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Complex Analysis
(MA317)

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Contents

Introduction

What is Complex Analysis?

The central object of study in complex analysis is a "complex differentiable" function $f : \mathbb{C} \to \mathbb{C}$. Since $\mathbb{C}$ is really $\mathbb{R}^2$, one might think of the function f as being a function from $\mathbb{R}^2$ to $\mathbb{R}^2$. So one might guess that the subject of complex analysis is similar to real analysis. But everything changes drastically if we assume f to be *complex differentiable*, that is,

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. The definition is similar to the definition of differentiability of f having a real argument, but now the numerator and denominator are *complex* numbers. This has far-reaching consequences. One might think that one can prove theorems for holomorphic functions via real variables, but actually this is a new subject, with its own proofs (often short). The core content of the course can be summarized in the following Grand Theorem[1]:

Theorem 0.1. *Let D be an open path connected set and let $f : D \to \mathbb{C}$. Then the following are equivalent:*

(1) *For all $z \in D$, $f'(z)$ exists.*

(2) *For all $z \in D$ and all $n \geq 0$, $f^{(n)}(z)$ exists.*

(3) *$u := \mathrm{Re}(f)$, $v := \mathrm{Im}(f)$ are continuously differentiable and $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$, $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ in D.*

(4) *For each simply connected subdomain S of D, there exists a holomorphic $F : S \to \mathbb{C}$ such that $F'(z) = f(z)$ for all $z \in S$.*

(5) *f is continuous on D and for all piecewise smooth closed paths γ in each simply connected subdomain of D, we have*

$$\int_\gamma f(z)dz = 0.$$

(6) *If $\{z \in \mathbb{C} : |z - z_0| \leq r\} \subset D$, then for all z with $|z - z_0| < r$, $f(z) = \dfrac{1}{2\pi i} \displaystyle\int_{|\zeta - z_0| = r} \dfrac{f(\zeta)}{\zeta - z} d\zeta$.*

(7) *If $\{z \in \mathbb{C} : |z - z_0| \leq r\} \subset D$, then there is a unique sequence $(c_n)_{n \geq 0}$ in $\mathbb{C}$ such that for all z with $|z - z_0| < r$,*

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n.$$

Furthermore, $c_n = \dfrac{1}{2\pi i} \displaystyle\int_{|\zeta - z_0| = r} \dfrac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta$ *and* $c_n = \dfrac{f^{(n)}(z_0)}{n!}$.

[1]Don't worry about the unfamiliar terms/notation here: that is what we will learn, besides the proof!

Why study complex analysis?

Complex analysis plays an important role in many branches of mathematics, and in applications. Here is a list of a few of them:

(1) **PDEs**. The real and analytic parts of a complex differentiable functions satisfy an important basic PDE, called the Laplace equation:

$$\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Many problems in applications give rise to this equation: for example gravitational fields, electrostatic fields, steady-state heat conduction, incompressible fluid flow, Brownian motion, computer animation[2] etc.

(2) **Real analysis**. To work integrals in $\mathbb{R}$. For example,

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2} dx.$$

Note that the problem is set in the reals, but one can solve it using complex analysis.

Moreover, one gains a deeper understanding of some real analysis facts via complex analysis. For example, it is easy to understand that the power series representation of the function

$$f(x) := \frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + \dots$$

is valid only for $x \in (-1, 1)$, since f itself has singularities at $x = 1$ and at $x = -1$. But if we look at the power series representation of the function

$$g(x) := \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + - \dots,$$

then this is again valid only for $x \in (-1, 1)$, despite there being no obvious reason from the formula for g for the series to break down at the points $x = -1$ and $x = +1$. The mystery will be resolved when we look at the complex functions

$$F(z) = \frac{1}{1-z^2} \text{ and } G(z) = \frac{1}{1+z^2}$$

(whose restriction to $\mathbb{R}$ are the functions f and g, respectively).

(3) **Applications.** Many tools used for solving problems in applications, such as the Fourier/Laplace/z-transform, rely on complex function theory. These tools in turn are useful for example to solve differential equations, such as the Black-Scholes equation for option pricing. (See for example Section 20.6 on pages 441-450 of Shaw's book [**S**].) Such tools are also important in applied subjects such as mathematical physics, biomedical engineering, control theory, signal processing and so on.

(4) **Analytic number theory**. Perhaps surprisingly, many questions about the natural numbers can be answered using complex analytic tools. For example, the Prime Number Theorem says that if $\pi(n)$ denotes the number of primes less than n, then

$$\frac{\pi(n)}{\frac{n}{\log n}} \stackrel{n\to\infty}{\longrightarrow} 1,$$

a proof of which can be given using complex analytic computations with the Riemann-zeta function.

[2] See `http://www.youtube.com/watch?v=egf4m6zVHUI` or `http://www.ams.org/samplings/feature-column/fcarc-harmonic`

Complex Analysis is not complex analysis!

Indeed, it is not very complicated, and there isn't much analysis. The analysis is "soft": there are fewer deltas and epsilons and difficult estimates, once a few key properties of complex differentiable functions are established. The Grand Theorem we mentioned earlier tells us that the subject is radically different from Real Analysis. Indeed, we have seen that a real-valued differentiable function on an open interval (a, b) need not have a continuous derivative. In contrast, a complex differentiable function on an open subset of $\mathbb{C}$ is infinitely many times differentiable! This miracle occurs because complex differentiable functions are "rigid" and they have a lot of structure that is imposed by the demand of complex differentiability. Nevertheless there are enough of them to make the subject nontrivial and interesting!

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Chapter 1

Complex numbers and their geometry

In this chapter, we set the stage for doing complex analysis. We study three main topics:

(1) We will introduce the set of complex numbers, and their arithmetic, making $\mathbb{C}$ into a field, "extending" the usual field of real numbers.

(2) Points in $\mathbb{C}$ can be depicted in the plane $\mathbb{R}^2$, and we will see that the arithmetic in $\mathbb{C}$ has geometric meaning in the plane. This correspondence between $\mathbb{C}$ and points in the plane also allows one to endow $\mathbb{C}$ with the usual Euclidean topology of the plane.

(3) Finally we will study a fundamental function in complex analysis, namely the exponential function. We also look at some elementary functions related to the exponential function, namely trigonometric functions and the logarithm.

1.1. The field of complex numbers

By definition, a *complex number* is an ordered pair of real numbers. The set $\mathbb{R} \times \mathbb{R}$ of all complex numbers is denoted by $\mathbb{C}$. Thus

$$\mathbb{C} = \{z = (x, y) : x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}.$$

For a complex number $z = (x, y) \in \mathbb{C}$, where $x, y \in \mathbb{R}$, the real number x is called the *real part of* z, and y is called the *imaginary part of* z.

We define the operations of *addition* and *multiplication* on $\mathbb{C}$ as follows:

$$\begin{aligned} (x_1, y_1) + (x_2, y_2) &= (x_1 + x_2, y_1 + y_2), \\ (x_1, y_1) \cdot (x_2, y_2) &= (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1), \end{aligned}$$

for complex numbers (x_1, y_1), (x_2, y_2). With these operations, $\mathbb{C}$ is a field, that is,

(F1) $(\mathbb{C}, +)$ is an Abelian group,

(F2) $(\mathbb{C} \setminus \{0\}, \cdot)$ is an Abelian group, and

(F3) the distributive law holds: for $a, b, c \in \mathbb{C}$, $(a + b) \cdot c = a \cdot c + b \cdot c$.

The additive identity is $(0, 0)$, the additive inverse of (x, y) is $(-x, -y)$. The multiplicative identity is $(1, 0)$, and the multiplicative inverse of $(x, y) \in \mathbb{C} \setminus \{(0, 0)\}$ is given by

$$\left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right). \tag{1.1}$$

Exercise 1.1. Check that (1.1) indeed is the inverse of $(x, y) \in \mathbb{C} \setminus \{(0, 0)\}$.

Proposition 1.2. $(\mathbb{C}, +, \cdot)$ *is a field.*

In fact, we can embed $\mathbb{R}$ inside $\mathbb{C}$, and view $\mathbb{R}$ as a subfield of $\mathbb{C}$, that is, one can show that the map

$$x \mapsto (x, 0)$$

sending the real number x to the complex number $(x, 0)$ is an injective field homomorphism. (This just means that the operations of addition and multiplication are preserved by this map, and distinct real numbers are sent to distinct complex numbers.)

But the advantage of working with $\mathbb{C}$ is that while in $\mathbb{R}$ there was no solution $x \in \mathbb{R}$ to the equation

$$x^2 + 1 = 0,$$

now with complex numbers we have

$$(0,1) \cdot (0,1) + (1,0) = (-1,0) + (1,0) = (0,0).$$

If we give a special symbol, say i, to the number $(0,1)$, then the above says that

$$i^2 + 1 = 0$$

(where we have made the usual identification of the real numbers 1 and 0 with their corresponding complex numbers $(1,0)$ and $(0,0)$).

Henceforth, for the complex number (x, y), where x, y are real, we write $x + yi$, since

$$(x,y) = \underbrace{(x,0)}_{\equiv x} + \underbrace{(y,0)}_{\equiv y} \cdot \underbrace{(0,1)}_{\equiv i} = x + yi.$$

Note that as $yi = (y,0) \cdot (0,1) = (0,y) = (0,1) \cdot (y,0) = iy$, we have $x + yi = x + iy$.

Exercise 1.3. Let $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Express $\dfrac{1 + i\tan\theta}{1 - i\tan\theta}$ in the form $x + yi$, where x, y are real.

1.1.1. Historical development of complex numbers. Contrary to popular belief, historically, it wasn't the need for solving *quadratic* equations that lead to complex numbers to be taken seriously, but rather it was *cubic* equations. The gist of this is the following. For the ancient Greeks, mathematics was synonymous with geometry, and in the 16th century, this conception was still dominant. Thus, when one wanted to solve

$$ax^2 + bx + c = 0,$$

one imagined that one was trying to solve geometrically the problem of finding the point of intersection of the parabola $y = ax^2$ with the line $y = -bx - c$. Thus it was easy to dismiss the lack of solvability in reals of a quadratic such as $x^2 + 1 = 0$, since that just reflected the geometric fact that parabola $y = x^2$ did not meet the line $y = -1$. See the picture on the left in Figure 1.

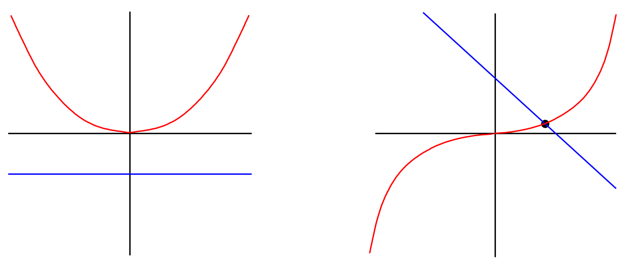


Figure 1. Lack of solvability in reals of $x^2 = -1$ versus the fact that $x^3 = 3px + 2q$ always has a real solution x.

Meanwhile, Cardano (1501-1576) gave a formula for solving the cubic $x^3 = 3px + 2q$, namely,

$$x = \sqrt[3]{q + \sqrt{q^2 - p^3}} + \sqrt[3]{q - \sqrt{q^2 - p^3}}.$$

For example, one can check that for the equation $x^3 = 6x + 6$, with $p = 2$ and $q = 3$, this yields one solution to be $x = \sqrt[3]{4} + \sqrt[3]{2}$. However, note that by the Intermediate Value Theorem, the cubic $y = x^3$ *always* intersects the line $y = 3px + 2q$. See the picture on the right in Figure 1. But for an equation like $x^3 = 15x + 4$, that is when $p = 5$ and $q = 2$, we have $q^2 - p^3 = -121 < 0$, and so Cardano's formula becomes meaningless with real numbers. On the other hand, it can be checked easily that $x = 4$ is a real solution to this equation.

It was some thirty years after Cardano's work appeared, that Bombelli had a "wild thought": that perhaps the solution $x = 4$ could, after all, be obtained from Cardano's formula, via a detour through the use of complex numbers. Note that Cardano's formula in this case gives

$$x = \sqrt[3]{2 + 11i} + \sqrt[3]{2 - 11i}.$$

Bombelli thought that if with the arithmetic of complex numbers it was the case that

$$\sqrt[3]{2 + 11i} = 2 + ni \text{ and } \sqrt[3]{2 - 11i} = 2 - ni,$$

for some real n, then indeed Cardano's formula gives 4. So we need to calculate $(2 + ni)^3$ and set it equal to $2 + 11i$ in order to find n, and this gives $n = 1$. (Check this!) It was in this manner that Bombelli's work on cubic equations established that complex numbers are important, since he showed that perfectly real problems needed complex number arithmetic for their solution.

Exercise 1.4. A field $\mathbb{F}$ is called *ordered* if there is a subset $P \subset \mathbb{F}$, called the *set of positive elements of* $\mathbb{F}$, satisfying the following:

(P1) For all $x, y \in P$, $x + y \in P$.

(P2) For all $x, y \in P$, $x \cdot y \in P$.

(P3) For each $x \in P$, one and only one of the following three statements is true:

$$\underline{1}^\circ \quad x = 0. \qquad \underline{2}^\circ \quad x \in P. \qquad \underline{3}^\circ \ -x \in P.$$

For example, the field of real numbers $\mathbb{R}$ is ordered, since $P := (0, \infty)$ is a set of positive elements of $\mathbb{R}$. (Once one has an ordered set of elements in a field, one can compare the elements of $\mathbb{F}$ by defining a relation $>_P$ in $\mathbb{F}$ by setting $y >_P x$ for $x, y \in \mathbb{F}$ if $y - x \in P$.) Show that $\mathbb{C}$ is not an ordered field.

Hint: Consider $x := i$, and first look at $x \cdot x$.

