

Consider $j \in \mathcal{D}(G, h)$ and the component to which j belongs. No other player in $\mathcal{D}(G, h)$ belongs to the component by (ii).

Partition players in the component based on distance from j: $N_j^0 = \{j\}$, and N_j^z consists of players whose shortest path to j contains z links.

Since only j defects in the component, for any $z \geq 0$ and $r \in N_j^z$:

$$d_{rk}(h) = \begin{cases} 0 & \text{if} \quad k \in N_r \backslash N_j^{z-1} \\ b_z(h) & \text{if} \quad k \in N_j^{z-1} \end{cases}$$



Impatient Players and Acyclic Graphs

First restrict the class of priors so that only acyclic graphs are feasible.

Let Π^{NC} be the set of prior beliefs such that if $f(G) > 0$, then G is acyclic.

Theorem (2)

If A1 holds and if δ is sufficiently high, the strategy profile ξ_N satisfies C, Π^{NC}-I, and Π^{NC}-S.

Π^{NC}-S is proven by induction on the number of players:

- it holds trivially with only two players;
- adding a link delays reversion to C by at most 2 periods.

Prove it!

Intuition Theorem 2

To prove Π^{NC}-I, set off-equilibrium beliefs so that player i at each history h_i attributes any observed deviation only to his neighbors:

$$\beta(G, h|h_i) > 0 \quad \Rightarrow \quad \mathcal{D}\left(G, h\right) \subseteq N_i \cup \{i\}$$

To do so, set trembles so that any finite # of deviations to D is:

- infinitely more likely than 1 deviation to C;
- infinitely more likely than 1 earlier deviation to D.

Beliefs in Π^{NC} imply that Lemma 2 holds. So, for any history i believes:

- $d_{jk} = 0$ for $j \in N_i$ and $k \in N_j \backslash \{i\}$;
- the action of $j \in N_i$ is solely determined by d_{ji}.

For such beliefs, the strategy is proven to be sequentially rational.

Robustness & Comments

All robustness checks of the previous section are met provided that the ordinal properties of the games are the same across relationships:

- uncertainty about the number of player
- heterogeneity in payoffs satisfying A1
- uncertainty about payoffs satisfying A1.

The strategy is also robust to heterogeneity in discount rates.

Properties obtain since defections cannot cycle on any admissible graph.

Impatient Players and Generic Beliefs

Next we extend the results obtained for acyclic networks to any environment in which players' beliefs have full support.

Let Π^{FS} be the set of prior beliefs such that $f(G) > 0$ for any G.

Theorem (3)

If A1 holds and if δ is sufficiently high, the strategy profile ξ_N satisfies C and Π^{FS}-I.

The proof proceeds by showing:

- how to construct trembles for which:

$$\beta(G, h|h_i) > 0 \quad \Rightarrow \quad \mathcal{D}(G, h) \subseteq N_i \cup \{i\}$$

- that for such trembles the same argument of Theorem 2 applies.

Although stability fails here, players will always believe in reversion to full cooperation in a finite time.

Intuition Theorem 3

The proof develops consistent beliefs for which i believes that:

I. player j's neighborhood contains only i whenever i observes a deviation from j;

II. all deviations are local.

Trembles are constructed so that:

- deviations by a players with $n_i = 1$ are infinitely more likely;
- more recent deviations are infinitely more likely than less recent ones.

Lemma (6)

If A3 holds, there exist beliefs β consistent with ξ_N such that, for any player i and observed history h_i, $\beta(G, h | h_i) = 0$:

(a) *if i observes a deviation from j and $n_j > 1$;*

(b) *if $\mathcal{D}(G, h) \not\subseteq N_i \cup \{i\}$.*

Prove it!

Robustness & Comments

Provided that the ordinal properties of the game coincide across relationships, all robustness checks of the previous section are met:

- uncertainty about the number of player
- heterogeneity in payoffs satisfying A1
- uncertainty about payoffs satisfying A1
- heterogeneity in discount rates

Properties obtain, since players expect defections not to cycle.

When $I > 0$ and $\delta < 1$:

- trigger strategies sustain cooperation for $\delta \in (\underline{\delta}, \overline{\delta})$;
- such strategies satisfy Π^A-I and C;
- cooperation can be extended to $\delta \in (\underline{\delta}/\overline{\delta}, 1)$ by partitioning the game into independent games.

