ERRATA AND COMMENTS FOR THE BOOK "A FRIENDLY APPROACH TO COMPLEX ANALYSIS", WORLD SCIENTIFIC, 2014

SARA MAAD SASANE AND AMOL SASANE

1. A list of typos

▷ Page 3, line -6: It should be "$x^2 + bx + c = 0$". (That is, no "a".)

▷ Page 19, line 3: Replace "$e^{(x_1+x_2)+i(y_1+y_2)}$" by "$\exp((x_1 + x_2) + i(y_1 + y_2))$".

▷ Page 25, line 19: Replace "$z = |z|e^{i\theta}$" by "$z = |z|\exp(i\theta)$".

▷ Page 26, last line: Replace "$a^b = e^{b\text{Log}(a)}$" by "$a^b = \exp(b \cdot \text{Log}(a))$"

▷ Page 29, line -2: Replace "$|f(z_0)|$" by "$|f'(z_0)|$".

▷ Page 55, first line: Replace "$z \mapsto e^z$" by "$z \mapsto \exp(z)$".
Also in lines 14-15, delete the redundant sentence "Moreover, we had ... in its domain.".

▷ Page 58, Example 2.16, first line of the display after "Moreover": On the right hand side of the equality, replace "$\dfrac{\partial}{\partial z}$" by "$\dfrac{\partial}{\partial \overline{z}}$".

▷ Page 64, line 9: Replace "$\widetilde{\gamma}(t) = \gamma(\varphi(t))$ for $t \in [a, b]$" by "$\widetilde{\gamma}(t) = \gamma(\varphi(t))$ for $t \in [c, d]$".
Also, in the display after the picture, it should be:

$$\begin{aligned}\int_{\widetilde{\gamma}} f(z)dz &= \int_c^d f(\widetilde{\gamma}(t))\widetilde{\gamma}'(t)dt = \int_c^d f(\gamma(\varphi(t)))\gamma'(\varphi(t))\varphi'(t)dt \\ &\overset{(\tau=\varphi(t))}{=} \int_a^b f(\gamma(\tau))\gamma'(\tau)d\tau = \int_\gamma f(z)dz.\end{aligned}$$

▷ Page 72, line 8: It should be "$\displaystyle\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}dt$".
(That is, dt is missing.)

▷ Page 78, line -7: Replace the first comma before "Intially" by a full stop.

▷ Page 125, line -2: It should be "$f(z) = (z - z_0)^m g(z)$".

▷ Page 150, item (3) in Exercise 4.35: It should be "If f is not identically zero and holomorphic ...".

▷ Page 152, line 5 in Example 4.17: Replace "$e^{i\theta}$" by "$\exp(i\theta)$".

▷ Page 164, display in the second row of the table: it should be $\dfrac{\partial u}{\partial t}-\dfrac{\partial^2 u}{\partial x^2}=0$ instead of $\dfrac{\partial^2 u}{\partial t}-\dfrac{\partial^2 u}{\partial x^2}=0$.

▷ Page 175, line 4: Replace "e^{-1/z^4}" by "$\exp(-1/z^4)$".

▷ Page 179: the location of $w := \cos\cos\dfrac{\pi}{3}+i\sin\dfrac{\pi}{3}$ is incorrect. The correct picture is shown below. The complex number w is the intersection point of the unit circle with centre $(0,0)$ and the vertical line passing through $(1/2,0)$.

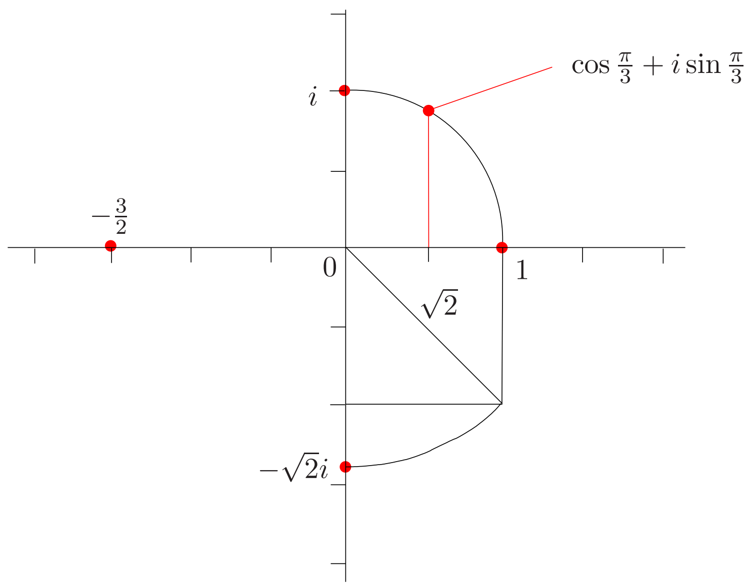

FIGURE 1. Location of the complex numbers 0, 1, $-3/2$, i, $-\sqrt{2}i$, $\cos\dfrac{\pi}{3}+i\sin\dfrac{\pi}{3}$.

▷ Page 218, fourth line in the Solution to Exercise 3.14: It should be

$$\int_\gamma \exp z\,dz=\int_0^1 \exp\left((a+ib)x\right)\cdot(a+ib)dx=\int_0^1 e^{ax}(\cos(bx)+i\sin(bx))(a+ib)dx.$$

2. A comment on the sufficiency of the Cauchy-Riemann equations

The hypothesis (2) in Theorem 2.2 on page 42 can be replaced by the equivalent (and more easily checkable) condition that u, v have continuous first partial derivatives in U, and then the sentence in the brackets after the display in (2.7) on page 42 can be ignored.

