

# Modelling Memory of Economic and Financial Time Series

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## Abstract

Much time series data are recorded on economic and financial variables. Statistical modelling of such data is now very well developed, and has applications in forecasting. We review a variety of statistical models from the viewpoint of 'memory', or strength of dependence across time, which is a helpful discriminator between different phenomena of interest. Both linear and nonlinear models are discussed.

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## 1. INTRODUCTION

Economic and financial time series data are often recorded at (almost) equally-spaced intervals of time, e.g. yearly, quarterly, monthly. Such data can often be viewed as representing observations on a continuous-time process. This might be modelled as a stochastic differential equation, say. But there are generally huge identification problems in trying to fit a continuous time model to discrete data (see e.g. Robinson, 1977). In this paper we consider modelling the discrete observations directly, reviewing a variety of models from the perspective of 'memory'.

Consider observations

$$y_t, \quad t = 1, \dots, n,$$

where  $y_t$  represents a financial or economic variable (e.g. GNP, asset price) at time  $t$ , and the unit interval can represent any constant time interval, e.g. 1 year, 1 second. A general model for  $y_t$  is

$$y_t = d_t + s_t \tag{1}$$

where:  $d_t$  is a deterministic component, e.g. a polynomial or cyclic function;  $s_t$  is a stochastic component, described by random variables. Note that (1) is an additive model. However, it could be obtained by taking logs in an initial multiplicative model.

Typically, both  $s_t$  and  $d_t$  are specified parametrically or nonparametrically by the econometrician. We will not discuss estimation. We focus on the modelling of  $s_t$ , and the (somewhat nebulous) issue of memory. We will not discuss  $d_t$  further, though there has been controversy as to whether trends are better described stochastically or deterministically.

Let  $x_t$  be a generic sequence of random variables, which could represent  $s_t$ . The notation

$$x_t \sim IID$$

means that the  $x_t$  are independent and identically distributed. Further, for  $\theta > 0$ ,

$x_t \sim IID(\theta)$  means that

$$x_t \sim IID \text{ and } E|x_t|^\theta < \infty. \quad (2)$$

For  $\theta \geq 1$  we will assume also that

$$E(x_t) = 0,$$

with no loss of generality when  $s_t = x_t$  because a non-zero mean could be introduced in  $d_t$ . In case (2) holds only for  $\theta < 1$ , an alternative location of the distribution of  $x_t$  would entail a zero median.

## 2. MODELS WITH SECOND MOMENT MEMORY

Often we assume

$$Ex_t^2 < \infty.$$

Here, a weaker concept than  $x_t \sim IID(2)$  is

$$x_t \sim UH,$$

i.e. the  $x_t$  are uncorrelated and homoscedastic. This means that

$$\begin{aligned} \text{var}(x_t) \text{ is constant over } t, \\ \text{cov}(x_t, x_{t+u}) = 0, \quad \text{all } u \neq 0. \end{aligned} \quad (3)$$

If  $x_t \sim IID(\theta)$ , some  $\theta > 0$ , we can say unambiguously that  $x_t$  has zero memory. If  $x_t \sim UH$  there is no memory with respect to 2nd moments (cf (3)). However, there could be memory with respect to higher moments, say.

The distinction between " $IID(2)$ " and " $UH$ " has become very important in econometrics and finance nowadays. We shall return to this, but we first discuss processes which have memory in 2nd moments.

A process  $x_t$  is covariance stationary if

$$\gamma_u = \text{cov}(x_t, x_{t+u})$$

depends on  $u$  only and is finite for all  $t$ .

If  $x_t \sim UH$ , then  $\gamma_u = 0$ , all  $u \neq 0$ . On the other hand, if  $\gamma_u \neq 0$  for some  $u \neq 0$ ,  $x_t$  has some (2nd moment) memory.