1.2. Geometric representation of complex numbers

Since $\mathbb{C} = \mathbb{R}^2$, we can identify complex numbers with points in the plane. See Figure 2.

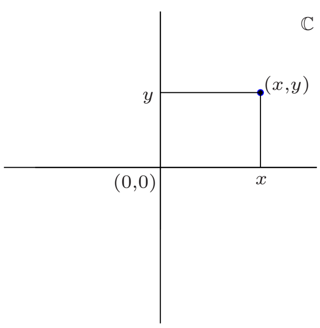


Figure 2. The complex number $x + iy$ in the complex plane.

The complex plane is sometimes called the *Argand*[1] *plane*.

Exercise 1.5. Locate the following points in the complex plane:

$$0, \quad 1, \quad -\frac{3}{2}, \quad i, \quad -\sqrt{2}i, \quad \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}.$$

With this identification of complex numbers with points in the plane, it is clear that complex addition is just addition of vectors in $\mathbb{R}^2$. See Figure 3.

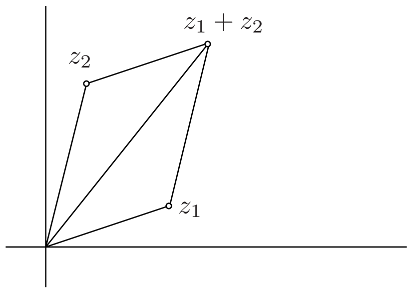


Figure 3. Addition of complex numbers is vector addition in $\mathbb{R}^2$.

We will now see the special geometric meaning of complex multiplication. In order to do this, it will be convenient to use polar coordinates. Thus, let the point $(x, y) \in \mathbb{R}^2$ have polar coordinates $r \geq 0$ and $\theta \in [\pi, \pi)$. Then

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta. \end{aligned}$$

r is the distance of (x, y) to $(0, 0)$, while θ is the angle made by the ray joining $(0, 0)$ to (x, y) with the positive real axis. (If (x, y) is itself $(0, 0)$, we set $\theta = 0$.) Thus we can express the complex number in terms of the polar coordinates (r, θ):

$$x + yi = r\cos\theta + (r\sin\theta)i = r(\cos\theta + i\sin\theta).$$

See Figure 4.

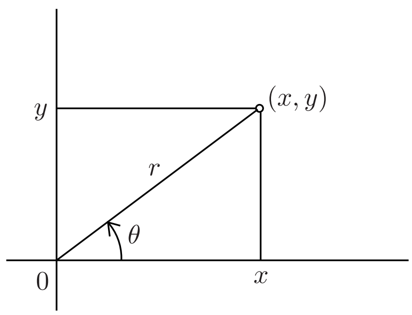


Figure 4. Polar coordinates (r, θ) of $(x, y) \in \mathbb{R}^2$.

Then for two complex numbers expressed in polar coordinates as

$$\begin{aligned} z_1 &= r_1(\cos\theta_1 + i\sin\theta_1), \\ z_2 &= r_2(\cos\theta_2 + i\sin\theta_2), \end{aligned}$$

[1]It is named after Jean-Robert Argand (1768-1822), although it was used earlier by Caspar Wessel (1745-1818).

we have that

$$\begin{aligned} z_1 \cdot z_2 &= r_1(\cos\theta_1 + i\sin\theta_1)\cdot r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1r_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\cos\theta_1\sin\theta_2 + \cos\theta_2\sin\theta_1)) \\ &= r_1r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)). \end{aligned}$$

Thus $z_1 \cdot z_2$ has the polar coordinates $(r_1r_2, \theta_1+\theta_2)$. In other words, the angles z_1 and z_2 make with the positive real axis are added in order to get the angle $z_1 \cdot z_2$ makes with the positive real axis, and the distances to the origin are multiplied to get the distance $z_1 \cdot z_2$ has to the origin. See Figure 5.

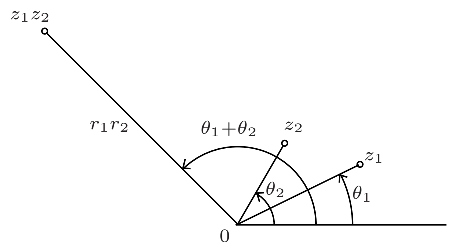


Figure 5. Geometric meaning of complex multiplication: angles get added, distances to the origin get multiplied.

In particular, we have

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

for all $n \in \mathbb{N}$. This is called *de Moivre's formula.*

Exercise 1.6. Recover the trigonometric equality $\cos(3\theta) = 4(\cos\theta)^3 - 3\cos\theta$ using de Moivre's formula.

Exercise 1.7. Express $(1+i)^{10}$ in the form $x+iy$ with real x, y without expanding!

Exercise 1.8. By considering the product $(2+i)(3+i)$, show that $\dfrac{\pi}{4} = \tan^{-1}\dfrac{1}{2} + \tan^{-1}\dfrac{1}{3}$.

Exercise 1.9. *Gaussian integers* are complex numbers of the form $m+in$, where m, n are integers. Thus they are integral lattice points in the complex plane. Show that it is impossible to draw an equilateral triangle such that all vertices are Gaussian integers.
Hint: Rotation of one of the sides should give the other. Recall that $\sqrt{3}$ is irrational.

De Moivre's formula gives an easy way of finding the *nth roots of a complex number z*, that is, complex numbers w that satisfy $w^n = z$. Indeed, we first write $z = r(\cos\theta + i\sin\theta)$ for some $r \geq 0$ and $\theta \in [0, 2\pi)$. Now if $w^n = z$, where $w = \rho(\cos\alpha + i\sin\alpha)$, then

$$w^n = \rho^n(\cos(n\alpha) + i\sin(n\alpha)) = r(\cos\theta + i\sin\theta) = z,$$

and so by equating the distance to the origin on both sides, we obtain $\rho^n = r$. Hence

$$\rho = \sqrt[n]{r}.$$

On the other hand, the angle made with the positive real axis is $n\alpha$, which is in the set

$$\{\cdots, \theta - 4\pi, \theta - 2\pi, \theta, \theta + 2\pi, \theta + 4\pi, \theta + 6\pi, \cdots\}$$

which yields distinct w for

$$\alpha \in \left\{\frac{\theta}{n}, \frac{\theta}{n} + \frac{2\pi}{n}, \frac{\theta}{n} + 2\cdot\frac{2\pi}{n}, \ldots, \frac{\theta}{n} + (n-1)\cdot\frac{2\pi}{n}\right\}.$$

In particular, if $z = 1$, we get the nth roots of unity, which are located at the vertices of a n-sided regular polygon inscribed in a circle. See Figure 6.

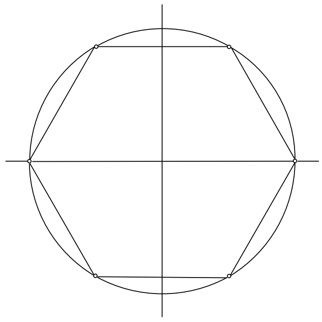


Figure 6. The six 6th roots of unity.

Exercise 1.10. Find all complex numbers w such that $w^4 = -1$. Depict these in the complex plane.

Exercise 1.11. Find all complex numbers z that satisfy $z^6 - z^3 - 2 = 0$.

Exercise 1.12. We know that if a, b, c are *real* numbers such that $a^2 + b^2 + c^2 = ab + bc + ca$, then they must be equal. Indeed, doubling both sides and rearranging gives $(a-b)^2 + (b-c)^2 + (c-a)^2 = 0$, and since each summand is nonnegative, it must be the case that each is 0. On the other hand, now show that if a, b, c are *complex* numbers such that $a^2 + b^2 + c^2 = ab + bc + ca$, then they must lie on the vertices of an equilateral triangle in the complex plane. Explain the real case result in light of this fact.
Hint: If ω is a nonreal cube root of unity, then calculate $((b-a)\omega + (b-c)) \cdot ((b-a)\omega^2 + (b-c))$.

Exercise 1.13. Show, using the geometry of complex numbers, that the line segments joining the centers of opposite external squares described on sides of an arbitrary convex quadrilateral are perpendicular and have equal lengths.

Exercise 1.14. The Binomial Theorem says that if a, b are real numbers and $n \in \mathbb{N}$, then

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}, \text{ where } \binom{n}{k} := \frac{n!}{k!(n-k)!}, \quad k = 0, 1, 2, \dots, n,$$

are the binomial coefficients. The algebraic reasoning leading to this is equally valid if a, b are complex numbers. Show that

$$\binom{3n}{0} + \binom{3n}{3} + \binom{3n}{6} + \cdots + \binom{3n}{3n} = \frac{2^{3n} + 2 \cdot (-1)^n}{3}.$$

Hint: Find $(1+1)^{3n} + (1+\omega)^{3n} + (1+\omega^2)^{3n}$, where ω denotes a nonreal cube root of unity.

The *absolute value* $|z|$ of the complex number $z = x + iy$, where $x, y \in \mathbb{R}$, is defined by

$$|z| = \sqrt{x^2 + y^2}.$$

Note that this is the distance of the complex number z to 0 in the complex plane. By expressing $z_1, z_2 \in \mathbb{C}$ in terms of polar coordinates, or by a direct calculation, it is clear that

$$|z_1 z_2| = |z_1| \cdot |z_2|.$$

Exercise 1.15. Verify the above property by expressing z_1, z_2 using Cartesian coordinates.

The *complex conjugate* $\overline{z}$ of $z = x + iy$ where $x, y \in \mathbb{R}$, is defined by

$$\overline{z} = x - iy.$$

In the complex plane, $\overline{z}$ is obtained by reflecting the point corresponding to z in the real axis.

The following properties are easy to check:

$$\overline{\overline{z}} = z, \quad z\overline{z} = |z|^2, \quad \mathrm{Re}(z) = \frac{z + \overline{z}}{2}, \quad \mathrm{Im}(z) = \frac{z - \overline{z}}{2i}.$$

Exercise 1.16. Verify that the four equalities above hold.

Exercise 1.17. Sketch the following sets in the complex plane:

(1) $\{z \in \mathbb{C} : |z-(1-i)| = 2\}$.
(2) $\{z \in \mathbb{C} : |z-(1-i)| < 2\}$.
(3) $\{z \in \mathbb{C} : 1 < |z-(1-i)| < 2\}$.
(4) $\{z \in \mathbb{C} : \mathrm{Re}(z-(1-i)) = 3\}$.
(5) $\{z \in \mathbb{C} : |\mathrm{Im}(z-(1-i))| < 3\}$.
(6) $\{z \in \mathbb{C} : |z-(1-i)| = |z-(1+i)|\}$.
(7) $\{z \in \mathbb{C} : |z-(1-i)| + |z-(1+i)| = 2\}$.
(8) $\{z \in \mathbb{C} : |z-(1-i)| + |z-(1+i)| < 3\}$.

Exercise 1.18. Prove that for all $z \in \mathbb{C}$, $|z| = |\overline{z}|$, $|\mathrm{Re}(z)| \le |z|$ and $|\mathrm{Im}(z)| \le |z|$. Give geometric interpretations of each.

Exercise 1.19. Suppose that $a, z \in \mathbb{C}$ are such that $|a| < 1$ and $|z| \le 1$. Prove that $\left|\dfrac{z-a}{1-\overline{a}z}\right| \le 1$.

Exercise 1.20. Consider the polynomial p given by $p(z) = c_0 + c_1 z + \cdots + c_d z^d$, where $c_0, c_1, \ldots, c_d \in \mathbb{R}$ and $c_d \neq 0$. Show that if $w \in \mathbb{C}$ is such that $p(w) = 0$, then also $p(\overline{w}) = 0$.

Exercise 1.21. Show that the area enclosed by the triangle formed by 0 and $a, b \in \mathbb{C}$ is given by $\left|\dfrac{\mathrm{Im}(a\overline{b})}{2}\right|$.

Exercise 1.22. Prove for arbitrary complex numbers z_1, z_2, z_3 that $i \det \begin{bmatrix} 1 & z_1 & \overline{z_1} \\ 1 & z_2 & \overline{z_2} \\ 1 & z_3 & \overline{z_3} \end{bmatrix}$ is real.

Exercise 1.23 (Enestrom's Theorem)**.** Let p be the polynomial given by $p(z) = c_0 + c_1 z + \cdots + c_d z^d$, where $d \ge 1$ and $c_0 \ge c_1 \ge c_2 \ge \cdots \ge c_d > 0$. Prove that the zeros of the polynomial p all lie outside the open unit disc with center 0 and radius 1.

Hint: Show that $(1-z)p(z) = 0$ implies that $c_0 = (c_0 - c_1)z + (c_1 - c_2)z^2 + \cdots + (c_{d-1} - c_d)z^d + c_d z^{d+1}$, which is impossible for $|z| < 1$.

1.3. Topology of $\mathbb{C}$

1.3.1. Metric on $\mathbb{C}$. In order to do calculus with complex numbers, we need a notion of distance between a pair of complex numbers. Since $\mathbb{C}$ is just $\mathbb{R}^2$, we use the usual Euclidean distance in $\mathbb{R}^2$ as the metric in $\mathbb{C}$. Thus, for complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, we have

$$d(z_1, z_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = |z_1 - z_2|.$$

By Pythagoras's Theorem, this is the length of the line segment joining the points $(x_1, y_1), (x_2, y_2)$ in $\mathbb{R}^2$.

In particular, the *triangle inequality* holds: $|z_1 + z_2| \le |z_1| + |z_2|$ for $z_1, z_2 \in \mathbb{C}$.

Exercise 1.24. Show that for all $z_1, z_2 \in \mathbb{C}$, $|z_1 - z_2| \ge ||z_1| - |z_2||$.

Just as we did in real analysis, we can now talk about open balls, open sets, closed sets in $\mathbb{C}$. We can also talk about convergent sequences in $\mathbb{C}$, and of compact subsets of $\mathbb{C}$.

