Sequential Changepoint Detection in Factor Models for Time Series

Ragvir Sabharwal

London School of Economics (LSE)

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I. Summary

Research Question

(1) How to Detect Changepoints in FMs

- Structural instabilities in factor models for time series:

1. Changes in Loadings (Λ); and/or
2. Changes in Number of Factors (r).

(2) ...on a Real-Time Basis?

- Unique? Existing literature only addresses offline setting.
- Necessary? Important for applications such as "Nowcasting".

My Contributions

(1) New Sequential Changepoint Estimator

- I propose to monitor the value of an eigenvalue ratio

$$\delta_{r+1}(\tau_s, \tau_e) = \frac{\mu^{\mathbf{x}}_{r+1}(\tau_s, \tau_e)}{\mu^{\mathbf{x}}_{r+1}(\tau_s - 1, \tau_e - 1)}$$

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using observations within a rolling window over time, and

- declare a changepoint when it breaches some threshold (H).
- So, with window of fixed length (z), my estimator is of type

$$\inf\{\tau_e > z : \delta_{r+1}(\tau_e - z + 1, \tau_e) \geq H\} - 1$$

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Post-κ: the $(r+1)^{th}$ eigenvalue diverges to infinity with N.

Ratio comparing successive values of this eigenvalue over time should spike at the changepoint and remain stable otherwise.

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  Simulations; Application to FTSE100 data (detect Brexit);

  Extension to emerging and disappearing factors.

Merits of Proposed Procedure

- Simple idea which works well in practice

  ...in fact, with no detection delay in FTSE100 data example;
- Allows us to detect different break types

  ...and distinguish among break types;
- Builds on standard modelling framework from the literature;
- Quick to implement.

II. Structural Instability

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- Consider a one-factor model with a structural break.

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- Define $g_{1t} = \begin{cases} f_t, t \leq \kappa \\ 0, t > \kappa \end{cases}$ and $g_{2t} = \begin{cases} 0, t \leq \kappa \\ f_t, t > \kappa \end{cases}$

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- $\mathbf{x_t} = \lambda_{\mathbf{1}} g_{1t} + \lambda_{\mathbf{2}} g_{2t} + \mathbf{e_t}$, an equivalent stable model.

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- Lemma 3.1 - Behaviour of eigenvalues of Common part;

  Lemma 3.2 - Behaviour of eigenvalues of Idiosyncratic part;

  Lemma 3.3 - Behaviour of eigenvalues of $\Sigma^{\mathbf{x}}(\tau_s, \tau_e)$.

Lemma 3.3 - Eigenvalue Behaviour under Instability

- For any $N \in \mathbb{N}$, there exist constants $\underline{M}_4$, $\overline{M}_4$, $\underline{M}_5$, $\overline{M}_5$ s.t.
  - (i) $0 < \underline{M}_4 \leq N^{-1}\mu_j^{\mathbf{x}}(\tau_s, \tau_e) \leq \overline{M}_4 < \infty$ for $j = 1, .., r^*$; and
  - (ii) $0 < \underline{M}_5 \leq \mu_{r^*+1}^{\mathbf{x}}(\tau_s, \tau_e) \leq \overline{M}_5 < \infty$

  where

$$r^* = \begin{cases} r, & \tau_s < \tau_e \leq \kappa \\ r + q, & \tau_s \leq \kappa < \tau_e \\ r, & \kappa < \tau_s < \tau_e \end{cases}$$

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  and $q \in \{1, ..., r\}$ is the # of breaking factors.
- $(r + q)$ eigenvalues diverge when window straddles κ; but only r eigenvalues diverge otherwise.

Theorem 3.1 - Upward Spike in Detection Statistic

- As $N \to \infty$,
  - (i) if $\kappa \neq (\tau_e - 1)$, then there exists a constant $\overline{M}_6$ s.t.
  $$\delta_{r+\rho}(\tau_s, \tau_e) \leq \overline{M}_6 < \infty;$$
  - (ii) but if $\kappa = (\tau_e - 1)$, then
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  $$\delta_{r+\rho}(\tau_s, \tau_e) \to \infty,$$
  for $\rho = 1, ..., q$.
- Proof is evident from analysis of Lemma 3.3.
- Corollary 3.1 - Useful secondary result (downward spike).

