Department of Mathematics, London School of Economics

Real Analysis
(MA203)

Amol Sasane

2023/24

Contents

Preface

What is Real Analysis?

First of all, ‘Analysis’ refers to the subdomain of Mathematics, which is roughly speaking an abstraction of the familiar subject of Calculus. Calculus arose as a box of tools enabling one to handle diverse problems in the applied sciences such as physics and engineering where quantities change (for example with time), and calculations based on ‘rates of change’ were needed. It soon became evident that the foundations of Calculus needed to be made mathematically precise. This is roughly the subject of Mathematical Analysis, where Calculus is made rigorous. But another byproduct of this rigorisation process is that mathematicians discovered that many of the things done in the set-up of usual calculus can be done in a much more general set up, enabling one to expand the domain of applications. We will study such things in this course.

Secondly, why do we use the adjective ‘Real’? We will start with the basic setting of making rigorous Calculus with *real* numbers, but we will also develop Calculus in more abstract settings, for example in $\mathbb{R}^n$. Using this adjective ‘Real’ also highlights that the subject is different from ‘Complex Analysis’ which is all about doing analysis in $\mathbb{C}$. (It turns out that Complex Analysis is a very specialised branch of analysis which acquires a somewhat peculiar character owing to the special geometric meaning associated with the multiplication of complex numbers in the complex plane.)

These notes were written by me in 2011. Thanks are due to Konrad Swanepoel for his help in correcting typos and updating the solutions. I would be grateful to hear about any remaining mistakes or other comments.

Amol Sasane

Chapter 1

Metric and normed spaces

We are familiar with concepts from calculus such as

(1) convergence of sequences of real numbers,

(2) continuity of a function $f : \mathbb{R} \to \mathbb{R}$,

(3) differentiability of a function $f : \mathbb{R} \to \mathbb{R}$.

Once these notions are available, one can prove useful results involving such notions. For example, we have seen the following:

Theorem 1.1. *If $f : [a, b] \to \mathbb{R}$ is continuous, then f has a minimiser on $[a, b]$.*

Theorem 1.2. *If $f : \mathbb{R} \to \mathbb{R}$ is such that $f''(x) \geqslant 0$ for all $x \in \mathbb{R}$ and $f'(x_0) = 0$, then x_0 is a minimiser of f.*

We will revisit these concepts in this course, and see that the same concepts can be defined in a much more general context, enabling one to prove results similar to the above in the more general set up. This means that we will be able to solve problems that arise in applications (such as optimisation and differential equations) that we wouldn't be able to solve earlier with our limited tools. Besides these immediate applications, concepts and results from real analysis are fundamental in mathematics itself, and are needed in order to study almost any topic in mathematics.

In this chapter, we wish to emphasise that the key idea behind defining the above concepts is that of a distance between points. In the case when one works with real numbers, this distance is provided by the absolute value of the difference between the two numbers: thus the distance between $x, y \in \mathbb{R}$ is taken as $|x - y|$. This coincides with our geometric understanding of distance when the real numbers are represented on the 'number line'. For instance, the distance between -1 and 3 is 4, and indeed $4 = |-1-3|$.

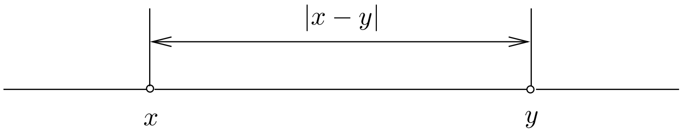


Figure 1. Distance between real numbers.

Recall for example, that a sequence $(a_n)_{n \in \mathbb{N}}$ is said to *converge with limit* $L \in \mathbb{R}$ if for every $\epsilon > 0$, there exists a $N \in \mathbb{N}$ such that whenever $n > N$, $|a_n - L| < \epsilon$. In other words, the sequence converges to L if no matter what *distance* $\epsilon > 0$ is given, one can guarantee that all the terms of the sequence beyond a certain index N are at a *distance* of at most ϵ away from L (this is the inequality $|a_n - L| < \epsilon$). So we notice that in this notion of 'convergence of a sequence' indeed the notion of distance played a crucial role. After all, we want to say that the terms of the sequence get 'close' to the limit, and to measure closeness, we use the distance between points of $\mathbb{R}$.

A similar thing happens with all the other notions listed at the outset. For example, recall that a function $f : \mathbb{R} \to \mathbb{R}$ is said to be *continuous at* $c \in \mathbb{R}$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $|x - c| < \delta$, $|f(x) - f(c)| < \epsilon$. Roughly, given any distance ϵ, I can find a distance δ such that whenever I choose an x not farther than a distance δ from c, I am guaranteed that $f(x)$ is not farther than a distance of ϵ from $f(c)$. Again notice the key role played by the distance in this definition.

1.1. Distance in $\mathbb{R}$

The distance between points $x, y \in \mathbb{R}$ is taken as $|x - y|$. Thus we have a map that associates to a pair $(x, y) \in \mathbb{R} \times \mathbb{R}$ of real numbers, the number $|x - y| \in \mathbb{R}$ which is the distance between x and y. We think of this map $(x, y) \mapsto |x - y| : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as the 'distance function' in $\mathbb{R}$.

Define $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $d(x, y) = |x - y|$ $(x, y \in \mathbb{R})$. Then it can be seen that this distance function d satisfies the following properties:

(D1) For all $x, y \in \mathbb{R}$, $d(x, y) \geqslant 0$. If $x \in \mathbb{R}$, then $d(x, x) = 0$. If $x, y \in \mathbb{R}$ are such that $d(x, y) = 0$, then $x = y$.

(D2) For all $x, y \in \mathbb{R}$, $d(x, y) = d(y, x)$.

(D3) For all $x, y, z \in \mathbb{R}$, $d(x, y) + d(y, z) \geqslant d(x, z)$.

It turns out that these are the key properties of the distance which are needed in developing analysis in $\mathbb{R}$. So it makes sense that when we want to generalise the situation with the set $\mathbb{R}$ being replaced by an arbitrary set X, we must define a distance function

$$d : X \times X \to \mathbb{R}$$

that associates a number (the distance!) to each pair of points $x, y \in X$, and which has the same properties (D1)-(D3) (with the obvious changes: $x, y, z \in X$). We do this in the next section.

1.2. Metric space

Definition 1.3. A *metric space* is a set X together with a function $d : X \times X \to \mathbb{R}$ satisfying the following properties:

(D1) (*Positive definiteness*) For all $x, y \in X$, $d(x, y) \geqslant 0$. For all $x \in X$, $d(x, x) = 0$. If $x, y \in X$ are such that $d(x, y) = 0$, then $x = y$.

(D2) (*Symmetry*) For all $x, y \in X$, $d(x, y) = d(y, x)$.

(D3) (*Triangle inequality*) For all $x, y, z \in X$, $d(x, y) + d(y, z) \geqslant d(x, z)$.

Such a d is referred to as a *distance function* or *metric*.

Let us consider some examples.

Example 1.4. $X := \mathbb{R}$, with $d(x, y) := |x - y|$ $(x, y \in \mathbb{R})$, is a metric space. $\diamond$

Example 1.5. For any nonempty set X, define

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases}$$

This d is called the *discrete metric*. Then d satisfies (D1)-(D3), and so X with the discrete metric is a metric space. $\diamond$

Note that in particular $\mathbb{R}$ with the discrete metric is a metric space as well. So the above two examples show that the distance function in a metric space is not unique, and what metric is to be used depends on the application one has in mind. Hence whenever we speak of a metric space, we always need to specify not just the set X but also the distance function d being considered. So often we say 'consider the metric space (X, d)', where X is the set in question, and $d : X \times X \to \mathbb{R}$ is the metric considered.

However, for some sets, there are some natural candidates for distance functions. One such example is the following one.

Example 1.6 (Euclidean space $\mathbb{R}^n$)**.** In $\mathbb{R}^2$ and $\mathbb{R}^3$, where we can think of vectors as points in the plane or points in the space, we can use the distance distance between two points as the length of the line segment joining these points. Thus (by Pythagoras's Theorem) in $\mathbb{R}^2$, we may use

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

as the distance between the points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{R}^2$. Similarly, in $\mathbb{R}^3$, one may use

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}$$

as the distance between the points $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in $\mathbb{R}^3$. See Figure 2.

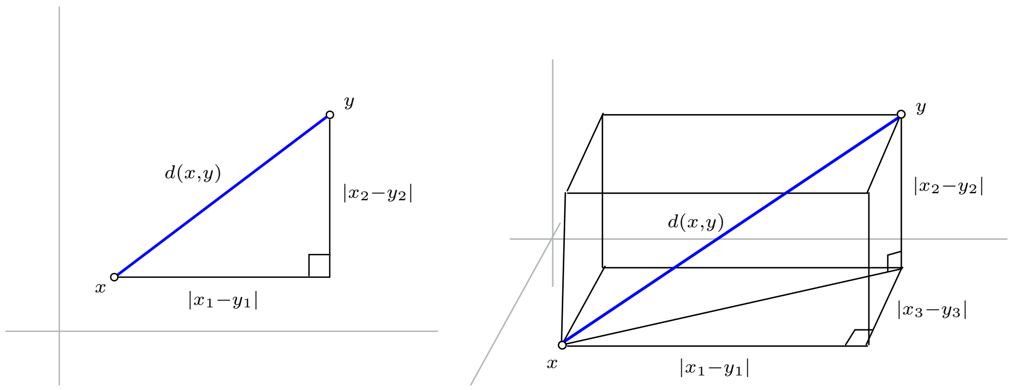


Figure 2. Distance in $\mathbb{R}^2$ and $\mathbb{R}^3$.

In an analogous manner to $\mathbb{R}^2$ and $\mathbb{R}^3$, more generally, for $x, y \in \mathbb{R}^n =: X$, we define the *Euclidean distance* by

$$d(x, y) = \sqrt{\sum_{k=1}^{n} (x_k - y_k)^2} = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$$

for

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n, \quad y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{R}^n.$$

Then $\mathbb{R}^n$ is a metric space with the Euclidean distance, and is referred to as the *Euclidean space*. The verification of (D3) can be done by using the *Cauchy-Schwarz inequality*: For real numbers $x_1, \dots, x_n$ and $y_1, \dots, y_n$, there holds that

$$\left(\sum_{k=1}^{n} x_k^2 \right) \left(\sum_{k=1}^{n} y_k^2 \right) \geqslant \left(\sum_{k=1}^{n} x_k y_k \right)^2.$$

This last property (D3) is sometimes referred to as the triangle inequality. The reason behind this is that, for triangles in Euclidean geometry of the plane, we know that the sum of the lengths of two sides of a triangle is at least as much as the length of the third side. If we now imagine the points $x, y, z \in \mathbb{R}^2$ as the three vertices of a triangle, then this is what (D3) says; see Figure 3.

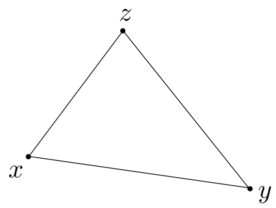


Figure 3. How the triangle inequality gets its name.

Throughout this course, whenever we refer to $\mathbb{R}^n$ as a metric space, unless specified otherwise, we mean that it is equipped with this Euclidean metric. Example 1.4 corresponds to the case when $n = 1$. ◇

Exercise 1.7. Verify that the d given in Example 1.5 does satisfy (D1)-(D3).

Exercise 1.8. One can show the Cauchy-Schwarz inequality as follows: Let x, y be vectors in $\mathbb{R}^n$ with the components $x_1, \dots, x_n$ and $y_1, \dots, y_n$, respectively. For a column vector $x \in \mathbb{R}^n$, $x^\top$ denotes its transpose. For $t \in \mathbb{R}$, consider the function

$$f(t) = (x + ty)^\top (x + ty) = x^\top x + 2t x^\top y + t^2 y^\top y.$$

From the rightmost expression, we see that f is a quadratic function of the variable t. It is clear from the middle expression that $f(t)$, being the sum of squares

$$\sum_{k=1}^{n} (x_k + t y_k)^2,$$

is nonnegative for all $t \in \mathbb{R}$. This means that the discriminant of f must be $\leqslant 0$, since otherwise, f would have two distinct real roots, and would then have negative values between these roots! Calculate the discriminant of the quadratic function and show that its nonpositivity yields the Cauchy-Schwarz inequality.

Normed space. Frequently in applications, one needs a metric not just in any old set X, but in a *vector space* X.

Recall that a (real) vector space X, is just a set X with the two operations of vector addition $+ : X \times X \to X$ and scalar multiplication $\cdot : \mathbb{R} \times X \to X$ which together satisfy the vector space axioms.

But now if one wants to also do analysis in a vector space X, there is so far no ready-made available notion of distance between vectors. One way of creating a distance in a vector space is to equip it with a 'norm' $\|\cdot\|$, which is the analogue of absolute value $|\cdot|$ in the vector space $\mathbb{R}$. The distance function is then created by taking the norm $\|x - y\|$ of the difference between pairs of vectors $x, y \in X$, just like in $\mathbb{R}$ the Euclidean distance between $x, y \in \mathbb{R}$ was taken as $|x - y|$.

Definition 1.9. Let $(X, +, \cdot)$ be a vector space. A function $\|\cdot\| : X \to \mathbb{R}$ is called a *norm* on X if it satisfies the following properties:

(N1) (*Positive definiteness*) For all $x \in X$, $\|x\| \geqslant 0$. If $x \in X$ is such that $\|x\| = 0$, then $x = \mathbf{0}$ (the zero vector in X).

(N2) (*Positive homogeneity*) For all $\alpha \in \mathbb{R}$ and all $x \in X$, $\|\alpha \cdot x\| = |\alpha| \, \|x\|$.

(N3) (*Triangle inequality*) For all $x, y \in X$, $\|x + y\| \leqslant \|x\| + \|y\|$.

A *normed space* is a vector space $(X, +, \cdot)$ together with a *norm*.

If X is a normed space then

$$d(x, y) := \|x - y\| \quad (x, y \in X)$$

satisfies (D1)-(D3) and makes X a metric space (Exercise 1.11). This distance is referred to as the *induced distance* in the normed space $(X, \|\cdot\|)$. Clearly then

$$\|x\| = \|x - \mathbf{0}\| = d(x, \mathbf{0}),$$

and so the norm of a vector is the induced distance to the zero vector in a normed space $(X, \|\cdot\|)$.

Example 1.10. $\mathbb{R}$ is a vector space with the usual operations of addition and multiplication. It is easy to see that the absolute value function

$$x \mapsto |x| \quad (x \in \mathbb{R})$$

satisfies (N1)-(N3), and so $\mathbb{R}$ is a normed space, and the induced distance is the usual Euclidean metric in $\mathbb{R}$.

More generally, in the vector space $\mathbb{R}^n$, with addition and scalar multiplication defined componentwise, we can introduce the 2-norm as

$$\|x\|_2 := \sqrt{x_1^2 + \cdots + x_n^2}$$

for vectors $x \in \mathbb{R}^n$ having components $x_1, \ldots, x_n$. Then $\|\cdot\|_2$ satisfies (N1)-(N3) and makes $\mathbb{R}^n$ a normed space. The induced metric is then the usual Euclidean metric in $\mathbb{R}^n$. $\diamond$

Exercise 1.11. Verify that if X is a normed space with norm $\|\cdot\|$, then $d \colon X \times X \to \mathbb{R}$ defined by $d(x, y) = \|x - y\|$ satisfies (D1)-(D3). *Hint:* Use each of the properties (N1), (N2) and (N3).

Exercise 1.12 (Reverse Triangle Inequality)**.** Let $(X, \|\cdot\|)$ be a normed space. Prove that for all $x, y \in X$, $|\,\|x\| - \|y\|\,| \leqslant \|x - y\|$.

Exercise 1.13. Verify that the norm $\|\cdot\|_2$ given on $\mathbb{R}^n$ in Example 1.10 does satisfy (N1)-(N3).

Exercise 1.14. Let X be a metric space with a metric d. Define $d_1 : X \times X \to \mathbb{R}$ by

$$d_1(x, y) = \frac{d(x, y)}{1 + d(x, y)} \quad (x, y \in X).$$

Note that $d_1(x, y) \leqslant 1$ for all $x, y \in X$. Show that d_1 is a metric on X. *Hint:* For the triangle inequality, write d_1 in a way in which d appears in just one place, e.g., $d_1 = 1 - \frac{1}{1+d}$ or $d_1 = \frac{1}{\frac{1}{d}+1}$, and use the triangle inequality for d.

Exercise 1.15. Consider the vector space $\mathbb{R}^{m \times n}$ of matrices with m rows and n columns of real numbers, with the usual entrywise addition and scalar multiplication. For $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$, let m_{ij} denote the entry in the i^{th} row and j^{th} column of M. Define for $M \in \mathbb{R}^{m \times n}$, the number

$$\|M\|_\infty := \max_{1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n} |m_{ij}|.$$

Show that $\|\cdot\|_\infty$ defines a norm on $\mathbb{R}^{m \times n}$.

Exercise 1.16. Let $C[a, b]$ denote the set of all continuous functions $f : [a, b] \to \mathbb{R}$. Then $C[a, b]$ is a vector space with addition and scalar multiplication defined pointwise. If $f \in C[a, b]$, define

$$\|f\|_\infty = \max_{x \in [a,b]} |f(x)|.$$

As $x \mapsto |f(x)| : [a, b] \to \mathbb{R}$ is continuous, by the Extreme Value Theorem, the above maximum exists.

(1) Show that $\|\cdot\|_\infty$ is a norm on $C[a, b]$.

(2) Let $f \in C[a, b]$ and let $\epsilon > 0$. Consider the set $B(f, \epsilon) := \{g \in C[a, b] : \|f - g\|_\infty < \epsilon\}$. Draw a picture to explain the geometric significance of the statement $g \in B(f, \epsilon)$.

Exercise 1.17. $C[a, b]$ can also be equipped with other norms. For example, prove that

$$\|f\|_1 := \int_a^b |f(x)|\, dx \quad (f \in C[a, b])$$

also defines a norm on $C[a, b]$.

Exercise 1.18. If (X, d) is a metric space, and if $Y \subset X$, then show that $(Y, d|_{Y\times Y})$ is a metric space. (Here $d|_{Y\times Y}$ denotes the restriction of d to the set $Y \times Y$, that is, $d|_{Y\times Y}(y_1, y_2) = d(y_1, y_2)$ for $y_1, y_2 \in Y$.) Hence every subset of a metric space is itself a metric space with the restriction of the original metric. The metric $d|_{Y\times Y}$ is referred to as the *induced metric on Y by d* or the *subspace metric on Y obtained from d*, and the metric space $(Y, d|_{Y\times Y})$ is called a *metric subspace* of (X, d).

Exercise 1.19. The set of integers $\mathbb{Z}$ $(\subset \mathbb{R})$ inherits the Euclidean metric from $\mathbb{R}$, but it also carries a very different metric, called the p-adic metric, where p is a prime number. For $n \in \mathbb{Z}$, the p-adic 'norm'[1] of n is $|n|_p := 1/p^k$, where k is the largest integer power of p that divides n. The norm of 0 is by definition 0. The more factors of p, the smaller the p-norm. The *p-adic metric* on $\mathbb{Z}$ is $d_p(x, y) := |x - y|_p$ $(x, y \in \mathbb{Z})$.

(1) Prove that if $x, y \in \mathbb{Z}$, then $|x + y|_p \leqslant \max\{|x|_p, |y|_p\}$.
(2) Show that d_p is a metric on $\mathbb{Z}$.

Exercise 1.20. Let ℓ^2 denote the set of all 'square summable' sequences of real numbers:

$$\ell^2 = \Big\{(a_n)_{n\in\mathbb{N}} : \sum_{n=1}^{\infty} |a_n|^2 < \infty\Big\}.$$

(1) Show that ℓ^2 is a vector space with addition and scalar multiplication defined termwise.
(2) Let $\|(a_n)_{n\in\mathbb{N}}\|_2 := \sqrt{\sum_{n=1}^{\infty} |a_n|^2}$ for $(a_n)_{n\in\mathbb{N}} \in \ell^2$. Prove that $\|\cdot\|_2$ defines a norm on ℓ^2.

So ℓ^2 is an infinite-dimensional analogue of the Euclidean space $(\mathbb{R}^n, \|\cdot\|_2)$.

Exercise 1.21. Let ℓ^∞ denote the set of all bounded sequences of real numbers:

$$\ell^\infty = \Big\{(a_n)_{n\in\mathbb{N}} : \sup_{n\in\mathbb{N}} |a_n| < \infty\Big\}.$$

(1) Show that ℓ^∞ is a vector space with addition and scalar multiplication defined termwise.
(2) Let $\|(a_n)_{n\in\mathbb{N}}\|_\infty := \sup_{n\in\mathbb{N}} |a_n|$ for $(a_n)_{n\in\mathbb{N}} \in \ell^\infty$. Prove that $\|\cdot\|_\infty$ defines a norm on ℓ^∞.

Exercise 1.22 (Hamming Distance)**.** Let $\mathbb{F}_2^n$ be the set of all ordered n-tuples of zeros and ones. For example, $\mathbb{F}_2^3 = \{000, 001, 010, 011, 100, 101, 110, 111\}$. For $x, y \in \mathbb{F}_2^n$, let

$$d(x, y) = \text{the number of places where } x \text{ and } y \text{ have different entries.}$$

For example, in $\mathbb{F}_2^3$, we have $d(110, 110) = 0$, $d(010, 110) = 1$ and $d(101, 010) = 3$. Show that $(\mathbb{F}_2^n, d)$ is a metric space. (This metric is used in the digital world, in coding and information theory.) *Hint:* For the triangle inequality, consider the function $\delta_k : \mathbb{F}_2^n \times \mathbb{F}_2^n \to \mathbb{R}$ defined by

$$\delta_k(x, y) = \begin{cases} 1 & \text{if the } k^{\text{th}} \text{ digit of } x \text{ and } y \text{ differ} \\ 0 & \text{otherwise,} \end{cases}$$

and note that $d(x, y) = \sum_{k=1}^{n} \delta_k(x, y)$.

1.3. Neighbourhoods and open sets

Let (X, d) be a metric space. With the metric d we can describe 'neighbourhoods' of points by considering sets which include all points whose distance to the given point is not too large.

Definition 1.23 (Open ball)**.** Let (X, d) be a metric space. If $x \in X$ and $r > 0$, we call the set

$$B(x, r) = \{y \in X : d(x, y) < r\}$$

the *open ball* centred at x with radius r.

The picture we have in mind is shown in Figure 4.

[1] Note that $\mathbb{Z}$ is not a real vector space and so this is not really a norm in the sense we have learnt.

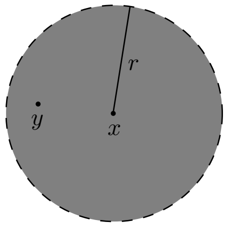
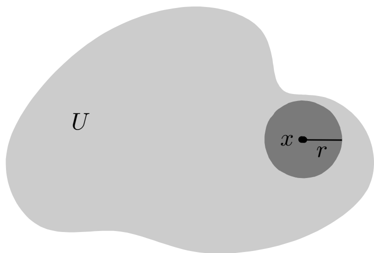


Figure 4. The open ball $B(x, r)$.

In the sequel, for example in our study of continuous functions, *open sets* will play an important role. Here is the definition.

Definition 1.24 (Open set)**.** Let (X, d) be a metric space. A set[2] $U \subset X$ is said to be *open* if for every $x \in U$, there exists an $r > 0$ such that $B(x, r) \subset U$.

Note that the radius r can depend on the choice of the point x. See the picture below. Roughly speaking, in a open set, no matter which point you take in it, there is always some 'room' around it consisting only of points of the open set.



Example 1.25. Let us show that the set (a, b) is open in $\mathbb{R}$. Given any $x \in (a, b)$, we have $a < x < b$. Motivated by Figure 5, let us take $r = \min\{x - a, b - x\}$. Then we see that $r > 0$ and whenever $|y - x| < r$, we have $-r < y - x < r$. So

$$a = x - (x - a) \leqslant x - r < y < x + r \leqslant x + (b - x) = b,$$

that is, $y \in (a, b)$. Hence $B(x, r) \subset (a, b)$. Consequently, (a, b) is open.

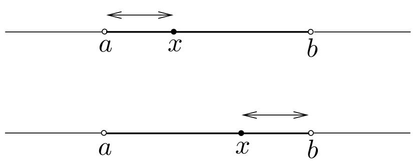


Figure 5. (a, b) is open in $\mathbb{R}$.

On the other hand, the interval $[a, b]$ is not open, because $x := a \in [a, b]$, but no matter how small an $r > 0$ we take, the set $B(a, r) = \{y \in \mathbb{R} : |y - a| < r\} = (a - r, a + r)$ contains points that do not belong to $[a, b]$: For example, $a - \frac{r}{2} \in B(a, r)$, but $a - \frac{r}{2} \notin [a, b]$. Figure 6 illustrates this. $\diamond$

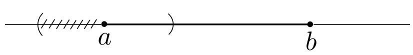


Figure 6. $[a, b]$ is not open in $\mathbb{R}$.

[2]Open sets are often denoted with the letter U since the word *umgebung* in German means 'neighbourhood'.

Example 1.26. The set X is open, since given an $x \in X$, we can take any $r > 0$, and notice that $B(x, r) \subset X$ trivially.

The empty set $\varnothing$ is also open ('vacuously'). Indeed, the reasoning is as follows: Can one show an x for which there is no $r > 0$ such that $B(x, r) \subset \varnothing$? And the answer is no, because there is no x in the empty set (let alone an x which has the extra property that there is no $r > 0$ such that $B(x, r) \subset \varnothing$!). ◇

Exercise 1.27. Let (X, d) be a metric space, $x \in X$ and $r > 0$. Show that the open ball $B(x, r)$ is an open set.

Lemma 1.28. *Any finite intersection of open sets is open.*

Proof. It is enough to consider two open sets, as the general case follows immediately by induction on the number of sets. Let U_1, U_2 be two open sets. Let $x \in U_1 \cap U_2$. Then there exist $r_1 > 0$, $r_2 > 0$ such that $B(x, r_1) \subset U_1$ and $B(x, r_2) \subset U_2$. Take $r = \min\{r_1, r_2\}$. Then $r > 0$, and we claim that $B(x, r) \subset U_1 \cap U_2$. To see this, let $y \in B(x, r)$. Then $d(x, y) < r \leqslant r_1$ and $d(x, y) < r \leqslant r_2$. So $y \in B(x, r_1) \cap B(x, r_1) \subset U_1 \cap U_2$. □

Example 1.29. The finiteness condition in the above lemma cannot be dropped. Here is an example. Consider the open sets in $\mathbb{R}$ given by

$$U_n := \left(-\frac{1}{n}, \frac{1}{n}\right) \quad (n \in \mathbb{N}).$$

Then we have $\bigcap_{n \in \mathbb{N}} U_n = \{0\}$, which is not open in $\mathbb{R}$. ◇

Lemma 1.30. *Any union of open sets is open.*

Proof. Let U_i $(i \in I)$ be a family of open sets indexed[3] by the set I. If

$$x \in \bigcup_{i \in I} U_i,$$

then $x \in U_{i_*}$ for some $i_* \in I$. But as U_{i_*} is open, there exists a $r > 0$ such that $B(x, r) \subset U_{i_*}$. Thus

$$B(x, r) \subset U_{i_*} \subset \bigcup_{i \in I} U_i.$$

Hence the union $\bigcup_{i \in I} U_i$ is open. □

Definition 1.31 (Closed set)**.** Let (X, d) be a metric space. A set[4] F is *closed* if its complement $X \backslash F$ is open.

Example 1.32. Let $a, b \in \mathbb{R}$ and $a < b$. Then $[a, b]$ is closed in $\mathbb{R}$: Indeed, its complement $\mathbb{R} \backslash [a, b]$ is the union of the two open sets $(-\infty, a)$ and (b, ∞). Hence $\mathbb{R} \backslash [a, b]$ is open, and $[a, b]$ is closed.

The set $(-\infty, b]$ is closed in $\mathbb{R}$. (Why?)

The sets $(a, b]$, $[a, b)$ are neither open nor closed in $\mathbb{R}$. (Why?) ◇

Example 1.33. X, $\varnothing$ are closed. ◇

Exercise 1.34. Show that arbitrary intersections of closed sets are closed. Prove that a finite union of closed sets is closed. Can the finiteness condition be dropped in the previous claim?

Exercise 1.35. We know that the segment $(0, 1)$ is open in $\mathbb{R}$. Show that the segment $(0, 1)$ considered as a subset of the plane, i.e., the set $I := (0, 1) \times \{0\} = \{(x, y) \in \mathbb{R}^2 : 0 < x < 1,\ y = 0\}$ is not open in $\mathbb{R}^2$.

---

[3]This means that we have a set I, and for each $i \in I$, there is a set U_i.

[4]Closed sets are often denoted with the letter F since the word *fermé* in French means 'closed'.

Exercise 1.36. Consider the following three metrics on $\mathbb{R}^2$: for $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$,

$$\begin{aligned} d_1(x, y) &:= |x_1 - y_1| + |x_2 - y_2|, \\ d_2(x, y) &:= \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}, \\ d_\infty(x, y) &:= \max\{|x_1 - y_1|, |x_2 - y_2|\}. \end{aligned}$$

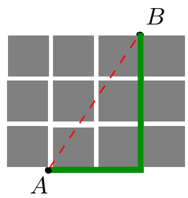
We already know that d_2 defines a metric on $\mathbb{R}^2$: It is just the Euclidean metric induced by the norm $\|\cdot\|_2$.

(1) Verify that d_1 and d_∞ are also metrics on $\mathbb{R}^2$.

(2) Sketch the 'unit balls' $B(\mathbf{0}, 1)$ in each of the metrics.

(3) Give a pictorial 'proof without words' to show that a set U is open in $\mathbb{R}^2$ in the Euclidean metric if and only if it is open when $\mathbb{R}^2$ is equipped with the metric d_1 or the metric d_∞. *Hint:* Inside every square you can draw a circle, and inside every circle, you can draw a square!

Remark: Note that $(\mathbb{R}^2, d_1)$, $(\mathbb{R}^2, d_2)$ and $(\mathbb{R}^2, d_\infty)$ are all *different* metric spaces. This illustrates the important fact that for a given set, we can obtain various metric spaces by choosing different metrics. What metric is considered depends on the particular application at hand. For example, imagine a city (like New York) in which there are streets and avenues with blocks in between, forming a square grid as shown in the picture below.



Then if we take a taxi/cab to go from point A to point B in the city, it is clear that it isn't the Euclidean norm in $\mathbb{R}^2$ which is relevant, but rather the $\|\cdot\|_1$-norm in $\mathbb{R}^2$. (It is for this reason that the $\|\cdot\|_1$-norm is sometimes called the *taxicab norm.*) So what norm one uses depends on the situation at hand, and is something that the modeller decides. It is not something that falls out of the sky!

Exercise 1.37. Determine if the following statements are true or false. Give reasons for your answers.

(1) If a set is not open, then it is closed.

(2) If a set is open, then it is not closed.

(3) There are sets which are both open and closed.

(4) There are sets which are neither open nor closed.

(5) $\mathbb{Q}$ is open in $\mathbb{R}$.

(6) $\mathbb{Q}$ is closed in $\mathbb{R}$.

(7) $\mathbb{Z}$ is closed in $\mathbb{R}$.

Exercise 1.38. Show that the unit sphere with centre $\mathbf{0}$ in $\mathbb{R}^3$, namely the set

$$\mathbb{S}^2 := \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$$

is closed in $\mathbb{R}^3$.

Exercise 1.39. Let (X, d) be a metric space. Show that a singleton (a subset of X containing precisely one element) is always closed. Conclude that every finite subset of X is closed.

Exercise 1.40. Let X be any nonempty set equipped with the discrete metric. Prove that every subset Y of X is both open and closed.

Exercise 1.41. A subset Y of a metric space (X, d) is said to be *dense* in X if for all $x \in X$ and all $\epsilon > 0$, there exists a $y \in Y$ such that $d(x, y) < \epsilon$. (That is, if we take any $x \in X$ and consider any ball $B(x, \epsilon)$ centred at x, it contains a point from Y. In everyday language, we may say for example that 'These woods have a dense growth of birch trees', and the picture we then have in mind is that in any small area of the woods, we find a birch tree. A similar thing is conveyed by the above: no matter what 'patch' (described by $B(x, \epsilon)$) we take in X (thought of as the woods), we can find an element of Y (analogous to birch trees) in that patch.) Show that $\mathbb{Q}$ is dense in $\mathbb{R}$ by proceeding as follows.

If $x, y \in \mathbb{R}$ and $x < y$, then show that there is a $q \in \mathbb{Q}$ such that $x < q < y$. *Hint:* By the Archimedean property[5] of $\mathbb{R}$, there is a positive integer n such that $n(y-x) > 1$. Next there are positive integers m_1, m_2 such that $m_1 > nx$ and $m_2 > -nx$ so that $-m_2 < nx < m_1$. Hence there is an integer m such that $m - 1 \leqslant nx < m$. Consequently $nx < m \leqslant 1 + nx < ny$, which gives the desired result.

Conclude that $\mathbb{Q}$ is dense in $\mathbb{R}$.

Exercise 1.42. Is the set $\mathbb{R}\backslash\mathbb{Q}$ of irrational numbers dense in $\mathbb{R}$? *Hint:* Take any $x \in \mathbb{R}$. If x is irrational itself, then we may just take y to be x and we are done; whereas if x is rational, then take $y = x + \frac{\sqrt{2}}{n}$ with a sufficiently large $n \in \mathbb{N}$.

Exercise 1.43 (Weierstrass's Approximation Theorem)**.** The aim of this exercise is to show that polynomials are dense in $(C[a,b], \|\cdot\|_\infty)$. By considering the map $\boldsymbol{x} \mapsto \boldsymbol{x}\big(a + \cdot(b-a)\big) : C[a,b] \to C[0,1]$, we see that there is no loss of generality in assuming that $a = 0$ and $b = 1$. For $\boldsymbol{x} \in C[0,1]$ and $n \in \mathbb{N}$, let $B_n\boldsymbol{x}$ be the polynomial given by

$$(B_n\boldsymbol{x})(t) := \sum_{k=0}^{n} \boldsymbol{x}\left(\tfrac{k}{n}\right)\tbinom{n}{k} t^k (1-t)^{n-k}, \quad t \in [0,1].$$

Introduce the auxiliary polynomials $\boldsymbol{p}_{n,k}(t) := \binom{n}{k} t^k (1-t)^{n-k}$, $t \in [0,1]$, $0 \leqslant k \leqslant n$, $n \in \mathbb{N}$. Show that

$$\sum_{k=0}^{n} \boldsymbol{p}_{k,n}(t) = 1, \quad \sum_{k=0}^{n} k\,\boldsymbol{p}_{k,n}(t) = nt, \quad \sum_{k=0}^{n} (k-nt)^2 \boldsymbol{p}_{k,n}(t) = nt(1-t).$$

The proof of Weierstrass's Approximation Theorem can now be completed as follows. For $\delta > 0$, we have

$$\sum_{k:|\frac{k}{n}-t|\geqslant\delta} \boldsymbol{p}_{n,k}(t) \leqslant \sum_{k:|\frac{k}{n}-t|\geqslant\delta} \boldsymbol{p}_{n,k}(t) \cdot \underbrace{\tfrac{(k-nt)^2}{\delta^2 n^2}}_{\geqslant 1} \leqslant \tfrac{1}{n^2\delta^2} \sum_{k=0}^{n} (k-nt)^2 \boldsymbol{p}_{k,n}(t) = \tfrac{t(1-t)}{n\delta^2} \leqslant \tfrac{1}{4n\delta^2},$$

where we used the observation $0 \leqslant (\sqrt{t} - \sqrt{1-t})^2 = 1 - 2\sqrt{t(1-t)}$ for all $t \in [0,1]$, in order to obtain the last inequality. Now for $\delta > 0$, set $\omega_\delta(\boldsymbol{x}) := \sup_{|t-s|\leqslant\delta} |\boldsymbol{x}(t) - \boldsymbol{x}(s)|$. Then we have

$$\begin{aligned}
|(B_n\boldsymbol{x})(t) - \boldsymbol{x}(t)| &= |(B_n\boldsymbol{x})(t) - \boldsymbol{x}(t)\sum_{k=0}^{n}\boldsymbol{p}_{n,k}(t)| = |\sum_{k=0}^{n}\boldsymbol{x}(\tfrac{k}{n})\boldsymbol{p}_{n,k}(t) - \boldsymbol{x}(t)\sum_{k=0}^{n}\boldsymbol{p}_{n,k}(t)| \\
&\leqslant \sum_{k=0}^{n} |\boldsymbol{x}(\tfrac{k}{n}) - \boldsymbol{x}(t)|\boldsymbol{p}_{n,k}(t) = \sum_{k:|\frac{k}{n}-t|<\delta} |\boldsymbol{x}(\tfrac{k}{n}) - \boldsymbol{x}(t)|\boldsymbol{p}_{n,k}(t) + \sum_{k:|\frac{k}{n}-t|\geqslant\delta} |\boldsymbol{x}(\tfrac{k}{n}) - \boldsymbol{x}(t)|\boldsymbol{p}_{n,k}(t) \\
&\leqslant \omega_\delta(\boldsymbol{x}) \sum_{k:|\frac{k}{n}-t|<\delta} \boldsymbol{p}_{n,k}(t) + 2\|\boldsymbol{x}\|_\infty \tfrac{1}{4n\delta^2} \leqslant \omega_\delta(\boldsymbol{x}) \cdot 1 + \tfrac{\|\boldsymbol{x}\|_\infty}{2n\delta^2}.
\end{aligned}$$

Let $\epsilon > 0$. Since $\boldsymbol{x}$ is 'uniformly continuous'[6], we can choose $\delta > 0$ such that $\omega_\delta(\boldsymbol{x}) < \epsilon/2$. Next choose $n > \|\boldsymbol{x}\|_\infty/(\epsilon\delta^2)$. Then it follows from the above that $\|B_n\boldsymbol{x} - \boldsymbol{x}\|_\infty < \epsilon$, completing the proof of the Weierstrass Approximation Theorem.

Exercise 1.44 (Separable spaces)**.** Recall that if S is an infinite set, then S is said to be *countable* if there is a bijective map from $\mathbb{N}$ onto S. If S is not countable, it is called *uncountable*. The set $\mathbb{Q}$ is countable; see the MA103 notes. On the other hand, the set A consisting of all $\{0,1\}$-valued sequences, is uncountable. (Indeed, if there exists an enumeration $f_1, f_2, f_3, \cdots$ of these sequences, we arrive at a contradiction by constructing an $f \in A$ which differs from each of these sequences: For $n \in \mathbb{N}$, set

$$f(n) = \begin{cases} 0 & \text{if } f_n(n) = 1, \\ 1 & \text{if } f_n(n) = 0. \end{cases}$$

Then $f \neq f_1$ since $f(1) \neq f_1(1)$, $f \neq f_2$ since $f(2) \neq f_2(2)$, $f \neq f_3$ since $f(3) \neq f_3(3)$, and so on, showing that f differs from each of $f_1, f_2, f_3, \cdots$, a contradiction.)

A metric space (X, d) is *separable* if it has a countable dense set, i.e., there exists $D := \{x_1, x_2, x_3, \cdots\} \subset X$ such that for every $r > 0$ and every $x \in X$, there exists an $x_n \in D$ such that $d(x_n, x) < r$. For example $\mathbb{R}$ is separable, since we can simply take $D = \mathbb{Q}$. Show that ℓ^∞ from Exercise 1.21 is not separable.

Hint: Consider the set $A \subset \ell^\infty$ of all sequences with each term equal to 0 or 1. Then the distance between any two distinct elements of A is at least 1. If $D = \{g_1, g_2, g_3, \cdots\}$ is a dense subset of ℓ^∞, then obtain an injective map from A to $\mathbb{N}$ by considering the ball $B(f, \frac{1}{3})$ for each $f \in A$. This contradicts the uncountability of A.

[5]The Archimedean property of $\mathbb{R}$ says that if $x, y \in \mathbb{R}$ and $x > 0$, then there exists an $n \in \mathbb{N}$ such that $y < nx$. See the notes for MA103.

[6]We will learn about uniform continuity in Chapter 4. Here $\boldsymbol{x} : [0,1] \to \mathbb{R}$ is *uniformly continuous* if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $t, s \in [0,1]$ satisfying $|t - s| < \delta$, there holds that $|\boldsymbol{x}(t) - \boldsymbol{x}(s)| < \epsilon$. We will learn Proposition 4.69, which implies that every continuous function on $[0,1]$ is uniformly continuous.

Exercise 1.45. A subset C of a normed space X is called *convex* if for all $x, y \in C$, and all $t \in (0,1)$, $(1-t)x + ty \in C$. (Geometrically, this means that for any pair of points in C, the 'line segment' joining them also lies in C.)

(1) Show that the open ball $B(\mathbf{0}, r)$ with centre $\mathbf{0} \in X$ and radius $r > 0$ is convex.
(2) Is the unit circle $\mathbb{S}^1 = \{x \in \mathbb{R}^2 : \|x\|_2 = 1\}$ a convex set in $\mathbb{R}^2$?
(3) Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Prove that the 'Linear Programming Simplex'[7]

$$\Sigma := \{x \in \mathbb{R}^n : Ax = b,\ x_1 \geqslant 0,\ \ldots,\ x_n \geqslant 0\}$$

is a convex set in $\mathbb{R}^n$.

Exercise 1.46. Define $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by

$$d(x,y) = \begin{cases} \|x\|_2 + \|y\|_2 & \text{if } x \neq y, \\ 0 & \text{if } x = y, \end{cases} \qquad (x, y \in \mathbb{R}^2).$$

We call this metric the 'express railway metric'. (For example in the British context, to get from A to B, travel via London, the origin.)

Show that the express railway metric is a metric on $\mathbb{R}^2$.

Exercise 1.47. Let (X, d) be a metric space, $x \in X$, $R > 0$. Show that $\overline{B(x,R)} := \{y \in X : d(y,x) \leqslant R\}$ is a closed set.

1.4. Notes (not part of the course)

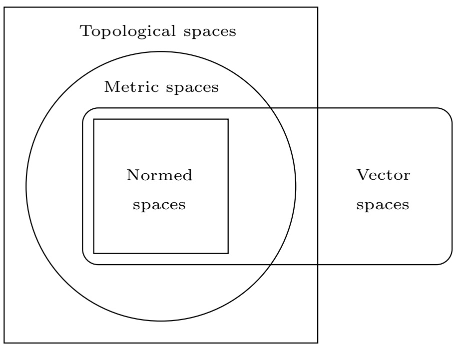
Topology. If we look at the collection $\mathcal{O}$ of open sets in a metric space (X, d), we notice that it has the following three properties:

(T1) $\varnothing, X \in \mathcal{O}$.

(T2) If U_i $(i \in I)$ is family of sets from $\mathcal{O}$ indexed by I, then $\bigcup_{i\in I} U_i \in \mathcal{O}$.

(T3) If $U_1, \ldots, U_n$ is a finite collection of sets from $\mathcal{O}$, then $\bigcap_{i=1}^{n} U_i \in \mathcal{O}$.

More generally, if X is any set (not necessarily one equipped with a metric), then any collection $\mathcal{O}$ of subsets of X that satisfy the properties (T1), (T2), (T3) is called a *topology on* X and $(X, \mathcal{O})$ is called a *topological space*. So for a metric space X, if we take $\mathcal{O}$ to be family of open sets in X, then we obtain a topological space. More generally, if one has a topological space $(X, \mathcal{O})$ given by the topology $\mathcal{O}$, we call each element of $\mathcal{O}$ *open*.



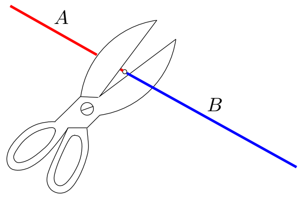
It turns out that one can in fact extend some of the notions from Real Analysis (such as convergence of sequences and continuity of maps) in the even more general set up of topological spaces, devoid of any metric, where the notion of closeness is specified by considering arbitrary open neighbourhoods provided by elements of $\mathcal{O}$. In some applications this is exactly the right thing needed, but we will not go into such abstractions in this course. In fact, this is a very broad subdiscipline of mathematics called *Topology*.

[7]This set arises as the 'feasible set' in a certain optimisation problem in $\mathbb{R}^n$, where the constraints are described by a bunch of linear inequalities.

Construction of the set of real numbers. In these notes, we treat the real number system $\mathbb{R}$ as a given. But one might wonder if we can take the existence of real numbers on faith alone. It turns out that a mathematical proof of its existence can be given. Roughly, we are already familiar with the natural numbers, the integers, and the rational numbers, and their rigorous mathematical construction is also relatively straightforward. However, the set $\mathbb{Q}$ of rational numbers has 'holes' (for example in MA103 we have seen that this manifests itself in the fact that $\mathbb{Q}$ does not possess the least upper bound property). The set of real numbers $\mathbb{R}$ is obtained by 'filling these holes'. There are several ways of doing this. One is by a general method called 'completion of metric spaces'. Another way, which is more intuitive, is via '(Dedekind) cuts', where we view real numbers as places where a line may be cut with scissors. More precisely, a *cut* $A|B$ in $\mathbb{Q}$ is a pair of subsets A, B of $\mathbb{Q}$ such that $A \bigcup B = \mathbb{Q}$, $A \neq \varnothing$, $B \neq \varnothing$, $A \bigcap B = \varnothing$, if $a \in A$ and $b \in B$ then $a < b$, and A contains no largest element. $\mathbb{R}$ is then taken as the set of all cuts $A|B$. Here are two examples of cuts:

$$\begin{aligned} A|B &= \{r \in \mathbb{Q} : r < 1\}|\{r \in \mathbb{Q} : r \geqslant 1\} \\ A|B &= \{r \in \mathbb{Q} : r \leqslant 0 \text{ or } r^2 < 2\}|\{r \in \mathbb{Q} : r > 0 \text{ and } r^2 \geqslant 2\}. \end{aligned}$$

It turns out that $\mathbb{R}$ is a field containing $\mathbb{Q}$, and it possesses the least upper bound property. The interested reader is referred to the Appendix to Chapter 1 in the classic textbook by Walter Rudin [**R**].



Although it is not a part of the course, we give the construction of $\mathbb{R}$ via the completion of $\mathbb{Q}$ in an Appendix (pp.101–111) to these notes.

Chapter 2

Sequences

In this chapter we study sequences in metric spaces. The notion of a convergent sequence is an important concept in Analysis. Besides its theoretical importance, it is also a natural concept arising in applications when one talks about better and better approximations to the solution of a problem using a numerical scheme. For example the method of Archimedes for finding the area of a circle by sandwiching it between the areas of a circumscribed and an inscribed regular polygon of ever increasing number of sides. There are also numerical schemes for finding a minimiser of a convex function (Newton's method), or for finding a solution to an ordinary differential equation (Euler's method), where convergence in more general metric spaces (such as $\mathbb{R}^n$ or $C[a, b]$) will play a role.

Before proceeding onto sequences in general metric spaces, let us first begin with (numerical) sequences in $\mathbb{R}$.

2.1. Sequences in $\mathbb{R}$

Let us recall the definition of a convergent sequence of real numbers.

Definition 2.1. A sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers is said to be *convergent with limit* $L \in \mathbb{R}$ if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $n > N$, $|a_n - L| < \epsilon$.

We have learnt that the limit of a convergent sequence $(a_n)_{n\in\mathbb{N}}$ is unique, and we denote it by

$$\lim_{n\to\infty} a_n.$$

We have also learnt the following important result[1]:

Theorem 2.2 (Bolzano-Weierstrass Theorem)**.** *Every bounded sequence of real numbers has a convergent subsequence.*

An important consequence of this result is the fact that in $\mathbb{R}$, the set of convergent sequences coincides with the set of Cauchy sequences. Let us first recall the definition of a Cauchy sequence.

Definition 2.3. A sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers is said to be a *Cauchy sequence* if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $m, n > N$, $|a_n - a_m| < \epsilon$.

Roughly speaking, we can make the terms of the sequence arbitrarily close to each other provided we go far enough in the sequence.

[1]See the MA103 lecture notes.

Example 2.4. The sequence $(\frac{1}{n})_{n\in\mathbb{N}}$ is Cauchy. Indeed, we have $|\frac{1}{n}-\frac{1}{m}| \leqslant \frac{1}{n}+\frac{1}{m} < \frac{1}{N}+\frac{1}{N} = \frac{2}{N}$ whenever $n, m > N$. Thus given $\epsilon > 0$, we can choose $N \in \mathbb{N}$ larger than $\frac{2}{\epsilon}$ so that we then have $|\frac{1}{n}-\frac{1}{m}| < \frac{2}{N} < \epsilon$ for all $n, m > N$. Consequently, $(\frac{1}{n})_{n\in\mathbb{N}}$ is Cauchy. ◇

Exercise 2.5. Show that if $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, then $(a_{n+1}-a_n)_{n\in\mathbb{N}}$ converges to 0.

Example 2.6. This example shows that for a sequence $(a_n)_{n\in\mathbb{N}}$ to be Cauchy, it is not enough that $(a_{n+1}-a_n)_{n\in\mathbb{N}}$ converges to 0. Take $a_n := \sqrt{n}$ $(n \in \mathbb{N})$. Then

$$a_{n+1}-a_n = \sqrt{n+1}-\sqrt{n} = \tfrac{1}{\sqrt{n+1}+\sqrt{n}} \stackrel{n\to\infty}{\longrightarrow} 0,$$

but $(a_n)_{n\in\mathbb{N}}$ is not Cauchy, since for any $n \in \mathbb{N}$, $|a_{4n}-a_n| = \sqrt{4n}-\sqrt{n} = \sqrt{n} \geqslant 1$. ◇

The next result says that Cauchyness is a *necessary* condition for a sequence to be convergent.

Lemma 2.7. *Every convergent sequence is Cauchy.*

Proof. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers that converges to L. Let $\epsilon > 0$. Then there exists an $N \in \mathbb{N}$ such that $|a_n - L| < \frac{\epsilon}{2}$. Thus for $n, m > N$, we have

$$|a_n - a_m| = |a_n - L + L - a_m| \leqslant |a_n - L| + |a_m - L| < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon.$$

So the sequence $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. □

Now we will prove the remarkable fact in $\mathbb{R}$, Cauchyness turns out to be also a *sufficient* condition for the sequence to be convergent. In other words, in $\mathbb{R}$, every Cauchy sequence is convergent. This is a very useful fact since, in order to prove that a sequence is convergent using the definition, we would need to guess what the limit is. In contrast, checking whether or not a sequence is Cauchy needs only knowledge of the terms of the sequence, and no guesswork regarding the limit is needed. So this is a powerful technique for proving existence results.

Theorem 2.8. *Every Cauchy sequence in $\mathbb{R}$ is convergent.*

Proof. There are three main steps. First we show that every Cauchy sequence is bounded. Then we use the Bolzano-Weierstrass theorem to conclude that it must have a convergent subsequence. Finally we show that a Cauchy sequence having a convergent subsequence must itself be convergent.

Step 1. Suppose that $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Choose any positive ϵ, say $\epsilon = 1$. Then there exists an $N \in \mathbb{N}$ such that for all $n, m > N$, $|a_n - a_m| < \epsilon$. In particular, with $m = N+1 > N$, and $n > N$, $|a_n - a_{N+1}| < \epsilon$. Hence by the triangle inequality, for all $n > N$,

$$|a_n| = |a_n - a_{N+1} + a_{N+1}| \leqslant |a_n - a_{N+1}| + |a_{N+1}| < 1 + |a_{N+1}|.$$

On the other hand, for $n \leqslant N$, $|a_n| \leqslant \max\{|a_1|, \dots, |a_N|, |a_{N+1}|+1\} =: M$. Consequently, $|a_n| \leqslant M$ for all $n \in \mathbb{N}$, that is, the sequence $(a_n)_{n\in\mathbb{N}}$ is bounded.

Step 2. By the Bolzano-Weierstrass Theorem, the bounded sequence $(a_n)_{n\in\mathbb{N}}$ has a convergent subsequence $(a_{n_k})_{k\in\mathbb{N}}$ that is convergent, to L, say.

Step 3. Finally we show that $(a_n)_{n\in\mathbb{N}}$ is also convergent with limit L. Let $\epsilon > 0$. Then there exists an $N \in \mathbb{N}$ such that for all $n, m > N$,

$$|a_n - a_m| < \tfrac{\epsilon}{2}. \tag{2.1}$$

Also, since $(a_{n_k})_{k\in\mathbb{N}}$ converges to L, we can find[2] an $n_K > N$ such that $|a_{n_K} - L| < \frac{\epsilon}{2}$. Taking $m = n_K$ in (2.1), for all $n > N$, $|a_n - L| = |a_n - a_{n_K} + a_{n_K} - L| \leqslant |a_n - a_{n_K}| + |a_{n_K} - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Thus $(a_n)_{n\in\mathbb{N}}$ is also convergent with limit L, and this completes the proof. □

[2] If $n_1 < n_2 < n_3 < \cdots$ is a strictly increasing sequence of natural numbers, then $n_k \geqslant k$. (Indeed, $n_1 \geqslant 1$, and if $n_k \geqslant k$ for some $k \in \mathbb{N}$, then $n_{k+1} > n_k \geqslant k$ gives $n_{k+1} \geqslant k+1$, and the claim follows by induction.) So here, we if K' is such that for $k > K'$, $|a_{n_k} - L| < \frac{\epsilon}{2}$, we may take $K = \max\{K', N\} + 1$ (since $n_K \geqslant K \geqslant N+1 > N$ and $n_K > K'$).

Exercise 2.9. Determine if the following statements are true or false. Give reasons for your answers.

(1) Every subsequence of a convergent real sequence is convergent.

(2) Every subsequence of a divergent real sequence is divergent.

(3) Every subsequence of a bounded real sequence is bounded.

(4) Every subsequence of an unbounded real sequence is unbounded.

(5) Every subsequence of a monotone real sequence is monotone.

(6) Every subsequence of a nonmonotone real sequence is nonmonotone.

(7) If every subsequence of a real sequence converges, the sequence itself converges.

(8) If for a real sequence $(a_n)_{n\in\mathbb{N}}$, the sequences $(a_{2n})_{n\in\mathbb{N}}$ and $(a_{2n+1})_{n\in\mathbb{N}}$ both converge, then $(a_n)_{n\in\mathbb{N}}$ converges.

(9) If for a real sequence $(a_n)_{n\in\mathbb{N}}$, the sequences $(a_{2n})_{n\in\mathbb{N}}$ and $(a_{2n+1})_{n\in\mathbb{N}}$ both converge to the same limit, then $(a_n)_{n\in\mathbb{N}}$ converges.

Exercise 2.10. Fill in the blanks in the following proof of the fact that *every bounded increasing sequence of real numbers converges.*

Let $(a_n)_{n\in\mathbb{N}}$ be a bounded increasing sequence of real numbers. Let M be the ______ upper bound of the set $\{a_n : n \in \mathbb{N}\}$. The existence of M is guaranteed by the ______ of the set of real numbers. We show that M is the ______ of $(a_n)_{n\in\mathbb{N}}$. Taking $\epsilon > 0$, we must show that there exists a positive integer N such that ______ for all $n > N$. Since $M - \epsilon < M$, $M - \epsilon$ is not ______ of $\{a_n : n \in \mathbb{N}\}$. Therefore there exists N with ______ $\geqslant a_N >$ ______. Since $(a_n)_{n\in\mathbb{N}}$ is ______, $|a_n - M| < \epsilon$ for all $n \geqslant N$. □

Exercise 2.11 (Euler's constant, e)**.** Consider the sequence $(a_n)_{n\in\mathbb{N}}$, where $a_n := 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$, $n \in \mathbb{N}$. Then $(a_n)_{n\in\mathbb{N}}$ is increasing, as $a_{n+1} - a_n = \frac{1}{(n+1)!} > 0$ for all $n \in \mathbb{N}$.

(1) Show that $(a_n)_{n\in\mathbb{N}}$ is bounded.

Hint: $a_n = 1 + 1 + \frac{1}{2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{2\cdot 3\cdots n} \leqslant 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} = 1 + \frac{1-\frac{1}{2^n}}{1-\frac{1}{2}} < 3.$

As $(a_n)_{n\in\mathbb{N}}$ is monotone and bounded, it is convergent, and we set $a := \lim\limits_{n\to\infty} a_n$.

Next, consider the sequence $(b_n)_{n\in\mathbb{N}}$, where $b_n := (1 + \frac{1}{n})^n$, $n \in \mathbb{N}$. Using the Binomial Theorem,

$$\begin{aligned} b_n &= 1 + n\tfrac{1}{n} + \tfrac{n(n-1)}{2!}\tfrac{1}{n^2} + \cdots + \tfrac{n(n-1)\cdots 2\cdot 1}{n!}\tfrac{1}{n^n} \\ &= 1 + 1 + \tfrac{1}{2!}(1 - \tfrac{1}{n}) + \cdots + \tfrac{1}{n!}(1 - \tfrac{1}{n})\cdots(1 - \tfrac{n-1}{n}). \end{aligned}$$

(2) Show by replacing n by $n + 1$ in factors of the type $(1 - \frac{k}{n})$ that $b_n \leqslant b_{n+1}$, $n \in \mathbb{N}$.

(3) Show that $b_n \leqslant a_n < 3$.

As $(b_n)_{n\in\mathbb{N}}$ is monotone and bounded, it is convergent, and we set $b := \lim\limits_{n\to\infty} b_n$.

(4) Fix $m \in \mathbb{N}$. Show that for $n \geqslant m$, $b_n \geqslant 1 + 1 + \frac{1}{2!}(1 - \frac{1}{n}) + \cdots + \frac{1}{m!}(1 - \frac{1}{n})\cdots(1 - \frac{m-1}{n})$.

(5) Conclude, by passing to the limit as $n \to \infty$ in the result from (4), that $b \geqslant a_m$. Show that $b \geqslant a$.

(6) Use part (3) to conclude that $b \leqslant a$. From parts (5) and (6), we get $b = a$.

We call this number *Euler's number*, denoted by $e \in \mathbb{R}$: $\lim\limits_{n\to\infty}(1 + \frac{1}{n})^n = e = \lim\limits_{n\to\infty}(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!})$.

Exercise 2.12. For each of the following sequences, determine whether it converges or not, and find the limit in case of convergence. Give reasons for your answers.

(1) $(\cos(\pi n))_{n\in\mathbb{N}}$ (2) $(1 + n^2)_{n\in\mathbb{N}}$ (3) $(\frac{\sin n}{n})_{n\in\mathbb{N}}$ (4) $(1 - \frac{3n^2}{n+1})_{n\in\mathbb{N}}$ (5) $(n^{\frac{1}{n}})_{n\in\mathbb{N}}$ (6) $0.9, 0.99, 0.999, \cdots$.

Exercise 2.13. Let $(a_n)_{n\in\mathbb{N}}$ be bounded. Define $\ell_k = \inf\{a_n : n \geqslant k\}$ and $u_k = \sup\{a_n : n \geqslant k\}$ $(k \in \mathbb{N})$.

(1) Show that $(\ell_n)_{n\in\mathbb{N}}$, $(u_n)_{n\in\mathbb{N}}$ are bounded and monotone, and hence convergent. Their respective limits are called the *limit superior* and *limit inferior*, respectively, and denoted by $\liminf\limits_{n\to\infty} a_n$ and $\limsup\limits_{n\to\infty} a_n$.

(2) Show that $\liminf\limits_{n\to\infty} a_n \leqslant \limsup\limits_{n\to\infty} a_n$. Given an example to show that there can be a strict inequality.

(3) Prove that $(a_n)_{n\in\mathbb{N}}$ is convergent if and only if $\liminf\limits_{n\to\infty} a_n = \limsup\limits_{n\to\infty} a_n$.

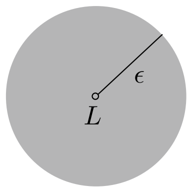
Moreover, then $\lim\limits_{n\to\infty} a_n = \liminf\limits_{n\to\infty} a_n = \limsup\limits_{n\to\infty} a_n$.

2.2. Sequences in metric spaces

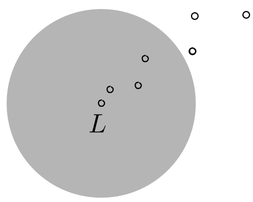
We now give the notion of convergence of a sequence in a general metric space. We will see that essentially the definition is the same as in $\mathbb{R}$, except that instead of having the distance between the n^{th} term and the limit L given by $|a_n - L|$, now we will replace it by $d(a_n, L)$ in a general metric space with metric d.

Definition 2.14. A sequence $(a_n)_{n\in\mathbb{N}}$ of points in a metric space (X, d) is said to be *convergent with limit* $L \in X$ if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $n > N$, $d(a_n, L) < \epsilon$.

Let us understand this definition pictorially. We have been given a sequence $(a_n)_{n\in\mathbb{N}}$ of points and a candidate L for its limit. We are allowed to say that this sequence converges to L if given any $\epsilon > 0$, that is, no matter how small a ball we consider around L,



there is an index N such that all the terms of the sequence beyond this index lie inside the ball.



Lemma 2.15. *The limit of a convergent sequence in a metric space is unique.*

Proof. Suppose that $(a_n)_{n\in\mathbb{N}}$ is a convergent sequence, and let it have two distinct limits L_1 and L_2. Then $d(L_1, L_2) > 0$. Set

$$\epsilon = \frac{1}{2} d(L_1, L_2) > 0.$$

Then there exists an N_1 such that for all $n > N_1$, $d(a_n, L_1) < \epsilon$. Also, there exists an N_2 such that for all $n > N_2$, $d(a_n, L_2) < \epsilon$. Hence for any $n > \max\{N_1, N_2\}$, we have

$$d(L_1, L_2) \leqslant d(L_1, a_n) + d(a_n, L_2) < \epsilon + \epsilon = d(L_1, L_2),$$

a contradiction. Thus the limit of $(a_n)_{n\in\mathbb{N}}$ is unique. $\square$

If $(a_n)_{n\in\mathbb{N}}$ is a convergent sequence, then we will denote its (unique) limit by $\lim\limits_{n\to\infty} a_n$.

Exercise 2.16. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in the Euclidean space $\mathbb{R}^d$. Show that $(a_n)_{n\in\mathbb{N}}$ is convergent with limit L if and only if for every $k \in \{1, \dots, d\}$, the sequence $(a_n^{(k)})_{n\in\mathbb{N}}$ in $\mathbb{R}$ formed by the k^{th} component of the terms of $(a_n)_{n\in\mathbb{N}}$ is convergent with limit $L^{(k)}$. (Here we use the notation $v^{(k)}$ for the k^{th} component of a vector $v \in \mathbb{R}^d$.)

Exercise 2.17. Consider the sequence $(a_n)_{n\in\mathbb{N}}$ in the Euclidean space $\mathbb{R}^2$:

$$a_n := \begin{bmatrix} \frac{n}{4n+2} \\ \frac{n^2}{n^2+1} \end{bmatrix} \quad (n \in \mathbb{N}).$$

Show that $(a_n)_{n\in\mathbb{N}}$ is convergent. What is its limit?

Exercise 2.18. Let X be a nonempty set equipped with the discrete metric. Show that a sequence $(a_n)_{n\in\mathbb{N}}$ is convergent if and only if it is eventually a constant sequence (that is, there is a $c \in X$ and an $N \in \mathbb{N}$ such that for all $n > N$, $a_n = c$).

Exercise 2.19. Let (X, d) be a metric space and let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be convergent sequences in X with limits a and b, respectively. Prove that $(d(a_n, b_n))_{n\in\mathbb{N}}$ is a convergent sequence in $\mathbb{R}$ with limit $d(a, b)$. *Hint:* $d(a_n, b_n) \leqslant d(a_n, a) + d(a, b) + d(b, b_n)$.

Exercise 2.20. Let $v_1 = (x_1, y_1) \in \mathbb{R}^2$ be such that $0 < x_1 < y_1$. Define

$$v_{n+1} = (x_{n+1}, y_{n+1}) = \left(\sqrt{x_n y_n}, \tfrac{x_n + y_n}{2}\right) \text{ for all } n \in \mathbb{N}.$$

(1) Show that $0 < x_n < x_{n+1} < y_{n+1} < y_n$ and that $y_{n+1} - x_{n+1} < y_{n+1} - x_n = \frac{y_n - x_n}{2}$.

(2) Conclude that $\lim\limits_{n\to\infty} v_n$ exists and equals (c, c) for some number $c \in \mathbb{R}$.
This value c is called the arithmetic-geometric mean[3], of x_1 and y_1, and is denoted by $\operatorname{agm}(x_1, y_1)$.

We can also define Cauchy sequences in a metric space analogous to the situation in $\mathbb{R}$.

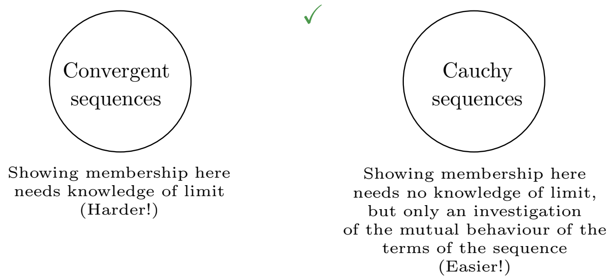
Definition 2.21. A sequence $(a_n)_{n\in\mathbb{N}}$ of points in a metric space (X, d) is said to be a *Cauchy sequence* if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $m, n > N$, $d(a_m, a_n) < \epsilon$.