Example 1.25. Let z be a complex number with $|z| < 1$. Then the sequence $(z^n)_{n\in\mathbb{N}}$ converges to 0. Indeed, we can write $z = |z|e^{i\theta}$, where $\theta \in (-\pi, \pi]$, and so if $|z| < 1$, we have $z^n = |z|^n e^{i\theta n}$, and $|z^n - 0| = |z|^n \to 0$ as $n \to \infty$. $\Diamond$

Exercise 1.26. Consider the polynomial p given by $p(z) = c_0 + c_1 z + \cdots + c_d z^d$, where $c_0, c_1, \ldots, c_d \in \mathbb{C}$ and $c_d \neq 0$. Show that there exist $M, R > 0$ such that $|p(z)| \ge M|z|^d$ for all $z \in \mathbb{C}$ such that $|z| > R$.

Exercise 1.27. Show that a sequence $(z_n)_{n\in\mathbb{N}}$ of complex numbers is convergent to z_* if and only if the two real sequences $(\mathrm{Re}(z_n))_{n\in\mathbb{N}}$ and $(\mathrm{Im}(z_n))_{n\in\mathbb{N}}$ are convergent respectively to $\mathrm{Re}(z_*)$ and $\mathrm{Im}(z_*)$.

Exercise 1.28. Show that a sequence $(z_n)_{n\in\mathbb{N}}$ of complex numbers is convergent to z_* if and only if $(\overline{z_n})_{n\in\mathbb{N}}$ converges to $\overline{z_*}$.

Exercise 1.29. Prove that $\mathbb{C}$ is complete.

Exercise 1.30. Prove that the map $z \mapsto \mathrm{Re}(z) : \mathbb{C} \to \mathbb{R}$ is continuous.

Exercise 1.31. Prove that $z \mapsto \overline{z} : \mathbb{C} \to \mathbb{C}$ is a homeomorphism.

1.3.2. Domains. In the sequel, the notion of a *path connected open set* will play an important role. We will call an open path-connected subset of $\mathbb{C}$ a *domain*. Let us first explain what we mean by a path connected set.

Definition 1.32. A *path* (or *curve*) in $\mathbb{C}$ is a continuous function $\gamma : [a, b] \to \mathbb{C}$.

A *stepwise path* is a path $\gamma : [a, b] \to \mathbb{C}$ such that there are points

$$t_0 = a < t_1 < \cdots < t_n < t_{n+1} = b$$

such that for each $k = 0, 1, \ldots, n$, the restriction $\gamma : [t_k, t_{k+1}] \to \mathbb{C}$ is a path with either a constant real part or constant imaginary part. See Figure 7.

An open set $U \subset \mathbb{C}$ is called *path connected* if for every $z_1, z_2 \in U$, there is a stepwise path $\gamma : [a, b] \to \mathbb{C}$ such that $\gamma(a) = z_1$, $\gamma(b) = z_2$, and for all $t \in [a, b]$, $\gamma(t) \in U$.

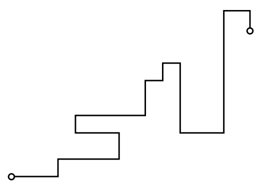


Figure 7. Stepwise path.

(Actually in the above definition of path connected open sets, the restriction of the paths being stepwise paths can be relaxed, but this is an unnecessary diversion for us. So we just live with this definition instead.)

Example 1.33. The open unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ is a domain.

For $r \in (0, 1)$, the open annulus $\mathbb{A}_r := \{z \in \mathbb{C} : r < |z| < 1\}$ is a domain.

The set $S := \{z \in \mathbb{C} : |z| \neq 1\}$ is not a domain, since although it is open, it is not path connected. Indeed, there is no path joining, say 0 and 2: for if there were one such path γ, then by the Intermediate Value Theorem applied to the function $t \mapsto |\gamma(t)| : [a, b] \to \mathbb{R}$, we see that since $|\gamma(a)| = 0 < 1 < 2 = |\gamma(b)|$, there must be a $t_* \in [a, b]$ such that $|\gamma(t_*)| = 1$, but then $\gamma(t_*) \notin S$. $\diamondsuit$

Exercise 1.34. Show that the set $\{z \in \mathbb{C} : \mathrm{Re}(z) \cdot \mathrm{Im}(z) > 1\}$ is open, but not a domain.

Exercise 1.35. Let D be a domain. Set $D^* := \{z \in \mathbb{C} : \overline{z} \in D\}$. Show that D^* is also a domain.

1.4. The exponential function and kith

In this last section, we discuss some basic complex functions: the exponential function, the trigonometric functions and the logarithm. They will serve as counterparts to the familiar functions from calculus, to which they reduce when restricted to the real axis. Some have new interesting properties not possessed by them when the argument is only allowed to be real. We begin with the exponential function.

1.4.1. The exponential e^z.

Definition 1.36 (The complex exponential)**.** For $z = x + iy \in \mathbb{C}$, where x, y are real, we define the *complex exponential*, denoted by e^z, as follows:

$$e^z = e^x(\cos y + i \sin y).$$

First we note that when $y = 0$, the right hand side is simply e^x. So this definition gives an extension of the usual exponential function of a real variable. We will see later on that this is a complex differentiable function everywhere in the complex plane, and it is the only possible extension of the real function $x \mapsto e^x$ having this property of being complex differentiable in the whole complex plane; see Exercise 4.38 on page 63. So we will eventually see that this mysterious looking definition is quite natural. Right now, let us check the following elementary properties:

Proposition 1.37.

(1) *For* $z_1, z_2 \in \mathbb{C}$, $e^{z_1+z_2} = e^{z_1}e^{z_2}$.

(2) *For* $z \in \mathbb{C}$, $e^z \neq 0$, *and* $(e^z)^{-1} = e^{-z}$.

(3) *For* $z \in \mathbb{C}$, $e^{z+2\pi i} = e^z$.

(4) *For* $z \in \mathbb{C}$, $|e^z| = e^{\mathrm{Re}(z)}$.

Proof. (1) If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$\begin{aligned} e^{z_1+z_2} &= e^{(x_1+x_2)+i(y_1+y_2)} = e^{x_1+x_2}(\cos(y_1+y_2) + i\sin(y_1+y_2)) \\ &= e^{x_1}e^{x_2}(\cos y_1 \cos y_2 - \sin y_1 \sin y_2 + i(\sin y_1 \cos y_2 + \cos y_1 \sin y_2)) \\ &= e^{x_1}(\cos y_1 + i \sin y_1)e^{x_2}(\cos y_2 + i \sin y_2) = e^{z_1}e^{z_2}. \end{aligned}$$

(2) From the previous part, we see that $1 = e^0 = e^{z-z} = e^z e^{-z}$, showing that $e^z \neq 0$ and $(e^z)^{-1} = e^{-z}$.

(3) We have $e^{z+2\pi i} = e^z e^{2\pi i} = e^z e^0(\cos(2\pi) + i\sin(2\pi)) = e^z \cdot 1 \cdot (1 + i \cdot 0) = e^z$.

(4) For $x, y \in \mathbb{R}$, $|e^x \cos y + ie^x \sin y| = \sqrt{e^{2x}((\cos y)^2 + (\sin y)^2)} = \sqrt{e^{2x}} = e^x$. So $|e^{x+iy}| = e^x$. $\square$

(3) above shows that $z \mapsto e^z$ is *not* one-to-one, but rather, it is periodic with period $2\pi i$. Figure 8 shows the effect of $z \mapsto e^z$ on horizontal (fixed imaginary part y) and vertical lines (fixed real part x).

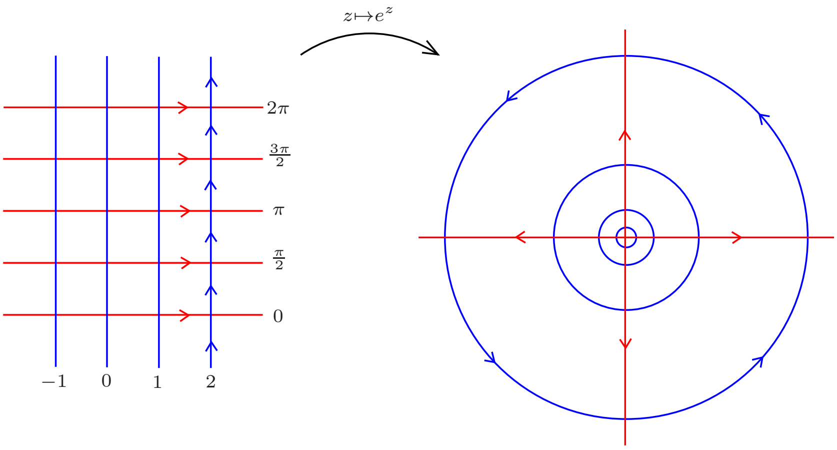


Figure 8. The image of horizontal and vertical lines under the exponential map.

Note that for $z = iy$, where y is real, we have

$$e^{iy} = \cos y + i \sin y.$$

This is the so-called *Euler's formula*. (A special case is the equation $e^{i\pi} + 1 = 0$, relating some of the fundamental numbers in mathematics.) Hence the polar form of a complex number can now be rewritten as

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}.$$

Exercise 1.38. Compute e^z for the following values of z: $i\dfrac{9\pi}{2}$, $3 + \pi i$.

Exercise 1.39. Find all $z \in \mathbb{C}$ that satisfy $e^z = \pi i$.

Exercise 1.40. Plot the curve $t \mapsto e^{it} : [0, 2\pi] \to \mathbb{C}$.

Exercise 1.41. Describe the image of the line $y = x$ under the map $z = x + iy \mapsto e^z$. Proceed as follows: Start with the parametric form $x = t$, $y = t$, and get an expression for the image curve in parametric form. Plot this curve, explaining what happens when t increases, and when $t \to \pm\infty$.

Exercise 1.42. Find the modulus and the real and imaginary parts of e^{z^2} and of $e^{\frac{1}{z}}$ in terms of the real and imaginary parts x, y of $z = x + iy$.

1.4.2. Trigonometric functions. Just as we extended the real exponential function, we now extend the familiar real trigonometric functions to complex trigonometric functions. From the Euler formula we established earlier, we have for real x that

$$\begin{aligned} e^{ix} &= \cos x + i \sin x, \\ e^{-ix} &= \cos x - i \sin x, \end{aligned}$$

which gives

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \text{ and } \sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

This prompts the following definitions: for $z \in \mathbb{C}$, we define

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \text{ and } \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

Several trigonometric identities continue to hold in the complex setting. For instance,

$$\cos(z_1 + z_2) = (\cos z_1)(\cos z_2) - (\sin z_1)(\sin z_2).$$

Indeed, we have

$$\begin{aligned} & (\cos z_1)(\cos z_2) - (\sin z_1)(\sin z_2) \\ = & \left(\frac{e^{iz_1} + e^{-iz_1}}{2}\right)\left(\frac{e^{iz_2} + e^{-iz_2}}{2}\right) - \left(\frac{e^{iz_1} - e^{-iz_2}}{2i}\right)\left(\frac{e^{iz_2} - e^{-iz_2}}{2i}\right) \\ = & \frac{2e^{i(z_1+z_2)} + 2e^{-i(z_1+z_2)}}{4} = \cos(z_1 + z_2). \end{aligned}$$

Exercise 1.43. Verify that for complex z_1, z_2, $\sin(z_1 + z_2) = (\sin z_1)(\cos z_2) + (\cos z_1)(\sin z_2)$.

Also, $(\sin z)^2 + (\cos z)^2 = 1$, since

$$(\sin z)^2 + (\cos z)^2 = \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 + \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 = \frac{(-e^{2iz} + 2 - e^{-2iz}) + (e^{2iz} + 2 + e^{-2iz})}{4} = 1.$$

However, as opposed to the real trigonometric functions which satisfy $|\sin x| \le 1$ and $|\cos x| \le 1$ for real x, $z \mapsto \sin z$ and $z \mapsto \cos z$ are *not* bounded. Indeed, for $z = iy$, where y is real, we have

$$\cos(iy) = \frac{e^{-y} + e^{y}}{2} \text{ and } \sin(iy) = \frac{e^{-y} - e^{y}}{2i},$$

and so $|\cos(iy)|, |\sin(iy)| \to \infty$ as $y \to \pm\infty$.

We will see later that $z \mapsto \cos z$ and $z \mapsto \sin z$ are complex differentiable everywhere in the complex plane.

Exercise 1.44. Show that for $z = x + iy$, where x, y are real,

$$\cos z = (\cos x)(\cosh y) + i(\sin x)(\sinh y) \quad \text{and} \quad |\cos z|^2 = (\cosh y)^2 - (\sin x)^2,$$

where $\cosh y := \dfrac{e^y + e^{-y}}{2}$ and $\sinh y := \dfrac{e^y - e^{-y}}{2}$.

Exercise 1.45. We know that the equation $\cos x = 3$ has no real solution x. However, show that there are complex z that satisfy $\cos z = 3$, and find them all.

1.4.3. Logarithm function. The logarithm function should be the inverse of the exponential function, that is, given a $z \neq 0$, we seek a complex number w such that

$$e^w = z,$$

and then the rule which sends z to w is the sought for logarithm function. However, the moment we find one w such that $e^w = z$, we know that there are infinitely many others, since $e^{w+2\pi in} = e^w = z$ for all $n \in \mathbb{Z}$. So we will fix a particular choice. To do this, let us note that the real part of w is uniquely determined by the equation $e^w = z$: Indeed, $e^{\text{Re}(w)} = |z|$, and so Re(w)= $\log |z|$. If $z = |z|e^{i\theta}$, where $\theta \in (-\pi, \pi]$, then

$$e^w = e^{\text{Re}(w)} e^{i\text{Im}(w)} = |z|e^{i\theta}.$$

Dividing by $e^{\text{Re}(w)} = |z|$ $(\neq 0)$, we obtain $e^{i\text{Im}(w)} = e^{i\theta}$, that is, $e^{i(\text{Im}(w)-\theta)} = 1$. Thus we have $\text{Im}(w) - \theta \in 2\pi\mathbb{Z}$. In particular, the choice $\text{Im}(w) = \theta \in (-\pi, \pi]$ is allowed, and so we have the following.

