Appendix to accompany: "Targeted transfers and the fiscal response to the great recession"

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A.1. Proof of proposition 1

Index the continuum of agents by i. Then, the family of all households wishes to maximize:

$$\mathbb{E}\int\sum_{t=0}^{\infty}\beta^t\left[\ln c_{it}-\chi(1-h_{it})n_{it}\right]di,$$

where each household receives the same weight since they were all *ex ante* identical at the start of time. The family can choose any value for $c_{it}\geq 0$ and $n_{it}\in\{0,1\}$ it wishes for each agent at each period in time, since it can transfer resources across members freely through the insurance payments. Integrating over all household's budget constraints in equation (3) in the main text gives the constraints of this maximization:

$$\begin{aligned} C_t+K_{t+1} &= (1-\delta+r_t)K_t+w_tL_t+d_t-G_t, \\ \int c_{it}di &= C_t \text{ and } \int s_{it}n_{it}di=L_t, \end{aligned}$$

for each period t.

Building the Lagrangian for this problem, with Lagrange multipliers ζ_{1t}, ζ_{2t}, ζ_{3t} for the three constraints, respectively, gives:

$$\begin{aligned} \mathcal{L} = \; & \mathbb{E}\sum_{t=0}^{\infty}\beta^t\left\{\int\left[\ln c_{it}-\chi(1-h_{it})n_{it}\right]di\right. \\ & +\zeta_{1t}\left[(1-\delta+r_t)K_t+w_tL_t+d_t-G_t-C_t-K_{t+1}\right] \\ & \left.+\,\zeta_{2t}\left(C_t-\int c_{it}di\right)+\zeta_{3t}\left(\int s_{it}n_{it}di-L_t\right)\right\}. \end{aligned}$$

The variables with respect to which to maximize are: $\{C_t, L_t, K_{t+1}, c_{it}, n_{it}\}$.

The first-order conditions with respect to individual and aggregate consumption are:

$$\frac{1}{c_{it}} = \zeta_{2t} \quad \text{and} \quad \zeta_{1t} = \zeta_{2t}.$$

Multiplying both sides by c_{it}, and integrating gives the solution for the multipliers: $\zeta_{1t} = \zeta_{2t} = 1/C_t$, as well as the sharing rule for individual consumption: $c_{it} = C_t$. All consume the same, since all were *ex ante* identical and they are all fully insured.

The optimality condition with respect to capital is:

$$\zeta_{1t} = \beta\zeta_{1t+1}(1 - \delta + r_{t+1}).$$

Replacing the Lagrange multiplier gives the Euler equation:

$$\frac{C_{t+1}}{C_t} = \beta(1 - \delta + r_{t+1}).$$

Finally, turn to the labor supply decision. It is clear from the structure of the problem that if $h_{it} = 1$, then $n_{it} = 1$ as there is no utility loss and only a positive wage gain from working. If $h_{it} < 1$, it should also be clear that $n_{it} = 1$ if and only if $h_{it} > h^*(s_{it})$, a threshold that depends on the salary offer of the agent. But then:

$$\begin{aligned}
\int \chi(1 - h_{it})n_{it}di &= \chi(1 - \pi)\int_s \int_0^\eta (1 - h_{it})n_{it}dF(h_t)dF(s_t) \\
&= \chi(1 - \pi)\int_s \left[\int_{h^*}^\eta (1 - h_{it})dh\right] dF(s_t) \\
&= \frac{\chi(1 - \pi)}{2}\left[\int_s (1 - h^*)^2 dF(s_t) - (1 - \eta)^2\right].
\end{aligned}$$

Using this result in the Lagrangian, the first-order conditions with respect to h^* and L_t are, respectively:

$$\chi(1 - h^*(s_{it})) = \zeta_{3t}s_{it} \quad \text{and} \quad \zeta_{1t}w_t = \zeta_{3t}.$$

Using the first-order condition for consumption to eliminate the Lagrange multipliers gives the optimal labor supply defining the $h^*(.)$ function:

$$1 - h^*(s_{it}) = \frac{w_t s_{it}}{\chi C_t}.$$

Recalling the definition of effective labor supply:

$$\begin{aligned}
L_t &= \int s_{it} n^*(k,s,h) di \\
&= \pi + (1-\pi) \int s_{it} \left(\eta - h^*(s_{it})\right) dF(s) \\
&= \pi - (1-\pi)(1-\eta) + (1-\pi) \int \frac{w_t s_{it}^2}{\chi C_t} dF(s) \\
&= \pi - (1-\pi)(1-\eta) + \frac{(1-\pi) w_t \mathbb{E}(s_{it}^2)}{\chi C_t}.
\end{aligned}$$

Collecting all the results, we are left with the Euler equation and the aggregate labor supply equation. These are identical to the two optimality conditions from the representative consumer problem in Proposition 1, proving the result.