Lemma 2.22. *Every convergent sequence is Cauchy.*

Proof. The proof is the same, mutatis mutandis[4], as the proof of Lemma 2.7. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of points in X that converges to $L \in X$. Let $\epsilon > 0$. Then there exists an $N \in \mathbb{N}$ such that $d(a_n, L) < \frac{\epsilon}{2}$. Thus for $n, m > N$, we have $d(a_n, a_m) \leqslant d(a_n, L) + d(L, a_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. So the sequence $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. ☐

In $\mathbb{R}$, we have seen that { convergent sequences } = { Cauchy sequences }. This raises the tempting question of whether this equality is true in general metric spaces too:

$$\{ \text{ convergent sequences } \} \;\underset{\subseteq\,\checkmark}{\overset{?\;\supseteq}{}}\; \{ \text{ Cauchy sequences } \}.$$



If the two sets coincide, then one can conclude that a sequence is convergent by just checking Cauchyness. This is the basis of many *existence* results in Analysis. For example, the convergence tests of series, the existence results for differential equations, etc. Once existence is known, (and after showing uniqueness, if valid), one can justify and use numerical approximations.

Unfortunately, the two sets do not always coincide. For example, consider the metric space $X = (0, 1]$ with the same Euclidean metric as in $\mathbb{R}$. Then the sequence $(\frac{1}{n})_{n\in\mathbb{N}}$ is easily seen to be Cauchy, but is not convergent in X, as there is a missing point in X, namely 0. However, in some other metric spaces, such as $\mathbb{R}$, the set of convergent sequences and the set of Cauchy sequences do coincide. So it makes sense to give such metric spaces a special name: they are called 'complete'.

Definition 2.23. A metric space in which every Cauchy sequence converges is called *complete*.

[3]Gauss observed that $I(a, b) := \int_{-\infty}^{\infty} \frac{1}{\sqrt{(x^2+a^2)(y^2+b^2)}} dx$ satisfies $I(a, b) = I(\frac{a+b}{2}, \sqrt{ab})$ with the help of the substitution $t = \frac{1}{2}(x - \frac{ab}{x})$, and using this, obtained the remarkable result that $\int_{-\infty}^{\infty} \frac{1}{\sqrt{(x^2+a^2)(y^2+b^2)}} dx = \frac{\pi}{\operatorname{agm}(a,b)}$.

[4]Latin phrase meaning 'by changing those things which need to be changed'.

Example 2.24. $\mathbb{R}$ with the Euclidean metric is complete. $\diamond$

Exercise 2.25. Let $X = (0, 1]$ be equipped with the same Euclidean metric as in $\mathbb{R}$. Show that the sequence $(\frac{1}{n})_{n\in\mathbb{N}}$ does not converge in X.

Exercise 2.26. Show that $\mathbb{Q}$ with the Euclidean metric is not complete. *Hint:* Revisit the solution to part (6) of Exercise 1.37.

Exercise 2.27. Let X be a metric space. If $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X which has a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$ with limit L, then show that $(x_n)_{n\in\mathbb{N}}$ is convergent with the same limit L.

Theorem 2.28. *$\mathbb{R}^d$ is complete.*

Proof. (Essentially, this is because $\mathbb{R}$ is complete, and one has d copies of $\mathbb{R}$ in $\mathbb{R}^d$.) Suppose that $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}^d$:

$$a_n = \begin{bmatrix} x_n^{(1)} \\ \vdots \\ x_n^{(d)} \end{bmatrix}.$$

We have $|x_n^{(k)} - x_m^{(k)}| \leqslant \|a_n - a_m\|_2$ ($n, m \in \mathbb{N}$, $k = 1, \dots, d$), from which it follows that each of the sequences $(x_n^{(k)})_{n\in\mathbb{N}}$, $k = 1, \dots, d$, is Cauchy in $\mathbb{R}$, and hence convergent, with respective limits, say $L^{(1)}, \dots, L^{(d)} \in \mathbb{R}$. So given $\epsilon > 0$, there exists a large enough N such that whenever $n > N$, we have $|x_n^{(k)} - L^{(k)}| < \frac{\epsilon}{\sqrt{d}}$, $k \in \{1, \dots, d\}$. Set

$$L = \begin{bmatrix} L^{(1)} \\ \vdots \\ L^{(d)} \end{bmatrix} \in \mathbb{R}^d.$$

Thus for $n > N$, $\|a_n - L\|_2 = \sqrt{\sum_{k=1}^{d} |x_n^{(k)} - L^{(k)}|^2} < \sqrt{\sum_{k=1}^{d} \frac{\epsilon^2}{d}} = \epsilon$. So $(a_n)_{n\in\mathbb{N}}$ converges to L. $\square$

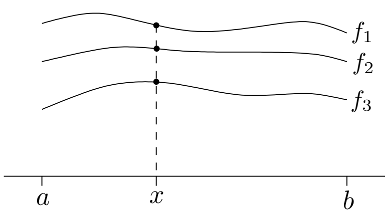
Exercise 2.29. $\mathbb{R}^{m\times n}$ with the metric induced by $\|\cdot\|_\infty$ is complete. (See Exercise 1.15 for the definition of the norm $\|\cdot\|_\infty$ on the vector space $\mathbb{R}^{m\times n}$.)

Exercise 2.30. Recall the normed space ℓ^∞ from Exercise 1.21. Show that ℓ^∞ is complete with the metric induced by $\|\cdot\|_\infty$.

The theorem below is important, and lies at the core of a result on the existence of solutions for Ordinary Differential Equations (ODEs). You can learn more about this in the course Differential Equations (MA209). (See Exercise 1.16 for the definition of the norm $\|\cdot\|_\infty$ on $C[a, b]$.)

Theorem 2.31. *$C[a, b]$ with the metric induced by $\|\cdot\|_\infty$ is complete.*

Proof. (*You may skip this proof.*) The idea behind the proof is similar to the proof of the completeness of $\mathbb{R}^d$. If $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, then we think of the $f_n(x)$ as being the 'components' of f_n indexed by $x \in [a, b]$. We first freeze an $x \in [a, b]$, and show that $(f_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$, and hence convergent to a number (which depends on x), and which we denote by $f(x)$. Next we show that the function $x \mapsto f(x)$ is continuous, and finally that $(f_n)_{n\in\mathbb{N}}$ does converge to f.



The Cauchy sequence $(f_n(x))_{n\in\mathbb{N}}$ obtained from the Cauchy sequence $(f_n)_{n\in\mathbb{N}}$ by freezing x.

Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence. Let $x \in [a,b]$. We claim that $(f_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. Let $\epsilon > 0$. Then there exists an $N \in \mathbb{N}$ such that for all $n, m > N$, $\|f_n - f_m\|_\infty < \epsilon$. But

$$|f_n(x) - f_m(x)| \leqslant \max_{y\in[a,b]} |f_n(y) - f_m(y)| = \|f_n - f_m\|_\infty < \epsilon,$$

for $n, m > N$. This shows that indeed $(f_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. But $\mathbb{R}$ is complete, and so the Cauchy sequence $(f_n(x))_{n\in\mathbb{N}}$ is in fact convergent, with a limit which depends on which $x \in [a,b]$ we had frozen at the outset. To highlight this dependence on x, we denote the limit of $(f_n(x))_{n\in\mathbb{N}}$ by $f(x)$. (Thus $f(a)$ is the number which is the limit of the convergent sequence $(f_n(a))_{n\in\mathbb{N}}$, $f(b)$ is the number which is the limit of the convergent sequence $(f_n(b))_{n\in\mathbb{N}}$, and so on.) So we have a function from $[a,b]$ to $\mathbb{R}$, so that

x is sent to the number which is the limit of the convergent sequence $(f_n(x))_{n\in\mathbb{N}}$.

We call this function f. This will serve as the limit of the sequence $(f_n)_{n\in\mathbb{N}}$. But first we have to see if it belongs to $C[a,b]$, that is, we need to check that this f is continuous on $[a,b]$.

Let $x \in [a,b]$. We will show that f is continuous at x. Recall that in order to do this, we have to show that for each $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $|y - x| < \delta$, we have $|f(y) - f(x)| < \epsilon$. Let $\epsilon > 0$. Choose N large enough so that for all $n, m > N$,

$$\|f_n - f_m\|_\infty < \tfrac{\epsilon}{3}.$$

Let $y \in [a,b]$. Then for $n > N$, $|f_n(y) - f_{N+1}(y)| \leqslant \|f_n - f_{N+1}\|_\infty < \frac{\epsilon}{3}$. Now let $n \to \infty$:

$$|f(y) - f_{N+1}(y)| = \lim_{n\to\infty} |f_n(y) - f_{N+1}(y)| \leqslant \tfrac{\epsilon}{3}.$$

As the choice of $y \in [a,b]$ was arbitrary, we have for all $y \in [a,b]$ that

$$|f(y) - f_{N+1}(y)| \leqslant \tfrac{\epsilon}{3}.$$

Now $f_{N+1} \in C[a,b]$. So there exists a $\delta > 0$ such that whenever $|y - x| < \delta$, we have

$$|f_{N+1}(y) - f_{N+1}(x)| < \tfrac{\epsilon}{3}.$$

Thus whenever $|y - x| < \delta$, we have

$$\begin{aligned} |f(y) - f(x)| &= |f(y) - f_{N+1}(y) + f_{N+1}(y) - f_{N+1}(x) + f_{N+1}(x) - f(x)| \\ &\leqslant |f(y) - f_{N+1}(y)| + |f_{N+1}(y) - f_{N+1}(x)| + |f_{N+1}(x) - f(x)| \\ &\leqslant \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} = \epsilon. \end{aligned}$$

So f is continuous at x. As the choice of $x \in [a,b]$ was arbitrary, f is continuous on $[a,b]$.

Finally, we show that $(f_n)_{n\in\mathbb{N}}$ does converge to f. Let $\epsilon > 0$. Choose N large enough so that for all $n, m > N$, $\|f_n - f_m\|_\infty < \epsilon$. Fix $n > N$. Let $x \in [a,b]$. Then for all $m > N$, $|f_n(x) - f_m(x)| \leqslant \|f_n - f_m\|_\infty < \epsilon$. Thus

$$|f_n(x) - f(x)| = \lim_{m\to\infty} |f_n(y) - f_m(y)| \leqslant \epsilon.$$

But $x \in [a,b]$ was arbitrary. Hence

$$\|f_n - f\|_\infty = \max_{x\in[a,b]} |f_n(x) - f(x)| \leqslant \epsilon.$$

But we could have fixed any $n > N$ at the outset and obtained the same result. So we have that for all $n > N$, $\|f_n - f\|_\infty \leqslant \epsilon$. Thus $\lim\limits_{n\to\infty} f_n = f$, and this completes the proof. □

The norm $\|\cdot\|_\infty$ is special in that $C[a,b]$ is complete with the corresponding induced metric. It turns out that $C[a,b]$ with the other natural norm met earlier, namely the $\|\cdot\|_1$-norm, is not complete. The objective in the following exercise is to demonstrate this.

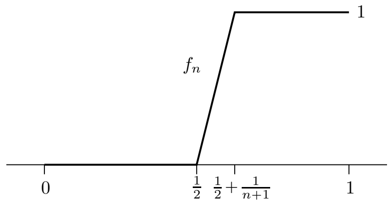
Exercise 2.32. Let $C[0,1]$ be equipped with the $\|\cdot\|_1$-norm given by $\|f\|_1 := \int_0^1 |f(x)|dx$ $(f \in C[0,1])$. Show that the corresponding metric space is not complete. For example, you may consider the sequence $(f_n)_{n\in\mathbb{N}}$ with the f_n as shown in Figure 2.2. Show that for $n, m > N$,

$$\|f_n - f_m\|_1 = \int_{\frac{1}{2}}^{\frac{1}{2}+\max\{\frac{1}{n+1},\frac{1}{m+1}\}} |f_n(x) - f_m(x)|dx \leqslant \frac{2}{N},$$

and so $(f_n)_{\in\mathbb{N}}$ is Cauchy. Prove that if $(f_n)_{n\in\mathbb{N}}$ converges to $f \in C[0,1]$, then f must satisfy

$$f(x) = \begin{cases} 0 & \text{for } x \in [0, \frac{1}{2}], \\ 1 & \text{for } x \in (\frac{1}{2}, 1], \end{cases}$$

which does not belong to $C[0,1]$, a contradiction.



Exercise 2.33. Show that any nonempty set X equipped with the discrete metric is complete.

Exercise 2.34. Prove that $\mathbb{Z}$ equipped with the Euclidean metric induced from $\mathbb{R}$ is complete.

2.3. Pointwise and uniform convergence

Convergence in $(C[a,b], \|\cdot\|_\infty)$ is referred to as *uniform convergence*. More generally, we have the following definition.

Definition 2.35. Let X be any set and $f, f_n : X \to \mathbb{R}$ $(n \in \mathbb{N})$ be functions.

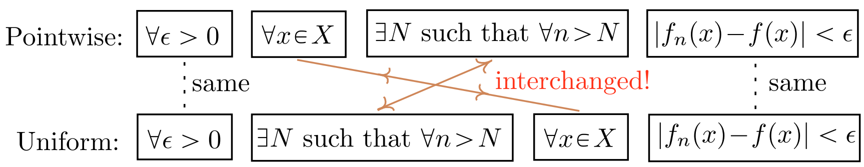
(1) The sequence $(f_n)_{n\in\mathbb{N}}$ is said to *converge uniformly to* f if

$$\forall \epsilon > 0, \ \exists N \in \mathbb{N} \text{ such that } \forall n > N, \ \forall x \in X, \ |f_n(x) - f(x)| < \epsilon.$$

(2) The sequence $(f_n)_{n\in\mathbb{N}}$ is said to *converge pointwise to* f if

$$\forall \epsilon > 0, \ \forall x \in X, \ \exists N \in \mathbb{N} \text{ such that } \forall n > N, \ |f_n(x) - f(x)| < \epsilon.$$

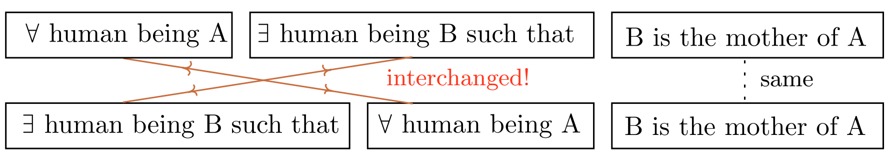
Pointwise versus uniform convergence. We now highlight the difference between pointwise and uniform convergence:



The difference between the two statements is the order of

$$\boxed{\forall x \in X} \text{ and } \boxed{\exists N \in \mathbb{N} \text{ such that } \forall n > N}.$$

Order of the phrases 'for every' and ' there exists' (called quantifiers) matters in mathematical statements. This seemingly small change of interchanging the order of quantifiers makes a world of difference. Indeed, even in everyday language, the two statements:



mean totally different things! In the latter, there is a person who is the mother to all human beings, a statement which is obviously false. The former statement is true, since it asserts for every person A we take, there exists (depending on *which* person A we have chosen) another person B who is the mother of A.

This is the same sort of a difference between the uniform convergence requirement, namely:

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } \forall n > N,\ \forall x \in X,\ |f_n(x) - f(x)| < \epsilon.$$

and the pointwise convergence requirement, namely

$$\forall \epsilon > 0,\ \forall x \in X,\ \exists N \in \mathbb{N} \text{ such that } \forall n > N,\ |f_n(x) - f(x)| < \epsilon.$$

In the former, the same N works for all $x \in X$, while in the latter, the N might depend on the x in question.

It is clear that if f_n converges uniformly to f, then f_n converges pointwise to f. (Indeed, if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n > N$ and for *all* $x \in X$, $|f_n(x) - f(x)| < \epsilon$, and we take *any particular fixed* $x_* \in X$, then also, we have that for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n > N$, $|f_n(x_*) - f(x_*)| < \epsilon$: In other words, for this $x_* \in X$,

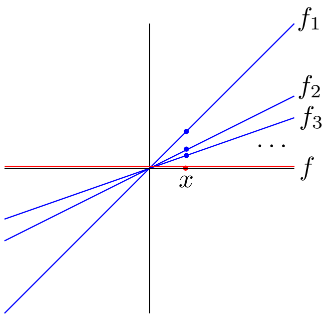
$$\lim_{n\to\infty} f_n(x_*) = f(x_*).$$

But the choice of $x_* \in X$ was arbitrary. So

$$\forall x \in X, \quad \lim_{n\to\infty} f_n(x) = f(x).$$

Hence $(f_n)_{n\in\mathbb{N}}$ converges pointwise to f.) But there are pointwise convergent sequences of functions which do not converge uniformly. Here is an example to illustrate this.

Example 2.36. Let $X = \mathbb{R}$, and for $x \in X = \mathbb{R}$, let $f(x) = 0$ and $f_n(x) = \frac{x}{n}$ $(n \in \mathbb{N})$. The picture below shows the graphs of the functions.



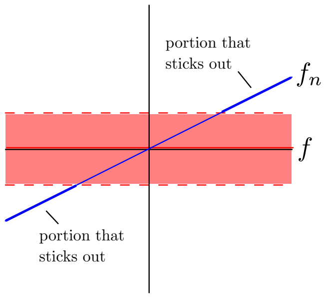
It is clear that if we fix any $x \in \mathbb{R}$, then

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{n} = x \lim_{n\to\infty} \frac{1}{n} = x \cdot 0 = 0 = f(x).$$

So $(f_n)_{n\in\mathbb{N}}$ converges pointwise to 0. Let us have a closer look at this. Let us fix an $x \in \mathbb{R}$. Let $\epsilon > 0$ be given. Take $N \in \mathbb{N}$ such that $N > \frac{|x|+1}{\epsilon}$. Then for $n > N$,

$$|f_n(x) - f(x)| = |\tfrac{x}{n} - 0| = \tfrac{|x|}{n} \leqslant \tfrac{|x|}{N} < \tfrac{|x|\epsilon}{|x|+1} < \epsilon.$$

Note that the N we required to guarantee that $[n > N \quad \Rightarrow \quad |f_n(x) - f(x)| < \epsilon]$ depends on the x fixed at the outset. (An $N < \frac{|x|}{\epsilon}$ won't do here!) In fact, From the picture below, it is visibly clear that $(f_n)_{n\in\mathbb{N}}$ does not converge uniformly to f. Indeed, whatever width of strip we look at around the graph of f, and no matter which n we take, it is not the case that the graph of f_n lies entirely inside the strip— some portion of the graph of f_n always 'sticks out'.



Here is a rigorous proof. Suppose that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f. Let $\epsilon = 1 > 0$. Then there exists an $N \in \mathbb{N}$ such that for all $x \in \mathbb{R}$, and all $n > N$, $|f_n(x) - f(x)| < 1$. Take $x = 2N+2$. Then the above gives us that for all $n > N$, $|\frac{2N+2}{n} - 0| < 1$. In particular, for $n = N+1$, $\frac{2N+2}{N+1} = 2 < 1$, that is, $2 < 1$, a contradiction! ◇

Example 2.37. Let $X = \mathbb{R}$, and for $x \in \mathbb{R}$, let $f_n(x) = \frac{\sin(nx)}{n}$ $(n \in \mathbb{N})$. Clearly for each $x \in \mathbb{R}$ we have $-\frac{1}{n} \leqslant \frac{\sin(nx)}{n} \leqslant \frac{1}{n}$ $(n \in \mathbb{N})$, and so by the Sandwich Theorem,

$$\lim_{n\to\infty} f_n(x) = 0 =: f(x),$$

where $f : \mathbb{R} \to \mathbb{R}$ is the constant function equal to 0 everywhere. So $(f_n)_{n\in\mathbb{N}}$ converges pointwise to f.

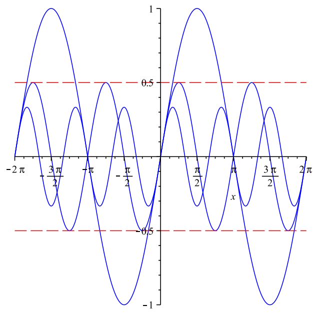


Figure 1. $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f: with $\epsilon = \frac{1}{2}$, we see that the graphs of $f_3, f_4, \cdots$ all lie in the strip of width ϵ about the graph of the zero function f.

Is the convergence uniform? We guess the answer is 'Yes', based on the Figure 1: Looking at a strip of an arbitrarily small width around the graph of the zero function f, it is clear that eventually the graphs of f_n lie in this strip. In fact, for all $x \in \mathbb{R}$, $|f_n(x) - f(x)| = \frac{|\sin(nx)|}{n} \leqslant \frac{1}{n}$. So given $\epsilon > 0$, if we choose $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon}$, then for $n > N$, we have that for all $x \in \mathbb{R}$, $|f_n(x) - f(x)| \leqslant \frac{1}{n} < \frac{1}{N} < \epsilon$. Hence, $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f. ◇

We know that if $(f_n)_{n\in\mathbb{N}}$ converges to f uniformly, then it converges pointwise to f. The next two exercises give a guide to investigate uniform convergence, knowing that $(f_n)_{n\in\mathbb{N}}$ is pointwise convergent:

(1) First for each $x \in X$, find $\lim\limits_{n\to\infty} f_n(x)$, and call the limit $f(x)$.

(2) Find a 'uniform bound' on $|f_n(x) - f(x)|$ (if it exists), namely $\sup\limits_{x\in X} |f_n(x) - f(x)| \leqslant a_n$. Then $(f_n)_{n\in\mathbb{N}}$ converges to f uniformly if $\lim\limits_{n\to\infty} a_n = 0$. See Exercise 2.38.

(3) If there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that $(|f_n(x_n) - f(x_n)|)_{n\in\mathbb{N}}$ does not converge to 0, then $(f_n)_{n\in\mathbb{N}}$ does not converge uniformly to f. See Exercise 2.39.

Exercise 2.38. Suppose that X is a nonempty set and $f_n : X \to \mathbb{R}$ $(n \in \mathbb{N})$ be a sequence which is pointwise convergent to $f : X \to \mathbb{R}$. Let the numbers $a_n := \sup\{|f_n(x) - f(x)| : x \in X\}$ $(n \in \mathbb{N})$ all exist. Prove that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f if and only if $\lim\limits_{n\to\infty} a_n = 0$.

Define $f_n : (0,\infty) \to \mathbb{R}$ by $f_n(x) = xe^{-nx}$, $x \in (0,\infty)$, $n \in \mathbb{N}$. Show that $(f_n)_{n\in\mathbb{N}}$ converges uniformly on $(0,\infty)$.

Exercise 2.39. Let X be a nonempty set and $f_n : X \to \mathbb{R}$ ($n \in \mathbb{N}$) be a sequence which is pointwise convergent to $f : X \to \mathbb{R}$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X such that $(|f_n(x_n) - f(x_n)|)_{n\in\mathbb{N}}$ does not converge to 0. Prove that $(f_n)_{n\in\mathbb{N}}$ does not converge uniformly to f.

For $n \in \mathbb{N}$, define $f_n : \mathbb{R} \to \mathbb{R}$ by $f_n(x) = 1$ if $x > n$, and $f_n(x) = 0$ if $x \leqslant n$. Show that $(f_n)_{n\in\mathbb{N}}$ converges pointwise to the function f which is 0 everywhere on $\mathbb{R}$. Prove that the convergence is not uniform.

Exercise 2.40. For $n \in \mathbb{N}$, let $f_n : [0, 1] \to \mathbb{R}$ be defined by $f_n(x) = \frac{x}{1+nx}$ ($x \in [0, 1]$). Does $(f_n)_{n\in\mathbb{N}}$ converge uniformly on $[0, 1]$?

Exercise 2.41. For $n \in \mathbb{N}$, let $f_n : (0, 1) \to \mathbb{R}$ be defined by $f_n(x) = x^n$, $x \in (0, 1)$.

(1) Does the sequence $(f_n)_{n\in\mathbb{N}}$ converge pointwise to some function?

(2) Is the convergence uniform?

(3) Sketch the graphs of the first few terms of $(f_n)_{n\in\mathbb{N}}$, and explain visually your answer to part (2) above.

Why bother with uniform convergence? Uniform convergence often implies that the limit function inherits the 'nice' properties possessed by the terms of the sequence. This is not guaranteed to happen if one has mere *pointwise* convergence. For instance, we will see later on that if a sequence $(f_n)_{n\in\mathbb{N}}$ of continuous functions f_n ($n \in \mathbb{N}$) converges uniformly to a function f, then f is also continuous; see Proposition 4.16. The reason nice things can happen with uniform convergence is that we can exchange two 'limiting processes', which is *not* always allowed when one just has pointwise convergence. The following exercises demonstrate the precariousness of exchanging limiting processes arbitrarily.

Exercise 2.42. Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = 1 - \frac{1}{(1+x^2)^n}$ ($x \in \mathbb{R}$, $n \in \mathbb{N}$). Show that the sequence $(f_n)_{n\in\mathbb{N}}$ of continuous functions converges pointwise to the function

$$f(x) = \begin{cases} 1 & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

which is discontinuous at 0.

Exercise 2.43. Let $a_{m,n} = \frac{m}{m+n}$, $mn, n \in \mathbb{N}$. Show that for each fixed n, $\lim\limits_{m\to\infty} a_{m,n} = 1$, while for each fixed m, $\lim\limits_{n\to\infty} a_{m,n} = 0$. Is $\lim\limits_{m\to\infty} \lim\limits_{n\to\infty} a_{m,n} = \lim\limits_{n\to\infty} \lim\limits_{m\to\infty} a_{m,n}$?

Exercise 2.44. Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = \frac{\sin(nx)}{\sqrt{n}}$ ($x \in \mathbb{R}$, $n \in \mathbb{N}$).

Show that $(f_n)_{n\in\mathbb{N}}$ converges pointwise to the zero function f.

Show that $(f_n')_{n\in\mathbb{N}}$ does not converge pointwise to (the zero function) f'.

Exercise 2.45. Let $f_n : [0, 1] \to \mathbb{R}$ ($n \in \mathbb{N}$) be defined by $f_n(x) = nx(1 - x^2)^n$ ($x \in [0, 1]$).

Show that $(f_n)_{n\in\mathbb{N}}$ converges pointwise to the zero function f.

Show that $\lim\limits_{n\to\infty} \int_0^1 f_n(x)dx = \frac{1}{2} \neq 0 = \int_0^1 \lim\limits_{n\to\infty} f(x)dx$.

Remark 2.46 (Not part of the course)**.** Besides Proposition 4.16, one also has the following results associated with uniform convergence, and we will see a proof of Proposition 2.48 later on when we study differentiation in Chapter 5, and a proof of Proposition 2.47 in Chapter 6.

Proposition 2.47. *If $f_n : [a, b] \to \mathbb{R}$ ($n \in \mathbb{N}$) is a sequence of Riemann-integrable functions on $[a, b]$ which converges uniformly to $f : [a, b] \to \mathbb{R}$, then f is also Riemann-integrable on $[a, b]$, and moreover $\int_a^b f(x)dx = \lim\limits_{n\to\infty} \int_a^b f_n(x)dx$.*

Proposition 2.48. *Let $f_n : (a, b) \to \mathbb{R}$ ($n \in \mathbb{N}$) be a sequence of differentiable functions on (a, b), such that there exists a point $c \in (a, b)$ for which $(f_n(c))_{n\in\mathbb{N}}$ converges. If the sequence $(f_n')_{n\in\mathbb{N}}$ converges uniformly to g on (a, b), then $(f_n)_{n\in\mathbb{N}}$ converges uniformly to a differentiable function f on (a, b), and moreover, $f'(x) = g(x)$ for all $x \in (a, b)$.*

2.4. Convergent sequences and closed sets

We have learnt that closed sets are ones whose complement is open. Here is another characterisation of closed sets.

Theorem 2.49. *A set F is closed if and only if for every convergent sequence $(a_n)_{n\in\mathbb{N}}$ such that $a_n \in F$ $(n \in \mathbb{N})$, we have that $\lim\limits_{n\to\infty} a_n \in F$.*

Proof. ('Only if' part:) Let F be closed. Let $(a_n)_{n\in\mathbb{N}}$ be a convergent sequence such that $a_n \in F$ $(n \in \mathbb{N})$ and denote its limit by L. Assume that $L \notin F$. Then $L \in \complement F$, the complement of F, which is open. So there exists an $r > 0$ such that the open ball $B(L, r)$ with center L and radius $r > 0$ is contained in $\complement F$, that is, $B(L, r)$ contains no points from F. As $(a_n)_{n\in\mathbb{N}}$ is convergent with limit L, we can choose a large enough n so that $d(a_n, L) < r$. This implies that $a_n \in B(L, r)$. But also $a_n \in F$, and so we have arrived at a contradiction. See Figure 2. This shows the 'only if' part.

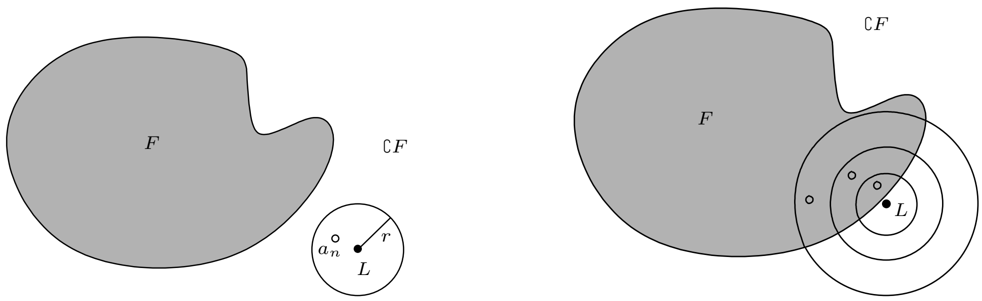


Figure 2. The left picture is for the 'only if' part, and the right one is for the 'if' part.

('If' part:) Suppose that the set F is not closed. Then its complement $\complement F$ is not open. This means that there is a point $L \in \complement F$ such that for every $r > 0$, the open ball $B(L, r)$ has at least one point from F. Now take successively $r = \frac{1}{n}$ $(n \in \mathbb{N})$, and choose a point $a_n \in F \cap B(L, \frac{1}{n})$. In this manner we obtain a sequence $(a_n)_{n\in\mathbb{N}}$ such that $a_n \in F$ for each n, and $d(a_n, L) < \frac{1}{n}$. The property $d(a_n, L) < \frac{1}{n}$ $(n \in \mathbb{N})$ implies that $(a_n)_{n\in\mathbb{N}}$ is a convergent sequence with limit L. So we have obtained the existence of a convergent sequence $(a_n)_{n\in\mathbb{N}}$ such that $a_n \in F$ $(n \in \mathbb{N})$, but for which the limit $\lim\limits_{n\to\infty} a_n = L \notin F$. See Figure 2. This completes the proof of the 'if' part. ☐

Exercise 2.50. We endow $\mathbb{R}^n$ with the Euclidean metric.

(1) Let $0 \neq a \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$. Show that the 'hyperplane' $H = \{x \in \mathbb{R}^n : a^\top x = \beta\}$ is closed in $\mathbb{R}^n$.

(2) Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$. Show that the set of solutions $S = \{x \in \mathbb{R}^n : Ax = b\}$ is a closed subset of $\mathbb{R}^n$.

(3) Show that the Linear Programming Simplex $\Sigma = \{x \in \mathbb{R}^n : Ax = b,\, x_1 \geqslant 0, \dots, x_n \geqslant 0\}$ is closed in $\mathbb{R}^n$.

Exercise 2.51. Let U be an open set and $(x_n)_{n\in\mathbb{N}}$ a sequence in a metric space. Show that if $(x_n)_{n\in\mathbb{N}}$ converges to $x \in U$, then there exists $N \in \mathbb{N}$ such that for all $n > N$, $x_n \in U$. (In words: If the limit of a convergent sequence lies in an open set, then the sequence eventually stays in the open set.)

Exercise 2.52. Recall the normed space ℓ^2 introduced in Exercise 1.20. Consider the subspace c_{00} of ℓ^2 consisting of all sequences with 'compact support' (that is sequences which have all terms equal to zero eventually). Show that c_{00} is not a closed subset of ℓ^2.

Exercise 2.53. Recall the normed space ℓ^∞ introduced in Exercise 1.21. Let c_0 be the subspace of ℓ^∞ consisting of all sequences convergent with limit 0. Show that c_0 is a closed subset of ℓ^∞.

Exercise 2.54. Let Y be a nonempty closed subset of a complete metric space (X, d). We endow Y with the induced metric $d|_{Y\times Y}$ from (X, d) (see Exercise 1.18). Show that $(Y, d|_{Y\times Y})$ is complete too.

2.5. Compact sets

In this section, we study an important class of subsets of a metric space, called compact sets. Before we learn the definition, let us give some motivation for this concept.

Of the different types of intervals in $\mathbb{R}$, perhaps the most important are those of the form $[a, b]$, where a, b are finite real numbers. Why are such intervals so important? This is not an easy question to answer, but we already know of one vital result, namely the Extreme Value Theorem, where such intervals play a vital role. Recall that the Extreme Value Theorem asserts that any continuous function $f : [a, b] \to \mathbb{R}$ attains a maximum and a minimum value on $[a, b]$. This result does not hold in general for continuous functions $f : I \to \mathbb{R}$ with $I = (a, b)$ or $I = [a, b)$ or $I = (a, \infty)$, and so on. Besides its theoretical importance in Analysis, the Extreme Value Theorem is also a fundamental result in Optimisation Theory. It turns out that when we want to generalise this result, the notion of 'compact sets' is pertinent, and we will learn (later on in Chapter 4) the following analogue of the Extreme Value Theorem: If K is a compact subset of a metric space X and $f : K \to \mathbb{R}$ is continuous, then f assumes a maximum and a minimum on K.

Here is the definition of a compact set.

Definition 2.55. Let (X, d) be a metric space. A subset K of X is said to be *compact* if every sequence in K has a convergent subsequence with limit in K, that is, if $(x_n)_{n\in\mathbb{N}}$ is a sequence such that $x_n \in K$ for each $n \in \mathbb{N}$, then there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ which converges to some $L \in K$.

Example 2.56. Let $a, b \in \mathbb{R}$ and $a < b$. The interval $[a, b]$ is a compact subset of $\mathbb{R}$. Indeed, every sequence $(a_n)_{n\in\mathbb{N}}$ contained in $[a, b]$ is bounded, and by the Bolzano-Weierstrass Theorem possesses a convergent subsequence, say $(a_{n_k})_{k\in\mathbb{N}}$, with limit L. But since

$$\text{for all } k \in \mathbb{N},\ a \leqslant a_{n_k} \leqslant b,$$

by letting $k \to \infty$, we obtain $a \leqslant L \leqslant b$, that is, $L \in [a, b]$. Hence $[a, b]$ is compact.

On the other hand, (a, b) is not compact, since the sequence $(a + \frac{b-a}{2n})_{n\in\mathbb{N}}$ is contained in (a, b), but it has no convergent subsequence whose limit belongs to (a, b). This is because the sequence is convergent, with limit a, and so every subsequence of this sequence is also convergent with limit a, which doesn't belong to (a, b).

$\mathbb{R}$ is not compact since the sequence $(n)_{n\in\mathbb{N}}$ cannot have a convergent subsequence. Indeed, if such a convergent subsequence existed, it would also be Cauchy, but the distance between any two terms with distinct indices is at least 1 (since the terms are distinct integers), contradicting the Cauchyness. ◇

In the above list of nonexamples, note that $\mathbb{R}$ is not bounded, and that (a, b) is not closed. On the other hand, the example $[a, b]$ is both bounded and closed. It turns out that compact sets are always closed and bounded. First we define exactly what we mean by a bounded subset of a metric space.

Definition 2.57. A subset S of a metric space X is said to be *bounded* if there exists an $M > 0$ such that for all $x, y \in S$, $d(x, y) \leqslant M$.

Exercise 2.58. Let (X, d) be a metric space and S be a nonempty subset of X. Show that the following are equivalent:

(1) There exists an $M > 0$ such that for all $x, y \in S$, $d(x, y) \leqslant M$.

(2) There exist an $R > 0$ and an $z_0 \in X$ such that for all $x \in S$, $d(x, z_0) \leqslant R$.

(3) For all $z \in X$, there exists an $R_z > 0$ such that for all $x \in S$, $d(x, z) \leqslant R_z$.

Thus S is bounded if and only if any one of the above statements hold. Also, a subset S in a normed space $(X, \|\cdot\|)$ is bounded if and only if there exists an $M > 0$ such that for all $x \in X$, $\|x\| \leqslant M$.

Exercise 2.59. Show that any convergent sequence $(a_n)_{n\in\mathbb{N}}$ in a metric space (X,d) is bounded, that is, the set $\{a_n : n \in \mathbb{N}\}$ is a bounded subset of X.

Theorem 2.60. *Any compact subset K of a metric space (X,d) is closed and bounded.*

Proof. We first show that K is closed. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in K that converges to $L \in X$. Then there is a convergent subsequence, say $(a_{n_k})_{k\in\mathbb{N}}$ that is convergent to a limit $L' \in K$. But as $(a_{n_k})_{k\in\mathbb{N}}$ is a subsequence of a convergent sequence with limit L, it is also convergent to L. By the uniqueness of limits, $L = L' \in K$. Thus, K is closed (by Theorem 2.49).

Next we show that K is bounded by contradiction. Suppose K is not bounded. Let $x_0 \in X$. Taking any $n \in \mathbb{N}$, it is not the case that for all $x \in K$, $d(x,x_0) \leqslant n$ (otherwise, K can be seen to be bounded taking $R := n$), and so for this n, there must be an $x \in K$, which we call a_n, such that $d(a_n, x_0) > n$. But this implies that no subsequence of $(a_n)_{n\in\mathbb{N}}$ is bounded. So no subsequence of $(a_n)_{n\in\mathbb{N}}$ can be convergent either. This contradicts the compactness of K. Thus our assumption was incorrect, that is, K is bounded. $\square$

The converse of the above theorem is, in general, false. That is, there exist metric spaces with subsets that are closed and bounded, but not compact, as shown by the following example. (However, as shown by Theorem 2.63 below, the converse is true for subsets of $\mathbb{R}^n$.)

Example 2.61 (The closed unit ball in $(\ell^2, \|\cdot\|_2)$ is *not* compact)**.** Recall the normed space introduced in Exercise 1.20. Consider closed the unit ball with centre $\mathbf{0} = (0)_{n\in\mathbb{N}}$ and radius 1 in the normed space ℓ^2:

$$\overline{B(\mathbf{0},1)} = \{x \in \ell^2 : \|x\|_2 \leqslant 1\}.$$

Then $\overline{B(\mathbf{0},1)}$ is bounded, it is closed (since its complement can be seen to be open), but $\overline{B(\mathbf{0},1)}$ is not compact, and this can be demonstrated as follows. Take the sequence $(\boldsymbol{e}_n)_{n\in\mathbb{N}}$, where $\boldsymbol{e}_n$ is the sequence with only the n^{th} term equal to 1, and all other terms are equal to 0:

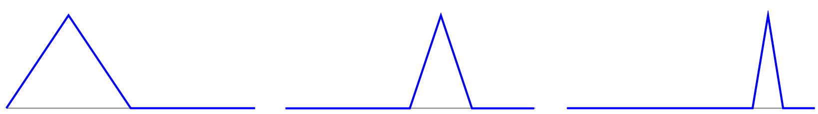
$$\boldsymbol{e}_n := (\;0\;,\;\cdots\;,\;0\;,\underbrace{\;1\;}_{n^{\text{th}}\text{ place}},\;0\;,\;\cdots\;) \in \overline{B(\mathbf{0},1)} \subset \ell^2$$

Then $(\boldsymbol{e}_n)_{n\in\mathbb{N}}$ in $\overline{B(\mathbf{0},1)} \subset \ell^2$ can have no convergent subsequence. Indeed, whenever $n \neq m$, $\|\boldsymbol{e}_n - \boldsymbol{e}_m\|_2 = \sqrt{2}$, and so no subsequence of $(\boldsymbol{e}_n)_{n\in\mathbb{N}}$ can be Cauchy, much less convergent! $\diamond$

Example 2.62. (The closed unit ball in $(C[0,1], \|\cdot\|_\infty)$ is *not* compact.) Consider the closed unit ball with centre $\mathbf{0}$ in $(C[0,1], \|\cdot\|_\infty)$ and radius 1:

$$\overline{B(\mathbf{0},1)} = \{\boldsymbol{x} \in C[0,1] : \|\boldsymbol{x}\|_\infty \leqslant 1\}.$$

Then $\overline{B(\mathbf{0},1)}$ is bounded, and also it is closed (since its complement is open). But $\overline{B(\mathbf{0},1)}$ is not compact, and this can be demonstrated by considering the sequence $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$, where the graphs of the terms $\boldsymbol{x}_n$ have 'narrowing' tents of height 1, with the supports of the tents moving to the right, on half of each remaining interval, as shown in the following picture:



Then this sequence does not have a convergent subsequence, since if it did, then the convergent subsequence would be Cauchy, but whenever $n \neq m$, $\|\boldsymbol{x}_n - \boldsymbol{x}_m\|_\infty = 1$, a contradiction to the Cauchyness. $\diamond$

We will now show the following important result.

Theorem 2.63. *A subset K of $\mathbb{R}^d$ is compact if and only if K is closed and bounded.*

Before showing this, we prove a technical result, which besides being interesting on its own, will also somewhat simplify the proof of the above theorem.

Lemma 2.64. *Every bounded sequence in $\mathbb{R}^d$ has convergent subsequence.*

Proof. We prove this using induction on d. Let us consider the case when $d = 1$. Then the statement is precisely the Bolzano-Weierstrass Theorem!

Now suppose that the result has been proved in $\mathbb{R}^d$ for some $d \geqslant 1$. We will show that it holds in $\mathbb{R}^{d+1}$. Let $(\boldsymbol{a}_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathbb{R}^{d+1}$. We split each $\boldsymbol{a}_n$ into its first d components (giving a vector in $\mathbb{R}^d$) and its last component in $\mathbb{R}$:

$$\boldsymbol{a}_n = \begin{bmatrix} \boldsymbol{\alpha}_n \\ \beta_n \end{bmatrix},$$

where $\boldsymbol{\alpha}_n \in \mathbb{R}^d$ and $\beta_n \in \mathbb{R}$. Clearly $\|\boldsymbol{\alpha}_n\|_2 \leqslant \|\boldsymbol{a}_n\|_2$, and so $(\boldsymbol{\alpha}_n)_{n\in\mathbb{N}}$ is a bounded sequence in $\mathbb{R}^d$. By the induction hypothesis, $(\boldsymbol{\alpha}_n)_{n\in\mathbb{N}}$ has a convergent subsequence, say $(\boldsymbol{\alpha}_{n_k})_{k\in\mathbb{N}}$ which converges, to say $\boldsymbol{\alpha} \in \mathbb{R}^d$. Consider now the sequence $(\beta_{n_k})_{k\in\mathbb{N}}$ in $\mathbb{R}$. Then $(\beta_{n_k})_{k\in\mathbb{N}}$ is bounded, and so by the Bolzano-Weierstrass Theorem, it has a convergent subsequence $(\beta_{n_{k_\ell}})_{\ell\in\mathbb{N}}$, with limit, say $\beta \in \mathbb{R}$. Then we have

$$\boldsymbol{a}_{n_{k_\ell}} = \begin{bmatrix} \boldsymbol{\alpha}_{n_{k_\ell}} \\ \beta_{n_{k_\ell}} \end{bmatrix} \xrightarrow{\ell\to\infty} \begin{bmatrix} \boldsymbol{\alpha} \\ \beta \end{bmatrix} =: L \in \mathbb{R}^{d+1}.$$

Thus the bounded sequence $(\boldsymbol{a}_n)_{n\in\mathbb{N}}$ has $(\boldsymbol{a}_{n_{k_\ell}})_{\ell\in\mathbb{N}}$ as a convergent subsequence. □

Now we return to the task of proving of Theorem 2.63.

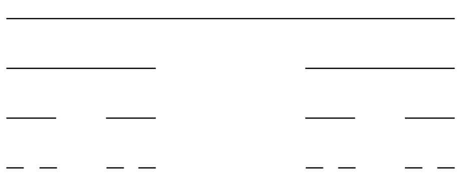
Proof. ('If' part.) Let K be closed and bounded. Let $(\boldsymbol{a}_n)_{n\in\mathbb{N}}$ be a sequence in K. Then $(\boldsymbol{a}_n)_{n\in\mathbb{N}}$ is bounded, and so it has a convergent subsequence, with limit $\boldsymbol{L} \in \mathbb{R}^d$. But since K is closed, and since each term of the sequence belongs to K, it follows that also $\boldsymbol{L} \in K$. So K is compact.

('Only if' part) This follows by Theorem 2.60. □

Example 2.65. The intervals $(a, b]$, $[a, b)$ are not compact, since although they are bounded, they are not closed. The intervals $(-\infty, b]$, $[a, \infty)$ are not compact, since although they are closed, they are not bounded. ◇

Let us consider an interesting compact subset of the real line, called the *Cantor set*.

Example 2.66 (Cantor set)**.** The Cantor set is constructed as follows. First, denote the closed interval $[0, 1]$ by F_1. Next, delete from F_1 the open interval $(\frac{1}{3}, \frac{2}{3})$ which is its middle third, and denote the remaining closed set by F_2. Clearly, $F_2 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Next, delete from F_2 the open intervals $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$, which are the middle thirds of its two pieces, and denote the remaining closed set by F_3. It is easy to see that $F_3 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$. If we continue this process, at each stage deleting the open middle third of each closed interval remaining from the previous stage, we obtain a sequence of closed sets F_n, each of which contains all of its successors. The picture below illustrates this.



The Cantor set is defined by $F = \bigcap\limits_{n=1}^{\infty} F_n$.

As F is an intersection of closed sets, it is closed. Moreover it is contained in $[0, 1]$ and so it is also bounded. Consequently it is compact. F consists of those points in the closed interval $[0, 1]$ which 'ultimately remain' after the removal of all the open intervals $(\frac{1}{3}, \frac{2}{3})$, $(\frac{1}{9}, \frac{2}{9})$, $(\frac{7}{9}, \frac{8}{9})$, ... What points do remain? F contains the end-points of the closed intervals which make up each set F_n:

$$0, 1, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{9}, \tfrac{2}{9}, \tfrac{7}{9}, \tfrac{8}{9}, \cdots .$$

Does F contain any other points? Actually, F contains many more points than the above list of end points. After all, the above list of endpoints is countable, but it can be shown that F is uncountable! It turns out that the Cantor set is a very intricate mathematical object, and is often a source of interesting examples/counterexamples in Analysis: For example, as the sum of the lengths of the intervals removed is

$$\tfrac{1}{3} + 2\tfrac{1}{3^2} + 4\tfrac{1}{3^3} + \cdots = 1,$$

(factor out $\frac{1}{3}$ and sum the resulting geometric series), the '(Lebesgue length) measure' of F is $1 - 1 = 0$. So this is an example of an uncountable set with 'Lebesgue measure' 0. ◇

Exercise 2.67. Determine if the following statements are true or false. Give reasons for your answers.

(1) If $S \subset \mathbb{R}$ is such that each convergent sequence in S has a convergent subsequence with limit in S, then S is compact.

(2) All closed and bounded sets are compact.

(3) If (X, d) is a metric space, Y is a nonempty subset of X equipped with the induced metric from (X, d), and K is a compact subset of $(Y, d|_{Y\times Y})$, then K is a compact subset of (X, d).

Exercise 2.68. Let K be a compact subset of $\mathbb{R}^d$. Let F be a closed subset of $\mathbb{R}^d$. Show that $F \cap K$ is compact.

Exercise 2.69. Show that the unit sphere with center $\mathbf{0}$ in $\mathbb{R}^d$, namely

$$\mathbb{S}^{d-1} := \{x \in \mathbb{R}^d : \|x\|_2 = 1\}$$

is compact.

Exercise 2.70. Show that $\{1, \frac{1}{2}, \frac{1}{3}, \dots\} \cup \{0\}$ is compact.

Exercise 2.71. Consider the metric space $(\mathbb{R}^{m\times m}, \|\cdot\|_\infty)$.
Is the subset (the 'General Linear' group[5]) $GL(m, \mathbb{R}) = \{A \in \mathbb{R}^{m\times m} : A \text{ is invertible}\}$ compact?

Exercise 2.72. In the metric space $(\mathbb{R}^{2\times 2}, \|\cdot\|_\infty)$. is the set of orthogonal matrices $O(2) = \{A \in \mathbb{R}^{2\times 2} : A^\top A = I_2\}$ compact?

Exercise 2.73. Consider the subset $H := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 x_2 = 1\}$ of $\mathbb{R}^2$. Show that H is not compact, but H is closed.

2.6. Notes (not part of the course)

Definition of compactness. The notion of a compact set that we have defined is really *sequential* compactness. In the context of the more general topological spaces, one defines the notion of compactness as follows.

Definition 2.74. Let X be a topological space with the topology given by the family of open sets $\mathcal{O}$. Let $Y \subset X$. A collection $\mathcal{C} = \{U_i : i \in I\}$ of open sets is said to be an *open cover of* Y if $Y \subset \bigcup_{i\in I} U_i$.

$K \subset X$ is said to be a *compact set* if every open cover of K has a finite subcover, that is, given any open cover $\mathcal{C} = \{U_i : i \in I\}$ of K, there exist finitely many indices $i_1, \dots, i_n \in I$ such that $K \subset U_{i_1} \cup \cdots \cup U_{i_n}$.

In the case of *metric spaces*, it can be shown that the set of compact sets *coincides* with the set of sequentially compact sets. But in general topological spaces, these may not be the same.

[5]The General Linear group is so named because the columns of an invertible matrix are linearly independent, hence the vectors they define are in 'general position' (linearly independent!), and matrices in the general linear group take points in general position to points in general position.

Chapter 3

Series

In this chapter we study series in normed spaces, but first we will begin with series in $\mathbb{R}$. Just as we learnt ways of deducing the convergence of sequences, we will learn about tests for checking convergence of series. Why bother learning about such things about series? It turns out that series play an important role in solutions to various problems that arise in Mathematics and applications to Mathematics in other disciplines. For example, in the theory of differential equations, in functional analysis, Fourier/harmonic analysis, complex analysis and so on.

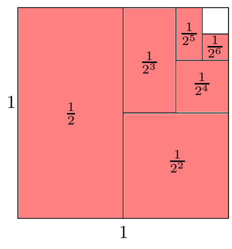
3.1. Series in $\mathbb{R}$

Given a sequence $(a_n)_{n\in\mathbb{N}}$, one can form a new sequence $(s_n)_{n\in\mathbb{N}}$ of its *partial sums*:

$$\begin{aligned} s_1 &:= a_1, \\ s_2 &:= a_1 + a_2, \\ s_3 &:= a_1 + a_2 + a_3, \\ &\vdots \end{aligned}$$

Definition 3.1. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence and let $(s_n)_{n\in\mathbb{N}}$ be the sequence of its partial sums. If $(s_n)_{n\in\mathbb{N}}$ converges, we say that the *series* $\sum\limits_{n=1}^{\infty} a_n$ *converges*, and we write $\sum\limits_{n=1}^{\infty} a_n = \lim\limits_{n\to\infty} s_n$. If the sequence $(s_n)_{n\in\mathbb{N}}$ does not converge we say that the series $\sum\limits_{n=1}^{\infty} a_n$ *diverges*.

Example 3.2. (1) The series $\sum\limits_{n=1}^{\infty} (-1)^n$ diverges. Indeed the sequence of partial sums is the sequence $-1, 0, -1, 0, \dots$ which is a divergent sequence.
(2) Let $(a_n)_{n\in\mathbb{N}}$ be the geometric sequence $(\frac{1}{2^n})_{n\in\mathbb{N}}$. Then $(s_n)_{n\in\mathbb{N}} = (1-\frac{1}{2^n})_{n\in\mathbb{N}}$ is convergent with limit 1. Thus $\sum\limits_{n=1}^{\infty} \frac{1}{2^n} = 1$. A pictorial proof is given below.



(3) The series $\sum\limits_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges. Its nth partial sum 'telescopes':

$$s_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}.$$

Since $\lim\limits_{n\to\infty} s_n = 1 - 0 = 1$, we have $\sum\limits_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$. ◇

Exercise 3.3 (Tantalising $\tan^{-1}$)**.** Show that $\sum\limits_{n=1}^{\infty} \tan^{-1} \frac{1}{2n^2} = \frac{\pi}{4}$. *Hint:* Write $\frac{1}{2n^2} = \frac{\frac{1}{2n-1} - \frac{1}{2n+1}}{1+\frac{1}{2n-1}\frac{1}{2n+1}}$ and use $\tan(a-b) = \frac{\tan a - \tan b}{1+\tan a \tan b}$.

Exercise 3.4. Show that for every real number $x > 1$, the series $\frac{1}{1+x} + \frac{2}{1+x^2} + \frac{4}{1+x^4} + \cdots + \frac{2^n}{1+x^{2^n}} + \ldots$ converges. *Hint:* Add $\frac{1}{1-x}$.

Exercise 3.5. Consider the Fibonacci sequence $(F_n)_{n\in\mathbb{N}}$ with $F_0 = F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \in \mathbb{N}$. Show that $\sum\limits_{n=1}^{\infty} \frac{1}{F_{n-1}F_{n+1}} = 1$.

In the above example of the divergent series $\sum\limits_{n=1}^{\infty} (-1)^n$, the sequence $(a_n)_{n\in\mathbb{N}} = ((-1)^n)_{n\in\mathbb{N}}$ was not convergent. In fact, we have the following necessary condition for convergence of a series.

Proposition 3.6. *If the series* $\sum\limits_{n=1}^{\infty} a_n$ *converges, then* $\lim\limits_{n\to\infty} a_n = 0$.

Proof. Let $s_n := a_1 + \cdots + a_n$. Since the series converges we have $\lim\limits_{n\to\infty} s_n = L$ for some $L \in \mathbb{R}$. But as $(s_{n+1})_{n\in\mathbb{N}}$ is a subsequence of $(s_n)_{n\in\mathbb{N}}$, it follows that $\lim\limits_{n\to\infty} s_{n+1} = L$. By the algebra of limits, $\lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} (s_{n+1} - s_n) = \lim\limits_{n\to\infty} s_{n+1} - \lim\limits_{n\to\infty} s_n = L - L = 0$. □

Exercise 3.7. Does the series $\sum\limits_{n=1}^{\infty} \cos \frac{1}{n}$ converge?

Exercise 3.8. Let $\sum\limits_{n=1}^{\infty} a_n$ converge.
(1) Show that for all $n \in \mathbb{N}$, the series[1] $\sum\limits_{k=n+1}^{\infty} a_k$ converges.
(2) Given any $\epsilon > 0$, show that there exists an $N \in \mathbb{N}$ such that for all $n > N$, we have $|\sum\limits_{k=n+1}^{\infty} a_k| < \epsilon$.

In Theorem 3.10, we will see an instance of a series which shows that although this condition is *necessary* for the convergence of a series, it is not *sufficient.* But first, let us see an important example of a convergent series. In fact, it lies at the core of most of the convergence results in Real Analysis.

Theorem 3.9. *Let* $r \in \mathbb{R}$*. The geometric series* $\sum\limits_{n=0}^{\infty} r^n$ *converges if and only if* $|r| < 1$.

Moreover, if $|r| < 1$*, then* $\sum\limits_{n=0}^{\infty} r^n = \frac{1}{1-r}$.

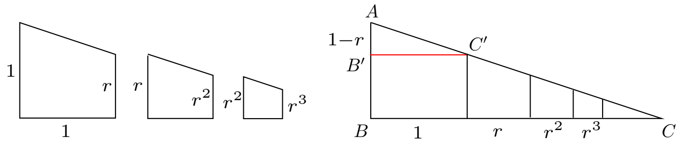
Proof. Let $|r| < 1$. First we will show that $\lim\limits_{n\to\infty} r^n = 0$. As $|r| < 1$, $|r| = \frac{1}{1+h}$ for $h := \frac{1}{|r|} - 1 > 0$. Then $(1+h)^n = 1 + \binom{n}{1}h + \cdots + h^n > nh$. Thus $0 \leqslant |r|^n = \frac{1}{(1+h)^n} < \frac{1}{nh}$, and so by the Sandwich Theorem, $\lim\limits_{n\to\infty} |r|^n = 0$. As $-|r|^n \leqslant r^n \leqslant |r|^n$, it follows again from the Sandwich Theorem that $\lim\limits_{n\to\infty} r^n = 0$.

Let $s_n := 1 + r + r^2 + \cdots + r^n = \frac{(1-r)(1+r+r^2+\cdots+r^n)}{1-r} = \frac{1-r^{n+1}}{1-r}$. As $\lim\limits_{n\to\infty} r^{n+1} = 0$, it follows that $\lim\limits_{n\to\infty} (1-r)s_n = 1$. Hence $\sum\limits_{n=1}^{\infty} r^n = \lim\limits_{n\to\infty} s_n = \frac{1}{1-r}$.

Now suppose that $|r| \geqslant 1$. If $r = 1$, then $\lim\limits_{n\to\infty} r^n = 1 \neq 0$, and so by Proposition 3.6, the series diverges. Similarly if $r = -1$, then $(r^n)_{n\in\mathbb{N}} = ((-1)^n)_{n\in\mathbb{N}}$ diverges, and so the series is divergent. Also if $|r| > 1$, then the sequence $(r^n)_{n\in\mathbb{N}}$ has the subsequence $(r^{2n})_{n\in\mathbb{N}}$ which is not bounded, and hence not convergent. Consequently $(r^n)_{n\in\mathbb{N}}$ diverges, and hence the series diverges. □

[1] Sometimes referred to as a 'tail of the series $\sum\limits_{n=1}^{\infty} a_n$'.

The name comes from the associated similarity in geometry.



Since the triangles $AB'C'$ and ABC are similar, $\frac{BC}{AB} = \frac{1+r+r^2+r^3+\cdots}{1} = \frac{B'C'}{AB'} = \frac{1}{1-r}$.

Theorem 3.10. *The harmonic series*[2] $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ *diverges.*

Proof. Let $s_n := 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. We have

$$\text{for all } n \in \mathbb{N},\ s_{2n} - s_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \geqslant n\frac{1}{2n} = \frac{1}{2}. \tag{3.1}$$

If the series converges, then $\lim\limits_{n\to\infty} s_n = L$ for some L. But then also $\lim\limits_{n\to\infty} s_{2n} = L$, and so $\lim\limits_{n\to\infty} (s_{2n} - s_n) = L - L = 0$, which contradicts (3.1). ☐

The n^{th} term of the above series satisfies $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1}{n} = 0$, showing that the condition given in Proposition 3.6 is necessary but *not* sufficient for the convergence of the series.

Theorem 3.11. *Let $s \in \mathbb{R}$. The series*[3] $\sum\limits_{n=1}^{\infty} \frac{1}{n^s}$ *converges if and only if $s > 1$.*

Proof. Let $S_n = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots + \frac{1}{n^s}$. Clearly $S_1 < S_2 < S_3 < \cdots$, so that $(S_n)_{n\in\mathbb{N}}$ is an increasing sequence.

Let $s > 1$. We have

$$\begin{aligned} S_{2n+1} &= 1 + \left(\tfrac{1}{2^s} + \tfrac{1}{4^s} + \cdots + \tfrac{1}{(2n)^s}\right) + \left(\tfrac{1}{3^s} + \tfrac{1}{5^s} + \cdots + \tfrac{1}{(2n+1)^s}\right) \\ &< 1 + \left(\tfrac{1}{2^s} + \tfrac{1}{4^s} + \cdots + \tfrac{1}{(2n)^s}\right) + \left(\tfrac{1}{2^s} + \tfrac{1}{4^s} + \cdots + \tfrac{1}{(2n)^s}\right) \\ &= 1 + \tfrac{2}{2^s}\left(1 + \tfrac{1}{2^s} + \cdots + \tfrac{1}{n^s}\right) = 1 + 2^{1-s} S_n \\ &< 1 + 2^{1-s} S_{2n+1}. \end{aligned}$$

As $s > 1$, we have $2^{1-s} < 1$, so that $S_{2n+1} < \frac{1}{1-2^{1-s}}$ $(n \in \mathbb{N})$. Also, $S_{2n} < S_{2n+1} < \frac{1}{1-2^{1-s}}$ $(n \in \mathbb{N})$. Thus $(S_n)_{n\in\mathbb{N}}$ is bounded. But an increasing sequence which is bounded above is convergent (to the supremum of its terms). Hence $\sum\limits_{n=1}^{\infty} \frac{1}{n^s}$ converges for $s > 1$.

If on the other hand $s \leqslant 1$, then the proof of divergence is similar to that of showing that the harmonic series diverges. Indeed, if the series converged, then $\lim\limits_{n\to\infty} (S_{2n} - S_n) = 0$, while

$$\text{for all } n \in \mathbb{N},\ S_{2n} - S_n = \frac{1}{(n+1)^s} + \frac{1}{(n+2)^s} + \cdots + \frac{1}{(2n)^s} \geqslant n\frac{1}{(2n)^s} \geqslant n\frac{1}{2n} = \frac{1}{2},$$

where we have used the fact that $s \leqslant 1$ in order to obtain the last inequality. ☐

For a sequence $(a_n)_{n\in\mathbb{N}}$ with nonnegative terms, we sometimes write $\sum\limits_{n=1}^{\infty} a_n < +\infty$ to mean that the series converges.

[2]Its name derives from the concept of overtones, or harmonics in music: The wavelengths of the overtones of a vibrating string are $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, and so on, of the string's fundamental wavelength.

[3]The function $s \mapsto \sum\limits_{n=1}^{\infty} \frac{1}{n^s}$ is called the *Riemann-zeta function*, which is an important function in number theory. The connection with number theory is brought out by Euler's identity, which says that $\zeta(s) := \sum\limits_{n=1}^{\infty} \frac{1}{n^s} = \prod\limits_{p \text{ prime}} \frac{1}{1-p^{-s}}$.

Exercise 3.12. Prove that if $a_1 \geqslant a_2 \geqslant a_3 \geqslant \cdots$ is a sequence of nonnegative numbers, and $\sum\limits_{n=1}^{\infty} a_n < +\infty$, then $\lim\limits_{n\to\infty} na_n = 0$. *Hint:* $s_{2n} - s_n = a_{n+1} + \cdots + a_{2n} \geqslant n \cdot a_{2n}$ and $a_{n+2} + \cdots + a_{2n+1} \geqslant n \cdot a_{2n+1}$. Show that the assumption $a_1 \geqslant a_2 \geqslant a_3 \geqslant \cdots$ above cannot be dropped by considering the *lacunary series* whose n^2th term is $\frac{1}{n^2}$ and all other terms are zero.

Exercise 3.13 (Astronomical patience!)**.** Suppose a computer is programmed to add 1 trillion terms of the harmonic series each second. Since the Big Bang (about 13.8 billion years ago), has enough time elapsed for the n^{th} partial sum to exceed 100? *Hint:* Compare the partial sum with $\int_1^n \frac{1}{x} dx$.

Exercise 3.14 (Infinitude of primes via divergence of the harmonic series)**.**

(1) Let $n \in \mathbb{N}$. For any prime p, show that $\frac{1}{1-\frac{1}{p}} \geqslant 1 + \frac{1}{p} + \cdots + \frac{1}{p^n}$.

(2) Let $\mathbb{N} \ni n \geqslant 2$ have a factorisation into primes given by $n = p_1^{\alpha_1} \cdots p_K^{\alpha_K}$, where $\alpha_1, \cdots, \alpha_K \in \mathbb{N}$ and $p_1, \cdots, p_K$ are primes. Show that $\alpha_k \leqslant n$ for all $1 \leqslant k \leqslant K$.

(3) If $p_1, \cdots, p_K$ are the only prime numbers, then show that for all $n \in \mathbb{N}$, $\prod\limits_{k=1}^{K} \frac{1}{1-\frac{1}{p_k}} \geqslant 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, and hence arrive at a contradiction (to the divergence of the harmonic series).

Exercise 3.15 ($\sum\limits_{p \text{ prime}} \frac{1}{p}$ diverges)**.** Let all the primes be $p_1 < p_2 < p_3 < \cdots$.

(1) For all $x \in [0, \frac{1}{2}]$, show that $2x - \log \frac{1}{1-x} \geqslant 0$.

(2) Show that if $n \in \mathbb{N}$, then $\prod\limits_{\text{prime } p \leqslant n} \frac{1}{1-\frac{1}{p}} \geqslant \prod\limits_{\text{prime } p \leqslant n} (1 + \frac{1}{p} + \cdots + \frac{1}{p^n}) \geqslant 1 + \frac{1}{2} + \cdots + \frac{1}{n}$.

(3) Conclude that $\sum\limits_{p \text{ prime}} \frac{1}{p}$ diverges.

Exercise 3.16. For $r \in \mathbb{R}$, consider the Arithmetic-Geometric Progression $1,\ 2r,\ 3r^2,\ 4r^3,\ \cdots$. Note that $1, 2, 3, 4, \cdots$ form an arithmetic progression, while $1, r, r^2, r^3, \cdots$ form a geometric progression. Show that if $|r| < 1$, then $1 + 2r + 3r^2 + \cdots = \frac{1}{(1-r)^2}$. *Hint:* Consider $s_n - rs_n$, where s_n is the n^{th} partial sum.

Definition 3.17. If the series $\sum\limits_{n=1}^{\infty} |a_n|$ converges, then we say that $\sum\limits_{n=1}^{\infty} a_n$ *converges absolutely.*

The name is justified, thanks to the following result.

Proposition 3.18. *If* $\sum\limits_{n=1}^{\infty} |a_n|$ *converges, then* $\sum\limits_{n=1}^{\infty} a_n$ *converges.*

Proof. Let $s_n := a_1 + \cdots + a_n$. We will show that $(s_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. For $n > m$,

$$\begin{aligned} |s_n - s_m| &= |(a_1 + \cdots + a_n) - (a_1 + \cdots + a_m)| = |a_{m+1} + \cdots + a_n| \\ &\leqslant |a_{m+1}| + \cdots + |a_n| = (|a_1| + \cdots + |a_n|) - (|a_1| + \cdots + |a_m|) = \sigma_n - \sigma_m, \end{aligned}$$

where $\sigma_k := |a_1| + \cdots + |a_k|$ $(k \in \mathbb{N})$. Since $\sum\limits_{n=1}^{\infty} |a_n| < +\infty$, its sequence of partial sums $(\sigma_n)_{n\in\mathbb{N}}$ is convergent, and in particular, Cauchy. This shows, from the above inequality $|s_n - s_m| \leqslant \sigma_n - \sigma_m$, that $(s_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$ and hence it is convergent. ☐

Exercise 3.19. Does the series $\sum\limits_{n=1}^{\infty} \frac{\sin n}{n^2}$ converge?