Definition 1.46. The *principal logarithm* Log z of $z \neq 0$ is defined by

$$\text{Log } z = \log |z| + i\text{Arg } z,$$

where Arg z is called the *principal argument of* z, defined to be a number in $(-\pi, \pi]$ such that

$$z = |z|e^{i\text{Arg } z}.$$

Of course, had we chosen the argument θ such that $z = |z|e^{i\theta}$ to lie in a different interval $(a, 2\pi + a]$ or $[a, 2\pi + a)$ for some other a, we would have obtained a different well-defined notion of the logarithm (which would also be equally legitimate).

One can now also talk about a^b, where a, b are complex numbers, and we define the *principal value of* a^b to be

$$a^b := e^{b\text{Log}(a)}.$$

Example 1.47. The principal value of i^i is $e^{i\text{Log}(i)} = e^{i(\log |i| + i\text{Arg } i)} = e^{i(0+i\frac{\pi}{2})} = e^{i(i\frac{\pi}{2})} = e^{-\frac{\pi}{2}}$. $\Diamond$

Exercise 1.48. Find Log$(1 + i)$.

Exercise 1.49. Find Log(-1) and Log(1). Show that Log(z^2) isn't always equal to 2Log(z).

Exercise 1.50. Find the image of the annulus $\{z \in \mathbb{C} : 1 < |z| < e\}$ under the principal logarithm.

Exercise 1.51. Show that the principal logarithm is not continuous on $\mathbb{C} \setminus \{0\}$. Later on we will see that it is complex differentiable in the domain $\mathbb{C} \setminus (-\infty, 0]$ (that is, the complex plane with a "cut" along the nonpositive real axis), and in particular, continuous there.

Exercise 1.52. Find the principal value of $(1 + i)^{1-i}$.

Exercise 1.53. Depict the points in the set $\left\{z \in \mathbb{C} : z \neq 0,\ \dfrac{\pi}{4} < \text{Arg}(z) < \dfrac{\pi}{3}\right\}$ in the complex plane.

Chapter 2

Complex differentiability

In this chapter we will learn three main things:

(1) The definition of complex differentiability.

(2) The Cauchy-Riemann equations: these are PDEs that are satisfied by the the real and imaginary parts of a holomorphic function.

(3) The geometric meaning of complex differentiability: locally the map is an “amplitwist”, namely an amplification together with a twist (a rotation).

2.1. Complex differentiability

Definition 2.1.

(1) Let U be an open subset of $\mathbb{C}$, and let $z_0 \in U$. A function $f : U \to \mathbb{C}$ is said to be *complex differentiable at* z_0 if there exists a complex number, say $f'(z_0) \in \mathbb{C}$, such that

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0),$$

that is, for every $\epsilon > 0$, there is a $\delta > 0$ such that whenever $0 < |z - z_0| < \delta$, $z \in U$ and we have

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \epsilon.$$

We sometimes denote $f'(z_0)$ instead by $\dfrac{df}{dz}(z_0)$.

(2) A function $f : U \to \mathbb{C}$ which is complex differentiable at all points of the open set U is called *holomorphic in* U.

(3) A function that is holomorphic in $\mathbb{C}$ is called *entire*.

Let us look at a simple example of an entire function.

Example 2.2. Consider the function $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = z^2$ ($z \in \mathbb{C}$). We show that f is entire, and that for $f'(z_0) = 2z_0$ for $z_0 \in \mathbb{C}$. Let $\epsilon > 0$. Set $\delta := \epsilon > 0$. Then whenever $z \in \mathbb{C}$ satisfies $0 < |z - z_0| < \delta$, we have

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - 2z_0 \right| = \left| \frac{z^2 - z_0^2}{z - z_0} - 2z_0 \right| = |z + z_0 - 2z_0| = |z - z_0| < \delta = \epsilon.$$

As $z_0 \in \mathbb{C}$ was arbitrary, f is entire. $\Diamond$

On the other hand, here is an example of a natural mapping which is *not* complex differentiable.

Example 2.3. Consider the function $g : \mathbb{C} \to \mathbb{C}$ defined by $g(z) = \overline{z}$ ($z \in \mathbb{C}$). We show that g is differentiable nowhere. Suppose that g is differentiable at $z_0 \in \mathbb{C}$. Let $\epsilon := \frac{1}{2} > 0$. Then there exists $\delta > 0$ such that whenever z satisfies $0 < |z - z_0| < \delta$, we have

$$\left|\frac{g(z) - g(z_0)}{z - z_0} - g'(z_0)\right| = \left|\frac{\overline{z} - \overline{z_0}}{z - z_0} - g'(z_0)\right| < \epsilon.$$

In particular, taking $z = z_0 + \dfrac{\delta}{2}$, we have $0 < |z - z_0| < \delta$, and so

$$\left|\frac{\overline{z} - \overline{z_0}}{z - z_0} - g'(z_0)\right| = \left|\frac{\delta/2}{\delta/2} - g'(z_0)\right| = |1 - g'(z_0)| < \epsilon. \tag{2.1}$$

Also, taking $z = z_0 + i\dfrac{\delta}{2}$, we have $0 < |z - z_0| < \delta$, and so

$$\left|\frac{\overline{z} - \overline{z_0}}{z - z_0} - g'(z_0)\right| = \left|\frac{-i\delta/2}{i\delta/2} - g'(z_0)\right| = |-1 - g'(z_0)| = |1 + g'(z_0)| < \epsilon. \tag{2.2}$$

It follows from (2.1) and (2.2) that

$$2 = |1 - g'(z_0) + 1 + g'(z_0)| \le |1 - g'(z_0)| + |1 + g'(z_0)| < \epsilon + \epsilon = 2\epsilon = 2 \cdot \frac{1}{2} = 1,$$

a contradiction. So g is not differentiable at z_0. $\Diamond$

Exercise 2.4. Show that $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = |z|^2$ for $z \in \mathbb{C}$, is complex differentiable at 0 and that $f'(0) = 0$. We will see later (in Exercise 2.28) that f is not complex differentiable at any nonzero complex number.

Exercise 2.5. Let D be a domain, and $f : D \to \mathbb{C}$ be holomorphic in D. Set $D^* := \{z \in \mathbb{C} : \overline{z} \in D\}$, and define $f^* : D^* \to \mathbb{C}$ by $f^*(z) = \overline{f(\overline{z})}$ ($z \in D^*$). Prove that f^* is holomorphic in D^*.

The following lemma is useful to prove elementary facts about complex differentiation. Roughly speaking, the result says that for a complex differentiable function f with complex derivative L at z_0, $f(z) - f(z_0) - L \cdot (z - z_0)$ goes to 0 "faster than $z - z_0$".

Lemma 2.6. *Let U be an open set in $\mathbb{C}$, $z_0 \in U$, and $f : U \to \mathbb{C}$. Then f is complex differentiable at z_0 if and only if there is a $r > 0$, a complex number L, and a complex valued function h defined on the open disc $D(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\} \subset U$ such that*

$$f(z) = f(z_0) + (L + h(z))(z - z_0) \quad \textit{for } |z - z_0| < r,$$

and $\lim\limits_{z \to z_0} h(z) = 0$. Moreover, then $f'(z_0) = L$.

Proof. (If) For $z \ne z_0$, we have, upon rearranging, that

$$\frac{f(z) - f(z_0)}{z - z_0} - L = h(z) \stackrel{z \to z_0}{\longrightarrow} 0$$

and so f is complex differentiable at z_0, and $f'(z_0) = L$.

(Only if) Now suppose that f is complex differentiable at z_0. Then there is a $\delta_1 > 0$ such that whenever $0 < |z - z_0| < \delta_1$, $z \in U$ and

$$\left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| < 1.$$

Set $r := \delta_1$, and define $h : D(z_0, r) \to \mathbb{C}$ by

$$h(z) = \begin{cases} \dfrac{f(z) - f(z_0)}{z - z_0} - f'(z_0) & \text{if } z \ne z_0, \\ 0 & \text{if } z = z_0. \end{cases}$$

Then clearly $f(z) = f(z_0) + (f'(z_0) + h(z))(z - z_0)$ holds whenever $|z - z_0| < r$. If $\epsilon > 0$, then there is a $\delta > 0$ (which can be chosen smaller than r) such that whenever $0 < |z - z_0| < \delta$, we have

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| \;(= |h(z) - 0|) < \epsilon.$$

This completes the proof. $\square$

For example, using this lemma, we see that holomorphic functions must be continuous.

Exercise 2.7. Let D be a domain in $\mathbb{C}$. Show that if $f : D \to \mathbb{C}$ is complex differentiable at $z_0 \in D$, then f is continuous at z_0. (Later on, we will show that if f is holomorphic in D, then in fact it is infinitely many times differentiable in D!)

Using Lemma 2.6, it is also easy to show the following.

Proposition 2.8. *Let U be an open subset of $\mathbb{C}$. Let $f, g : U \to \mathbb{C}$ be complex differentiable functions at $z_0 \in U$.*

(1) *$f + g$ is complex differentiable at z_0 and $(f + g)'(z_0) = f'(z_0) + g'(z_0)$.*
(Here $f + g : U \to \mathbb{C}$ is the function defined by $(f + g)(z) = f(z) + g(z)$ for $z \in U$.)

(2) *If $\alpha \in \mathbb{C}$, then $\alpha \cdot f$ is complex differentiable and $(\alpha \cdot f)'(z_0) = \alpha f'(z_0)$.*
(Here $\alpha \cdot f : U \to \mathbb{C}$ is the function defined by $(\alpha \cdot f)(z) = \alpha f(z)$ for $z \in U$.)

(3) *fg is complex differentiable at z_0 and $(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$.*
(Here $fg : U \to \mathbb{C}$ is the function defined by $(fg)(z) = f(z)g(z)$ for $z \in U$.)

Remark 2.9. Let U be an open subset of $\mathbb{C}$, and let $H(U)$ denote the set of all holomorphic functions in U. Then it follows from the above that $H(U)$ is a complex vector space with pointwise operations.

On the other hand, the third statement above shows that the pointwise product of two holomorphic functions is again holomorphic, and so $H(U)$ also has the structure of a ring with pointwise addition and multiplication.

Example 2.10. It is easy to see that if $f(z) := z$ $(z \in \mathbb{C})$, then $f'(z) = 1$. Using the rule for complex differentiation of a pointwise product of two holomorphic functions, it follows by induction that for all $n \in \mathbb{N}$, $z \mapsto z^n$ is entire, and

$$\frac{d}{dz} z^n = n z^{n-1}.$$

In particular all polynomials are entire. $\Diamond$

Exercise 2.11. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $H(\mathbb{D})$ denote the complex vector space of all holomorphic functions in $\mathbb{D}$ with pointwise operations. Is $H(\mathbb{D})$ finite dimensional?

Exercise 2.12. Let U be an open subset of $\mathbb{C}$, and let $f : U \to \mathbb{C}$ be such that $f(z) \neq 0$ for $z \in U$ and f is holomorphic in U. Prove that the function

$$\frac{1}{f} : U \to \mathbb{C}$$

defined by $\left(\frac{1}{f}\right)(z) = \frac{1}{f(z)}$ for all $z \in U$, is holomorphic, and that $\left(\frac{1}{f}\right)'(z) = -\frac{f'(z)}{(f(z))^2}$ $\quad (z \in U)$.

Exercise 2.13. Show that in $\mathbb{C} \setminus \{0\}$, for each $m \in \mathbb{Z}$, $\frac{d}{dz} z^m = m z^{m-1}$.

Proposition 2.14 (Chain rule)**.** *Let D_f, D_g be domains, and let $f : D_f \to \mathbb{C}$ and $g : D_g \to \mathbb{C}$ be holomorphic functions such that $f(D_f) \subset D_g$. Then their composition $g \circ f : D_f \to \mathbb{C}$, defined by*

$$(g \circ f)(z) = g(f(z)) \quad (z \in D_f),$$

is holomorphic in D_f and $(g \circ f)'(z) = g'(f(z))f'(z)$ for all $z \in D_f$.

Proof. Let $z_0 \in D_f$. Then $f(z_0) \in D_g$. From the complex differentiability of f at z_0 and that of g at $f(z_0)$, we know that there are functions h_f and h_g, defined in the discs $D(z_0, r_f) \subset D_f$ and $D(f(z_0), r_g) \subset D_g$ such that

$$\begin{aligned} f(z) - f(z_0) &= (f'(z_0) + h_f(z))(z - z_0), \\ g(w) - g(f(z_0)) &= (g'(f(z_0)) + h_g(w))(w - f(z_0)), \end{aligned}$$

and

$$\lim_{z \to z_0} h_f(z) = 0, \qquad \lim_{w \to f(z_0)} h_g(w) = 0.$$

But it follows from the continuity of f at z_0 that when z is close to z_0, $w := f(z)$ is close to $f(z_0)$, and so if $z \neq z_0$, but close to z_0, we have

$$\frac{(g \circ f)(z) - (g \circ f)(z_0)}{z - z_0} = (g'(f(z_0)) + h_g(f(z)))(f'(z_0) + h_f(z)),$$

and so the claim follows. □

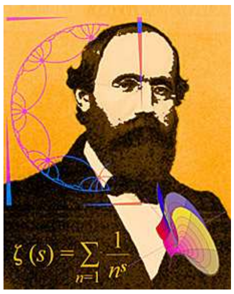
Exercise 2.15. Assuming that e^z is entire and that

$$\frac{d}{dz} e^z = e^z$$

(we will prove this later), show that $e^{-\frac{1+z}{1-z}}$ is holomorphic in the unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, and find its derivative.

