Comment on "Quantile Autoregression" by R. Koenker and Z. Xiao.

P.M. Robinson[*]
London School of Economics

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My remarks about this paper are organised within four points. The first of these notes the close connection between the authors' random coefficient model and some earlier models that were not formulated in terms of "random coefficients". The second comments on the model's identification. The third point queries the transition from the authors' basic model (1) to their quantile version (2). Finally we request a clarification of the sufficient conditions for the theorems justifying quantile-based inference.

1. Write (1)/(5) as

   $$y_t = \mu_0 + \sum_{j=1}^{p} a_j y_{t-j} + u_t + \sum_{j=1}^{p} \eta_{jt} y_{t-j},$$

   where $a_j = E\theta_j(u_t)$, $\eta_{jt} = \theta_j(U_t) - a_j$. The conditional mean of y_t given $\mathcal{F}_{t-1}$ is

   $$E(y_t \,|\mathcal{F}_{t-1}) = \mu_0 + \sum_{j=1}^{p} a_j y_{t-j}. \tag{A}$$

   The conditional mean for both standard, constant-coefficient, $AR(p)$ models, and the "usual" random coefficient $AR(p)$ models (in which coefficients are independent of errors) is also of form (A). The "usual" random coefficient models go back at least as far as Anděl (1976), but my impression is that they have not been greatly used in practice; if I am correct here, a partial explanation may be the inability to distinguish them from constant-coefficient models at the conditional mean level (typically, using second moment information).

   On the other hand, since the "usual" random coefficient models were first developed, conditional heteroscedasticity and asymmetry (despite possibly symmetric innovations) have emerged as an important feature of many time series, notably financial ones. Dependence of coefficients on errors

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[*] Tooke Professor of Economic Science and Statistics, London School of Economics, Houghton Street, London WC2A 2AE, UK.

allows the authors' model to describe both phenomena, thereby distinguishing it from both the constant-coefficient and "usual" random coefficient models. However, the authors' model overlaps with others already in the literature. Assuming $E\theta_j^2(U_t) < \infty,\ \ j = 0, 1, ..., p$, the conditional variance of y_t given $\mathcal{F}_{t-1}$ is

$$V(y_t\,|\mathcal{F}_{t-1}) = \gamma_{00} + 2\sum_{j=1}^{p} \gamma_{0j} y_{t-j} + \sum_{j=1}^{p}\sum_{k=1}^{p} \gamma_{jk} y_{t-j} y_{t-k}, \tag{B}$$

where $\gamma_{jk} = cov\,(\theta_j(U_t), \theta_k(U_t))$. Suppose first that

$$\theta_j(\tau) = b_j \theta_0(\tau), 1 \le j \le p \tag{C}$$

for some constants b_j, and let $V(\theta_0(U_t)) = 1$ with no loss of generality. Then the $(p+1) \times (p+1)$ matrix with (j,k)-th element $\gamma_{j-1,k-1}$ has unit rank, and (B) reduces to

$$V(y_t\,|\mathcal{F}_{t-1}) = \left(b_0 + \sum_{j=1}^{p} b_j y_{t-j}\right)^2. \tag{D}$$

This is a special case of a model proposed by Robinson (1991, formula (16)). With respect to the authors' discussion of asymmetry, Robinson (1991) pointed out that in (D), with martingale difference levels y_t, $cov\left(y_{t+j}^2, y_t\right)$ can be non-zero for $j \ge 1$ even when $\theta_0(U_t)$ is symmetrically distributed. His model was formulated principally in terms of the conditional variance property, but subsequently Giraitis, Leipus, Robinson and Surgailis (2004) considered a corresponding dynamic model for levels that is analogous to a special case of the current authors', calling

$$y_t = \left(b_0 + \sum_{j=1}^{p} b_j y_{t-j}\right) \eta_{0t} \tag{E}$$

a linear ARCH or "LARCH" model; they studied in detail its leverage effect, as well as conditions for stationarity. There have been other developments of this kind of model. Giraitis, Robinson and Surgailis (2000) established a long memory property of powers generated by an infinite autoregressive extension of (E) which started from the long memory version that Robinson (1991) described for his conditional variance form. Giraitis and Surgailis (2002) then replaced the left side of (E) by an infinite autoregressive structure, to model also (long memory) autocorrelation in levels; despite the focus on long memory in the latter reference, its model (1.1) formally covers short memory. I emphasize that all these models are based on the structure (C), which is special relative to the authors' $\theta_j(\tau)$. However, Sentana (1995) extended (D) to a "quadratic ARCH" model with conditional variance function of identical generality to (B), which is a consequence of the authors' model (1) without the restriction (C), and also discussed asymmetry.

2. The formulae (A), (B) and (D) also prompt a point about model choice. In (1) and (2), p represents a maximal order, due to the possibility of some a_j vanishing, or degeneracy of some $\theta_j(U_t)$ (if this is known, it should be imposed for the sake of parsimony). In their first empirical example, the authors apparently apply a rule for selecting p that pertains to (A) and is based on second moments; if the order in (B) is greater than in (A), this rule will tend to underfit.

3. Now I want to move on from (1) to ask about precisely how it leads to the conditional quantile function (2), which is then employed in constructing rules of large sample inference. The authors assume that the right hand side of (1) is monotone increasing in U_t. However, it also depends on $y_{t-1}, ..., y_{t-p}$. Certainly, non-negativity of all $y_{t-1}, ..., y_{t-p}$ is sufficient for monotone increase of all $\theta_j(\tau)$ to imply the same of the right hand side of (1). But monotone increase of $\theta_j(\tau)$ implies $\theta_j(\tau)y_{t-j}$ is monotone decreasing when $y_{t-j} < 0$, and in general for some configurations of $y_{t-1}, ..., y_{t-p}$ monotone increase of all the $\theta_j(\tau)$ does not imply the same for the right side of (1). Much raw data are non-negative, but negative values can occur when transformations (such as logs) are used. Gaussian errors $\theta_0(U_t)$, as in the authors' special case (4), would imply that arbitrarily large negative y_{t-j} can occur with positive probability.

4. The authors discuss in Section 4 monotonicity issues in relation to their quantile autoregression (2), but I could not see there or elsewhere in the paper an answer to the following query, which relates to the preceding point. In their statements of Theorem 2.1 and Corollaries 2.1 and 3.2, which concern the behaviour of partial sums of y_t, the authors complete their Assumptions A.1-A.3 by indicating the underlying model that is assumed, respectively (1), (6) and (9). By contrast, there are no such model assumptions in the statements of Theorem 3.1 or of the theorems in Section 5. These results give useful asymptotic properties of quantile estimates and consequent test statistics, and are surely the main theoretical results in the paper of interest to the practitioner. I would thus be grateful if the authors would complete their statements, by confirming sufficient conditions on the $\theta_j(\tau)$. Non-negativity of the y_t can be seen in simple cases to follow from certain restrictions on the support of the $\theta_j(\tau)$, but it would be nice to elucidate conditions for the general model (1). But since the authors don't mention non-negativity of y_t then perhaps they can characterize a more general setting .

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