Exercise 3.20. If $\sum\limits_{n=1}^{\infty} a_n$ converges absolutely, then show that $|\sum\limits_{n=1}^{\infty} a_n| \leqslant \sum\limits_{n=1}^{\infty} |a_n|$.

Example 3.21. The series $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n}$ does not converge absolutely, since $\sum\limits_{n=1}^{\infty} |\frac{(-1)^n}{n}| = \sum\limits_{n=1}^{\infty} \frac{1}{n}$, and we have seen that the harmonic series diverges.

A series of the form $\sum\limits_{n=1}^{\infty} (-1)^n a_n$ with $a_n \geqslant 0$ for all $n \in \mathbb{N}$ is called an *alternating series.* The series above, namely $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n}$ is an alternating series $\sum\limits_{n=1}^{\infty} (-1)^n a_n$ with $a_n := \frac{1}{n}$ $(n \in \mathbb{N})$.

We'll now learn a result, called the Leibniz Alternating Series Theorem, allowing us to conclude that this alternating series is in fact convergent (since the sufficiency conditions for convergence in the Leibniz Alternating Series Theorem are satisfied: $a_1 = 1 \geqslant a_2 = \frac{1}{2} \geqslant a_3 = \frac{1}{3} \geqslant \cdots$ and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1}{n} = 0$). ◇

Theorem 3.22 (Leibniz Alternating Series Theorem)**.** *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence such that*

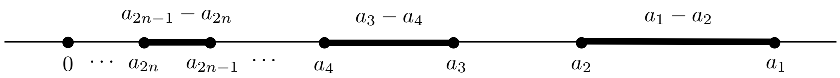
(1) *it has nonnegative terms ($a_n \geqslant 0$ for all n),*

(2) *it is decreasing ($a_1 \geqslant a_2 \geqslant a_3 \geqslant \dots$), and*

(3) $\lim\limits_{n\to\infty} a_n = 0$.

Then the series $\sum\limits_{n=1}^{\infty}(-1)^n a_n$ *converges.*

A pictorial 'proof without words' is shown below. The sum of the lengths of the disjoint dark intervals is at most the length of $(0, a_1)$.



Proof. We may just as well prove the convergence of $\sum\limits_{n=1}^{\infty}(-1)^{n+1}a_n$ $(= -\sum\limits_{n=1}^{\infty}(-1)^n a_n)$.

Let $s_n = a_1 - a_2 + a_3 - + \cdots + (-1)^{n-1}a_n$. Clearly

$$\begin{aligned} s_{2n+1} &= s_{2n-1} - a_{2n} + a_{2n+1} \leqslant s_{2n-1}, \\ s_{2n+2} &= s_{2n} + a_{2n+1} - a_{2n+2} \geqslant s_{2n}, \end{aligned}$$

and so the sequence $s_2, s_4, s_6, \dots$ is increasing, while the sequence $s_3, s_5, s_7, \dots$ is decreasing. Also,

$$s_{2n} \leqslant s_{2n} + a_{2n+1} = s_{2n+1} \leqslant s_{2n-1} \leqslant \cdots \leqslant s_3.$$

So $(s_{2n})_{n\in\mathbb{N}}$ is a bounded ($s_2 \leqslant s_{2n} \leqslant s_3$ for all n), increasing sequence, and hence it is convergent. But as $(a_{2n+1})_{n\in\mathbb{N}}$ is also convergent with limit 0, it follows that $(s_{2n+1})_{n\in\mathbb{N}}$ is convergent too, and

$$\lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty}(s_{2n} + a_{2n+1}) = \lim_{n\to\infty} s_{2n}.$$

Hence $(s_n)_{n\in\mathbb{N}}$ is convergent, and so the series converges. □

Exercise 3.23. Let $s > 0$. Show that $\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{n^s}$ converges.

Exercise 3.24. Prove that $\sum\limits_{n=1}^{\infty}(-1)^n\frac{\sqrt{n}}{n+1}$ converges.

Exercise 3.25. Prove that $\sum\limits_{n=1}^{\infty}(-1)^n \sin\frac{1}{n}$ converges.

One might tend to think of a series as an 'infinite sum', and hence be tempted to attribute to it the usual properties associated with finite sums such as grouping and changing the order of terms. The next two exercises show that this is fraught with dangers, and one ought to go back to the definitions in order to check if the manipulation at hand is allowed.

Exercise 3.26 (Inserting parenthesis)**.**

(1) Show that if a series converges, then the new series one obtains by 'inserting parentheses' in the original one (that is, adding up finite blocks of consecutive terms) converges to the same sum.

(2) Show by means of an example that a divergent series may become convergent by inserting parenthesis.

Exercise 3.27 (Rearrangement)**.** A bijective mapping $p : \mathbb{N} \to \mathbb{N}$ is called a *permutation* (of $\mathbb{N}$). The series $\sum\limits_{n=1}^{\infty}a_{p(n)}$ is called a *rearrangement of the series* $\sum\limits_{n=1}^{\infty}a_n$.

(1) Show that $1 - 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + - \cdots$ is convergent with sum 0, but its rearrangement given by $1 + \frac{1}{2} - 1 + \frac{1}{3} + \frac{1}{4} - \frac{1}{2} + \frac{1}{5} + \frac{1}{6} - \frac{1}{3} + \cdots$ has a postive sum.

(2) Let p be any permutation of $\mathbb{N}$. If the series $\sum\limits_{n=1}^{\infty}a_n$ is absolutely convergent, then so is $\sum\limits_{n=1}^{\infty}a_{p(n)}$, and moreover, their sums coincide. *Hint:* First consider all terms being nonnegative, and show that their respective sums must be bounded by each other. For the general case, begin with $\sum\limits_{n=1}^{\infty}(|a_n| - a_n)$.

3.1.1. Comparison, Ratio, Root. We will now learn three important tests for the convergence of a series:

(1) the comparison test (where we compare with a series whose convergence status is known)
(2) the ratio test (where we look at the behaviour of the ratio of terms $\frac{a_{n+1}}{a_n}$)
(3) the root test (where we look at the behaviour of $\sqrt[n]{|a_n|}$)

We summarise them in the table below.

		Comparison	Ratio	Root
Absolute convergence	$\Leftarrow$	$\lvert a_n\rvert \leqslant c_n$ for all large n; $\sum\limits_{n=1}^{\infty} c_n$ converges.	$\lvert\frac{a_{n+1}}{a_n}\rvert \leqslant r < 1$ for all large n.	$\sqrt[n]{\lvert a_n\rvert} \leqslant r < 1$ for all large n.
Divergence	$\Leftarrow$	$a_n \geqslant d_n \geqslant 0$ for all large n; $\sum\limits_{n=1}^{\infty} d_n$ diverges.	$\lvert\frac{a_{n+1}}{a_n}\rvert \geqslant 1$ for all large n.	$\sqrt[n]{\lvert a_n\rvert} \geqslant 1$ infinitely often.

Theorem 3.28 (Comparison test)**.**

(1) *If $(a_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ are such that there exists an $N \in \mathbb{N}$ such that $|a_n| \leqslant c_n$ for all $n \geqslant N$, and $\sum\limits_{n=1}^{\infty} c_n$ converges, then $\sum\limits_{n=1}^{\infty} a_n$ converges absolutely.*

(2) *If $(a_n)_{n\in\mathbb{N}}$, $(d_n)_{n\in\mathbb{N}}$ are such that there exists an $N \in \mathbb{N}$ such that $a_n \geqslant d_n \geqslant 0$ for all $n \geqslant N$, and $\sum\limits_{n=1}^{\infty} d_n$ diverges, then $\sum\limits_{n=1}^{\infty} a_n$ diverges.*

Proof. Let $s_n := |a_1| + \cdots + |a_n|$ and $\sigma_n := c_1 + \cdots + c_n$. For $n > m$, we have

$$|s_n - s_m| = |a_{m+1}| + \cdots + |a_n| \leqslant c_{m+1} + \cdots + c_n = |\sigma_n - \sigma_m|.$$

As $(\sigma_n)_{n\in\mathbb{N}}$ is Cauchy, $(s_n)_{n\in\mathbb{N}}$ is Cauchy. So $(s_n)_{n\in\mathbb{N}}$ is convergent, i.e., $\sum\limits_{n=1}^{\infty} a_n$ converges absolutely.

The second claim follows from the first one. For if $\sum\limits_{n=1}^{\infty} a_n$ converges, so must $\sum\limits_{n=1}^{\infty} d_n$. □

Example 3.29. Let us revisit Exercise 3.19, where we showed that the series $\sum\limits_{n=1}^{\infty} \frac{\sin n}{n^2}$ converges. Since $|\frac{\sin n}{n^2}| \leqslant \frac{1}{n^2}$ for all $n \in \mathbb{N}$, and as $\sum\limits_{n=1}^{\infty} \frac{1}{n^2} < +\infty$, it follows from the Comparison Test that $\sum\limits_{n=1}^{\infty} \frac{\sin n}{n^2}$ converges absolutely, and hence it is convergent. ◇

Example 3.30. $\sum\limits_{n=2}^{\infty} \frac{1}{\log n}$ diverges. For all $n \in \mathbb{N}$, $\log n \leqslant n$. (By the Mean Value Theorem[4], there exists a $c \in (1, n)$ such that $\frac{\log n - \log 1}{n-1} = \frac{\log n}{n-1} = \frac{1}{c} < 1$, and by rearranging, $\log n < n - 1 < n$ for $n > 1$.) So for all $n \geqslant 2$, $\frac{1}{\log n} \geqslant \frac{1}{n} =: d_n$. As $\sum\limits_{n=2}^{\infty} \frac{1}{n}$ diverges, it follows from the Comparison Test that $\sum\limits_{n=2}^{\infty} \frac{1}{\log n}$ diverges too. ◇

Theorem 3.31 (Ratio test)**.** *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of nonzero terms.*

(1) *If there exists an $r \in (0, 1)$ and there exists an $N \in \mathbb{N}$ such that for all $n > N$, $|\frac{a_{n+1}}{a_n}| \leqslant r$, then $\sum\limits_{n=1}^{\infty} a_n$ converges absolutely.*

(2) *If there exists an $N \in \mathbb{N}$ such that for all $n > N$, $|\frac{a_{n+1}}{a_n}| \geqslant 1$, then $\sum\limits_{n=1}^{\infty} a_n$ diverges.*

[4]See Theorem 5.13 in Chapter 5.

Proof. (1) We have

$$\begin{aligned} |a_{N+1}| &\leqslant r|a_N|, \\ |a_{N+2}| &\leqslant r|a_{N+1}| \leqslant r^2|a_N|, \\ |a_{N+3}| &\leqslant r|a_{N+2}| \leqslant r^3|a_N|, \\ &\vdots \end{aligned}$$

Since the geometric series $\sum\limits_{n=1}^{\infty} r^n$ converges, we obtain $\sum\limits_{n=N+1}^{\infty} |a_n| < +\infty$ by the Comparison Test. By adding the finitely sum $|a_1| + \cdots + |a_N|$ to each partial sum of this last series, we see that also $\sum\limits_{n=1}^{\infty} |a_n|$ converges. This completes the proof of the claim in (1).

(2) The given condition implies that

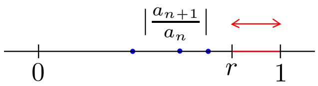
$$\cdots \geqslant |a_{N+3}| \geqslant |a_{N+2}| \geqslant |a_{N+1}|, \tag{3.2}$$

If the series $\sum\limits_{n=1}^{\infty} a_n$ was convergent, then $0 = \lim\limits_{n\to\infty} a_n = \lim\limits_{k\to\infty} a_{N+k}$. Hence $\lim\limits_{k\to\infty} |a_{N+k}| = 0$ as well. But by the (3.2), we see that $\lim\limits_{k\to\infty} |a_{N+k}| \geqslant |a_{N+1}| > 0$, a contradiction. □

It does not suffice for convergence of the series that for all sufficiently large n, $|\frac{a_{n+1}}{a_n}| < 1$.

For example, for the harmonic series $|\frac{a_{n+1}}{a_n}| = |\frac{\frac{1}{n+1}}{\frac{1}{n}}| = \frac{n}{n+1} < 1$, but $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges.

So the ratios have to uniformly separated from 1 (by a positive distance $1 - r$).



In the case of the Harmonic Series, there is no $r \in (0, 1)$ such that $|\frac{a_{n+1}}{a_n}| = \frac{n}{n+1} \leqslant r < 1$ for all large n, since if there were such an r, then $\lim\limits_{n\to\infty} \frac{n}{n+1} = 1 \leqslant r < 1$, a contradiction.

Corollary 3.32. *Suppose that the terms of the sequence $(a_n)_{n\in\mathbb{N}}$ are all nonzero. If $\lim\limits_{n\to\infty} |\frac{a_{n+1}}{a_n}| < 1$, then $\sum\limits_{n=1}^{\infty} a_n$ converges absolutely.*

Proof. Let $L := \lim\limits_{n\to\infty} |\frac{a_{n+1}}{a_n}| \in [0, 1)$. Then $\epsilon := \frac{1-L}{2} > 0$. Choose $N \in \mathbb{N}$ such that for $n > N$,

$$\Big(\, |\tfrac{a_{n+1}}{a_n}| - L \leqslant \,\Big) \quad \Big||\tfrac{a_{n+1}}{a_n}| - L\Big| < \epsilon = \tfrac{1-L}{2},$$

and so $|\frac{a_{n+1}}{a_n}| < \frac{1+L}{2} =: r < \frac{1+1}{2} = 1$. The claim follows from Theorem 3.31(1). □

Example 3.33 (The exponential series)**.** Let $a \in \mathbb{R}$. The series $e^a := \sum\limits_{n=0}^{\infty} \frac{1}{n!} a^n$ converges.

For $a = 0$, $e^0 = 1$. For $a \neq 0$, convergence follows from the ratio test: $|\frac{\frac{a^{n+1}}{(n+1)!}}{\frac{a^n}{n!}}| = \frac{|a|}{n+1} \stackrel{n\to\infty}{\longrightarrow} 0$. ◇

Exercise 3.34. Suppose that the terms of the sequence $(a_n)_{n\in\mathbb{N}}$ are all nonzero. If $\limsup\limits_{n\to\infty} |\frac{a_{n+1}}{a_n}| < 1$, then show that $\sum\limits_{n=1}^{\infty} a_n$ converges absolutely.

Theorem 3.35 (Root test)**.**

(1) *If there exists an $r \in (0, 1)$ and there exists an $N \in \mathbb{N}$ such that for all $n > N$, $\sqrt[n]{|a_n|} \leqslant r$, then $\sum\limits_{n=1}^{\infty} a_n$ converges absolutely.*

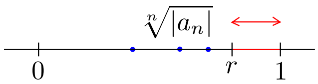
(2) *If for infinitely many n, $\sqrt[n]{|a_n|} \geqslant 1$, then $\sum\limits_{n=1}^{\infty} a_n$ diverges.*

Proof. (1) We have $|a_n| \leqslant r^n$ for all $n > N$, so that by the Comparison Test, $\sum\limits_{n=N+1}^{\infty} |a_n|$ converges.

(2) Suppose that for the subsequence $(a_{n_k})_{k\in\mathbb{N}}$, we have $\sqrt[n_k]{|a_{n_k}|} \geqslant 1$. Then $|a_{n_k}| \geqslant 1$. If the series was convergent, then $\lim\limits_{n\to\infty} a_n = 0$, and so also $\lim\limits_{n\to\infty} |a_{n_k}| = 0$, a contradiction. ☐

It does not suffice for convergence of the series that for all sufficiently large n, $\sqrt[n]{|a_n|} < 1$. For example, for the harmonic series $\sqrt[n]{|a_n|} = \frac{1}{\sqrt[n]{n}} < 1$, but $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges.

One needs the *uniform* separation from 1 (by a positive distance $1 - r$).



Example 3.36 (Ratio Test inconclusive; but Root Test decisive). $\sum\limits_{n=1}^{\infty} \frac{1}{2^{n+(-1)^n}}$ converges. We have:

$$\begin{array}{r|ccccccccccccc}
n: & 1 & \xrightarrow{1} & 2 & \xrightarrow{1} & 3 & \xrightarrow{1} & 4 & \xrightarrow{1} & 5 & \xrightarrow{1} & 6 & \cdots \\
(-1)^n: & -1 & \xrightarrow{2} & 1 & \xrightarrow{-2} & -1 & \xrightarrow{2} & 1 & \xrightarrow{-2} & -1 & \xrightarrow{2} & 1 & \cdots \\
\hline
n+(-1)^n: & \cdot & \xrightarrow{3} & \cdot & \xrightarrow{-1} & \cdot & \xrightarrow{3} & \cdot & \xrightarrow{-1} & \cdot & \xrightarrow{3} & \cdot & \cdots
\end{array}$$

So $|\frac{a_{n+1}}{a_n}|$ alternates between $2^{-3} = \frac{1}{8}$ and $2^1 = 2$, and the Ratio Test is inconclusive. But

$$\sqrt[n]{|a_n|} = \frac{1}{2^{\frac{n+(-1)^n}{n}}} = \frac{1}{2^{1+\frac{(-1)^n}{n}}} \xrightarrow{n\to\infty} \frac{1}{2^{1+0}} = \frac{1}{2} < 1,$$

and so, by the Root Test, $\sum\limits_{n=1}^{\infty} \frac{1}{2^{n+(-1)^n}}$ converges. ◇

Corollary 3.37. *If* $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} < 1$, *then* $\sum\limits_{n=1}^{\infty} a_n$ *converges absolutely.*

Proof. Let $L := \lim\limits_{n\to\infty} \sqrt[n]{|a_n|} \in [0, 1)$. Then $\epsilon := \frac{1-L}{2} > 0$. Choose $N \in \mathbb{N}$ such that for $n > N$,

$$\left(\sqrt[n]{|a_n|} - L \leqslant \right) \quad \left| \sqrt[n]{|a_n|} - L \right| < \epsilon = \tfrac{1-L}{2},$$

and so $\sqrt[n]{|a_n|} < \frac{1+L}{2} =: r < \frac{1+1}{2} = 1$. The claim follows from Theorem 3.35(1). ☐

Exercise 3.38.

(1) If $\limsup\limits_{n\to\infty} \sqrt[n]{|a_n|} < 1$, then show that $\sum\limits_{n=1}^{\infty} a_n$ converges absolutely.

(2) If $\limsup\limits_{n\to\infty} \sqrt[n]{|a_n|} > 1$, then show that $\sum\limits_{n=1}^{\infty} a_n$ diverges.

(3) If $\limsup\limits_{n\to\infty} \sqrt[n]{|a_n|} = 1$, then show by examples $\sum\limits_{n=1}^{\infty} a_n$ can converge or diverge.

Exercise 3.39. Determine if the following series are convergent or not.

(1) $\sum\limits_{n=1}^{\infty} \frac{n^2}{2^n}$.

(2) $\sum\limits_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$.

(3) $\sum\limits_{n=1}^{\infty} (\frac{4}{5})^n n^5$.

Exercise 3.40. Prove that $\sum\limits_{n=1}^{\infty} \frac{n}{n^4+n^2+1}$ converges, and find its value. *Hint:* $n^4 + n^2 + 1 = (n^2+1)^2 - n^2$.

Exercise 3.41. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that the series $\sum\limits_{n=1}^{\infty} a_n^4$ converges. Show that $\sum\limits_{n=1}^{\infty} a_n^5$ converges. *Hint:* First conclude that for large n, $|a_n| < 1$.

Exercise 3.42. Determine if the following statements are true or false. Give reasons for your answers.

(1) If $\sum_{n=1}^{\infty} |a_n|$ is convergent, then so is $\sum_{n=1}^{\infty} a_n^2$.

(2) If $\sum_{n=1}^{\infty} a_n$ is convergent, then so is $\sum_{n=1}^{\infty} a_n^2$.

(3) If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

(4) If $\lim_{n\to\infty} (a_1 + \cdots + a_n) = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

(5) $\sum_{n=1}^{\infty} \log \frac{n+1}{n}$ converges.

(6) If $a_n > 0$ $(n \in \mathbb{N})$ and the partial sums of $(a_n)_{n\in\mathbb{N}}$ are bounded above, then $\sum_{n=1}^{\infty} a_n$ converges.

(7) If $a_n > 0$ $(n \in \mathbb{N})$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} \frac{1}{a_n}$ diverges.

Exercise 3.43 (Fourier series)**.** In order to understand a complicated situation, it is natural to try to break it up into simpler things. For example, from Calculus we learn that an analytic function can be expanded into a Taylor series, where we break it down into the simplest possible analytic functions, namely monomials $1, x, x^2, \dots$ as follows: $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots$.

The idea behind the Fourier series is similar. In order to understand a complicated *periodic* function, we break it down into the simplest periodic functions, namely sines and cosines. Thus if $T \geqslant 0$ and $f : \mathbb{R} \to \mathbb{R}$ is T*-periodic*, that is, $f(x) = f(x + T)$ $(x \in \mathbb{R})$, then one tries to find coefficients $a_0, a_1, a_2, \dots$ and $b_1, b_2, b_3, \dots$ such that

$$f(x) = a_0 + \sum_{n=1}^{\infty} \Big(a_n \cos(\tfrac{2\pi n}{T}x) + b_n \sin(\tfrac{2\pi n}{T}x)\Big). \tag{3.3}$$

(1) Let the Fourier series (3.3) converge pointwise to f on $\mathbb{R}$.
Show that if $\sum_{n=1}^{\infty} (|a_n| + |b_n|) < \infty$, then in fact the series converges uniformly.

(2) The aim of this part of the exercise is to give experimental evidence for two things. Firstly, the plausibility of the Fourier expansion, and secondly, that the uniform convergence might fail if the condition in the previous part of this exercise does not hold. Consider the *square wave* $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} 1 & \text{if } x \in [n, n+1) \text{ for } n \text{ even}, \\ -1 & \text{if } x \in [n, n+1) \text{ for } n \text{ odd}. \end{cases}$$

Then f is 2-periodic. From the theory of Fourier Series, which we will not discuss here, the coefficients can be calculated, and they happen to be $0 = a_0 = a_1 = a_2 = a_3 = \dots$ and

$$b_n = \begin{cases} \frac{4}{n\pi} & \text{if } n \text{ is odd}, \\ 0 & \text{if } n \text{ is even}. \end{cases}$$

Write a Maple program to plot the graphs of the partial sums of the series in (3.3) with, say, 3, 33, 333 terms. Discuss your observations.

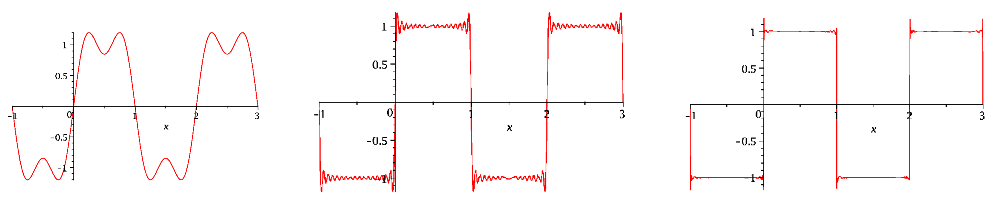


Figure 1. Partial sums of the Fourier series for the square wave considered in Exercise 3.43.

Exercise 3.44. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence with nonnegative terms.
Show that if $\sum_{n=1}^{\infty} a_n$ converges, then so does $\sum_{n=1}^{\infty} \sqrt{a_n a_{n+1}}$.

Exercise 3.45. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence with nonnegative terms.
Prove that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ converges.

Exercise 3.46. Let ℓ^1 be defined by $\ell^1 = \{(a_n)_{n\in\mathbb{N}} : \sum_{n=1}^{\infty} |a_n| < \infty\}$. Show that $\ell^1 \subset \ell^2$. Is $\ell^1 = \ell^2$? (The normed space ℓ^2 was defined in Exercise 1.20 on page 6.)

Exercise 3.47. Since the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, we have that the reciprocal $\frac{1}{s_n}$ of the n^{th} partial sum $s_n := 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, approaches 0 as $n \to \infty$. So the necessary condition for the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{s_n}$ is satisfied. But we don't know yet whether or not it actually converges. It is clear that the harmonic series diverges very slowly, which means that $\frac{1}{s_n}$ decreases very slowly, and this prompts the guess that this series diverges. Show that in fact our guess is correct. *Hint:* $s_n \leqslant n$.

Exercise 3.48. Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^n}$ converges.

Exercise 3.49. Define the Fibonacci sequence $(F_n)_{n\in\mathbb{N}}$ by $F_0 = F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \in \mathbb{N}$. Show that $\sum_{n=0}^{\infty} \frac{1}{F_n} < +\infty$. *Hint:* $F_{n+1} = F_n + F_{n-1} \geqslant F_{n-1} + F_{n-1} = 2F_{n-1}$. So $F_{2n} \geqslant 2^n$ and $F_{2n+1} \geqslant 2^n$.

Exercise 3.50. Determine if the series $\sum_{n=1}^{\infty} (\sqrt{1+n^2} - n)$ is convergent or not.

Exercise 3.51. Show that $\sum_{n=1}^{\infty} \sin(\pi\sqrt{n^4+1})$ converges absolutely.

Exercise 3.52 (Dirichlet series)**.** In Analytic Number Theory, one encounters *Dirichlet series*, which is a series of the form $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$, where $(a_n)_{n\in\mathbb{N}}$ is a real sequence, and $s \in \mathbb{R}$. An example is the Riemann zeta function, where each $a_n = 1$, and we have seen that the series converges for all $s > 1$, but diverges if $s = 1$. In this exercise we consider two examples, one of a Dirichlet series that converges for all $s \in \mathbb{R}$, and another which diverges for each $s \in \mathbb{R}$.

(1) Show that for all $s \in \mathbb{R}$, $\sum_{n=1}^{\infty} \frac{1}{n!n^s}$ converges.

(2) Show that for all $s \in \mathbb{R}$, $\sum_{n=1}^{\infty} \frac{n!}{n^s}$ diverges.

3.1.2. Power series. Let $(c_n)_{n\in\mathbb{N}}$ be a real sequence (thought of as a sequence of 'coefficients'). An expression of the type

$$\sum_{n=0}^{\infty} c_n x^n$$

is called a *power series* in the variable $x \in \mathbb{R}$.

This is generalization of the familiar polynomial function $c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_d x^d$. Indeed, all polynomial expressions are (finite) power series, with the coefficients being eventually all zeros. For example, $1 + 399x - x^3 = \underset{c_0}{1} + \underset{c_1}{399}x + \underset{c_2}{0}x^2 + \underset{c_3}{(-1)}x^3 + \underset{c_4}{0}x^4 + \underset{c_5}{0}x^5 + \underset{c_6}{0}x^6 + \cdots$.

$\sum_{n=0}^{\infty} x^n$, $\sum_{n=0}^{\infty} \frac{1}{n!}x^n$ are examples of power series, which are not polynomials.

Power series arise naturally in applications. For example, it can be shown that the following boundary value problem for the Ordinary Differential Equation (ODE)

$$f''(x) + xf'(x) + x^2 f(x) = 0 \text{ with } f(0) = 1,\ f(1) = 0$$

has the following 'power series solution':

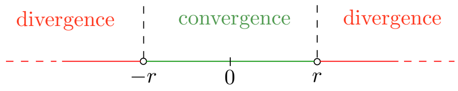
$$f(x) = 1 - \tfrac{1}{12}x^4 + \tfrac{1}{90}x^6 + \tfrac{1}{3360}x^8 + \cdots, \quad x \in [0,1].$$

So questions about the convergence of power series are also natural.

Note that we have not said anything about the set of $x \in \mathbb{R}$ where the power series converges. Of course the power series $\sum_{n=0}^{\infty} c_n x^n$ always converges for $x = 0$.

For which $x \in \mathbb{R}$ does $\sum_{n=0}^{\infty} c_n x^n$ converge?

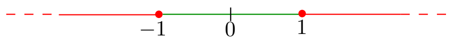
We will discover that the answer is: For all x in an interval like this:



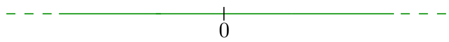
It turns out that there is a maximal open interval $B(0, r) = (-r, r)$ centred at 0 of radius r where the power series converges absolutely, and we call the radius r as the *radius of convergence* of the power series. If the power series converges for all $x \in \mathbb{R}$, that is, if the above maximal interval is $(-\infty, \infty)$, we say that the power series has *infinite radius of convergence*.

Example 3.53.

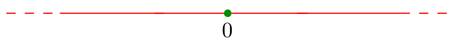
The radius of convergence of $\sum\limits_{n=0}^{\infty} x^n$ is 1. Indeed, the geometric series converges for $x \in (-1, 1)$ and diverges whenever $|x| \geqslant 1$.



The radius of convergence of $\sum\limits_{n=0}^{\infty} \frac{1}{n!} x^n$ is infinite, since it converges for every $x \in \mathbb{R}$.



The radius of convergence of $\sum\limits_{n=0}^{\infty} n^n x^n$ is zero. Indeed, whenever $x \neq 0$, $\sqrt[n]{|n^n x^n|} = n|x| > 1$ for all n large enough. By the Root test, the power series diverges for all nonzero real numbers.



$\diamond$

Theorem 3.54. *Let $(c_n)_{n \geqslant 0}$ be a real sequence. Then*

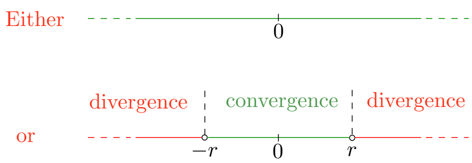
either $\sum\limits_{n=0}^{\infty} c_n x^n$ *is absolutely convergent for all $x \in \mathbb{R}$*

or *there exists a unique $r \geqslant 0$ such that*

(1) $\sum\limits_{n=0}^{\infty} c_n x^n$ *is absolutely convergent for $x \in (-r, r)$ and*

(2) $\sum\limits_{n=0}^{\infty} c_n x^n$ *diverges for $x \notin [-r, r]$.*

That is:



Proof. Let $S := \left\{ y \in [0, \infty) : \exists x \in \mathbb{R} \text{ such that } y = |x| \text{ and } \sum\limits_{n=0}^{\infty} c_n x^n \text{ converges} \right\}$. Clearly $0 \in S$. Only two cases are possible:

1° S is not bounded above (in which case we'll show '$r = \infty$').

2° S is bounded above (in which case we'll show $r = \sup S$).

1° Suppose that S is not bounded above. Let $x \in \mathbb{R}$. Then $|x|$ can't be an upper bound for S. So there must be an element $y \in S$ that prevents $|x|$ from being an upper bound, that is, we can find a $y = |x_0| \in S$ such that $\sum_{n=0}^{\infty} c_n x_0^n$ converges, and $|x| < |x_0|$. It follows that the n^{th} term goes to 0 as $n \to \infty$, and in particular, the sequence of terms is bounded: $|c_n x_0^n| \leqslant M$. Then noting that $|x_0| > 0$ (because $|x_0| = y > |x| \geqslant 0$), we have with $\rho := \frac{|x|}{|x_0|}$ (< 1), that $|c_n x^n| = |c_n x_0^n| (\frac{|x|}{|x_0|})^n \leqslant M\rho^n$ $(n \in \mathbb{N})$. As the geometric series $\sum_{n=0}^{\infty} M\rho^n$ converges, by the Comparison Test, $\sum_{n=0}^{\infty} c_n x^n$ is absolutely convergent. As $x \in \mathbb{R}$ was arbitrary, the claim follows.

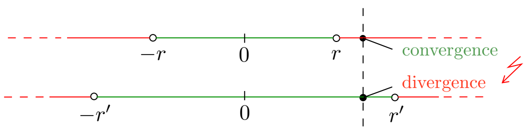
2° Now suppose that S is bounded above.

(1) If $x \in \mathbb{R}$ and $|x| < \sup S$, then by the definition of supremum, there exists a $y \in S$ such that $|x| < y$. Then we repeat the proof in 1° above as follows.

Since $y \in S$, there exists an $x_0 \in \mathbb{R}$ such that $y = |x_0|$ and $\sum_{n=0}^{\infty} c_n x^n$ converges. Hence $|c_n x_0^n| \stackrel{n\to\infty}{\longrightarrow} 0$, and in particular, there exists an $M > 0$ such that for all n, $|c_n x_0^n| \leqslant M$. Then with $\rho := \frac{|x|}{|x_0|}$ (< 1), we have $|c_n x^n| = |c_n x_0^n| (\frac{|x|}{|x_0|})^n \leqslant M\rho^n$ $(n \in \mathbb{N})$. As $\rho < 1$, $\sum_{n=0}^{\infty} M\rho^n$ converges. By the Comparison Test, $\sum_{n=0}^{\infty} c_n x^n$ is absolutely convergent.

(2) If $x \in \mathbb{R}$ and $|x| > \sup S$, then setting $y := |x|$, we see that $y \notin S$.

So by the definition of S, $\sum_{n=0}^{\infty} c_n x^n$ diverges (for otherwise $y \in S$).



The uniqueness of the radius of convergence is obvious, since if r, r' are distinct numbers having the property described in the theorem and $r < r'$, then $r < \rho := \frac{r+r'}{2} < r'$, and as $0 < \rho < r'$, $\sum_{n=1}^{\infty} c_n \rho^n$ ought to converge, while as $0 < \rho < r$, it ought to diverge, a contradiction. □

If r is the radius of convergence of a power series, then $(-r, r)$ is called the *interval of convergence* of that power series. We note that the interval of convergence is the empty set if $r = 0$, and we set the interval of convergence to be $\mathbb{R}$ when the radius of convergence is infinite.

The calculation of the radius of convergence is facilitated in some cases by the following two results.

Theorem 3.55. *Consider the power series* $\sum_{n=0}^{\infty} c_n x^n$.

If $L := \lim_{n\to\infty} |\frac{c_{n+1}}{c_n}|$ *exists, then* $r = \frac{1}{L}$ *if* $L \neq 0$*, and the radius of convergence is infinite if* $L = 0$.

Proof. Let $L \neq 0$. We have that for all nonzero x such that $|x| < r = \frac{1}{L}$, there exists a $q < 1$ and a N large enough such that $\frac{|c_{n+1}x^{n+1}|}{|c_n x^n|} = |\frac{c_{n+1}}{c_n}||x| \leqslant q < 1$ for all $n > N$. (This is because $|\frac{c_{n+1}}{c_n} x| \stackrel{n\to\infty}{\longrightarrow} L|x| < 1$. So we may take for example $q = \frac{L|x|+1}{2} < 1$.) Thus by the Ratio Test, the power series converges absolutely for such x.

If $L = 0$, then for any nonzero $x \in \mathbb{R}$, we can guarantee that $\frac{|c_{n+1}x^{n+1}|}{|c_n x^n|} = |\frac{c_{n+1}}{c_n}||x| \leqslant q < 1$ for all $n > N$. (This is because $|\frac{c_{n+1}}{c_n} x| \stackrel{n\to\infty}{\longrightarrow} 0|x| = 0 < 1$. So we may take for example $q = \frac{1}{2} < 1$.) Thus by the Ratio Test, the power series converges absolutely for such x.

If $L \neq 0$ and $|x| > \frac{1}{L}$, then there exists a N large enough such that $\frac{|c_{n+1}x^{n+1}|}{|c_n x^n|} = |\frac{c_{n+1}}{c_n}||x| > 1$ for all $n > N$. This is because $|\frac{c_{n+1}}{c_n} x| \stackrel{n\to\infty}{\longrightarrow} L|x| > 1$. By the Ratio Test, the power series diverges. □

Theorem 3.56. *Consider the power series* $\sum_{n=0}^{\infty} c_n x^n$.

If $L := \lim_{n\to\infty} \sqrt[n]{|c_n|}$ *exists, then* $r = \frac{1}{L}$ *if* $L \neq 0$, *and the radius of convergence is infinite if* $L = 0$.

Proof. Let $L \neq 0$. We have that for all nonzero x such that $|x| < r = \frac{1}{L}$, there exists a $q < 1$ and a N large enough such that $\sqrt[n]{|c_n x^n|} = \sqrt[n]{|c_n|}\,|x| \leqslant q < 1$ for all $n > N$. (This is because $\sqrt[n]{|c_n|}\,|x| \overset{n\to\infty}{\longrightarrow} L|x| < 1$. So we may take for example $q = \frac{L|x|+1}{2} < 1$.) Thus by the Root Test, the power series converges absolutely for such x.

If $L = 0$, then for any nonzero $x \in \mathbb{R}$, we can guarantee that $\sqrt[n]{|c_n x^n|} = \sqrt[n]{|c_n|}\,|x| \leqslant q < 1$ for all $n > N$. (This is because $\sqrt[n]{|c_n|}\,|x| \overset{n\to\infty}{\longrightarrow} 0|x| = 0 < 1$. So we may take for example $q = \frac{1}{2} < 1$.) Thus by the Root Test, the power series converges absolutely for such x.

If $L \neq 0$ and $|x| > \frac{1}{L}$, then there exists a N large enough such that $\sqrt[n]{|c_n x^n|} = \sqrt[n]{|c_n|}\,|x| > 1$ for all $n > N$. This is because $\sqrt[n]{|c_n|}\,|x| \overset{n\to\infty}{\longrightarrow} L|x| > 1$. By the Root Test, the power series diverges. ☐

Note that whether or not the power series converges at $x = r$ and $x = -r$ is not answered by Theorem 3.54. In fact this is a delicate issue, and either convergence or divergence can take place at these points, as demonstrated by the following examples.

Example 3.57. We have the following:

Power series	Radius of convergence	Set of x's for which the power series converges
$\sum_{n=1}^{\infty} x^n$	1	$(-1, 1)$
$\sum_{n=1}^{\infty} \frac{x^n}{n^2}$	1	$[-1, 1]$
$\sum_{n=1}^{\infty} \frac{x^n}{n}$	1	$[-1, 1)$
$\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n}$	1	$(-1, 1]$

◇

Exercise 3.58. Check all the claims in Example 3.57.

Exercise 3.59. Find the radius of convergence for each of the following power series:

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \cdots \quad \text{and} \quad \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \cdots .$$

Exercise 3.60. Let the power series $\sum_{n=0}^{\infty} c_n x^n$ have radius of convergence r.

(1) If $(\sqrt[n]{|c_n|})_{n\in\mathbb{N}}$ is not bounded, then show that $r = 0$.

(2) If $(\sqrt[n]{|c_n|})_{n\in\mathbb{N}}$ is bounded, and we define $M_n := \sup\{\sqrt[m]{|c_m|} : m \geqslant n\}$ $(n \in \mathbb{N})$, then we know that $(M_n)_{n\in\mathbb{N}}$ is convergent since it is decreasing and bounded. Set $L := \lim_{n\to\infty} M_n = \limsup_{n\to\infty} \sqrt[n]{|c_n|}$.

If $L = 0$, then show that $r = \infty$.
If $L \neq 0$, then show that $r = \frac{1}{L}$.

Power series are infinitely differentiable. We will now show that just like polynomials, power series are infinitely many times differentiable in their respective intervals of convergence, and moreover the derivative is again given by a power series, obtained by termwise differentiation of the original series, and this power series for the derivative has a radius of convergence at least as big as the original series. One can also relate the coefficients of the power series with the successive derivatives of the function defined by the power series at 0.

Let $\sum\limits_{n=0}^{\infty} c_n x^n$ have radius of convergence $r > 0$ and let

$$f(x) := \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots \text{ for all } x \in (-r, r).$$

If termwise differentiation were allowed, then

$$f'(x) = 0 + c_1 \cdot 1 + c_2 \cdot 2x + c_3 \cdot 3x^2 + \cdots = \sum_{n=1}^{\infty} n c_n x^{n-1} \text{ for all } x \in (-r, r).$$

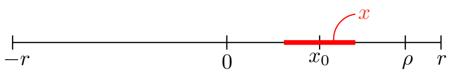
We justify this now.

Theorem 3.61. *Let $r > 0$, and let the power series $f(x) := \sum\limits_{n=0}^{\infty} c_n x^n$ converge for $x \in (-r, r)$. Then f is differentiable in $(-r, r)$, and $f'(x) = \sum\limits_{n=1}^{\infty} n c_n x^{n-1}$ for $x \in (-r, r)$.*

Proof. (*You may skip this proof.*)

Step 1. First we show that the power series $g(x) := \sum\limits_{n=1}^{\infty} n c_n x^{n-1} = c_1 + 2c_2 x + \cdots + n c_n x^{n-1} + \ldots$ is absolutely convergent in $(-r, r)$. Fix $x \in (-r, r)$ and let ρ satisfy $|x| < \rho < r$. By hypothesis, $\sum\limits_{n=0}^{\infty} c_n \rho^n$ converges, and so $\lim\limits_{n\to\infty} c_n \rho^n = 0$. In particular, $(c_n \rho^n)_{n\in\mathbb{N}}$ is bounded, and there is some positive number M such that $|c_n \rho^n| < M$ for all n. Now let $\alpha := \frac{|x|}{\rho}$. Then $0 \leqslant \alpha < 1$, and we have $|n c_n x^{n-1}| = |c_n \rho^n| \cdot \frac{1}{\rho} \cdot n |\frac{x}{\rho}|^{n-1} \leqslant \frac{M n \alpha^{n-1}}{\rho}$. But as $\alpha \in [0, 1)$, by Exercise 3.16, $\sum\limits_{n=1}^{\infty} n\alpha^{n-1} = \frac{1}{(1-\alpha)^2}$. By the Comparison Test, it follows that $\sum\limits_{n=1}^{\infty} n c_n x^{n-1}$ converges absolutely.

Step 2. Now we show that $f'(x_0) = g(x_0)$ for $|x_0| < r$, that is, $\lim\limits_{x\to x_0} \left(\frac{f(x)-f(x_0)}{x-x_0} - g(x_0)\right) = 0$. As before, let ρ be such that $|x_0| < \rho < r$. Below we consider $x \in (-r, r)$ satisfying $|x| < \rho$.



Let $\epsilon > 0$. As $\sum\limits_{n=1}^{\infty} n c_n \rho^{n-1}$ converges absolutely, there is an N such that

$$\sum_{n=N}^{\infty} |n c_n \rho^{n-1}| < \frac{\epsilon}{4}. \tag{3.4}$$

Keep N fixed. We have $f(x) - f(x_0) = \sum\limits_{n=1}^{\infty} c_n (x^n - x_0^n)$, and so for $x \neq x_0$,

$$\frac{f(x)-f(x_0)}{x-x_0} = \sum_{n=1}^{\infty} c_n \frac{x^n - x_0^n}{x - x_0} = \sum_{n=1}^{\infty} c_n (x^{n-1} + x^{n-2} x_0 + \cdots + x x_0^{n-2} + x_0^{n-1}).$$

Thus $\frac{f(x)-f(x_0)}{x-x_0} - g(x_0) = \sum\limits_{n=1}^{\infty} c_n (x^{n-1} + x^{n-2} x_0 + \cdots + x x_0^{n-2} + x_0^{n-1} - n x_0^{n-1})$. We let S_1 be the sum of the first $N-1$ terms of this series (that is, from $n = 1$ to $n = N-1$) and S_2 be the sum of the remaining terms (from $n = N$ to ∞). Then since $|x|, |x_0| < \rho$, it follows that

$$|S_2| \leqslant \sum_{n=N}^{\infty} |c_n| \big(\underbrace{\rho^{n-1} + \rho^{n-1} + \cdots + \rho^{n-1}}_{n \text{ terms}} + n\rho^{n-1} \big) = \sum_{n=N}^{\infty} 2n |c_n| \rho^{n-1} < \frac{\epsilon}{2}.$$

The last inequality holds by (3.4). Also, $S_1 = \sum\limits_{n=1}^{N} c_n (x^{n-1} + x^{n-2} x_0 + \cdots + x x_0^{n-2} + x_0^{n-1} - n x_0^{n-1})$ is a polynomial in x and so

$$\lim_{x\to x_0} S_1 = \sum_{n=1}^{N} c_n \big(x_0^{n-1} + x_0^{n-2} x_0 + \cdots + x_0 x_0^{n-2} + x_0^{n-1} - n x_0^{n-1}\big) = \sum_{n=1}^{N} c_n \left(n x_0^{n-1} - n x_0^{n-1}\right) = 0.$$

So there is a $\delta > 0$ such that whenever $|x - x_0| < \delta$, we have $|S_1| < \frac{\epsilon}{2}$. Thus for $|x| < \rho$ and $0 < |x - x_0| < \delta$, we have $|\frac{f(x)-f(x_0)}{x-x_0} - g(x_0)| \leqslant |S_1| + |S_2| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. This means that $f'(x_0) = g(x_0)$, as wanted. □

By a repeated application of the previous result, we have the following.

Corollary 3.62. *Let $r>0$ and let $f(x) := \sum\limits_{n=0}^{\infty} c_n x^n$ converge for $|x|<r$.*

Then for $k \geqslant 1$, $f^{(k)}(x) = \sum\limits_{n=k}^{\infty} n(n-1)(n-2)\cdots(n-k+1)c_n x^{n-k}$ for $|x|<r$. $(\star)$

In particular, for $n \geqslant 0$, $c_n = \frac{1}{n!} f^{(n)}(0)$.

Proof. A repeated application of Theorem 3.61 gives this: For $n, k \in \mathbb{N} \cup \{0\}$,

$$\frac{d^k}{dx^k} x^n = \begin{cases} n(n-1)\cdots(n-(k-1))x^{n-k} & \text{for } 0 \leqslant k \leqslant n, \\ 0 & \text{for } k > n. \end{cases}$$

For the last claim, we have $f(0) = c_0$, and for the $n \in \mathbb{N}$ cases, set $x=0$ in $(\star)$:

$$f^{(k)}(0) = k(k-1)\cdots 1\, c_k + x \sum_{n=k+1}^{\infty} n(n-1)\cdots(n-k+1)c_n x^{n-k-1}\big|_{x=0} = k!\, c_k. \qquad \square$$

Remark 3.63. There is nothing special about taking power series centered at 0. One can also consider $\sum\limits_{n=0}^{\infty} c_n (x-a)^n$, where a is a fixed real number, and get analogous results to the foregoing.

Exercise 3.64. It can be shown that the power series $f(x) := \sum\limits_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$ and $g(x) := \sum\limits_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$ both have an infinite radius of convergence. Show that for all $x \in \mathbb{R}$, $f'(x) = g(x)$ and $g'(x) = f(x)$.
Show that $(f(x))^2 - (g(x))^2 = 1$, $x \in \mathbb{R}$. *Hint:* Differentiate $f^2 - g^2$ to show constancy, and evaluate at 0.

Exercise 3.65. Find $1 + \frac{2^2}{1!} + \frac{3^2}{2!} + \frac{4^2}{3!} + \cdots$.

Exercise 3.66 (Power series method for solving differential equations)**.** Assuming that the solution to the differential equation $f'(x) = 2xf(x)$ has a power series expansion $f(x) = \sum\limits_{n=0}^{\infty} c_n x^n$, $x \in \mathbb{R}$, find f.

Exercise 3.67 (Generalised Binomial Theorem)**.**
Let $a \in (0,1)$. Show that the radius of convergence of the power series

$$1 + ax + \frac{a(a-1)}{2!}x^2 + \frac{a(a-1)(a-2)}{3!}x^3 + \cdots$$

is 1. Let $f : (-1,1) \to \mathbb{R}$ be the sum of the above power series. Prove that $(1+x)f'(x) = af(x)$ in $(-1,1)$. Calculate $((1+\cdot)^{-a} f)'$ in $(-1,1)$, and hence show that $f(x) = (1+x)^a$, $x \in (-1,1)$.

Exercise 3.68 (Pathological Taylor series)**.**
We have seen that power series define infinitely differentiable functions in the respective regions of convergence. Now suppose that we *start* with an infinitely differentiable function f in an interval $(-r,r)$. Then does it have a 'power series expansion'? We can certainly form the power series $\sum\limits_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$.
Now we may ask: If this series converges for an $x \neq 0$, then is its sum equal to $f(x)$? The answer is, rather surprisingly, 'Not always!'. There exist infinitely differentiable functions f for which the power series converges for $x \neq 0$, but the sum of the series is different from $f(x)$. Consider for example the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

We will show below that $f^{(n)}(0) = 0$ for all $n \geqslant 0$. Hence the power series $\sum\limits_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \equiv 0$, which does not equal $f(x)$ for any nonzero x.

(1) Sketch the graph of f.

(2) Prove that for every $n \in \mathbb{N}$, $\lim\limits_{x \to 0} \frac{f(x)}{x^n} = 0$.

(3) Show that for each $n \in \mathbb{N}$, there is a polynomial p_n such that for all $x \neq 0$, $f^{(n)}(x) = e^{-\frac{1}{x^2}} p_n(\frac{1}{x})$.

(4) Prove that $f^{(n)}(0) = 0$ for all $n \geqslant 1$.

Exercise 3.69. By termwise differentiating the geometric series in its region of convergence, rederive the result in Exercise 3.16: If if $|r| < 1$, then $1 + 2r + 3r^2 + \cdots = \frac{1}{(1-r)^2}$.

3.2. Series in normed spaces

We can't define series in a general metric space, since we need to *add* terms. But in the setting of a normed space, addition of vectors is available, and so we can define the notion of convergence of a series in a normed space.

Definition 3.70. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in a normed space $(X, \|\cdot\|)$.
The sequence $(s_n)_{n\in\mathbb{N}}$ of *partial sums* is defined by $s_n = a_1 + \cdots + a_n \in X$ $(n \in \mathbb{N})$.
The series $\sum\limits_{n=1}^{\infty} a_n$ is called *convergent* if $(s_n)_{n\in\mathbb{N}}$ converges in $(X, \|\cdot\|)$. Then we write $\sum\limits_{n=1}^{\infty} a_n = \lim\limits_{n\to\infty} s_n$.
If the sequence $(s_n)_{n\in\mathbb{N}}$ does not converge we say that the series $\sum\limits_{n=1}^{\infty} a_n$ *diverges.*

It turns out the convergence in a *complete* normed space is guaranteed by the convergence of an associated real series (of the norms of its terms). We first introduce the notion of *absolute convergence* of a series in a normed space, analogous to the absolute convergence of a real series.

Definition 3.71. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in a normed space $(X, \|\cdot\|)$.
We say that the series $\sum\limits_{n=1}^{\infty} a_n$ *converges absolutely* if the (real) series $\sum\limits_{n=1}^{\infty} \|a_n\|$ converges.

Theorem 3.72. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in a complete normed space $(X, \|\cdot\|)$ and $\sum\limits_{n=1}^{\infty} \|a_n\| < +\infty$. Then $\sum\limits_{n=1}^{\infty} a_n$ converges in X.*

Proof. (The proof is the same, mutatis mutandis, as the proof of the fact that absolutely convergent real series converge. The only change is we use norms instead of absolute values, and use the completeness of X in order to conclude that when the partial sums form a Cauchy sequence, they converge to a limit in X.) Let $s_n := a_1 + \cdots + a_n$. We will show that $(s_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Let $\sigma_k := \|a_1\| + \cdots + \|a_k\|$, $k \in \mathbb{N}$. For $n > m$, we have

$$\begin{aligned}\|s_n - s_m\| &= \|(a_1 + \cdots + a_n) - (a_1 + \cdots + a_m)\| = \|a_{m+1} + \cdots + a_n\| \\ &\leqslant \|a_{m+1}\| + \cdots + \|a_n\| = (\|a_1\| + \cdots + \|a_n\|) - (\|a_1\| + \cdots + \|a_m\|) = \sigma_n - \sigma_m.\end{aligned}$$

Since the series $\sum\limits_{n=1}^{\infty} \|a_n\|$ converges (given!), its sequence of partial sums is convergent, and in particular, Cauchy. From the inequality $\|s_n - s_m\| \leqslant \sigma_n - \sigma_m$ above, it follows that $(s_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X. As X is complete, $(s_n)_{n\in\mathbb{N}}$ is convergent. ☐

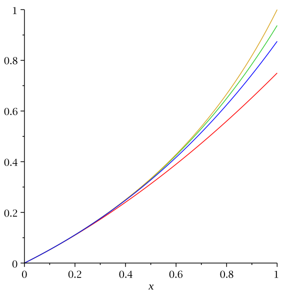
Example 3.73. Let the sequence $(f_n)_{n\in\mathbb{N}}$ in the normed space $(C[0,1], \|\cdot\|_\infty)$ be given by

$$f_n(x) = (\tfrac{x}{2})^n \quad (x \in [0,1],\ n \in \mathbb{N}).$$

The series $\sum\limits_{n=1}^{\infty} f_n$ converges in $(C[0,1], \|\cdot\|_\infty)$ since $\|f_n\|_\infty = \max\limits_{x\in[0,1]} |(\frac{x}{2})^n| = \frac{1}{2^n}$, and the series $\sum\limits_{n=1}^{\infty} \|f_n\|_\infty = \sum\limits_{n=1}^{\infty} \frac{1}{2^n}$ converges. In fact, one can see directly that, since

$$s_n(x) = f_1(x) + \cdots + f_n(x) = \frac{\frac{x}{2} - (\frac{x}{2})^{n+1}}{1 - (\frac{x}{2})} = \frac{x}{2-x}(1 - \frac{x^n}{2^n}),$$

with f defined by $f(x) = \frac{x}{2-x}$ $(x \in [0,1])$, we have $\|s_n - f\|_\infty = \max\limits_{x\in[0,1]} \frac{x}{2-x}\frac{x^n}{2^n} \leqslant \frac{1}{2^n} \stackrel{n\to 0}{\longrightarrow} 0$, and so the series $\sum\limits_{n=1}^{\infty} f_n$ converges to f in $(C[0,1], \|\cdot\|_\infty)$.



The picture above shows plots of the partial sums and their limit f. $\diamond$

The following result plays a central role in Differential Equation theory.

Theorem 3.74. *Let $A \in \mathbb{R}^{d\times d}$. Then the exponential series $e^A := I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$ converges in $(\mathbb{R}^{d\times d}, \|\cdot\|_\infty)$.*

Proof. It is easy to see that if $A, B \in \mathbb{R}^{d\times d}$, then $\|AB\|_\infty \leqslant d\|A\|_\infty\|B\|_\infty$. This follows from:

$$|(AB)_{ij}| = |\sum_{k=1}^{d} A_{ik}B_{kj}| \leqslant \sum_{k=1}^{d} |A_{ik}||B_{kj}| \leqslant \sum_{k=1}^{d} \|A\|_\infty\|B\|_\infty = d\|A\|_\infty\|B\|_\infty.$$

Hence by induction, we have $\|A^n\|_\infty \leqslant d^{n-1}\|A\|_\infty^n \leqslant (d\|A\|_\infty)^n$. Thus $\|\frac{1}{n!}A^n\|_\infty \leqslant \frac{1}{n!}(d\|A\|_\infty)^n$. As $e^{d\|A\|_\infty} = \sum_{n=0}^{\infty} \frac{1}{n!}(d\|A\|_\infty)^n$ converges, by the Comparison Test, the real series $\sum_{n=0}^{\infty} \|\frac{1}{n!}A^n\|_\infty$ converges. Since $(\mathbb{R}^{d\times d}, \|\cdot\|_\infty)$ is complete, $e^A := I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$ converges in $(\mathbb{R}^{d\times d}, \|\cdot\|_\infty)$. $\square$

Exercise 3.75. Determine:

(1) $e^{\mathbf{0}}$, where $\mathbf{0}$ denotes the $d \times d$ matrix with all entries equal to 0.

(2) e^I, where I denotes the $d \times d$ identity matrix.

(3) e^D, where D is the diagonal matrix $D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_d \end{bmatrix}$, where $\lambda_1, \cdots, \lambda_d \in \mathbb{R}$.

Exercise 3.76. Recall Exercise 2.52. We equip c_{00} with the same $\|\cdot\|_2$ norm as for ℓ^2. Consider the sequence $(\frac{1}{n^2}\boldsymbol{e}_n)_{n\in\mathbb{N}}$ in c_{00}, where $\boldsymbol{e}_n$ is the sequence with all terms zeroes, except for the n^{th} one, which ie equal to 1. Show that the series $\sum_{n=1}^{\infty} \|\frac{1}{n^2}\boldsymbol{e}_n\|_2 < \infty$, but that the series $\sum_{n=1}^{\infty} \frac{1}{n^2}\boldsymbol{e}_n$ does not converge in ℓ^2. Conclude that $(c_{00}, \|\cdot\|_2)$ is not a complete normed space.

Exercise 3.77. Let X be a normed space in which every series $\sum_{n=1}^{\infty} a_n$ for which there holds $\sum_{n=1}^{\infty} \|a_n\| < +\infty$, is convergent in X. Prove that X is complete. *Hint:* Given a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$, construct a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ satisfying $\|x_{n_{k+1}} - x_{n_k}\| < \frac{1}{2^k}$. Then take $a_1 = x_{n_1}$, $a_2 = x_{n_2} - x_{n_1}$, $a_3 = x_{n_3} - x_{n_2}$, and so on, and use the fact that a Cauchy sequence possessing a convergent subsequence must itself be convergent, which was a result established in Exercise 2.27.

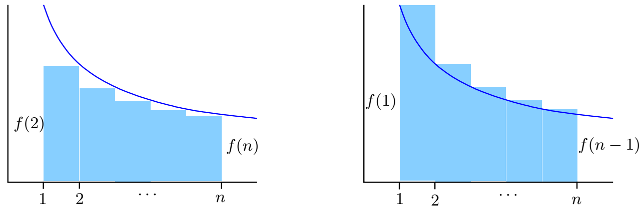
3.3. Notes (not part of the course)

Erdös conjecture on APs. In connection with the divergence of the harmonic series, we mention the *Erdös conjecture on arithmetic progressions* (APs) : If the sums of the reciprocals of the numbers of a set A of natural numbers diverges, then A contains arbitrarily long arithmetic progressions. That is, if $\sum_{n\in A} \frac{1}{n}$ diverges, then A contains APs of any given length. We know that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, and in this case the claim is trivially true. In Exercise 3.15, we have seen that $\sum_{p \text{ prime}} \frac{1}{p}$ diverges. So one may ask: Does the claim hold in this special case? The answer is 'Yes', and this is the Green-Tao Theorem proved in 2004. Terence Tao was awarded the Fields Medal in 2006, among other things, for this result.

Integral Test. It can sometimes be easy to determine whether or not the improper integral $\int_1^\infty f(x)dx$ converges or diverges, and this can be used to deduce the convergence status of the series $\sum_{n=1}^{\infty} f(n)$. This result is known as the *Integral Test*.

Suppose that $f : [1, \infty) \to [0, \infty)$ is decreasing, and that f is Riemann integrable on $[1, n]$ for all $n \in \mathbb{N}$. Let us first show that the following inequalities hold: $\sum_{k=2}^{n} f(k) \leqslant \int_1^n f(x)dx \leqslant \sum_{k=1}^{n-1} f(k)$, for all $n \in \mathbb{N}$.

Consider the interval $[1, n]$, and let $\underline{\sigma}_n$ and $\overline{\sigma}_n$ be the step functions defined by $\underline{\sigma}_n(x) = f(k+1)$ and $\overline{\sigma}_n(x) = f(k)$, for $x \in [k, k+1)$, $k \in \{1, \cdots, n\}$. Since f is decreasing, for all $x \in [1, n]$, $\underline{\sigma}_n(x) \leqslant f(x) \leqslant \overline{\sigma}_n(x)$. Thus $\sum_{k=2}^{n} f(k) = \int_1^n \underline{\sigma}_n(x)dx \leqslant \int_1^n f(x)dx \leqslant \int_1^n \overline{\sigma}_n(x)dx = \sum_{k=1}^{n} f(k)$.



$\sum_{k=2}^{n} f(k) \leqslant \int_1^n f(x)dx$ $\qquad$ $\int_1^n f(x)dx \leqslant \sum_{k=1}^{n-1} f(k)$

We claim that $\sum_{n=1}^{\infty} f(n)$ converges if and only if $\int_1^\infty f(x)dx$ does. We have the following cases:

1° $\int_1^\infty f(x)dx$ converges. The first inequality above shows that the partial sums $\sum_{k=1}^{n} f(k)$ are bounded above by $f(1) + \int_1^\infty f(x)dx$. As $f(k) \geqslant 0$ for all k, the partial sums are increasing. So $\sum_{n=1}^{\infty} f(n)$ converges.

2° $\int_1^\infty f(x)dx$ diverges. Since for all $n \in \mathbb{N}$, $\int_1^n f(x)dx \leqslant \sum_{k=1}^{n-1} f(k)$, it follows that the partial sums $\sum_{k=1}^{n-1} f(k)$ can't form a bounded sequence, and so $\sum_{n=1}^{\infty} f(n)$ diverges.

Example 3.78 ($\sum_{n=2}^{\infty} \frac{1}{n \log n}$ diverges)**.** Let $f(x) = \frac{1}{x \log x}$, $x \geqslant 2$. Then $f : [2, \infty) \to (0, \infty)$ is decreasing. Using the substitution $u = \log x$ (so that $du = \frac{1}{x}dx$, and when $x = 2$, $u = \log 2$, while if $x = y$, then $u = \log y$), we have $\int_2^y \frac{1}{x \log x}dx = \int_{\log 2}^{\log y} \frac{1}{u}du = \log u\big|_{\log 2}^{\log y} = \log(\log y) - \log(\log 2)$. As $\log y \stackrel{y\to\infty}{\longrightarrow} \infty$ it follows that $\log(\log y) \stackrel{y\to\infty}{\longrightarrow} \infty$, and so $\int_2^\infty \frac{1}{x \log x}dx$ does not converge. Hence by the Integral Test $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ diverges too. (Note that we start the sum with $n = 2$ to avoid n being 1 when $\log n = 0$.) ◇

Example 3.79 ($\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2} < +\infty$)**.** Let $g(x) = \frac{1}{x(\log x)^2}$, $x \geqslant 2$. Then $g : [2, \infty) \to (0, \infty)$ is decreasing. Using the substitution $u = \log x$ (so that $du = \frac{1}{x}dx$, and when $x = 2$, $u = \log 2$, while if $x = y$, then $u = \log y$), we have $\int_2^y \frac{1}{x(\log x)^2}dx = \int_{\log 2}^{\log y} \frac{1}{u^2}du = -\frac{1}{u}\big|_{\log 2}^{\log y} = -\frac{1}{\log y} + \frac{1}{\log 2}$. Thus

$$\int_2^\infty \frac{1}{x(\log x)^2}dx = \lim_{y\to\infty} \int_2^y \frac{1}{x(\log x)^2}dx = \lim_{y\to\infty} \left(-\frac{1}{\log y} + \frac{1}{\log 2}\right) = 0 + \frac{1}{\log 2} = \frac{1}{\log 2}.$$

As the improper integral $\int_2^\infty \frac{1}{x(\log x)^2}dx$ converges, by the Integral Test, $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$ converges too. ◇

The Riemann Rearrangement Theorem. In light of Exercise 3.27, one might wonder what happens with series that are convergent, but not absolutely convergent. (Such series are sometimes called *conditionally convergent*.) The behaviour is radically different, as demonstrated by the following result. It is suprising enough that the naive expectation of 'commutativity' fails, but even more striking is the fact that the rearrangement can be done so as to get any limit whatsoever!

Theorem 3.80 (Riemann Rerrangement Theorem)**.** *Let $\sum_{n=1}^{\infty} a_n$ be a conditionally convergent series.*

(1) *If $L \in \mathbb{R}$, then there exists a permutation $p_L : \mathbb{N} \to \mathbb{N}$ such that $\sum_{n=1}^{\infty} a_{p_L(n)} = L$.*

(2) *There exist permutations p_∞ and $p_{-\infty}$ such that $\sum_{n=1}^{\infty} a_{p_\infty(n)}$ and $-\sum_{n=1}^{\infty} a_{p_{-\infty}(n)}$ diverge to $+\infty$.*

In (2), 'diverges to $+\infty$' means that if $(s_n)_{n\in\mathbb{N}}$ is the sequence of partial sums, then for all $M \in \mathbb{R}$, there exists an index $N \in \mathbb{N}$ such that for every $n > N$, $s_n > M$. The interested reader is referred to [**R**] for a proof of the above result.

Chapter 4

Continuous functions

Let X and Y be metric spaces. As there is a notion of distance between pairs of elements in either space, one can talk about continuity of maps. Within the huge collection of all maps, the class of continuous maps form an important subset. We are interested in continuous maps as they possess some useful properties. Before discussing maps between metric spaces, let us first of all recall the notion of continuity of a function $f : \mathbb{R} \to \mathbb{R}$.

4.1. Continuity of functions from $\mathbb{R}$ to $\mathbb{R}$

Recall that continuity is a 'local' concept, and we have the following notion of the continuity of a function at a point.

Definition 4.1. Let I be an interval and let $c \in I$. A function $f : I \to \mathbb{R}$ is *continuous at* c if for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $x \in I$ satisfies $|x - c| < \delta$, we have $|f(x) - f(c)| < \epsilon$.

f is said to be *continuous on* I if for every $c \in I$, f is continuous at c.

We have seen that if $f, g : \mathbb{R} \to \mathbb{R}$ are continuous on $\mathbb{R}$, then their *composition* $f \circ g : \mathbb{R} \to \mathbb{R}$, given by $(f \circ g)(x) := f(g(x))$ $(x \in \mathbb{R})$, is also continuous on $\mathbb{R}$.

Recall also the following important properties possessed by continuous functions: They preserve convergent sequences, the Intermediate Value Theorem and the Extreme Value Theorem.

Theorem 4.2. *Let I be an interval, $c \in I$, and $f : I \to \mathbb{R}$. Then the following are equivalent:*

(1) *f is continuous at c.*

(2) *For every sequence $(x_n)_{n\in\mathbb{N}}$ contained in I such that $(x_n)_{n\in\mathbb{N}}$ converges to c, the sequence $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(c)$.*

In other words, f is continuous at c if and only if f 'preserves' convergent sequences.