2.2. Cauchy-Riemann equations

We now prove the main result in this chapter, which says roughly that a function $f = u + iv$ is holomorphic if and only if its real and imaginary parts (viewed as real valued functions living in an open subset of $\mathbb{R}^2$) satisfy a pair of partial differential equations, called the Cauchy-Riemann equations.

Cauchy-Riemann

Theorem 2.16. *Let U be an open subset of $\mathbb{C}$ and let $f : U \to \mathbb{C}$ be complex differentiable at $z_0 = x_0 + iy_0 \in U$. Then the functions*

$$\begin{aligned} (x, y) &\mapsto u(x, y) := \mathrm{Re}(f(x + iy)) : U \to \mathbb{R} \text{ and} \\ (x, y) &\mapsto v(x, y) := \mathrm{Im}(f(x + iy)) : U \to \mathbb{R} \end{aligned}$$

are differentiable at (x_0, y_0) and

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) \quad \text{and} \quad \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0). \tag{2.3}$$

(The two PDEs in (2.3) are called the *Cauchy-Riemann equations*.)

Proof. (The idea of the proof is easy, we just let (x, y) tend to (x_0, y_0) by first keeping y fixed at y_0, and then by keeping x fixed at x_0, and look at what this gives us.)

Let $z_0 = (x_0, y_0) \in U$. Let $\epsilon > 0$. Then there is $\delta > 0$ such that whenever $0 < |z - z_0| < \delta$, we have $z \in U$ and

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \epsilon.$$

Hence if $x \in \mathbb{R}$ is such that $0 < |x - x_0| < \delta$, then $z := x + iy_0$ satisfies $z - z_0 = x - x_0$ and so $0 < |z - z_0| < \delta$. Thus

$$\begin{aligned} \left| \frac{u(x, y_0) - u(x_0, y_0)}{x - x_0} - \mathrm{Re}(f'(z_0)) \right| &= \left| \mathrm{Re} \left(\frac{f(x + iy_0) - f(x_0 + iy_0)}{z - z_0} - f'(z_0) \right) \right| \\ &\leq \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \epsilon. \end{aligned}$$

Thus the partial derivative $\dfrac{\partial u}{\partial x}(x_0, y_0) = \lim\limits_{x \to x_0} \dfrac{u(x, y_0) - u(x_0, y_0)}{x - x_0} = \mathrm{Re}(f'(z_0))$.

Similarly, we also have $\dfrac{\partial v}{\partial x}(x_0, y_0) = \lim\limits_{x \to x_0} \dfrac{v(x, y_0) - v(x_0, y_0)}{x - x_0} = \mathrm{Im}(f'(z_0))$. Thus

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i \frac{\partial v}{\partial x}(x_0, y_0). \tag{2.4}$$

Now take $y \in \mathbb{R}$ such that $0 < |y - y_0| < \delta$, then $z := x_0 + iy$ satisfies $z - z_0 = i(y - y_0)$ and so $0 < |z - z_0| < \delta$. Thus

$$\begin{aligned} \left| \frac{u(x_0, y) - u(x_0, y_0)}{y - y_0} + \mathrm{Im}(f'(z_0)) \right| &= \left| \mathrm{Im} \left(-\frac{f(x_0 + iy) - f(x_0 + iy_0)}{z - z_0} + f'(z_0) \right) \right| \\ &\leq \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \epsilon. \end{aligned}$$

Thus the partial derivative

$$\frac{\partial u}{\partial y}(x_0, y_0) = \lim_{y \to y_0} \frac{u(x_0, y) - u(x_0, y_0)}{y - y_0} = -\mathrm{Im}(f'(z_0)).$$

Similarly,

$$\begin{aligned} \left| \frac{v(x_0, y) - v(x_0, y_0)}{y - y_0} - \mathrm{Re}(f'(z_0)) \right| &= \left| \mathrm{Re} \left(\frac{f(x_0 + iy) - f(x_0 + iy_0)}{z - z_0} - f'(z_0) \right) \right| \\ &\leq \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \epsilon. \end{aligned}$$

Thus the partial derivative $\dfrac{\partial v}{\partial y}(x_0, y_0) = \lim\limits_{y \to y_0} \dfrac{v(x_0, y) - v(x_0, y_0)}{y - y_0} = \mathrm{Re}(f'(z_0))$. Hence

$$f'(z_0) = \frac{\partial v}{\partial y}(x_0, y_0) - i \frac{\partial u}{\partial y}(x_0, y_0). \tag{2.5}$$

From (2.4) and (2.5), it follows that

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) \quad \text{and} \quad \frac{\partial v}{\partial x}(x_0, y_0) = -\frac{\partial u}{\partial y}(x_0, y_0).$$

Finally, we have for $z=(x,y)$ satisfying $0<|z-z_0|<\delta$ that

$$\begin{aligned}
&\frac{\left|u(x,y)-u(x_0,y_0)-\left(\frac{\partial u}{\partial x}(x_0,y_0)\right)\cdot(x-x_0)-\left(\frac{\partial u}{\partial y}(x_0,y_0)\right)\cdot(y-y_0)\right|}{\|(x,y)-(x_0,y_0)\|_2}\\
=&\frac{\left|u(x,y)-u(x_0,y_0)-\left(\left(\frac{\partial u}{\partial x}(x_0,y_0)\right)\cdot(x-x_0)-\left(\frac{\partial v}{\partial x}(x_0,y_0)\right)\cdot(y-y_0)\right)\right|}{\|(x,y)-(x_0,y_0)\|_2}\\
=&\frac{|\mathrm{Re}(f(z)-f(z_0)-f'(z_0)(z-z_0))|}{|z-z_0|}<\epsilon.
\end{aligned}$$

Thus u is differentiable at (x_0,y_0). Similarly, v is also differentiable at (x_0,y_0). □

Remark 2.17. We will see later on that in fact the real and imaginary parts of a holomorphic function are infinitely many times differentiable.

Let us revisit Example 2.3.

Example 2.18. For the function $g:\mathbb{C}\to\mathbb{C}$ defined by $g(z)=\overline{z}$ $(z\in\mathbb{C})$, we have that

$$\begin{aligned}
u(x,y) &= \mathrm{Re}(g(x+iy))=\mathrm{Re}(x-iy)=x,\\
v(x,y) &= \mathrm{Im}(g(x+iy))=\mathrm{Im}(x-iy)=-y.
\end{aligned}$$

Thus $\frac{\partial u}{\partial x}(x,y)=1\neq -1=\frac{\partial v}{\partial y}(x,y)$. This shows that the Cauchy-Riemann equations can't hold at any point. So we recover our previous observation that g is differentiable nowhere. ◊

Exercise 2.19. Consider Exercise 2.4 again. Show that f is not differentiable at any point of the open set $\mathbb{C}\setminus\{0\}$.

One also has the following converse to Theorem 2.16. This is a very useful result to check the holomorphicity of functions.

Theorem 2.20. *Let U be an open subset of $\mathbb{C}$. Let $f:U\to\mathbb{C}$ be such that the functions*

$$\begin{aligned}
(x,y) &\mapsto u(x,y):=\mathrm{Re}(f(x+iy)):U\to\mathbb{R} \text{ and}\\
(x,y) &\mapsto v(x,y):=\mathrm{Im}(f(x+iy)):U\to\mathbb{R}
\end{aligned}$$

are continuously differentiable in U and the Cauchy-Riemann equations hold:

$$\text{for all } (x,y)\in U,\quad \frac{\partial u}{\partial x}(x,y)=\frac{\partial v}{\partial y}(x,y)\quad\text{and}\quad \frac{\partial u}{\partial y}(x,y)=-\frac{\partial v}{\partial x}(x,y).$$

Then f is holomorphic in U and $f'(x+iy)=\frac{\partial u}{\partial x}(x,y)+i\frac{\partial v}{\partial x}(x,y)$ for $x+iy\in U$.

Proof. Let $z_0=x_0+iy_0\in U$. Let $\epsilon>0$. Let $\delta>0$ be such that whenever $|z-z_0|<\delta$, we have $z=x+iy\in U$ and

$$\left|\frac{\partial u}{\partial x}(x,y)-\frac{\partial u}{\partial x}(x_0,y_0)\right|+\left|\frac{\partial v}{\partial x}(x,y)-\frac{\partial v}{\partial x}(x_0,y_0)\right|<\epsilon. \tag{2.6}$$

Fix such a z distinct from z_0. Define $\varphi:(-1,1)\to\mathbb{R}^2$ by

$$\varphi(t)=\begin{bmatrix} u(x_0+t(x-x_0),y_0+t(y-y_0))\\ v(x_0+t(x-x_0),y_0+t(y-y_0))\end{bmatrix}.$$

Then with $p(t) := (x_0 + t(x - x_0), y_0 + t(y - y_0))$, we have

$$\varphi'(t) = \begin{bmatrix} \frac{\partial u}{\partial x}(p(t))(x-x_0) + \frac{\partial u}{\partial y}(p(t))(y-y_0) \\ \frac{\partial v}{\partial x}(p(t))(x-x_0) + \frac{\partial v}{\partial y}(p(t))(y-y_0) \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x}(p(t)) & -\frac{\partial v}{\partial x}(p(t)) \\ \frac{\partial v}{\partial x}(p(t)) & \frac{\partial u}{\partial x}(p(t)) \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix},$$

where we have used the Cauchy-Riemann equations to get the last expression. Thus

$$\varphi'(t) = \begin{bmatrix} \mathrm{Re}\left(\left(\frac{\partial u}{\partial x}(p(t)) + i\frac{\partial v}{\partial x}(p(t))\right)(z - z_0)\right) \\ \mathrm{Im}\left(\left(\frac{\partial u}{\partial x}(p(t)) + i\frac{\partial v}{\partial x}(p(t))\right)(z - z_0)\right) \end{bmatrix}.$$

Let φ_1, φ_2 be the two scalar-valued components of φ. Setting

$$A := \int_0^1 \frac{\partial u}{\partial x}(p(t))dt \quad \text{and} \quad B := \int_0^1 \frac{\partial v}{\partial x}(p(t))dt,$$

we have using the Fundamental Theorem of Integral Calculus that

$$\varphi_1(1) - \varphi_1(0) = \int_0^1 \varphi_1'(t)dt = \mathrm{Re}((A + iB)(z - z_0)).$$

Similarly, also $\varphi_2(1) - \varphi_2(0) = \mathrm{Im}((A + iB)(z - z_0))$. Thus

$$f(z) - f(z_0) = \varphi_1(1) - \varphi_1(0) + i(\varphi_2(1) - \varphi_2(0)) = (A + iB)(z - z_0),$$

and so

$$\begin{aligned} &\frac{f(z) - f(z_0)}{z - z_0} - \left(\frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0)\right) \\ = \; &\int_0^1 \left(\frac{\partial u}{\partial x}(p(t)) - \frac{\partial u}{\partial x}(p(0))\right) dt + i\int_0^1 \left(\frac{\partial v}{\partial x}(p(t)) - \frac{\partial v}{\partial x}(p(0))\right) dt. \end{aligned}$$

By (2.6), it follows that

$$\left|\frac{f(z) - f(z_0)}{z - z_0} - \left(\frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0)\right)\right| < \epsilon.$$

This holds for all z satisfying $0 < |z - z_0| < \delta$, and so f is complex differentiable at z_0 and

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0).$$

This completes the proof. □

Let us revisit Example 2.2 again.

Example 2.21. For the function $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = z^2$ ($z \in \mathbb{C}$) we have that

$$\begin{aligned} u(x, y) &= \mathrm{Re}(f(x + iy)) = \mathrm{Re}(x^2 - y^2 + 2xyi) = x^2 - y^2, \\ v(x, y) &= \mathrm{Im}(f(x + iy)) = \mathrm{Im}(x^2 - y^2 + 2xyi) = 2xy. \end{aligned}$$

Thus

$$\begin{aligned} \frac{\partial u}{\partial x}(x, y) &= 2x = \frac{\partial v}{\partial y}(x, y), \\ \frac{\partial u}{\partial y}(x, y) &= -2y = -\frac{\partial v}{\partial x}(x, y), \end{aligned}$$

which shows that the Cauchy-Riemann equations hold in $\mathbb{C}$. So we recover our previous observation that f is entire, and since

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0) = 2x_0 + 2y_0 i = 2z_0,$$

we also obtain that $f'(z) = 2z$ for $z \in \mathbb{C}$. ◊

Example 2.22. For the function $g : \mathbb{C} \to \mathbb{C}$ defined by $g(z) = e^z$ $(z \in \mathbb{C})$ we have that

$$\begin{aligned} u(x,y) &= \operatorname{Re}(g(x+iy)) = \operatorname{Re}(e^x(\cos y + i \sin y)) = e^x \cos y, \\ v(x,y) &= \operatorname{Im}(g(x+iy)) = \operatorname{Im}(e^x(\cos y + i \sin y)) = e^x \sin y. \end{aligned}$$

Thus

$$\begin{aligned} \frac{\partial u}{\partial x}(x,y) &= e^x \cos y &= \frac{\partial v}{\partial y}(x,y), \\ \frac{\partial u}{\partial y}(x,y) &= -e^x \sin y &= -\frac{\partial v}{\partial x}(x,y), \end{aligned}$$

which shows that the Cauchy-Riemann equations hold in $\mathbb{C}$. So we arrive at the important result that e^z is entire, and since

$$g'(z_0) = \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0) = e^x \cos y + ie^x \sin y = e^z,$$

we also obtain that $\frac{d}{dz}e^z = e^z$ for $z \in \mathbb{C}$. Hence from Proposition 2.14, also the trigonometric functions

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

are entire functions, and moreover,

$$\begin{aligned} \frac{d}{dz}\sin z &= \frac{ie^{iz} - (-i)e^{-iz}}{2i} = \frac{e^{iz} + e^{-iz}}{2} = \cos z, \text{ and} \\ \frac{d}{dz}\cos z &= \frac{ie^{iz} + (-i)e^{-iz}}{2} = -\frac{e^{iz} - e^{-iz}}{2i} = -\sin z. \end{aligned}$$

$\Diamond$

Example 2.23. We will show that $\frac{d}{dz}\operatorname{Log}(z) = \frac{1}{z}$ for $z \in \mathbb{C} \setminus (-\infty, 0]$.