The proof is an adaptation of an argument first used by Ellison (1994).

A similar argument was developed independently by Xue (2011) in a model in which the network is common knowledge and players can communicate.

Theorem (4)

If A1 holds, $I > 0$ and if δ is sufficiently high, a strategy profile that satisfies C and Π^A-I exists. Prove it!

Robustness & Comments

- Theorem 4 is robust to uncertainty about the number of players.
- Theorem 4 is **not robust** to heterogeneity in δ.
- Theorem 4 is **not robust** to heterogeneity in payoffs, since g must be common to all relationships.
- Similarly, the values of l and η_{ij} can be private information.
- Theorem 4 violates Π^A-S, since no player ever reverts to full cooperation after observing a deviation.

Conclusions

The main result for **patient** players shows that:

1. SE satisfying C, I and S exist.

The main results for **impatient** players instead, show that:

2. SE exist satisfying C, I and S, if monitoring is acyclic;
3. SE exist satisfying C and I, if beliefs have full support;
4. SE exist satisfying C and I, in specific PD games.

An impossibility result for C, I and S with general graphs and impatience lies within future objectives.

Banach-Mazur Limits Back

Let ℓ_∞ denote the set of bounded sequences of real numbers

A Banach-Mazur limit is a linear functional $\Lambda : \ell_\infty \to \mathbb{R}$ such that:

1. $\Lambda(e) = 1$ if $e = \{1, 1, ...\}$
2. $\Lambda(x^1, x^2, ...) = \Lambda(x^2, x^3, ...)$ for $\forall \{x^t\} \in \ell_\infty$

It can be shown that, for any sequence $\{x^t\} \in \ell_\infty$:

$$\liminf x^t \leq \Lambda\left(x^t\right) \leq \limsup x^t$$

Remark

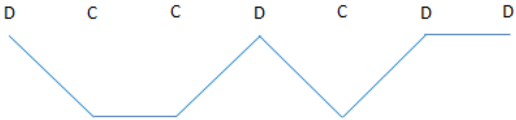
For simplicity, we restrict players to use pure strategies. This ensures that expectation of the Banach-Mazur limit is the same as the Banach-Mazur limit of the expectation. Our analysis can be extended to mixed strategies with infinite supports by medial limits, which can be shown to exists under the continuum hypothesis (see Abdou and Mertens (1989)).

Intuition Lemma 1

To prove (1) observe that transitions always satisfy:

a_i	D	D	C	C
a_j	D	C	D	C
Δe_{ij}	0	-1	1	0

Thus a simple counting argument establishes (1) for any path.



Given (1), two simple induction arguments establish (2) and (3).

To prove (1) observe that transitions always satisfy:

a_i	D	D	C	C
a_j	D	C	D	C
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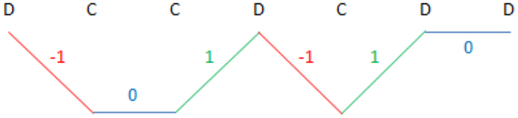


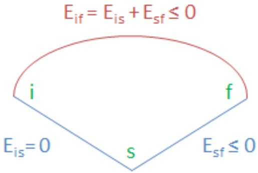
Given (1), two simple induction arguments establish (2) and (3).

Intuition Lemma 1 Back

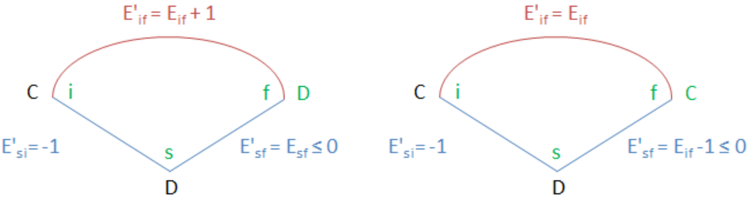
Also (4) is proven by induction on the history length:

1. The claim holds for the empty history.
2. Suppose that $S(h)$ is non-empty for h.
3. Consider $h' = (h, a)$ and $i \in S(h)$.
4. If $i \in S(h')$, the claim holds.
5. If $i \notin S(h')$, there exists a player s such that:
   - $E_{is}(h) = 0$, since $i \in S(h)$;
   - $E_{is}(h') = 1$, since $i \notin S(h')$.
6. If so, $s \in S(h')$ and the claim still holds.