Exercise 4.3. Show that the statement (2) in Theorem 4.2 can be weakened to the following:

($2'$) For every sequence $(x_n)_{n\in\mathbb{N}}$ contained in I such that $(x_n)_{n\in\mathbb{N}}$ converges to c, the sequence $(f(x_n))_{n\in\mathbb{N}}$ converges.

Theorem 4.4 (Intermediate Value Theorem)**.** *If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, and $y \in \mathbb{R}$ is such that $f(a) \leqslant y \leqslant f(b)$ or $f(b) \leqslant y \leqslant f(a)$ (that is, if y lies between $f(a)$ and $f(b)$), then there exists a $c \in [a, b]$ such that $f(c) = y$.*

In other words, a continuous function attains all real values between the values of the function attained at the endpoints.

Finally, we recall the Extreme Value Theorem.

Theorem 4.5 (Extreme Value Theorem)**.** *Let* $f : [a, b] \to \mathbb{R}$ *be continuous on* $[a, b]$*. There exists a* $c \in [a, b]$ *and there exists a* $d \in [a, b]$ *such that*

$$\begin{aligned} f(c) &= \sup\{f(x) : x \in [a, b]\}, \\ f(d) &= \inf\{f(x) : x \in [a, b]\}. \end{aligned}$$

Since $c, d \in [a, b]$, we have $f(c), f(d) \in \{f(x) : x \in [a, b]\}$, and so the supremum and infimum are in fact maximum and minimum, respectively:

$$\begin{aligned} f(c) &= \sup\{f(x) : x \in [a, b]\} = \max\{f(x) : x \in [a, b]\}, \\ f(d) &= \inf\{f(x) : x \in [a, b]\} = \min\{f(x) : x \in [a, b]\}. \end{aligned}$$

We observe that in our definition of continuity of a function at a point, they key idea is that:

'We are guaranteed that $f(x)$ stays close to $f(c)$ for all x close enough to c.'

But 'closeness' is something we know not just in $\mathbb{R}$ but in the context of general metric spaces! We will now learn that indeed continuity can in fact be defined in a quite abstract setting, when we have maps between metric spaces. We will also gain insights into the above properties of continuous functions when we study analogues of the above results in our more general setting.

Exercise 4.6. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \begin{cases} x & \text{if } x \text{ is rational,} \\ -x & \text{if } x \text{ is irrational.} \end{cases}$
Prove that f is continuous only at 0. *Hint:* For every real number, there is a sequence of irrational numbers that converges to it, and a sequence of rational numbers that converges to it (see Exercises 1.41, 1.42).

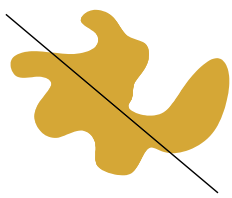
Exercise 4.7. Every nonzero rational number q can be *uniquely* written as $q = \frac{n}{d}$, where n, d denote integers without any common divisors and $d > 0$. When $r = 0$, we take $d = 1$ and $n = 0$.

Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational,} \\ \frac{1}{d} & \text{if } x \ (= \frac{n}{d}) \text{ is rational.} \end{cases}$

Prove that f is discontinuous at every rational number, and continuous at every irrational number.

Hint: For an irrational number x, given any $\epsilon > 0$, and any interval $(N, N + 1)$ containing x, show that there are just finitely many rational numbers r in $(N, N + 1)$ for which $f(r) \geqslant \epsilon$. Use this to show the continuity at irrationals.

Exercise 4.8. Consider a flat pancake of arbitrary shape. Show that there is a straight line cut that divides the pancake into two parts having equal areas. Can the direction of the straight line cut be chosen arbitrarily?



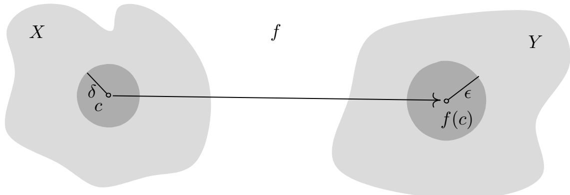
Exercise 4.9. A curve (in the plane) is a map $[0, 1] \ni t \mapsto (x(t), y(t)) \in \mathbb{R} \times \mathbb{R}$, where $x, y : [0, 1] \to \mathbb{R}$ are continuous functions.

(1) Show that any curve $\gamma : [0, 1] \to \mathbb{R} \times \mathbb{R}$ such that $\gamma(0) = (0, 0)$ and $\gamma(1) = (2, 0)$ meets the circle $\mathbb{T} := \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 = 1\}$ at some point, that is, there exists a $c \in (0, 1)$ such that $\gamma(c) \in \mathbb{T}$. *Hint*: If $\gamma(t) = (x(t), y(t))$, $t \in [0, 1]$, then consider $t \mapsto (x(t))^2 + (y(t))^2$.

(2) Suppose $\mu : [0, 1] \to \mathbb{R} \times \mathbb{R}$ is a curve which does not meet the origin, that is, for all $t \in [0, 1]$, $\mu(t) \neq (0, 0)$. Prove that there exist positive real numbers r, R such that the image of μ lies in the 'annulus' $\mathbb{A} := \{(x, y) \in \mathbb{R} \times \mathbb{R} : r^2 < x^2 + y^2 < R^2\}$.

4.2. Continuity of maps between metric spaces

Definition 4.10. Let (X, d_X), (Y, d_Y) be metric spaces, $c \in X$ and $f : X \to Y$ be a map. Then f is said to be *continuous at* c if for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $x \in X$ satisfies $d_X(x, c) < \delta$, we have $d_Y(f(x), f(c)) < \epsilon$.



f is said to be *continuous on* X if for every $c \in X$, f is continuous at c.

First of all we notice that if I is an interval in $\mathbb{R}$, and we take $X = I$, $Y = \mathbb{R}$, both equipped with the Euclidean metric, then the above definition of continuity of a function $f : I \to \mathbb{R}$ at a $c \in \mathbb{R}$ coincides with our earlier Definition 4.1.

We remark that although X may be equal to Y (as sets), they might be equipped with different metrics; see as an extreme example, Exercise 4.12 below.

Exercise 4.11. Show that $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(\mathbf{x}) = \left\{ \begin{array}{cl} \frac{x_1 x_2}{\sqrt{x_1^2 + x_2^2}} & \text{if } \mathbf{x} = (x_1, x_2) \neq \mathbf{0} \\ 0 & \text{if } \mathbf{x} = \mathbf{0} \end{array} \right\}$ is continuous at $\mathbf{0}$.

Exercise 4.12. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 0 & \text{if } x \leqslant 0, \\ 1 & \text{if } x > 0. \end{cases}$

(1) Suppose both the domain $X = \mathbb{R}$ and the codomain $Y = \mathbb{R}$ are equipped with the Euclidean metric. Show that f is not continuous at 0.

(2) Equip the domain $X = \mathbb{R}$ with the discrete metric, and the codomain $Y = \mathbb{R}$ with the Euclidean metric. Prove that f is continuous at 0.

Exercise 4.13. Let (X, d) be a metric space, and let $p \in X$. Show that the distance to p is a continuous map, that is, prove that the function $f : X \to \mathbb{R}$ defined by $f(x) := d(x, p)$ $(x \in X)$ is continuous.

Exercise 4.14. Show that addition $(x, y) \mapsto x + y$ and multiplication $(x, y) \mapsto xy$ are continuous maps from $\mathbb{R}^2$ to $\mathbb{R}$ with the usual Euclidean metrics.

Exercise 4.15. Consider the normed space $(C[0, 1], \|\cdot\|_\infty)$, and let $S : C[0, 1] \to C[0, 1]$ be defined by $(S(f))(x) = (f(x))^2$ $(x \in [0, 1],\ f \in C[0, 1])$. Show that S is continuous.

Proposition 4.16. *Let (X, d_X) be a metric space, and let $f_n : X \to \mathbb{R}$ $(n \in \mathbb{N})$ be a sequence of continuous functions that converges uniformly to $f : X \to \mathbb{R}$. Then f is continuous.*

Proof. Let $c \in X$ and $\epsilon > 0$. Choose an $N \in \mathbb{N}$ such that that for all $x \in X$, $|f_N(x) - f(x)| < \frac{\epsilon}{3}$. As f_N is continuous, there exists a $\delta > 0$ such that for all $x \in X$ satisfying $d_X(x, c) < \delta$, we have $|f_N(x) - f_N(c)| < \frac{\epsilon}{3}$. For all $x \in X$ satisfying $d_X(x, c) < \delta$, we have, using the triangle inequality, that $|f(x) - f(c)| \leqslant |f(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(c)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$. Hence f is continuous at c. Since the choice of $c \in X$ was arbitrary, it follows that f is continuous on X. ☐

Exercise 4.17. A subset S of $\mathbb{R}^n$ is *path connected* if for all $x, y \in \mathbb{R}^n$, there exists a continuous function $\gamma : [0, 1] \to S$ such that $\gamma(0) = x$ and $\gamma(1) = y$. (Think of γ as a 'path' beginning at x and ending at y.)

(1) Show that every convex set C is path connected. (See Exercise 1.45 for the definition of convex sets.)

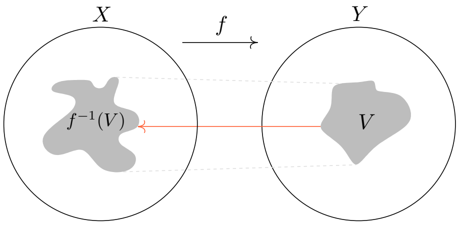
(2) Define the relation R on S by setting $x\,R\,y$ if there is a path $\gamma : [0, 1] \to S$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Prove that R is an equivalence relation on S. The equivalence classes of S under R are called the *path components* of S. So a path connected S has a unique path component, namely S.

(3) Which of the following subsets of $\mathbb{R}^2$ are path connected? For a set that is not path connected, determine its path components. $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, $\{(x, y) \in \mathbb{R}^2 : xy = 0\}$, $\{(x, y) \in \mathbb{R}^2 : xy = 1\}$.

4.3. Continuous maps and open sets

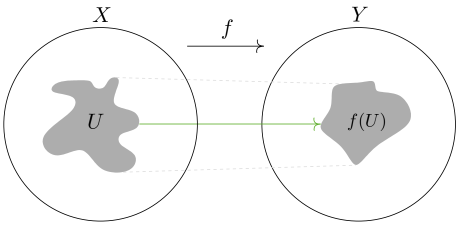
We will now learn an important property of continuous functions, namely that 'inverse images' of open sets under a continuous map are open. In fact, we will see that this property is a characterisation of continuity.

But first we fix some standard notation. Let $f : X \to Y$ be a map, and let $V \subset Y$. Then we set $f^{-1}(V) := \{x \in X : f(x) \in V\}$, and call it the *inverse image of V under f*. See the picture below. Clearly $f^{-1}(Y) = X$ and $f^{-1}(\varnothing) = \varnothing$.



Exercise 4.18. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \cos x$ $(x \in \mathbb{R})$. Find $f^{-1}(V)$, where $V = \{-1, 1\}$, $V = \{1\}$, $V = \{3\}$, $V = [-1, 1]$, $V = \mathbb{R}$, $V = (-\frac{1}{2}, \frac{1}{2})$.

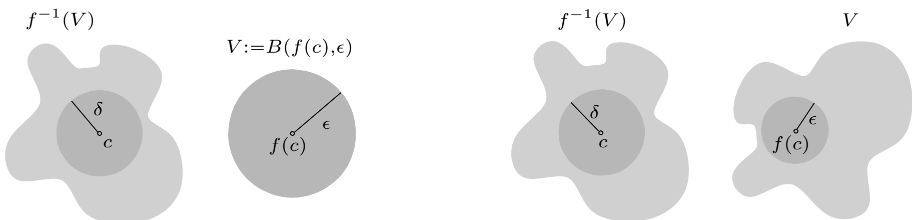
If $U \subset X$, then we set $f(U) := \{f(x) \in Y : x \in U\}$, and call it the *image of U under f*.



Exercise 4.19. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \cos x$ $(x \in \mathbb{R})$. Find $f(U)$, where $U = \mathbb{R}$, $U = [0, 2\pi]$, $U = [\delta, \delta + 2\pi]$ where δ is any positive number.

Theorem 4.20. *Let (X, d_X), (Y, d_Y) be metric spaces and $f : X \to Y$ be a map. Then f is continuous on X if and only if for every V open in Y, $f^{-1}(V)$ is open in X.*

Proof. (If) Let $c \in X$, and let $\epsilon > 0$. Consider the open ball $B(f(c), \epsilon)$ with center $f(c)$ and radius ϵ in Y. We know that this open ball $V := B(f(c), \epsilon)$ is an open set in Y. Thus we also know that $f^{-1}(V) = f^{-1}(B(f(c), \epsilon))$ is an open set in X. But the point $c \in f^{-1}(B(f(c), \epsilon))$, because $f(c) \in B(f(c), \epsilon)$ (indeed, $d_Y(f(c), f(c)) = 0 < \epsilon$!). So by the definition of an open set, there is a $\delta > 0$ such that $B(c, \delta) \subset f^{-1}(B(f(c), \epsilon))$. In other words, whenever $x \in X$ satisfies $d_X(x, c) < \delta$, we have that $x \in f^{-1}(B(f(c), \epsilon))$, that is, $f(x) \in B(f(c), \epsilon)$, which implies $d_Y(f(x), f(c)) < \epsilon$. Hence f is continuous at c. But the choice of $c \in X$ was arbitrary. Consequently f is continuous on X. See the picture on the left-hand side below.



(If) (Only if)

(Only if) Now suppose that f is continuous, and let V be an open subset of Y. We would like to show that $f^{-1}(V)$ is open. So let $c \in f^{-1}(V)$. Then $f(c) \in V$. As V is open, there is a small open ball $B(f(c), \epsilon)$ with center $f(c)$ and radius $\epsilon > 0$ that is contained in V. By the continuity of f at c, there is a $\delta > 0$ such that whenever $d_X(x, c) < \delta$, we have $d_Y(f(x), f(c)) < \epsilon$, that is, $f(x) \in V$. But this means that $B(c, \delta) \subset f^{-1}(V)$. Indeed, if $x \in B(c, \delta)$, then $d_X(x, c) < \delta$ and so by the above, $f(x) \in V$, that is, $x \in f^{-1}(V)$. Consequently, $f^{-1}(V)$ is open in X. See the picture on the right-hand side at the bottom of page 50. ☐

Note that the theorem does *not* claim that for every U open in X, $f(U)$ is open in Y. Consider for example $X = Y = \mathbb{R}$ equipped with the Euclidean metric, and the constant function $f(x) = c$ ($x \in \mathbb{R}$). Then $X = \mathbb{R}$ is open in $X = \mathbb{R}$, but $f(X) = \{c\}$ is not open in $Y = \mathbb{R}$.

Corollary 4.21. *Let (X, d_X), (Y, d_Y) be metric spaces and $f : X \to Y$ be a map. Then f is continuous on X if and only if for every F closed in Y, $f^{-1}(F)$ is closed in X.*

Proof. If $F \subset Y$, then $f^{-1}(Y \backslash F) = X \backslash (f^{-1}(F))$. ☐

Exercise 4.22. Fill in the details of the proof of Corollary 4.21.

Theorem 4.23. *Let $(X, d_X), (Y, d_Y)$, (Z, d_Z) be metric spaces, $f : X \to Y$ and $g : Y \to Z$ be continuous maps. Then the composition map $g \circ f : X \to Z$, defined by $(g \circ f)(x) := g(f(x))$ ($x \in X$), is continuous.*

Proof. Let W be open in Z. Then since g is continuous, $g^{-1}(W)$ is open in Y. Also, since f is continuous, $f^{-1}(g^{-1}(W))$ is open in X. Finally, we note that $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$. Consequently, $g \circ f$ is continuous. ☐

Exercise 4.24. In the proof of Theorem 4.23, we used $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$. Check this.

Exercise 4.25. Let X be a metric space and $f : X \to \mathbb{R}$ be a continuous map. Determine if the following statements are true or false. Justify your answers.

(1) $\{x \in X : f(x) < 1\}$ is an open set.

(2) $\{x \in X : f(x) > 1\}$ is an open set.

(3) $\{x \in X : f(x) = 1\}$ is an open set.

(4) $\{x \in X : f(x) \leqslant 1\}$ is a closed set.

(5) $\{x \in X : f(x) = 1\}$ is a closed set.

(6) $\{x \in X : f(x) = 1 \text{ or } f(x) = 2\}$ is a closed set.

(7) $\{x \in X : f(x) = 1\}$ is a compact set.

Analogous to Theorem 4.2, we have the following characterisation of continuous maps in terms of convergence of sequences.

Theorem 4.26. *Let $(X, d_X), (Y, d_Y)$ be metric spaces, $c \in X$, and let $f : X \to Y$ be a map. Then following two statements are equivalent:*

(1) *f is continuous at c.*

(2) *For every sequence $(x_n)_{n \in \mathbb{N}}$ in X that converges to c, $(f(x_n))_{n \in \mathbb{N}}$ converges to $f(c)$.*

Proof. (1) $\Rightarrow$ (2): Suppose that f is continuous at c. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in X such that $(x_n)_{n \in \mathbb{N}}$ converges to c. Let $\epsilon > 0$. Then there exists a $\delta > 0$ such that for all $x \in X$ satisfying $d_X(x, c) < \delta$, we have $d_Y(f(x), f(c)) < \epsilon$. As the sequence $(x_n)_{n \in \mathbb{N}}$ converges to c, for this $\delta > 0$, there exists an $N \in \mathbb{N}$ such that whenever $n > N$, $d_X(x_n, c) < \delta$. But then by the above, $d_Y(f(x_n), f(c)) < \epsilon$. So we have shown that for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that for all $n > N$, $d_Y(f(x_n), f(c)) < \epsilon$. In other words, the sequence $(f(x_n))_{n \in \mathbb{N}}$ converges to $f(c)$.

(2) $\Rightarrow$ (1): Suppose that f is not continuous at c. Then

$$\neg\big(\forall \epsilon > 0,\ \exists \delta > 0 \text{ such that } \forall x \in X \text{ satisfying } d_X(x,c) < \delta,\ \text{we have } d_Y(f(x), f(c)) < \epsilon\big).$$

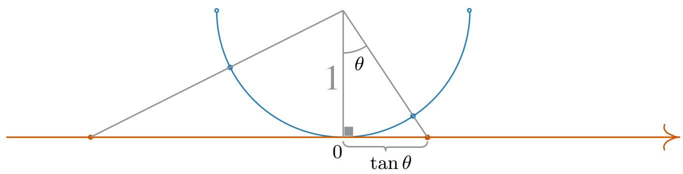
Thus $\exists \epsilon > 0$ such that $\forall \delta > 0$, $\exists x \in X$ satisfying $d_X(x,c) < \delta$, but for which $d_Y(f(x), f(c)) \geqslant \epsilon$. We will use this latter statement to construct a sequence $(x_n)_{n\in\mathbb{N}}$ for which the conclusion in (2) does not hold. For $\delta = \frac{1}{n}$ ($n \in \mathbb{N}$), denote the corresponding x as x_n: Thus, $d_X(x_n, c) < \delta = \frac{1}{n}$, but $d_Y(f(x_n), f(c)) \geqslant \epsilon$. Clearly the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent with limit c, but $(f(x_n))_{n\in\mathbb{N}}$ does not converge to $f(c)$ since $d_Y(f(x_n), f(c)) \geqslant \epsilon$ for all $n \in \mathbb{N}$. So (2) does not hold. We have shown that if (1) does not hold, then (2) does not hold. Consequently, (2) $\Rightarrow$ (1). ☐

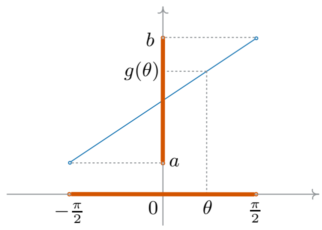
Exercise 4.27. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that for all $x, y \in \mathbb{R}$, $f(x+y) = f(x) + f(y)$. Show that there exists a real number a such that for all $x \in \mathbb{R}$, $f(x) = ax$. *Hint:* Show first that for natural numbers n, $f(n) = nf(1)$. Extend this to integers n, and then to rational numbers $\frac{n}{d}$, $n \in \mathbb{Z}$, $d \in \mathbb{N}$. Finally use the density of $\mathbb{Q}$ in $\mathbb{R}$ to prove the claim.

Exercise 4.28. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$, $f(x) + f(2x) = 0$.
Hint: Show that $f(x) = -f(\frac{x}{2}) = f(\frac{x}{4}) = -f(\frac{x}{8}) = \cdots$.

Exercise 4.29. Define the multiplication function $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(\boldsymbol{x}) = x_1 x_2$ for all $\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2$. Show that f is continuous on $\mathbb{R}^2$ using the characterisation of continuous functions in terms of preservation of convergent sequences. Compare this with Exercise 4.14.

Exercise 4.30. Two metric spaces are called *homeomorphic* if there exists a bijection $f : X \to Y$ such that $f : X \to Y$ and $f^{-1} : Y \to X$ are both continuous. The map f is then called a *homeomorphism*. For example, $f : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ given by $f(\theta) = \tan\theta$ for all $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, is a homeomorphism between the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ and $\mathbb{R}$, both equipped with the Euclidean metric. (This bijection is based on the left-hand side picture below which gives a one-to-one correspondence between points of the semicircular arc of radius 1 and the real line, but can also be checked directly. Based on the continuity of tan on $(-\frac{\pi}{2}, \frac{\pi}{2})$, and the fact that $\tan\theta \to \pm\infty$ as $\theta \to \pm\frac{\pi}{2}$, it follows from the Intermediate Value Theorem that f is surjective. It is also injective, because it can be shown that $f'(\theta) = \frac{1}{(\cos\theta)^2} > 0$, showing that f is strictly increasing on $(-\frac{\pi}{2}, \frac{\pi}{2})$. Hence f is a bijection. Moreover, $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ and $\tan^{-1} : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ are continuous.)





It follows from here than for any real numbers a, b with $a < b$, the open interval (a, b) is homeomorphic to $\mathbb{R}$. This is because there is a homeomorphism $g : (-\frac{\pi}{2}, \frac{\pi}{2}) \to (a, b)$, e.g. using the right-hand side picture above (and then the bijection g is given explicitly by $g(\theta) = (\theta + \frac{\pi}{2})\frac{(b-a)}{\pi} + a$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$).

A natural question is whether the continuity of f^{-1} is actually implied by the continuity of a bijection $f : X \to Y$. This is not true in general. For example, the map $f : [0, 2\pi) \to \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ given by $f(\theta) = (\cos\theta, \sin\theta)$ for all $\theta \in [0, 2\pi)$ can be seen to be a continuous bijection, but its inverse is not continuous at $(1, 0)$: In deed, we have that the sequence $\big(f(2\pi - \frac{1}{n})\big)_{n\in\mathbb{N}}$ converges to $(1, 0)$, but $\big(f^{-1}(f(2\pi - \frac{1}{n}))\big)_{n\in\mathbb{N}} = (2\pi - \frac{1}{n})_{n\in\mathbb{N}}$ does not converge to $0 = f^{-1}(1, 0)$.

The aim of this exercise is to give another example of a continuous bijection whose inverse is not continuous. Recall Exercise 1.21, and let c_{00} denote the subspace of ℓ^∞ consisting of all sequences that have all terms equal to 0 eventually. Consider $c_{00} \subset \ell^\infty$ as normed space with the norm $\|\cdot\|_\infty$, and the map $f : c_{00} \to c_{00}$ given by $f(x_1, x_2, x_3, \cdots) = (x_1, \frac{x_2}{2}, \frac{x_3}{3}, \cdots)$ for all $\boldsymbol{x} = (x_n)_{n\in\mathbb{N}} \in c_{00}$. Show that f is a continuous bijection, whose inverse is not continuous. *Hint:* It can be shown that f^{-1} is not continuous at 0 by using the fact that f is linear and $f(\boldsymbol{e}_m) = \frac{1}{m}\boldsymbol{e}_m$, where $\boldsymbol{e}_m$ is the sequence all of whose terms are zeroes, except for the m^{th} one which is equal to 1.

Exercise 4.31. A 'manifold' is a topological space that locally resembles the Euclidean space. More precisely, we will call a subset M of $\mathbb{R}^n$ a *manifold* (*of dimension*[1] k) if for every $x \in M$, there is an open set O_x containing x such that O_x is homeomorphic to an open subset U of $\mathbb{R}^k$. Locally the surface of the Earth, which is a sphere, looks flat, and so we expect that the sphere in $\mathbb{R}^3$ is a manifold of dimension 2. Give an argument, based on pictures, that the unit sphere $\mathbb{S}^2 := \{x \in \mathbb{R}^3 : \|x\|_2 = 1\}$ is indeed a manifold of dimension 2. (This explains why one uses the superscript '2' on top of $\mathbb{S}$ in the (standard) notation for the unit sphere in $\mathbb{R}^3$. Similarly the circle $\mathbb{S}^1 := \{x \in \mathbb{R}^2 : \|x\|_2 = 1\}$ in $\mathbb{R}^2$ is a manifold of dimension 1, and is denoted by $\mathbb{S}^1$. More generally, it can be shown that the unit sphere $\mathbb{S}^d := \{x \in \mathbb{R}^{d+1} : \|x\|_2 = 1\}$ in $\mathbb{R}^{d+1}$ is a manifold of dimension d.)

Exercise 4.32. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(\boldsymbol{x}) = \begin{cases} \frac{x_1^2 - x_2^2}{x_1^2 + x_2^2} & \text{if } \boldsymbol{x} = (x_1, x_2) \neq (0,0), \\ c & \text{if } \boldsymbol{x} = (0,0). \end{cases}$

Show that no matter what $c \in \mathbb{R}$ we take in the above, f is not continuous at $(0,0)$.

Exercise 4.33. Show that the determinant function $M \mapsto \det M$ from $(\mathbb{R}^{2\times 2}, \|\cdot\|_\infty)$ to $(\mathbb{R}, |\cdot|)$ is continuous. Prove that the set of invertible matrices is open in $(\mathbb{R}^{2\times 2}, \|\cdot\|_\infty)$. *Hint:* Consider $\det^{-1}\{0\}$.

Exercise 4.34. Give an example of a continuous function $f : X \to Y$, where X, Y are metric spaces, and a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ for which $(f(x_n))_{n\in\mathbb{N}}$ is not a Cauchy sequence in Y.

Exercise 4.35. Let X, Y be metric spaces. A map $f : X \to Y$ is called *open* if for every open subset U of X, $f(U)$ is open in Y. Equip $X = \mathbb{R}$ with the usual Euclidean metric, and $Y = \mathbb{R}$ with the discrete metric. Consider the identity map $f : X \to Y$ defined by $f(x) = x$ for all $x \in \mathbb{R}$. Show that f is open, but for each $x \in \mathbb{R}$, f is not continuous at x.

Exercise 4.36. Define $f, g : \mathbb{R}^2 \to \mathbb{R}$ by $f(\mathbf{0}) = g(\mathbf{0}) = 0$, and $f(x,y) = \frac{xy^2}{x^2+y^4}$, $g(x,y) = \frac{xy^2}{x^2+y^6}$ for $(x,y) \neq \mathbf{0}$.

(1) Show that f is bounded on $\mathbb{R}^2$, that is, $\exists M \in \mathbb{R}$ such that for all $(x,y) \in \mathbb{R}^2$, $|f(x,y)| \leqslant M$.

(2) Prove that g is unbounded in every ball centred at $\mathbf{0} = (0,0)$.

(3) Show that f is not continuous at $(0,0)$.

(4) Prove that g is not continuous at $(0,0)$.

(5) If $Y \subset X$ and if Φ is a function defined on X, the *restriction* of Φ to Y is the function φ whose domain is Y, and such that $\varphi(y) = \Phi(y)$ $(y \in Y)$. Show that the restrictions of both f and g to every straight line in $\mathbb{R}^2$ are continuous!

For functions from the Euclidean space $\mathbb{R}^n$ to the Euclidean space $\mathbb{R}^m$, we have the following simplification.

Proposition 4.37. *A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous if and only if each of its components $f_1, \dots, f_m : \mathbb{R}^n \to \mathbb{R}$ are continuous. (Here for $k \in \{1, \cdots, m\}$, $f_k(x) := e_k^\top f(x)$, $x \in \mathbb{R}^n$, where $e_1, \cdots, e_n$ are the standard basis vectors.)*

Proof. For all $x, y \in \mathbb{R}^n$, we have that $|f_k(x) - f_k(y)| \leqslant \sqrt{\sum\limits_{i=1}^n |f_i(x) - f_i(y)|^2} = \|f(x) - f(y)\|_2$.

So if f is continuous, then each of its components is continuous too.

Vice versa, if $f_1, \cdots, f_m$ are continuous and $(x_n)_{n\in\mathbb{N}}$ converges to $c \in \mathbb{R}^n$, then $(f_k(x_n))_{n\in\mathbb{N}}$ converges to $f_k(c)$ for all $k \in \{1, \cdots, d\}$, and so it follows that $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(c)$ in $\mathbb{R}^n$. Thus f is continuous at c. As $c \in \mathbb{R}^n$ was arbitrary, f is continuous. ☐

Another case when checking continuity becomes considerably simpler is in the case of *linear transformations* between *normed* spaces.

Proposition 4.38. *Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be normed spaces and let $T : X \to Y$ be a linear transformation. Then the following are equivalent:*

(1) *T is continuous.*

(2) *T is continuous at* 0.

(3) *There exists an $M > 0$ such that for all $x \in X$, $\|Tx\|_Y \leqslant M\|x\|_X$.*

[1] It can be shown that this is a well-defined notion.

Proof. (1) ⇒ (2) follows from the definition. Let us show that (2) ⇒ (3). As T is continuous at 0, we have that given $\epsilon := 1 > 0$, there is a $\delta > 0$ such that whenever $\|x - 0\|_X = \|x\|_X < \delta$, we have $\|Tx - T0\|_Y = \|Tx - 0\|_Y = \|Tx\|_Y < 1$. Define $M := \frac{2}{\delta}$. Then:

1° If $x = 0$, then $\|Tx\|_Y = \|T0\|_Y = \|0\|_Y = 0 = \frac{2}{\delta}0 = M\|0\|_X = M\|x\|_X$.

2° If $x \neq 0$, then with $y := \frac{\delta}{2\|x\|_X}x$, we have $\|y\| = \frac{\delta}{2} < \delta$, and so $\|Ty\|_Y < 1$.

Thus $1 > \|Ty\|_Y = \|T(\frac{\delta}{2\|x\|_X}x)\|_Y = \frac{\delta}{2\|x\|_X}\|Tx\|_Y$. Rearranging, we get $\|Tx\|_Y < \frac{2}{\delta}\|x\|_X = M\|x\|_X$.

Consequently (3) holds.

Finally, we show that (3) ⇒ (1). Let $c \in X$, and $\epsilon > 0$. Let $\delta := \frac{\epsilon}{M} > 0$. Then for all $x \in X$ satisfying $\|x - c\|_X < \delta = \frac{\epsilon}{M}$, we have $\|Tx - Tc\|_Y = \|T(x - c)\|_Y \leqslant M\|x - c\|_X < M\delta = M\frac{\epsilon}{M} = \epsilon$. Hence f is continuous at c. But the choice of $c \in X$ was arbitrary, and so f is continuous. □

Example 4.39. Consider the map $I : C[a, b] \to \mathbb{R}$ from the normed space $(C[a, b], \|\cdot\|_\infty)$ to $\mathbb{R}$ given by $I(f) = \int_a^b f(x)dx$ for all $f \in C[a, b]$. Then clearly I is a linear transformation. Moreover, since for every $f \in C[a, b]$ we have $|I(f)| = |\int_a^b f(x)dx| \leqslant \int_a^b |f(x)|dx \leqslant \int_a^b \|f\|_\infty dx = \|f\|_\infty(b - a)$, it follows that I is continuous. ◇

Example 4.40. Let $A \in \mathbb{R}^{n\times m}$. Consider the map T_A from the Euclidean space $\mathbb{R}^n$ to the Euclidean space $\mathbb{R}^m$, given by matrix multiplication: $T_Ax = Ax$ $(x \in \mathbb{R}^n)$. Then T_A is a linear transformation, and it is continuous, since

$$\|T_Ax\|_2 = \|Ax\|_2 = \sqrt{\sum_{i=1}^{m}(\sum_{j=1}^{n}a_{ij}x_j)^2} \leqslant \sqrt{\sum_{i=1}^{m}(\sum_{j=1}^{n}a_{ij}^2)(\sum_{k=1}^{n}x_k^2)} \leqslant \sqrt{\sum_{i=1}^{m}n\|A\|_\infty^2\|x\|_2^2} = \sqrt{mn}\,\|A\|_\infty\|x\|_2.$$

(The first inequality follows from the Cauchy-Schwarz inequality.) Hence T_A is continuous. ◇

Exercise 4.41. Show that if $A \in \mathbb{R}^{n\times m}$, then $\ker A = \{x \in \mathbb{R}^m : Ax = 0\}$ is a closed subspace of $\mathbb{R}^m$.

Exercise 4.42. Prove that *every* subspace of $\mathbb{R}^n$ is closed. *Hint:* Construct a linear transformation whose kernel is the given subspace.

Exercise 4.43. A metric space X is called *connected* if X is not the union of two disjoint nonempty open sets. Let X be a connected metric space, Y be a metric space, and $f : X \to Y$ be a surjective map. Prove that Y is connected.

Exercise 4.44. Suppose $f \in C[a, b]$ is such that $\int_a^b x^n f(x)dx = 0$ for all $n \in \mathbb{N}$. Prove that f is identically zero on $[a, b]$. *Hint:* Use the density of polynomials in $(C[a, b], \|\cdot\|_\infty)$ shown in Exercise 1.43.

Exercise 4.45. Let X, Y be normed spaces, $T : X \to Y$ be a linear transformation, and $c \in X$. Show that T is continuous at c if and only if T is continuous at 0.

Exercise 4.46. Let $C^1[0, 1] := \{f \in C[0, 1] : \forall t \in [0, 1],\ f'(t) \text{ exists, and } f' \in C[0, 1]\}$. Then $C^1[0, 1]$ is a subspace of the vector space $C[0, 1]$. Define $D : C^1[0, 1] \to C[0, 1]$ by $(Df)(t) = f'(t)$ for all $t \in [0, 1]$ and $f \in C^1[0, 1]$. Then D is a linear transformation.

(1) Show that if $C[0, 1]$ and $C^1[0, 1]$ are both given the norm $\|\cdot\|_\infty$, then D is not continuous at any point.

(2) Let $\|f\|_{1,\infty} = \|f\|_\infty + \|f'\|_\infty$ for all $f \in C^1[0, 1]$. Check that $\|\cdot\|_{1,\infty}$ does define a norm on $C^1[0, 1]$.

(3) Show that D is continuous if $C^1[0, 1]$ bears the $\|\cdot\|_{1,\infty}$ norm, and $C[0, 1]$ has the $\|\cdot\|_\infty$ norm.

Limits and continuity.

Definition 4.47. Let U be an open subset of $\mathbb{R}^n$, $c \in U$, $L \in \mathbb{R}^m$ and $f : U\backslash\{c\} \to \mathbb{R}^m$. We write

$$f(x) \stackrel{x\to c}{\longrightarrow} L \quad \textit{or} \quad \lim_{x\to c} f(x) = L$$

if for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $x \in U$ satisfies $0 < \|x - c\|_2 < \delta$, we have $\|f(x) - L\|_2 < \epsilon$. We then say that f *has a limit at* c, and call L its *limit*.

We can recast this definition in terms of sequences.

Theorem 4.48. *Let U be an open subset of $\mathbb{R}^n$, $c \in U$, $L \in \mathbb{R}^m$ and $f : U\backslash\{c\} \to \mathbb{R}^m$.*
Then the following are equivalent:

(1) $\lim\limits_{x\to c} f(x) = L$.

(2) *For every sequence $(x_n)_{n\in\mathbb{N}}$ contained in U such that for all $n \in \mathbb{N}$, $x_n \neq c$, and $\lim\limits_{n\to\infty} x_n = c$, we have $\lim\limits_{n\to\infty} f(x_n) = L$.*

Proof. (1) $\Rightarrow$ (2): Let $\lim\limits_{x\to c} f(x) = L$, and that $(x_n)_{n\in\mathbb{N}}$ is a sequence contained in U such that $x_n \neq c$ ($n \in \mathbb{N}$), $\lim\limits_{n\to\infty} x_n = c$. Let $\epsilon > 0$. There exists a $\delta > 0$ such that whenever $x \in U$ satisfies $0 < \|x - c\|_2 < \delta$, we have $\|f(x) - L\|_2 < \epsilon$. There exists an $N \in \mathbb{N}$ such that for all $n > N$, $0 < \|x_n - c\| < \delta$. Consequently, for $n > N$ we have $\|f(x_n) - L\|_2 < \epsilon$. Hence $\lim\limits_{n\to\infty} f(x_n) = L$.

(2) $\Rightarrow$ (1): Suppose that $\lim\limits_{x\to c} f(x) = L$ does not hold. Then

$$\neg\big(\forall \epsilon > 0,\ \exists \delta > 0 \text{ such that } \forall x \in U \text{ satisfying } 0 < \|x - c\|_2 < \delta, \text{ we have } \|f(x) - L\|_2 < \epsilon$$

i.e., there is an $\epsilon > 0$ such that for every $\delta > 0$, there is a point $x \in U$ (depending on δ), for which $0 < \|x - c\|_2 < \delta$, but $\|f(x) - L\|_2 \geqslant \epsilon$. Taking δ successively to be $\frac{1}{n}$ ($n \in \mathbb{N}$), we can thus find a sequence $(x_n)_{n\in\mathbb{N}}$ contained in U such that for all $n \in \mathbb{N}$, $x_n \neq c$, $(x_n)_{n\in\mathbb{N}}$ converges to c, and $\|f(x_n) - L\|_2 \geqslant \epsilon$. This last condition means that $\lim\limits_{n\to\infty} f(x_n) = L$ does not hold. So (2) does not hold. Hence we have shown that if (1) does not hold, then (2) does not hold, i.e., (2) $\Rightarrow$ (1). ☐

Corollary 4.49. *Let U be an open subset of $\mathbb{R}^n$, $c \in U$ and $f : U\backslash\{c\} \to \mathbb{R}^m$.*
If f has a limit at c, then it is unique.

Proof. We use Theorem 4.48, and the fact that convergent sequences have unique limits. ☐

Using the algebra of limits for real sequences, it follows that the same result carries over to limits of real-valued functions.

Corollary 4.50. *Let U be an open subset of $\mathbb{R}^n$, and $c \in U$.*
Suppose that $f, g : U\backslash\{c\} \to \mathbb{R}$ and $\lim\limits_{x\to c} f(x) = L_f$ and $\lim\limits_{x\to c} g(x) = L_g$, where $L_f, L_g \in \mathbb{R}$.
Define $f+g, fg : U\backslash\{c\} \to \mathbb{R}$ by $(f+g)(x) = f(x)+g(x)$ and $(fg)(x) = f(x)g(x)$ ($x \in U\backslash\{c\}$).
Then:

(1) $\lim\limits_{x\to c}(f + g)(x) = L_f + L_g = \lim\limits_{x\to c} f(x) + \lim\limits_{x\to c} g(x)$.

(2) $\lim\limits_{x\to c}(fg)(x) = L_f L_g = \big(\lim\limits_{x\to c} f(x)\big)\big(\lim\limits_{x\to c} g(x)\big)$.

The following result is clear from the definitions.

Theorem 4.51. *Let U be an open subset of $\mathbb{R}^n$, and $c \in U$.*
Then $f : U \to \mathbb{R}^m$ is continuous at c if and only if $\lim\limits_{x\to c} f(x) = f(c)$.

4.4. Compactness and continuity

In this section we will learn about a very useful result in Optimisation Theory, on the existence of global minimisers of real-valued continuous functions on compact sets.

Theorem 4.52. *Let K be a compact subset of a metric space X, Y be a metric space, and $f : K \to Y$ be a continuous function. Then $f(K)$ is a compact subset of Y.*

Proof. Suppose that $(y_n)_{n\in\mathbb{N}}$ is a sequence in contained in $f(K)$. Then for each $n \in \mathbb{N}$, there exists an $x_n \in K$ such that $y_n = f(x_n)$. Thus we obtain a sequence $(x_n)_{n\in\mathbb{N}}$ in the set K. As K is compact, there exists a convergent subsequence, say $(x_{n_k})_{k\in\mathbb{N}}$, with limit $L \in K$. As f is continuous, it preserves convergent sequences. So $(f(x_{n_k}))_{k\in\mathbb{N}} = (y_{n_k})_{k\in\mathbb{N}}$ is convergent with limit $f(L) \in f(K)$. Consequently, $f(K)$ is compact. □

Now we prove the aforementioned result which turns out to be very useful in Optimisation Theory, namely that a real-valued continuous function on a compact set attains its maximum/minimum. This is a generalisation of the Extreme-Value Theorem we had learnt earlier, where the compact set in question was just the interval $[a, b]$.

Theorem 4.53 (Weierstrass's theorem)**.**
Let K be a nonempty compact subset of a metric space X, and let $f : K \to \mathbb{R}$ be continuous. Then there exists a $c \in K$ such that $f(c) = \sup\{f(x) : x \in K\}$.

Since $c \in K$, $f(c) \in \{f(x) : x \in K\}$, and so the supremum above is actually a maximum:

$$f(c) = \sup\{f(x) : x \in K\} = \max\{f(x) : x \in K\}.$$

Also, under the same hypothesis of the above result, there exists a minimiser in K, that is, there exists a $d \in K$ such that

$$f(d) = \inf\{f(x) : x \in K\} = \min\{f(x) : x \in K\}.$$

This follows from the above result by just looking at $-f$, that is, by applying the above result to the continuous function $g : K \to \mathbb{R}$ given by $g(x) = -f(x)$ $(x \in K)$.

Proof of Theorem 4.53. We know that the image of K under f, namely the set $f(K)$ is compact and hence bounded. So $\{f(x) : x \in K\}$ is bounded. It is also nonempty since K is nonempty. But by the least upper bound property of $\mathbb{R}$, a nonempty bounded subset of $\mathbb{R}$ has a least upper bound. Thus $M := \sup\{f(x) : x \in K\} \in \mathbb{R}$. Now consider $M - \frac{1}{n}$ $(n \in \mathbb{N})$. This number cannot be an upper bound for $\{f(x) : x \in K\}$. So there must be an $x_n \in K$ such that $f(x_n) > M - \frac{1}{n}$. In this manner we get a sequence $(x_n)_{n\in\mathbb{N}}$ in K. As K is compact, $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$ with limit, say c, belonging to K. As f is continuous, $(f(x_{n_k}))_{k\in\mathbb{N}}$ is convergent as well with limit $f(c)$. But from the inequalities $f(x_n) > M - \frac{1}{n}$ $(n \in \mathbb{N})$, it follows that $f(c) \geqslant M$. On the other hand, from the definition of M, we also have that $f(c) \leqslant M$. Hence $f(c) = M$. □

Example 4.54. Since the set $K = \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$ is compact in $\mathbb{R}^3$ and since the function $x \mapsto x_1 + x_2 + x_3$ is continuous on $\mathbb{R}^3$, it follows that the optimisation problem

$$\begin{array}{ll} \text{minimise} & x_1 + x_2 + x_3 \\ \text{subject to} & x_1^2 + x_2^2 + x_3^2 = 1 \end{array}$$

has a minimiser. ◇

Remark 4.55. In Optimisation Theory, one often meets *necessary* conditions for an optimal solution, that is, results of the following form:

> If $\widehat{x}$ is an optimal solution to the optimisation problem $\left\{\begin{array}{ll} \text{maximise} & f(x) \\ \text{subject to} & x \in \mathcal{F}\ (\subset \mathbb{R}^n) \end{array}\right\}$,
> then $\widehat{x}$ satisfies $\boxed{***}$.

(Where $\boxed{***}$ are certain mathematical conditions, such as the Lagrange multiplier equations.) Now such a result has limited use as such since even if we find all $\widehat{x}$(s) which satisfy $\boxed{***}$, we can't conclude that there is one that is optimal. But now suppose that we know that $f : \mathcal{F} \to \mathbb{R}$ is continuous and that $\mathcal{F}$ is compact. Then we know that an optimal solution exists, and so we know that among the $\widehat{x}$(s) that satisfy $\boxed{***}$, there is at least one which is an optimal solution.

Exercise 4.56. Let X be a compact metric space and let $f : X \to \mathbb{Z}$ be a continuous function. Here $\mathbb{Z}$ has the Euclidean topology induced from $\mathbb{R}$. Prove that f can assume only finitely many values.

Exercise 4.57. Let $N : \mathbb{R}^d \to \mathbb{R}$ be any norm on $\mathbb{R}^d$. The aim of this exercise is to show that N is 'equivalent to[2]' $\|\cdot\|_2$, i.e., there are constants $M, m > 0$ such that for all $\boldsymbol{x} \in \mathbb{R}^d$, $m\|\boldsymbol{x}\|_2 \leqslant N(\boldsymbol{x}) \leqslant M\|\boldsymbol{x}\|_2$.

(1) Let $\boldsymbol{e}_1 = (1, 0, \dots, 0), \cdots, \boldsymbol{e}_d = (0, \dots, 0, 1)$ be the standard basis vectors in $\mathbb{R}^d$. Thus the vector $\boldsymbol{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$ is the linear combination $x_1\boldsymbol{e}_1 + \cdots + x_d\boldsymbol{e}_d$. Show, using the triangle inequality and the Cauchy-Schwarz inequality, that there is an $M > 0$ such that for all $\boldsymbol{x} \in \mathbb{R}^d$, $N(\boldsymbol{x}) \leqslant M\|\boldsymbol{x}\|_2$.

(2) Prove using the triangle inequality for N that for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$, $|N(\boldsymbol{x}) - N(\boldsymbol{y})| \leqslant N(\boldsymbol{x} - \boldsymbol{y})$. Conclude that the map $N : (\mathbb{R}^d, \|\cdot\|_2) \to \mathbb{R}$ is continuous.

(3) Consider the compact set $K := \{\boldsymbol{x} \in \mathbb{R}^d : \|\boldsymbol{x}\|_2 = 1\}$ and use Weierstrass's theorem to prove the existence of $m > 0$ such that for all $\boldsymbol{x} \in \mathbb{R}^d$, $m\|\boldsymbol{x}\|_2 \leqslant N(\boldsymbol{x})$.

Exercise 4.58. In each case, give an example of a continuous function $f : S \to T$, such that $f(S) = T$ or else explain why there can be no such f. (We use the usual metrics, for example $(0, 1)$ in the first part has the Euclidean metric of $\mathbb{R}$.)

(1) $S = (0, 1)$, $T = (0, 1]$.

(2) $S = (0, 1)$, $T = (0, 1) \cup (1, 2)$.

(3) $S = \mathbb{R}$, $T = \mathbb{Q}$.

(4) $S = [0, 1] \cup [2, 3]$, $T = \{0, 1\}$.

(5) $S = [0, 1] \times [0, 1]$, $T = \mathbb{R}^2$.

(6) $S = [0, 1] \times [0, 1]$, $T = (0, 1) \times (0, 1)$.

(7) $S = (0, 1) \times (0, 1)$, $T = \mathbb{R}^2$.

Exercise 4.59. Let (X, d) be a metric space and let $f : X \to X$ be a function that satisfies

$$\text{for all } x, y \in X \text{ such that } x \neq y, \quad d(f(x), f(y)) < d(x, y). \tag{4.1}$$

(1) Prove that f has at most one fixed point (that is, a point $c \in X$ such that $f(c) = c$).

(2) Let $X = (0, \frac{1}{2})$ with the usual metric, and define $f : X \to X$ by $f(x) = x^2$ for all $x \in (0, \frac{1}{2})$. Show that f satisfies (4.59), but it has no fixed point.

(3) Show that the function $g : X \to \mathbb{R}$ given by $g(x) = d(x, f(x))$ $(x \in X)$ is continuous.

(4) Prove that if X is compact, then f has exactly one fixed point. *Hint: g attains a minimum on X.*

Exercise 4.60. Recall the notion of homeomorphism from Exercise 4.30.

(1) Show that $[0, 1]$ and $(0, 1)$ are homeomorphic when both spaces are equipped with the discrete metric.

(2) Show that $[0, 1]$ and $(0, 1)$ are not homeomorphic when both spaces are equipped with the Euclidean metric.

4.5. Uniform continuity

Roughly speaking, we use the adjective 'uniform' in Analysis whenever 'the same thing works everywhere'. We have already seen one instance of this when we discussed *uniform* convergence of a sequence of functions. Now we will learn about uniform continuity.

Recall that if (X, d_X) and (Y, d_Y) are metric spaces, then a function $f : X \to Y$ is said to be continuous at a point $c \in X$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $x \in X$ satisfies $d_X(x, c) < \delta$, we have $d_Y(f(x), f(c)) < \epsilon$. And f is called continuous if for every $c \in X$, f is continuous at c, that is:

$$\forall \epsilon > 0, \ \forall c \in X, \ \exists \delta > 0 \text{ such that if } x \in X \text{ satisfies } d_X(x, c) < \delta, \text{ then } d_Y(f(x), f(c)) < \epsilon.$$

In the above statement, the choice of δ might depend on which $c \in X$ we consider. For a 'uniformly' continuous function on I, it doesn't! That is, given an $\epsilon > 0$, the same δ (depending only on ϵ)

[2]The fuss about equivalent norms on a vector space X is that whenever two norms N_1, N_2 are equivalent, the open sets in (X, N_1) coincide with the ones in (X, N_2), and so as topological spaces, they are the same!

works everywhere in X, irrespective of which $c \in X$ we have considered. We now give the precise definition.

Definition 4.61. Let $(X, d_X), (Y, d_Y)$ be metric spaces. A map $f : X \to Y$ is said to be *uniformly continuous* if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in X$ satisfying $d_X(x, y) < \delta$, we have $d_Y(f(x), f(y)) < \epsilon$.

Note that in the definition we are introducing the notion of uniform continuity of a function *on a set*, and not *at a point*.

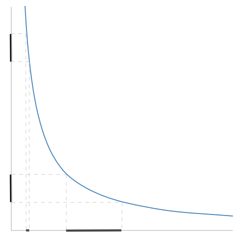
Proposition 4.62. *Let $(X, d_X), (Y, d_Y)$ be metric spaces. If $f : X \to Y$ is uniformly continuous, then f is continuous.*

Proof. Let $c \in X$. Suppose $\epsilon > 0$. By the uniform continuity of f, there exists a $\delta > 0$ such that for all $x, y \in X$ satisfying $d_X(x, y) < \delta$, there holds that $d_Y(f(x), f(y)) < \epsilon$. In particular, if $x \in X$ satisfies $d_X(x, c) < \delta$, we have $d_Y(f(x), f(c)) < \epsilon$. Thus f is continuous at c. But the choice of c was arbitrary, and so f is continuous on X. □

The following example shows that uniform continuity is a strictly stronger notion than continuity, that is, there are continuous functions that are not uniformly continuous.

Example 4.63. The map $f : (0, 1) \to \mathbb{R}$ given by $f(x) = \frac{1}{x}$ $(0 < x < 1)$ is continuous on $(0, 1)$: If $(x_n)_{n \in \mathbb{N}}$ is a convergent sequence in $(0, 1)$ with limit $L \in (0, 1)$, then $(f(x_n))_{n \in \mathbb{N}} = (\frac{1}{x_n})_{n \in \mathbb{N}}$ is a convergent sequence with limit $\frac{1}{L} = f(L)$.

However, f is not uniformly continuous. Suppose it is. Then given $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $|x - y| < \delta$, we have $|\frac{1}{x} - \frac{1}{y}| < \epsilon$. Consider $x = \frac{1}{n}$ and $y = \frac{1}{2n}$. Then $|x - y| = \frac{1}{2n}$, and so $|x - y| < \delta$ for all n large enough, but $|\frac{1}{x} - \frac{1}{y}| = n > \epsilon$, for n large enough.



The lack of uniform continuity is also clear in an intuitive manner pictorially. ◇

Exercise 4.64. Show that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ $(x \in \mathbb{R})$ is continuous, but not uniformly continuous. *Hint:* Consider $x = n$ and $y = n + \frac{1}{n}$ for large n.

Exercise 4.65. Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$ $(x \in \mathbb{R})$ is uniformly continuous.

Exercise 4.66. Let (X, d) be a metric space and let $c \in X$. Define $f : X \to \mathbb{R}$ by $f(x) = d(x, c)$. Prove that f is uniformly continuous on X.

Exercise 4.67. Let X, Y be metric spaces, and let $f : X \to Y$ be uniformly continuous. Show that if $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in X, then $(f(x_n))_{n \in \mathbb{N}}$ is a Cauchy sequence in Y. Compare this with Exercise 4.34.

Exercise 4.68. Let $f, g : I \to \mathbb{R}$ be uniformly continuous functions on the interval I.

(1) Show that if $f + g$ is also uniformly continuous on I.

(2) Is fg also always uniformly continuous on I?

(3) In addition to the assumed uniform continuity of f, g, if f, g are also bounded, then show that fg is uniformly continuous on I.

In Example 4.63, we have seen that there are continuous functions which aren't uniformly continuous. But the following result tells us that if we are working with a *compact* domain, then *mere* continuity is enough to conclude (the *stronger* property) of uniform continuity.

Proposition 4.69. *Let $(X, d_X), (Y, d_Y)$ be metric spaces, and suppose that X is compact. If $f : X \to Y$ is continuous, then f is also uniformly continuous.*

Proof. We will prove this by contradiction. So let us suppose that f is not uniformly continuous:

$$\neg\big(\forall \epsilon > 0,\ \exists \delta > 0 \text{ such that } \forall x, y \in X \text{ satisfying } d_X(x, y) < \delta, \text{ we have } d_Y(f(x), f(y)) < \epsilon\big).$$

Then there exists an $\epsilon > 0$ such that for every $\delta > 0$, there are some $x, y \in X$ such that $d_X(x, y) < \delta$, but $d_Y(f(x), f(y)) \geqslant \epsilon$. In particular, taking $\delta = \frac{1}{n}$ ($n \in \mathbb{N}$), there exist $x_n, y_n \in X$ such that $d_X(x_n, y_n) < \frac{1}{n}$ but $d_Y(f(x_n), f(y_n)) \geqslant \epsilon$. By using the compactness of X, and considering subsequences if necessary, we may assume that $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are convergent, with limits say $x, y \in X$, respectively[3]. Since $d_X(x_n, y_n) < \frac{1}{n}$, we obtain $d_X(x, y) \leqslant 0$, and so $x = y$. Also, by the continuity of f, we have that $(f(x_n))_{n\in\mathbb{N}}$ and $(f(y_n))_{n\in\mathbb{N}}$ converge to $f(x)$ and $f(y)$, respectively. Hence[4] $(d_Y(f(x_n), f(y_n)))_{n\in\mathbb{N}}$ converges to $d_Y(f(x), f(y)) = 0$ (as $x = y$!). But on the other hand, from $d_Y(f(x_n), f(y_n)) \geqslant \epsilon$, we obtain $d_Y(f(x), f(y)) \geqslant \epsilon > 0$, a contradiction. ☐

Exercise 4.70. Show that $f : [0, \infty) \to \mathbb{R}$ given by $f(x) = \sqrt{x}$ ($x \geqslant 0$) is uniformly continuous.

Definition 4.71. Let $(X, d_X), (Y, d_Y)$ be metric spaces. A function $f : X \to Y$ is called *Lipschitz* if there exists a number $L > 0$ such that for all $x, y \in X$, $d(f(x), f(y)) \leqslant L\, d(x, y)$.

Proposition 4.72. *Let $(X, d_X), (Y, d_Y)$ be metric spaces. If $f : X \to Y$ is Lipschitz, then f is uniformly continuous.*

Exercise 4.73. Prove Proposition 4.72.

Exercise 4.74.

(1) Show that the function $f : [-1, 1] \to \mathbb{R}$ defined by $f(x) = x^2$ is Lipschitz.

(2) Explain why the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is not Lipschitz.

(3) Show that the function $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$ is not Lipschitz continuous. (Compare with Exercise 4.70. Thus there are uniformly continuous functions that are not Lipschitz.)

4.6. Notes (not part of the course)

Weierstrass's Theorem can be used to prove:

Theorem 4.75 (Fundamental Theorem of Algebra)**.** *Every polynomial p with complex coefficients of degree at least 1 has a zero in $\mathbb{C}$.*

Recall that a *polynomial* p of degree $d \in \mathbb{N}$ is a function $p : \mathbb{C} \to \mathbb{C}$ such that there exist $c_0, \cdots, c_d \in \mathbb{C}$, with $c_d \neq 0$, such that for all $z \in \mathbb{C}$, $p(z) = c_0 + c_1 z + \cdots + c_d z^d$. A complex number $\zeta \in \mathbb{C}$ is a *zero* of p if $p(\zeta) = 0$. We note that for $z = (x, y) \in \mathbb{C} = \mathbb{R}^2$, the complex absolute value of z is $|z| = \sqrt{x^2 + y^2} = \|(x, y)\|_2$, the Euclidean norm of $(x, y) \in \mathbb{R}^2$. We will first show the following:

Lemma 4.76. *If p is a zero-free polynomial of degree at least 1, then $|p|$ has no minimiser in $\mathbb{C}$.*

Proof. Let $z_0 \in \mathbb{C}$ be a minimiser of $|p|$. Set $q(z) = \frac{p(z+z_0)}{p(z_0)}$. Then $q(0) = 1$. Replacing p by q, we may assume that $z_0 = 0$ and $p(0) = 1$. We will show that there exists a $w \in \mathbb{C}$ such that $|p(w)| < 1$, giving the desired contradiction. Let $d \in \mathbb{N}$ and $c_1, \cdots, c_d \in \mathbb{C}$ be such that $c_d \neq 0$ and $p(z) = 1 + c_1 z + \cdots + c_d z^d$ ($z \in \mathbb{C}$). Let $m \in \{1, \cdots, d\}$ be the least index such that $c_m \neq 0$. Then $p(z) = 1 + c_m z^m + r(z)$, where $r(z) = 0$ if $m = d$, and $r(z) := z^{m+1}(c_{m+1} + c_{m+2} z + \cdots + c_d z^{d-m-1})$ if $m < d$. Let

$$r_1 := \frac{|c_m|}{|c_{m+1}| + \cdots + |c_d|} \text{ and } r_2 := \frac{1}{\sqrt[m]{|c_m|}}.$$

[3]Since X is compact, the sequence $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence, say $(x_{n_k})_{k\in\mathbb{N}}$, converging to, say $x \in X$. Also, the sequence $(y_{n_k})_{k\in\mathbb{N}}$ has a convergent subsequence, say $(y_{n_{k_\ell}})_{\ell\in\mathbb{N}}$, converging to, say $y \in Y$.

[4]See Exercise 2.19.

If $m = d$, we have $|r(z)| = 0 < 1$ for all $z \in \mathbb{C}$, and if $m < d$, then we have

$$\begin{aligned} |r(z)| &\leqslant |z|^{m+1}(|c_{m+1}| + |c_{m+2}||z| + \cdots + |c_d||z^{d-m-1}|) \\ &< |z|^{m+1}(|c_{m+1}| + |c_{m+2}| + \cdots + |c_d|) && \text{if } 0 < |z| < 1 \\ &< |z|^m|c_m| = |c_m z^m| && \text{if moreover } |z| < r_1 \\ &< 1 && \text{if moreover } |z| < r_2. \end{aligned}$$

So if $0 < |z| \leqslant \frac{1}{2}\min\{1, r_1, r_2\} =: R$, then $|r(z)| < |c_m z^m| < 1$. Let $w = Rv$, where v is an m^{th} root of $\frac{-\overline{c_m}}{|c_m|}$. Then $|v| = 1$, $0 < |w| = R$, and $c_m w^m = c_m R^m(\frac{-\overline{c_m}}{|c_m|}) = -|c_m|R^m$.

Thus $|p(w)| \leqslant |1 + c_m w^m| + |r(w)| < |1 - |c_m|R^m| + |c_m w^m| = 1 - |c_m|R^m + |c_m|R^m = 1 = p(0)$. $\square$

Proof of Theorem 4.75. Let $d \in \mathbb{N}$ and $c_0, \cdots, c_d \in \mathbb{C}$ be such that $p(z) = c_0 + c_1 z + \cdots + c_d z^d$ $(z \in \mathbb{C})$, and p is zero-free. Replacing p by $\frac{1}{c_d}p$, we may assume that $c_d = 1$. Then $p(z) = z^d(1 + r(z))$, where $r(z) := \frac{c_0}{z^d} + \frac{c_1}{z^{d-1}} + \cdots + \frac{c_{d-1}}{z}$. Let $R := 2d\max\{|c_0|, \cdots, |c_d|\}$. Then $R \geqslant 2d|c_d| = 2d \cdot 1 = 2d > 1$. For $|z| \geqslant R$ and $0 \leqslant k \leqslant d-1$, we have that $|\frac{c_k}{z^{d-k}}| = \frac{|c_k|}{|z|^{d-k}} \leqslant \frac{|c_k|}{R^{d-k}} \leqslant \frac{|c_k|}{R} \leqslant \frac{|c_k|}{2d|c_k|} = \frac{1}{2d}$, which yields $|r(z)| \leqslant |\frac{c_0}{z^d}| + |\frac{c_1}{z^{d-1}}| + \cdots + |\frac{c_{d-1}}{z}| \leqslant \frac{1}{2d}d = \frac{1}{2}$. Hence for $|z| \geqslant R$, $|1 + r(z)| \geqslant 1 - |r(z)| \geqslant 1 - \frac{1}{2} = \frac{1}{2}$, and so $|p(z)| = |z^d(1 + r(z))| \geqslant \frac{R^d}{2}$. Since $R \geqslant 2d|c_0| = 2d|p(0)|$, we have

$$\text{for all } z \text{ with } |z| \geqslant R, \;\; |p(0)| \leqslant \frac{R}{2d} \leqslant \frac{R^d}{2} \leqslant |p(z)|. \qquad (\star)$$

By Weierstrass's Theorem, the real-valued continuous function $|p|$ assumes a minimum value on the (closed and bounded and hence) compact set $K := \{w \in \mathbb{C} : |w| \leqslant R\}$, say at $z_0 \in K$. The inequality $(\star)$ implies that this z_0 must be a minimiser of $|p|$ on all of $\mathbb{C}$. This contradicts Lemma 4.76. $\square$

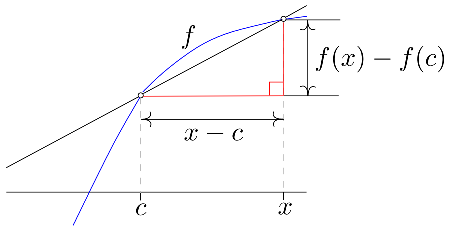
Chapter 5

Differentiation

For a function $f : (a, b) \to \mathbb{R}$, and a point $c \in (a, b)$, the *difference quotient* for $x \in (a, b)$, $x \neq c$, is

$$\frac{f(x) - f(c)}{x - c}.$$

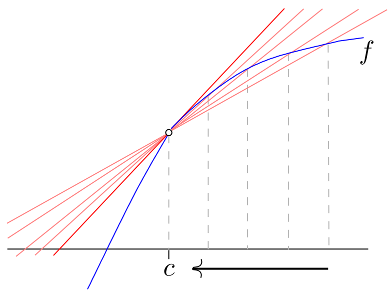
Geometrically, this number represents the slope of the chord passing through the points $(c, f(c))$ and $(x, f(x))$ on the graph of f:



Suppose that as x goes to c, the difference quotients approach a number, say L, that is,

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = L.$$

In other words, for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $x \in (a, b)$ satisfies $0 < |x - c| < \delta$, we have $|\frac{f(x)-f(c)}{x-c} - L| < \epsilon$. Then we say that f is *differentiable at the point* c. See the picture below, where we see the geometric interpretation of L: It is the slope of the tangent to the graph of f at the point c. Notice also that if we 'zoom into' the graph of f around the point $(c, f(c))$, the graph seems to coincide with the tangent line. In other words, the tangent line is a 'linear approximation' of f near the point c.



The number L is unique, and we denote this unique number by $f'(c)$. We call $f'(c)$ the *derivative of* f at c. If f is differentiable at every $c \in (a, b)$, then we say that f is *differentiable on* (a, b).

Theorem 5.1. *Let $f : (a, b) \to \mathbb{R}$ be differentiable at $c \in (a, b)$. Then f is continuous at c.*

Proof. Let $\epsilon > 0$. Let $\delta' > 0$ be such that for all $x \in (a, b)$ such that $0 < |x - c| < \delta'$, $|\frac{f(x)-f(c)}{x-c} - f'(c)| < 1$. Then rearrangement (using the triangle inequality) gives

$$\begin{aligned}|f(x) - f(c)| &= |f(x) - f(c) - f'(c)(x - c) + f'(c)(x - c)|\\ &\leqslant |f(x) - f(c) - f'(c)(x - c)| + |f'(c)||x - c| < (1 + |f'(c)|)|x - c|.\end{aligned}$$

Define $\delta := \min\{\delta', \frac{\epsilon}{1+|f'(c)|}\}$. Then for all $x \in (a, b)$ such that $0 < |x - c| < \delta$, we have

$$|f(x) - f(c)| < (1 + |f'(c)|)|x - c| \leqslant (1 + |f'(c)|)\frac{\epsilon}{1+|f'(c)|} = \epsilon.$$

Consequently f is continuous at c. □

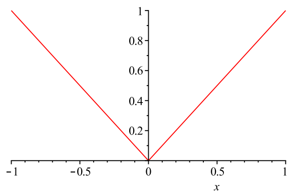
The converse of the theorem is not true, and the following example demonstrates this.

Example 5.2. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = |x|$ $(x \in \mathbb{R})$ is (uniformly) continuous since for all $x, y \in \mathbb{R}$, we have $|f(x) - f(y)| = \big||x| - |y|\big| \leqslant |x - y|$.