First note that when $z, z_0 \in \mathbb{C} \setminus (-\infty, 0]$ are distinct, then $\operatorname{Log}(z) \neq \operatorname{Log}(z_0)$. (Why?) Let $\epsilon > 0$. Set

$$\epsilon_1 := \min\left\{\frac{|z_0|}{2}, \frac{|z_0|^2}{2}\epsilon\right\}.$$

Since e^w is differentiable at $w_0 := \operatorname{Log}(z_0)$, there is a $\delta_1 > 0$ such that whenever

$$0 < |w - w_0| = |w - \operatorname{Log}(z_0)| < \delta_1,$$

we have

$$\left|\frac{e^w - e^{w_0}}{w - w_0} - e^{w_0}\right| = \left|\frac{e^w - z_0}{w - \operatorname{Log}(z_0)} - z_0\right| < \epsilon_1.$$

But by the continuity and injectivity of Log in $\mathbb{C} \setminus (-\infty, 0]$, there exists a $\delta > 0$ such that whenever $0 < |z - z_0| < \delta$, we have

$$0 < |\operatorname{Log}(z) - \operatorname{Log}(z_0)| < \delta_1.$$

Thus with $w := \operatorname{Log}(z)$, where z satisfies $0 < |z - z_0| < \delta$, we have $0 < |w - w_0| < \delta_1$, and so

$$\left|\frac{z - z_0}{\operatorname{Log}(z) - \operatorname{Log}(z_0)} - z_0\right| < \epsilon_1.$$

But then $\left|\frac{z - z_0}{\operatorname{Log}(z) - \operatorname{Log}(z_0)}\right| \geq |z_0| - \epsilon_1 \geq \frac{|z_0|}{2}$. Hence whenever $0 < |z - z_0| < \delta$, we have

$$\left|\frac{\operatorname{Log}(z) - \operatorname{Log}(z_0)}{z - z_0} - \frac{1}{z_0}\right| < \frac{\epsilon_1}{|z_0|\left(\left|\frac{z - z_0}{\operatorname{Log}(z) - \operatorname{Log}(z_0)}\right|\right)} \leq \frac{\epsilon_1}{|z_0|\frac{|z_0|}{2}} = \frac{2\epsilon_1}{|z_0|^2} < \epsilon.$$

Thus Log is holomorphic in $\mathbb{C} \setminus (-\infty, 0]$ and moreover, $\frac{d}{dz}\operatorname{Log}(z) = \frac{1}{z}$. $\Diamond$

We now consider an example illustrating the fact that the assumption of differentiability of u, v in Theorem 2.20 (as opposed to mere existence of partial derivatives of u, v satisfying the Cauchy-Riemann equations), is not superfluous.

Example 2.24. Consider the function $f : \mathbb{C} \to \mathbb{C}$ given by

$$f(x+iy) = \frac{xy(x+iy)}{x^2+y^2}$$

if $x + iy \neq 0$, and $f(0) = 0$. We have that for nonzero $(x, y) \in \mathbb{R}^2$,

$$\begin{aligned} u(x,y) &= \operatorname{Re}(f(x+iy)) = \frac{x^2y}{x^2+y^2}, \\ v(x,y) &= \operatorname{Im}(f(x+iy)) = \frac{xy^2}{x^2+y^2}, \end{aligned}$$

and $u(0, 0) = v(0, 0) = 0$. Thus

$$\begin{aligned} \frac{\partial u}{\partial x}(0,0) &= 0 = \frac{\partial v}{\partial y}(0,0), \\ \frac{\partial u}{\partial y}(0,0) &= 0 = -\frac{\partial v}{\partial x}(0,0), \end{aligned}$$

which shows that the Cauchy-Riemann equations hold at the point $(0, 0)$. However, the function is not complex differentiable at 0, since if it were, we would have

$$f'(0) = \frac{\partial u}{\partial x}(0,0) + i\frac{\partial v}{\partial x}(0,0) = 0 + i0 = 0,$$

and for $\epsilon = \dfrac{1}{4}$, there would exist a corresponding δ such that whenever $0 < |z - 0| = |x + iy| < \delta$, we would have

$$\left| \frac{f(z)-f(0)}{z-0} - f'(0) \right| = \left| \frac{xy}{x^2+y^2} \right| < \epsilon,$$

but taking $x + iy = \dfrac{\delta}{2} + i\dfrac{\delta}{2}$, we arrive at the contradiction that

$$\frac{1}{2} = \left| \frac{xy}{x^2+y^2} \right| < \epsilon = \frac{1}{4}.$$

This shows that f is not complex differentiable at 0.

We note that there is no contradiction to Theorem 2.20, since for example u is not differentiable at $(0, 0)$. It it were, its derivative at $(0, 0)$ would have to be the linear transformation

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} \frac{\partial u}{\partial x}(0,0) & \frac{\partial u}{\partial x}(0,0) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0.$$

But then with $\epsilon := \frac{1}{3} > 0$, there must exist a $\delta > 0$ such that whenever $0 < \|(x, y) - (0, 0)\|_2 < \delta$, we would have

$$\frac{|u(x,y) - u(0,0) - 0((x,y)-(0,0))|}{\|(x,y)-(0,0)\|_2} = \frac{x^2y}{(x^2+y^2)^{\frac{3}{2}}} < \epsilon = \frac{1}{3}.$$

But then with $(x, y) = \left(\dfrac{\delta}{2}, \dfrac{\delta}{2}\right)$, we have $\|(x, y) - (0, 0)\|_2 = \dfrac{\delta}{\sqrt{2}} < \delta$, and so we must have

$$\frac{x^2y}{(x^2+y^2)^{\frac{3}{2}}} = \frac{\frac{\delta^2}{4} \cdot \frac{\delta}{2}}{\left(\frac{\delta^2}{4} + \frac{\delta^2}{4}\right)^{\frac{3}{2}}} = \frac{1}{\sqrt{8}} < \epsilon = \frac{1}{3} = \frac{1}{\sqrt{9}},$$

a contradiction. So u is not differentiable at $(0, 0)$. $\Diamond$

The Cauchy-Riemann equations can also be used to prove some interesting facts, for example the following one, which highlights the "rigidity" of holomorphic functions alluded to earlier. See also Exercise 2.29 below.

Example 2.25 (Holomorphic function with constant modulus on a disc is a constant)**.** Consider the disc $D = \{z \in \mathbb{C} : |z - z_0| < r\}$. We will show using the Cauchy-Riemann equations that if $f : D \to \mathbb{C}$ is holomorphic in D, with the property that there is a $c \in \mathbb{R}$ such that $|f(z)| = c$ for all $z \in D$, then f is constant.

Let u, v denote the real and imaginary parts of f. By assumption, $c^2 = |f|^2 = u^2 + v^2$, and so by differentiation,

$$\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} &= 0, \\ u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} &= 0. \end{aligned}$$

Now use $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$ in the first equation and $\dfrac{\partial v}{\partial y} = \dfrac{\partial u}{\partial x}$ in the second equation to get

$$u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} = 0, \tag{2.7}$$

$$u\frac{\partial u}{\partial y} + v\frac{\partial u}{\partial x} = 0. \tag{2.8}$$

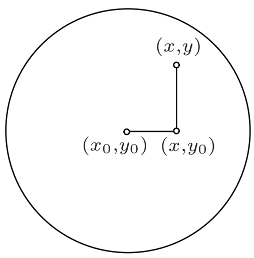
To get rid of $\frac{\partial u}{\partial y}$, multiply (2.7) by u, (2.8) by v, and add. Similarly, to eliminate $\frac{\partial u}{\partial x}$, multiply (2.7) by $-v$, (2.8) by u, and add. This yields

$$\begin{aligned} (u^2 + v^2)\frac{\partial u}{\partial x} &= 0, \\ (u^2 + v^2)\frac{\partial u}{\partial y} &= 0. \end{aligned}$$

If $c = 0$ so that $u^2 + v^2 = c^2 = 0$, then $u = v = 0$ and so $f = 0$ in D. If $c \neq 0$, then the above gives

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0,$$

and by the Cauchy-Riemann equations, it now follows also that $\dfrac{\partial v}{\partial x} = \dfrac{\partial v}{\partial y} = 0$.



By the Fundamental Theorem of Integral Calculus, it follows that

$$\begin{aligned} u(x, y_0) - u(x_0, y_0) = \int_{x_0}^{x} \frac{\partial u}{\partial x}(\xi, y_0)d\xi &= 0, \\ u(x, y) - (x, y_0) = \int_{y_0}^{y} \frac{\partial u}{\partial y}(x, \eta)d\eta &= 0. \end{aligned}$$

Hence the value of u at any (x, y) is the same as the value of u at $z_0 = (x_0, y_0)$. Hence u is constant in D. Similarly, v is constant in D, too. Consequently, $f = u + iv$ is constant in D. $\Diamond$

Exercise 2.26. Show that $z \mapsto z^3$ is entire using the Cauchy-Riemann equations.

Exercise 2.27. Show that $z \mapsto \text{Re}(z)$ is complex differentiable nowhere.

Exercise 2.28. Show that $z \mapsto |z|^2$ is not complex differentiable at any nonzero complex number.

Exercise 2.29. Let $D \subset \mathbb{C}$ be a domain. Show, using the Cauchy-Riemann equations, that if $f : D \to \mathbb{C}$ is holomorphic in D, with the property that $f(z) \in \mathbb{R}$ for all $z \in D$, then f is constant in D.

Exercise 2.30. Let $D \subset \mathbb{C}$ be a domain. Show that if $f : D \to \mathbb{C}$ is holomorphic in D, with the property that $f'(z) = 0$ for all $z \in D$, then f is constant in D.

Exercise 2.31. Let k be a fixed real number, and let f be defined by $f(z) = (x^2-y^2)+kxyi$ for $z = x+iy$, $x, y \in \mathbb{R}$. Show that f is entire if and only if $k = 2$.

2.3. Geometric meaning: local amplitwist and conformality

In calculus we have learnt the geometric meaning of the derivative of a real valued function at a point in the interior of an interval. It gives the slope of the line which is tangential to the graph of the function at that point. In other words, it describes the local behaviour of the function by a linear function. One can imagine magnifying the graph of the function using lenses of greater and greater magnifying power, and then the graph looks like a straight line.

One might ask analogously: What is the geometric meaning of the complex derivative? The answer turns out to be that the complex derivative at a point describes the action of the complex differentiable function locally by an "amplitwist", namely a rotation together with a scaling. Holomorphic functions are precisely those whose local effect is an amplitwist: all the "infinitesimal" complex numbers emanating from a single point are amplified and twisted the same amount.

To see this, let us first note that since our map is from U ($\subset \mathbb{C}$) to $\mathbb{C}$, in order to draw a graph of this function, we would need 4 dimensions, and so we can't visualize it that way. So we will think of a picture of U as part of the plane, and we have that the function will map points from this part of the plane to the complex plane. For example, Figure 1 shows how the function $z \mapsto z^2$ transforms the square grid.

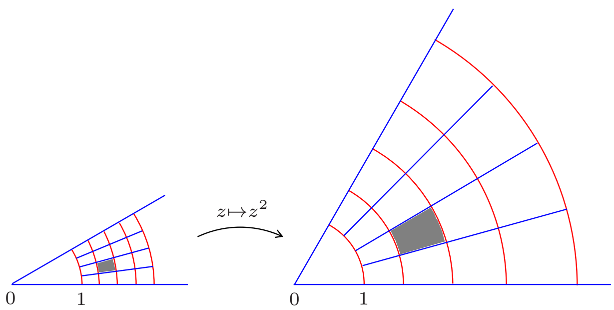


Figure 1. Note that just as in the domain, also in the image, the blue and red lines are mutually perpendicular.

Thus the map f maps the point (x, y) to $(u(x, y), v(x, y))$, where $u := \text{Re}(f)$ and $v := \text{Im}(f)$. If the map

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}$$

is differentiable, then we know that its derivative is the linear transformation

$$A := \begin{bmatrix} \dfrac{\partial u}{\partial x}(x,y) & \dfrac{\partial u}{\partial y}(x,y) \\ \dfrac{\partial v}{\partial x}(x,y) & \dfrac{\partial v}{\partial y}(x,y) \end{bmatrix}$$

which describes the local action. But if f is complex differentiable at a point, then we know that the Cauchy-Riemann equations are satisfied, and so we have

$$\frac{\partial u}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y) =: a, \quad \text{and} \quad -\frac{\partial v}{\partial x}(x,y) = \frac{\partial u}{\partial y}(x,y) =: b,$$

so that

$$A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} = \sqrt{a^2+b^2} \begin{bmatrix} \dfrac{a}{\sqrt{a^2+b^2}} & \dfrac{b}{\sqrt{a^2+b^2}} \\ -\dfrac{b}{\sqrt{a^2+b^2}} & \dfrac{a}{\sqrt{a^2+b^2}} \end{bmatrix} =: r \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

This shows that A is not any old linear transformation, but a anticlockwise rotation through an angle θ followed by a scaling by r. Hence a key difference from merely differentiable functions from $\mathbb{R}^2$ to $\mathbb{R}^2$ is that *complex* differentiable maps are those which have these special types of linear transformations as derivatives when viewed as mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$. Since the local behaviour of the map is described by means of the linear transformation A, it follows that the local action of a holomorphic function is indeed an amplitwist.

