# The Analytics of Monetary Non-Neutrality in the Sidrauski Model

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**Abstract**

This note analytically characterizes the equilibrium dynamics of the Sidrauski model and reaches three conclusions regarding monetary policy: (i) it is typically not neutral, (ii) in some cases, it is not neutral even in the steady state, and (iii) a policy that has the nominal interest rate falling over time may sustain higher output and consumption forever.

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## 1. Introduction

Sidrauski's (1967) model of money in the utility function in a neoclassical growth model is quite popular. It is often the first monetary model in courses and textbooks, and it is the starting point for most monetary models that feature capital accumulation. The super-neutrality result, that steady-state real output is independent of the growth rate of money supply, is the common benchmark in studies of monetary policy. However, the effects of other monetary policies and the dynamics of the model outside of the steady state are more complicated. Fischer (1979), using a linear approximation around the steady state, found that money was not neutral but had to conclude that: "In brief, a convincing intuitive explanation of the basic result is not yet available." (Fischer, 1979: 1439).

This note shows how to analytically write the equilibrium dynamics of the Sidrauski model in a way that makes transparent the effects of monetary policy. By writing monetary policy as setting a path for the nominal interest rate, I am able to completely characterize the effects of policy, both on and off the steady state, with a clear economic intuition. I use this to show that: (i) monetary policy is typically not neutral, (ii) there are steady states where monetary policy is not neutral, and (iii) in some cases, a declining path for nominal interest rates can sustain higher levels of output and consumption for a long period of time, possibly forever. Recent and future models of monetary policy that have capital accumulation and money in the utility function will likely share these properties.

## 2. The Sidrauski model

The representative private agent solves the problem:

$$\max_{\{c_t, m_t\}} \int_0^\infty e^{-\rho t} u(c_t, m_t) dt, \text{ s.t.} \tag{1}$$

$$\dot{a}_t = f(k_t) - \delta k_t - \pi_t m_t + v_t - c_t, \tag{2}$$

$$\lim_{t\to\infty} e^{-\rho t} a_t \geq 0. \tag{3}$$

The notation is standard: $c_t$ is consumption, $m_t$ real money balances, $k_t$ capital, $a_t = k_t + m_t$ assets, $\pi_t$ the inflation rate, $v_t$ lump-sum government transfers, $\rho$ the rate of time preference, $\delta$ the depreciation rate. I make the standard assumptions on utility: $u_c > 0$, $u_{cc} < 0$, $u_m \geq 0$, $u_{mm} \leq 0$, $u_{cc}u_{mm} \geq u_{cm}^2$, and assume that money and consumption are complements, $u_{cm} \geq 0$, consistent with the idea that money makes consumption easier. The production function is neoclassical: $f_k > 0$, $f_{kk} < 0$, $f(0) = 0$, $\lim_{k\to 0} f_k = +\infty$, $\lim_{k\to\infty} f_k = 0$, and the agent starts with a positive $k_0$.

The other agent is the government, that prints money at rate $\mu_t$ and runs a balanced budget rebating seigniorage revenues immediately to the consumer: $v_t = \mu_t m_t$. I assume it sets a non-negative path for nominal interest rates $R_t \equiv f_k - \delta + \pi_t$ by choosing an appropriate path for money growth.[1]

Optimal behavior by the private agent implies the set of necessary optimality conditions:

$$u_c(.) = \lambda_t, \tag{4}$$

$$u_m(.) = \lambda_t R_t, \tag{5}$$

$$\frac{\dot{\lambda}_t}{\lambda_t} = \rho - (f_k(.) - \delta), \tag{6}$$

$$\lim_{t\to\infty} e^{-\rho t} a_t \lambda_t \leq 0, \tag{7}$$

where $\lambda_t$ is the marginal value in utility units of having one more unit of assets. The equilibrium of the economy at any point in time can be represented as follows:

**Proposition 1.** *Equilibrium $c_t$ and $k_t$ solve the first-order differential equations:*

$$\dot{k}_t = f(k_t) - \delta k_t - c_t, \tag{8}$$

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta_t}\left(f_k(k_t) - \delta - \rho - \xi_t \eta_t \frac{\dot{R}_t}{R_t}\right), \tag{9}$$

*subject to the boundary conditions $k_0$ and $\lim_{t\to\infty} e^{-(f_k-\delta)t}(k_t + m_t) = 0$, and given a path for $R_t$. Real output equals $f(k_t)$, while real money balances $m_t = \varphi(c_t, R_t)$, where $\varphi(.)$ is the money demand function implicitly defined by $u_m(c, \varphi(c, R))/u_c(c, \varphi(c, R)) = R$. The new notation stands for $\theta_t = -u_{cc}c/u_c - u_{cm}\varphi_c c/u_c$, $\eta_t = -\varphi_R R/\varphi$, and $\xi_t = u_{cm}m/u_c$.*

**Proof.** Combining (4) and (5) by replacing for $\lambda_t$ defines $\varphi(.)$. Using it to substitute for $m_t$ in (4), taking time derivatives, and using (6) to replace for $\lambda_t$, gives (9). Using $v_t = \mu_t m_t$ to replace for $v_t$ in (2) leads to (8). Finally, combining the definition of assets, condition (3), and the transversality condition (7), and substituting for $\lambda_t$ using (6), gives the terminal condition. □

The new notation is: $\theta_t$ the positive inverse of the intertemporal elasticity of substitution, $\eta_t$ the non-negative interest rate elasticity of money demand, and $\xi_t$ the non-negative elasticity of the marginal utility of consumption with respect to real money balances.

The key new term relative to the standard growth model is $\xi_t \eta_t \dot{R}_t/R_t$ in (9). Changes in the growth rate of nominal interest rates lead to changes in the demand for money, which in turn change the marginal utility of consumption, and thus the attractiveness of consuming today versus tomorrow. If the nominal interest rate is expected to increase (decrease) in the future, the agent foresees she will desire less (more) money in the future and thus also less (more) consumption. She therefore saves less (more) by choosing a flatter (steeper) path for consumption. This is the channel by which money and monetary policy affect individual

[1] As it is well-known, setting nominal interest rates determines inflation but leaves the price level indeterminate. There are well-known slight modifications of the policy rule that ensure determinacy, and this indeterminacy is inconsequential for the paths of real variables that I focus on.

decisions in the Sidrauski model.

## 3. The effects of monetary policy

A policy is neutral if changes in its level do not affect consumption, capital or output, and super-neutral if the same applies to changes in its growth rate.

**Proposition 2.** *Monetary policy affects consumption, capital, and output as long as $\xi_t \eta_t$ is different from zero. A nominal interest rate policy is neutral but not super-neutral.*

The intuition is the following. As long as money demand is interest rate elastic ($\eta_t$ is not zero) a policy-induced change in the path of nominal interest rates leads to an adjustment on the desired time path of real money balance holdings. As long as money affects the marginal utility of consumption ($\xi_t$ is not zero), the relative value of consuming today vis-a-vis consuming tomorrow will change. The consumer will respond by changing his saving and consumption decisions, which in turn will affect capital accumulation and output. Only changes in the growth rate but not in the level of nominal interest rates matter because only these change the relative prices of money and consumption over time.

This result applies globally, whether in the steady state or not, requires no linearizations, and has a simple intuition. Only if money and consumption are separable in the utility function ($\xi_t = 0$), which is empirically rejected by Koenig's (1990) Euler equation estimates, or if money demand is interest inelastic ($\eta_t = 0$), which is rejected by almost all studies of money demand, is there super-neutrality.