Let us now show that f is not differentiable at 0. If it were, then given $\epsilon = \frac{1}{2} > 0$, there exists a $\delta > 0$ such that whenever $0 < |x| < \delta$, we have

$$|\tfrac{|x|}{x} - f'(0)| < \epsilon = \tfrac{1}{2}.$$

Taking $x = \frac{\delta}{2}$, we obtain $|1 - f'(0)| < \frac{1}{2}$. Taking $x = -\frac{\delta}{2}$, we also get $| - 1 - f'(0)| < \frac{1}{2}$. Thus $2 = | - 1 - 1| = | - 1 - f'(0) + f'(0) - 1| \leqslant | - 1 - f'(0)| + |f'(0) - 1| < \frac{1}{2} + \frac{1}{2} = 1$, a contradiction.



(The lack of differentiability of $|\cdot|$ at 0 is visually obvious, since one can't draw a tangent at the 'corner' to the graph at $(0, 0)$.) ◇

The following result gives rules for differentiating the sum and product of differentiable functions.

Proposition 5.3. *Let $f, g : (a, b) \to \mathbb{R}$ be differentiable at $c \in (a, b)$. Then:*

(1) *The sum $f + g : (a, b) \to \mathbb{R}$ defined by $(f + g)(x) = f(x) + g(x)$ $(x \in (a, b))$ is differentiable at c, and $(f + g)'(c) = f'(c) + g'(c)$.*

(2) *The product $fg : (a, b) \to \mathbb{R}$ defined by $(fg)(x) = f(x) \cdot g(x)$ $(x \in (a, b))$ is differentiable at c, and $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$.*

Proof. These claims follow from the algebra of limits, namely Theorem 4.50. Indeed we have

$$\lim_{x\to c} \frac{(f + g)(x) - (f + g)(c)}{x - c} = \lim_{x\to c} \frac{f(x) - f(c)}{x - c} + \lim_{x\to c} \frac{g(x) - g(c)}{x - c} = f'(c) + g'(c),$$

which proves (1). Also, (2) follows from the following:

$$\begin{aligned}\lim_{x\to c} \frac{(fg)(x) - (fg)(c)}{x - c} &= \lim_{x\to c} \frac{f(x)g(x) - f(c)g(x) + f(c)g(x) - f(c)g(c)}{x - c}\\ &= \lim_{x\to c} \frac{f(x) - f(c)}{x - c} \cdot g(x) + \lim_{x\to c} f(c) \cdot \frac{g(x) - g(c)}{x - c}\\ &= \lim_{x\to c} \frac{f(x) - f(c)}{x - c} \cdot \lim_{x\to c} g(x) + f(c) \lim_{x\to c} \frac{g(x) - g(c)}{x - c}\\ &= f'(c)g(c) + f(c)g'(c).\end{aligned}$$

This completes the proof. □

Example 5.4. The derivative of a constant function is clearly zero. It is also easy to see that if f is defined by $f(x) = x$ ($x \in \mathbb{R}$), then $f'(x) = 1$. Repeated application of (2) above shows that the derivative of x^n ($n \in \mathbb{N}$) is nx^{n-1}. Thus every polynomial function is differentiable. ◇

Henceforth, we will take for granted the standard results on differentiating elementary functions such as sin that the student is familiar from ordinary calculus.

Exercise 5.5. Use the definition to find $f'(x)$, where $f(x) := \sqrt{x^2+1}$, $x \in \mathbb{R}$.

Exercise 5.6. Let $f : (0, \infty) \to \mathbb{R}$ be a function and let $c > 0$. Show that f is differentiable at c if and only if $\lim\limits_{k\to 1} \frac{f(kc)-f(c)}{k-1}$ exists. Moreover, then $f'(c) = \frac{1}{c} \lim\limits_{k\to 1} \frac{f(kc)-f(c)}{k-1}$.

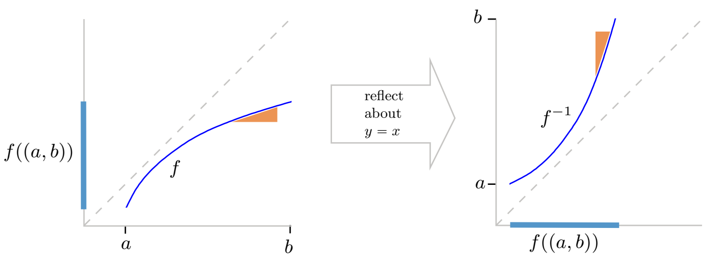
Exercise 5.7. Let $f : (-a, a) \to \mathbb{R}$ be differentiable, and even (i.e., for all $x \in (-a, a)$, $f(-x) = f(x)$). Show that f' is an odd function (i.e., for all $x \in (-a, a)$, $f'(-x) = -f'(x)$). What is $f'(0)$?

Exercise 5.8. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R}\backslash\mathbb{Q}. \end{cases}$ Show that f is differentiable at 0. What can you say about the differentiability of f at nonzero real numbers?

Exercise 5.9. If $f : (a, b) \to \mathbb{R}$ is differentiable at $c \in (a, b)$, then show that $\lim\limits_{h\to 0} \frac{f(c+h)-f(c-h)}{2h}$ exists and equals $f'(c)$. Is the converse true, that is, if $\lim\limits_{h\to 0} \frac{f(c+h)-f(c-h)}{2h}$ exists, then must f be differentiable at c?

Exercise 5.10. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(0) = 0$ and for $x \neq 0$, $f(x) = x^2 \sin \frac{1}{x}$. Prove that f is differentiable, but f' is not continuous at 0.

Exercise 5.11 (Differentiable Inverse Theorem)**.** Let $f : (a, b) \to \mathbb{R}$ be injective on (a, b). Then we can define its inverse $f^{-1} : f((a, b)) \to \mathbb{R}$.



By looking at the fate of the little triangle when we reflect in the 45° line, we can guess what happens to the derivatives: $(f^{-1})'(f(c)) = \frac{1}{f'(c)}$. This is the content of the Differentiable Inverse Theorem:

If $f : (a, b) \to \mathbb{R}$ is such that f is strictly increasing[1], f is continuous, f is differentiable at $c \in (a, b)$, and $f'(c) \neq 0$, then $f^{-1} : f((a, b)) \to \mathbb{R}$ is differentiable at $f(c)$ and $(f^{-1})'(f(c)) = \frac{1}{f'(c)}$.

The goal of this exercise is to prove this result.

(1) Show that $f((a, b))$ is open.

(2) Show that $f^{-1} : f((a, b)) \to \mathbb{R}$ is strictly increasing and continuous.

(3) We want to show that $\lim\limits_{y\to f(c)} \frac{f^{-1}(y) - f^{-1}(f(c))}{y - f(c)} = \frac{1}{f'(c)}$. We will use Theorem 4.48 to show this.

Let $(y_n)_{n\in\mathbb{N}}$ be any sequence with terms belonging to $f((a, b))\backslash\{f(c)\}$, that converges to $f(c)$.

So $y_n = f(x_n)$, $n\in\mathbb{N}$, for some $(x_n)_{n\in\mathbb{N}} \in (a, b)\backslash\{c\}$. We want $\lim\limits_{n\to\infty} \frac{f^{-1}(y_n) - f^{-1}(f(c))}{y_n - f(c)} = \lim\limits_{n\to\infty} \frac{x_n - c}{f(x_n) - f(c)} = \frac{1}{f'(c)}$.

Use the continuity of f^{-1} to show that $(x_n)_{n\in\mathbb{N}}$ converges to c.

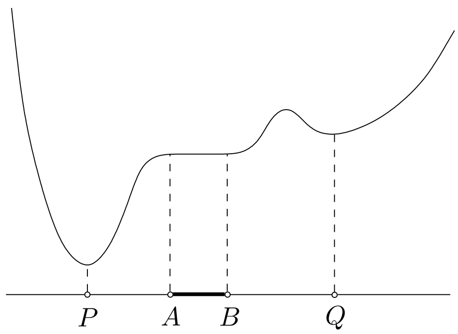
Use the differentiability of f at c and Theorem 4.48 again to conclude that $\lim\limits_{n\to\infty} \frac{x_n - c}{f(x_n) - f(c)} = \frac{1}{f'(c)}$.

Remark. See the appendix to this chapter (page 77) for an analogue of this result in $\mathbb{R}^n$.

[1] Or strictly decreasing, but here we just treat the strictly increasing case.

5.1. Local minimisers and derivatives

Intuitively, we expect that when a function $f : (a, b) \to \mathbb{R}$ has a local bump or a local trough, then at the highest or lowest point x_* of the bump/trough, the tangent line should be horizontal, that is, the slope $f'(x_*) = 0$. We will prove this result below. We say that $f : (a, b) \to \mathbb{R}$ has a *local minimum* at $c \in (a, b)$ if there exists a $\delta > 0$ such that whenever $x \in (a, b)$ satisfies $|x - c| < \delta$, we have $f(x) \geqslant f(c)$. In other words, 'locally' around c, the value assumed by f at c is the smallest. Local maximisers are defined likewise. See the picture below, in which the points P, Q and all points in the interior of the line segment AB are all local minimisers.



Theorem 5.12. *Let $f : (a, b) \to \mathbb{R}$ be such that f has a local minimum at $c \in (a, b)$, and f is differentiable at c. Then $f'(c) = 0$.*

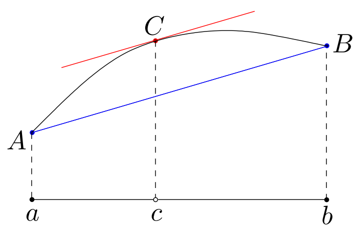
An analogous result holds for a local maximiser.

Proof. Let $\delta > 0$ be such that $a < c - \delta < c < c + \delta < b$ and $f(x) \geqslant f(c)$ for x satisfying $|x - c| < \delta$. Given an $\epsilon > 0$, we can also ensure (by making δ smaller if required) that for all x satisfying $0 < |x - c| < \delta$, we have $|\frac{f(x)-f(c)}{x-c} - f'(c)| < \epsilon$. Hence we have for all x satisfying $c < x < c + \delta$ that $0 - f'(c) \leqslant \frac{f(x)-f(c)}{x-c} - f'(c) \leqslant |\frac{f(x)-f(c)}{x-c} - f'(c)| < \epsilon$. (In order to obtain the first inequality, we have used the fact that $x - c > 0$ and $f(x) \geqslant f(c)$.) Similarly, for x satisfying $c - \delta < x < c$,we have $0 + f'(c) \leqslant -\frac{f(x)-f(c)}{x-c} + f'(c) \leqslant |\frac{f(x)-f(c)}{x-c} - f'(c)| < \epsilon$. Consequently, we have $|f'(c)| < \epsilon$. But the choice of $\epsilon > 0$ was arbitrary, and hence $f'(c) = 0$. □

5.2. Mean Value Theorem

Theorem 5.13 (Mean-Value Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there is a point $c \in (a, b)$ such that $\frac{f(b)-f(a)}{b-a} = f'(c)$.*

This result has a simple geometric interpretation. If we look at the chord AB in the plane which joins the end points $A \equiv (a, f(a))$ and $B \equiv (b, f(b))$ of the graph of f, then there is a point $c \in (a, b)$, where the tangent to f at the point $C \equiv (c, f(c))$ is parallel to the chord AB.



Why 'Mean Value'? If we think of $[a, b]$ as a time interval and $f(t)$ as being the position at time t of a particle moving along the real line, then

$$\frac{f(b)-f(a)}{b-a} = \frac{\text{total displacement}}{\text{time taken}} = \text{average or } \textit{mean} \text{ speed over } [a, b].$$

At some time instances, the instantaneous speed could have been *more* than this mean speed, while at other times *less* than the mean speed. The Mean Value Theorem says that at some time instance c, the instantaneous speed $f'(c)$ was exactly *equal* to the mean speed[2]!

Proof of Theorem 5.13. Define $\varphi : [a, b] \to \mathbb{R}$ by $\varphi(x) = (f(b) - f(a))x - (b - a)f(x)$ for all $x \in (a, b)$. Then φ is continuous on $[a, b]$, differentiable on (a, b) and

$$\varphi(a) = (f(b) - f(a))a - (b - a)f(a) = f(b)a - bf(a) = (f(b) - f(a))b - (b - a)f(b) = \varphi(b).$$

Moreover, for $x \in (a, b)$, we have $\varphi'(x) = f(b) - f(a) - (b - a)f'(x)$, and so in order to prove the theorem, it suffices to show that $\varphi'(c) = 0$ for some $c \in (a, b)$.

1° If φ is constant, then this holds for all $x \in (a, b)$.

2° Suppose there exists an $x \in (a, b)$ such that $\varphi(x) < \varphi(a) = \varphi(b)$. Let $c \in [a, b]$ be a minimiser of φ (Extreme Value Theorem!). Then since $\varphi(b) = \varphi(a)$, we conclude that $c \in (a, b)$. By the necessary condition for a local minimiser, we have $\varphi'(c) = 0$.

3° Suppose there exists an $x \in (a, b)$ such that $\varphi(x) > \varphi(a) = \varphi(b)$. Let $c \in [a, b]$ be a maximiser of φ (Extreme Value Theorem!). Then since $\varphi(b) = \varphi(a)$, we conclude that $c \in (a, b)$. By the necessary condition for a local maximiser, we have $\varphi'(c) = 0$. □

Corollary 5.14 (Rolle's theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.*

Exercise 5.15 (Cauchy's theorem)**.** If $f, g : [a, b] \to \mathbb{R}$ are continuous on $[a, b]$ and differentiable on (a, b), then show that there is a point $c \in (a, b)$ such that $(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c)$.

Hint: Apply Rolle's Theorem to φ given by $\varphi(x) = \det \begin{bmatrix} f(x) & g(x) & 1 \\ f(a) & g(a) & 1 \\ f(b) & g(b) & 1 \end{bmatrix}$ $(x \in [a, b])$.

Corollary 5.16. *Suppose that $f : (a, b) \to \mathbb{R}$ is differentiable on (a, b). Then:*

(1) *If $f'(x) > 0$ for all $x \in (a, b)$, then f is strictly increasing.*

(2) *If f is strictly increasing, then $f'(x) \geqslant 0$ for all $x \in (a, b)$.*

(3) *$f'(x) \geqslant 0$ for all $x \in (a, b)$ if and only if f is increasing.*

(4) *$f'(x) = 0$ for all $x \in (a, b)$ if and only if f is constant.*

Proof. (1) For $x_1, x_2 \in (a, b)$, with $x_1 < x_2$, it follows by the Mean Value Theorem, that

$$f(x_2) - f(x_1) = \underbrace{f'(x)}_{>0}\underbrace{(x_2 - x_1)}_{>0}$$

for some x between x_1 and x_2, and so $f(x_2) > f(x_1)$. Hence f is strictly increasing.

(2) Let $c \in (a, b)$. The sequence $(c + \frac{1}{n})_{n\in\mathbb{N}}$ converges to c. For all large enough n, $c + \frac{1}{n} \in (a, b)$, and $f(c + \frac{1}{n}) - f(c) > 0$. By Theorem 4.48, $f'(c) = \lim\limits_{x\to c} \frac{f(x)-f(c)}{x-c} = \lim\limits_{n\to\infty} \frac{f(c+\frac{1}{n})-f(c)}{(c+\frac{1}{n})-c} \geqslant 0$.
As the choice of $c \in (a, b)$ was arbitrary, the claim follows.

(3) The proof is analogous to (1) and (2).

(4) If f is constant, then clearly f' is pointwise 0.
Vice versa, if f' is identically 0, then for any pair of numbers $x_1, x_2 \in (a, b)$, it follows by the Mean Value Theorem, that $f(x_2) - f(x_1) = f'(x)(x_2 - x_1) = 0(x_2 - x_1) = 0$, giving $f(x_2) = f(x_1)$. Hence f is constant on (a, b). □

[2]After learning about the Fundamental Theorem of Calculus, we will also see that $\frac{f(b)-f(a)}{b-a} = \frac{1}{b-a}\int_a^b f'(t)dt$, and we may view the right hand side as an average/mean of all instantaneous speeds $f'(t)$ for t in $[a, b]$.

In (2), it may happen that f' is zero at some points, and it may fail to be positive. For example, consider the function x^3 on $\mathbb{R}$. It is strictly increasing, but $\frac{d}{dx}x^3|_{x=0} = 3 \cdot 0^2 = 0$. A similar version holds with 'decreasing' instead of 'increasing':

Corollary 5.17. *Suppose that $f : (a, b) \to \mathbb{R}$ is differentiable on (a, b). Then:*

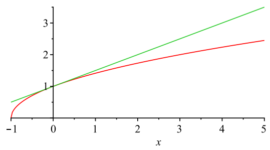
(1) *If $f'(x) < 0$ for all $x \in (a, b)$, then f is strictly decreasing.*

(2) *If f is strictly decreasing, then $f'(x) \leqslant 0$ for all $x \in (a, b)$.*

(3) *$f'(x) \leqslant 0$ for all $x \in (a, b)$ if and only if f is decreasing.*

The Mean Value Theorem can be used to prove interesting inequalities; here is an example.

Example 5.18. Let us show that for all $x > 0$, $\sqrt{1+x} < 1 + \frac{1}{2}x$.



Consider the function $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{1+x}$. Then f is continuous on $[0, \infty)$ and differentiable on $(0, \infty)$. If $x > 0$, then applying the Mean Value Theorem to f on the interval $[0, x]$, we obtain the existence of a c such that $0 < c < x$ and

$$\frac{f(x)-f(0)}{x-0} = \frac{\sqrt{1+x}-1}{x} = f'(c) = \frac{1}{2\sqrt{1+c}} < \frac{1}{2}.$$

Rearranging, we obtain the desired inequality. $\diamond$

Exercise 5.19. Suppose that $f : \mathbb{R} \to \mathbb{R}$ has the property that for all $x, y \in \mathbb{R}$, $|f(x) - f(y)| \leqslant (x-y)^2$. Prove that f is constant.

Exercise 5.20. Let $f : (a, b) \to \mathbb{R}$ be differentiable on (a, b) and suppose that there is number M such that for all $x \in (a, b)$, $|f'(x)| \leqslant M$. Show that f is Lipschitz, hence uniformly continuous, on (a, b).

Exercise 5.21. Show that for every real $a, b \in \mathbb{R}$, $|\cos a - \cos b| \leqslant |a - b|$.

Exercise 5.22. Recall that for $g : \mathbb{R} \to \mathbb{R}$, we write $\lim\limits_{x\to\infty} g(x) = L$ if for every $\epsilon > 0$, there exists an $R > 0$ such that for all $x > R$, we have $|g(x) - L| < \epsilon$. If $f : \mathbb{R} \to \mathbb{R}$ is differentiable, and there exist $L, L' \in \mathbb{R}$ such that $\lim\limits_{x\to\infty} f(x) = L$ and $\lim\limits_{x\to\infty} f'(x) = L'$, then prove that $L' = 0$.

Exercise 5.23. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable, $|f'(x)| \leqslant 1$ for all $x \in \mathbb{R}$, and that there exists an $a > 0$ such that $f(-a) = -a$, $f(a) = a$. Show that $f(0) = 0$.

Exercise 5.24. Let $c \in (a, b)$, and let $f : (a, b) \to \mathbb{R}$ be such that f is differentiable on $(a, b) \setminus \{c\}$, continuous on (a, b), and $\lim\limits_{x\to c} f'(x)$ exists. Show that f is differentiable at c, and $f'(c) = \lim\limits_{x\to c} f'(x)$. Contrast this situation with with the case of the function $x \mapsto |x|$ with $c = 0$.

Exercise 5.25. Prove that if $c_0, \cdots, c_d$ are any real numbers satisfying $\frac{c_0}{1} + \frac{c_1}{2} + \cdots + \frac{c_d}{d+1} = 0$, then the polynomial $c_0 + c_1 x + \cdots + c_d x^d$ has a zero in $(0, 1)$.

Exercise 5.26. Show that there are exactly two real values of x such that $x^2 = x \sin x + \cos x$ and that they lie in $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Higher order derivatives. If f has a derivative $f'(x)$ at each $x \in (a, b)$, then we can consider the *derivative function*, namely the map f' given by $x \mapsto f'(x)$ on the interval (a, b). Suppose now that f' is itself differentiable on (a, b). Then we may consider the derivative function f'' of f'. One can continue in this manner (provided of course that each successive function obtained is again differentiable), and obtain the functions f', f'', $f^{(3)}, \cdots$, $f^{(n)}$, each of which is the derivative of the previous one. $f^{(n)}$ is called the n^{th} derivative, or the derivative or order n, of f.

Example 5.27. All polynomials have derivatives of all orders, and eventually all high order derivatives are the zero function. ◇

Exercise 5.28. Let $f : \mathbb{R} \to \mathbb{R}$. We call $x \in \mathbb{R}$ a *fixed point* of f if $f(x) = x$.

(1) If f is differentiable, and for all $x \in \mathbb{R}$, $f'(x) \neq 1$, then prove that f has at most one fixed point.

(2) Let the sequence $(x_n)_{n\in\mathbb{N}}$ be generated by taking an arbitrary x_1, and setting $x_{n+1} = f(x_n)$ for $n \in \mathbb{N}$. Show that if there exists an $M < 1$ such that for all $x \in \mathbb{R}$, $|f'(x)| \leqslant M$, then there is a fixed point x_* of f, and that $x_* = \lim\limits_{n\to\infty} x_n$.

(3) Visualise the process in (2) via the zig-zag path $(x_1, x_2) \to (x_2, x_2) \to (x_2, x_3) \to (x_3, x_3) \to (x_3, x_4) \to \cdots$.

(3) Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x + \frac{1}{1+e^x}$ $(x \in \mathbb{R})$ has no fixed point, although $0 < f'(x) < 1$ for all $x \in \mathbb{R}$. Is this a contradiction to the result in part (2) above? Explain.

Exercise 5.29. Let I be an open interval, and $f, g : I \to \mathbb{R}$.

(1) Show that if f, g are twice differentiable, then $(fg)''(x) = f''(x)\,g(x) + 2\,f'(x)\,g'(x) + f(x)\,g''(x)$, $x \in I$.

(2) Show that if f, g are infinitely differentiable, then $(fg)^{(n)}(x) = \sum\limits_{k=0}^{n} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x)$, $x \in I$.

(3) For $x \in \mathbb{R}$ and $n \in \mathbb{N} \cup \{0\}$, define $x^{[n]} := x(x-1)\cdots(x-n+1)$. Show that if $x, y \in \mathbb{R}$, then

$$(x+y)^{[n]} = \sum_{k=0}^{n} \binom{n}{k} x^{[k]} y^{[n-k]}.$$

Hint: Differentiate t^{x+y} n times with respect to $t \in I := (0, \infty)$.

Exercise 5.30. Let $I \subset \mathbb{R}$ be an open interval. A function $f : I \to \mathbb{R}$ is said to be *convex* if for all $x, y \in I$ and all $t \in (0,1)$, $f((1-t)x + ty) \leqslant (1-t)f(x) + tf(y)$.

(1) Draw a picture and explain the geometric meaning of the inequality above.

(2) Let f be twice differentiable on I. Show that if $f''(x) \geqslant 0$ for all $x \in I$, then f is convex.
Hint: If $x < y$, then apply the Mean Value Theorem to f on $[x, (1-t)x + ty]$ and on $[(1-t)x + ty, y]$.

(3) Prove that if f is differentiable on I and convex, then f' is increasing.
Hint: If $x < u < y$, then using the convexity, derive the inequalities $\frac{f(u)-f(x)}{u-x} \leqslant \frac{f(y)-f(x)}{y-x} \leqslant \frac{f(y)-f(u)}{y-u}$ and pass to the limits $u \to x$ from above, and $u \to y$ from below.

(Combining this with (2), a twice differentiable f is convex *if and only if* $f''(x) \geqslant 0$ for all $x \in I$.)

(4) Prove that if f is differentiable on I, convex, and $f'(x_0) = 0$ for an $x_0 \in I$, then x_0 is a minimiser of f.

Exercise 5.31 (Arithmetic Mean-Geometric Mean Inequality)**.**

(1) Let $f : I \to \mathbb{R}$ be a convex function on an interval $I \subset \mathbb{R}$. If $n \in \mathbb{N}$, and $x_1, \cdots, x_n \in I$, then show that $f(\frac{x_1+\cdots+x_n}{n}) \leqslant \frac{f(x_1)+\cdots+f(x_n)}{n}$.

(2) Show that $-\log : (0, \infty) \to \mathbb{R}$ is convex.

(3) Prove the Arithmetic Mean-Geometric Mean Inequality: For $a_1, \cdots, a_n \in (0, \infty)$, $\frac{a_1+\cdots+a_n}{n} \geqslant \sqrt[n]{a_1 \cdots a_n}$.
(The left-hand side is the *arithmetic mean of* $a_1, \cdots, a_n$, and the right-hand side their *geometric mean.*)

Exercise 5.32 (Taylor's formula)**.** For a polynomial p given by $p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_d x^d$, we have $p^{(k)}(0) = c_k(k!)$ for $k \in \{0, 1, \cdots, d\}$, and $p^{(k)}(0) = 0$ for all $k > d$. So there is a special relationship between the coefficients c_k and the successive derivatives of p at 0. Now suppose that we *start* with a smooth enough function $f : \mathbb{R} \to \mathbb{R}$ and form a related d degree (Taylor) polynomial p given by $p(x) := f(0) + \frac{f'(0)}{1!}x + \cdots + \frac{f^{(d)}(0)}{d!}x^d$, $x \in \mathbb{R}$. Then $\frac{p^{(k)}(0)}{k!} = \frac{f^{(k)}(0)}{k!}$ for all $k \in \{0, 1, \cdots, d\}$, and so p matches very well with f at 0. It is thus natural to ask: How big is the error $E(x) := f(x) - p(x)$ when $x \neq 0$? Taylor's Formula answers this question:

If $f : \mathbb{R} \to \mathbb{R}$ is $(d+1)$ times differentiable, p is the degree-d Taylor polynomial of f, and $x \neq 0$, then there exists a ξ in the open interval with endpoints 0 and x, such that $E(\xi) = \frac{f^{(d+1)}(\xi)}{(d+1)!}x^{d+1}$.

(1) If $g : \mathbb{R} \to \mathbb{R}$ is differentiable, $g(0) = 0$, $m \in \mathbb{N}$, and $x \neq 0$, then there exists a ξ in the open interval with endpoints 0 and x, such that $\frac{g(x)}{x^m} = \frac{g'(\xi)}{m\xi^{m-1}}$. *Hint:* Use Rolle's theorem on $h(t) = t^m g(x) - x^m g(t)$.

(2) Show Taylor's theorem by applying the result from part (1) successively to get the that existence of $\xi_1, \cdots, \xi_d, \xi$ such that $\frac{E(x)}{x^{d+1}} = \frac{E'(\xi_1)}{(d+1)\xi_1^d} = \frac{E''(\xi_2)}{(d+1)d\xi_2^{d-1}} = \cdots = \frac{E^{(d)}(\xi_d)}{(d+1)!\xi_d^1} = \frac{E^{(d+1)}(\xi)}{(d+1)!} = \frac{f^{(d+1)}(\xi)}{(d+1)!}$.

(3) Applying Taylor's formula to the exponential function $f = (x \mapsto e^x)$, and the estimates $0 < e < 4$, show that $0 < e - (1 + \frac{1}{1!} + \cdots + \frac{1}{n!}) < \frac{4}{(n+1)!}$ for all $n \in \mathbb{N}$. Use this to conclude that e is not rational.

5.3. Uniform convergence and differentiation

When we studied uniform convergence, we had mentioned that interchanging limits is facilitated by uniform convergence. An instance of this is the possibility of differentiating a uniformly convergent series termwise; as shown in the Corollary 5.34 below. This relies on the following result, which in turn is an application of the Mean Value Theorem.

Proposition 5.33. *Let $f_n : (a, b) \to \mathbb{R}$ ($n \in \mathbb{N}$) be a sequence of differentiable functions on (a, b), such that there exists a point $c \in (a, b)$ for which $(f_n(c))_{n\in\mathbb{N}}$ converges. If the sequence $(f_n')_{n\in\mathbb{N}}$ converges uniformly to g on (a, b), then $(f_n)_{n\in\mathbb{N}}$ converges uniformly to a differentiable function f on (a, b), and moreover, $f'(x) = g(x)$ for all $x \in (a, b)$.*

Proof. (*You may skip reading this proof.*) Let $\epsilon > 0$. Let $N_1 \in \mathbb{N}$ be such that for all $m, n > N_1$, for all $x \in (a, b)$, we have $|f_m'(x) - f_n'(x)| < \min\{\frac{\epsilon}{3}, \frac{\epsilon}{3(b-a)}\}$, and also $|f_m(c) - f_n(c)| < \frac{\epsilon}{3}$. Let $x \in (a, b)$. Applying the Mean Value Theorem to $f_m - f_n$ on the interval with the endpoints x, c, we get $f_m(x) - f_n(x) = f_m(c) - f_n(c) + (x - c)(f_m'(y) - f_n'(y))$ for some y (depending on m, n, x, c) between x, c. Hence we obtain $|f_m(x) - f_n(x)| \leqslant |f_m(c) - f_n(c)| + (b - a)|f_m'(y) - f_n'(y)| < \frac{2}{3}\epsilon < \epsilon$ for all $x \in (a, b)$ and all $m, n > N_1$. Consequently, $(f_n)_{n\in\mathbb{N}}$ is uniformly convergent on (a, b). Let $f : (a, b) \to \mathbb{R}$ be its limit. As each f_n is continuous, so is f.

To show that f is differentiable at a point $x_0 \in (a, b)$, we apply the Mean Value Theorem once again to the function $f_m - f_n$ on the interval with endpoints $x_0, x \in (a, b)$, and $x \neq x_0$. Then we obtain that $(f_m(x) - f_n(x)) - (f_m(x_0) - f_n(x_0)) = (x - x_0)(f_m'(y) - f_n'(y))$ for some y (depending on x, x_0, m, n) between x and x_0. Dividing by $x - x_0$, and taking absolute values, we get

$$\left|\frac{f_m(x)-f_m(x_0)}{x-x_0} - \frac{f_n(x)-f_n(x_0)}{x-x_0}\right| \leqslant |f_m'(y) - f_n'(y)| < \frac{\epsilon}{3}$$

for all $m, n > N_1$ and $x \in (a, b)\backslash\{x_0\}$. Passing to the limit as $m \to \infty$ yields for all $x \in (a, b)\backslash\{x_0\}$:

$$\left|\frac{f(x)-f(x_0)}{x-x_0} - \frac{f_n(x)-f_n(x_0)}{x-x_0}\right| \leqslant \frac{\epsilon}{3} \qquad (\star)$$

for all $n > N_1$. Now let $N_2 \in \mathbb{N}$ be such that $|f_n'(x_0) - g(x_0)| < \frac{\epsilon}{3}$ for all $n > N_2$. Let $N = \max\{N_1, N_2\} + 1$, and let $\delta > 0$ be such that $0 < |x - x_0| < \delta$ implies

$$\left|\frac{f_N(x)-f_N(x_0)}{x-x_0} - f_N'(x_0)\right| < \frac{\epsilon}{3}. \qquad (\star\star)$$

Then combining the inequalities $(\star)$ and $(\star\star)$, we get that $\left|\frac{f(x)-f(x_0)}{x-x_0} - g(x_0)\right| < \epsilon$ for all $x \in (a, b)$ satisfying $0 < |x - x_0| < \delta$. As the choice of $\epsilon > 0$ was arbitrary, it follows that f is differentiable at x_0 and $f'(x_0) = g(x_0)$. □

The condition that the sequence of functions converges somewhere is needed for the conclusion of to hold. For example, let $f_n(x) = (-1)^n$ for all $n \in \mathbb{N}$ and $x \in (a, b)$. Clearly, for each $x \in (a, b)$, $(f_n(x))_{n\in\mathbb{N}} = ((-1)^n)_{n\in\mathbb{N}}$ does not converge. But as $f_n'(x) = 0$ for all $x \in (a, b)$ and all $n \in \mathbb{N}$, it follows that $(f_n')_{n\in\mathbb{N}}$ converges uniformly: It is the constant sequence, each of whose terms is the constant function taking value 0 everywhere. So the conclusion of this proposition, that $(f_n)_{n\in\mathbb{N}}$ converges uniformly, doesn't hold, because it doesn't even converge pointwise.

Corollary 5.34. *Let $f_n : (a, b) \to \mathbb{R}$, $n \in \mathbb{N}$, be a sequence of differentiable functions such that*

(1) $\sum\limits_{n=1}^{\infty} f_n(c)$ *converges for some $c \in (a, b)$, and*

(2) $g := \sum\limits_{n=1}^{\infty} f_n'$ *is uniformly convergent on (a, b).*

Then $f := \sum\limits_{n=1}^{\infty} f_n$ is uniformly convergent on (a, b), and $f' = g$ on (a, b), i.e., $(\sum\limits_{n=1}^{\infty} f_n)' = \sum\limits_{n=1}^{\infty} f_n'$.

We had seen that power series can be differentiated termwise in Theorem 3.61. The following exercise shows that we can also recover that result using Corollary 5.34, by just using Step 1 of the proof of Theorem 3.61 and using the absolute convergence of the termwise differentiated power series to get uniform convergence in any closed interval within the interval of convergence.

Exercise 5.35. Suppose that the power series $\sum\limits_{n=0}^{\infty} c_n x^n$ has a radius of convergence $r > 0$. From Step 1 of the proof of Theorem 3.61, $\sum\limits_{n=1}^{\infty} n c_n x^n$ converges absolutely for all $x \in (-r, r)$.

(1) Let $a \in (0, r)$. Show that $\sum\limits_{n=1}^{\infty} n c_n x^n$ converges uniformly on $(-a, a)$.

(2) Use Corollary 5.34 to show that $(\sum\limits_{n=0}^{\infty} c_n x^n)' = \sum\limits_{n=1}^{\infty} n c_n x^{n-1}$ for all $x \in (-r, r)$.

(3) We know that $f(x) := \sum\limits_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all $x \in \mathbb{R}$. Show that f satisfies $\frac{d}{dx} f(x) = f(x)$ $(x \in \mathbb{R})$.

Exercise 5.36. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \sum\limits_{n=1}^{\infty} \frac{1}{x^2+n^2}$ for all $x \in \mathbb{R}$. Prove that f is continuously differentiable on $\mathbb{R}$.

5.4. Derivative of maps from $\mathbb{R}^n$ to $\mathbb{R}^m$

In order to differentiate a function whose domain is $\mathbb{R}^n$ (or an open subset of $\mathbb{R}^n$) and takes values in $\mathbb{R}^m$, we first look at the familiar $n = m = 1$ case, and recast the old definition in a manner that will naturally lend itself for extension to the case when n or m is > 1.

We defined $f : (a, b) \to \mathbb{R}$ to be differentiable at $c \in (a, b)$ if $f'(c) := \lim\limits_{x \to c} \frac{f(x)-f(c)}{x-c}$ exists. In other words, for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $x \in (a, b)$ satisfies $0 < |x - c| < \delta$, we have $|\frac{f(x)-f(c)}{x-c} - f'(c)| < \epsilon$, i.e., $\frac{|f(x)-f(c)-f'(c)(x-c)|}{|x-c|} < \epsilon$.

If now f is instead a map from an open set $U \subset \mathbb{R}^n$ to $\mathbb{R}^m$, then bearing in mind that the $\|\cdot\|_2$-norm is a generalisation of the absolute value in $\mathbb{R}$, we may try mimicking the above definition, and replace the denominator in the inequality above by $\|x-c\|_2$. Similarly, the numerator absolute value can be replaced by the $\|\cdot\|_2$-norm in $\mathbb{R}^m$ (since we see that $f(x) - f(c)$ lives in $\mathbb{R}^m$). But what object should be there in the box below?:

$$\frac{\|f(x) - f(c) - \boxed{f'(c)}(x - c)\|_2}{\|x - c\|_2} < \epsilon.$$

Since $f(x), f(c)$ live in $\mathbb{R}^m$, we expect the term $f'(c)(x - c)$ to be also in $\mathbb{R}^m$. As $x - c$ is in $\mathbb{R}^n$, $f'(c)$ should take this into $\mathbb{R}^m$. So we see that it is natural that we should not expect $f'(c)$ to be a number (as was the case when $n = m = 1$), but rather it we expect it should be a certain mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$. We will in fact want it to be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. Why? We will see this in detail soon, but a short answer is that in the $n = m = 1$ case, the term $f'(c)(x - c)$ can indeed be viewed as the action of the linear transformation $L : \mathbb{R} \to \mathbb{R}$ given by $\mathbb{R} \ni v \mapsto f'(c)v$ on the vector $v := x - c \in \mathbb{R}$. We will then see that with our generalised definition, we can prove analogous theorems from ordinary calculus, and we can use these theorems in applications to solve real-life problems. After this rough motivation, let us now see the precise definition.

Definition 5.37. Let $U \subset \mathbb{R}^n$ be open, $c \in U$ and $f : U \to \mathbb{R}^m$. Then we say that f is *differentiable at* c if there exists a linear transformation $L : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{x \to c} \frac{\|f(x) - f(c) - L(x - c)\|_2}{\|x - c\|_2} = 0, \tag{5.1}$$

that is, for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $x \in U$ satisfies $0 < \|x - c\|_2 < \delta$, we have

$$\frac{\|f(x) - f(c) - L(x - c)\|_2}{\|x - c\|_2} < \epsilon.$$

Then L is called the *derivative of* f *at* c, and we write $f'(c) = L$.

The relation (5.1) can be expressed by saying that $f(x) - f(c) = f'(c)(x-c) + r(x)$, where the remainder r satisfies $\lim\limits_{x \to c} \frac{\|r(x)\|_2}{\|x-c\|_2} = 0$. (See Exercise 5.43.) So we can interpret this as saying that $f'(c)$ is that linear transformation which has the property that for x close to c, $f(x) - f(c)$ is approximately equal to its action on $x - c$. Next we show that in fact there can be only one such linear transformation.

Lemma 5.38. *Let $U \subset \mathbb{R}^n$ be open, $c \in U$ and $f : U \to \mathbb{R}^m$ be differentiable at c. Then the derivative of f at c is unique.*

Proof. Let L_1, L_2 be linear transformations such that

$$\lim_{x \to c} \frac{\|f(x)-f(c)-L_1(x-c)\|_2}{\|x-c\|_2} = 0 = \lim_{x \to c} \frac{\|f(x)-f(c)-L_2(x-c)\|_2}{\|x-c\|_2}.$$

Thus given $\epsilon > 0$, we can choose a $\delta > 0$ such that whenever $0 < \|x-c\|_2 < \delta$, we have

$$\frac{\|f(x)-f(c)-L_1(x-c)\|_2}{\|x-c\|_2} < \epsilon \quad \text{and} \quad \frac{\|f(x)-f(c)-L_2(x-c)\|_2}{\|x-c\|_2} < \epsilon.$$

Using the triangle inequality and the above two inequalities, we obtain

$$\frac{\|L_2(x-c)-L_1(x-c)\|_2}{\|x-c\|_2} = \frac{\|f(x)-f(c)-L_1(x-c)-\big(f(x)-f(c)-L_2(x-c)\big)\|_2}{\|x-c\|_2} < 2\epsilon,$$

that is, $\|L_2(x-c) - L_1(x-c)\|_2 \leqslant 2\epsilon\|x-c\|_2$ whenever $0 < \|x-c\|_2 < \delta$. Given any nonzero $h \in \mathbb{R}^n$, defining $x := c + \frac{\delta}{2\|h\|_2} h$, we have $0 < \|x-c\|_2 = \frac{\delta}{2} < \delta$, and so $\|L_2 h - L_1 h\|_2 \leqslant 2\epsilon\|h\|_2$. But the choice of $\epsilon > 0$ was arbitrary, and so $L_2 h = L_1 h$ for all nonzero $h \in \mathbb{R}^n$. Thus $L_1 = L_2$. ☐

Before we see simple illustrative examples on the calculation of the derivative, let us check that we have a genuine extension of the notion of differentiability from ordinary calculus. Over there the concept of derivative was very simple, and $f'(x_0)$ was just a number. But now we will see that over there too, it was actually a linear transformation, but it just so happens that any linear transformation from $\mathbb{R}$ to $\mathbb{R}$ is given by multiplication by a fixed number. We explain this below.

Coincidence of the new definition with the old one when $n = m = 1$, $f : \mathbb{R} \to \mathbb{R}$, $c \in \mathbb{R}$.

(1) Differentiable in the old sense $\Rightarrow$ differentiable in the new sense.

Let $\lim\limits_{x \to c} \frac{f(x)-f(c)}{x-c}$ exist and equals the number $f'_{\text{old}}(c) \in \mathbb{R}$. Define $L : \mathbb{R} \to \mathbb{R}$ by $L(v) = f'_{\text{old}}(c)\, v$ for all $v \in \mathbb{R}$. Then L is a linear transformation because

(L1) For every $v_1, v_2 \in \mathbb{R}$, $L(v_1 + v_2) = f'_{\text{old}}(c)\,(v_1 + v_2) = f'_{\text{old}}(c)\, v_1 + f'_{\text{old}}(c)\, v_2 = L(v_1) + L(v_2)$.

(L2) For every $\alpha \in \mathbb{R}$ and every $v \in V$, $L(\alpha \cdot v) = f'_{\text{old}}(c)\,(\alpha v) = \alpha\,(f'_{\text{old}}(c)\, v) = \alpha \cdot L(v)$.

We know $\lim\limits_{x \to c} \frac{f(x)-f(c)}{x-c} = f'_{\text{old}}(c)$, i.e., for all $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \in \mathbb{R}$ satisfies $0 < |x-c| < \delta$, then $|\frac{f(x)-f(c)}{x-c} - f'_{\text{old}}(c)| = \frac{|f(x)-f(c)-f'_{\text{old}}(c)\,(x-c)|}{|x-c|} < \epsilon$, i.e., $\frac{|f(x)-f(c)-L(x-c)|}{|x-c|} < \epsilon$.

So f is differentiable in the new sense too, and $f'_{\text{new}}(c) = L$, i.e., $f'_{\text{new}}(c)(v) = f'_{\text{old}}(c)\, v$, $v \in \mathbb{R}$.

(2) Differentiable in the new sense $\Rightarrow$ differentiable in the old sense.

Suppose there is a linear transformation $f'_{\text{new}}(c) : \mathbb{R} \to \mathbb{R}$ such that for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $x \in \mathbb{R}$ satisfies $0 < |x-c| < \delta$, we have $\frac{|f(x)-f(c)-f'_{\text{new}}(c)(x-c)|}{|x-c|} < \epsilon$, Define $f'_{\text{old}}(c) := f'_{\text{new}}(c)(1) \in \mathbb{R}$. Then for $x \in \mathbb{R}$ we have

$$f'_{\text{new}}(c)(x-c) = f'_{\text{new}}(c)((x-c)\cdot 1) = (x-c) \cdot f'_{\text{new}}(c)(1) = f'_{\text{old}}(c)\,(x-c).$$

So there exists a number, namely $f'_{\text{old}}(c)$, such that for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $x \in \mathbb{R}$ satisfies $0 < |x-c| < \delta$, we have

$$\left|\frac{f(x)-f(c)}{x-c} - f'_{\text{old}}(c)\right| = \frac{|f(x)-f(c)-f'_{\text{old}}(c)\,(x-c)|}{|x-c|} = \frac{|f(x)-f(c)-f'_{\text{new}}(c)(x-c)|}{|x-c|} < \epsilon.$$

Consequently, f is differentiable at c in the old sense, and $f'_{\text{old}}(c) = f'_{\text{new}}(c)(1)$.

Example 5.39. Let $A \in \mathbb{R}^{m\times n}$. Consider the map $T_A : \mathbb{R}^n \to \mathbb{R}^m$ given by $T_A x = Ax$ ($x \in \mathbb{R}^n$). If $c \in \mathbb{R}^n$, then is T_A differentiable at c? If so, then what is its derivative? The answers turn out to be very simple. We note that for $x \in \mathbb{R}^n$, we have $T_A(x) - T_A(c) = T_A(x-c)$, and so

$$\lim_{x\to c} \frac{\|T_A(x) - T_A(c) - T_A(x-c)\|_2}{\|x-c\|_2} = \lim_{x\to c} \frac{0}{\|x-c\|_2} = \lim_{x\to c} 0 = 0.$$

So T_A is differentiable at $c \in \mathbb{R}^n$, and $T_A'(c) = T_A$! (This is analogous to the observation in ordinary calculus that a linear function $x \mapsto ax$ has the same slope at all points, namely the number a.) $\diamond$

Exercise 5.40. Suppose that the function $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $c \in \mathbb{R}^n$. Define the new function $g : \mathbb{R}^n \to \mathbb{R}$ by $g(x) = (f(x))^2$ ($x \in \mathbb{R}^n$). Show that $g : \mathbb{R}^n \to \mathbb{R}$ is differentiable at c too.
Hint: $(f(x))^2 - (f(c))^2 = (f(x) + f(c))(f(x) - f(c)) \approx 2f(c)f'(c)(x-c)$ for x near c.

Exercise 5.41. Let $Q \in \mathbb{R}^{n\times n}$ be a symmetric matrix, that is, $Q = Q^\top$. Define $q : \mathbb{R}^n \to \mathbb{R}$ by $q(x) = x^\top Q x$ ($x \in \mathbb{R}^n$). Prove that q is differentiable at each $c \in \mathbb{R}^n$ and that $q'(c) : \mathbb{R}^n \to \mathbb{R}$ is given by $q'(c)v = 2c^\top Q v$ ($v \in \mathbb{R}^n$).

Exercise 5.42. Consider the map $f : \mathbb{R}^n \to \mathbb{R}$ given by $f(x) = \|x\|_2^4$ ($x \in \mathbb{R}^n$). Calculate $f'(c)$ for $c \in \mathbb{R}^n$.
Hint: Use the results in Exercises 5.40 and 5.41.

Exercise 5.43. Let U be an open set in $\mathbb{R}^n$, $f : U \to \mathbb{R}^m$, and $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Show that f is differentiable at $c \in U$ with $f'(c) = L$ if and only if there exists $r : U \to \mathbb{R}^m$ such that $f(x) = f(c) + L(x-c) + r(x)$ for all $x \in U$ and $\lim\limits_{x\to c} \frac{\|r(x)\|_2}{\|x-c\|_2} = 0$.

Exercise 5.44. Suppose that U is an open set in $\mathbb{R}^n$, and that $f : U \to \mathbb{R}^m$ is differentiable at $c \in U$. Prove that f is continuous at c.

Theorem 5.45 (Chain Rule)**.** *Let $U \subset \mathbb{R}^n$ be open, $f : U \to \mathbb{R}^m$, f be differentiable at $c \in U$, $V \subset \mathbb{R}^m$ be an open set such that $f(U) \subset V$, and $g : V \to \mathbb{R}^\ell$ be differentiable at $f(c) \in V$. Then $g \circ f : U \to \mathbb{R}^\ell$ is differentiable at c, and $(g\circ f)'(c) = g'(f(c)) \circ f'(c)$.*

Proof. (*You may skip this proof.*) In light of Exercise 5.43, by the differentiability of f at c, and of g at $f(c)$, there exist functions r_f, r_g such that

$$\left.\begin{aligned} f(x) - f(c) &= f'(c)(x-c) + r_f(x) && \text{for all } x \in U \\ g(y) - g(f(c)) &= g'(f(c))(y - f(c)) + r_g(y) && \text{for all } y \in V \end{aligned}\right\} \qquad (\star)$$

and $\lim\limits_{x\to c} \frac{\|r_f(x)\|_2}{\|x-c\|_2} = 0$, $\lim\limits_{y\to f(c)} \frac{\|r_g(y)\|_2}{\|y-f(c)\|_2} = 0$. Define $r_{g\circ f} : U \to \mathbb{R}^m$ by

$$r_{g\circ f}(x) := (g\circ f)(x) - (g\circ f)(c) - \big(g'(f(c)) \circ f'(c)\big)(x-c), \quad x \in U.$$

Then we have using $(\star)$ that

$$\begin{aligned} r_{g\circ f}(x) &= (g\circ f)(x) - (g\circ f)(c) - \big(g'(f(c)) \circ f'(c)\big)(x-c) \\ &= g(f(x)) - g(f(c)) - \big(g'(f(c)) \circ f'(c)\big)(x-c) \\ &= g'(f(c))(f(x) - f(c)) + r_g(f(x)) - g'(f(c))(f'(c)(x-c)) \\ &= g'(f(c))(f'(c)(x-c) + r_f(x)) + r_g(f(x)) - g'(f(c))(f'(c)(x-c)) \\ &= g'(f(c))r_f(x) + r_g(f(x)). \end{aligned}$$

Let $\epsilon > 0$. Let $\delta_1 > 0$ be such that whenever $x \in U$ satisfies $0 < \|x-c\|_2 < \delta_1$, we have

$$\frac{\|r_f(x)\|_2}{\|x-c\|_2} < \min\{\frac{\epsilon}{2(1+\sqrt{m\ell}\|g'(f(c))\|_\infty)}, 1\}.$$

Let $\delta_2 > 0$ be such that whenever $y \in V$ satisfies $0 < \|y - f(c)\|_2 < \delta_2$, we have

$$\frac{\|r_g(y)\|_2}{\|y-f(c)\|_2} < \frac{\epsilon}{2(\sqrt{nm}\|f'(c)\|_\infty + 1)}.$$

Let $\delta_3 > 0$ be such that whenever $x \in U$ satisfies $\|x-c\|_2 < \delta_3$, we have $\|f(x)-f(c)\|_2 < \delta_2$. With $\delta = \min\{\delta_1, \delta_3\} > 0$, for $x \in U$ satisfying $0 < \|x-c\|_2 < \delta$, we have $\|f(x)-f(c)\|_2 < \delta_2$, and

$$\begin{aligned}
\frac{\|r_{g\circ f}(x)\|_2}{\|x-c\|_2} &= \frac{\|g'(f(c))r_f(x)+r_g(f(x))\|_2}{\|x-c\|_2} \\
&\leqslant \frac{\|g'(f(c))r_f(x)\|_2}{\|x-c\|_2} + \frac{\|r_g(f(x))\|_2}{\|x-c\|_2} \\
&\leqslant \frac{\sqrt{m\ell}\|g'(f(c))\|_\infty \|r_f(x)\|_2}{\|x-c\|_2} + \frac{\frac{\epsilon}{2(\sqrt{nm}\|f'(c)\|_\infty+1)}\|f(x)-f(c)\|_2}{\|x-c\|_2} \\
&\leqslant \sqrt{m\ell}\|g'(f(c))\|_\infty \frac{\epsilon}{2(1+\sqrt{m\ell}\|g'(f(c))\|_\infty)} + \frac{\epsilon}{2(\sqrt{nm}\|f'(c)\|_\infty+1)} \frac{\|f'(c)(x-c)+r_f(x)\|_2}{\|x-c\|_2} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2(\sqrt{nm}\|f'(c)\|_\infty+1)} \frac{\sqrt{nm}\|f'(c)\|_\infty\|x-c\|_2 + \|r_f(x)\|_2}{\|x-c\|_2} \\
&\leqslant \frac{\epsilon}{2} + \frac{\epsilon}{2(\sqrt{nm}\|f'(c)\|_\infty+1)} \left(\sqrt{nm}\|f'(c)\|_\infty + 1\right) = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

The claim now follows from Exercise 5.43. □

Exercise 5.46. Recover the result in Exercise 5.40 by using the Chain Rule.

Exercise 5.47. Let $x_1, x_2 \in \mathbb{R}^n$ be distinct points. Define $\gamma : \mathbb{R} \to \mathbb{R}^n$ by $\gamma(t) = (1-t)x_1 + tx_2$ for all $t \in \mathbb{R}$. Prove that if $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\gamma(t_0)$ for some $t_0 \in \mathbb{R}$, then $f \circ \gamma : \mathbb{R} \to \mathbb{R}$ is differentiable at t_0 and

$$\tfrac{d}{dt}(f \circ \gamma)(t_0) = f'(\gamma(t_0))(x_2 - x_1).$$

Deduce that if $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable and $f'(x) = \mathbf{0}$ at every $x \in \mathbb{R}^n$, then f is constant. Here $\mathbf{0} : \mathbb{R}^n \to \mathbb{R}$ is the zero linear transformation which sends each $v \in \mathbb{R}^n$ to $0 \in \mathbb{R}$.

5.5. Partial derivatives

Suppose that U is an open subset of $\mathbb{R}^n$, and let $\boldsymbol{f} : U \to \mathbb{R}^m$ be a function. Let the components of $\boldsymbol{f}$ be denoted by $f_1, \dots, f_m$. Thus for $i = 1, \dots, m$,

$$f_i(x) := \boldsymbol{e}_i^\top \boldsymbol{f}(\boldsymbol{x}) \quad (\boldsymbol{x} \in U),$$

where $\boldsymbol{e}_1, \dots, \boldsymbol{e}_m$ denote the standard basis vectors in $\mathbb{R}^m$, that is,

$$\boldsymbol{e}_1 := \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \cdots, \boldsymbol{e}_m := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Let $\boldsymbol{c} \in U$. If

$$\frac{\partial f_i}{\partial x_j}(\boldsymbol{c}) := \lim_{x_j \to c_j} \frac{f_i(c_1, \cdots, c_{j-1}, x_j, c_{j+1}, \cdots, c_n) - f_i(c_1, \cdots, c_{j-1}, c_j, c_{j+1}, \cdots, c_n)}{x_j - c_j}$$

exists, then we call $\frac{\partial f_i}{\partial x_j}(\boldsymbol{c})$ the (i,j)th partial derivative f at $\boldsymbol{c}$. Thus, we look only at the i^{th} component $f_i : U \to \mathbb{R}$, keep all the variables $x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n$ as fixed, with values $c_1, \cdots, c_{j-1}, c_{j+1}, \cdots, c_n$, respectively, and differentiate the function

$$x_j \mapsto f_i(c_1, \cdots, c_{j-1}, x_j, c_{j+1}, \cdots, c_n)$$

with respect to x_j at c_j.

Example 5.48. Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x_1, x_2) = \begin{bmatrix} x_1^2 + x_2^2 \\ x_1 x_2 \end{bmatrix}$, for $\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2$. Then

$$\begin{aligned}
\frac{\partial f_1}{\partial x_1}(c_1, c_2) &= 2c_1, & \frac{\partial f_1}{\partial x_2}(c_1, c_2) &= 2c_2, \\
\frac{\partial f_2}{\partial x_1}(c_1, c_2) &= c_2, & \frac{\partial f_2}{\partial x_2}(c_1, c_2) &= c_1.
\end{aligned}$$

◇

Theorem 5.49. *Let U be an open subset of $\mathbb{R}^n$, and $\boldsymbol{c} \in U$. If $\boldsymbol{f} : U \to \mathbb{R}^m$ is differentiable at $\boldsymbol{c}$, then all the partial derivatives of $\boldsymbol{f}$ at $\boldsymbol{c}$, namely, $\frac{\partial f_i}{\partial x_j}(\boldsymbol{c})$ $(i = 1, \cdots, m,\ j = 1, \cdots, n)$ exist, and the matrix $[f'(\boldsymbol{c})]$ of the linear transformation $f'(\boldsymbol{c})$ with respect to the standard bases for $\mathbb{R}^n$ and $\mathbb{R}^m$ is given by*

$$[f'(\boldsymbol{c})] = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\boldsymbol{c}) & \cdots & \frac{\partial f_1}{\partial x_n}(\boldsymbol{c}) \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}(\boldsymbol{c}) & \cdots & \frac{\partial f_m}{\partial x_n}(\boldsymbol{c}) \end{bmatrix},$$

that is, $f'(\boldsymbol{c}) : \mathbb{R}^n \to \mathbb{R}^m$ is the linear transformation given by $f'(\boldsymbol{c})\boldsymbol{x} = [f'(\boldsymbol{c})]\boldsymbol{x}$ for all $\boldsymbol{x} \in \mathbb{R}^n$.

Proof. Let $\epsilon > 0$. As $\boldsymbol{f}$ is differentiable at $\boldsymbol{c}$, there exists a $\delta > 0$ such that for all $\boldsymbol{x} \in U$ such that $0 < \|\boldsymbol{x} - \boldsymbol{c}\|_2 < \delta$, we have $\frac{\|f(\boldsymbol{x}) - f(\boldsymbol{c}) - f'(\boldsymbol{c})(\boldsymbol{x}-\boldsymbol{c})\|_2}{\|\boldsymbol{x}-\boldsymbol{c}\|_2} < \epsilon$. Let x_j be such that $0 < |x_j - c_j| < \delta$. Define $\boldsymbol{x} := (c_1, \cdots, c_{j-1}, x_j, c_{j+1}, \cdots, c_n)$. Then $\boldsymbol{x} - \boldsymbol{c} = (0, \cdots, 0, x_j - c, 0, \cdots, 0) = (x_j - c) \cdot \boldsymbol{e}_j$, and so $\|\boldsymbol{x} - \boldsymbol{c}\|_2 = |x_j - c|\|\boldsymbol{e}_j\|_2 = |x_j - c_j|1 = |x_j - c_j|$. Thus for such vectors $\boldsymbol{x}$, we have $\frac{\|f(\boldsymbol{x}) - f(\boldsymbol{c}) - f'(\boldsymbol{c})(\boldsymbol{x}-\boldsymbol{c})\|_2}{\|\boldsymbol{x}-\boldsymbol{c}\|_2} < \epsilon$. Also, $f_i(\boldsymbol{x}) - f_i(\boldsymbol{c}) - (\boldsymbol{e}_i^\top f'(\boldsymbol{c})\boldsymbol{e}_j)(x_j - c_j) = \boldsymbol{e}_i^\top(f(\boldsymbol{x}) - f(\boldsymbol{c}) - f'(\boldsymbol{c})(\boldsymbol{x}-\boldsymbol{c}))$, and so $|f_i(\boldsymbol{x}) - f_i(\boldsymbol{c}) - (\boldsymbol{e}_i^\top f'(\boldsymbol{c})\boldsymbol{e}_j)(x_j - c_j)| \leqslant \|f(\boldsymbol{x}) - f(\boldsymbol{c}) - f'(\boldsymbol{c})(\boldsymbol{x} - \boldsymbol{c})\|_2$. Hence for numbers x_j satisfying $0 < |x_j - c_j| < \delta$, we have

$$\left|\frac{f_i(c_1, \cdots, c_{j-1}, x_j, c_{j+1}, \cdots, c_n) - f_i(c_1, \cdots, c_{j-1}, c_j, c_{j+1}, \cdots, c_n)}{x_j - c_j} - \boldsymbol{e}_i^\top \boldsymbol{f}'(\boldsymbol{c})\boldsymbol{e}_j\right| < \epsilon.$$

So $\frac{\partial f_i}{\partial x_j}(\boldsymbol{c}) = \boldsymbol{e}_i^\top \boldsymbol{f}'(\boldsymbol{c})\boldsymbol{e}_j$. Set $A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\boldsymbol{c}) & \cdots & \frac{\partial f_1}{\partial x_n}(\boldsymbol{c}) \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}(\boldsymbol{c}) & \cdots & \frac{\partial f_m}{\partial x_n}(\boldsymbol{c}) \end{bmatrix} = \begin{bmatrix} \boldsymbol{e}_1^\top \boldsymbol{f}'(\boldsymbol{c})\boldsymbol{e}_1 & \cdots & \boldsymbol{e}_1^\top \boldsymbol{f}'(\boldsymbol{c})\boldsymbol{e}_n \\ \vdots & & \vdots \\ \boldsymbol{e}_m^\top \boldsymbol{f}'(\boldsymbol{c})\boldsymbol{e}_1 & \cdots & \boldsymbol{e}_m^\top \boldsymbol{f}'(\boldsymbol{c})\boldsymbol{e}_n \end{bmatrix}$. Then

$$\begin{aligned} f'(\boldsymbol{c})\boldsymbol{x} &= f'(\boldsymbol{c})\Big(\sum_{j=1}^n x_j \boldsymbol{e}_j\Big) = \sum_{j=1}^n x_j f'(\boldsymbol{c})\boldsymbol{e}_j = \sum_{j=1}^n x_j \begin{bmatrix} \boldsymbol{e}_1^\top \boldsymbol{f}'(\boldsymbol{c})\boldsymbol{e}_j \\ \vdots \\ \boldsymbol{e}_m^\top \boldsymbol{f}'(\boldsymbol{c})\boldsymbol{e}_j \end{bmatrix} = \begin{bmatrix} \boldsymbol{e}_1^\top \boldsymbol{f}'(\boldsymbol{c})\boldsymbol{e}_1 x_1 + \cdots + \boldsymbol{e}_1^\top \boldsymbol{f}'(\boldsymbol{c})\boldsymbol{e}_n x_n \\ \vdots \\ \boldsymbol{e}_m^\top \boldsymbol{f}'(\boldsymbol{c})\boldsymbol{e}_1 x_1 + \cdots + \boldsymbol{e}_m^\top \boldsymbol{f}'(\boldsymbol{c})\boldsymbol{e}_n x_n \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{e}_1^\top \boldsymbol{f}'(\boldsymbol{c})\boldsymbol{e}_1 & \cdots & \boldsymbol{e}_1^\top \boldsymbol{f}'(\boldsymbol{c})\boldsymbol{e}_n \\ \vdots & & \vdots \\ \boldsymbol{e}_m^\top \boldsymbol{f}'(\boldsymbol{c})\boldsymbol{e}_1 & \cdots & \boldsymbol{e}_m^\top \boldsymbol{f}'(\boldsymbol{c})\boldsymbol{e}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = A\boldsymbol{x}, \end{aligned}$$

for all $\boldsymbol{x} \in \mathbb{R}^n$. □

The above result says that for the derivative to exist, it is necessary that the partial derivatives exist. Surprisingly, this is not a sufficient condition.

Example 5.50. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(0,0) = 0$, and for $(x_1, x_2) \neq (0,0)$, $f(x_1, x_2) = \frac{x_1 x_2}{x_1^2 + x_2^2}$.
Claim: Though $\frac{\partial f}{\partial x_1}(0,0)$ and $\frac{\partial f}{\partial x_2}(0,0)$ exist, $f'(0,0)$ doesn't, i.e., f is not differentiable at $(0,0)$.

For $x_1 \neq 0$, $f(x_1, 0) - f(0,0) = 0 - 0 = 0$, and so $\frac{\partial f}{\partial x_1}(0,0) = \lim\limits_{x_1 \to 0} \frac{f(x_1, 0) - f(0,0)}{x_1 - 0} = \lim\limits_{x_1 \to 0} 0 = 0$.
Similarly, $\frac{\partial f}{\partial x_2}(0,0) = \lim\limits_{x_2 \to 0} \frac{f(0, x_2) - f(0,0)}{x_2 - 0} = \lim\limits_{x_1 \to 0} \frac{0 - 0}{x_2} = \lim\limits_{x_1 \to 0} 0 = 0$.

Thus all the partial derivatives of f exist at $(0,0)$. However, we will now show that $f'(0,0)$ does not exist. Suppose that $f'(0,0)$ exists. By Theorem 5.49, $[f'(0,0)] = \left[\, \frac{\partial f}{\partial x_1}(0,0)\ \frac{\partial f}{\partial x_2}(0,0) \,\right] = [\, 0\ 0 \,]$. Let $\epsilon > 0$. Then there exists a $\delta > 0$ such that for all $\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2$ satisfying $0 < \|\boldsymbol{x} - \boldsymbol{0}\|_2 < \delta$, we have $\frac{\|f(\boldsymbol{x}) - f(\boldsymbol{0}) - f'(\boldsymbol{0})(\boldsymbol{x} - \boldsymbol{0})\|_2}{\|\boldsymbol{x} - \boldsymbol{0}\|_2} < \epsilon$, that is, $\frac{|x_1 x_2|}{(x_1^2 + x_2^2)\sqrt{x_1^2 + x_2^2}} < \epsilon$. For all $n \in \mathbb{N}$ large enough, with $x_1 := x_2 := \frac{1}{n}$, we have that $0 < \|(x_1, x_2) - (0,0)\|_2 = \frac{\sqrt{2}}{n} < \delta$, and so

$$\frac{n}{2\sqrt{2}} = \frac{\frac{1}{n^2}}{\frac{2}{n^2}\frac{\sqrt{2}}{n}} = \frac{|x_1 x_2|}{(x_1^2 + x_2^2)\sqrt{x_1^2 + x_2^2}} < \epsilon,$$

for all large n, a contradiction. ◇

Remark: (Not part of the course.) A sufficient condition for differentiability in an open set is that all partials are continuous; see Theorem 5.64.

Let U be an open subset of $\mathbb{R}^n$. We say that $f : U \to \mathbb{R}$ has a *local minimum* at $\boldsymbol{c} \in U$ if there exists a $\delta > 0$ such that whenever $\boldsymbol{x} \in U$ satisfies $\|\boldsymbol{x} - \boldsymbol{c}\|_2 < \delta$, we have $f(\boldsymbol{x}) \geqslant f(\boldsymbol{c})$. Local maximisers are defined analogously.

Corollary 5.51. *Let U be an open subset of $\mathbb{R}^n$. Let $f : U \to \mathbb{R}$ be such that f has a local minimum at $\boldsymbol{c} \in U$, and f is differentiable at $\boldsymbol{c}$. Then $f'(\boldsymbol{c}) = \mathbf{0}$.*

Proof. If $\boldsymbol{c}$ is a local minimiser for f, then each of the functions $x_i \mapsto f(c_1, \dots, c_{i-1}, x_i, c_{i+1}, \dots, c_n)$ has a local minimum at c_i, and so by the one variable result, we have $\frac{\partial f}{\partial x_i}(\boldsymbol{c}) = 0$ for each $i \in \{1, \cdots, m\}$. Consequently, Theorem 5.49 yields $f'(\boldsymbol{c}) = \mathbf{0}$. □

An analogous result holds for a local maximiser.