The above explanation was via real analysis and the Cauchy-Riemann equations. A more direct way of seeing this is to observe that from the definition of the complex derivative $f'(z_0)$, we have that for z near z_0, $f(z_0+h) - f(z_0) \approx f'(z_0)h$, and so if we imagine a tiny change in z_0, the effect is that of changing $f(z_0)$ by $f'(z_0)h$, and this latter term is obtained from h by rotating h by the argument of $f'(z_0)$ and scaling h by $|f'(z_0)|$. See Figure 2.

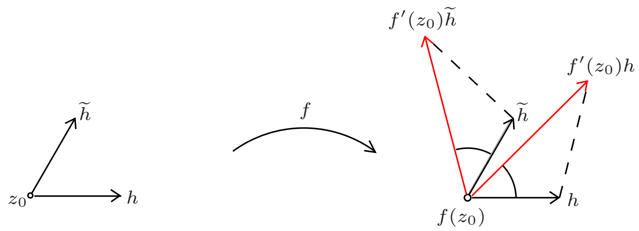


Figure 2. Geometric local meaning of the complex derivative.

Exercise 2.32. Figure 3 shows the shaded interior of a curve being mapped by an entire function to the exterior of the image curve. If z travels round the curve in the domain in an anticlockwise manner, then which way does its image w travel round the image curve? *Hint:* Draw some infinitesimal arrows emanating from z, including one in the direction of motion.

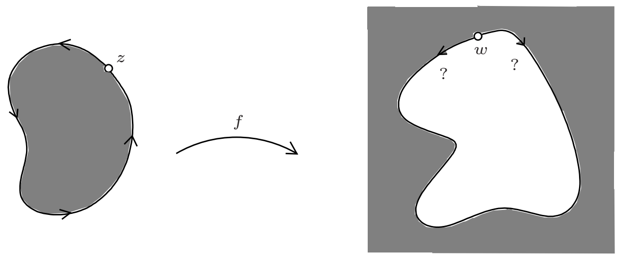


Figure 3. Which way?

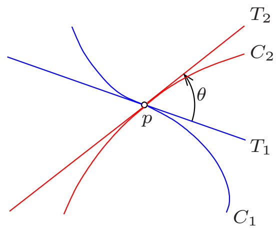


Figure 4. Angle between curves.

2.3.1. Conformality. We now highlight an important consequence of the local amplitwist property of holomorphic functions, namely that such mappings are *conformal*: they "preserve angle between curves". Let us explain what we mean by this. Imagine two smooth curves C_1, C_2, intersecting at a point p. Since the curves are smooth (that is continuously differentiable), we can draw their tangent lines T_1, T_2, at p, as shown in Figure 4.

We now define the *angle between* C_1 *and* C_2 *at* p to be the acute angle from T_1 to T_2. Thus this angle has a sign attached to it. The angle between C_2 and C_1 at p is thus *minus* the angle between C_1 and C_2 at p. If we now apply a sufficiently smooth mapping f to the curves, then the image curves will again possess tangents at the image $f(p)$ of p, and so there will be a well-defined angle between the image curves $f(C_1)$ and $f(C_2)$ at $f(p)$. If the angle between the image curves is the same as the angle between the original curves through p, then we say that the map f has "preserved" the angle at p. See Figure 5.

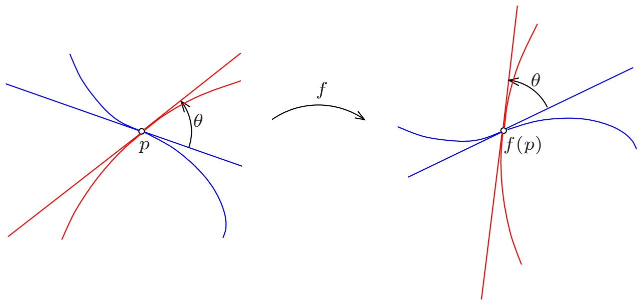


Figure 5. Preservation of angles at p by f.

It is perfectly possible that the map f preserves the angle between *some* pair of curves through p, but not *every* pair of curves through p. However, if it does preserve the angle between every pair of curves through p, we say it is *conformal at* p. We stress that this means that both the magnitude *and the sign* of the angle are preserved; see Subsection 2.3.3 below. If the map f is conformal at every point of its domain, we simply call it *conformal*. Holomorphic functions are conformal (at all points where $f'(z_0) \neq 0$), since each of the tangent vectors will be twisted by the same amount by the holomorphic function f. See Figure 6.

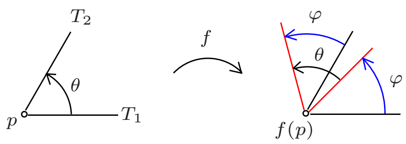


Figure 6. Conformality of a holomorphoc f. Here $\varphi = \mathrm{Arg}(f'(p))$

Look at Figure 1 on page 23 and Figure 8 on page 9, which show the action of the entire mappings z^2 and e^z, respectively. In each of these pictures, we see that just like in the domain, also in the image, the blue and red lines are mutually perpendicular, illustrating a special case of the conformality.

2.3.2. Visual differentiation of $z \mapsto z^2$. In order to find the derivative of the squaring map $z \mapsto z^2$, we will understand its local behaviour, and find the amplitwist by pictures. This will tell us what the derivative is!

Look at Figure 7. Consider the point P corresponding to $z_0 = r(\cos\theta + i\sin\theta)$. We imagine the length of the segment PQ to be $\epsilon \ll r$. Then $\angle POQ =: d\theta \approx 0$.

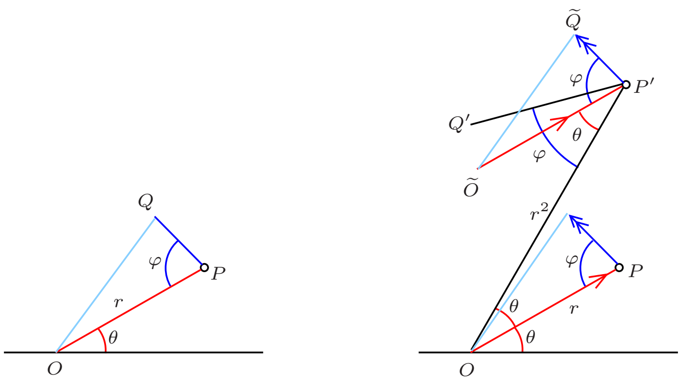


Figure 7. Visual differentiation of the squaring map $z \mapsto z^2$.

First we determine the twist produced on PQ, which is mapped to $P'Q'$. If we draw the ray $P'\widetilde{O}$ which is parallel to OP, and the ray $P'\widetilde{Q}$ parallel to PQ, as shown in Figure 7, then we note that

$$\text{“twist”} = \angle Q'P'\widetilde{Q} = \angle \widetilde{O}P'\widetilde{Q} - \angle \widetilde{O}P'Q' = \angle OPQ - (\angle OP'Q' - \angle OP'\widetilde{O}) = \varphi - (\varphi - \theta) = \theta.$$

Let us now determine the length of the segment $P'Q'$. To do this, we note that $\angle Q'OP'$ is $2d\theta$, and we construct the angle bisector of this angle, meeting the side $P'Q'$ in R'. See Figure 8.

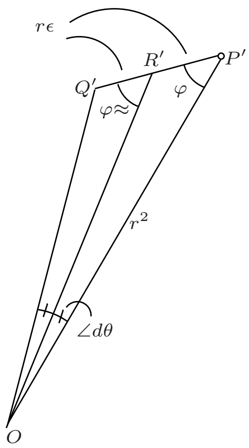


Figure 8. Visual differentiation of the squaring map $z \mapsto z^2$.

Now $\Delta R'OP'$ is similar to ΔQOP. Thus

$$\frac{\ell(OP')}{\ell(OP)} = \frac{\ell(P'R')}{\ell(PQ)}$$

and so $\ell(R'P') = r\epsilon$. Also, since $d\theta$ is small, $\Delta P'OR'$ is congruent to $\Delta R'OQ'$, and so $\ell(Q'R') = r\epsilon$. Hence

$$\text{“ampli”} = \frac{\ell(P'Q')}{\ell(PQ)} = \frac{2r\epsilon}{\epsilon} = 2r.$$

Consequently, the amplitwist is $2r(\cos\theta + i\sin\theta) = 2z_0$, and so $f'(z_0) = 2z_0$.

Exercise 2.33. We know that the power function $z \mapsto z^n$, $n \in \mathbb{N}$, is entire. Find its complex derivative by a pictorial argument.

Hint: Since the amplitwist is the same for all arrows, one can simplify the argument by looking at what happens to an infinitesimal vector *perpendicular* to the ray joining 0 to z_0.

Exercise 2.34. We know that the exponential function $z \mapsto e^z$ is entire. Find its complex derivative using a pictorial argument.

Hint: Take a typical point $x + iy$, and move it vertically up through a distance of δ. Look at the image to determine the amplification. Similarly, by moving $x + iy$ horizontally through δ, determine the “twist”.

2.3.3. Visual nonholomorphicity of $z \mapsto \overline{z}$.

Figure 9 shows that the map $z \mapsto \overline{z}$ is not conformal. Indeed, although the *magnitude* of the angle between C_1 and C_2 is preserved, the complex conjugation map does not preserve the angle, since the *sign* of the angle is opposite in the image.

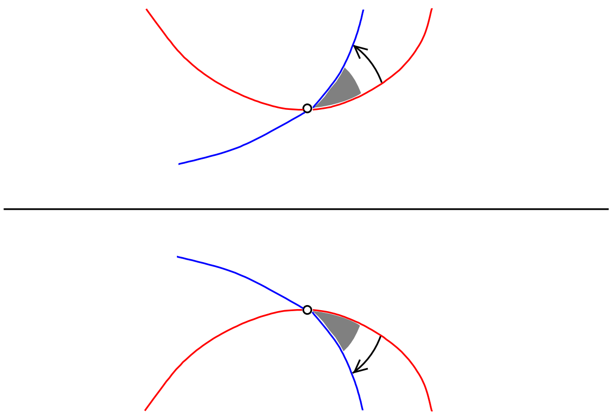


Figure 9. $z \mapsto \overline{z}$ is not holomorphic.

Exercise 2.35. Give a visual argument to show that the map $z \mapsto \mathrm{Re}(z)$ is not holomorphic in $\mathbb{C}$.

2.4. The d-bar operator

The two Cauchy-Riemann equations can be written as a single equation by introducing what is called the “d-bar operator” $\dfrac{\partial}{\partial \overline{z}}$.

Let us define the differential operators[1]

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \text{ and } \frac{\partial}{\partial \overline{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Then the Cauchy-Riemann equations for f in a domain can be rewritten simply as

$$\frac{\partial}{\partial \overline{z}} f = 0$$

since $\frac{\partial}{\partial \overline{z}} f = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right) + i\frac{1}{2}\left(\frac{\partial v}{\partial x} + i\frac{\partial v}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + i\frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0.$

Also, the derivative

$$f' = \frac{\partial}{\partial z} f,$$

because

$$\begin{aligned}\frac{\partial}{\partial z} f = \frac{1}{2}\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right) + i\frac{1}{2}\left(\frac{\partial v}{\partial x} - i\frac{\partial v}{\partial y}\right) &= \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + i\frac{1}{2}\left(-\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ &= \frac{1}{2}\left(2\frac{\partial u}{\partial x}\right) + i\frac{1}{2}\left(2\frac{\partial v}{\partial x}\right) = f'.\end{aligned}$$

So philosophically, we ought to think of holomorphic functions as ones which are "independent of $\overline{z}$". For example, $z + \overline{z}$ is not holomorphic since

$$\frac{\partial}{\partial \overline{z}}(z + \overline{z}) = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(2x) = 1 \neq 0.$$

Exercise 2.36. Show that $4\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} = \Delta$, where $\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplacian operator.

[1] These expressions are motivated by the following meaningless calculation: We have $x = \frac{z+\overline{z}}{2}$, $y = \frac{z-\overline{z}}{2i}$ and

$$\begin{aligned}\frac{\partial}{\partial z} &= \frac{\partial}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial}{\partial y}\frac{\partial y}{\partial z} = \frac{\partial}{\partial x}\frac{1}{2} + \frac{\partial}{\partial y}\frac{1}{2i}, \\ \frac{\partial}{\partial \overline{z}} &= \frac{\partial}{\partial x}\frac{\partial x}{\partial \overline{z}} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \overline{z}} = \frac{\partial}{\partial x}\frac{1}{2} + \frac{\partial}{\partial y}\frac{-1}{2i}.\end{aligned}$$

Chapter 3

Cauchy Integral Theorem and consequences

Having become familiar with complex differentiation, we now turn to integration. The importance of integration in the complex plane stems from the fact that it will lead to a greater understanding of holomorphic functions. For example, the fundamental fact that holomorphic functions are infinitely many times complex differentiable! In this chapter we will learn the following main topics:

(1) Contour integration and its properties.

(2) The Fundamental Theorem of Contour Integration.

(3) The Cauchy Integral Theorem.

(4) Consequences of the Cauchy Integral Theorem:
   (a) Existence of a primitive.
   (b) Infinite differentiability of holomorphic functions.
   (c) Liouville's Theorem and the Fundamental Theorem of Algebra.
   (d) Morera's Theorem.

3.1. Definition of the contour integral

In ordinary calculus, the symbol

$$\int_a^b f(x)dx \tag{3.1}$$

has a clear meaning. Now suppose we wish to generalize this in the complex setting: given z, w complex numbers, want to give meaning to something like

$$\int_z^w f(\zeta)\,d\zeta.$$

Then a first question is:

How do we get from z to w?