## 4. Monetary policy in the steady state

**Definition.** *A steady-state of the real economy is an equilibrium in which consumption, capital, and output do not change over time.*

Note that this definition does not require that the steady state lasts forever or that money or interest rates have to be constant. The effects of monetary policy on the steady state are described by:

**Proposition 3.** *As long as policy is able to set the nominal interest rate to satisfy:*

$$\dot{R}_t / R_t = -x / \xi_t \eta_t, \tag{10}$$

where $x$ *is a policy-chosen constant, it can keep the economy in any steady state* $(c^*, k^*)$ *it wishes. For the economy to be in this steady state forever,* $x$ *must be lower than the rate of time preference and* $\lim_{t\to\infty} e^{-(\rho-x)t} \varphi(c^*, R_t) = 0$.

**Proof.** In a steady state $\dot{c} = \dot{k} = 0$, so equations (9) and (8) imply that

$$\begin{aligned} c^* &= f(k^*) - \delta k^*, && (11) \\ f_k(k^*) &= \delta + \rho + \xi_t \eta_t \frac{\dot{R}_t}{R_t}. && (12) \end{aligned}$$

As long as (10) holds, all terms in these equations are independent of time, so the economy remains in a steady state. For this steady state to persist forever, $x$ must respect the transversality condition. If $x \geq \rho$, steady-state $k^*$ is higher than the golden rule level, the real interest rate is non-positive, and the transversality condition is violated. If $x < \rho$, then $\lim_{t\to\infty} e^{-(\rho - x)t} k^* = 0$, so the transversality condition reduces to the condition on $\varphi(.)$. $\square$

The monetary authority chooses $x$ and can affect the steady state as long as $\xi_t \eta_t \neq 0$ and $R_t \geq 0$. If $x = 0$, the economy is in the same steady state as if there was no money ($u_m = 0$). If $x$ is negative, nominal interest rates rise, people save less, and steady-state consumption and output are lower. If $x$ is positive, consumption and output are higher, potentially all the way until just below their golden rule levels, and nominal interest rates are falling. In the case where $\xi_t \eta_t$ is a positive constant, policy is very powerful: by picking an arbitrarily small initial $R_0$ and having $R_t$ fall proportionally at the rate $x/\xi\eta$ towards zero without ever reaching it, it can raise consumption and output forever.[2]

To illustrate this result, consider two specific utility functions

$$u(c, m) = \frac{\left[c^{1-\gamma} m^{\gamma}\right]^{1-\theta}}{1-\theta}, \text{ with } \theta < 1,\ \gamma \in (0, 1); \tag{13}$$

$$u(c, m) = \frac{\left[(1-\alpha)^{\frac{1}{\eta}} c^{\frac{\eta-1}{\eta}} + \alpha^{\frac{1}{\eta}} m^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}(1-\theta)}}{1-\theta}, \text{ with } \theta \neq 1. \tag{14}$$

The first was used by Fischer (1979) and implies $\eta = 1$ and $\xi = \gamma(1-\theta)$. Figure 1 plots the path of the nominal interest rate needed to raise steady state consumption by 1% forever relative to the economy without money, starting from $R_0 = 4\%$. Nominal interest rates fall proportionately over time at the rate $x/\gamma(1-\theta)$. The second utility function, used by Lucas (2000), implies a constant $\eta$, not necessarily 1. To keep the economy in the steady state, the nominal interest rate must follow a Bernoulli differential equation:

$$\dot{R}_t = -x - x\frac{1-\alpha}{\alpha} R_t^{\eta}. \tag{15}$$

If $\eta \geq 1$, the dynamics are similar to those in figure 1. If $\eta < 1$, starting from some positive $R_0$, the nominal interest rate falls and hits zero at date $\hat{t} = \ln\left(1 + \frac{\alpha}{1-\alpha} R_0^{1-\eta}\right)^{\frac{1}{(1-\eta)x}}$, as in figure 2.[3] Following the policy rule in (15) only keeps the economy in a steady state for $\hat{t}$ periods, after which it converges to the steady state of the economy without money. Still, by choosing a large enough $R_0$, $\hat{t}$ can be arbitrarily large.[4]

[2] For $\xi_t \eta_t$ to be constant in the steady state, $(u_{mm} u_c)/(u_{cm} u_m)$ must not depend on $m_t$.