Theorem 5.52. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable and convex (i.e., for all $t \in (0,1)$, and $x, y \in \mathbb{R}^n$, $f((1-t)x+ty) \leqslant (1-t)f(x)+tf(y))$. If $x_* \in \mathbb{R}^n$ is such that $f'(x_*) = \mathbf{0}$, then f has a minimum at x_*.*

Proof. (*May be skipped.*) Let $x_0 \in \mathbb{R}^n$ and $f(x_0) < f(x_*)$. Define the function $\varphi : \mathbb{R} \to \mathbb{R}$ by $\varphi(t) = f(tx_0 + (1-t)x_*)$, $t \in \mathbb{R}$. Then φ is convex, since for $\alpha \in (0,1)$ and $t_1, t_2 \in \mathbb{R}$, we have

$$\begin{aligned}
\varphi((1-\alpha)t_1 + \alpha t_2) &= f\big(((1-\alpha)t_1 + \alpha t_2)x_0 + (1 - (1-\alpha)t_1 - \alpha t_2)x_*\big) \\
&= f\big((1-\alpha)(t_1 x_0 - t_1 x_*) + \alpha(t_2 x_0 - t_2 x_*) + x_*\big) \\
&= f\big((1-\alpha)(t_1 x_0 - t_1 x_*) + \alpha(t_2 x_0 - t_2 x_*) + (1-\alpha)x_* + \alpha x_*\big) \\
&= f\big((1-\alpha)(t_1 x_0 - t_1 x_* + x_*) + \alpha(t_2 x_0 - t_2 x_* + x_*)\big) \\
&= f\big((1-\alpha)(t_1 x_0 + (1-t_1)x_*) + \alpha(t_2 x_0 + (1-t_2)x_*)\big) \\
&\leqslant (1-\alpha)f\big(t_1 x_0 + (1-t_1)x_*\big) + \alpha f\big(t_2 x_0 + (1-t_2)x_*\big) \\
&= (1-\alpha)\varphi(t_1) + \alpha\varphi(t_2).
\end{aligned}$$

From Exercise 5.47, φ is differentiable at 0, and $\varphi'(0) = f'(x_*)(x_0 - x_*) = \mathbf{0}(x_0 - x_*) = 0$. We have $\varphi(1) = f(x_0) < f(x_*) = \varphi(0)$. By the Mean Value Theorem that there exists a $\theta \in (0,1)$ such that $\varphi'(\theta) = \frac{\varphi(1)-\varphi(0)}{1-0} < 0 = \varphi'(0)$. This contradicts the convexity of φ (indeed, by Exercise 5.30, φ' must be increasing). Thus there cannot exist an $x_0 \in X$ such that $f(x_0) < f(x_*)$. □

Exercise 5.53. Let $\tau, \xi \in \mathbb{R}$ satisfy $\tau - \xi^2 = 0$. Show that $f(x,t) := e^{\tau t + \xi x}$ satisfies the diffusion equation $\frac{\partial f}{\partial t}(x,t) - \frac{\partial^2 f}{\partial x^2}(x,t) = 0$ in $\mathbb{R}^2$. (Here $\frac{\partial^2 f}{\partial x^2}$ denotes the partial derivative with respect to x of $(x,t) \mapsto \frac{\partial f}{\partial x}(x,t)$.)

Exercise 5.54. Let $f : [0, \frac{\pi}{2}] \to \mathbb{R}^2$ given by $f(t) = (\cos t, \sin t)$, $t \in [0, \frac{\pi}{2}]$. Show the failure of the Mean Value Theorem for f, by proving there is no $c \in (0, \frac{\pi}{2})$ such that $f(\frac{\pi}{2}) - f(0) = (\frac{\pi}{2} - 0)[f'(c)]$.

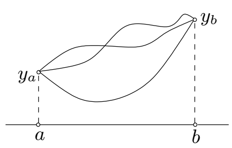
Exercise 5.55. Suppose $\boldsymbol{f} : [a,b] \to \mathbb{R}^d$ is continuous, and $\boldsymbol{f}$ is differentiable in (a,b). For $t \in (a,b)$, the matrix of the linear transformation $\boldsymbol{f}'(t) : \mathbb{R} \to \mathbb{R}^d$ is identified with a (column) vector $[\boldsymbol{f}'(t)] \in \mathbb{R}^d$.

(1) Put $\boldsymbol{z} = \boldsymbol{f}(b) - \boldsymbol{f}(a)$, and define $\varphi : [a,b] \to \mathbb{R}$ by $\varphi(t) = \boldsymbol{z}^\top \boldsymbol{f}(t)$ ($t \in [a,b]$). Show that φ is continuous, and φ is differentiable in (a,b), with $\varphi'(t) = \boldsymbol{z}^\top [\boldsymbol{f}'(t)]$ for all $t \in (a,b)$.

(2) Applying the Mean Value Theorem for φ, prove the *Mean Value Inequality*: There exists a $c \in (a,b)$ such that $\|\boldsymbol{f}(b) - \boldsymbol{f}(a)\|_2 \leqslant (b-a)\|[\boldsymbol{f}'(c)]\|_2$.

Exercise 5.56. Let $U \subset \mathbb{R}^n$ be open, $\boldsymbol{c} \in U$, and $\boldsymbol{f} : U \to \mathbb{R}^m$ have the components $f_1, \cdots, f_m : U \to \mathbb{R}$. Show that $\boldsymbol{f}$ is differentiable at $\boldsymbol{c}$ if and only if for all $i \in \{1, \cdots, m\}$, f_i is differentiable at $\boldsymbol{c}$.

Exercise 5.57. In the subject of 'Calculus of Variations', the following type of optimisation problem is studied: Minimise $f(\boldsymbol{x}) := \int_a^b F(\boldsymbol{x}(t), \boldsymbol{x}'(t), t)dt$. Here f is the cost function, and the integrand F is a function $\mathbb{R}^3 \ni (\alpha, \beta, \gamma) \mapsto F(\alpha, \beta, \gamma) \in \mathbb{R}$. The domain of f is the set of continuously differentiable functions $\boldsymbol{x} : [a,b] \to \mathbb{R}$ such that $\boldsymbol{x}(a) = y_a$ and $\boldsymbol{x}(b) = y_b$. Hence we observe that this is an optimisation problem in which the domain of the cost function f is itself a set of functions.



A central result in Calculus of Variations is that if $\boldsymbol{x}_*$ is a minimiser, then it must satisfy the following 'Euler-Lagrange' equation: $\frac{\partial F}{\partial \alpha}(\boldsymbol{x}_*(t), \boldsymbol{x}_*'(t), t) - \frac{d}{dt}(\frac{\partial F}{\partial \beta}(\boldsymbol{x}_*(t), \boldsymbol{x}_*'(t), t)) = 0$ $(t \in [a, b])$. Consider for example the problem of maximising the profit, given by $f(\boldsymbol{x}) := \int_0^T (P - a\boldsymbol{x}(t) - b\boldsymbol{x}'(t))\boldsymbol{x}'(t)dt$, associated with a possible choice of operation $\boldsymbol{x} : [0, T] \to \mathbb{R}$ over the time interval $[0, T]$ satisfying $\boldsymbol{x}(0) = 0$ and $\boldsymbol{x}(T) = Q$. Here T, P, a, b, Q are given positive constants. Assuming that an optimal operation $\boldsymbol{x}_*$ exists, find it using the Euler-Lagrange equation.

Exercise 5.58. Let $f(x_1, x_2) := x_1^4 - 12x_1x_2 + x_2^4$, $(x_1, x_2) \in \mathbb{R}^2$. Find all (global) minimisers of f on $\mathbb{R}^2$.

Exercise 5.59. Find the derivative of the multiplication map $(x, y) \mapsto xy : \mathbb{R}^2 \to \mathbb{R}$ at (x_0, y_0) in $\mathbb{R}^2$.

Exercise 5.60.

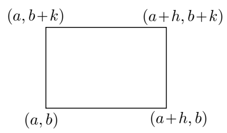
(1) Verify that $\frac{\partial^2 f}{\partial x \partial y}(x, y) = \frac{\partial^2 f}{\partial y \partial x}(x, y)$ for all $(x, y) \in \mathbb{R}^2$, where $f(x, y) := 3x^3 + 9y^2 - 9x^3y$ $((x, y) \in \mathbb{R}^2)$.

(2) Show that $\frac{\partial^2 f}{\partial x \partial y}(x, y) = \frac{\partial^2 f}{\partial y \partial x}(x, y)$ does not hold at $(0, 0)$ if $f(x, y) := \begin{cases} xy\frac{x^2-y^2}{x^2+y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$

In Exercise 5.60, we saw a real function for which the order of taking partial derivatives mattered. The following result gives a sufficient condition for it to be irrelevant.

Theorem 5.61. *Let $U \subset \mathbb{R}^2$ be open and $f : U \to \mathbb{R}$ be such that $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ exist at each point of U. If $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ are continuous at a point $(a, b) \in U$, then $\frac{\partial^2 f}{\partial x \partial y}(a, b) = \frac{\partial^2 f}{\partial y \partial x}(a, b)$.*

Proof. (*May be skipped.*) Let $h, k > 0$ be such that the rectangle with corners (a, b), $(a + h, b)$, $(a, b + k)$, $(a + h, b + k)$ lies in U. Let $D(h, k) := f(a + h, b + k) - f(a + h, b) - f(a, b + k) + f(a, b)$.



Define $G(x) = f(x, b + k) - f(x, b)$, $x \in [a, a + h]$. Then $D(h, k) = G(a + h) - G(a)$. By the Mean Value Theorem for G on $[a, a+h]$, there exists an $x_1 \in (a, a+h)$ such that $G(a+h)-G(a) = hG'(x_1)$, and so $D(h, k) = G(a + h) - G(a) = hG'(x_1) = h(\frac{\partial f}{\partial x}(x_1, b + k) - \frac{\partial f}{\partial x}(x_1, b))$. Applying the Mean Value Theorem to the function $\frac{\partial f}{\partial x}(x_1, \cdot)$ on $[b, b + k]$, there exists a $y_1 \in (b, b + k)$ such that $\frac{\partial f}{\partial x}(x_1, b + k) - \frac{\partial f}{\partial x}(x_1, b) = k\frac{\partial^2 f}{\partial y \partial x}(x_1, y_1)$. Thus

$$D(h, k) = hk\frac{\partial^2 f}{\partial y \partial x}(x_1, y_1). \qquad (\star)$$

Define $H(y) = f(a + h, y) - f(a, y)$, $y \in [b, b + k]$. Then $D(h, k) = H(b + k) - H(b)$. By the Mean Value Theorem for H on $[b, b+k]$, there exists an $y_2 \in (b, b+k)$ such that $H(b+k)-H(b) = kH'(y_2)$, and so $D(h, k) = H(b + k) - H(b) = kH'(y_2) = k(\frac{\partial f}{\partial y}(a + h, y_2) - \frac{\partial f}{\partial y}(a, y_2))$. Applying the Mean Value Theorem to the function $\frac{\partial f}{\partial y}(\cdot, y_2)$ on $[a, a + h]$, there exists a $x_2 \in (a, a + h)$ such that $\frac{\partial f}{\partial y}(a + h, y_2) - \frac{\partial f}{\partial y}(a, y_2) = h\frac{\partial^2 f}{\partial x \partial y}(x_2, y_2)$. Thus

$$D(h, k) = hk\frac{\partial^2 f}{\partial x \partial y}(x_2, y_2). \qquad (\star\star)$$

From $(\star)$ and $(\star\star)$,

$$\frac{\partial^2 f}{\partial y \partial x}(x_1, y_1) = \frac{\partial^2 f}{\partial x \partial y}(x_2, y_2). \qquad (*)$$

Now we take $h = k = \frac{1}{n}$ for all $n \in \mathbb{N}$ large enough, say $n > N$, so that the aforementioned rectangle lies in U. We use the notation $(x_1^{(n)}, y_1^{(n)})$ and $(x_2^{(n)}, y_2^{(n)})$ instead of (x_1, y_1) and (x_2, y_2) to highlight the dependence on the n at hand. We note that the sequences $(x_1^{(k+N)}, y_1^{(k+N)})_{k \in \mathbb{N}}$ and $(x_2^{(k+N)}, y_2^{(k+N)})_{k \in \mathbb{N}}$ both converge to (a, b), by virtue of the following inequalities: For $n > N$,

$$a < x_1^{(n)} < a + \tfrac{1}{n}, \quad b < y_1^{(n)} < b + \tfrac{1}{n}, \quad a < x_2^{(n)} < a + \tfrac{1}{n}, \quad b < y_2^{(n)} < b + \tfrac{1}{n}.$$

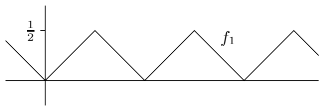
From $(*)$ and the continuity at (a, b) of $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$, it follows that $\frac{\partial^2 f}{\partial y \partial x}(a, b) = \frac{\partial^2 f}{\partial x \partial y}(a, b)$. $\square$

Exercise 5.62 (A law of conservation of symbols). Let $f : \mathbb{R}^2 \to \mathbb{R}$ has continuous partial derivatives up to order 2, and $\frac{\partial f}{\partial x}\frac{\partial f}{\partial y} \equiv 0$. Show that $\frac{\partial^2 f^2}{\partial x \partial y} \equiv 0$.

Exercise 5.63. Let $d \in \mathbb{N}$, and $f : \mathbb{R}^d \to \mathbb{R}$ be such that for all $t \in \mathbb{R}$ and all $\mathbf{x} \in \mathbb{R}^d$, $f(t \cdot \mathbf{x}) = tf(\mathbf{x})$. Show that $f(\mathbf{0}) = 0$. Suppose moreover that f is differentiable at $\mathbf{0}$. Show that f is a linear transformation. *Hint:* For a nonzero $\mathbf{h} \in \mathbb{R}^d$ and $\delta > 0$, $\|t\mathbf{h} - \mathbf{0}\|_2 < \delta$ for all $|t|$ small enough.

5.6. Notes (not part of the course)

Continuous everywhere, differentiable nowhere functions. In connection with Theorem 5.1, we might wonder how badly behaved continuous functions can be with respect to the notion of differentiability. It turns out that there are functions that are continuous *everywhere*, but differentiable *nowhere*. One construction is that of the *blancmange function* obtained by taking the basic *sawtooth* function f_1,



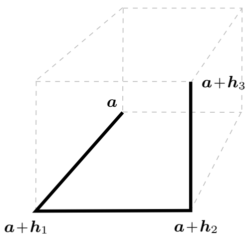
and constructing $f_2, f_3, \cdots$ by setting $f_2(x) = \frac{f_1(2x)}{2}$, $f_3(x) = \frac{f_1(4x)}{4}, \cdots$, $f_n(x) = \frac{f_1(2^{n-1}x)}{2^{n-1}}, \cdots$, and adding these: $b(x) = \sum_{n=1}^{\infty} f_n(x)$, $x \in \mathbb{R}$. Then it can be shown that b is continuous on $\mathbb{R}$, but not differentiable at any $x \in \mathbb{R}$.

Sufficient condition for differentiability. We have seen in Example 5.50 that even though all the partial derivatives exist at a point, the function may not be differentiable at that point. However, the following result says that if the partial derivatives are *continuous* in a neighbourhood of the point, then the function is differentiable in that neighbourhood. Here is the precise statement of the result.

Theorem 5.64. *Let $U \subset \mathbb{R}^n$ be open, and $f : U \to \mathbb{R}^m$ be such that $\frac{\partial f_i}{\partial x_j}$, $i \in \{1, \cdots, m\}$, $j \in \{1, \cdots, n\}$, are continuous on U. Then f is differentiable on U.*

Proof. By Exercise 5.56, it suffices to take $m = 1$. Let $\boldsymbol{a} = (a_1, \cdots, a_n) \in U$. As U is open, there exists an $r > 0$ such that $B(\boldsymbol{a}, r) \subset U$. Given $\boldsymbol{h} = (h_1, \cdots, h_n)$ with $\|\boldsymbol{h}\|_2 < r$, define $\boldsymbol{h_0} = \mathbf{0}$, and $\boldsymbol{h}_i = (h_1, \cdots, h_i, 0, \cdots, 0)$, $i \in \{1, \cdots, n\}$. Then

$$f(\boldsymbol{a} + \boldsymbol{h}) - f(\boldsymbol{a}) = \sum_{i=1}^{n} \big(f(\boldsymbol{a} + \boldsymbol{h}_i) - f(\boldsymbol{a} + \boldsymbol{h}_{i-1})\big).$$



By the Mean Value Theorem applied to $[a_i, a_i + h_i] \ni x \mapsto f(a_1 + h_1, \cdots, a_{i-1} + h_{i-1}, x, a_{i+1}, \cdots, a_n)$, there exists a $b_i \in (a_i, a_i + h_i)$ such that

$$\begin{aligned} f(\boldsymbol{a} + \boldsymbol{h}_i) - f(\boldsymbol{a} + \boldsymbol{h}_{i-1}) &= \ f(a_1 + h_1, \cdots, a_{i-1} + h_{i-1}, a_i + h_i, a_{i+1}, \cdots, a_n) \\ &\quad -f(a_1 + h_1, \cdots, a_{i-1} + h_{i-1}, a_i, a_{i+1}, \cdots, a_n) \\ &= h_i \frac{\partial f}{\partial x_i}(a_1 + h_1, \cdots, a_{i-1} + h_{i-1}, b_i, a_{i+1}, \cdots, a_n) \\ &= h_i \frac{\partial f}{\partial x_i}(\boldsymbol{b}_i), \end{aligned}$$

where $\boldsymbol{b}_i := (a_1 + h_1, \cdots, a_{i-1} + h_{i-1}, b_i, a_{i+1}, \cdots, a_n)$. We note that $\|\boldsymbol{b}_i - \boldsymbol{a}\|_2 \leqslant \|\boldsymbol{h}\|_2$. So for $\boldsymbol{h} \neq \mathbf{0}$,

$$\frac{|f(\boldsymbol{a}+\boldsymbol{h}) - f(\boldsymbol{a}) - \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(\boldsymbol{a})|}{\|\boldsymbol{h}\|_2} = \frac{|\sum_{i=1}^{n} h_i \big(\frac{\partial f}{\partial x_i}(\boldsymbol{b}_i) - \frac{\partial f}{\partial x_i}(\boldsymbol{a})\big)|}{\|\boldsymbol{h}\|_2} \leqslant \sum_{i=1}^{n} \Big|\frac{\partial f}{\partial x_i}(\boldsymbol{b}_i) - \frac{\partial f}{\partial x_i}(\boldsymbol{a})\Big| \frac{|h_i|}{\|\boldsymbol{h}\|_2} \leqslant \sum_{i=1}^{n} \Big|\frac{\partial f}{\partial x_i}(\boldsymbol{b}_i) - \frac{\partial f}{\partial x_i}(\boldsymbol{a})\Big|.$$

By the continuity of $\frac{\partial f}{\partial x_i}$ in U and noting that $\|\boldsymbol{b}_i - \boldsymbol{a}\|_2 \leqslant \|\boldsymbol{h}\|_2$, $i \in \{1, \cdots, m\}$, it follows that given any $\epsilon > 0$, we can choose a $\delta \in (0, r)$ such that whenever $\boldsymbol{x} \in U$ satisfies $0 < \|\boldsymbol{x} - \boldsymbol{a}\|_2 < \delta$, we have with $\boldsymbol{h} := \boldsymbol{x} - \boldsymbol{a}$ that the left-hand side above is $< \epsilon$. This completes the proof. So f is differentiable at $\boldsymbol{a}$. As $\boldsymbol{a} \in U$ was arbitrary, f is differentiable in U. □

Inverse Function Theorem and Implicit Function Theorem. A useful analogue of the Differentiable Inverse Theorem we met in Exercise 5.11 is the following result.

Theorem 5.65 (Inverse Function Theorem)**.**
Let $O \subset \mathbb{R}^n$ be an open set and $\boldsymbol{f} : O \to \mathbb{R}^n$ be a continuously differentiable function on O. Suppose that $\boldsymbol{f}'(\boldsymbol{a})$ is invertible for some $\boldsymbol{a} \in O$. Then there exist open sets U and V in $\mathbb{R}^n$ such that $\boldsymbol{a} \in U$, $\boldsymbol{f}(\boldsymbol{a}) \in V$, f is injective on U, $\boldsymbol{f}(U) = V$, and $\boldsymbol{f}^{-1} : V \to \mathbb{R}^n$ is continuously differentiable on V, with $(\boldsymbol{f}^{-1})'(\boldsymbol{f}(\boldsymbol{x})) = (\boldsymbol{f}'(\boldsymbol{x}))^{-1}$ for all $\boldsymbol{x} \in U$.

The above result can be used to derive further corollaries, for example, the Implicit Function Theorem, stated below. In order to motivate this result, consider a curve $\{(x, y) \in \mathbb{R}^2 : F(x, y) = 0\}$ and the question of whether there exists a local description of the curve around a point (a, b) of the form $(g(y), y)$ with the 'parameter' y belonging to some open interval containing b. The Implicit Function Theorem answers this question. Before we state this result, we introduce some convenient notation. For a linear transformation $T : \mathbb{R}^{n+m} \to \mathbb{R}^n$, we define $T_x : \mathbb{R}^n \to \mathbb{R}^n$ and $T_y : \mathbb{R}^m \to \mathbb{R}^n$ by

$$\begin{aligned} T_x \boldsymbol{h} &= T(\boldsymbol{h}, \boldsymbol{0}) \quad (\boldsymbol{h} \in \mathbb{R}^n), \\ T_y \boldsymbol{k} &= T(\boldsymbol{0}, \boldsymbol{k}) \quad (\boldsymbol{k} \in \mathbb{R}^m). \end{aligned}$$

Theorem 5.66 (Implicit Function Theorem)**.**
Let $O \subset \mathbb{R}^{n+m}$ be an open set and $\boldsymbol{f} : O \to \mathbb{R}^n$ be a continuously differentiable function on O. Let $(\boldsymbol{a}, \boldsymbol{b}) \in O$ be such that $\boldsymbol{f}(\boldsymbol{a}, \boldsymbol{b}) = \boldsymbol{0}$ and the linear transformation $(\boldsymbol{f}'(\boldsymbol{a}, \boldsymbol{b}))_x : \mathbb{R}^n \to \mathbb{R}^n$ is invertible. Then there exist open sets $U \subset \mathbb{R}^{n+m}$ and $V \subset \mathbb{R}^m$, with $(\boldsymbol{a}, \boldsymbol{b}) \in U$ and $\boldsymbol{b} \in V$ such that for every $\boldsymbol{y} \in V$, there is a unique vector $\boldsymbol{g}(\boldsymbol{y})$ such that $(\boldsymbol{g}(\boldsymbol{y}), \boldsymbol{y}) \in U$ and $\boldsymbol{f}(\boldsymbol{g}(\boldsymbol{y}), \boldsymbol{y}) = \boldsymbol{0}$. The map $\boldsymbol{g} : V \to \mathbb{R}^n$ is continuously differentiable, $\boldsymbol{g}(\boldsymbol{b}) = \boldsymbol{a}$ and $\boldsymbol{g}'(\boldsymbol{b}) = -((\boldsymbol{f}'(\boldsymbol{a}, \boldsymbol{b}))_x)^{-1}(\boldsymbol{f}'(\boldsymbol{a}, \boldsymbol{b}))_y$.

For a proof of the Inverse Function Theorem and the Implicit Function Theorem, see for example [**R**]. The Implicit Function Theorem in turn is very useful, for example to show the Lagrange Multiplier Theorem in constrained optimisation, see e.g. [**A**].

Chapter 6

Integration

One traditional topic in real analysis that we haven't covered yet in these notes is Integration Theory. There are two important types of integrals: Riemann integrals and Lebesgue integrals. Riemann integration has the advantage that it is intuitive and easy to follow, and multivariable Riemann integration will be covered in the course MA212 *Further Mathematical Methods*. However, it turns out that Riemann integration is not amenable to certain natural limiting processes. For example, it turns out that the functional analogue of the Euclidean space $\mathbb{R}^n$, namely the space $C[a,b]$ equipped with the norm

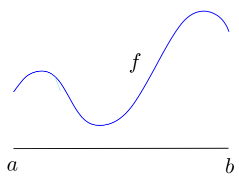
$$\|f\|_2 := \sqrt{\int_a^b |f(x)|^2 dx} \quad (f \in C[a,b])$$

is not complete, which turns out to be awkward when one wants to deal with applications. To remedy this, a more general integral called the Lebesgue integral can be defined, which rescues this situation. The interested reader is referred to the book by Rudin [**R**] for these matters.

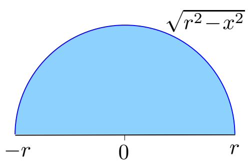
In this last chapter, we study the foundations of Riemann integration in the case of a function $f : [a,b] \to \mathbb{R}$. We study the definition, elementary properties and finish with establishing the Fundamental Theorem of Integral Calculus.

6.1. Motivation and definition of the Riemann integral

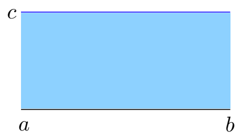
Let $f : [a,b] \to \mathbb{R}$ be a 'nice' function, and consider its graph:



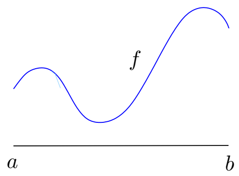
It is a basic problem in geometry to calculate the area under the graph of such a function f. Let us (for now) denote this area by $A(f)$. For example, when $f : [-r,r] \to \mathbb{R}$ is given by $f(x) = \sqrt{r^2 - x^2}$ $(-r \leqslant x \leqslant r)$, then we would like to calculate the area $A(f)$ under the graph of f, which is the area of the semicircular region:



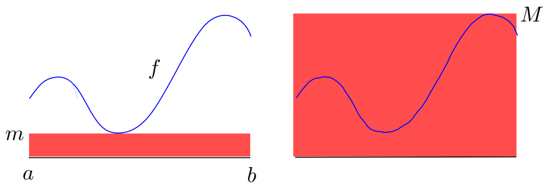
But what do we mean by 'area' and for which $f : [a, b] \to \mathbb{R}$ does $A(f)$ exist? Consider first a very simple case, namely when $f : [a, b] \to \mathbb{R}$ is a constant function $f(x) = c$, $x \in [a, b]$. Clearly, the area $A(f)$ under the graph of f should be the area of the shaded rectangle, given by $A(f) = c \cdot (b - a)$ (the product of the length with the breadth of the rectangle).



But what if f is not constant, and instead looks like this?



If there are numbers M, m such that $m \leqslant f(x) \leqslant M$ for all $x \in [a, b]$, then clearly we should have $m \cdot (b - a) \leqslant A(f) \leqslant M \cdot (b - a)$. This is illustrated by the picture below: The area $A(f)$ under the graph of f is flanked by the areas of the two shaded rectangles, that is, it satisfies $m(b - a) \leqslant A(f) \leqslant M(b - a)$.



This gives us the idea that we can estimate the area $A(f)$ by considering little rectangles, as shown in Figure 1 below, and we anticipate that if we make the rectangles finer and finer, then we should be able to approximate $A(f)$ better and better.

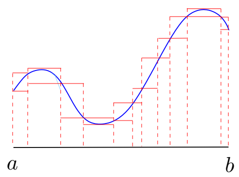
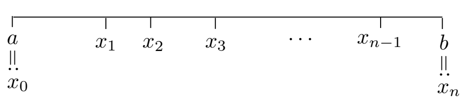


Figure 1. The area $A(f)$ under the graph of f satisfies $\underline{S} \leqslant A(f) \leqslant \overline{S}$, where $\underline{S}$ is the sum of all the areas of the rectangles shown above which lie *below* the graph of f, and $\overline{S}$ is the sum of all the areas of the rectangles shown in the picture which lie *above* the graph of f.

In order to make this precise, we introduce the notions of

- a *partition* P of an interval $[a, b]$, and
- an *upper/lower sum* associated with a partition P of $[a, b]$ and a bounded function $f : [a, b] \to \mathbb{R}$.

Definition 6.1 (Partition of an interval)**.** A *partition* (of an interval $[a, b] \subset \mathbb{R}$) is a finite set $P = \{x_0, x_1, \cdots, x_{n-1}, x_n\}$ such that $x_0 := a < x_1 < x_2 < x_3 < \cdots < x_{n-1} < b =: x_n$. The collection of all partitions of $[a, b]$ is denoted by $\mathcal{P}_{[a,b]}$.



Example 6.2. The sets $\{a, b\}$, $\{a, \frac{a+b}{2}, b\}$, $\{a, a+\frac{b-a}{3}, b\}$, $\{a, a+\frac{b-a}{n}, a+2\frac{b-a}{n}, \cdots, a+(n-1)\frac{b-a}{n}, b\}$ ($n \in \mathbb{N}$), are examples of partitions of $[a, b]$, and all of these belong to $\mathcal{P}_{[a,b]}$. ◇

Exercise 6.3. Which of the following statements is true?

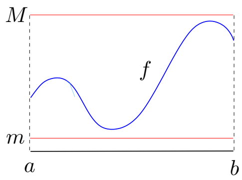
(1) $\{0, 1, \frac{1}{2}, \frac{1}{3}, \cdots\}$ is a partition of $[0, 1]$.

(2) Every interval $[a, b]$ has an infinite number of partitions.

(3) $\{0, 1, 2, 3\}$ is a partition of $[0, \infty)$.

(4) $\{\frac{1}{3}, \frac{1}{2}, \frac{3}{4}\}$ is a partition of $[0, 1]$.

Definition 6.4 (Bounded function)**.** A function $f : [a, b] \to \mathbb{R}$ is said to be *bounded* if there exist M, m such that for all $x \in [a, b]$, $m \leqslant f(x) \leqslant M$.



Pictorially, if we imagine a light source at '$x = +\infty$', sending parallel light rays to the left, then the 'shadow of the graph of f on the y-axis' is a bounded set. The following are equivalent:

(1) $f : [a, b] \to \mathbb{R}$ is bounded.

(2) There exists an $M \geqslant 0$ such that for all $x \in [a, b]$, $|f(x)| \leqslant M$.

(3) The range of f, namely the set $\{f(x) : x \in [a, b]\}$ is a bounded set.

That (2) and (3) are equivalent follows from Exercise 2.58. For (1)$\Rightarrow$(2), if for all $x \in [a, b]$ we have $\widetilde{m} \leqslant f(x) \leqslant \widetilde{M}$, then also $-f(x) \leqslant -\widetilde{m}$, giving $|f(x)| \leqslant \max\{\widetilde{M}, -\widetilde{m}\} =: M$, i.e., (2) holds. Vice versa, if (2) holds, then for all $x \in [a, b]$, $m := -M \leqslant f(x) \leqslant M$, i.e., (1) holds.

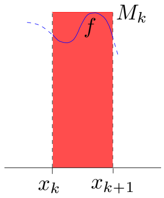
Example 6.5. The function $f : [0, 1] \to \mathbb{R}$ given by $f(x) = x^2$, $x \in \mathbb{R}$, is bounded. Indeed, for all $x \in [0, 1]$, we have $m := 0 \leqslant f(x) = x^2 \leqslant 1 =: M$.

On the other hand, the function $g : [0, 1] \to \mathbb{R}$ given by $g(x) = \left\{ \begin{array}{ll} \frac{1}{x} & \text{if } 0 < x \leqslant 1 \\ 0 & \text{if } x = 0 \end{array} \right\}$ is not bounded: If there exists an $M \in \mathbb{R}$ such that $g(x) \leqslant M$ for all $x \in [0, 1]$, then in particular, for all $n \in \mathbb{N}$, with $x := \frac{1}{n} \in [0, 1]$, we would have $g(x) = \frac{1}{\frac{1}{n}} = n \leqslant M$, ($n \in \mathbb{N}$), which is impossible by the Archimedean Property of $\mathbb{R}$. ◇

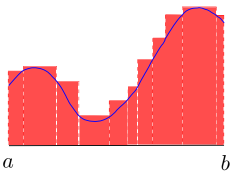
Definition 6.6 (Upper sum)**.** Let $f : [a, b] \to \mathbb{R}$ be bounded, and P be a partition of $[a, b]$.

The *upper sum* $\overline{S}(f, P)$ of f associated with a partition P is $\overline{S}(f, P) := \sum\limits_{k=0}^{n-1} M_k (x_{k+1} - x_k)$, where $M_k := \sup\limits_{x \in [x_k, x_{k+1}]} f(x)$, and $k \in \{0, 1, \cdots, n-1\}$.

The set $\{f(x) : x \in [x_k, x_{k+1}]\}$, namely the range of f restricted to the subinterval $[x_k, x_{k+1}]$ of $[a, b]$ is nonempty and bounded above (by any upper bound for the range of f on $[a, b]$). So M_k above makes sense for all indices k. The upper sum is formed by the addition of the various terms $M_k (x_{k+1} - x_k)$ for the different indices k. Each one of such terms is the area of the rectangle with base as the interval $[x_k, x_{k+1}]$ and height M_k for the various indices k, i.e., it is the area of the shortest rectangle lying above the graph of f in the interval $[x_k, x_{k+1}]$.



The rationale behind the notation $\overline{S}(f, P)$ is that S is for 'sum' (of areas of rectangles), the $\overline{\cdot}$ reminds us that the rectangles have their upper edges lying *above* the graph of f, and the (f, P) tells us *which* function f and partition P of $[a, b]$ we are forming the upper sum for.



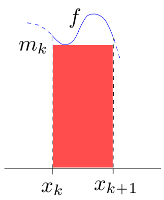
Example 6.7. For $n \in \mathbb{N}$, let P_n be the partition $P_n := \{0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \cdots, \frac{n-1}{n}, 1\}$ of the interval $[0, 1]$, and let $f : [0, 1] \to \mathbb{R}$ be the squaring function $f(x) = x^2$, $x \in [0, 1]$. As f is increasing, we have $M_k := \sup\limits_{x \in [\frac{k}{n}, \frac{k+1}{n}]} f(x) = \frac{(k+1)^2}{n^2}$. Thus the upper sum $\overline{S}(f, P_n)$ associated with f and P_n is

$$\begin{aligned}\overline{S}(f, P_n) &= \sum_{k=0}^{n-1} M_k \left(\tfrac{k+1}{n} - \tfrac{k}{n}\right) = \sum_{k=0}^{n-1} M_k \tfrac{1}{n} = \sum_{k=0}^{n-1} \tfrac{(k+1)^2}{n^2} \tfrac{1}{n} = \tfrac{1}{n^3} \sum_{k=0}^{n-1} (k+1)^2 \\ &= \tfrac{1}{n^3}(1^2 + 2^2 + 3^2 + \cdots + n^2) \overset{(*)}{=} \tfrac{1}{n^3} \tfrac{n(n+1)(2n+1)}{6} = \tfrac{1}{6}(1 + \tfrac{1}{n})(2 + \tfrac{1}{n}).\end{aligned}$$

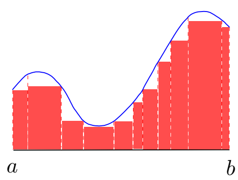
In $(*)$, we used the fact that for all $n \in \mathbb{N}$, $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$, which can be proved using induction on n. $\diamond$

Definition 6.8 (Lower sum)**.** Let $f : [a, b] \to \mathbb{R}$ be bounded, and P be a partition of $[a, b]$. The *lower sum* $\underline{S}(f, P)$ of f associated with a partition P is $\underline{S}(f, P) := \sum\limits_{k=0}^{n-1} m_k (x_{k+1} - x_k)$, where $m_k := \inf\limits_{x \in [x_k, x_{k+1}]} f(x)$, and $k \in \{0, 1, \cdots, n-1\}$.

The set $\{f(x) : x \in [x_k, x_{k+1}]\}$, namely the range of f restricted to the subinterval $[x_k, x_{k+1}]$ of $[a, b]$ is nonempty and bounded below (by any lower bound for the range of f on $[a, b]$). So m_k above makes sense for all indices k. The lower sum is obtained by adding the various terms $m_k (x_{k+1} - x_k)$ for the different indices k. Each of such term is the area of the rectangle with base as the interval $[x_k, x_{k+1}]$ and height m_k, i.e., it is the area of the tallest rectangle lying below the graph of f in the interval $[x_k, x_{k+1}]$.



The rationale behind the notation $\underline{S}(f, P)$ is that S is for 'sum' (of areas of rectangles), the $\underline{\cdot}$ reminds us that the rectangles have their upper edges lying *below* the graph of f, and the (f, P) tells us *which* function f and partition P of $[a, b]$ we are forming the lower sum for.



Example 6.9. For $n \in \mathbb{N}$, let P_n be the partition $P_n := \{0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \cdots, \frac{n-1}{n}, 1\}$ of the interval $[0,1]$, and let $f : [0,1] \to \mathbb{R}$ be the squaring function $f(x) = x^2$, $x \in [0,1]$. As f is increasing, we have $m_k := \sup\limits_{x \in [\frac{k}{n}, \frac{k+1}{n}]} f(x) = \frac{k^2}{n^2}$. Thus the lower sum $\underline{S}(f, P_n)$ associated with f and P_n is

$$\begin{aligned}\underline{S}(f, P_n) &= \sum_{k=0}^{n-1} m_k \left(\tfrac{k+1}{n} - \tfrac{k}{n}\right) = \sum_{k=0}^{n-1} m_k \frac{1}{n} = \sum_{k=0}^{n-1} \frac{k^2}{n^2}\frac{1}{n} = \frac{1}{n^3}\sum_{k=0}^{n-1} k^2 \\ &= \tfrac{1}{n^3}(0^2 + 1^2 + 2^2 + \cdots + (n-1)^2) = \tfrac{1}{n^3}\tfrac{(n-1)n(2n-1)}{6} = \tfrac{1}{6}(1 - \tfrac{1}{n})(2 - \tfrac{1}{n}). \qquad \diamond\end{aligned}$$

In order to arrive at a sensible definition of the integral of $f : [a,b] \to \mathbb{R}$, that is, of the area $A(f)$ under the graph of f, we first make the following observations, which will help us to formulate this sought after definition:

(1) For *any* partition P, we expect the area $A(f)$ under the graph of f to satisfy $A(f) \leqslant \overline{S}(f, P)$, and so the number $A(f)$ should be a lower bound for the set of all upper sums $\overline{S}(f, P)$ where P belongs to the collection $\mathcal{P}_{[a,b]}$ of all partitions of $[a,b]$. Thus

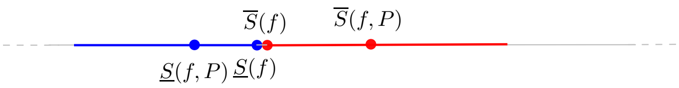
$$A(f) \leqslant \overline{S}(f) := \inf_{P \in \mathcal{P}_{[a,b]}} \overline{S}(f, P). \tag{6.1}$$

(2) For *any* partition P, we expect the area $A(f)$ under the graph of f to satisfy $\underline{S}(f, P) \leqslant A(f)$, and so the number $A(f)$ should be an upper bound for the set of all lower sums $\underline{S}(f, P)$ where P belongs to the collection $\mathcal{P}_{[a,b]}$ of all partitions of $[a,b]$. Thus

$$\sup_{P \in \mathcal{P}_{[a,b]}} \underline{S}(f, P) =: \underline{S}(f) \leqslant A(f). \tag{6.2}$$

(3) Putting (6.1) and (6.2) together, we see that our notion of the integral must satisfy

$$\underline{S}(f) \leqslant A(f) \leqslant \overline{S}(f).$$



Also, as our partitions P get finer, we expect that for nice functions f (for which we *can* define the area under its graph), $\underline{S}(f, P) \approx \overline{S}(f, P)$, and so for such nice functions, we would then expect that $\underline{S}(f) = A(f) = \overline{S}(f)$. And this motivates the following definition.

Definition 6.10 (Riemann integral of a Riemann integrable function)**.**

Let $\mathcal{P}_{[a,b]}$ be the collection of all partitions of $[a,b]$, and let $f : [a,b] \to \mathbb{R}$ be bounded.

Then f is said to be *Riemann integrable* (on $[a,b]$) if $\underline{S}(f) = \overline{S}(f)$, and the *Riemann integral*, denoted by $\int_a^b f(x)dx$ is defined to be this common value: $\int_a^b f(x)dx = \underline{S}(f) = \overline{S}(f)$.

The set of all Riemann integrable functions on $[a,b]$ is denoted by $RI[a,b]$.

In the notation $\int_a^b f(x)dx$, the $\int$ symbol is really an elongated S from 'sum', and the '$f(x)dx$' reminds us that we are taking in the upper and lower sums, we have areas of little rectangles, whose base length is an elemental change dx in x, and height is $f(x)$. The a and b at the bottom and top simple indicate what interval $[a,b]$ we are working with. The function f is often referred to as the *integrand*.

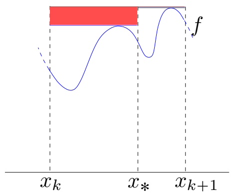
We will soon show that in general for *any* bounded function $f : [a,b] \to \mathbb{R}$ (Riemann integrable or not), we have $\overline{S}(f) \geqslant \underline{S}(f)$. For non-Riemann integrable functions, one has a strict inequality, and for Riemann integrable functions, one has an equality. In order to prove the inequality, we will need to investigate what happens to upper and lower sums when points are added to a partition. The new partition obtained by the process of adding extra points is called a refinement.

Definition 6.11 (Refinement of a partition).
If P, P_* are partitions of $[a, b]$ such that $P \subset P_*$, then P_* is called a *refinement of* P.

When a partition is refined, one can imagine that the approximations to the area under the graph of f becomes better, and so lower sums ought to increase, and upper sums ought to decrease. This is exactly what happens, and this is the content of the next result.

Lemma 6.12 (Refinement Lemma).
If P, P_* are partitions of $[a, b]$ with $P \subset P_*$, and $f : [a, b] \to \mathbb{R}$ is bounded, then $\overline{S}(f, P_*) \leqslant \overline{S}(f, P)$, and $\underline{S}(f, P_*) \geqslant \underline{S}(f, P)$.

Proof. Let $P = \{x_0, x_1, \cdots, x_{n-1}, x_n\}$. First suppose that P_* has just *one* extra point x_*, occurring in some subinterval $[x_k, x_{k+1}]$.



If we compare $\overline{S}(f, P)$ with $\overline{S}(f, P_*)$, we notice that most of the terms in the two sums are identical, except for the terms involving the interval $[x_k, x_{k+1}]$. (From the picture above, we see that $\overline{S}(f, P) - \overline{S}(f, P_*)$ essentially is the nonnegative area of the shaded rectangle.) We have

$$\begin{aligned}\overline{S}(f,P)-\overline{S}(f,P_*) &= \Big(\sup_{x\in[x_k,x_{k+1}]} f(x)\Big)(x_{k+1}-x_k) - \Big(\sup_{x\in[x_k,x_*]} f(x)\Big)(x_*-x_k) - \Big(\sup_{x\in[x_*,x_{k+1}]} f(x)\Big)(x_{k+1}-x_*)\\ &= \Big(\sup_{x\in[x_k,x_{k+1}]} f(x)\Big)(x_{k+1}-x_*+x_*-x_k) - \Big(\sup_{x\in[x_k,x_*]} f(x)\Big)(x_*-x_k) - \Big(\sup_{x\in[x_*,x_{k+1}]} f(x)\Big)(x_{k+1}-x_*)\\ &= \Big(\sup_{x\in[x_k,x_{k+1}]} f(x) - \sup_{x\in[x_k,x_*]} f(x)\Big)(x_*-x_k) + \Big(\sup_{x\in[x_k,x_{k+1}]} f(x) - \sup_{x\in[x_*,x_{k+1}]} f(x)\Big)(x_{k+1}-x_*)\\ &\geqslant 0+0=0.\end{aligned}$$

If P_* has several additional points (instead of just one additional point), then we repeat the argument several times, considering one extra point in each step to obtain

$$\overline{S}(f, P_*) \leqslant \cdots \leqslant \overline{S}(f, P_2) \leqslant \overline{S}(f, P_1) \leqslant \overline{S}(f, P)$$

where

P_1	is a refinement of P having one more point than P,
P_2	is a refinement of P_1 having one more point than P_1, and two extra points than P,
$\cdots$	and so on.

Thus $\overline{S}(f, P_*) \leqslant \overline{S}(f, P)$. The proof of $\underline{S}(f, P_*) \geqslant \underline{S}(f, P)$ is analogous. □

Corollary 6.13. *If $f : [a, b] \to \mathbb{R}$ is bounded then $\overline{S}(f) \geqslant \underline{S}(f)$.*

Proof. If P, P' are any two refinements of $[a, b]$, then $P \cup P'$ is a refinement of P as well as P', and so by the Refinement Lemma, we have

$$\overline{S}(f, P) \geqslant \overline{S}(f, P \cup P') \geqslant \underline{S}(f, P \cup P') \geqslant \underline{S}(f, P').$$

Thus $\overline{S}(f, P) \geqslant \underline{S}(f, P')$, for *any* two partitions P, P'. (So **any** upper sum is always bigger than **any** lower sum!) Let P be a fixed partition. For any partition $P' \in \mathcal{P}_{[a,b]}$, $\overline{S}(f, P) \geqslant \underline{S}(f, P')$. So $\overline{S}(f, P) \geqslant \sup\limits_{P'\in\mathcal{P}_{[a,b]}} \underline{S}(f, P') = \underline{S}(f)$. As $P \in \mathcal{P}_{[a,b]}$ was arbitrary, $\inf\limits_{P\in\mathcal{P}_{[a,b]}} \overline{S}(f, P) = \overline{S}(f) \geqslant \underline{S}(f)$. □

Let us now show that the squaring function is Riemann integrable on $[0,1]$, and calculate its value.

Example 6.14. Consider the bounded function $f:[0,1]\to\mathbb{R}$ given by $f(x)=x^2$, $x\in[0,1]$. We will show that $f\in RI[0,1]$ and that $\int_0^1 x^2dx=\frac{1}{3}$. Rather than considering *all* partitions, it turns out that we can be efficient and consider just the special partitions $P_n=\{0,\frac{1}{n},\frac{2}{n},\frac{3}{n},\cdots,\frac{n-1}{n},1\}$, $n\in\mathbb{N}$. From Examples 6.7 and 6.9, $\overline{S}(f,P_n)=\frac{1}{6}(1+\frac{1}{n})(1+\frac{2}{n})$ and $\underline{S}(f,P_n)=\frac{1}{6}(1-\frac{1}{n})(1-\frac{2}{n})$. Thus

$$\begin{aligned}\overline{S}(f) &= \inf_{P\in\mathcal{P}_{[0,1]}}\overline{S}(f,P)\leqslant\inf_{n\in\mathbb{N}}\overline{S}(f,P_n)=\inf_{n\in\mathbb{N}}\tfrac{1}{6}(1+\tfrac{1}{n})(2+\tfrac{1}{n})\overset{(*)}{=}\tfrac{1}{3},\ \text{and}\\ \underline{S}(f) &= \sup_{P\in\mathcal{P}_{[0,1]}}\underline{S}(f,P)\geqslant\sup_{n\in\mathbb{N}}\underline{S}(f,P_n)=\sup_{n\in\mathbb{N}}\tfrac{1}{6}(1-\tfrac{1}{n})(2-\tfrac{1}{n})\overset{(**)}{=}\tfrac{1}{3}.\end{aligned}$$

For the justification of $(*)$ and $(**)$, note that the sequence with n^{th} term $\frac{1}{6}(1+\frac{1}{n})(2+\frac{1}{n})$ is decreasing and bounded below by 0, and hence convergent to $\inf_{n\in\mathbb{N}}\frac{1}{6}(1+\frac{1}{n})(2+\frac{1}{n})$. On the other hand, from the Algebra of Limits, the limit must be $\frac{1}{6}(1+\lim_{n\to 0}\frac{1}{n})(2+\lim_{n\to 0}\frac{1}{n})=\frac{1}{6}(1+0)(2+0)=\frac{1}{3}$. The proof of $(**)$ is analogous.

Hence $\frac{1}{3}\geqslant\overline{S}(f)\geqslant\underline{S}(f)\geqslant\frac{1}{3}$, and so $\overline{S}(f)=\underline{S}(f)=\frac{1}{3}$. Thus $f\in RI[0,1]$ and $\int_0^1 x^2dx=\frac{1}{3}$. ◇

In the above example, we had to work rather hard to find the integral of a simple function. But we will soon learn about the Fundamental Theorem of Calculus, which will enable us to avoid such complicated calculations with partitions, lower and upper sums, infimums and supremums etc. Indeed, the Fundamental Theorem of Calculus says that if the integrand f is the derivative of a function F, then $\int_a^b f(x)dx=F(b)-F(a)$! In light of this result, we can now easily evaluate our previous example for the squaring function. Indeed, we simply note that the integrand $f:=x^2$ is the derivative of $F:=\frac{x^3}{3}$, and so $\int_0^1 x^2dx=\frac{1^3}{3}-\frac{0^3}{3}=\frac{1}{3}$. But before we establish the Fundamental Theorem of Calculus, we will first learn about a few basic, but important properties of the Riemann integral in the next section.

Are *all* bounded functions $f:[a,b]\to\mathbb{R}$ Riemann integrable? The answer is no, and here is an example.

Example 6.15. ($\mathbf{1}_\mathbb{Q}\notin RI[0,1]$.) Consider the indicator function[1] $\mathbf{1}_\mathbb{Q}$ of the rationals:

$$\mathbf{1}_\mathbb{Q}(x)=\begin{cases}1 & \text{if } x\in\mathbb{Q},\\ 0 & \text{if } x\notin\mathbb{Q}.\end{cases}$$

Clearly $\mathbf{1}_\mathbb{Q}$ is bounded: for all x, $0\leqslant\mathbf{1}_\mathbb{Q}(x)\leqslant 1$. We will show that (the restriction of) $\mathbf{1}_\mathbb{Q}$ on $[0,1]$ is not Riemann integrable on $[0,1]$ by showing that $\overline{S}(\mathbf{1}_\mathbb{Q})\geqslant 1>0\geqslant\underline{S}(\mathbf{1}_\mathbb{Q})$. Let $P=\{x_0=0,x_1,\cdots,x_{n-1},x_n=1\}$ be any partition of $[0,1]$. Then by the density of $\mathbb{Q}$ in $\mathbb{R}$, each $[x_k,x_{k+1}]$ contains a rational number, say $\alpha_k\in\mathbb{Q}$, and an irrational number, say $\beta_k\notin\mathbb{Q}$. Thus

$$M_k:=\sup_{x\in[x_k,x_{k+1}]}f(x)\geqslant f(\alpha_k)=1,\ \text{ and }\ m_k:=\inf_{x\in[x_k,x_{k+1}]}f(x)\leqslant f(\beta_k)=0.$$

Hence

$$\begin{aligned}\overline{S}(\mathbf{1}_\mathbb{Q},P) &= \sum_{k=0}^{n-1}M_k\,(x_{k+1}-x_k)\geqslant\sum_{k=0}^{n-1}1\,(x_{k+1}-x_k)\\ &= (x_1-x_0)+(x_2-x_1)+\cdots+(x_n-x_{n-1})=x_n-x_0=1-0=1.\end{aligned}$$

Similarly $\underline{S}(\mathbf{1}_\mathbb{Q},P)=\sum_{k=0}^{n-1}m_k\,(x_{k+1}-x_k)\leqslant\sum_{k=0}^{n-1}0\,(x_{k+1}-x_k)=0$.

So $\overline{S}(\mathbf{1}_\mathbb{Q})=\inf_{P\in\mathcal{P}_{[0,1]}}\overline{S}(\mathbf{1}_\mathbb{Q},P)\geqslant 1>0\geqslant\sup_{P\in\mathcal{P}_{[0,1]}}\overline{S}(\mathbf{1}_\mathbb{Q},P)=\underline{S}(\mathbf{1}_\mathbb{Q})$, and $\mathbf{1}_\mathbb{Q}\notin RI[0,1]$. ◇

Hence we have $RI[a,b]\subsetneq B[a,b]$, where $B[a,b]$ denotes the set of all bounded functions on $[a,b]$.

[1] If S is a subset of $\mathbb{R}$, then the *indicator function* $\mathbf{1}_S$ is defined by $\mathbf{1}_S(x)=1$ if $x\in S$ and 0 if $x\notin S$.

Exercise 6.16. Define $f : [0,1] \to \mathbb{R}$ by $f(x) = \begin{Bmatrix} 0 & \text{if } x \in [0,1]\backslash\mathbb{Q} \\ x & \text{if } x \in [0,1] \cap \mathbb{Q} \end{Bmatrix}$. Is f Riemann integrable on $[0,1]$?
Hint: For any partition $P = \{x_0 = 0 < x_1 < \cdots < x_{n-1} < x_n = 1\}$, $x_{k+1} \geqslant \frac{x_{k+1}+x_k}{2}$, $k \in \{0, \cdots, n-1\}$. Use this to find a positive lower bound on upper sums.

Let us now show that there is an ample supply of Riemann integrable functions: all continuous functions are Riemann integrable, that is, $C[a,b] \subset RI[a,b]$.

Theorem 6.17. *Every continuous function on $[a,b]$ is Riemann integrable on $[a,b]$.*

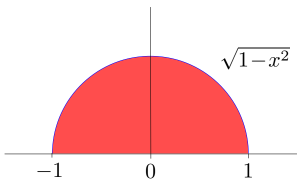
Proof. As f is continuous on $[a,b]$ and since $[a,b]$ is a compact interval, f is also *uniformly* continuous on $[a,b]$. Let $\epsilon > 0$. Then there exists a $\delta > 0$ such that whenever $x, y \in [a,b]$ satisfy $|x-y| < \delta$, we have $|f(x) - f(y)| < \epsilon$. Consider any partition $P_* = \{x_0, x_1, \cdots, x_{n-1}, x_n\}$ such that $\max\limits_{k \in \{0,1,\cdots,n-1\}} |x_{k+1} - x_k| < \delta$. Let $M_k := \sup\limits_{x \in [x_k, x_{k+1}]} f(x)$ and $m_k := \inf\limits_{x \in [x_k, x_{k+1}]} f(x)$. By the Extreme Value Theorem, $M_k = f(c_k)$ and $m_k = f(d_k)$ for some $c_k, d_k \in [x_k, x_{k+1}]$. Thus

$$\begin{aligned} \overline{S}(f, P_*) - \underline{S}(f, P_*) = \sum_{k=0}^{n-1} (M_k - m_k)(x_{k+1} - x_k) &= \sum_{k=0}^{n-1} (f(c_k) - f(d_k))(x_{k+1} - x_k) \\ &\leqslant \sum_{k=0}^{n-1} \epsilon\, (x_{k+1} - x_k) = \epsilon\, (b-a). \end{aligned}$$

Thus $0 \leqslant \overline{S}(f) - \underline{S}(f) \leqslant \overline{S}(f, P_*) - \underline{S}(f, P_*) \leqslant \epsilon(b-a)$. Since $\epsilon > 0$ was arbitrary, it follows that $\overline{S}(f) = \underline{S}(f)$, that is, $f \in RI[a,b]$. ☐

Example 6.18. All polynomial functions, being continuous, are Riemann integrable on every compact interval $[a,b]$. ◇

Example 6.19 (Definition of π)**.** Consider the continuous function $f : [-1,1] \to \mathbb{R}$ defined by $f(x) = \sqrt{1-x^2}$, $x \in [-1,1]$. Then f is Riemann integrable. We *define* the number $\pi \in \mathbb{R}$ by $\pi := 2\int_{-1}^{1} \sqrt{1-x^2}dx =$ two times the area of the semicircular disc of radius 1.



(It can be shown that for a circle of radius r, the area enclosed by it is πr^2; see Exercise 6.48.) ◇

Exercise 6.20. (The aim of this exercise is twofold: first, to show that $C[a,b] \subsetneq RI[a,b]$, and secondly, to point out that the Riemann integral gives the *signed* area under the graph of f, so that if the graph lies below the x-axis, then the area is attributed a *negative* sign.) Let $f : [0,2] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 1 & \text{if } x \in [0,1], \\ -1 & \text{if } x \in (1,2]. \end{cases}$$

Show that $f \in RI[0,2] \backslash C[0,2]$ and $\int_0^2 f(x)dx = 0$. *Hint:* Consider the partitions $P_n = \{0, 1, 1+\frac{1}{n}, 2\}$, $n \in \mathbb{N}$.

Exercise 6.21. For a partition $P = \{x_0 = a, x_1, \cdots, x_{n-1}, x_n = b\}$ of $[a,b]$, with $x_k < x_{k+1}$ for all $k \in \{0, \cdots, n-1\}$, define $\Phi(P) := \max\{x_{k+1} - x_k : k = 0, \cdots, n-1\}$. Which of the following is always true for any continuous function $f : [a,b] \to \mathbb{R}$?

(A) If P_2 is a refinement of P_1 (that is, $P_1 \subset P_2$), then $\Phi(P_2) \leqslant \Phi(P_1)$.
(B) If $\Phi(P_2) \leqslant \Phi(P_1)$, then $\overline{S}(f, P_2) \leqslant \overline{S}(f, P_1)$.
(C) If $\Phi(P_2) \leqslant \Phi(P_1)$, then $\underline{S}(f, P_2) \leqslant \underline{S}(f, P_1)$.
(D) If $\Phi(P_2) \leqslant \Phi(P_1)$, then $\underline{S}(f, P_2) \leqslant \overline{S}(f, P_1)$.

6.2. Properties of the Riemann integral

Theorem 6.22.[2] *If $f, g \in RI[a,b]$ and $\alpha \in \mathbb{R}$, then $f+g \in RI[a,b]$, $\alpha\cdot f \in RI[a,b]$. Moreover, $\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$, and $\int_a^b (\alpha\cdot f)(x)dx = \alpha \int_a^b f(x)dx$.*

Proof. (*May be skipped.*) Let $\epsilon > 0$. Then there exist partitions P_f, P_g of $[a,b]$ such that $\overline{S}(f,P_f) < \overline{S}(f) + \frac{\epsilon}{2}$ and $\overline{S}(g,P_g) < \overline{S}(g) + \frac{\epsilon}{2}$. Then $P := P_f \cup P_g =: \{x_0, x_1, \cdots, x_{n-1}, x_n\}$ is a refinement of P_f and P_g, and

$$\begin{aligned}\overline{S}(f+g) &\leqslant \overline{S}(f+g,P) = \sum_{k=0}^{n-1}\Big(\sup_{x\in[x_k,x_{k+1}]}(f(x)+g(x))\Big)(x_{k+1}-x_k)\\ &\leqslant \sum_{k=0}^{n-1}\Big(\sup_{x\in[x_k,x_{k+1}]} f(x) + \sup_{x\in[x_k,x_{k+1}]} g(x)\Big)(x_{k+1}-x_k) = \overline{S}(f,P)+\overline{S}(g,P)\\ &\leqslant \overline{S}(f,P_f)+\overline{S}(g,P_g) < \overline{S}(f)+\tfrac{\epsilon}{2}+\overline{S}(g)+\tfrac{\epsilon}{2} = \overline{S}(f)+\overline{S}(g)+\epsilon.\end{aligned}$$

As $\epsilon > 0$ was arbitrary, it follows that $\overline{S}(f+g) \leqslant \overline{S}(f)+\overline{S}(g)$.

In a similar manner we can show that $\underline{S}(f,g) \geqslant \underline{S}(f)+\underline{S}(g)$. Here are the details. Let $\epsilon > 0$. There are partitions P_f, P_g of $[a,b]$ so that $\underline{S}(f,P_f) > \underline{S}(f) - \frac{\epsilon}{2}$ and $\underline{S}(g,P_g) > \underline{S}(g) - \frac{\epsilon}{2}$. Then $P := P_f \cup P_g =: \{x_0, x_1, \cdots, x_{n-1}, x_n\}$ is a refinement of P_f, P_g, and

$$\begin{aligned}\underline{S}(f+g) &\geqslant \underline{S}(f+g,P) = \sum_{k=0}^{n-1}\Big(\inf_{x\in[x_k,x_{k+1}]}(f(x)+g(x))\Big)(x_{k+1}-x_k)\\ &\geqslant \sum_{k=0}^{n-1}\Big(\inf_{x\in[x_k,x_{k+1}]} f(x) + \inf_{x\in[x_k,x_{k+1}]} g(x)\Big)(x_{k+1}-x_k) = \underline{S}(f,P)+\underline{S}(g,P)\\ &\geqslant \underline{S}(f,P_f)+\underline{S}(g,P_g) > \underline{S}(f)-\tfrac{\epsilon}{2}+\underline{S}(g)-\tfrac{\epsilon}{2} = \underline{S}(f)+\underline{S}(f)-\epsilon.\end{aligned}$$

As $\epsilon > 0$ was arbitrary, we obtain $\underline{S}(f,g) \geqslant \underline{S}(f)+\underline{S}(g)$.

From $\overline{S}(f+g) \leqslant \overline{S}(f)+\overline{S}(g)$ and $\underline{S}(f,g) \geqslant \underline{S}(f)+\underline{S}(g)$, we have

$$\underline{S}(f)+\underline{S}(g) \leqslant \underline{S}(f+g) \leqslant \overline{S}(f+g) \leqslant \overline{S}(f)+\overline{S}(g). \qquad (\star)$$

Since $f, g \in RI[a,b]$, we have $\underline{S}(f) = \overline{S}(f)$ and $\underline{S}(g) = \overline{S}(g)$. Thus the first and last terms in $(\star)$ are equal. Consequently, $\underline{S}(f+g) = \overline{S}(f+g)$, that is, $f+g \in RI[a,b]$. Moreover, we have that $\int_a^b (f(x)+g(x))dx = \overline{S}(f+g) = \overline{S}(f)+\overline{S}(g) = \int_a^b f(x)dx + \int_a^b g(x)dx$.

For the second claim, consider the three possible cases $\alpha > 0$, $\alpha = 0$ and $\alpha < 0$ separately:

1° $\alpha > 0$. For every partition P of $[a,b]$, we have

$$\begin{aligned}\overline{S}(\alpha\cdot f,P) &= \sum_{k=0}^{n-1}(\sup_{x\in[x_k,x_{k+1}]}(\alpha f(x)))(x_{k+1}-x_k) = \sum_{k=0}^{n-1}\alpha(\sup_{x\in[x_k,x_{k+1}]} f(x))(x_{k+1}-x_k) = \alpha\overline{S}(f,P),\\ \underline{S}(\alpha\cdot f,P) &= \sum_{k=0}^{n-1}(\inf_{x\in[x_k,x_{k+1}]}(\alpha f(x)))(x_{k+1}-x_k) = \sum_{k=0}^{n-1}\alpha(\inf_{x\in[x_k,x_{k+1}]} f(x))(x_{k+1}-x_k) = \alpha\underline{S}(f,P).\end{aligned}$$

Thus

$$\begin{aligned}\overline{S}(\alpha\cdot f) &= \inf_{P\in\mathcal{P}_{[a,b]}} \overline{S}(\alpha\cdot f,P) = \inf_{P\in\mathcal{P}_{[a,b]}} \alpha\overline{S}(f,P) = \alpha \inf_{P\in\mathcal{P}_{[a,b]}} \overline{S}(f,P) = \alpha\overline{S}(f)\\ &= \alpha\underline{S}(f) = \alpha \sup_{P\in\mathcal{P}_{[a,b]}} \underline{S}(f,P) = \sup_{P\in\mathcal{P}_{[a,b]}} \alpha\underline{S}(f,P) = \sup_{P\in\mathcal{P}_{[a,b]}} \underline{S}(\alpha\cdot f,P) = \underline{S}(\alpha\cdot f).\end{aligned}$$

Hence $\alpha\cdot f \in RI[a,b]$ and $\int_a^b (\alpha\cdot f)(x)dx = \overline{S}(\alpha\cdot f) = \alpha\overline{S}(f) = \alpha\int_a^b f(x)dx$.

2° $\alpha = 0$. Then $\alpha f(x) = 0$ for all $x \in [a,b]$, and so for every partition P of $[a,b]$, we have $\overline{S}(\alpha\cdot f,P) = 0 = \underline{S}(\alpha\cdot f,P)$, so that $\overline{S}(\alpha\cdot f) = 0 = \underline{S}(\alpha\cdot f)$. Hence $\alpha\cdot f \in RI[a,b]$ and $\int_a^b (\alpha\cdot f)(x)dx = 0 = 0\int_a^b f(x)dx = \alpha\int_a^b f(x)dx$.

[2] The content of this result can be expressed in linear algebraic language by saying that $RI[a,b]$ forms a *vector space* with operations of addition and scalar multiplication defined pointwise, and that the map $f \mapsto \int_a^b f(x)dx : RI[a,b] \to \mathbb{R}$ is a *linear transformation*.

3° $\alpha < 0$. Let $P = \{x_0, x_1, \cdots, x_{n-1}, x_n\}$ be any partition of $[a,b]$. First let $\alpha = -1$. Then

$$\overline{S}(-f,P) = \sum_{k=0}^{n-1}\Big(\sup_{x\in[x_k,x_{k+1}]} -f(x)\Big)(x_{k+1}-x_k) = \sum_{k=0}^{n-1}\Big(-\inf_{x\in[x_k,x_{k+1}]} f(x)\Big)(x_{k+1}-x_k) = -\underline{S}(f,P).$$

By replacing f by $-f$, we obtain from the above that $\underline{S}(-f,P) = -\overline{S}(f,P)$. Hence we have

$$\begin{aligned}\overline{S}(-f) &= \inf_{P\in\mathcal{P}_{[a,b]}} \overline{S}(-f,P) = \inf_{P\in\mathcal{P}_{[a,b]}} -\underline{S}(f,P) = -\sup_{P\in\mathcal{P}_{[a,b]}} \underline{S}(f,P) = -\underline{S}(f)\\ \underline{S}(-f) &= \sup_{P\in\mathcal{P}_{[a,b]}} \underline{S}(-f,P) = \sup_{P\in\mathcal{P}_{[a,b]}} -\overline{S}(f,P) = -\inf_{P\in\mathcal{P}_{[a,b]}} \overline{S}(f,P) = -\overline{S}(f).\end{aligned}$$

Thus $\overline{S}(-f) = -\underline{S}(f) = -\overline{S}(f) = \underline{S}(-f)$, and so $-f \in RI[a,b]$. Moreover, we have that $\int_a^b -f(x)dx = \overline{S}(-f) = -\overline{S}(f) = -\int_a^b f(x)dx$.