Now define the lag operator  $L$ , such that  $Lx_t = x_{t-1}$ . Our first model example is as follows (see e.g. Box and Jenkins, 1971):

**Example 1** *Moving average (MA) process (of order 1)*

$$x_t = (1 + \alpha L)\varepsilon_t, \quad \alpha \neq 0,$$

where  $\varepsilon_t \sim UH$ . (Often  $|\alpha| < 1$  is prescribed for invertibility or identifiability reasons.)

For this process

$$\begin{aligned} \gamma_u &\neq 0, \quad u = 1 \\ &= 0, \quad u > 1. \end{aligned}$$

Our next example (see e.g. Box and Jenkins, 1971) is:

**Example 2** *Autoregressive (AR) process (of order 1)*

$$(1 - \alpha L)x_t = \varepsilon_t, \quad 0 < |\alpha| < 1,$$

where  $\varepsilon_t \sim UH$ .

For this model  $\gamma_u \neq 0$ , for all  $u$ , but  $\gamma_u$  decays exponentially to 0 as  $u \rightarrow \infty$ .

Both Examples 1 and 2 illustrate short memory models. They can be significantly generalized, to allow for additional lags, and combined (to form mixed autoregressive

moving average (ARMA) models), but still retain the property of eventual cutout, or exponential decay, of  $\gamma_u$ .

However, we adopt a much less stringent definition of short memory, that covers many other processes. We say  $x_t$  has short memory (in 2nd moments) if

$$\sum_{u=-\infty}^{\infty} |\gamma_u| < \infty. \quad (4)$$

It is convenient to consider this restriction alongside properties of the spectral density, which is given by

$$f(\lambda) = \frac{1}{2\pi} \sum_{u=-\infty}^{\infty} \gamma_u \cos(\lambda u), \quad -\pi < \lambda \leq \pi.$$

Clearly,  $f(\lambda)$  is well-defined under (4). If  $x_t \sim UH$

$$f(\lambda) = \frac{\gamma_0}{2\pi} = \text{constant}.$$

But otherwise  $f(\lambda)$  varies. By way of interpretation, if  $f(\lambda)$  is large for small  $\lambda$  there are substantial long term movements in the series.

The summability condition (4) on the  $\gamma_u$  is related to smoothness conditions on  $f(\lambda)$ . In particular if  $f(\lambda) \sim Lip(\eta)$ , for some  $\eta > \frac{1}{2}$  then (4) holds (see Zygmund, 1979); the Lipschitz condition is stronger than continuity of  $f(\lambda)$  but weaker than differentiability. Since  $f(\lambda)$  is periodic of period  $2\pi$ , it is implied that  $f(\lambda)$  is bounded.

However, there has been considerable interest in processes  $x_t$  which do not satisfy (4), and have unbounded  $f(\lambda)$ . Empirically, smoothed nonparametric estimates of  $f(\lambda)$  are sometimes very peaked near  $\lambda = 0$ , say, possibly suggesting that

$$f(0) = \infty,$$

i.e.

$$\sum_{u=-\infty}^{\infty} \gamma_u = \infty. \quad (5)$$

If (5) holds we say  $x_t$  has long memory (in 2nd moments). Note that  $f(\lambda)$  could instead diverge at one or more non-zero  $\lambda$ , when there is long memory of a cyclic or seasonal type. However, we will not discuss such phenomena.

An  $x_t$  that satisfies (5) is as follows

**Example 3**  $I(d)$  model (Adenstedt, 1974)

$$(1 - L)^d x_t = \varepsilon_t, \quad \varepsilon_t \sim UH, \quad |d| < \frac{1}{2}.$$

For such  $x_t$ ,

$$\begin{aligned} f(\lambda) &= \frac{\text{var}(\varepsilon_t)}{2\pi} |1 - e^{i\lambda}|^{-2d}, \quad -\pi < \lambda \leq \pi \\ &\sim C\lambda^{-2d}, \quad \text{as } \lambda \rightarrow 0+. \end{aligned}$$

For  $d = 0$ ,  $x_t = \varepsilon_t$ , i.e. has short memory,  $0 < f(0) < \infty$ . For  $0 < d < \frac{1}{2}$ ,  $x_t$  has long memory,  $f(0) = \infty$ . For  $-\frac{1}{2} < d < 0$ ,  $x_t$  has negative memory,  $f(0) = 0$ .