[3] To prove this result, change variables to $z_t = \frac{1-\alpha}{\alpha} + R_t^{1-\eta}$, and solve the resulting linear ordinary differential equation with constant coefficients. Evaluating the solution when $R_{\hat{t}} = 0$ gives $\hat{t}$.

[4] Figures 1 and 2 are for $\rho = 0.04$, $\delta = 0.08$, $\theta = 0.5$, $\gamma = 0.5$, $f(k) = k^{1/3}$, $\alpha = 0.5$, and $\eta = 0.5$. In figure 1, $x \leq \rho\gamma(1-\theta)/(1 + \gamma(1-\theta))$ so the steady state can last forever. In figure 2, $\hat{t} = 64.18$.

Figure 1: Path of nominal interest rates with utility function (13).

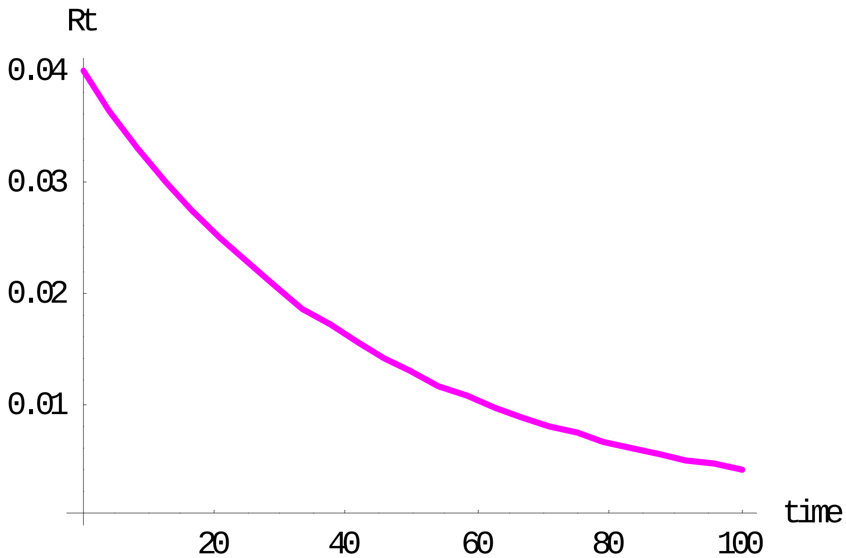
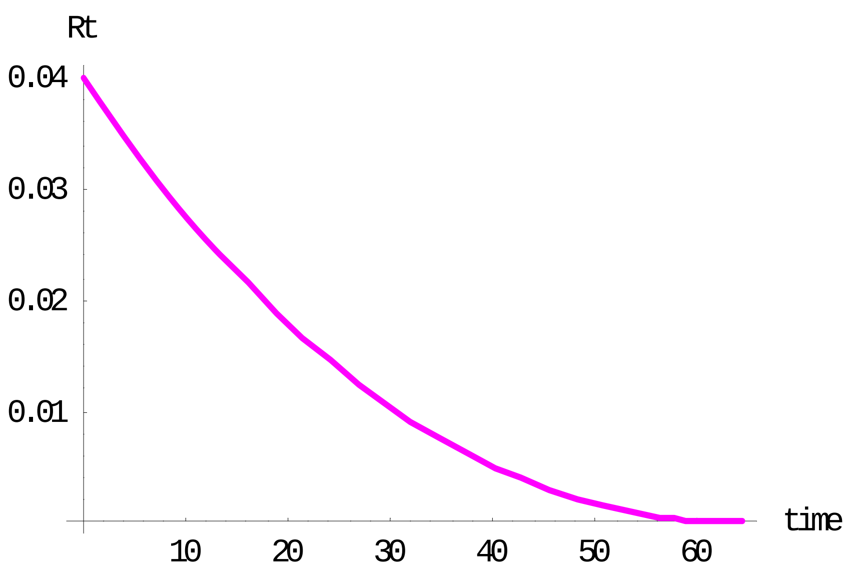



Figure 2: Path of nominal interest rates with utility function (14).