Observe that $i \in S(h)$ implies:



and therefore $s \in S(h')$:



Lemma (2)

The strategy profile ζ_N satisfies Π^A-S. Prove It!

The proof shows that the set $S(h)$ always expands in equilibrium:

(A) $S(h^t) \subseteq S(h^{t+1})$
(B) $S(h^t) \subset S(h^{t+k})$ for some $k > 0$ if $S(h^t) \subset N$

The intuition follows by observing that:

1. if $i \in S(h)$, then (d_{ij}, d_{ji}) declines for $j \in N_i$
2. if $i \in S(h)$, and $(d_{ij}, d_{ji}) = (0,0)$, then player $j \in S(h)$

By Π^A-S, the payoff in every relationship converges to 1.

To prove Π^A-I we show that no player can deviate to play infinitely more D's than his opponents.

Let $n_{ij}^t(a_i, a_j|T)$ denote the number of times in which (a_i, a_j) is played on link ij between periods t and $t+T$.

Let $\Delta^t(T) \equiv n_{ij}^t(D, C|T) - n_{ij}^t(C, D|T)$ and observe that:

$$\begin{aligned} T+1 &\geq n_{ij}^t(C, C|T) + n_{ij}^t(C, D|T) + n_{ij}^t(D, C|T) \\ &= n_{ij}^t(C, C|T) + 2n_{ij}^t(C, D|T) + \Delta^t(T) \end{aligned}$$

Intuition Theorem 1 Back

By A1, the payoff of player i in relationship ij satisfies:

$$\begin{aligned}\sum_{t}^{t+T} u_{ij}(\theta_i,\zeta_j^h) &= n_{ij}^t(C,C|T) + (1+g-l)\,n_{ij}^t(C,D|T) + (1+g)\Delta^t(T)\\ &\leq n_{ij}^t(C,C|T) + 2n_{ij}^t(C,D|T) + (1+g)\Delta^t(T)\\ &\leq (T+1) + g\Delta^t(T)\end{aligned}$$

Note that:

1. $\Delta^t(T) = e_{ij}^{t-1} - e_{ij}^{t+T}$;
2. $e_{ij}^s < 0$ implies that j plays D and thus $e_{ij}^{s+1} \geq e_{ij}^s$;
3. hence $\Delta^t(T)$ is bounded above since $e_{ij}^{t+T} \geq -1$.

Since $\Delta^t(T)$ is bounded $\Lambda_T \left(\sum_t^{t+T} u_{ij}(\theta_i,\zeta_j^h)/T+1\right) \leq 1$.

Consider any player $j \in \mathcal{D}(G, h) \setminus \{i\}$.

Let $(N(G_j), G_j)$ denote the component of $G \setminus \{ij\}$ to which j belongs.

By (ii), such component cannot include i and players in $N_i \setminus \{j\}$.

Partition $N(G_j)$ based on the distance from j:

- $N_j^0 = \{j\}$;
- N_j^z consists of players whose shortest path to j contains z links.

We show that, since $\mathcal{D}(G_j, h) = \{j\}$, for any $z \geq 0$, $r \in N_j^z$, and $rk \in G_j$:

$$d_{rk}(h) = \begin{cases} 0 & \text{if} \quad k \in N_r \setminus N_j^{z-1} \\ b_z(h) & \text{if} \quad k \in N_j^{z-1} \end{cases} \quad (1)$$

Condition (1) holds if $h = \emptyset$, as $d_{rk}(\emptyset) = 0$ for any $rk \in G_j$.

Observe that for $z > 0$ and $m \in N_j^z$:

- $N_m \subset N_j^{z-1} \cup N_j^z \cup N_j^{z+1}$
- $N_m \cap N_j^{z-1} \neq \emptyset$

We show that if (1) holds for a history h of length up to T, it holds for the history (h, a) of length $T + 1$.

For any $z > 0$ and any $r \in N_j^z$:

$$a_r = D \Leftrightarrow d_{rk}(h) > 0 \text{ for } k \in N_j^{z-1} \tag{2}$$

since $r \notin \mathcal{D}(G, h)$ and since $d_{rk}(h) = 0$ for any $k \in N_r \backslash N_j^{z-1}$.