For general $\alpha < 0$, we have $\alpha = -|\alpha|$, and as $f \in RI[a,b]$, it follows from 1° that $|\alpha|\cdot f \in RI[a,b]$. From the above, we now obtain that $-|\alpha|\cdot f \in RI[a,b]$, that is, $\alpha\cdot f \in RI[a,b]$. Also,

$$\int_a^b \alpha f(x)dx = \int_a^b -|\alpha| f(x)dx = -\int_a^b |\alpha| f(x)dx = -|\alpha|\int_a^b f(x)dx = \alpha\int_a^b f(x)dx. \qquad \square$$

Example 6.23. For $n \in \mathbb{N}$, the map $x \mapsto x^n : \mathbb{R} \to \mathbb{R}$ is continuous, and so $x^n \in RI[a,b]$ for all a, b. Thus the polynomial $p \in RI[a,b]$, where $p(x) := c_0 + c_1 x + \cdots + c_d x^d$, and moreover, we have $\int_a^b p(x)dx = c_0\int_a^b 1dx + c_1\int_a^b xdx + \cdots + c_d\int_a^b x^d dx$. After learning the Fundamental Theorem of Calculus, we will know that $\int_a^b x^n dx = \int_a^b \frac{d}{dx}\frac{x^{n+1}}{n+1}dx = \frac{b^{n+1}-a^{n+1}}{n+1}$, for all $n = 0, 1, 2, 3, \cdots$. So $\int_a^b p(x)dx = c_0(b-a) + c_1\frac{b^2-a^2}{2} + \cdots + c_d\frac{b^{d+1}-a^{d+1}}{d+1}$. $\diamond$

The following result will play an important role in the sequel.

Theorem 6.24 (Riemann Condition)**.** *Let $f : [a,b] \to \mathbb{R}$ be bounded. Then we have:*

$f \in RI[a,b] \Leftrightarrow$ $\boxed{\text{for all } \epsilon > 0, \text{ there exists a partition } P_\epsilon \in \mathcal{P}_{[a,b]} \text{ such that } \overline{S}(f,P_\epsilon) - \underline{S}(f,P_\epsilon) < \epsilon.}$

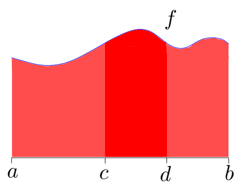
Proof. (*May be skipped.*)

($\Leftarrow$) For all $\epsilon > 0$, $0 \leqslant \overline{S}(f) - \underline{S}(f) \leqslant \overline{S}(f,P_\epsilon) - \underline{S}(f,P_\epsilon) < \epsilon$. It follows that $\overline{S}(f) = \underline{S}(f)$.

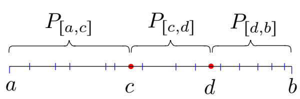
($\Rightarrow$) Suppose $f \in RI[a,b]$. Let $\epsilon > 0$. Then there exists a partition P_1 of $[a,b]$ such that $\overline{S}(f,P_1) < \overline{S}(f) + \frac{\epsilon}{2}$. Similarly, there exists a partition P_2 such that $\underline{S}(f,P_2) > \underline{S}(f) - \frac{\epsilon}{2}$. Consider the refinement $P_\epsilon := P_1 \cup P_2$ of P_1 and P_2. Then $\overline{S}(f,P_\epsilon) \leqslant \overline{S}(f,P_1) < \overline{S}(f) + \frac{\epsilon}{2}$, and $\underline{S}(f,P_\epsilon) \geqslant \underline{S}(f,P_2) > \underline{S}(f) - \frac{\epsilon}{2}$. So $0 \leqslant \overline{S}(f,P_\epsilon) - \underline{S}(f,P_\epsilon) < \underbrace{\overline{S}(f) - \underline{S}(f)}_{=0} + \epsilon = \epsilon$. $\square$

Let us now show that restrictions of Riemann integrable functions are Riemann integrable.

Theorem 6.25. *If $[c,d] \subset [a,b]$ and $f \in RI[a,b]$, then $f \in RI[c,d]$.*



Proof. (*May be skipped.*) Let $\epsilon > 0$. As $f \in RI[a,b]$, by the Riemann Condition, there exists a partition P_ϵ of $[a,b]$ such that $\overline{S}(f,P_\epsilon) - \underline{S}(f,P_\epsilon) < \epsilon$. Let $P'_\epsilon := P_\epsilon \cup \{c,d\} = P_{[a,c]} \cup P_{[c,d]} \cup P_{[d,b]}$, where $P_{[a,c]}$ is a partition of $[a,c]$, $P_{[c,d]}$ is a partition of $[c,d]$, and $P_{[d,b]}$ is a partition of $[d,b]$.



We know that

$$\overline{S}(f, P_\epsilon) \geqslant \overline{S}(f, P'_\epsilon) = \overline{S}(f, P_{[a,c]}) + \overline{S}(f, P_{[c,d]}) + \overline{S}(f, P_{[d,b]}),$$
$$\underline{S}(f, P_\epsilon) \leqslant \underline{S}(f, P'_\epsilon) = \underline{S}(f, P_{[a,c]}) + \underline{S}(f, P_{[c,d]}) + \underline{S}(f, P_{[d,b]}).$$

Thus

$$\epsilon > \overline{S}(f,P_\epsilon) - \underline{S}(f,P_\epsilon) \geqslant \overline{S}(f,P_{[a,c]}) + \overline{S}(f,P_{[c,d]}) + \overline{S}(f,P_{[d,b]}) - (\underline{S}(f,P_{[a,c]}) + \underline{S}(f,P_{[c,d]}) + \underline{S}(f,P_{[d,b]}))$$
$$\geqslant \overline{S}(f,P_{[a,c]}) - \underline{S}(f,P_{[a,c]}) + \overline{S}(f,P_{[c,d]}) - \underline{S}(f,P_{[c,d]}) + \overline{S}(f,P_{[d,b]}) - \underline{S}(f,P_{[d,b]})$$
$$\geqslant 0 + \overline{S}(f,P_{[c,d]}) - \underline{S}(f,P_{[c,d]}) + 0 = \overline{S}(f,P_{[c,d]}) - \underline{S}(f,P_{[c,d]}).$$

Hence by the Riemann Condition, $f \in RI[c, d]$. □

Exercise 6.26. Let $f : [a, b] \to \mathbb{R}$, $a < c < b$, $f \in RI[a, c]$ and $f \in RI[c, b]$. Then $f \in RI[a, b]$ and moreover $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.

Exercise 6.27. Let $f : [a, b] \to \mathbb{R}$ be a bounded function, such that f has only one discontinuity at $c \in (a, b)$. Show that $f \in RI[a, b]$. Extend the result to a finite number of discontinuities of f in (a, b).

Theorem 6.28. *If $f, g \in RI[a, b]$, then $f \cdot g \in RI[a, b]$.*

Proof. (*May be skipped.*) Let $\epsilon > 0$. Let $M_f, M_g > 0$ be such that $|f(x)| < M_f$ and $|g(x)| < M_g$ for all $x \in [a, b]$. Since $f \in RI[a, b]$, by the Riemann Condition, there exists a partition P_f of $[a, b]$ such that $\overline{S}(f, P_f) - \underline{S}(f, P_f) < \frac{\epsilon}{2M_g}$. As $g \in RI[a, b]$, there exists a partition P_g of $[a, b]$ such that $\overline{S}(f, P_g) - \underline{S}(f, P_g) < \frac{\epsilon}{2M_f}$. Consider the refinement $P := P_f \cup P_g =: \{x_0, x_1, \cdots, x_{n-1}, x_n\}$ of P_f and P_g. For a bounded function φ on $[a, b]$ and a $k \in \{0, 1, \cdots, n-1\}$, we use the notation $M_{\varphi,k} := \sup_{x\in[x_k,x_{k+1}]} \varphi(x)$, and $m_{\varphi,k} := \inf_{x\in[x_k,x_{k+1}]} \varphi(x)$. Then for $x, y \in [x_k, x_{k+1}]$,

$$\begin{aligned}(f \cdot g)(x) - (f \cdot g)(y) &= f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)\\ &= f(x)(g(x) - g(y)) + (f(x) - f(y))g(y)\\ &\leqslant |f(x)||g(x) - g(y)| + |g(y)||f(x) - f(y)|\\ &\leqslant M_f(M_{g,k} - m_{g,k}) + M_g(M_{f,k} - m_{f,k}).\end{aligned}$$

As $x, y \in [x_k, x_{k+1}]$ were arbitrary, $M_{f\cdot g,k} - m_{f\cdot g,k} \leqslant M_f(M_{g,k} - m_{g,k}) + M_g(M_{f,k} - m_{f,k})$. Thus

$$\begin{aligned}\overline{S}(f \cdot g) - \underline{S}(f \cdot g) &\leqslant \overline{S}(f \cdot g, P) - \underline{S}(f \cdot g, P) \leqslant M_f(\overline{S}(g, P) - \underline{S}(g, P)) + M_g(\overline{S}(f, P) - \underline{S}(f, P))\\ &\leqslant M_f(\overline{S}(g, P_g) - \underline{S}(g, P_g)) + M_g(\overline{S}(f, P_f) - \underline{S}(f, P_f)) \leqslant M_f\tfrac{\epsilon}{2M_f} + M_g\tfrac{\epsilon}{2M_g} = \epsilon.\end{aligned}$$

By the Riemann Condition, we conclude that $f \cdot g \in RI[a, b]$. □

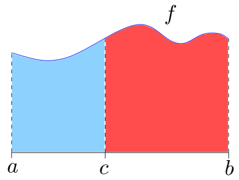
Some conventions. When defining $\int_a^b f(x)dx$, we assumed that $a < b$.

To simplify matters in what is to follow, we will adopt the following new definitions:

(1) If $a = b$, then *every* $f : [a, b] \to \mathbb{R}$ is Riemann integrable, and we define $\int_a^a f(x)dx := 0$.

(2) If $a > b$ and $f : [b, a] \to \mathbb{R}$ is Riemann integrable, then we define $\int_a^b f(x)dx := -\int_b^a f(x)dx$.

Theorem 6.29 (Domain additivity)**.** *Suppose that $f \in RI[a, b]$ and let c lie between a and b. Then $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.*



Proof. As restrictions of Riemann integrable functions are Riemann integrable, $f \in RI[a,c]$ and $f \in RI[c,b]$. The claim now follows immediately from Exercise 6.26. ☐

Some useful inequalities associated with Riemann integration.

Theorem 6.30. *Let $a < b$ and $f, g \in RI[a,b]$. Then we have:*

(1) *If for all $x \in [a,b]$, $f(x) \geqslant 0$, then $\int_a^b f(x)dx \geqslant 0$.*

(2) *If for all $x \in [a,b]$, $f(x) \geqslant g(x)$, then $\int_a^b f(x)dx \geqslant \int_a^b g(x)dx$.*

(3) *$|f| \in RI[a,b]$ and $|\int_a^b f(x)dx| \leqslant \int_a^b |f(x)|dx$.*

(4) *Let $f \in C[a,b]$ and for all $x \in [a,b]$ $f(x) \geqslant 0$.*
If $\int_a^b f(x)dx = 0$, then $f \equiv 0$ on $[a,b]$, that is, f is identically zero on $[a,b]$.

Proof. (*May be skipped.*)

(1) We have $\int_a^b f(x)dx = \underline{S}(f) = \sup_{P \in \mathcal{P}_{[a,b]}} \underline{S}(f,P) \geqslant \underline{S}(f,\{a,b\}) = (\underbrace{\inf_{x \in [a,b]} f(x)}_{\geqslant 0})(b-a) \geqslant 0.$

(2) We apply (1) to $h := f - g$. Clearly, $h(x) \geqslant 0$ for all $x \in [a,b]$, and $h = f - g \in RI[a,b]$. So $\int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b (f(x) - g(x))dx = \int_a^b h(x)dx \geqslant 0$. Thus $\int_a^b f(x)dx \geqslant \int_a^b g(x)dx$.

(3) Let $\epsilon > 0$. By the Riemann Condition, there exists a partition $P_\epsilon = \{x_0, x_1, \cdots, x_{n-1}, x_n\}$ of $[a,b]$ such that $\overline{S}(f, P_\epsilon) - \underline{S}(f, P_\epsilon) < \epsilon$.

Claim: $\overline{S}(|f|, P_\epsilon) - \underline{S}(|f|, P_\epsilon) < \epsilon$.

For any fixed $k \in \{0, 1, \cdots, n-1\}$, let $x, y \in [x_k, x_{k+1}]$. With $M_k := \sup_{x \in [x_k, x_{k+1}]} f(x)$ and $m_k := \inf_{x \in [x_k, x_{k+1}]} f(x)$, we have that $f(x) - f(y) \leqslant M_k - m_k$, and $f(y) - f(x) \leqslant M_k - m_k$. Hence $|f(x) - f(y)| \leqslant M_k - m_k$. So $|f|(x) - |f|(y) = |f(x)| - |f(y)| \leqslant |f(x) - f(y)| \leqslant M_k - m_k$. Thus $\sup_{x \in [x_k, x_{k+1}]} |f|(x) - \inf_{y \in [x_k, x_{k+1}]} |f|(y) \leqslant M_k - m_k = \sup_{x \in [x_k, x_{k+1}]} f(x) - \inf_{y \in [x_k, x_{k+1}]} f(y)$. So $\overline{S}(|f|, P_\epsilon) - \underline{S}(|f|, P_\epsilon) \leqslant \overline{S}(f, P_\epsilon) - \underline{S}(f, P_\epsilon) < \epsilon$. This completes the proof of the claim.

By the Riemann Condition, $|f| \in RI[a,b]$.

Moreover, for all $x \in [a,b]$, $f(x) \leqslant |f(x)|$ and $-f(x) \leqslant |f(x)|$. So $\int_a^b f(x)dx \leqslant \int_a^b |f(x)|dx$ and $-\int_a^b f(x)dx \leqslant \int_a^b |f(x)|dx$. Thus $|\int_a^b f(x)dx| \leqslant \int_a^b |f(x)|dx$.

(4) Let $\neg(f \equiv 0 \text{ on } [a,b])$. Then there exists a $c \in [a,b]$ such that $f(c) \neq 0$. As $f \geqslant 0$, $f(c) > 0$. For $\epsilon := \frac{f(c)}{2} > 0$, by the continuity of f at c, there exists a $\delta > 0$ such that whenever $x \in [a,b]$ satisfies $|x - c| < \delta$, we have $|f(x) - f(c)| < \epsilon = \frac{f(c)}{2}$, and so

$$f(c) - f(x) \leqslant |f(c) - f(x)| = |f(x) - f(c)| < \frac{f(c)}{2}.$$

Hence $f(x) > f(c) - \frac{f(c)}{2} = \frac{f(c)}{2} > 0$ for $x \in [a,b] \cup (c - \delta, c + \delta)$. This also shows that if $c = a$ or $c = b$, then there are other values of c where f is positive. Thus there is no loss of generality in assuming that $c \in (a,b)$. Also, by reducing δ if necessary, we may assume that $a < c - \delta < c + \delta < b$. With $P_* := \{a, c - \delta, c + \delta, b\}$, we have

$$\begin{aligned} \int_a^b f(x)dx = \underline{S}(f) \geqslant \underline{S}(f, P_*) &= (\inf_{x \in [a, c-\delta]} f(x))(c - \delta - a) + (\inf_{x \in [c-\delta, c+\delta]} f(x))2\delta + (\inf_{x \in [c-\delta, b]} f(x))(b - c - \delta) \\ &\geqslant 0 + \frac{f(c)}{2} 2\delta + 0 = \delta f(c) > 0, \end{aligned}$$

a contradiction. ☐

Exercise 6.31.

(1) Let $f, g \in RI[a,b]$. Show that $\max\{f,g\}$, $\min\{f,g\} \in RI[a,b]$, where $\max\{f,g\} := \max\{f(x), g(x)\}$ and $\min\{f,g\} := \min\{f(x), g(x)\}$, for all $x \in [a,b]$. *Hint:* $\max\{a,b\} = \frac{a+b+|a-b|}{2}$ for $a, b \in \mathbb{R}$.

(2) The aim of this exercise is twofold: Firstly, to show that the pointwise supremum of a sequence of Riemann integrable functions need not be Riemann integrable, and secondly, to demonstrate that the pointwise limit of Riemann integrable functions need not be Riemann integrable.
Let $r_1, r_2, r_3, \cdots$ be an enumeration of the rationals in $[0,1]$.
Define $f_n : [0,1] \to \mathbb{R}$ by $f_n(x) = \begin{cases} 1 & \text{if } x \in \{r_1, \cdots, r_n\}, \\ 0 & \text{otherwise.} \end{cases}$
Is each $f_n \in RI[0,1]$? Let $(\sup\limits_{n\in\mathbb{N}} f_n)(x) := \sup\limits_{n\in\mathbb{N}} f_n(x)$, $x \in [0,1]$. Is $\sup\limits_{n\in\mathbb{N}} f_n \in RI[0,1]$?

Exercise 6.32. We have seen in Theorems 6.22, 6.28 and 6.30(3) that if $f, g \in RI[a,b]$, then so is their pointwise sum, product and their respective modulus. Give examples of bounded $f, g : [0,1] \to \mathbb{R}$ that are *not* Riemann integrable, but for which the functions $|f|$, $f+g$, fg *are* all Riemann integrable on $[0,1]$.

Exercise 6.33 (An integral mean value result)**.** Let $f \in C[a,b]$, $\varphi \in RI[a,b]$, and let ρ be pointwise nonnegative. (We may interpret ρ as the 'mass density' of a rod, along the interval $[a,b]$, made of a possibly inhomogeneous material. If $\rho \equiv c$, a constant, then the rod has uniform density along its length.) Show that there is a $c \in [a,b]$ such that $\int_a^b f(x)\rho(x)dx = f(c)\int_a^b \rho(x)dx$. (So for $\rho \equiv 1$, $\frac{1}{b-a}\int_a^b f(x)dx = f(c)$.)
If $f(x) = x$, then we can interpret the position c as the 'center of mass' of the horizontal (inhomogeneous) rod, namely the place about which if the rod is pivoted, it will remain balanced, since the moments about that point due to the weight of the the constituent particles of the rod add up to 0. If the rod is homogeneous, then the centre of mass c is given by $\frac{b^2-a^2}{2} = \int_a^b x \cdot 1\ dx = c\int_a^b 1\ dx = c(b-a)$, that is, $c = \frac{a+b}{2}$, as expected based on our physical intuition.
Give an example to show that the assumption $f \in C[a,b]$ cannot be dropped for the conclusion to hold. Moreover, provide an example to show that the nonnegativity of ρ is also a necessary condition.

Exercise 6.34 (Cantor set)**.** The aim of this exercise is to show that there exist Riemann integrable functions which have infinitely many points of discontinuity. Indeed, we will show that the indicator function $\mathbf{1}_C$ of the Cantor set (see Example 2.66) is Riemann integrable on $[0,1]$. Proceed as follows.

(1) As $C \subset F_n$, clearly $\mathbf{1}_C \leqslant \mathbf{1}_{F_n}$. Since $\mathbf{1}_{F_n}$ has only finitely many discontinuities, $\mathbf{1}_{F_n} \in RI[0,1]$.
Show that $\int_0^1 \mathbf{1}_{F_n}(x)dx =$ length of the intervals in $F_n = (\frac{2}{3})^n$.
Conclude that if $\epsilon > 0$, then there exists a partition P of $[0,1]$ such that $\overline{S}(\mathbf{1}_{F_n}, P) < (\frac{2}{3})^n + \epsilon$.
Deduce that $\overline{S}(\mathbf{1}_C) \leqslant 0$.

(2) As $\mathbf{1}_C \geqslant 0$, it is clear that $\underline{S}(\mathbf{1}_C, P) \geqslant 0$ for all partitions P of $[0,1]$, and so $\underline{S}(\mathbf{1}_C) \geqslant 0$.

(3) Conclude from Parts (1) and (2) that $\mathbf{1}_C \in RI[0,1]$, and that $\int_0^1 \mathbf{1}_C(x)dx = 0$.

Exercise 6.35. Can the assumption that $f \in C[0,1]$ in Theorem 6.30·(4) be replaced by the condition that $f \in RI[0,1]$?

Exercise 6.36 (Dirac δ)**.** For doing quantum mechanical computations, the physicist Paul Dirac introduced the δ 'function' (as eigenstates of the position operator). The aim of this exercise is to show that such function does not exist[3]. Show that there is no function $\delta : \mathbb{R} \to \mathbb{R}$ such that for all $a > 0$,
(1) $\delta|_{[-a,a]}$ is bounded, and $\delta \in RI[-a,a]$.
(2) For every $\varphi \in C[-a,a]$, $\int_{-a}^a \delta(x)\varphi(x)dx = \varphi(0)$.

Exercise 6.37 (Cauchy-Schwarz)**.** If $f, g \in RI[a,b]$, then show that

$$(\int_a^b f(x)g(x)dx)^2 \leqslant (\int_a^b (f(x))^2 dx)(\int_a^b (g(x))^2 dx)$$

by proceeding as follows. For $t \in \mathbb{R}$, define $\varphi(t) = \int_a^b ((f + t\cdot g)(x))^2 dx$. Then φ is a quadratic function of the variable t, and $\varphi(t) \geqslant 0$ for all $t \in \mathbb{R}$. This means that the discriminant of φ must be $\leqslant 0$, since otherwise, f would have two distinct real roots, and would then have negative values between these roots! Calculate the discriminant of φ and show that its nonpositivity yields the desired inequality.

Exercise 6.38. Define $\|\cdot\|_2 := \int_a^b |f(x)|^2 dx$, $f \in C[a,b]$. Show that $(C[a,b], \|\cdot\|_2)$ is a normed space.

[3]However, the mathematician Laurent Schwartz later gave a mathematical foundation to the Dirac δ by viewing it as a 'distribution', in which one thinks of δ as a (linear) map $\delta : C_0^\infty(\mathbb{R}) \to \mathbb{R}$, which sends $\varphi \in C_0^\infty(\mathbb{R})$ to the number $\varphi(0)$. Here $C_0^\infty(\mathbb{R})$ denotes the set of all functions $\varphi : \mathbb{R} \to \mathbb{R}$, which are infinitely many times differentiable and vanishing outside some compact interval (depending on φ). Distributions play a fundamental role in the study of partial differential equations.

Exercise 6.39. The aim of this exercise is to prove Proposition 2.47. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $RI[a,b]$ and $f:[a,b]\to\mathbb{R}$ be such that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f. We want to show that $f\in RI[a,b]$ and $\int_a^b f(x)dx = \lim\limits_{n\to\infty}\int_a^b f_n(x)dx$. Proceed as follows:

(1) Show that f is bounded and so for each $n\in\mathbb{N}$, $M_n := \sup\limits_{x\in[a,b]} |f_n(x)-f(x)|$ is well-defined.

(2) Prove that for all $n\in\mathbb{N}$ and all $x\in[a,b]$, $f_n(x)-M_n \leqslant f(x)\leqslant f_n(x)+M_n$.

(3) Use (2) to conclude that for all $n\in\mathbb{N}$, $\int_a^b(f_n(x)-M_n)dx \leqslant \underline{S}(f)\leqslant \overline{S}(f)\leqslant \int_a^b(f_n(x)+M_n)dx$.

(4) Use (3) to show for all $n\in\mathbb{N}$, $0\leqslant \overline{S}(f)-\underline{S}(f)\leqslant 2M_n(b-a)$, and Exercise 2.38 to show $f\in RI[a,b]$.

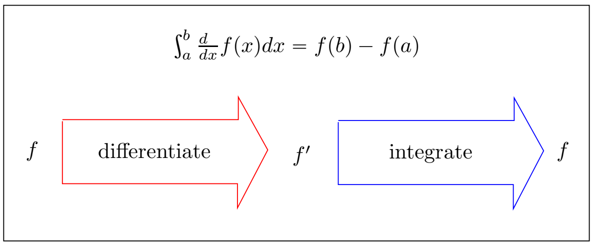
(5) It follows from (3) that for all $n\in\mathbb{N}$, $\overline{S}(f)-M_n(b-a)\leqslant \int_a^b f_n(x)dx \leqslant M_n(b-a)+\underline{S}(f)$.
Use the Sandwich Theorem to conclude that $\int_a^b f(x)dx = \lim\limits_{n\to\infty}\int_a^b f_n(x)dx$.

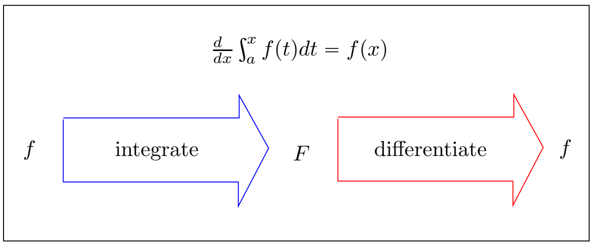
6.3. Fundamental Theorem of Calculus

Calculus has two components:

Differentiation	Integration
Local process: Derivative at a point depends only on values of the function near the point.	Global process: Takes into account values of the function in the entire interval.

But now we will learn about a bridge between these two seemingly different worlds of differentiation and integration, namely the Fundamental Theorem of Calculus, which says, roughly that the two processes of differentiation and integration are inverses of each other.





Before stating the Fundamental Theorem of Calculus, we give the following definition.

Definition 6.40 (Primitive of a function)**.** Let $f:[a,b]\to\mathbb{R}$. Then a function $F:[a,b]\to\mathbb{R}$ is called a *primitive of* f if

(1) F is differentiable on $[a,b]$ and

(2) for every[4] $x\in[a,b]$, $F'(x)=f(x)$.

[4]The derivative at the boundary point a is the number $L =: F'(a)$ (if it exists) such that for every $\epsilon>0$, there exists a $\delta>0$ such that whenever $x\in[a,b]$ satisfies $0<x-a<\delta$, we have $|\frac{F(x)-F(a)}{x-a}-L|<\epsilon$. Similarly, the derivative $F'(b)$ at the boundary point b is the number such that for every $\epsilon>0$, there exists a $\delta>0$ such that whenever $x\in[a,b]$ satisfies $0<b-x<\delta$, we have $|\frac{F(x)-F(b)}{x-b}-F'(b)|<\epsilon$. A straightforward adaptation of the proof Theorem 5.1 shows that if $F'(a)$ exists, then F is continuous at a, and if $F'(b)$ exists, then F is continuous at b.

Example 6.41 (Primitives are not unique)**.** Both the functions $\frac{x^2}{2}$ and $\frac{x^2}{2} - 399$ are primitives of x. In fact, any function $x^2 + C$, where C is an arbitrary constant, is a primitive of x. ◇

The previous example shows that primitives are not unique. But we will show later on that they are unique 'up to additive constants', that is, for any two primitives $F, \widetilde{F}$ of f, there is a constant C (depending on the pair $F, \widetilde{F}$) such that $\widetilde{F} = F + C$ on $[a, b]$.

Theorem 6.42 (Fundamental Theorem of Calculus)**.** *Let* $f \in RI[a, b]$. *Then:*

(1) *If* f *has a primitive* F, *then* $\int_a^x f(t)dt = F(x) - F(a)$ *for all* $x \in [a, b]$.

(2) *Define* $F : [a, b] \to \mathbb{R}$ *by* $F(x) := \int_a^x f(t)dt$ *for all* $x \in [a, b]$.
If f *is continuous at* $c \in [a, b]$, *then* F *is differentiable at* c, *and* $F'(c) = f(c)$.
In particular, if $f \in C[a, b]$, *then* F *is a primitive of* f.

Proof. (of Part (1):)
(If $x = a$, the both the left hand side and right hand side are 0, and so the result holds. So let us assume that $x > a$.) Let $P = \{x_0, x_1, \cdots, x_{n-1}, x_n\}$ be *any* partition of $[a, x]$. By the Mean Value Theorem, $\frac{F(x_{k+1})-F(x_k)}{x_{k+1}-x_k} = f(c_k)$, for some $c_k \in (x_k, x_{k+1})$. Thus

$$\begin{aligned}\overline{S}(f, P) &= \sum_{k=0}^{n-1}\big(\sup_{x\in[x_k,x_{k+1}]} f(x)\big)(x_{k+1} - x_k)\\ &\geqslant \sum_{k=0}^{n-1} f(c_k)(x_{k+1} - x_k) = \sum_{k=0}^{n-1}(F(x_{k+1}) - F(x_k))\\ &= F(x_1) - F(x_0) + F(x_2) - F(x_1) + \cdots + F(x_n) - F(x_{n-1})\\ &= F(x_n) - F(x_0) = F(x) - F(a).\end{aligned}$$

Thus, for any partition P of $[a, x]$, we have $\overline{S}(f, P) \geqslant F(x) - F(a)$, and so

$$\overline{S}(f) \geqslant F(x) - F(a). \tag{6.3}$$

Similarly,

$$\underline{S}(f, P) = \sum_{k=0}^{n-1}\big(\inf_{x\in[x_k,x_{k+1}]} f(x)\big)(x_{k+1} - x_k) \leqslant \sum_{k=0}^{n-1} f(c_k)(x_{k+1} - x_k) = \sum_{k=0}^{n-1}(F(x_{k+1}) - F(x_k)) = F(x) - F(a).$$

Hence, for any partition P, we have $\underline{S}(f, P) \leqslant F(x) - F(a)$, giving

$$\underline{S}(f) \leqslant F(x) - F(a). \tag{6.4}$$

From (6.3) and (6.4), we obtain $\int_a^x f(t)dt = \underline{S}(f) \leqslant F(x) - F(a) \leqslant \overline{S}(f) = \int_a^x f(t)dt$. Consequently, $F(x) - F(a) = \int_a^x f(t)dt$. This finishes the proof of Part (1). □

Before moving on to the proof of Part (2), here is an example illustrating Part (1).

Example 6.43. With $F := \frac{x^3}{3}$ and $f := x^2$, we have $F' = f$ on $\mathbb{R}$. Since $f \in C[0, 1] \subset RI[0, 1]$, it follows from the above and the Fundamental Theorem of Calculus that $\int_0^1 x^2 dx = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$. Note the remarkable simplicity obtained (as opposed to the calculation done earlier in Example 6.14), thanks to the Fundamental Theorem of Calculus. ◇

Now let us continue with the proof of Part (2) of the Fundamental Theorem of Calculus.

Proof. (of Part (2)): Let $\epsilon > 0$. As f is continuous at c, there exists a $\delta > 0$ such that whenever $t \in [a, b]$ satisfies $|t - c| < \delta$, we have $|f(t) - f(c)| < \epsilon$. Let $x \in [a, b]\backslash\{c\}$. Then by the definition of F and the result on Domain Additivity, we obtain

$$\frac{F(x)-F(c)}{x-c} = \frac{1}{x-c}\Big(\int_a^x f(t)dt - \int_a^c f(t)dt\Big) = \frac{1}{x-c}\int_c^x f(t)dt. \tag{6.5}$$

Also, by Part (1) of the Fundamental Theorem of Calculus,

$$\int_c^x f(c)dt = \int_c^x (f(c)\cdot t)'dt = f(c)\cdot x - f(c)\cdot c = f(c)\cdot(x-c),$$

and so for $x \in [a,b]\backslash\{c\}$,

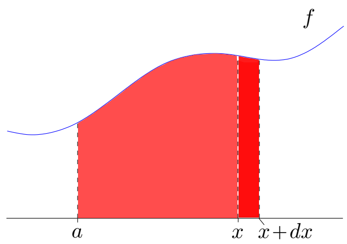
$$f(c) = \frac{1}{x-c}\int_c^x f(c)dt. \tag{6.6}$$

From (6.5) and (6.6), $|\frac{F(x)-F(c)}{x-c} - f(c)| = |\frac{1}{x-c}\int_c^x f(t)dt - \frac{1}{x-c}\int_c^x f(c)dt| = \frac{1}{|x-c|}|\int_c^x \big(f(t)-f(c)\big)dt|$ for all $x \in [a,b]\backslash\{c\}$. So for $x \in [a,b]$ satisfying $0 < |x-c| < \delta$, we have

$$|\frac{F(x)-F(c)}{x-c} - f(c)| = \frac{1}{|x-c|}|\int_c^x \big(f(t)-f(c)\big)dt| \leqslant \frac{1}{|x-c|}\int_{\min\{c,x\}}^{\max\{c,x\}} |f(t)-f(c)|dt$$
$$\leqslant \frac{1}{|x-c|}\overline{S}(|f(\cdot)-f(c)|,\{c,x\}) \leqslant \frac{1}{|x-c|}\epsilon|x-c| = \epsilon.$$

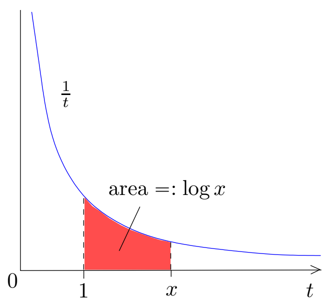
Consequently, $F'(c) = f(c)$. □

Geometric interpretation of the Fundamental Theorem of Calculus. The plausibility of Part (2) of the Fundamental Theorem of Calculus can be illustrated geometrically. See the figure above, in which we have depicted the graph of a Riemann integrable function f.



Let F be defined by $F(x) = \int_a^x f(t)dt$ for all $x \geqslant a$. Then $F(x)$ is the area under the graph of f from a to x. Consider an $x \geqslant a$, and imagine increasing x by a tiny amount dx. The area of the little strip created is $F(x+dx) - F(x) \approx f(x)\cdot dx$, and dividing throughout by dx, we obtain $F'(x) \approx \frac{F(x+dx)-F(x)}{dx} \approx f(x)$.

Example 6.44. For $n \in \mathbb{Z}\backslash\{-1\}$, $(\frac{x^{n+1}}{n+1})' = x^n$, $x \neq 0$. If $b > a > 0$, then by the Fundamental Theorem of Calculus, $\int_a^b x^n dx = \frac{x^{n+1}}{n+1}\big|_a^b := \frac{b^{n+1}-a^{n+1}}{n+1}$. (The notation $F(x)|_a^b$ means $F(b) - F(a)$.) What if $n = -1$? Define 'logarithm' function $\log : (0,\infty) \to \mathbb{R}$ by $\log x := \int_1^x \frac{1}{t}dt$, $x > 0$.



By the Fundamental Theorem of Calculus, $(\log x)' = (\int_1^x \frac{1}{t}dt)' = \frac{1}{x}$, $x > 0$. ◇

Exercise 6.45. Suppose that $f : [a,b] \to \mathbb{R}$ is bounded, and that $f \in RI[a,b]$. Define $F : [a,b] \to \mathbb{R}$ by $F(x) = \int_a^x f(t)dt$ for all $x \in [a,b]$. Show that F is uniformly continuous on $[a,b]$.

Exercise 6.46 (Leibniz's Rule for Integrals)**.** If $f \in C[a,b]$ and u, v are differentiable on $[c,d]$ and $u([c,d]) \subset [a,b]$, $v([c,d]) \subset [a,b]$, then $\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)dt = f(v(x))\,v'(x) - f(u(x))\,u'(x)$, $x \in [c,d]$.

Exercise 6.47 (Integration by Parts)**.** Let $f, g, G : [a, b] \to \mathbb{R}$ be such that f, G are continuously differentiable on $[a, b]$, and $G' = g$. Prove that $\int_a^b f(x)g(x)dx = f(x)G(x)|_a^b - \int_a^b f'(x)G(x)dx$.
Use integration by parts to show that $\int_1^3 \log x\, dx = 3\log 3 - 2$.

Exercise 6.48 (Integration by Substitution/Change of Variables)**.** Let φ be continuously differentiable on $[\alpha, \beta]$, $\varphi([\alpha, \beta]) = [a, b]$, and $f \in C[a, b]$. Show that $\int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\, dx = \int_\alpha^\beta f(\varphi(t))\, \varphi'(t)\, dt$.
Show that if $r > 0$, then $\int_{-r}^{r} \sqrt{r^2 - x^2}\, dx = \frac{\pi r^2}{2}$. (Recall that $\pi := 2\int_{-1}^{1} \sqrt{1 - x^2}\, dx$.)

Exercise 6.49. Let $f : [0, \infty) \to \mathbb{R}$ be continuous. Find f in each of the cases below if the given equation is known to hold for all $x \geqslant 0$, or if no such f exists, justify why not.

(1) $\int_0^{x^2} f(t)dt = e^{-x^2}$. (2) $\int_0^{f(x)} t^2 dt = e^{-x^2}$. (3) $\int_0^{e^{-x^2}} f(t)dt = x^2$.

Exercise 6.50. Let f be a continuous function on $\mathbb{R}$ and $\lambda \neq 0$. Consider $y(x) = \frac{1}{\lambda}\int_0^x f(t)\sin(\lambda(x - t))dt$ for $x \in \mathbb{R}$. Show that y is a solution to the inhomogeneous differential equation $y''(x) + \lambda^2 y(x) = f(x)$ for all $x \in \mathbb{R}$ and with the initial conditions $y(0) = 0$ and $y'(0) = 0$.

Exercise 6.51. Using the binomial fomula $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$, and the Fundamental Theorem of Calculus, show that $\sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k} = \frac{2^{n+1}-1}{n+1}$.

Exercise 6.52. Let $e^y := \sum_{n=0}^{\infty} \frac{1}{n!} y^n$, $y \in \mathbb{R}$. From Example 3.33 that this series converges for all $y \in \mathbb{R}$. Note that $e^0 = 1$, and from Exercise 5.35(3), we also know that $\frac{d}{dy}e^y = e^y$, $y \in \mathbb{R}$.

(1) Show that $\frac{d}{dy}(e^y e^{-y}) \equiv 0$ using the Product Rule and the Chain Rule for differentiation.
Conclude that $e^y e^{-y} = 1$ for all $y \in \mathbb{R}$.

(2) Show that log defined in Example 6.44 is strictly increasing, and that its range is $\mathbb{R}$.

So $\log : (0, \infty) \to \mathbb{R}$ is bijective. Denote its inverse by $f : \mathbb{R} \to (0, \infty)$. As $\log 1 = \int_1^1 \frac{1}{t}dt = 0$, $f(0) = 1$.

(3) Use the Differentiable Inverse Theorem (Exercise 5.11) to show that $f'(y) = f(y)$ for all $y \in \mathbb{R}$.

(4) Show that the initial value problem $\left\{ \begin{array}{l} g'(y) = g(y) \quad (y \in \mathbb{R}) \\ g(0) = C \end{array} \right\}$ has the unique solution $g(y) = Ce^y$.
Hint: Differentiate $\frac{d}{dy}(e^{-y}g)$.

(5) Show that the inverse f of log is the exponential function: $f(y) = e^y$, $y \in \mathbb{R}$.
Hint: Consider the initial value problem above with $C = 1$.

(6) By considering the initial value problem above with $C = e^a$, and the two functions $g(y) = e^{y+a}$ and $\tilde{g}(y) = e^a e^y$, show that $e^{a+b} = e^a e^b$ for all $a, b \in \mathbb{R}$.

(7) Show that $\log(x_1 x_2) = \log x_1 + \log x_2$ for all $x_1, x_2 \in (0, \infty)$.

(8) Let $e := \sum_{n=0}^{\infty} \frac{1}{n!} = f(1)$. Then $\log e = 1$. Show that $2 \leqslant e \leqslant 4$.

(9) Let $a > 0$ and $b \in \mathbb{R}$. Define $a^b = e^{b \log a}$. Show that if $n \in \mathbb{N}$, then $a^n = a \cdots a$ (n times).
Prove that if $c \in \mathbb{R}$, then $(a^b)^c = a^{bc}$.

6.4. Notes (not part of the course)

In this section, we give a summary of the Lebesgue integral in one dimension. This is of course no substitute for a thorough exposition to the subject. We will work in one dimension, although one can more generally work in $\mathbb{R}^d$ in an analogous manner.

The extended real number system. For several reasons (e.g. handling limiting processes), it will be useful to extend the real number system by adding two symbols ∞ and $-\infty$. The set $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty, -\infty\}$ is called the *extended real number system.* We extend the order $<$ from $\mathbb{R}$ to $\overline{\mathbb{R}}$ by defining $-\infty < x < \infty$ for all $x \in \mathbb{R}$. Every subset of $\overline{\mathbb{R}}$ has an upper bound ∞, and a lower bound $-\infty$. Then every subset of $\overline{\mathbb{R}}$ has a least upper bound and a greatest lower bound in $\overline{\mathbb{R}}$. For example, $\sup \mathbb{R} = \infty$ and $\sup \varnothing = -\infty$. We also define the following:

For $x \in \mathbb{R}$, $x + \infty = \infty = \infty + x$, $x + (-\infty) = -\infty = (-\infty) + x$.
If $x > 0$, then $x \cdot \infty = \infty = \infty \cdot x$ and $x \cdot (-\infty) = -\infty = (-\infty) \cdot x$.
If $x < 0$, then $x \cdot \infty = -\infty = \infty \cdot x$ and $x \cdot (-\infty) = \infty = (-\infty) \cdot x$.

Measurable sets. The length of an interval $I \subset \mathbb{R}$ is defined to be

$$\lambda(I) := \begin{cases} b-a & \text{if } I = [a,b],\ (a,b),\ (a,b],\ [a,b) \text{ and } -\infty < a \leqslant b < \infty, \\ \infty & \text{if } I \text{ is unbounded.} \end{cases}$$

We shall now associate a 'measure' to more general subsets of $\mathbb{R}$ following a method originally due to Henri Lebesgue (1875-1941). The more general sets which possess a measure will be called measurable sets, and we will denote the measure of a measurable set $A \subset \mathbb{R}$ by $\lambda(A)$. The associated integral, which we will define in the next section is called the Lebesgue integral.

Step 1: Compact sets. Let $K \subset \mathbb{R}$ be a compact set, that is, closed and bounded. Let K be covered by intervals $I_1, \cdots, I_n \subset \mathbb{R}$, $n \in \mathbb{N}$. Then we expect $\lambda(K)$ to satisfy $\lambda(K) \leqslant \sum\limits_{k=1}^{n} \lambda(I_k)$. This should hold for every such cover of K, and we expect the right-hand side above to be close to the left-hand side when the 'overlap' of the covering intervals becomes smaller. This motivates the following definition. We define $\lambda(K) := \inf\{\sum\limits_k \lambda(I_k) : K \subset \bigcup\limits_k I_k\}$, where the infimum is taken over all covers of K by a finite number of intervals I_k. We note that if $K = [a,b]$, then our definition above delivers $\lambda(K) = b - a$, which is indeed the length of the interval $[a,b]$. Also, we note that $\lambda(K) < \infty$ for compact K.

Step 2: Open sets. The measure of an open set $U \subset \mathbb{R}$ is $\lambda(U) := \sup\{\lambda(K) : K \subset U,\ K \text{ compact}\}$. For open sets U, $0 \leqslant \lambda(U) \leqslant \infty$. If $U = (a,b)$, then $\lambda(U) = b - a$ for finite a, b, and is ∞ if $a = \infty$ or $b = \infty$.

Step 3: Bounded measurable sets. Let $A \subset \mathbb{R}$ be a bounded set. Consider all compact sets $K \subset A$ and all open sets $U \supset A$. Then we have $\lambda(K) \subset \lambda(U)$. Thus $\sup\limits_{\text{compact } K \subset A} \lambda(K) \leqslant \inf\limits_{\text{open } U \supset A} \lambda(U)$. We say that the bounded set A is measurable if there is equality above, and define its measure $\lambda(A)$ to be the common value, that is, $\lambda(A) := \sup\limits_{\text{compact } K \subset A} \lambda(K) = \inf\limits_{\text{open } U \supset A} \lambda(U)$. If A is compact, then this definition coincides with the ones from Step 1. Also, if A is open and bounded, then this definition coincides with the one from Step 2. It can be shown (invoking Zorn's Lemma) that there exist bounded subsets $A \subset \mathbb{R}$ that are not measurable.

Step 4: Measurable sets. Let $A \subset \mathbb{R}$. We call A *measurable* if for every compact set $K \subset \mathbb{R}$, the bounded set $A \cap K$ is measurable, and we define the *measure* $\lambda(A)$ *of* A by $\lambda(A) := \sup\limits_{K \text{ compact}} \lambda(A \cap K)$. If A is bounded, then this definition coincides with the one from Step 3.

This is how the (Lebesgue) measure $\lambda(A)$ is defined for (Lebesgue) measurable subsets A of $\mathbb{R}$. We have:

(1) If A is measurable, then $\mathbb{R} \backslash A$ is also measurable.

(2) Let A be measurable and $x \in \mathbb{R}$. Set $x + A := \{x + a : a \in A\}$ and $xA := \{xa : a \in A\}$. Then $x + A$ and xA are measurable, and $\lambda(x + A) = \lambda(A)$ and $\lambda(xA) = |x|\lambda(A)$.

(3) If A_1, A_2 are measurable and $A_1 \subset A_2$, then $\lambda(A_1) \leqslant \lambda(A_2)$.

Now suppose that $(A_n)_{n\in\mathbb{N}}$ is a sequence of measurable sets.

(4) $\bigcup\limits_{n\in\mathbb{N}} A_n$ is measurable, and $\lambda(\bigcup\limits_{n\in\mathbb{N}} A_n) \leqslant \sum\limits_{n=1}^{\infty} \lambda(A_n)$.
If $A_i \cap A_j = \varnothing$ whenever $i \neq j$, then $\lambda(\bigcup\limits_{n\in\mathbb{N}} A_n) = \sum\limits_{n=1}^{\infty} \lambda(A_n)$.
If $A_1 \subset A_2 \subset A_3 \subset \cdots$, then $\lambda(\bigcup\limits_{n\in\mathbb{N}} A_n) = \sup\limits_{n\in\mathbb{N}} \lambda(A_n)$.

(5) $\bigcap\limits_{n\in\mathbb{N}} A_n$ is measurable.

Sets of measure zero. Sets of measure 0 play an important role in measure theory (for example, they underlie the notions of 'almost everywhere' and 'for almost all', as we shall see). For example:

- $A = \{a\}$, a singleton, because A is then an interval in $\mathbb{R}$, with $\lambda(A) = a - a = 0$.
- $A = \{a_1, a_2, a_3, \cdots\} = \bigcup\limits_{n\in\mathbb{N}} \{a_n\}$, a countable set. Then $\lambda(A) = \sum\limits_{n=1}^{\infty} \lambda(\{a_n\}) = \sum\limits_{n=1}^{\infty} 0 = 0$.

There are uncountable sets with Lebesgue measure 0, for example, the Cantor set, recalled below.

Example 6.53 (Cantor set)**.** Recall the Cantor set C from Example 2.66. Let us show that C is uncountable. We will prove that there is a one-to-one correspondence between points of C and the points of $[0,1]$. Any point x in C is associated with a sequence of letters 'L' or 'R' as follows. Let $x \in C$. Then for any n, $x \in F_n$, and when the middle thirds of each subinterval in F_n is removed, x is present either in the left part or the right part of the subinterval, and the n^{th} term in the sequence of letters is L or R accordingly. For example,

$$\begin{aligned} 0 &\equiv \text{L,L,L,L,L,L,} \cdots \\ 1 &\equiv \text{R,R,R,R,R,R,} \cdots \\ \tfrac{1}{3} &\equiv \text{L,R,R,R,R,R,} \cdots \\ \tfrac{2}{9} &\equiv \text{L,R,L,L,L,L,} \cdots \\ \tfrac{20}{27} &\equiv \text{R,L,R,L,L,L,} \cdots . \end{aligned}$$

But points in $[0,1]$ are also in one to one correspondence with such sequences. Indeed,

$$\begin{aligned} [0,1] &= [0,\tfrac{1}{2}] \cup (\tfrac{1}{2},1] \\ &= [0,\tfrac{1}{4}] \cup (\tfrac{1}{4},\tfrac{1}{2}] \cup (\tfrac{1}{2},\tfrac{3}{4}] \cup (\tfrac{3}{4},1] \\ &= [0,\tfrac{1}{8}] \cup (\tfrac{1}{8},\tfrac{1}{4}] \cup (\tfrac{1}{4},\tfrac{3}{8}] \cup (\tfrac{3}{8},\tfrac{1}{2}] \cup (\tfrac{1}{2},\tfrac{5}{8}] \cup (\tfrac{5}{8},\tfrac{3}{4}] \cup (\tfrac{3}{4},\tfrac{7}{8}] \cup (\tfrac{7}{8},1] \\ &\quad \cdots . \end{aligned}$$

If $x \in [0,1]$, then for each n, we can look at the n^{th} equality, and see if x falls in the left or the right part, when each subinterval in the right-hand side of the n^{th} equality is divided into two parts, and this gives the $(n+1)^{\text{st}}$ term of the sequence of Ls and Rs associated with x: for example,

$$\begin{aligned} 0 &\equiv \text{L,L,L,L,L,L,} \cdots \\ 1 &\equiv \text{R,R,R,R,R,R,} \cdots \\ \tfrac{1}{2} &\equiv \text{L,R,R,R,R,R,} \cdots . \end{aligned}$$

As $[0,1]$ is uncountable, it follows that so is C.

As the sum of the lengths of the intervals removed is $\frac{1}{3} + 2\frac{1}{3^2} + 4\frac{1}{3^3} + \cdots = 1$, the measure of F is $1 - 1 = 0$. So this is an example of an uncountable set with measure 0. ◇

Any subset of a measurable set of measure 0 is also measurable with measure 0. We say that two functions $\boldsymbol{x}_1, \boldsymbol{x}_2 : A \to \mathbb{R}$ defined on a measurable set A are equal *almost everywhere* if there exists a measurable set N with $\lambda(N) = 0$ such that $\boldsymbol{x}_1(t) = \boldsymbol{x}_2(t)$ for all $t \in A \backslash N$. Sometimes then we also say that $\boldsymbol{x}_1(t) = \boldsymbol{x}_2(t)$ for *almost all* $t \in A$.

Measurable functions. Let A be a measurable subset of $\mathbb{R}$. A function $\boldsymbol{x} : A \to \mathbb{R} \cup \{-\infty, \infty\}$ is called *measurable* if $\boldsymbol{x}$ has any of the following equivalent properties:

(M1) For all $y \in \mathbb{R}$, $\{t \in A : \boldsymbol{x}(t) < y\}$ is measurable.

(M2) For all $y \in \mathbb{R}$, $\{t \in A : \boldsymbol{x}(t) \leqslant y\}$ is measurable.

(M3) For all $y \in \mathbb{R}$, $\{t \in A : \boldsymbol{x}(t) > y\}$ is measurable.

(M4) For all $y \in \mathbb{R}$, $\{t \in A : \boldsymbol{x}(t) \geqslant y\}$ is measurable.

Practically *all* functions are measurable, and they are abundant:

(1) All continuous functions are measurable.

(2) All functions that are continuous outside a set of measure 0.
For example if $\boldsymbol{x}(t) := \left\{ \begin{array}{cl} 1 & \text{if } t = \frac{1}{n\pi},\ n \in \mathbb{Z}\backslash\{0\}, \text{ or } t = 0 \\ \frac{1}{\sin \frac{1}{t}} & \text{otherwise} \end{array} \right\}$, then $\boldsymbol{x}$ is measurable.
Such functions are called *continuous almost everywhere*.

(3) All monotone functions are measurable.

(4) If A is a measurable set, then its indicator function $\mathbf{1}_A$, given by $\mathbf{1}_A(t) = \left\{ \begin{array}{ll} 1 & \text{if } t \in A \\ 0 & \text{if } t \notin A \end{array} \right\}$, is a measurable function, since $\{t \in \mathbb{R} : \mathbf{1}_A(t) \geqslant y\} = \begin{cases} \mathbb{R} & \text{if } y \leqslant 0, \\ A & \text{if } 0 < y \leqslant 1, \\ \varnothing & \text{if } y > 1. \end{cases}$

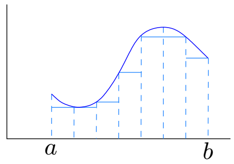
(5) The sum, product and (if well-defined) the quotient of measurable functions are all measurable.

(6) If $\boldsymbol{x}$ is measurable, then so is $|\boldsymbol{x}|$.
Hence[5] if $\boldsymbol{x}_1, \boldsymbol{x}_2$ are measurable, then $\max\{\boldsymbol{x}_1, \boldsymbol{x}_2\}$ and $\min\{\boldsymbol{x}_1, \boldsymbol{x}_2\}$ are also measurable.

(7) If $(\boldsymbol{x}_n)_{n \in \mathbb{N}}$ is a sequence of measurable functions, such that their pointwise limit, say $\boldsymbol{x}$, exists, then $\boldsymbol{x}$ is measurable.

[5] For real a, b, $\max\{a, b\} = \frac{a+b+|a-b|}{2}$, and $\min\{a, b\} = a + b - \max\{a, b\}$.

The integral of measurable functions. While defining the *Riemann* integral, we consider upper and lower sums corresponding to a partition $P = \{a = t_0, t_1, \cdots, t_{n-1}, t_n = b\}$ of the *domain* $[a, b]$ of the function $\boldsymbol{x}$, for example the lower sum $\underline{S}(\boldsymbol{x}, P) = \sum_{k=0}^{n-1} (\inf_{t \in [t_k, t_{k+1}]} \boldsymbol{x}(t))(t_{k+1} - t_k)$. This is really the Riemann integral of a *step function*, which assumes finitely many values, and is constant on intervals.



While defining the *Lebesgue* integral, we shall consider *simple* functions. A simple function assumes finitely many values (just as before, with step functions), but now is constant (more generally than the case of step functions) on *measurable* sets (instead of mere intervals). Roughly speaking, such simple functions arise from a partition of the *range* (rather than a partition of the *domain* for the step functions considered when defining the Riemann integral). Every step function is a simple function (as every interval is measurable), but not every simple function is a step function (since not every measurable set is an interval).

Now let A be a measurable set, and let $\boldsymbol{s} : A \to \mathbb{R}$ be a simple function. This means that $\boldsymbol{s}$ assumes finitely many values, which we arrange in increasing order: $-\infty < y_1 < y_2 < \cdots < y_n < \infty$, and let $A_k = \{t \in A : \boldsymbol{s}(t) = y_k\}$, $1 \leqslant k \leqslant n$. Thus we may write $\boldsymbol{s} = y_1 \cdot \mathbf{1}_{A_1} + \cdots + y_n \cdot \mathbf{1}_{A_n}$. If $\boldsymbol{s}(t) \geqslant 0$ for all $t \in A$, then $y_1 \geqslant 0$, and in this case, we define $\int_A \boldsymbol{s}(t)dt := y_1 \cdot \lambda(A_1) + \cdots + y_n \cdot \lambda(A_n)$. The right-hand side is either a nonnegative real number or ∞ (if one of the sets A_k has infinite measure).

The collection of all nonnegative simple functions on A is denoted by $S_+(A)$. For each $\boldsymbol{s} \in S_+(A)$, we have defined $\int_A \boldsymbol{s}(t)dt$. If A is a set of measure 0, then for all $\boldsymbol{s} \in S_+(A)$, $\int_A \boldsymbol{s}(t)dt = 0$. Indeed, since every subset of a set of measure 0 is also a measurable set of measure 0, it follows, with the notation from the previous paragraph, that $\lambda(A_k) = 0$ for all $1 \leqslant k \leqslant n$. The claim follows by the definition of the integral.

Let $\boldsymbol{x} : A \to \mathbb{R} \cup \{-\infty, \infty\}$ be a measurable function, and $\boldsymbol{x}(t) \geqslant 0$ for all $t \in A$. Then we define

$$\int_A \boldsymbol{s}(t)dt = \sup_{\boldsymbol{x} \geqslant \boldsymbol{s} \in S_+(A)} \int_A \boldsymbol{s}(t)dt.$$

The right-hand side is either a nonnegative real number or ∞. In this sense, we can say that for nonnegative measurable functions, their Lebesgue integral always exists, but this is not the case with Riemann integrals.

Example 6.54. Let $A = [0, 1]$, and let $\boldsymbol{x} : [0, 1] \to \mathbb{R}$ be defined by $\boldsymbol{x}(t) = \begin{Bmatrix} 1 & \text{if } t \text{ is irrational} \\ 0 & \text{if } t \text{ is rational} \end{Bmatrix}$.

The sets $A_0 = \{t \in [0, 1] : \boldsymbol{x}(t) = 0\}$ and $A_1 = \{t \in [0, 1] : \boldsymbol{x}(t) = 1\}$ are measurable. Since A_0 is countable, $\lambda(A_0) = 0$. On the other hand, $\lambda(A_1) = \lambda(A \backslash A_0) = \lambda(A) - \lambda(A_0) = 1 - 0 = 1$. Since $\boldsymbol{x} = \mathbf{1}_{A_1}$ is a simple function, $\int_A \boldsymbol{x}(t)dt = 1 \cdot \lambda(A_1) = 1$. But we had seen earlier that $\boldsymbol{x}$ is not Riemann integrable. ◇

Let A be a measurable set. Suppose that all the functions appearing in the list below are defined on A, take values in $[0, \infty) \cup \{\infty\}$, and are measurable. Then we have:

(1) $\int_A (\boldsymbol{x}_1(t) + \boldsymbol{x}_2(t))dt = \int_A \boldsymbol{x}_1(t)dt + \int_A \boldsymbol{x}_2(t)dt$.

(2) For $\alpha \geqslant 0$, $\int_A \alpha\, \boldsymbol{x}(t)dt = \alpha \int_A \boldsymbol{x}(t)dt$.

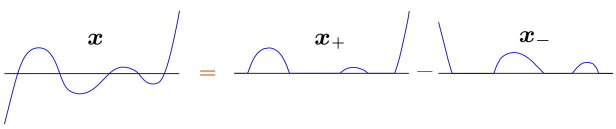
(3) If for all $t \in A$, $\boldsymbol{x}_1(t) \leqslant \boldsymbol{x}_2(t)$, then $\int_A \boldsymbol{x}_1(t)dt \leqslant \int_A \boldsymbol{x}_2(t)dt$.

(4) (Monotone Convergence Theorem).
If $0 \leqslant \boldsymbol{x}_1(t) \leqslant \boldsymbol{x}_2(t) \leqslant \cdots$, and $\boldsymbol{x}(t) := \lim_{n\to\infty} \boldsymbol{x}_n(t)$. Then $\int_A \boldsymbol{x}(t)dt = \lim_{n\to\infty} \int_A \boldsymbol{x}_n(t)dt$.

(5) If $\lambda(A) = 0$, then $\int_A \boldsymbol{x}(t)dt = 0$.

(6) If $\int_A \boldsymbol{x}(t)dt < \infty$, then there exists a set N of measure zero such that $\boldsymbol{x}(t) < \infty$ for all $t \in A \backslash N$.

Let $\boldsymbol{x} : A \to \mathbb{R} \cup \{-\infty, \infty\}$ be a measurable function defined on the measurable set A. Note that $\boldsymbol{x}$ is no longer assumed to be nonnegative. We can, nevertheless, write $\boldsymbol{x}$ as a *difference*, $\boldsymbol{x} = \boldsymbol{x}_+ - \boldsymbol{x}_-$, of the two nonnegative (and measurable) functions $\boldsymbol{x}_+ := \max\{\boldsymbol{x}, \mathbf{0}\}$ and $\boldsymbol{x}_- := \max\{-\boldsymbol{x}, \mathbf{0}\} = -\min\{\boldsymbol{x}, \mathbf{0}\}$.



We say that $\boldsymbol{x}$ is (*absolutely*) *integrable on* A if $\int_A |\boldsymbol{x}(t)|dt < \infty$. Then $\int_A \boldsymbol{x}(t)dt := \int_A \boldsymbol{x}_+(t)dt - \int_A \boldsymbol{x}_-(t)dt$. Since $0 \leqslant \boldsymbol{x}_\pm(t) \leqslant |\boldsymbol{x}(t)|$, and thanks to assumption that $\int_A |\boldsymbol{x}(t)|dt < \infty$, it follows from (3) on page 98, that $\int_A \boldsymbol{x}_+(t)dt$, $\int_A \boldsymbol{x}_-(t)dt < \infty$, and so their difference, $\int_A \boldsymbol{x}(t)dt$, is finite too.

The set of all absolutely integrable functions on A is denoted by $\mathcal{L}^1(A)$. For $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x} \in \mathcal{L}^1(A)$ and $\alpha \in \mathbb{R}$, we have the following:

(1) $\boldsymbol{x}_1 + \boldsymbol{x}_2 \in \mathcal{L}^1(A)$ and $\int_A (\boldsymbol{x}_1 + \boldsymbol{x}_2)(t)dt = \int_A \boldsymbol{x}_1(t)dt + \int_A \boldsymbol{x}_2(t)dt$.

(2) $\alpha \cdot \boldsymbol{x} \in \mathcal{L}^1(A)$ and $\int_A (\alpha \cdot \boldsymbol{x})(t)dt = \alpha \int_A \boldsymbol{x}(t)dt$.

(3) $|\boldsymbol{x}| \in \mathcal{L}^1(A)$ and $\int_A |\boldsymbol{x}(t)|dt \leqslant |\int_A \boldsymbol{x}(t)dt|$.

(4) If $\int_A |\boldsymbol{x}(t)|dt = 0$, then there exists a set $N \subset A$ of measure 0 such that $\boldsymbol{x}(t) = 0$ for all $t \in A \backslash N$.

(5) Let $A = B \cup C$, where B, C are measurable too and $B \cap C = \varnothing$.
Then $\boldsymbol{x} \in \mathcal{L}^1(B)$, $\boldsymbol{x} \in \mathcal{L}^1(C)$ and $\int_A \boldsymbol{x}(t)dt = \int_B \boldsymbol{x}(t)dt + \int_C \boldsymbol{x}(t)dt$.

(6) If $\boldsymbol{y} : A \to \mathbb{R} \cup \{-\infty, +\infty\}$ is measurable and $|\boldsymbol{y}(t)| \leqslant \boldsymbol{x}(t)$ for almost all $t \in A$, then $\boldsymbol{y} \in \mathcal{L}^1(A)$ and $|\int_A \boldsymbol{y}(t)dt| \leqslant \int_A |\boldsymbol{y}(t)|dt \leqslant \int_A \boldsymbol{x}(t)dt$.

The parts (1), (2) assert that $\mathcal{L}^1(A)$ is a real vector space, and the integral $\boldsymbol{x} \mapsto \int_A \boldsymbol{x}(t)dt : \mathcal{L}^1(A) \to \mathbb{R}$ is a linear transformation (or a linear functional, since the co-domain is the field of scalars $\mathbb{R}$).

We also remark that in part (4), under the given hypothesis, we cannot in general conclude that $\boldsymbol{x} \equiv 0$ on *all* of A. Indeed, $\int_0^1 \mathbf{1}_{\mathbb{Q}\cap[0,1]}(t)dt = \lambda(\mathbb{Q} \cap [0,1]) = 0$, as $\mathbb{Q}$ is countable, however the integrand is not identically zero: for example, its value at $\frac{1}{2}$ is 1. On the other hand, if in (4), we are also given that $\boldsymbol{x}$ is continuous, then we can safely conclude that $\boldsymbol{x} \equiv 0$ on A.

The Dominated Convergence Theorem says that if there is an $\mathcal{L}^1$-majorant for all the terms $\boldsymbol{x}_n$ in a sequence of functions, then assuming that their pointwise limit $\boldsymbol{x}$ exists almost everywhere, this pointwise limit is also an element of $\mathcal{L}^1(A)$.

Dominated Convergence Theorem. Let A be measurable. Let $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{L}^1(A)$, and $\boldsymbol{x} : A \to \mathbb{R}$, be such that $\lim\limits_{n\to\infty} \boldsymbol{x}_n(t) = \boldsymbol{x}(t)$ for almost all $t \in A$. Let $\boldsymbol{y} \in \mathcal{L}^1(A)$ be such that for all $n \in \mathbb{N}$, $|\boldsymbol{x}_n(t)| \leqslant \boldsymbol{y}(t)$ for almost all $t \in [0,1]$. Then $\boldsymbol{x} \in \mathcal{L}^1(A)$, and $\lim\limits_{n\to\infty} \int_A \boldsymbol{x}_n(t)dt = \int_A \lim\limits_{n\to\infty} \boldsymbol{x}_n(t)dt = \int_A \boldsymbol{x}(t)dt$.

We remark that the hypothesis of the existence of an $\mathcal{L}^1$ majorant is essential, as demonstrated by the following two examples.

Example 6.55. (Lacking an $\mathcal{L}^1$ majorant). Let $A = \mathbb{R}$.

(1) Let $\boldsymbol{x}_n = \mathbf{1}_{[-n,n]}$, $\boldsymbol{x} = \mathbf{1}$. Then $\boldsymbol{x}_n \in \mathcal{L}^1(\mathbb{R})$, $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$ converges pointwise everywhere on $\mathbb{R}$ to $\boldsymbol{x}$, but $\boldsymbol{x} \notin \mathcal{L}^1(\mathbb{R})$.

(2) Let $\boldsymbol{x}_n = \mathbf{1}_{[n,n+1]}$, $\boldsymbol{x} = \mathbf{0}$. Then $\boldsymbol{x}_n \in \mathcal{L}^1(\mathbb{R})$, $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$ converges pointwise everywhere on $\mathbb{R}$ to $\boldsymbol{x}$, but $\int_\mathbb{R} \boldsymbol{x}(t)dt = 0 \neq 1 = \lim\limits_{n\to\infty} \int_\mathbb{R} \boldsymbol{x}_n(t)dt$. ◇

Link with the Riemann integral. Let $\boldsymbol{x} \in C[a,b]$. Then $\boldsymbol{x} \in \mathcal{L}^1[a,b]$ and

$$\text{(the Lebesgue integral)} \int_a^b \boldsymbol{x}(t)dt = \text{(the Riemann integral)} \int_a^b \boldsymbol{x}(t)dt.$$

That $\boldsymbol{x} \in \mathcal{L}^1[a,b]$ follows from the fact that $\boldsymbol{x}$ is measurable (since it is continuous), and it is bounded (Extreme Value Theorem).