## 5. The link to Sidrauski's results

The results so far may seem novel in comparison with those in Sidrauski (1967). To understand the link, consider two new constraints:

**Condition A** *The money supply expands at a constant rate over time.*

**Condition B** *In a steady state, real money balances must also be constant over time.*

The first condition constrains the admissible policies, whereas the second condition imposes a stricter requirement on the definition of a steady state. Either of them leads to:

**Proposition 4.** *If either condition A or condition B hold, then $\dot{R}_t = 0$ in the steady state and monetary policy is super-neutral.*

**Proof.** At the steady state $m_t = \varphi(c^*, R_t)$. If condition A holds, taking time derivatives and using the definition of $m_t$: $-\eta\dot{R}_t/R_t = \mu - \pi_t$. Multiplying both sides by $\xi_t$ and using (10), this becomes $\xi_t(\mu - \pi_t) = x$. Finally, substituting out for $\pi_t$ as the difference between nominal and real interest rates and using the steady state real interest rate: $\xi_t(\mu - R_t + \rho - x) = x$. This condition must hold in the steady state, but the left-hand side depends on $R_t$ both directly and through $\xi_t$ (via $m_t$), while the right hand-side does not. Aside from the very special case in which preferences and $x$ are such that $\xi_t(\mu - R_t + \rho - x)$ is independent of $R_t$, then $R_t$ must be constant. If condition B holds, since $m^* = \varphi(c^*, R_t)$, then $R_t$ must be constant. □

Sidrauski (1967) assumed both conditions; thus, his super-neutrality result. If condition A holds, an increase in $\mu$ at steady state raises nominal interest rates proportionately once and for all. Because there are no future expected changes in nominal interest rates, the channel for monetary policy in proposition 1 is shut off. Yet this restriction is neither true in the data, nor is it a priori a necessary requirement for policy.

The intuition for the effect of condition B is that, if real money balances have to be constant, then so is the marginal utility of consumption. Thus, policy cannot affect incentives to save so it is super-neutral. The question is whether restriction B is appealing. The definition of the steady state used in this note implies that, with technical progress, consumption, capital, and output grow over time at a constant rate, roughly in line with the U.S. experience. The data for real money balances is not clear about its trend: it is either negative (for currency or M0), close to zero (for M2), or positive (for broad measures of money that include the deepening of financial markets). Moreover, the trend of real money balances clearly depends on the monetary policy chosen, which has not been that in proposition 3. There isn't a strong reason to impose condition B.[5]

[5] I have not found any counterpart to propositions 1-4 in the literature. One might wonder why this is so, given the attention that this model has received. The reason seems to be that most research has focused on money growth policies and imposed condition A. The closest articles are Cohen (1985), who studies the dynamics of the nominal interest rate in the Sidrauski model but still imposes condition A, and Begg (1980), who notes the importance of $\eta_t$ and $\xi_t$ for the neutrality of money in a steady state, but in the context of an IS-LM-AS model.

## 6. Policy implications

This note showed that, in the Sidrauski model, monetary policy can have a powerful effect on the real economy. But should it try to? In the model in this note, the optimal policy is Friedman's $R_t = 0$, so that holding money is costless and the equilibrium replicates that of the economy without money. In economies with other distortions, this need not be the case. For instance, if there are externalities to capital accumulation as in Romer (1986), the equilibrium is inefficient, with too little capital and too high real interest rates because agents do not internalize the full marginal return of saving an extra unit. As long as the conditions in proposition 3 hold, monetary policy can push the economy to the Pareto optimal levels of consumption, capital, and output. Moreover, if preferences imply a constant $\xi\eta$, then $R_0$ can be very close to zero in order to be arbitrarily close to the optimal level of real money balances. In this case, a monetary policy that has slightly positive nominal interest rates that asymptotically approach zero can have a large impact on welfare.

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