Thus, all players in N_j^z choose the same action since $N_j^{z-1} \cap N_r \neq \emptyset$.

Hence, for any player $r \in N_j^z$ and any link $rk \in G_j$:

$$\begin{aligned} d_{rk}(h,a) &= 0 \text{ if } k \in N_i^z \text{ and } z > 0 \\ d_{rk}(h,a) &= 0 \text{ if } k \in N_j^{z+1} \text{ and } z \geq 0 \\ d_{rk}(h,a) &= b_z(h,a) \text{ if } k \in N_j^{z-1} \text{ and } z > 0 \end{aligned}$$

where the three conditions respectively hold since:

1. $d_{rk}(h) = d_{kr}(h) = 0$ and $a_r = a_k$.
2. $d_{rk}(h) = 0$, and because (2) implies that $d_{rk}(h,a) = 0$.
3. $d_{rk}(h) = b_z(h)$, and because $a_l = a_m$ for any $l, m \in N_j^s$ for $s \geq 0$.

Proof Lemma 3 Back

Thus, (1) holds for any history in which only j has deviated in G_j.

Thus, (i) and (ii) imply $d_{jk}(h) = 0$ for $j \in \mathcal{D}(G,h) \setminus \{i\}$ and $k \in N_j \setminus \{i\}$.

To conclude the proof consider players in $N_i \setminus \mathcal{D}(G,h)$.

Denote by $(N(G_i), G_i)$ the component of the graph to which i belongs when links between i and players in $\mathcal{D}(G,h)$ have been removed from G.

Clearly, $N_i \setminus \mathcal{D}(G,h) \subset N(G_i)$ and $\mathcal{D}(G_i,h) = \{i\}$.

Hence, the previous argument applies and $d_{jk}(h) = 0$ for any $j \in N_i \setminus \mathcal{D}(G,h)$ and any $k \in N_j \setminus \{i\}$. ∎

Lemma (4)

The strategy profile ξ_N satisfies Π^{NC}-S.

Consider any tree G and any history.

If players adhere to ξ_N, the transitions are:

$l \geq 0$

d_{ij}	0	0	0	0	0	0	+
d_{ji}	0	0	0	0	+	+	+
a_i	D	D	C	C	D	C	D
a_j	D	C	D	C	D	D	D
Δd_{ij}	0	0	2	0	0	0	-1
Δd_{ji}	0	2	0	0	0	-1	-1

$l \leq 0$

d_{ij}	0	0	0	0	0	0	+
d_{ji}	0	0	0	0	+	+	+
a_i	D	D	C	C	D	C	D
a_j	D	C	D	C	D	D	D
Δd_{ij}	0	0	1	0	0	0	-1
Δd_{ji}	0	1	0	0	-1	-1	-1

Proof Theorem 2 Back

The claim is proven by induction on the number of players:

- Π^{NC}-S holds for $n = 2$. So, suppose that $n > 2$.
- Consider a terminal node j and his unique neighbor i.
- If $d_{ij} = 0$, it remains so for the remainder of the game.
- If so, relationship ij is superfluous for the play of i:

$$\xi_i = D \quad \Leftrightarrow \quad d_{ik} > 0 \text{ for } k \in N_i \backslash \{j\}$$

- By induction Π^{NC}-S holds in the network $G \backslash \{ij\}$.
- Thus, Π^{NC}-S must hold also in G as $d_{ji} \leq 2$.
- If $d_{ij} > 0$, $d_{ij} = 0$ in two periods, as j's only neighbor is i.
- And the above argument applies again. ■

C is obvious. To prove Π^{NC}-I, set off-equilibrium beliefs so that i at each history h_i attributes any observed deviation only to players in N_i.

Such beliefs can be derived by assuming that the recent deviations to D are infinitely more likely than earlier deviations, eg:

1. if $\max_j d_{ij} = 0$, a deviation to D by i in period t occurs with probability ε^{α^t} for $\alpha \in (0, 1/(n+1))$
2. if $\max_j d_{ij} > 0$, a deviation to C by i in period t occurs with probability ε^2

As $\varepsilon \rightarrow 0$, any finite # of deviations to D is infinitely more likely:

- than 1 deviation to C;
- than 1 earlier deviation to D.