The space $L^1[0,1]$. Consider on $\mathcal{L}^1[0,1]$ the candidate norm $\|\boldsymbol{x}\|_1 := \int_0^1 |\boldsymbol{x}(t)|dt$ for all $\boldsymbol{x} \in \mathcal{L}^1[0,1]$. This map $\|\cdot\|_1$ fails to be a norm because functions that are almost everywhere 0 (e.g. $\mathbf{1}_{\mathbb{Q}\cap[0,1]}$) have zero norm. Hence we should essentially 'consider such functions to be also the zero vector in the vector space $\mathcal{L}^1[0,1]$'. This intuitive remark can be made rigorous by considering the following relation on $\mathcal{L}^1[0,1]$. We say that $\boldsymbol{x} \sim \boldsymbol{y}$ if there exists a set[6] $N \subset [0,1]$ of measure 0, such that $\boldsymbol{x}(t) = \boldsymbol{y}(t)$ for all $t \in [0,1] \backslash N$. It can be seen that $\sim$ is an equivalence relation on $\mathcal{L}^1[0,1]$, that is,

(ER1) (*Reflexivity*). $\boldsymbol{x} \sim \boldsymbol{x}$ for all $\boldsymbol{x} \in \mathcal{L}^1[0,1]$.

(ER2) (*Symmetry*). If $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{L}^1[0,1]$ and $\boldsymbol{x} \sim \boldsymbol{y}$, then $\boldsymbol{y} \sim \boldsymbol{x}$.

(ER3) (*Transitivity*). If $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathcal{L}^1[0,1]$, $\boldsymbol{x} \sim \boldsymbol{y}$ and $\boldsymbol{y} \sim \boldsymbol{z}$, then $\boldsymbol{x} \sim \boldsymbol{z}$.

[6] depending in general on $\boldsymbol{x}$ and $\boldsymbol{y}$

Let $[\boldsymbol{x}]$ denote the equivalence class of $\boldsymbol{x}$: $[\boldsymbol{x}] = \{\boldsymbol{y} \in \mathcal{L}^1[0,1] : \boldsymbol{x} \sim \boldsymbol{y}\}$. Thus $[\boldsymbol{x}]$ is the collection of all elements of $\mathcal{L}^1[0,1]$ that are almost everywhere equal to $\boldsymbol{x}$ on $[0,1]$. Define $L^1[0,1] := \{[\boldsymbol{x}] : \boldsymbol{x} \in \mathcal{L}^1[0,1]\}$. Then we can endow a vector space structure on $L^1[0,1]$ by setting $[\boldsymbol{x}] + [\boldsymbol{y}] = [\boldsymbol{x} + \boldsymbol{y}]$ and $\alpha \cdot [\boldsymbol{x}] = [\alpha \cdot \boldsymbol{x}]$ for $[\boldsymbol{x}], [\boldsymbol{y}] \in \mathcal{L}^1[0,1]$ and $\alpha \in \mathbb{R}$. It can also be seen that these operations $+, \cdot$ are well-defined, that is, they do not depend on the chosen representatives $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{L}^1[0,1]$ for the equivalence classes $[\boldsymbol{x}], [\boldsymbol{y}]$, respectively.

We now define the map $\|\cdot\|_1 : L^1[0,1] \to \mathbb{R}$ by $\|[\boldsymbol{x}]\|_1 := \int_0^1 \boldsymbol{x}(t)dt$ for all $[\boldsymbol{x}] \in L^1[0,1]$. Then it can be checked that $\|\cdot\|_1$ defines a norm on $L^1[0,1]$. In particular, now if $\|[\boldsymbol{x}]\|_1 = 0$, then it follows that $\boldsymbol{x}(t) = 0$ for almost all $t \in [0,1]$, and so $[\boldsymbol{x}] = [\mathbf{0}]$, that is, $[\boldsymbol{x}]$ is the zero vector from the vector space $L^1[0,1]$, as desired.

The normed space $L^1[0,1]$ is complete, and in this sense $L^1[0,1]$ is 'better' than $C[0,1]$ with the $\|\cdot\|_1$ norm (see Exercise 2.32). We supply a sketch of the proof. Let $([\boldsymbol{x}_n])_{n\in\mathbb{N}}$ be a Cauchy sequence in $L^1[0,1]$. To prove its convergence, it is enough to show the convergence of a subsequence. So we may assume (by passing to a subsequence if necessary) that $\|[\boldsymbol{x}_{n+1}] - [\boldsymbol{x}_n]\|_1 < \frac{1}{2^n}$ $(n \in \mathbb{N})$. Let $\boldsymbol{x}_0 := \mathbf{0}$. Set

$$\boldsymbol{y}_n(t) := \sum_{k=0}^{n} |\boldsymbol{x}_{k+1}(t) - \boldsymbol{x}_k(t)| \text{ and } \boldsymbol{y}(t) := \sum_{k=0}^{\infty} |\boldsymbol{x}_{k+1}(t) - \boldsymbol{x}_k(t)|.$$

By the Triangle Inequality, we have

$$\|[\boldsymbol{y}_n]\|_1 = \int_0^1 |\boldsymbol{y}_n(t)|dt \leqslant \sum_{k=0}^{n} \|[\boldsymbol{x}_{k+1}] - [\boldsymbol{x}_k]\|_1 \leqslant \|[\boldsymbol{x}_1] - [\boldsymbol{x}_0]\|_1 + \sum_{k=1}^{n} \frac{1}{2^k}.$$

By the Monotone Convergence Theorem,

$$\|[\boldsymbol{y}]\|_1 = \int_0^1 |\boldsymbol{y}(t)|dt = \lim_{n\to\infty} \int_0^1 |\boldsymbol{y}_n(t)|dt \leqslant \|[\boldsymbol{x}_1]\|_1 + 1 < \infty.$$

Hence the function $\boldsymbol{y}$ is finite almost everywhere on $[0,1]$. So the series $\sum\limits_{k=0}^{\infty} (\boldsymbol{x}_{k+1}(t) - \boldsymbol{x}_k(t))$ is absolutely convergent for almost all $t \in [0,1]$. For such t, we set

$$\boldsymbol{x}(t) = \sum_{k=0}^{\infty} (\boldsymbol{x}_{k+1}(t) - \boldsymbol{x}_k(t)).$$

But $\boldsymbol{x}_n(t) = \sum\limits_{k=0}^{n-1} (\boldsymbol{x}_{k+1}(t) - \boldsymbol{x}_k(t))$ for all $t \in [0,1]$, and so $\lim\limits_{n\to\infty} \boldsymbol{x}_n(t) = \boldsymbol{x}(t)$. Furthermore,

$$|\boldsymbol{x}_n(t)| \leqslant \sum_{k=0}^{n-1} |\boldsymbol{x}_{k+1}(t) - \boldsymbol{x}_k(t)| \leqslant \boldsymbol{y}(t) \text{ for almost all } t \in [0,1].$$

By the Dominated Convergence Theorem,

$$\int_0^1 |\boldsymbol{x}(t)|dt = \lim_{n\to\infty} \int_0^1 |\boldsymbol{x}_n(t)|dt \leqslant \int_0^1 \boldsymbol{y}(t)dt < \infty.$$

Hence $[\boldsymbol{x}] \in L^1[0,1]$. Also, note that $[|\boldsymbol{x}|] + [\boldsymbol{y}] \in L^1[0,1]$, and furthermore $|\boldsymbol{x} - \boldsymbol{x}_n| \leqslant |\boldsymbol{x}| + \boldsymbol{y}$ for all n. The Dominated Convergence Theorem again gives

$$\lim_{n\to\infty} \|[\boldsymbol{x}_n] - [\boldsymbol{x}]\|_1 = \lim_{n\to\infty} \int_0^1 |\boldsymbol{x}_n(t) - \boldsymbol{x}(t)|dt = 0,$$

showing that $([\boldsymbol{x}_n])_{n\in\mathbb{N}}$ converges to $[\boldsymbol{x}]$ in $(L^1[0,1], \|\cdot\|_1)$. Consequently, $(L^1[0,1], \|\cdot\|_1)$ is complete.

Appendix: The real number system

Equivalence relations

Definition A.1 (Relation).
A *relation* R on a set S is a subset of the $S \times S := \{(a,b) : a,b \in S\}$. If $(a,b) \in R$, then we write aRb.

For example, if we take S to be the set of all human beings, then

$$R_{\text{sibling}} := \{(a,b) \in S \times S : a,b \text{ have the same biological parents}\}$$

is a relation. As another example, we can take the set $S = \mathbb{Z}$, the set of all integers, and $R_{\text{mod } 2} = \{(m,n) \in \mathbb{Z} \times \mathbb{Z} : m - n \text{ is divisible by } 2\}$. Sometimes we use the symbol $\sim$ to denote a relation, and then instead of aRb, we will write $a \sim b$.

Definition A.2 (Equivalence relation).
A relation R on a set S is called an *equivalence relation* if it satisfies the following:

(ER1) R is *reflexive*, that is, for all $a \in S$, aRa.

(ER2) R is *symmetric*, that is, if aRb, then bRa.

(ER3) R is *transitive*, that is, if aRb and bRc, then aRc.

In our example above, where $S = \{\text{all human beings}\}$, R_{sibling} can easily be checked to be an equivalence relation[7]. Similarly $R_{\text{mod } 2}$ is an equivalence relation on $\mathbb{Z}$.

Why are equivalence relations useful? They help 'partition' the set into 'equivalence classes', and help to break down the big set into smaller subsets, such that all the elements in each subset are related to each other, and hence 'equivalent' in some way. For example, R_{sibling} enables one to partition the set of human beings into equivalence classes consisting of groups of brothers/sisters. On the other hand, $R_{\text{mod } 2}$ partitions $\mathbb{Z}$ into the sets $\{\text{even integers}\}$ and $\{\text{odd integers}\}$.

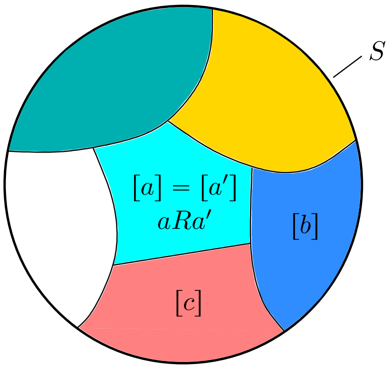
Definition A.3 (Equivalence class).
If R is an equivalence relation of a set S, then the *equivalence class of* a, denoted by $[a]$, is defined to be the set $[a] = \{b \in S : aRb\}$.

Given any $a, b \in S$, either $[a] = [b]$ or $[a] \cap [b] = \varnothing$. Indeed, let $[a] \cap [b] \neq \varnothing$. Suppose $c \in [a] \cap [b]$, that is, aRc and bRc. By symmetry, cRb. As aRc and cRb, by transitivity, we obtain aRb, and again by symmetry, bRa. If $d \in [a]$, then aRd. As bRa and aRd, by transitivity, bRd. So $d \in [b]$ too. So we have shown that $[a] \subset [b]$. In the same way, one can show $[b] \subset [a]$ as well. So $[a] = [b]$.

Clearly, $\bigcup\limits_{a \in S} [a] \subset S$. For $a \in S$, aRa (reflexivity), and so $a \in [a]$. Thus $S \subset \bigcup\limits_{a \in S} [a]$. So $S = \bigcup\limits_{a \in S} [a]$.

[7] Here we accept that a person is one's own sibling.

As any two distinct equivalence classes do not overlap at all, it follows that S is partitioned into equivalence classes by R, as shown in the schematic picture below.



So the idea is that an equivalence relation is really an 'attention focusing device', where we have chosen to ignore other distinguishing features of objects which are related, and have put them together in an equivalence class. So an equivalence relation gives one a 'pair of glasses' through which we 'clump together' things which are 'essentially the same' (equivalent under the relation) and see them as one object! For example, if our set is the collection of children in a school bus and we consider the equivalence relation R_1 of 'having the same sex', then through these glasses, we see only two equivalence classes: boys and girls. On the other hand, if we consider the equivalence relation R_2 of 'having the same age', then through these glasses, we see groups of children sorted by age.

Real numbers

Finally we have reached the point where we can learn about the construction of the most important number system from the point of view of Mathematical Analysis, namely the real number system $\mathbb{R}$. Roughly speaking, the set of real numbers are the numbers to which Cauchy sequences in $\mathbb{Q}$ 'want to converge to'. As these limits may not be rational, we just name/label these numbers by the whole Cauchy sequence in $\mathbb{Q}$ itself! But then two Cauchy sequences in $\mathbb{Q}$ might want to converge to the same thing (e.g. think of $(a_n)_{n\in\mathbb{N}}$ and $(a_n + \frac{1}{n})_{n\in\mathbb{N}}$), and so we ought not to distinguish between such two Cauchy sequences. So we must build an equivalence relation $\sim$ on Cauchy sequences, so that

$$(a_n)_{n\in\mathbb{N}} \sim (b_n)_{n\in\mathbb{N}} \text{ if } \lim_{n\to\infty}(a_n - b_n) = 0,$$

and consider the real numbers as equivalence classes of Cauchy sequences under this equivalence relation. Since we are trying to construct the reals, we are only allowed to use rational numbers. So we need to restrict ourselves to ϵ that are rational in the definition of convergence. We do this carefully below.

Definition A.4.

- A *Cauchy sequence in* $\mathbb{Q}$ is a sequence $(a_n)_{n\in\mathbb{N}}$ of rational numbers such that for every rational $\epsilon>0$, there exists an $N\in\mathbb{N}$ such that whenever $m,n>N$, we have $|a_n - a_m| < \epsilon$.
- The set of all Cauchy sequences in $\mathbb{Q}$ is denoted by $\mathcal{C}$.
- Let $r\in\mathbb{Q}$. A sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{Q}$ *converges to* r *in* $\mathbb{Q}$ if for every rational $\epsilon > 0$, there exists an $N\in\mathbb{N}$ such that for all $n > N$, $|a_n - r| < \epsilon$.
- The relation $\sim$ on $\mathcal{C}$ is defined as follows:

$$(a_n)_{n\in\mathbb{N}} \sim (b_n)_{n\in\mathbb{N}} \text{ if the sequence } (a_n - b_n)_{n\in\mathbb{N}} \text{ converges to 0 in } \mathbb{Q}.$$

Exercise A.5. Let $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ be Cauchy sequences in $\mathbb{Q}$. Show that $(a_n+b_n)_{n\in\mathbb{N}}$ is Cauchy sequence in $\mathbb{Q}$ too.

Proposition A.6. *Every Cauchy sequence in $\mathbb{Q}$ is bounded.*

Proof. Let $(a_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathbb{Q}$. Choose a rational $\epsilon > 0$, say $\epsilon = 1$. Then there exists an $N \in \mathbb{N}$ such that for all $n, m > N$, we have $|a_n - a_m| < \epsilon = 1$. In particular, with $m = N+1 > N$, and $n > N$, $|a_n - a_{N+1}| < 1$. Hence by the Triangle Inequality[8] in $\mathbb{Q}$, for all $n > N$,

$$|a_n| = |a_n - a_{N+1} + a_{N+1}| \leqslant |a_n - a_{N+1}| + |a_{N+1}| < 1 + |a_{N+1}|.$$

On the other hand, for $n \leqslant N$, $|a_n| \leqslant \max\{|a_1|, \dots, |a_N|, |a_{N+1}| + 1\} =: M > 0$. Consequently, $|a_n| \leqslant M$ $(n \in \mathbb{N})$, that is, the sequence $(a_n)_{n\in\mathbb{N}}$ is bounded. □

Exercise A.7. Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be Cauchy sequences in $\mathbb{Q}$. Show that $(a_nb_n)_{n\in\mathbb{N}}$ is Cauchy sequence in $\mathbb{Q}$ too.

Exercise A.8. Suppose that $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are sequences in $\mathbb{Q}$ such that $(a_n)_{n\in\mathbb{N}}$ (respectively $(b_n)_{n\in\mathbb{N}}$) converges in $\mathbb{Q}$ to $r_a \in \mathbb{Q}$ (respectively $r_b \in \mathbb{Q}$).

(1) Show that the limit is unique: If $(a_n)_{n\in\mathbb{N}}$ converges in $\mathbb{Q}$ to $r'_a \in \mathbb{Q}$, then $r_a = r'_a$.

(2) Show that $(-a_n)_{n\in\mathbb{N}}$ converges to $-r_a$.

(3) Show that $(a_n + b_n)_{n\in\mathbb{N}}$ converges to $r_a + r_b$.

Exercise A.9. Show that $\sim$ is an equivalence relation on $\mathcal{C}$.

Definition A.10 (The set of real numbers)**.**
A *real number* is an equivalence class of $\mathcal{C}$ under the relation $\sim$. If $(a_n)_{n\in\mathbb{N}} \in \mathcal{C}$, then $[(a_n)_{n\in\mathbb{N}}]$ denotes the real number which is the equivalence class of $\mathcal{C}$ containing the sequence $(a_n)_{n\in\mathbb{N}}$. The set of all real numbers is denoted by $\mathbb{R}$.

The set of real numbers is supposed to be an *extension* of the rational numbers $\mathbb{Q}$, that is, we want to see that $\mathbb{Q}$ '$\subset$' $\mathbb{R}$. Given a rational number $r \in \mathbb{Q}$, the constant sequence $r, r, r, \cdots$, that is, $(r)_{n\in\mathbb{N}}$, is a Cauchy sequence in $\mathbb{Q}$. Thus $[(r)_{n\in\mathbb{N}}]$ is a real number. We have the following.

Proposition A.11. *The map $\mathbb{Q} \ni r \mapsto [(r)_{n\in\mathbb{N}}] \in \mathbb{R}$ is injective.*

Proof. Let $r, s \in \mathbb{Q}$ be such that $[(r)_{n\in\mathbb{N}}] = [(s)_{n\in\mathbb{N}}]$. Then $(r)_{n\in\mathbb{N}} \sim (s)_{n\in\mathbb{N}}$. So

$$\lim_{n\to\infty} (r - s) = 0.$$

But the constant sequence $r - s, r - s, r - s, \cdots$ converges in $\mathbb{Q}$ to $r - s$. By the uniqueness of limits, $r - s = 0$, that is, $r = s$. □

Addition and multiplication

If addition in $\mathbb{R}$ is to respect the addition in $\mathbb{Q}$, we expect that for $r, s \in \mathbb{Q}$, $[(r)_{n\in\mathbb{N}}]+[(s)_{n\in\mathbb{N}}]$ equals $[(r + s)_{n\in\mathbb{N}}]$. Similarly, $[(r)_{n\in\mathbb{N}}] \cdot [(s)_{n\in\mathbb{N}}]$ should equal $[(rs)_{n\in\mathbb{N}}]$. This motivates the following.

Definition A.12. The *sum* of the real numbers $[(a_n)_{n\in\mathbb{N}}]$ and $[(b_n)_{n\in\mathbb{N}}]$ is given by

$$[(a_n)_{n\in\mathbb{N}}] + [(b_n)_{n\in\mathbb{N}}] = [(a_n + b_n)_{n\in\mathbb{N}}].$$

The *product* of the real numbers $[(a_n)_{n\in\mathbb{N}}]$ and $[(b_n)_{n\in\mathbb{N}}]$ is defined by

$$[(a_n)_{n\in\mathbb{N}}] \cdot [(b_n)_{n\in\mathbb{N}}] = [(a_nb_n)_{n\in\mathbb{N}}].$$

[8]The proof of the Triangle Inequality is exactly the same, replacing 'real/$\mathbb{R}$' everywhere by 'rational/$\mathbb{Q}$'. Note that we are not allowed to use reals yet, and so we can't just specialise the Triangle Inequality for $\mathbb{R}$ to the rationals.

As usual, we have to check well-definedness. We leave this as an exercise for addition, but give an argument below for multiplication. Let $[(a_n)_{n\in\mathbb{N}}] = [(a'_n)_{n\in\mathbb{N}}] \in \mathbb{R}$ and $[(b_n)_{n\in\mathbb{N}}] = [(b'_n)_{n\in\mathbb{N}}] \in \mathbb{R}$. The idea is to use the inequality

$$|a'_n b'_n - a_n b_n| = |a'_n b'_n - a'_n b_n + a'_n b_n - a_n b_n| \leqslant |a'_n||b'_n - b_n| + |a'_n - a_n||b_n|$$

and the boundedness of the terms a'_n, b_n to show $(a_n b_n)_{n\in\mathbb{N}} \sim (a'_n b'_n)_{n\in\mathbb{N}}$. We carry out the details below.

As $(a'_n)_{n\in\mathbb{N}}$ is Cauchy, it is bounded, and let $A' > 0$ be a rational number such that $|a'_n| < A'$ for all $n \in \mathbb{N}$. Similarly, $(b_n)_{n\in\mathbb{N}}$ is bounded, and let $B > 0$ be a rational number such that $|b_n| < B$ for all $n \in \mathbb{N}$. Let $\epsilon > 0$ be a rational number. As $(a_n)_{n\in\mathbb{N}} \sim (a'_n)_{n\in\mathbb{N}}$, we have that $(a_n - a'_n)_{n\in\mathbb{N}}$ converges in $\mathbb{Q}$ to 0. So for the rational $\frac{\epsilon}{2B} > 0$, there exists an $N_a \in \mathbb{N}$ such that $|a'_n - a_n| < \frac{\epsilon}{2B}$. Similarly, as $(b_n)_{n\in\mathbb{N}} \sim (b'_n)_{n\in\mathbb{N}}$, we have that for the rational $\frac{\epsilon}{2A'} > 0$, there exists an $N_b \in \mathbb{N}$ such that $|b'_n - b_n| < \frac{\epsilon}{2A'}$. Set $N = N_a + N_b$. For all $n > N$, we have

$$\begin{aligned} |a'_n b'_n - a_n b_n| &= |a'_n b'_n - a'_n b_n + a'_n b_n - a_n b_n| \leqslant |a'_n||b'_n - b_n| + |a'_n - a_n||b_n| \\ &\leqslant A'|b'_n - b_n| + |a'_n - a_n|B < A'\frac{\epsilon}{2A'} + \frac{\epsilon}{2B}B = \epsilon. \end{aligned}$$

Thus $(a_n b_n)_{n\in\mathbb{N}} \sim (a'_n b'_n)_{n\in\mathbb{N}}$.

Exercise A.13 (Addition is well-defined)**.** Let $(a_n)_{n\in\mathbb{N}} \sim (a'_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}} \sim (b'_n)_{n\in\mathbb{N}}$. Show that $(a_n + b_n)_{n\in\mathbb{N}} \sim (a'_n + b'_n)_{n\in\mathbb{N}}$.

Example A.14 (The real numbers **0** and **1**)**.** We define the real numbers $\mathbf{0} = [(0)_{n\in\mathbb{N}}]$ and $\mathbf{1} = [(1)_{n\in\mathbb{N}}]$. Then for every real number $\boldsymbol{x} \in \mathbb{R}$, we have

$$\begin{aligned} \mathbf{0} + \boldsymbol{x} = \boldsymbol{r} = \boldsymbol{x} + \mathbf{0}, &\text{ and} \\ \mathbf{1} \cdot \boldsymbol{x} = \boldsymbol{r} = \boldsymbol{x} \cdot \mathbf{1}. \end{aligned}$$

Thus **0** serves as the additive identity and **1** serves as the multiplicative identity. Clearly $\mathbf{1} \neq \mathbf{0}$ because the sequence $(1-0)_{n\in\mathbb{N}}$ converges in $\mathbb{Q}$ to $1 \neq 0$. ◇

The set $\mathbb{R}$, together with the operations $+, \cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ forms a 'field', i.e., the following hold.

$$+\left\{\begin{array}{lll} \text{(F1)} & \text{(Associativity)} & \text{For all } \boldsymbol{x},\boldsymbol{y},\boldsymbol{z} \in \mathbb{R},\ \boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z}) = (\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z}. \\ \text{(F2)} & \text{(Additive identity)} & \text{For all } \boldsymbol{x} \in \mathbb{R},\ \boldsymbol{x} + \mathbf{0} = \boldsymbol{x} = \mathbf{0} + \boldsymbol{x}. \\ \text{(F3)} & \text{(Inverses)} & \text{For all } \boldsymbol{x} \in \mathbb{R},\ \text{there exists } -\boldsymbol{x} \in \mathbb{R} \\ & & \ \text{such that } \boldsymbol{x} + (-\boldsymbol{x}) = \mathbf{0} = -\boldsymbol{x} + \boldsymbol{x}. \\ \text{(F4)} & \text{(Commutativity)} & \text{For all } \boldsymbol{x},\boldsymbol{y} \in \mathbb{R},\ \boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x}. \end{array}\right.$$

$$\cdot\left\{\begin{array}{lll} \text{(F5)} & \text{(Associativity)} & \text{For all } \boldsymbol{x},\boldsymbol{y},\boldsymbol{z} \in \mathbb{R},\ \boldsymbol{x} \cdot (\boldsymbol{y} \cdot \boldsymbol{z}) = (\boldsymbol{x} \cdot \boldsymbol{y}) \cdot \boldsymbol{z}. \\ \text{(F6)} & \text{(Multiplicative identity)} & \mathbf{1} \neq \mathbf{0} \text{ and for all } x \in \mathbb{R},\ \boldsymbol{x} \cdot \mathbf{1} = \boldsymbol{x} = \mathbf{1} \cdot \boldsymbol{x}. \\ \text{(F7)} & \text{(Inverses)} & \text{For all } \boldsymbol{x} \in \mathbb{R}\backslash\{\mathbf{0}\},\ \text{there exists } \boldsymbol{x}^{-1} \in \mathbb{R} \\ & & \ \text{such that } \boldsymbol{x} \cdot \boldsymbol{x}^{-1} = \mathbf{1} = \boldsymbol{x}^{-1} \cdot \boldsymbol{x}. \\ \text{(F8)} & \text{(Commutativity)} & \text{For all } \boldsymbol{x},\boldsymbol{y} \in \mathbb{R},\ \boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{y} \cdot \boldsymbol{x}. \end{array}\right.$$

$$+,\cdot\left\{\ \text{(F9)} \quad \text{(Distributivity)} \quad \text{For all } \boldsymbol{x},\boldsymbol{y},\boldsymbol{z} \in \mathbb{R},\ \boldsymbol{x} \cdot (\boldsymbol{y} + \boldsymbol{z}) = \boldsymbol{x} \cdot \boldsymbol{y} + \boldsymbol{x} \cdot \boldsymbol{z}.\right.$$

In fact, if we replace everywhere $\mathbb{R}$ by $\mathbb{Q}$ (and $\mathbf{1}, \mathbf{0}$ by the rational numbers $1, 0$, respectively), then the set $\mathbb{Q}$ of rational numbers with their addition and multiplication, also satisfy the same properties. We say that $(\mathbb{Q}, +, \cdot)$ is also a field. (However, $(\mathbb{Z}, +, \cdot)$ is not a field, because multiplicative inverses don't always exist: The equation $2x = 1$ has no solution $x \in \mathbb{Z}$.)

We will not check each the above, as they essentially follow by 'termwise verifications', and by using the corresponding properties from the field of rationals. We remark that the additive inverse of $\boldsymbol{x} = [(a_n)_{n\in\mathbb{N}}]$ is $-\boldsymbol{x} := [(-a_n)_{n\in\mathbb{N}}]$. Let us show the existence of multiplicative inverses for nonzero reals. First we prove the following lemma.

Lemma A.15. *Let $\boldsymbol{x} \in \mathbb{R}$ be such that $\boldsymbol{x} \neq \mathbf{0}$. If $(a_n)_{n\in\mathbb{N}} \in \boldsymbol{x}$, then there exists a rational $d > 0$ and an $N \in \mathbb{N}$ such that for all $n > N$, $|a_n| > d$.*

Proof. As $[(a_n)_{n\in\mathbb{N}}] = \boldsymbol{x} \neq \mathbf{0} = [(0)_{n\in\mathbb{N}}]$, we have $\neg\big((a_n)_{n\in\mathbb{N}} \sim (0)_{n\in\mathbb{N}}\big)$, i.e.,

$$\neg\big((a_n - 0)_{n\in\mathbb{N}} \text{ converges in } \mathbb{Q} \text{ to } 0\big), \text{ i.e.,}$$
$$\neg\big(\forall \text{ rational } \epsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } \forall n > N,\ |a_n - 0| < \epsilon\big), \text{ i.e.,}$$

Thus

$$\exists \text{ rational } \epsilon > 0 \text{ such that } \forall N \in \mathbb{N},\ \exists n > N \text{ such that } |a_n - 0| \geqslant \epsilon. \qquad (\star)$$

Since $(a_n)_{n\in\mathbb{N}} \in \mathcal{C}$, for the rational $\epsilon/2 > 0$, there exists an $N_* \in \mathbb{N}$ such that for all $n, m > N_*$, $|a_n - a_m| < \frac{\epsilon}{2}$. From $(\star)$, taking $N = N_*$, there exists $n_* > N_*$ such that $|a_{n_*} - 0| \geqslant \epsilon$. Hence for $n > N_*$, we have

$$|a_n| = |a_n - a_{n_*} + a_{n_*}| \geqslant |a_{n_*}| - |a_n - a_{n_*}| \geqslant \epsilon - \frac{\epsilon}{2} = \frac{\epsilon}{2} =: d. \qquad \square$$

Proposition A.16. *Let the real number $\boldsymbol{x} \neq \mathbf{0}$. Then there exists an $\boldsymbol{x}^{-1} \in \mathbb{R}$ such that*

$$\boldsymbol{x} \cdot \boldsymbol{x}^{-1} = \mathbf{1} = \boldsymbol{x}^{-1} \cdot \boldsymbol{x}.$$

Proof. Let $\boldsymbol{x} = [(a_n)_{n\in\mathbb{N}}]$. By Lemma A.15 there exists a rational $d > 0$ and an $N \in \mathbb{N}$ such that $|a_n| > d$ for all $n > N$. In particular, $a_n \neq 0$ for all $n > N$. Set[9]

$$b_n := \begin{cases} 0 & \text{if } 1 \leqslant n \leqslant N, \\ a_n^{-1} & \text{if } n > N. \end{cases}$$

Then $(b_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{Q}$. Firstly, for $n, m > N$,

$$|b_n - b_m| = \Big|\frac{1}{a_n} - \frac{1}{a_m}\Big| = \frac{|a_n - a_m|}{|a_n||a_m|} \leqslant \frac{|a_n - a_m|}{d^2}.$$

Secondly, as $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{Q}$, given a rational $\epsilon > 0$, there exists an $M \in \mathbb{N}$ such that for all $n, m > M$, $|a_n - a_m| < \epsilon d^2$. Hence for all $n, m > N + M$,

$$|b_n - b_m| \leqslant \frac{|a_n - a_m|}{d^2} < \frac{\epsilon d^2}{d^2} = \epsilon.$$

Consequently, $(b_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{Q}$. We have[10]

$$a_n b_n := \begin{cases} 0 \text{ if } 1 \leqslant n \leqslant N, \\ 1 \text{ if } n > N. \end{cases}$$

Hence $(a_n b_n)_{n\in\mathbb{N}}$ converges in $\mathbb{Q}$ to 1. So $[(a_n b_n)_{n\in\mathbb{N}}] = \mathbf{1}$. Set $\boldsymbol{x}^{-1} := [(b_n)_{n\in\mathbb{N}}]$. Then we have $\boldsymbol{x} \cdot \boldsymbol{x}^{-1} = \mathbf{1} = \boldsymbol{x}^{-1} \cdot \boldsymbol{x}$. $\square$

Exercise A.17 (Distributive law)**.** Let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}$. Prove that $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$.

Order

To compare real numbers $\boldsymbol{x} = [(a_n)_{n\in\mathbb{N}}]$ and $\boldsymbol{y} = [(b_n)_{n\in\mathbb{N}}]$, we would like to use the order relation $<$ on $\mathbb{Q}$. If we try to define $\boldsymbol{x} < \boldsymbol{y}$ by saying that for *all* $n \in \mathbb{N}$, $a_n < b_n$, then this will not be a well-defined notion. Indeed, changing the first few terms of $(a_n)_{n\in\mathbb{N}}$ we could easily violate this, without changing $[(a_n)_{n\in\mathbb{N}}]$. Intuitively, $\boldsymbol{x}$ is the real number that $(a_n)_{n\in\mathbb{N}}$ converges to. So thinking formally

$$\text{‘}\boldsymbol{x} = \lim a_n\text{’}, \quad \text{‘}\boldsymbol{y} = \lim b_n\text{’},$$

we would say $\boldsymbol{x} < \boldsymbol{y}$ if ‘$\lim a_n < \lim b_n$’, that is,

$$\text{‘}\lim(b_n - a_n) > 0\text{’}.$$

---

[9] Although we set $b_n = 0$ for $1 \leqslant n \leqslant N$ here, any arbitrary N rational numbers can be specified here.

[10] Had we specified $b_1, \cdots, b_N$ arbitrarily, we would get a bunch of initial terms $a_n b_n$ for $1 \leqslant n \leqslant N$, but this won't affect the rest of the proof.

But from our former intuition with limits, we know that this means that for all *large enough* $n \in \mathbb{N}$, $b_n - a_n$ stays away from 0 by some positive distance d, say. This motivates the following.

Definition A.18. Let $\boldsymbol{x} = [(a_n)_{n\in\mathbb{N}}]$ and $\boldsymbol{y} = [(b_n)_{n\in\mathbb{N}}]$ be real numbers. Then $\boldsymbol{x} < \boldsymbol{y}$ if there exists a rational number $d > 0$ and an $N \in \mathbb{N}$ such that for all $n > N$, $b_n - a_n > d$. If $\boldsymbol{x} < \boldsymbol{y}$, we write equivalently $\boldsymbol{y} > \boldsymbol{x}$.

Let us show that this is a well-defined notion.

Proposition A.19. *Let $[(a_n)_{n\in\mathbb{N}}] = [(a'_n)_{n\in\mathbb{N}}] \in \mathbb{R}$ and $[(b_n)_{n\in\mathbb{N}}] = [(b'_n)_{n\in\mathbb{N}}] \in \mathbb{R}$. Suppose that there exists a rational number $d > 0$ and an $N \in \mathbb{N}$ such that for all $n > N$, $b_n - a_n > d$. Then there exists a rational number $d' > 0$ and an $N' \in \mathbb{N}$ such that for all $n > N'$, $b'_n - a'_n > d'$.*

Proof. As $(a_n)_{n\in\mathbb{N}} \sim (a'_n)_{n\in\mathbb{N}}$, we know that $(a_n - a'_n)_{n\in\mathbb{N}}$ converges in $\mathbb{Q}$ to 0. So for the rational $d/4 > 0$, there exists an $N_a \in \mathbb{N}$ such that for all $n > N_a$, $|a_n - a'_n| < d/4$, i.e., $-d/4 < a_n - a'_n < d/4$. In particular

$$a_n - a'_n > -\frac{d}{4} \text{ for all } n > N_a. \qquad (*)$$

Similarly, $(b_n)_{n\in\mathbb{N}} \sim (b'_n)_{n\in\mathbb{N}}$ yields the existence of an $N_b \in \mathbb{N}$ such that

$$b'_n - b_n > -\frac{d}{4} \text{ for all } n > N_b. \qquad (**)$$

Set $N' = N_a + N_b + N$. Then for all $n > N'$, using $(*)$ and $(**)$, we have

$$b'_n - a'_n = b_n - a_n + b'_n - b_n + a_n - a'_n > d - \frac{d}{4} - \frac{d}{4} = \frac{d}{2} > 0.$$

So with the rational $d' := \frac{d}{2} > 0$, for all $n > N'$, we have $b'_n - a'_n > d'$. $\square$

Exercise A.20. Show that if $r, s \in \mathbb{Q}$ and $r < s$, then $[(r)_{n\in\mathbb{N}}] < [(s)_{n\in\mathbb{N}}]$. (In particular, for the real numbers $\mathbf{0}, \mathbf{1}$, we have $\mathbf{0} < \mathbf{1}$.)

Exercise A.21 (Transitivity of $<$)**.** Let $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}$ be such that $\boldsymbol{x} < \boldsymbol{y}$ and $\boldsymbol{y} < \boldsymbol{z}$. Prove that $\boldsymbol{x} < \boldsymbol{z}$.

Theorem A.22 (Trichotomy Law)**.** *Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}$. Then one and exactly one of the following hold:*

1° $\boldsymbol{x} < \boldsymbol{y}$. $\quad 2^\circ$ $\boldsymbol{x} = \boldsymbol{y}$. $\quad 3^\circ$ $\boldsymbol{x} > \boldsymbol{y}$.

Proof. Let $\boldsymbol{x} = [(a_n)_{n\in\mathbb{N}}]$ and $\boldsymbol{y} = [(b_n)_{n\in\mathbb{N}}]$. If $\boldsymbol{x} = \boldsymbol{y}$, then $(a_n - b_n)_{n\in\mathbb{N}}$ converges in $\mathbb{Q}$ to 0. Let us show that $\neg(\boldsymbol{x} > \boldsymbol{y})$. Indeed, otherwise there exists a rational $d > 0$ and an $N \in \mathbb{N}$ such that $a_n - b_n > d$ for all $n > N$. But then taking the rational $\epsilon := d/2 > 0$, we get, thanks to the convergence of $(a_n - b_n)_{n\in\mathbb{N}}$, that there is an $N' \in \mathbb{N}$ such that for all $n > N'$, $|a_n - b_n| < d/2$. So with $n = N + N'$, we arrive at the contradiction that $d < a_n - b_n \leqslant |a_n - b_n| < d/2$. So if $\boldsymbol{x} = \boldsymbol{y}$, then $\neg(\boldsymbol{x} > \boldsymbol{y})$. Interchanging the roles of $\boldsymbol{x}, \boldsymbol{y}$, we also have that if $\boldsymbol{x} = \boldsymbol{y}$, then $\neg(\boldsymbol{x} < \boldsymbol{y})$. Let us also note that if $\boldsymbol{x} < \boldsymbol{y}$, then $\neg(\boldsymbol{y} < \boldsymbol{x})$: Otherwise there exist rational $d, d' > 0$ and $N, N' \in \mathbb{N}$ such that for all $n > N$ we have $b_n - a_n > d$, and for all $n > N'$, we have $a_n - b_n > d'$, so that with $n := N + N'$, we get $d' < a_n - b_n < -d$, giving $0 > d + d' > d + 0 = d$, a contradiction.

Let $[(a_n)_{n\in\mathbb{N}}] := \boldsymbol{x} \neq \boldsymbol{y} =: [(b_n)_{n\in\mathbb{N}}]$. Then it is not the case that the sequence $(a_n - b_n)_{n\in\mathbb{N}}$ converges in $\mathbb{Q}$ to 0. Thus there exists a rational $\epsilon_* > 0$ such that

$$\text{for all } N \in \mathbb{N}, \text{ there exists an } n > N \text{ such that } |a_n - b_n| \geqslant \epsilon_*. \qquad (\star)$$

As $(a_n)_{n\in\mathbb{N}}$ is Cauchy, there exists an $N_a \in \mathbb{N}$ such that for all $m, n > N_a$, we have $|a_n - a_m| < \epsilon_*/4$, i.e., $-\epsilon_*/4 < a_n - a_m < \epsilon_*/4$. In particular, $a_n - a_m > -\epsilon_*/4$ for all $n, m > N$. Similarly, as $(b_n)_{n\in\mathbb{N}}$ is Cauchy, there exists an $N_b \in \mathbb{N}$ such that for all $m, n > N_b$, $|b_n - b_m| < \epsilon_*/4$, giving in particular $b_n - b_m > -\epsilon_*/4$. Now take $N = N_a + N_b$ in $(\star)$. Then there exists an $n_* > N$ such that $|a_{n_*} - b_{n_*}| \geqslant \epsilon_* > 0$. In particular, $a_{n_*} - b_{n_*} \neq 0$. So by the trichotomy law for $<$ in $\mathbb{Q}$, we have the following two mutually exclusive possible cases:

1° $a_{n_*} - b_{n_*} > 0$. Then $a_{n_*} - b_{n_*} = |a_{n_*} - b_{n_*}| \geqslant \epsilon_*$. For all $m > n_*$ ($> N = N_a + N_b$),

$$a_m - b_m = a_{n_*} - b_{n_*} + a_m - a_{n_*} + b_{n_*} - b_m > \epsilon_* - \tfrac{\epsilon_*}{4} - \tfrac{\epsilon_*}{4} = \tfrac{\epsilon_*}{2} =: d.$$

So for all $m > n_*$, we have that $a_m - b_m > d$, showing $\boldsymbol{x} > \boldsymbol{y}$.

2° $a_{n_*} - b_{n_*} < 0$. Then $b_{n_*} - a_{n_*} = |a_{n_*} - b_{n_*}| \geqslant \epsilon_*$. For all $m > n_*$ ($> N = N_a + N_b$),

$$b_m - a_m = b_{n_*} - a_{n_*} + b_m - b_{n_*} + a_{n_*} - a_m > \epsilon_* - \tfrac{\epsilon_*}{4} - \tfrac{\epsilon_*}{4} = \tfrac{\epsilon_*}{2} =: d.$$

So for all $m > n_*$, we have that $b_m - a_m > d$, showing $\boldsymbol{y} > \boldsymbol{x}$. □

Definition A.23 (The set $\mathbb{P}$ of positive reals)**.** We define $\mathbb{P} := \{\boldsymbol{x} \in \mathbb{R} : \boldsymbol{x} > \boldsymbol{0}\}$.

Exercise A.24. Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{P}$. Show that $\boldsymbol{x} + \boldsymbol{y} \in \mathbb{P}$ and $\boldsymbol{x} \cdot \boldsymbol{y} \in \mathbb{P}$.

Exercise A.25. Let $\boldsymbol{x} \in \mathbb{R}$ be such that $\boldsymbol{x} > \boldsymbol{0}$. Prove that there exists an $r \in \mathbb{Q}$ such that $\boldsymbol{0} < [(r)_{n\in\mathbb{N}}] < \boldsymbol{x}$. We write this succinctly as $\boldsymbol{0} < r < \boldsymbol{x}$.

Exercise A.26. Let $(a_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathbb{Q}$. Suppose there exists an $N \in \mathbb{N}$ such that for all $n > N$, we have $a_n \geqslant 0$. Show that the real number $\boldsymbol{x} = [(a_n)_{n\in\mathbb{N}}] \geqslant \boldsymbol{0}$.

Exercise A.27 (No order for $\mathbb{C}$)**.** A field $\mathbb{F}$ is called *ordered* if there is a subset $P \subset \mathbb{F}$, called the *set of positive elements of* $\mathbb{F}$, satisfying the following:

(P1) For all $x, y \in P$, $x + y \in P$.

(P2) For all $x, y \in P$, $x \cdot y \in P$.

(P3) For each $x \in P$, one and only one of the following three cases is true:

1° $x = 0$. 2° $x \in P$. 3° $-x \in P$.

(Once one has an ordered set of elements in a field, one can compare the elements of $\mathbb{F}$ by defining a relation $>_P$ in $\mathbb{F}$ by setting $y >_P x$ for $x, y \in \mathbb{F}$ if $y - x \in P$.)

Show that $\mathbb{C}$ is not an ordered field. *Hint:* Consider $x := i$, and first look at $x \cdot x$.

The least upper bound property of $\mathbb{R}$

Finally we are ready to prove the ultimate goal, namely the least upper bound property of $\mathbb{R}$, i.e., we show the following:

Theorem A.28 (Least upper bound property of $\mathbb{R}$)**.**
Every nonempty subset of $\mathbb{R}$ *which is bounded above has a supremum.*

We first give an example to show that $\mathbb{Q}$ does not possess the Least Upper Bound Property.

Example A.29 ($\mathbb{Q}$ does not possess the Least Upper Bound Property)**.**
Consider the set $S := \{x \in \mathbb{Q} : x^2 \leqslant 2\}$. Clearly S is a subset of $\mathbb{Q}$ and it is nonempty since $1 \in S$: $1^2 = 1 \leqslant 2$. Let us show that S is bounded above. In fact, 2 serves as an upper bound of S. Since if $x > 2$, then $x^2 > 4 > 2$. Thus if $x \in S$, then $x^2 \leqslant 2$, and so $x \leqslant 2$.

If $\mathbb{Q}$ has the Least Upper Bound Property, then the above nonempty subset of $\mathbb{Q}$ which is bounded above must possess a least upper bound $u_* := \sup S \in \mathbb{Q}$. We will show that this $u_* \in \mathbb{Q}$ must satisfy that $u_*^2 = 2$. But we know that this is impossible as we know that there is no rational number whose square is 2.

Firstly, $u_* \geqslant 1$ (as u_* is in particular an upper bound of S and $1 \in S$). Now define

$$r := u_* - \frac{u_*^2 - 2}{u_* + 2} = \frac{2(u_* + 1)}{u_* + 2} > 0. \tag{A.1}$$

As $u_* \in \mathbb{Q}$, the rightmost expression for r shows that $r \in \mathbb{Q}$ as well. Then

$$r^2 - 2 = \frac{2(u_*^2 - 2)}{(u_* + 2)^2}. \tag{A.2}$$

1° Suppose $u_*^2 < 2$. Then (A.2) implies that $r^2 - 2 < 0$, and so $r \in S$. But from (A.1), $r > u_*$, contradicting the fact that u_* is an upper bound of S.

2° Suppose that $u_*^2 > 2$. If $r' > r$ (> 0), then $r'^2 = r' \cdot r' > r \cdot r' > r \cdot r = r^2$. From (A.2), $r^2 > 2$, and so from the above, we know that $r'^2 > 2$ as well. Hence $r' \notin S$. So we have shown that if $r' \in S$, then $r' \leqslant r$. This means that r is an upper bound of S. But this is impossible, since (A.1) shows that $r < u_*$, and u_* is the *least* upper bound of S.

So it must be the case that $u_*^2 = 2$. But this is impossible. Hence $\mathbb{Q}$ does not possess the Least Upper Bound Property. ◇

This 'analytical flaw' of the rational number system is remedied by the set of real numbers. Moreover, we had seen that not all Cauchy sequences in $\mathbb{Q}$ converge in $\mathbb{Q}$. In contrast, we have shown, using the Least Upper Bound Property of $\mathbb{R}$, that we the following happy situation in $\mathbb{R}$: $\{$Cauchy sequences in $\mathbb{R}\} = \{$convergent sequences in $\mathbb{R}\}$.

We will show that every nonempty subset of $\mathbb{R}$ which is bounded above has a supremum.

Lemma A.30 ('Baby' Archimedean Principle)**.** *If $\boldsymbol{x} \in \mathbb{R}$, then there exists a natural number $n \in \mathbb{N}$ such that $n > \boldsymbol{x}$.*

We cannot use the Archimedean Principle to prove the above, since that earlier result was proved using the Least Upper Bound Property of $\mathbb{R}$, which we haven't established yet!

Proof. If $\boldsymbol{x} \leqslant \mathbf{0}$, then take $n = 1$, since $\mathbf{0} < \mathbf{1}$ gives by transitivity that $\boldsymbol{x} < \mathbf{1}$.

Let $\boldsymbol{x} = [(a_n)_{n\in\mathbb{N}}] > \mathbf{0}$. We have seen that every Cauchy sequence in $\mathbb{Q}$ is bounded. So there exists a rational $A > 0$ such that for all $n \in \mathbb{N}$, $a_n \leqslant A$. This implies $\boldsymbol{x} < [(A+1)_{n\in\mathbb{N}}]$ (since $A + 1 - a_n \geqslant A + 1 - A = 1 > 0$ for all $n \in \mathbb{N}$). Write $A + 1 = [(\frac{p}{q})]$, where $p, q \in \mathbb{N}$. Set $n = p + 1$. Then $A + 1 = [(\frac{p}{q})] < [(\frac{n}{1})]$ (since $p < p + 1 \leqslant (p + 1)q = nq$). So $\boldsymbol{x} < [(A + 1)_{n\in\mathbb{N}}] < [(\frac{n}{1})]$. Succinctly, $\boldsymbol{x} < n$. □

Lemma A.31 (Density of $\mathbb{Q}$ in $\mathbb{R}$ redone)**.** *Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}$ be such that $\boldsymbol{x} < \boldsymbol{y}$. Then there exists an $r \in \mathbb{Q}$ such that $\boldsymbol{y} < r < \boldsymbol{x}$.*

Proof. As $\boldsymbol{y} - \boldsymbol{x} > \mathbf{0}$, we have in particular $\boldsymbol{y} - \boldsymbol{x} \neq 0$, and so $(\boldsymbol{y} - \boldsymbol{x})^{-1}$ exists in $\mathbb{R}$. By Lemma A.30, there exists an $n \in \mathbb{N}$ such that $n > (\boldsymbol{y} - \boldsymbol{x})^{-1}$, and so $n(\boldsymbol{y} - \boldsymbol{x}) > \mathbf{1}$, i.e., $n\boldsymbol{x} + \mathbf{1} < n\boldsymbol{y}$.

By Lemma A.30, there exists an $m_1 \in \mathbb{N}$ such that $m_1 > n\boldsymbol{x}$, and there exists an $m_2 \in \mathbb{N}$ such that $m_2 > -n\boldsymbol{x}$. So $-m_2 < n\boldsymbol{x} < m_1$ for some integers m_1, m_2. Among the finitely many integers $k \in \mathbb{Z}$ such that $-m_2 \leqslant k \leqslant m_1$, we take as $\lfloor n\boldsymbol{x} \rfloor$ the largest one such that it is also $\leqslant n\boldsymbol{x}$.

Let $m := \lfloor n\boldsymbol{x} \rfloor + 1$. Then $\lfloor n\boldsymbol{x} \rfloor \leqslant n\boldsymbol{x} < \lfloor n\boldsymbol{x} \rfloor + 1$, that is, $m - 1 \leqslant n\boldsymbol{x} < m$. So

$$\boldsymbol{x} < \frac{m}{n} \leqslant \frac{n\boldsymbol{x} + 1}{n} < \frac{n\boldsymbol{y}}{n} = \boldsymbol{y}.$$

With $r := \dfrac{m}{n} \in \mathbb{Q}$, the proof is complete. □

Lemma A.32. *Let $\boldsymbol{x} = [(a_n)_{n\in\mathbb{N}}] \in \mathbb{R}$. Given any rational $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n > N$, $a_n + \epsilon \geqslant \boldsymbol{x} \geqslant a_n - \epsilon$.*

Proof. Let a rational $\epsilon > 0$ be given. As $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{Q}$, there exists an $N \in \mathbb{N}$ such that for all $n, m > N$, $|a_n - a_m| < \epsilon/2$, i.e., $-\frac{\epsilon}{2} < a_n - a_m < \frac{\epsilon}{2}$. Fix an $n > N$. For all $m > N$, $a_n + \epsilon = a_n - a_m + a_m + \epsilon > -\frac{\epsilon}{2} + a_m + \epsilon = a_m + \frac{\epsilon}{2}$, i.e.,

$$(a_n + \epsilon) - a_m > \tfrac{\epsilon}{2} =: d > 0.$$

Thus $[(a_n + \epsilon, a_n + \epsilon, a_n + \epsilon, \cdots)] > [(a_m)_{m\in\mathbb{N}}] = \boldsymbol{x}$. For all $m > N$, we also have

$$a_n - \epsilon = a_n - a_m + a_m - \epsilon < \tfrac{\epsilon}{2} + a_m - \epsilon = a_m - \tfrac{\epsilon}{2},$$

i.e., $a_m - (a_n - \epsilon) > \frac{\epsilon}{2} > 0$. So $\boldsymbol{x} = [(a_m)_{m\in\mathbb{N}}] > [(a_n - \epsilon, a_n - \epsilon, a_n - \epsilon, \cdots)]$. □

Lemma A.33. *Let $\boldsymbol{x} = [(a_n)_{n\in\mathbb{N}}]$. Suppose that there exist $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}$ and $N \in \mathbb{N}$ such that for all $n > N$, $\boldsymbol{\alpha} \leqslant a_n \leqslant \boldsymbol{\beta}$. Then $\boldsymbol{\alpha} \leqslant \boldsymbol{x} \leqslant \boldsymbol{\beta}$.*

Proof. By the density of $\mathbb{Q}$ in $\mathbb{R}$, for each $n \in \mathbb{N}$, there exist $\alpha_n, \beta_n \in \mathbb{Q}$ such that

$$\boldsymbol{\alpha} - \frac{1}{n} < \alpha_n < \boldsymbol{\alpha}, \text{ and } \boldsymbol{\beta} < \beta_n < \boldsymbol{\beta} + \frac{1}{n}.$$

We claim that $(\alpha_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{Q}$. Indeed, for any $n, m \in \mathbb{N}$

$$\boldsymbol{\alpha} - \frac{1}{n} < \alpha_n < \boldsymbol{\alpha}, \quad \text{and} \quad -\boldsymbol{\alpha} < -\alpha_m < -\boldsymbol{\alpha} + \frac{1}{m},$$

which together give $-\frac{1}{n} < \alpha_n - \alpha_m < \frac{1}{m}$. Given any rational $\epsilon > 0$, let $N' \in \mathbb{N}$ be such that $N' > \epsilon^{-1}$. Then for $n, m > N'$,

$$-\frac{1}{N'} < -\frac{1}{n} < \alpha_n - \alpha_m < \frac{1}{m} < \frac{1}{N'},$$

so that $|\alpha_n - \alpha_m| < 1/N' < \epsilon$. So $(\alpha_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{Q}$. A similar proof shows that also $(\beta_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{Q}$.

We now show that $\boldsymbol{\alpha} = [(\alpha_n)_{n\in\mathbb{N}}]$. To do this we eliminate the other possibilities, namely $\boldsymbol{\alpha} < [(\alpha_n)_{n\in\mathbb{N}}]$ or $\boldsymbol{\alpha} > [(\alpha_n)_{n\in\mathbb{N}}]$. Let $\boldsymbol{\alpha} = [(\widetilde{\alpha}_n)_{n\in\mathbb{N}}]$.

1° Suppose $\boldsymbol{\alpha} < [(\alpha_n)_{n\in\mathbb{N}}]$. Then there exists a rational $d > 0$ and an $M \in \mathbb{N}$ such that for all $n > M$, $\alpha_n - \widetilde{\alpha}_n > d$. By Lemma A.32 with $\epsilon = d/2$, there exists an $M' \in \mathbb{N}$ such that for all $n > M'$, $\widetilde{\alpha}_n + \epsilon = \widetilde{\alpha}_n + d/2 \geqslant \boldsymbol{\alpha}$. Thus for $n > M + M'$, $d < \alpha_n - \widetilde{\alpha}_n \leqslant \alpha_n - \boldsymbol{\alpha} + \frac{d}{2} < 0 + \frac{d}{2} = \frac{d}{2}$, a contradiction.

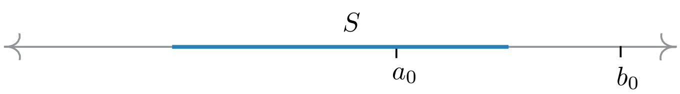
2° Suppose $\boldsymbol{\alpha} > [(\alpha_n)_{n\in\mathbb{N}}]$. Then there exists a rational $d > 0$ and an $M \in \mathbb{N}$ such that for all $n > M$, $\widetilde{\alpha}_n - \alpha_n > d$. By Lemma A.32 with $\epsilon = d/4$, there exists an $M' \in \mathbb{N}$ such that for all $n > M'$, $\widetilde{\alpha}_n - \epsilon = \widetilde{\alpha}_n - d/4 \leqslant \boldsymbol{\alpha}$. Finally, there exists an $M'' \in \mathbb{N}$ such that $M'' > 4/d$. Then for all $n > M + M' + M''$, we have $d < \widetilde{\alpha}_n - \alpha_n \leqslant \frac{d}{4} + \boldsymbol{\alpha} - \alpha_n < \frac{d}{4} + \frac{1}{n} < \frac{d}{4} + \frac{d}{4} = \frac{d}{2}$, a contradiction.

Thus $\boldsymbol{\alpha} = [(\alpha_n)_{n\in\mathbb{N}}]$. In a similar manner, we also have $\boldsymbol{\beta} = [(\beta_n)_{n\in\mathbb{N}}]$.

Since for all $n > N$ we have $a_n - \alpha_n \geqslant \boldsymbol{\alpha} - \alpha_n > 0$, and $\beta_n - a_n \geqslant \beta_n - \boldsymbol{\beta} > 0$, it follows from Exercise A.26 that $[(a_n - \alpha_n)_{n\in\mathbb{N}}] \geqslant \mathbf{0}$ and $[(\beta_n - a_n)_{n\in\mathbb{N}}] \geqslant \mathbf{0}$, that is, $\boldsymbol{x} - \boldsymbol{\alpha} \geqslant \mathbf{0}$ and $\boldsymbol{\beta} - \boldsymbol{x} \geqslant \mathbf{0}$. Rearranging, we obtain $\boldsymbol{\alpha} \leqslant \boldsymbol{x} \leqslant \boldsymbol{\beta}$. ☐

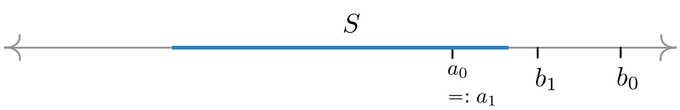
Theorem A.34. *Every nonempty subset of $\mathbb{R}$, bounded above, has a supremum.*

Proof. Let $S \subset \mathbb{R}$ be a nonempty subset, which is bounded above. Since S is nonempty, there exists an element $a_0 \in S$, and as S is bounded above, there exists an upper bound $b_0 \in \mathbb{R}$, that is, $a \leqslant b_0$ for all $a \in S$.



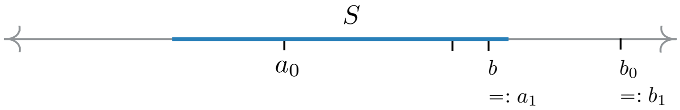
We define a_1, b_1 as follows:

1° If $\frac{a_0+b_0}{2}$ is an upper bound of S, then define $a_1 := a_0$ and $b_1 := \frac{a_0+b_0}{2}$.



Then $a_0 \leqslant a_1$, $b_0 \geqslant b_1$, $a_1 \in S$, b_1 is an upper bound of S, $0 \leqslant b_1 - a_1 \leqslant \frac{b_0-a_0}{2}$.

2° If $\frac{a_0+b_0}{2}$ is not an upper bound of S, then there exists a $b \in S$ such that $\frac{a_0+b_0}{2} < b$, and taking any such b, we define $a_1 := b$ and $b_1 = b_0$.



Then $a_0 \leqslant a_1$, $b_0 \geqslant b_1$, $a_1 \in S$, b_1 is an upper bound of S, $0 \leqslant b_1 - a_1 \leqslant \frac{b_0 - a_0}{2}$.

Suppose for some $n \in \mathbb{N}$,

- $a_0, a_1, \cdots, a_{n-1} \in S$ and
- $b_0, b_1, \cdots, b_{n-1}$, upper bounds for S,

have been constructed such that

- $a_0 \leqslant a_1 \leqslant \cdots \leqslant a_{n-1}$
- $b_0 \geqslant b_1 \geqslant \cdots \geqslant b_{n-1}$, and
- $0 \leqslant b_k - a_k \leqslant \frac{b_0 - a_0}{2^k}$, $k \in \{1, \cdots, n-1\}$.

Now we construct a new $a_n \in S$ and a new upper bound b_n of S.

1° If $\frac{a_{n-1}+b_{n-1}}{2}$ is an upper bound of S, then $a_n := a_{n-1}$ and $b_n := \frac{a_{n-1}+b_{n-1}}{2}$. Then $a_{n-1} \leqslant a_n$, $b_{n-1} \geqslant b_n$, $a_n \in S$, b_n is an upper bound of S, and

$$0 \leqslant b_n - a_n = \frac{b_{n-1} - a_{n-1}}{2} \leqslant \frac{b_0 - a_0}{2 \cdot 2^{n-1}} = \frac{b_0 - a_0}{2^n}.$$

2° If $\frac{a_{n-1}+b_{n-1}}{2}$ is not an upper bound of S, then there exists a $b \in S$ such that $\frac{a_{n-1}+b_{n-1}}{2} < b$, and taking any such b, define $a_n := b$ and $b_n = b_{n-1}$. Then $a_{n-1} = \frac{a_{n-1}+a_{n-1}}{2} \leqslant \frac{a_{n-1}+b_{n-1}}{2} < b = a_n$, $b_{n-1} \geqslant b_n$, $a_n \in S$, b_n is an upper bound of S, and

$$0 \leqslant b_n - a_n = b_{n-1} - b < b_{n-1} - \frac{b_{n-1}+a_{n-1}}{2} = \frac{b_{n-1} - a_{n-1}}{2} \leqslant \frac{b_0 - a_0}{2^n}.$$

So we get sequences $a_0, a_1, \cdots$ in S, and $b_0, b_1, \cdots$ of upper bounds of S, such that

- $a_0 \leqslant a_1 \leqslant \cdots$,
- $b_0 \geqslant b_1 \geqslant \cdots$, and
- $0 \leqslant b_n - a_n \leqslant \frac{b_0 - a_0}{2^n}$, $n \in \mathbb{N}$.

If for some $n \geqslant 0$, $a_n = b_n$, then we claim that $u_* := b_n$ is the supremum of S. Indeed, firstly, $u_* = b_n$ is an upper bound of S by construction. Moreover, for any $u < u_* = b_n$, u cannot be an upper bound of S (because $u < u_* = b_n = a_n \in S$).

So we now have to consider the case that for all $n \geqslant 0$, $a_n < b_n$. By the density of $\mathbb{Q}$ in $\mathbb{R}$ (Lemma A.31), for each $n \in \mathbb{N}$, there exists an $r_n \in \mathbb{Q}$ such that $a_n < r_n < b_n$. We claim that $(r_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{Q}$. To see this, let $\epsilon > 0$ be a given rational number. By the 'baby' Archimedean principle (Lemma A.30), there exists an $N \in \mathbb{N}$ such that $N > \frac{b_0 - a_0}{\epsilon}$, and so

$$\frac{b_0 - a_0}{2^N} \leqslant \frac{b_0 - a_0}{N} < \epsilon$$

(thanks to the inequality $n < 2^n$ for $n \in \mathbb{N}$: Indeed, we have $1 < 2^1$, and if $n < 2^n$, then $n+1 < 2^n + 1 < 2^n + 2^n = 2 \cdot 2^n = 2^{n+1}$). Now if $n > m > N$, then $a_m < r_m < b_m$, $a_n < r_n < b_n$, $a_n \geqslant a_m$, which together give

$$r_m - r_n < b_m - r_n < b_m - a_n \leqslant b_m - a_m.$$

As $b_n \leqslant b_m$, we have

$$r_m - r_n > a_m - r_n > a_m - b_n \geqslant a_m - b_m.$$

Hence

$$|r_m - r_n| < b_m - a_m \leqslant \frac{b_0 - a_0}{2^m} < \frac{b_0 - a_0}{2^N} \leqslant \frac{b_0 - a_0}{N} < \epsilon.$$

So $(r_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{Q}$, and $u_* := [(r_n)_{n \in \mathbb{N}}] \in \mathbb{R}$.

We will now show that u_* is the supremum of S. First, for every fixed m, we have for all $n \geqslant m$ that $a_m \leqslant a_n < r_n < b_n \leqslant b_m$, and so by Lemma A.33,

$$a_m \leqslant u_* \leqslant b_m. \qquad (\star)$$

Now suppose that u_* is not an upper bound of S. Then there exists an $a \in S$ such that $a > u_*$. By the density of $\mathbb{Q}$ in $\mathbb{R}$, there exists an $r \in \mathbb{Q}$ such that

$$0 < r < a - u_*. \qquad (\star\star)$$

By the 'baby' Archimedean Principle, there exists an $m \in \mathbb{N}$ such that $m > \frac{b_0 - a_0}{r}$. So

$$0 \leqslant b_m - a_m \leqslant \frac{b_0 - a_0}{2^m} \leqslant \frac{b_0 - a_0}{m} < r.$$

Hence using ($\star$) and ($\star\star$),

$$b_m < a_m + r \leqslant u_* + r < a,$$

a contradiction to the fact that b_m is an upper bound of S.

Next, suppose that $u < u_*$. Let $r \in \mathbb{Q}$ be such that $0 < r < u_* - u$. In the same manner as above, there exists an $m \in \mathbb{N}$ such that $0 \leqslant b_m - a_m < r$. Then

$$a_m > b_m - r \overset{(\star)}{\geqslant} u_* - r > u,$$

showing u is not an upper bound of S. So u_* is the least upper bound of S. ☐

Bibliography

[A] T.M. Apostol. *Mathematical Analysis.* Second Edition. Addison-Wesley, 1974.
(This is a good book in real (and complex) analysis and includes a wealth of material.)

[B1] K.G. Binmore. *The Foundations of Analysis: a Straightforward Introduction. Book 1. Logic, Sets and Numbers.* Cambridge University Press, 1980.
(This book is an informal introduction to logic and set theory. As the title suggests, the properties of the real numbers are emphasised in order to prepare the student for courses in analysis. Examples, exercises, and diagrams are abundant.)

[B2] K.G. Binmore. *The Foundations of Analysis: a Straightforward Introduction. Book 2. Topological Ideas.* Cambridge University Press, 1981.
(This is a leisurely introduction to the basic notions of analysis. Clear motivation is provided for the results to follow.)

[R] W. Rudin. *Principles of Mathematical Analysis.* Third edition. McGraw-Hill, 1976.
(This book is tough to read, but having ploughed through it, with the exercises, gives a solid foundation.)

[S] G.E. Shilov. *Elementary Real and Complex Analysis.* Revised English edition translated from the Russian and edited by Richard A. Silverman. Corrected reprint of the 1973 English edition. Dover Publications, 1996.
(This book is easier to read than Rudin's book, and offers the reader a leisurely but systematic introduction to Analysis. The Dover edition is affordable.)

Index