A.2. Proof of proposition 2

Combining the optimality conditions in section 3.2, without nominal rigidities:

$$r_t = \alpha A_t \left(\frac{K_t}{L_t}\right)^{\alpha-1} \quad \text{and} \quad \mu w_t = (1-\alpha) A_t \left(\frac{K_t}{L_t}\right)^{\alpha}.$$

Defining $\mu = 1 + \tau$ gives immediately the result.

A.3. Proof of proposition 3

Combining Propositions 1 and 2, all that remains is to check the market clearing condition: $M_t = d_t - G_t$. But with flexible prices $d_t = (\mu - 1) w_t L_t$. Using the definition of taxes in Proposition 2, $M_t = \tau w_t L_t - G_t$. Finally, to solve for employment:

$$\begin{aligned}
E_t &= \int n_{it} di = \pi + (1-\pi) \int \left(\eta - h^*(s_{it})\right) dF(s) \\
&= \pi + (1-\pi)\eta - (1-\pi) \int \left(1 - \frac{w_t s_{it}}{\chi C_t}\right) dF(s) \\
&= \pi - (1-\pi)(1-\eta) + \frac{(1-\pi) w_t}{\chi C_t}.
\end{aligned}$$

Combining with the expression for L_t in the proof of Proposition 1 gives the expression for E_t.

A.4. Numerical solution of the full model

We solve the household problem in the Bellman equations (1)-(5) in the main text by numerically by value function iteration. For the first few iterations, we discretize the state space, but once we are close to the solution, we switch to interpolating the value function linearly, and using a golden section search algorithm for the maximization. It is possible to reduce the dimension of the state space from 3 to 2, by re-defining variables, but after extensive experimentation we found that surprisingly this did not materially speed up the calculations.

As for the production sector, the optimality conditions were described in section 3.2. In the steady state, where all firms are perfectly informed of the current state of affairs that has been lasting for an indefinitely long time, given values for X_0 and r_0, we can sequentially find the other variables by solving in order the system of equations:

$$\begin{aligned} K_0 &= \left(\frac{\alpha A_0}{r_0}\right)^{1/(1-\alpha)} X_0 \text{ and } w_0 = \frac{(1-\alpha)A_0}{\mu}\left(\frac{K_0}{X_0}\right)^{\alpha}, \\ L_0 &= X_0 \text{ and } \ d_0 = (\mu-1)w_0 L_0. \end{aligned}$$

Following a shock in period 1, only a fraction Λ_t of the firms know about it in period t. Since prices are being set according to equations (10)-(11) in the main text, the price index for intermediate goods in equation (8) equals:

$$p = \mu\left[\Lambda_t w_t^{\frac{1}{1-\mu}} + (1-\Lambda_t)\, w_0^{\frac{1}{1-\mu}}\right]^{1-\mu} = (1-\alpha)A\left(\frac{K}{X}\right)^{\alpha},$$

where the second equality comes from equation (7).

In turn, letting X_t^A be the output of attentive firms, that have learned about the change, and X_t^I be the output of inattentive firms:

$$X_t^{1/\mu} = \Lambda_t X_t^{A1/\mu} + (1-\Lambda_t)\, X_t^{I1/\mu}.$$

Of the following two expressions, the first comes from combining the production function in equation (9), with the labor market clearing condition in equation (15), and the second from dividing the demand functions in (8):

$$\begin{aligned} L_t &= \Lambda_t X_t^A + (1-\Lambda_t)\, X_t^I, \\ X_t^A / X_t^I &= (w_t/w_0)^{-\mu/(\mu-1)}. \end{aligned}$$