Let β denote the system of beliefs obtained as $\varepsilon \to 0$.

For any history h_i observed by $i \in N$:

$$\beta(G, h | h_i) > 0 \quad \Rightarrow \quad \mathcal{D}(G, h) \subseteq N_i \cup \{i\}$$

The latter observation and A2 imply that Lemma 3 holds.

Hence, for any history h, player i believes:

- $d_{jk}(h) = 0$ for $j \in N_i$ and $k \in N_j \backslash \{i\}$;
- the action of $j \in N_i$ is solely determined by $d_{ji}(h)$.

First suppose that $l \geq 0$.

Only seven states are possible since $d_{ij} \in \{0, 1, 2\}$.

If $\max_j d_{ij}(h_i) = 0$ and δ is high, i prefers to comply and play C since his expected payoff with any $j \in N_i$ satisfies:

	Equilibrium: C			Deviation: D		
(d_{ij},d_{ji})	t	t+1	t+2	t	t+1	t+2
(0,0)	1	1	1	1+g	-l	-l
(0,1)	-l	1	1	0	-l	1
(0,2)	-l	-l	1	0	-l	-l

Player i strictly prefers to comply with any neighbor.

If $\max_j d_{ij}(h_i) = 1$ and δ is high, i prefers to comply since:

	Equilibrium: D				Deviation: C			
(d_{ij}, d_{ji})	t	t+1	t+2	t+3	t	t+1	t+2	t+3
(0,0)	1+g	-l	-l	1	1	1+g	-l	-l
(0,1)	0	-l	1	1	-l	1+g	-l	-l
(1,0)	1+g	1	1	1	1	0	1	1
(1,1)	0	1	1	1	-l	0	1	1
(0,2)	0	-l	-l	1	-l	0	-l	1

The first and the last stream converge.

Equilibrium is strictly preferred in the remaining scenarios.

Since $\max_j d_{ij}(h_i) = 1$, a neighbor exists with whom player i strictly loses, when δ is close to 1.

Proof Theorem 2 Back

If $\max_j d_{ij}(h_i) = 2$ and δ is high, i prefers to comply since:

	Equilibrium: D					Deviation: C				
(d_{ij}, d_{ji})	t	t+1	t+2	t+3	t+4	t	t+1	t+2	t+3	t+4
(0,0)	1+g	0	-l	-l	1	1	1+g	0	-l	-l
(0,1)	0	0	-l	1	1	-l	1+g	0	-l	-l
(1,0)	1+g	1+g	-l	-l	1	1	0	1+g	-l	-l
(1,1)	0	1+g	-l	-l	1	-l	0	1+g	-l	-l
(0,2)	0	0	-l	-l	1	-l	0	0	-l	1
(2,0)	1+g	1+g	1	1	1	1	0	0	1	1
(2,2)	0	0	1	1	1	-l	0	0	1	1

An argument similar to the preceding applies.

The strategy profile ξ_N thus trivially satisfies EP, since the incentives to comply are not affected by the beliefs about the graph.

Now suppose that $l \leq 0$.

Only five states are possible since $d_{ij} \in \{0, 1, 2\}$.

If $\max_j d_{ij}(h_i) = 0$ and δ is high, i prefers to comply and play C since his expected payoff with any $j \in N_i$ satisfies:

	Equilibrium: C			Deviation: D		
(d_{ij},d_{ji})	t	t+1	t+2	t	t+1	t+2
(0,0)	1	1	1	1+g	-l	1
(0,1)	-l	1	1	0	1	1

Player i weakly prefers to comply with any neighbor.

Proof Theorem 2 Back

If $\max_j d_{ij}(h_i) = 1$ and δ is high, i prefers to comply since:

	Equilibrium: D				Deviation: C			
(d_{ij},d_{ji})	t	t+1	t+2	t+3	t	t+1	t+2	t+3
(0,0)	1+g	-l	1	1	1	1+g	0	1
(0,1)	0	1	1	1	-l	1+g	0	1
(1,0)	1+g	1	1	1	1	0	0	1
(1,1)	0	1	1	1	-l	0	0	1

Equilibrium is weakly preferred in state $(0,0)$.

Equilibrium is strictly preferred in the remaining scenarios.