The restriction  $d < \frac{1}{2}$  indicates covariance stationarity, the restriction  $d > -\frac{1}{2}$  indicates invertibility. The  $I(d)$  model can be extended to allow  $\varepsilon_t$  to be a stationary and invertible AR, MA or ARMA, without affecting this memory classification. Such "fractional" models form a convenient bridge from (short memory) stationary to nonstationary models. There is also interest in fractional nonstationary models (where  $d \geq \frac{1}{2}$ ), as well as fractional noninvertible ones (where  $d \leq -\frac{1}{2}$ ). We will discuss only the former.

For nonstationary models  $\gamma_u$  and  $f(\lambda)$  are not strictly defined. However, we can introduce a truncation, modifying Example 3 as follows.

**Example 4**  $I(d)$  model,  $d \geq \frac{1}{2}$ ,

$$\begin{aligned} (1 - L)^d x_t &= \varepsilon_t, \quad t \geq 1, \\ x_t &= 0, \quad t \leq 0, \end{aligned}$$

where  $\varepsilon_t \sim UH$ .

For this model  $x_t$  has variance that is finite for all  $t$ , but changes as  $t \rightarrow \infty$ . We can say that  $d$  measures the memory of  $x_t$ ;  $d$  is sometimes called the memory parameter. As a special case for  $d = 1$  we have the familiar unit root process

$$(1 - L)x_t = \varepsilon_t, \quad t \geq 1.$$

This can also be obtained from the AR

$$(1 - \alpha L)x_t = \varepsilon_t,$$

putting  $\alpha = 1$  (to violate the stationarity restriction in Example 2). But the "fractional" class is "smoother" with respect to departures from the unit root, in the sense that asymptotic distributions of, for example, statistics for testing for a unit root directed against fractional alternatives are of standard ( $\chi^2$ ) form, whereas ones directed against autoregressive alternatives are of non-standard form (see Dickey and Fuller, 1979, Robinson, 1994).

We focus on univariate processes  $x_t$ , but the vector case is also important. For example, we can cover (fractional) cointegration, between two or more related economic series, e.g. consumption and income (Engle and Granger, 1987). Here, the observable series  $x_t$  and  $y_t$  both have memory  $d$  but for some  $\beta$

$$y_t - \beta x_t$$

has memory  $c < d$ .

If  $x_t$  is Gaussian and stationary then it suffices to model it in terms of  $\gamma_u$  (or equivalently  $f(\lambda)$ ). But otherwise not all the information is contained in 2nd moments. One way of modelling such non-Gaussian  $x_t$  is as follows (see e.g. Hannan, 1970):

**Example 5** *Linear process*

$$x_t = \alpha(L)\varepsilon_t,$$

where the  $\varepsilon_t$  are IID with some non-normal distribution and

$$\alpha(L) = 1 + \sum_{j=1}^{\infty} \alpha_j L^j.$$

For example, in the MA special case

$$\alpha(L) = 1 + \alpha L.$$

If  $\varepsilon_t \sim IID(2)$  then we can include models with either short memory or long memory in 2nd moments. But we can also study other properties. And if  $\varepsilon_t \sim IID(\theta)$ ,  $\theta < 2$ , this can be a convenient model for heavy-tailed data.

An alternative way of modelling non-Gaussian series is via non-linear models.

**Example 6** *Nonlinear AR (e.g. Jones, 1978)*

$$x_t = g(x_{t-1}) + \varepsilon_t,$$

where  $g$  is some non-linear function and  $\varepsilon_t \sim IID$ .