The two expressions can be used above to replace for X_t^A and X_t^I to obtain:

$$L_t = X_t\left[\frac{\Lambda_t\left(\frac{w_t}{w_0}\right)^{\frac{\mu}{1-\mu}} + 1 - \Lambda_t}{\left[\Lambda_t\left(\frac{w_t}{w_0}\right)^{\frac{1}{1-\mu}} + 1 - \Lambda_t\right]^{1-\mu}}\right].$$

As for dividends, note that:

$$\begin{aligned} d_t &= \Lambda_t d_t^A + (1-\Lambda_t) d_t^I \\ &= \Lambda_t(\mu-1) w_t X_t^A + (1-\Lambda_t)\left(\frac{\mu w_0}{w_t} - 1\right) w_t X_t^I, \end{aligned}$$

where the second equality comes from equation (17). Again, we can replace for X_t^A and X_t^I just as in the previous paragraph.

Combining all of the previous results then, given values for X_t and r_t the variables in the production sector K_t, w_t, l_t, d_t solve, again sequentially, the system of equations:

$$\begin{aligned} K_t &= X_t \left(\frac{\alpha A_t}{r_t}\right)^{\frac{1}{1-\alpha}}, \\ w_t &= w_0 \left(\frac{\left[\frac{(1-\alpha)A_t}{w_0 \mu}\left(\frac{K_t}{X_t}\right)^{\alpha}\right]^{\frac{1}{1-\mu}} + \Lambda_t - 1}{\Lambda_t} \right)^{1-\mu}, \\ l_t &= X_t \left[\frac{\Lambda_t \left(\frac{w_t}{w_0}\right)^{\frac{\mu}{1-\mu}} + 1 - \Lambda_t}{\left[\Lambda_t \left(\frac{w_t}{w_0}\right)^{\frac{1}{1-\mu}} + 1 - \Lambda_t\right]^{1-\mu}} \right], \\ d_t &= (\mu-1) w_t l_t \left[\frac{\Lambda_t \left(\frac{w_t}{w_0}\right)^{\frac{\mu}{1-\mu}} + (1-\Lambda_t)\left(\frac{\frac{\mu w_0}{w_t} - 1}{\mu - 1}\right)}{\Lambda_t \left(\frac{w_t}{w_0}\right)^{\frac{\mu}{1-\mu}} + 1 - \Lambda_t} \right]. \end{aligned}$$

Combining all of the results gives the following algorithm, drawn from the original work of Aiyagari (1994) to find the steady state:

1. Guess values for X and r.

2. Compute sequentially K, w, l, d using the steady-state optimality conditions for the production sector.

3. Solve the decision problem of the household to obtain $k'^*(k,s,h)$ and $n^*(k,s,h)$.

4. Use this decision function and the exogenous transition function for s to build $F(k,s,h)$.

5. Obtain new guesses for X and r sequentially from:

$$\begin{aligned} X &= \left(\int s^{1/\mu} n^*(k,s,h) dF(k,s,h) \right)^{\mu}, \\ r &= \alpha \left(\frac{\int k'(k,s,h) dF(k,s,h)}{X} \right)^{\alpha-1}, \end{aligned}$$

and iterate until convergence.

For the transition dynamics to shocks, we follow the approach of Conesa and Krueger (1999) starting from the programs of Heer and Maussner (2005). We adapt this previous work to deal with transitory shocks (they had permanent shocks) as follows. First, we pick a finite T and assume that by that time the transitory shock to the exogenous variables has disappeared and all of the endogenous variables have converged back to their steady state. In the implementation, $T = 120$, and increasing it led to no noticeable differences in the paths. Then, start by guessing the path: $\{r_t, X_t\}_{t=1}^{T}$. The optimality conditions in the production sector in section 3.2 deliver the implied paths for $\{K_t, w_t, l_t, d_t\}_{t=1}^{T}$. Knowing that the value function at period $T+1$ is the one at the steady-state, applying steps 2-4 of the algorithm for the steady state above gives the decision rules and value functions at date T. Repeating this gives the decision rules at date $T-1$, and so on until date 1. Finally, we use the decision rules to calculate $\{X_t, r_t\}_{t=1}^{T}$ as in step 5 of the steady-state algorithm. Iterating this procedure until convergence gives the transitional dynamics.

References

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