Since $\max_j d_{ij}(h_i) = 1$, a neighbor exists with whom player i strictly loses, when δ is close to 1.

If $\max_j d_{ij}(h_i) = 2$ and δ is high, i prefers to comply since:

	Equilibrium: D				Deviation: C			
(d_{ij},d_{ji})	t	t+1	t+2	t+3	t	t+1	t+2	t+3
(0,0)	1+g	0	1	1	1	1+g	0	1
(0,1)	0	1+g	-l	1	-l	1+g	0	1
(1,0)	1+g	1+g	-l	1	1	0	0	1
(1,1)	0	1+g	-l	1	-l	0	0	1
(2,2)	0	0	1	1	-l	0	0	1

An argument similar to the preceding applies.

The strategy profile ξ_N thus trivially satisfies EP, since the incentives to comply are not affected by the beliefs about the graph. ■

For any i, consider trembles such that a deviation in period t occurs:

(i) with probability ε^{α^t}, if $n_i = 1$, for $\alpha < 1/(n+1)$;

(ii) with probability ε^2, if $n_i > 1$.

Let $\theta^\varepsilon(G, h)$ be the probability of node (G, h).

The conditional belief of node $(G, h) \in U(h_i)$ is:

$$\beta^\varepsilon(G, h|h_i) = \frac{\theta^\varepsilon(G, h)}{\sum_{(G', h') \in U(h_i)} \theta^\varepsilon(G', h')}$$

Let $\beta(G, h|h_i) = \lim_{\varepsilon \to 0} \beta^\varepsilon(G, h|h_i)$.

Let G_i^* denote incomplete star network (with i as hub and N_i periphery) and a disjoint totally connected component.

Let $h^*(h_i)$ be the history in which players not in $N_i \cup \{i\}$ always plays according to ξ_N.

To establish (a), consider any h_i and $j \in \mathcal{D}\left(G_i^*, h^*\left(h_i\right)\right)$.

Since at $\left(G_i^*, h^*\left(h_i\right)\right)$ all deviations are of type (i):

$$\theta^{\varepsilon}\left(G_i^*, h^*\left(h_i\right)\right) \geq f\left(G_i^*\right)(1-\varepsilon)^{nT} \varepsilon, \text{ since } n\sum_{t=1}^{T} \alpha^t < 1.$$

Consider $(G, h) \in U\left(h_i\right)$ such that $n_j > 1$:

1. If $j \in \mathcal{D}(G, h)$, j's deviation is of type (ii) and $\theta^{\varepsilon}(G, h) \leq \varepsilon^2$.

   Thus, for ε close to zero there exists $q > 0$ such that:

$$\beta^{\varepsilon}\left(G, h \mid h_i\right) \leq \frac{\theta^{\varepsilon}(G, h)}{\theta^{\varepsilon}\left(G_i^*, h^*\left(h_i\right)\right)} \leq \frac{\varepsilon}{q} \longrightarrow 0$$

2. if $j \notin \mathcal{D}(G, h)$, let t^* denote the first period in which:

$$\mathcal{D}(G_i^*, h^*(h_i), t) \neq \mathcal{D}(G, h, t).$$

Part 1 and Lemma 5 then yield $\mathcal{D}(G_i^*, h^*(h_i), t^*) \subset \mathcal{D}(G, h, t^*)$.

If $K(t)$ denotes # of players in $\mathcal{D}(G, h, t)$:

$$\begin{aligned} \theta^{\varepsilon}(G, h) &\leq \varepsilon^{\sum_{t=1}^{t^*} K(t)\alpha^t} \\ \theta^{\varepsilon}(G_i^*, h^*(h_i)) &\geq f(G_i^*)(1-\varepsilon)^{nT} \varepsilon^{-\left(1-n\frac{\alpha}{1-\alpha}\right)\alpha^{t^*} + \sum_{t=1}^{t^*} K(t)\alpha^t} \end{aligned}$$

Thus, for ε close to zero there exists $k > 0$ such that:

$$\beta^{\varepsilon}(G, h|h_i) \leq \frac{\theta^{\varepsilon}(G, h)}{\theta^{\varepsilon}(G_i^*, h^*(h_i))} \leq \frac{\varepsilon^{\left(1-n\frac{\alpha}{1-\alpha}\right)\alpha^{t^*}}}{k} \longrightarrow 0$$

This establishes (a) and implies that:

$$\beta(G, h|h_i) > 0 \quad \Rightarrow \quad \mathcal{D}(G, h) \subseteq N_i \cup \{i\}$$

To prove (b), observe that by (a) we can focus on networks such that:

$$n_j = 1 \text{ for any } j \in \mathcal{D}\left(G_i^*, h^*\left(h_i\right)\right) \setminus \{i\}$$

We prove the claim by contradiction.