For such models, we can again look at 2nd moment memory, but also at other properties, bringing us to our next topic.

### 3. MODELS WITH NO SECOND MOMENT MEMORY BUT WITH MEMORY IN NONLINEAR FUNCTIONS

For some financial data, an important class of models starts from the contention that

$$x_t \sim UH$$

may be reasonable, but not

$$x_t \sim IID(2).$$

**Example 7** *ARCH model (Engle, 1982)*

$$x_t = \varepsilon_t(1 + \alpha x_{t-1}^2)^{\frac{1}{2}}, \quad 0 < \alpha < 1,$$

where  $\varepsilon_t \sim IID(2)$ .

The ARCH model implies that

$$cov(x_t, x_{t+u}) = 0, \quad \text{all } u \neq 0$$

but

$$var(x_t | x_{t-1}) = var(\varepsilon_t)(1 + \alpha x_{t-1}^2).$$

Such a model is said to possess conditional heteroscedasticity. It is implied that the sequence  $x_t$  has zero (2nd moment) memory but the sequence  $x_t^2$  has short (2nd moment) memory. Such models have been extended and greatly used in practice. In some versions of the model  $var(x_t) = \infty$ . In more,  $Ex_t^4 = \infty$ , agreeing with some empirical evidence. However, ARCH models can be hard to handle theoretically, and they may not explain all features of the data. One such feature is leverage:  $cov(x_t^2, x_{t-u}) < 0$ , for some  $u \geq 1$ . Another such feature is long memory in  $x_t^2$ .

One model that overcomes both these drawbacks is as follows:

**Example 8** *LARCH (Robinson, 1991):*

$$\begin{aligned} x_t &= \varepsilon_t(\mu + \alpha(L)\varepsilon_t), \\ \varepsilon_t &\sim IID(2), \quad \alpha(L) = \sum_{j=1}^{\infty} \alpha_j L^j. \end{aligned}$$

However, unlike ARCH, the estimation of LARCH has not been adequately discussed, so it is not presently a very viable tool for empirical analysis.

Far more popular alternatives to ARCH are stochastic volatility (SV) models. A particular version that is often studied is as follows.

**Example 9** *SV model (Taylor, 1986).*

$$x_t = \varepsilon_t e^{\mu + \alpha \eta_t},$$

where  $\varepsilon_t$  is IID and  $\eta_t$  is a stationary Gaussian process.

Distributional assumptions are often imposed also on  $\varepsilon_t$ , and properties are affected depending on whether  $\eta_t$  is independent of  $\varepsilon_s$ , for all  $s > t$ , or  $\eta_t$  is independent of  $\varepsilon_s$ , for all  $s, t$ . In any case whereas

$$x_t \sim UH,$$

we have

$$\text{cov}(|x_t|^\theta, |x_{t+u}|^\theta) \neq 0, \quad u \neq 0, \theta > 0,$$

(e.g. when  $\theta = 2$ ). Moreover, if we choose  $\eta_t$  to have long memory then  $|x_t|^\theta$  can also have long memory. Further, we can generalize to models such as

$$x_t = f_1(\varepsilon_t) f_2(\eta_t)$$

where  $\varepsilon_t$  and  $\eta_t$  can both be vector processes with long or short memory. We can then study the memory of quantities such as  $|x_t|^\theta$  (see Robinson, 2001).

#### 4. FINAL COMMENTS

Versions of the models we have discussed involving finitely many unknown parameters are commonly estimated. But they can also be the basis for nonparametric

modelling. In either case, finite-sample properties of estimates and test statistics are generally intractable. However, asymptotic (as sample size  $\rightarrow \infty$ ) properties are well developed in some of the models, less so in others. In many cases we have a normal approximation for the estimates, leading to convenient hypothesis testing and interval estimation. An important application of the estimated model is in forecasting.

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