III. Changepoint Detection

Lemma 3.4/Theorem 3.2 - Estimation

Lemma 3.4

For any $N \in \mathbb{N}$ and $j \in \{1, ..., N\}$,

$$\left| \frac{\hat{\mu}_j^{\mathbf{x}}(\tau_s, \tau_e)}{N} - \frac{\mu_j^{\mathbf{x}}(\tau_s, \tau_e)}{N} \right| = O_p\left((\tau_e - \tau_s + 1)^{-1/2} \right).$$

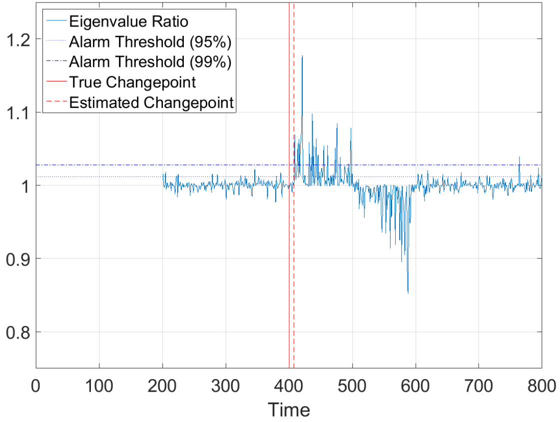
Theorem 3.2

For $j \in \{1, ..., N\}$,

$$\hat{\delta}_j(\tau_s, \tau_e) \xrightarrow{p} \delta_j(\tau_s, \tau_e) \text{ as } (\tau_e - \tau_s + 1) \to \infty.$$

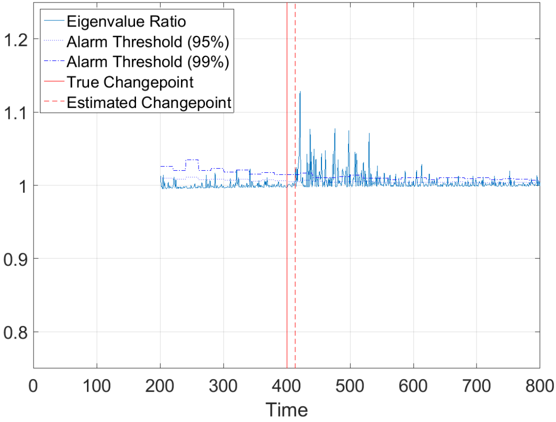
Simulation using Rolling Window Methodology

Figure 3.1



Simulation using Expanding Window Methodology

Figure 3.4



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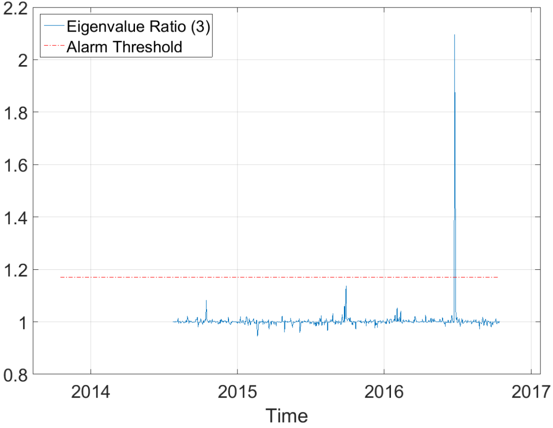
- Overlapping blocks resampling scheme from Kunsch(1989).
- Rolling Window:
  (i) bootstrap from training period; choose $100(1-\alpha)^{th}$ pctile;
  (ii) generate a single threshold for use over time.
- Expanding Window:
  (i) ongoing bootstrap every (or every w) period(s);
  (ii) resample from training period with equal probability AND from observations thereafter with geometrically declining probability;
  (iii) choose $100(1-\alpha)^{th}$ pctile;
  (iv) relevant threshold declines step-wise over time.

June 23, 2016: Changepoint Detected!

Figure 3.8



IV. Concluding Remarks

Research Plan

High Frequency Data

- Jump Detection in continuous-time models.
- Pelger (2015) and Ait-Sahalia and Xiu (2015).
- Adapt existing method (eigenvalue-based criterion)?
- Develop new method (test statistic)?

Thank you, Kostas and Matteo!

RECAP:

Real-time Detection of Changepoints in FMs:

- Introduced detection statistic based on eigenvalue ratio;
- Provided theoretical justification for changepoint estimator;
- Developed a sequential changepoint detection procedure;
- Tested procedure using simulations and real-world data.

...and a special thanks to Matteo for all his help with my research!