Let t^* be the earliest period t such that

$$\mathcal{D}\left(G_i^*, h^*\left(h_i\right), t\right) \neq \mathcal{D}\left(G, h, t\right).$$

Observe that the argument in (a) implies that

$$\mathcal{D}\left(G_i^*, h^*\left(h_i\right), t^*\right) \subset \mathcal{D}\left(G, h, t^*\right)$$

and the claim is proved analogously. ∎

Consider a profile of grim trigger strategies such that:

1. player i plays C if $\forall j \in N_i$ played C in every previous period;
2. player i plays D otherwise.

Consider the sets $\mathcal{C}_i \subset N_i$, $\mathcal{D}_i = N_i \backslash \mathcal{C}_i$, and δ such that

$$\begin{aligned} 1 &> (1-\delta)(1+g) & \text{(a)} \\ \sum_{j \in \mathcal{C}_i} \eta_{ij} (1+g) &> (1+\delta(1+g)) \sum_{j \in \mathcal{C}_i} \eta_{ij} - l \sum_{j \in \mathcal{D}_i} \eta_{ij} & \text{(b)} \end{aligned}$$

(a) $\Rightarrow$ if all players are in state 1, no player prefers to deviate from state 1.

(b) $\Rightarrow$ if a player believes that players in $\mathcal{C}_i$ are in state 1 and players in $\mathcal{D}_i$ and himself are in state 2, he prefers not to deviate from state 2.

The two inequalities reduce to:

$$\frac{g}{g+1} < \delta < \frac{g}{g+1} + \frac{l \sum_{j \in \mathcal{D}_i} \eta_{ij}}{(g+1) \sum_{j \in \mathcal{C}_i} \eta_{ij}}$$

The upper-bound is decreasing in $\sum_{j \in \mathcal{C}_i} \eta_{ij}$ and increasing in $\sum_{j \in \mathcal{D}_i} \eta_{ij}$.

Recall that, $\eta_{ij} > 0$ for any ij, $i \neq j$ and let:

$$\eta = \frac{\min_{ij \in G} \eta_{ij}}{(n-1) \max_{ij \in G} \eta_{ij}}$$

and suppose that

$$\delta \in \left(\frac{g}{g+1}, \frac{g}{g+1} + \frac{l\eta}{(g+1)} \right) \tag{3}$$

If so, if a player believes that $\mathcal{D}_i \neq \emptyset$, playing D is strictly optimal; otherwise, playing C is strictly optimal.

Proof Theorem 4 Back

The above strategy is a sequential equilibrium, since consistent beliefs are such that:

i. if every player $j \in N_i$ played C in every previous period, player i believes that all players in the graph are in state 1;
ii. if a player $j \in N_i$ played D in a previous period, player i believes that at least one of his neighbors is in state 2.

The strategy satisfies Π^A-S, since it is optimal for any belief about the underlying information network.

The theorem is proven, if the upper bound of the interval in (3) is greater or equal to one.

Otherwise...

Otherwise, consider an open interval $(a, b) \subset (0, 1)$.

If $\delta \in \left(\frac{a}{b}, 1\right)$, then $\delta^T \in (a, b)$ for some positive integer T.

Let $a = \frac{g}{g+1}$ and $b = \frac{g}{g+1} + \frac{l\eta}{(g+1)}$.

If so, partitioning the game into $T-1$ independent games played every T periods (as in Ellison (1994)) yields a discount rate:

$$\delta^T \in \left(\frac{g}{g+1}, \frac{g}{g+1} + \frac{l\eta}{(g+1)}\right)$$

Thus, the modified strategy satisfies Π^A-I and C when players are sufficiently patient.

Naturally, cooperation is sustained at the expense of Π^A-S.

A player defecting in one of the T games never returns to full cooperation.