Also, we remark that in Example 2.10, the remark at the end of page 47 can be ignored, since although correct, it can be percieved to be emphasizing differentiability of u, v as being necessary, which is not the relevant thing here: indeed, the Looman-Menchoff theorem (not discussed in this book, but can be found for example in Narasimhan's book *Complex Analysis in One Variable*, Birkhäuser, 2001), states that a *continuous* complex-valued function defined in an open set of the complex plane is holomorphic if it satisfies the Cauchy-Riemann equations.

A somewhat clearer (pointwise) version of the sufficiency of the Cauchy-Riemann equations for complex differentiability is given below.

Theorem 2.2. Let

(1) U be an open subset of $\mathbb{C}$,
(2) $u, v : U \to \mathbb{R}$ be real differentiable at $(x_0, y_0) \in U$, and
(3) u, v satisfy the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) \text{ and } \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0).$$

Then $f := u + iv : U \to \mathbb{C}$ is complex differentiable at $x_0 + iy_0$ and

$$f'(x_0 + iy_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0).$$

Remark: Recall that if $u, v : U \to \mathbb{R}$ have partial derivatives in U

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y},$$

which moreover are continuous on U, then u, v are real differentiable in U.

Proof. Let

$$A := \frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0), \qquad B := -\frac{\partial u}{\partial y}(x_0, y_0) = \frac{\partial v}{\partial x}(x_0, y_0).$$

Let $\epsilon > 0$. As u is real differentiable at the point (x_0, y_0), there exists a $\delta_1 > 0$ such that if $(x, y) \in U$ satisfies $0 < \|(x, y) - (x_0, y_0)\|_2 < \delta_1$, then

$$\begin{aligned} &\frac{\left|u(x, y) - u(x_0, y_0) - \begin{bmatrix} A & -B \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}\right|}{\|(x, y) - (x_0, y_0)\|_2} \\ = \quad &\frac{\left|u(x, y) - u(x_0, y_0) - \begin{bmatrix} \frac{\partial u}{\partial x}(x_0, y_0) & \frac{\partial u}{\partial y}(x_0, y_0) \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}\right|}{\|(x, y) - (x_0, y_0)\|_2} < \frac{\epsilon}{2}. \end{aligned}$$

Similarly there exists a $\delta_2 > 0$ such that whenever $(x, y) \in U$ satisfies $0 < \|(x, y) - (x_0, y_0)\|_2 < \delta_2$, we have

$$\begin{aligned} & \frac{\left| v(x,y) - v(x_0,y_0) - \left[\begin{array}{cc} B & A \end{array} \right] \left[\begin{array}{c} x - x_0 \\ y - y_0 \end{array} \right] \right|}{\|(x,y) - (x_0,y_0)\|_2} \\ = & \frac{\left| v(x,y) - v(x_0,y_0) - \left[\begin{array}{cc} \frac{\partial v}{\partial x}(x_0,y_0) & \frac{\partial v}{\partial y}(x_0,y_0) \end{array} \right] \left[\begin{array}{c} x - x_0 \\ y - y_0 \end{array} \right] \right|}{\|(x,y) - (x_0,y_0)\|_2} < \frac{\epsilon}{2}. \end{aligned}$$

Let $z_0 := x_0 + iy_0$. Set $\delta := \min\{\delta_1, \delta_2\} > 0$. If $z = x + iy$, where $x, y \in \mathbb{R}$, satisfies $0 < |z - z_0| < \delta$, then we have

$$\begin{aligned} \Big| \frac{f(z) - f(z_0)}{z - z_0} - & \Big(\frac{\partial u}{\partial x}(x_0,y_0) + i \frac{\partial v}{\partial x}(x_0,y_0) \Big) \Big| \\ = & \frac{|f(z) - f(z_0) - (A + iB)(z - z_0)|}{|z - z_0|} \\ \leq & \frac{|\mathrm{Re}(f(z) - f(z_0) - (A + iB)(z - z_0))|}{|z - z_0|} \\ & + \frac{|\mathrm{Im}(f(z) - f(z_0) - (A + iB)(z - z_0))|}{|z - z_0|} \\ = & \frac{|u(x,y) - u(x_0,y_0) - (A(x - x_0) - B(y - y_0))|}{|(x - x_0) + i(y - y_0)|} \\ & + \frac{|(v(x,y) - v(x_0,y_0) - (B(x - x_0) + A(y - y_0))|}{|(x - x_0) + i(y - y_0)|} \\ = & \frac{\left| u(x,y) - u(x_0,y_0) - \left[\begin{array}{cc} A & -B \end{array} \right] \left[\begin{array}{c} x - x_0 \\ y - y_0 \end{array} \right] \right|}{\|(x,y) - (x_0,y_0)\|_2} \\ & + \frac{\left| u(x,y) - u(x_0,y_0) - \left[\begin{array}{cc} B & A \end{array} \right] \left[\begin{array}{c} x - x_0 \\ y - y_0 \end{array} \right] \right|}{\|(x,y) - (x_0,y_0)\|_2} \\ < & \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Thus f is complex differentiable at z_0 and

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0,y_0) + i \frac{\partial v}{\partial x}(x_0,y_0).$$

This completes the proof. $\square$

Center for Mathematical Sciences, LTH, Lund University, 22100 Lund, Sweden

E-mail address: sara@maths.lth.se

Mathematics Department, London School of Economics, Houghton Street, London WC2A 2AE, United Kingdom.

E-mail address: sasane@lse.ac.uk