Proof Lemma 2 Back

Fix a network G and a history h of length z.

Suppose that players comply with ζ_N after h.

In any relationship ij, the states transition according to:

d_{ij}	0	0	0	0	0	0	+
d_{ji}	0	0	0	0	+	+	+
a_i	D	D	C	C	D	C	D
a_j	D	C	D	C	D	D	D
Δd_{ij}	0	0	1	0	0	0	-1
Δd_{ji}	0	1	0	0	0	-1	-1

(4)

Let $T(h) = \max_{ij\in G}\{\min\{d_{ij}(h), d_{ji}(h)\}\}$.

Let h^t denote the history $t - z$ periods longer than h generated by ζ_N.

Proof Lemma 2

If players comply with the strategy after h for any $t > T(h)$ and ij:

$$\min\left\{d_{ij}^t, d_{ji}^t\right\} = 0$$

Therefore, **if** for any $t > T(h)$

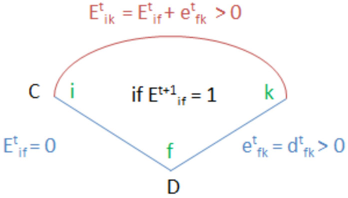
$$\begin{array}{l} \text{(A) } S^t \subseteq S^{t+1} \\ \text{(B) } S^t \neq S^{t+k} \text{ for some } k > 0 \text{ if } S^t \neq N \end{array}$$

then Π^A-S holds, since $S^t = N$ for some $t > T(h)$ implies

$$\max\left\{d_{ij}^t, d_{ji}^t\right\} = 0.$$

(A) holds, since $i \in S^t \backslash S^{t+1}$ implies that:

- $E^t_{if} = 0$ and $E^{t+1}_{if} = 1$ for some f;
- thus $\zeta^t_f = D$ and $d^t_{fk} > 0$ for some k;
- which contradicts $i \in S^t$.



(B) holds, since $i \in S^t$ implies $d^t_{ij} = 0$ for any j and thus

- $d^{t+1}_{ji} = \max\{d^t_{ji} - 1, 0\}$;
- $j \in S^{t+z}$ for z large enough.

Lemma (5)

Consider a node $(G, h) \in U(h_i)$ *where history* h *is of length* T*. If*

(i) $\mathcal{D}(G_i^*, h^*(h_i), t) = \mathcal{D}(G, h, t)$ *for any* $t < T$*, and*

(ii) $N_j = \{i\}$ *for any* $j \in \mathcal{D}(G, h^{T-1}) \backslash \{i\}$*,*

then $\mathcal{D}(G_i^*, h^*(h_i), T) \subseteq \mathcal{D}(G, h, T)$*.*

Proof

For $h \in H$, let h^t denote the sub-history of length $t < T$.

Suppose that (i) and (ii) hold and recall that:

$$\mathcal{D}(G_i^*, h^*(h_i), t) \subseteq N_i \cup \{i\}.$$

Lemma 3 applies and establishes that for $t < T$ and for any $j \in N_i$:

$$d_{jk}(h^t) = 0 \text{ for } k \in N_j \backslash \{i\}$$

Moreover, for any $t < T$ and $j \in N_i$,

$$d_{ji}\left(h^t\right) = d_{ji}\left(h^*\left(h_i\right)^t\right) \text{ and } d_{ij}\left(h^t\right) = d_{ij}\left(h^*\left(h_i\right)^t\right).$$

Thus, $i \in \mathcal{D}\left(G_i^*, h^*\left(h_i\right), T\right) \Rightarrow i \in \mathcal{D}\left(G, h, T\right)$.

Instead, if $j \in \mathcal{D}\left(G_i^*, h^*\left(h_i\right), T\right) \backslash \{i\}$ and if, at period T, j plays:

- C then $d_{ji}\left(h^*\left(h_i\right)^{T-1}\right) > 0$,
  and $j \in \mathcal{D}\left(G, h, T\right)$ since $d_{ji}\left(h^{T-1}\right) > 0$;
- D then $d_{ji}\left(h^*\left(h_i\right)^{T-1}\right) = 0$,
  and $j \in \mathcal{D}\left(G, h, T\right)$ since $d_{jk}\left(h^{T-1}\right) = 0$ for $k \in N_j$. ∎