In $\mathbb{R}$, if $a < b$, then there is just one way of going from the real number a to the real number b, but now z and w are points in the complex plane, and so there are many possible connecting paths along which we could integrate. See Figure 1.

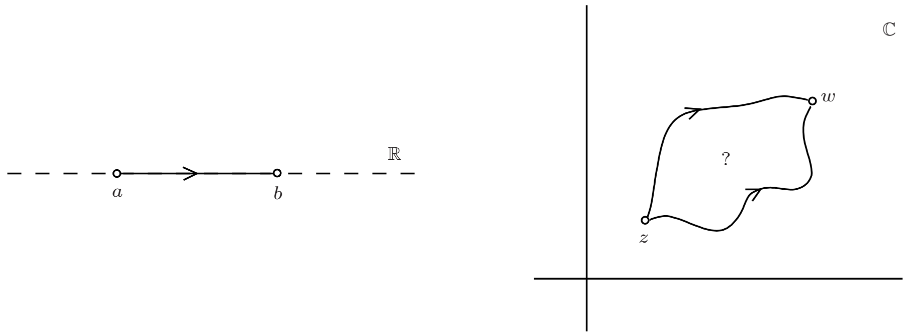


Figure 1. Which path?

So in the complex setting, besides specifying the end points z and w, we will also specify the path γ taken to go from z to w, and we will replace the above expression (3.1) in the real case now by an expression which looks like this in the complex setting:

$$\int_\gamma f(z)dz.$$

We call such an expression a "contour" integral, for the computation of which we need the following data:

(1) A domain D ($\subset \mathbb{C}$).

(2) A continuous function $f : D \to \mathbb{C}$.

(3) A smooth path $\gamma : [a, b] \to D$.

The precise definition is given below.

Definition 3.1. Let D be a domain, and let $f : D \to \mathbb{C}$ be a continuous function. If $\gamma : [a, b] \to D$ is a smooth path (that is, a continuously differentiable function), then we define

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt. \tag{3.2}$$

Remark 3.2. It is very common and convenient to refer to the *range* $\{\gamma(t) : t \in [a, b]\}$ of a path $\gamma : [a, b] \to \mathbb{C}$ as the *path/curve* itself. With this usage, a path becomes a concrete geometric object (as opposed to being a mapping), such as a circle or a straight line segment in the complex plane and hence can be easily visualized. The difficulty with this abuse of terminology is that several *different* paths can have the same image, and so it causes ambiguity.

Exercise 3.3. Consider the three paths $\gamma_1, \gamma_2, \gamma_3 : [0, 2\pi] \to \mathbb{C}$ defined by

$$\begin{aligned} \gamma_1(t) &= e^{it}, \\ \gamma_2(t) &= e^{2it}, \\ \gamma_3(t) &= e^{-it}, \end{aligned}$$

for $t \in [0, 2\pi]$. Show that their images are the same, but the three contour integrals

$$\int_{\gamma_1} \frac{1}{z}dz, \quad \int_{\gamma_2} \frac{1}{z}dz, \quad \int_{\gamma_3} \frac{1}{z}dz$$

are all different.

Exercise 3.4. Let f be holomorphic in a domain and let $\gamma : [0, 1] \to D$ be a smooth path. Show that

$$\frac{d}{dt}f(\gamma(t)) = f'(\gamma(t)) \cdot \gamma'(t) \text{ for all } t \in [0, 1].$$

The integral on the right hand side of (3.2) exists, since if we decompose γ and f into their real and imaginary parts, namely $\gamma(t) = x(t) + iy(t)$ and $f(z) = u(z) + iv(z)$, then

$$\int_a^b f(\gamma(t))\gamma'(t)dt = \int_a^b \Big(u(\gamma(t))x'(t) - v(\gamma(t))y'(t)\Big)dt + i\int_a^b \Big(u(\gamma(t))y'(t) + v(\gamma(t))x'(t)\Big)dt$$

and each of the latter integrals is just the usual real Riemann integral of continuous integrands.

Note that $\gamma'(t)dt$ is an "incremental segment" of the curve, and so the integral is just like the Riemann sum in the real setting; see Figure 2.

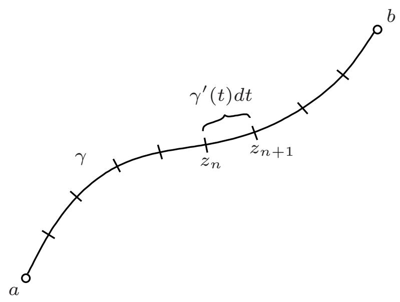


Figure 2. Riemann sum $\displaystyle\sum_{n=1}^{N} f(z_n)(z_{n+1} - z_n)$.

We also note that the definition is independent of the parametrization of the curve, that is, if $\widetilde{\gamma} : [c, d] \to \mathbb{C}$ is such that there is a continuously differentiable function $\varphi : [a, b] \to [c, d]$ such that $c = \varphi(a)$, $d = \varphi(b)$ and $\gamma(t) = \widetilde{\gamma}(\varphi(t))$ for $t \in [a, b]$, then by the chain rule it follows that

$$\begin{aligned}
\int_\gamma f(z)dz &= \int_a^b f(\gamma(t))\gamma'(t)dt = \int_a^b f(\widetilde{\gamma}(\varphi(t)))\widetilde{\gamma}'(\varphi(t))\varphi'(t)dt \\
&\overset{(\tau=\varphi(t))}{=} \int_c^d f(\widetilde{\gamma}(\tau))\widetilde{\gamma}'(\tau)d\tau \\
&= \int_{\widetilde{\gamma}} f(z)dz.
\end{aligned}$$

We extend the definition above to paths with "corners". A path $\gamma : [a, b] \to \mathbb{C}$ is called a *piecewise smooth path/curve* if there exist points $c_1, \dots, c_n$ such that $a < c_1 < \dots < c_n < b$ and such that γ is continuously differentiable on $[a, c_1]$, $[c_1, c_2]$, $\dots$, $[c_{n-1}, c_n]$, $[c_n, b]$. For such a path, we define

$$\int_\gamma f(z)dz = \int_a^{c_1} f(\gamma(t))\gamma'(t)dt + \int_{c_1}^{c_2} f(\gamma(t))\gamma'(t)dt + \dots + \int_{c_{n-1}}^{c_n} f(\gamma(t))\gamma'(t)dt + \int_{c_n}^{b} f(\gamma(t))\gamma'(t)dt.$$

In general the value of the contour integral *will* depend on the route chosen. Here is an example.

Example 3.5. Consider the two paths γ_1, γ_2 from 0 to $1 + i$ given by

$$\begin{aligned}
\gamma_1(t) &= (1+i)t, \quad t \in [0,1], \\
\gamma_2(t) &= \begin{cases} t & \text{if } t \in [0,1] \\ 1 + (t-1)i & \text{if } t \in (1,2]. \end{cases}
\end{aligned}$$

See Figure 3.

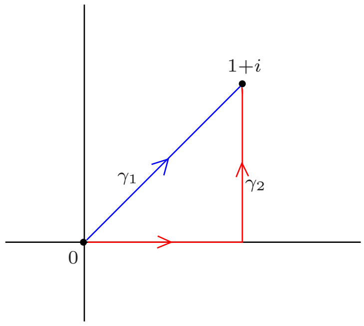


Figure 3. The two paths γ_1 and γ_2.

Then we have

$$\int_{\gamma_1} \overline{z}\, dz = \int_0^1 \overline{(1+i)t}(1+i)dt = \int_0^1 (1-i)(1+i)tdt = \int_0^1 2tdt = 1,$$

while

$$\int_{\gamma_2} \overline{z}\, dz = \int_0^1 \overline{t}1dt + \int_1^2 \overline{(1+(t-1)i)}idt = \int_0^1 tdt + \int_1^2 (1-(t-1)i)idt = \frac{1}{2} + \frac{1}{2} + i = 1+i.$$

Thus the integral depends on the path for the nonholomorphic integrand $z \mapsto \overline{z}$. ◊

The main goal in this chapter (beyond the definition of contour integration) will be to show that the contour integration of a holomorphic function along two paths from z to w is the *same* provided that the map is holomorphic everywhere in the region between the two paths. It turns out that this is a fundamental result (called the Cauchy Integral Theorem) in complex analysis because many further results follow from this.

Example 3.6. Consider the same two contours γ_1, γ_2 considered in Example 3.5 above. But instead of the nonholomorphic map $z \mapsto \overline{z}$, consider now the entire function z: Then we have

$$\int_{\gamma_1} z\, dz = \int_0^1 (1+i)t(1+i)dt = \int_0^1 2itdt = i,$$

and

$$\int_{\gamma_2} z\, dz = \int_0^1 t1dt + \int_1^2 (1+(t-1)i)idt = \int_0^1 tdt + \int_1^2 (i-(t-1))dt = \frac{1}{2} - \frac{1}{2} + i = i.$$

We note that the answer is the same for γ_1 and for γ_2. ◊

Exercise 3.7. Integrate the following functions over the circle $|z| = 2$, oriented anticlockwise:

(1) $z + \overline{z}$.
(2) $z^2 - 2z + 3$.
(3) xy, where $z = x + iy$, $x, y \in \mathbb{R}$.

Exercise 3.8. Evaluate $\displaystyle\int_\gamma \text{Re}(z)dz$, where γ is:

(1) The straight line segment from 0 to $1+i$.
(2) The short circular arc with center i and radius 1 joining 0 to $1+i$.
(3) The part of the parabola $y = x^2$ from $x = 0$ to $x = 1$ joining 0 to $1+i$.

3.1.1. A far reaching little integral. Let r be any positive real number, and γ_r be the path along a circle of radius r centered at the origin, traversed once anticlockwise, defined by $\gamma_r(t) = re^{it}$ ($t \in [0, 2\pi]$). Let n be any integer, and let f_n be defined by $f_n(z) = z^n$, $z \in \mathbb{C} \setminus \{0\}$. Then

$$\begin{aligned} \int_{\gamma_r} f_n(z)\,dz &= \int_0^{2\pi} (re^{it})^n rie^{it} dt = ir^{n+1} \int_0^{2\pi} e^{i(n+1)t} dt \\ &= ir^{n+1} \left\{ \begin{array}{ll} 2\pi & \text{if } n = -1 \\ 0 & \text{otherwise} \end{array} \right\} = \left\{ \begin{array}{ll} 2\pi i & \text{if } n = -1, \\ 0 & \text{otherwise.} \end{array} \right. \end{aligned}$$

Thus the contour integral

$$\int_{\gamma_r} f_n(z)\,dz = \left\{ \begin{array}{ll} 2\pi i & \text{if } n = -1, \\ 0 & \text{otherwise} \end{array} \right.$$

is independent of r, and excluding the case $n = -1$, always 0.

We will see later that this has enormous consequences. For instance, if we have an f that has a representation in terms of integral powers of z:

$$f(z) = \sum_{n \in \mathbb{Z}} a_n z^n,$$

then formally we have $\displaystyle\frac{f(z)}{z^{m+1}} = \sum_{n \in \mathbb{Z}} a_n z^{n-m-1}$, and so $\displaystyle\frac{1}{2\pi i} \int_{\gamma_r} \frac{f(z)}{z^{m+1}} dz = a_m$.

In the above, we assumed that the sum passes through integration over γ_r. We will make precise the details later, but things work out essentially as suggested by this calculation.

Exercise 3.9. Let C be the circular path with center 0 and radius 1 traversed in the anticlockwise direction. Show that for $0 \le k \le n$,

$$\binom{n}{k} = \frac{1}{2\pi i} \int_C \frac{(1+z)^n}{z^{k+1}} dz.$$

3.2. Properties of contour integration

The following properties are easy to show.

Proposition 3.10. *Let D be a domain in $\mathbb{C}$, $\gamma : [a, b] \to D$ be a piecewise smooth path, and $f, g : D \to \mathbb{C}$ be continuous. Then:*

(1) $\displaystyle\int_\gamma (f+g)(z)\,dz = \int_\gamma f(z)\,dz + \int_\gamma g(z)\,dz.$

(2) *If $\alpha \in \mathbb{C}$, then* $\displaystyle\int_\gamma (\alpha f)(z)\,dz = \alpha \int_\gamma f(z)\,dz.$

(3) *Let $-\gamma : [a, b] \to D$ be the path defined by $(-\gamma)(t) = \gamma(a + b - t)$ for $t \in [a, b]$. (Thus $-\gamma$ is just γ traversed in the opposite direction.) Then*

$$\int_{-\gamma} f(z)\,dz = -\int_\gamma f(z)\,dz.$$

(4) *Let $\gamma_1 : [a_1, b_1] \to D$ and $\gamma_2 : [a_2, b_2] \to D$ be two paths such that $\gamma_1(b_1) = \gamma_2(a_2)$ (so that γ_2 starts where γ_1 ends). Define $\gamma_1 + \gamma_2 : [a_1, b_1 + b_2 - a_2]$ to be their "concatenation" by:*

$$(\gamma_1 + \gamma_2)(t) = \left\{ \begin{array}{ll} \gamma_1(t) & \text{for } a_1 \le t \le b_1, \\ \gamma_2(t - b_1 + a_2) & \text{for } b_1 \le t \le b_1 + b_2 - a_2. \end{array} \right.$$

Then $\displaystyle\int_{\gamma_1+\gamma_2} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz.$

Proof. These are straightforward, and follow using the definition and properties of the real integral. We just prove (4):

$$\begin{aligned}
\int_{\gamma_1+\gamma_2} f(z)\,dz &= \int_{a_1}^{b_1+b_2-a_2} f((\gamma_1+\gamma_2)(t))(\gamma_1+\gamma_2)'(t)dt \\
&= \int_{a_1}^{b_1} f((\gamma_1+\gamma_2)(t))(\gamma_1+\gamma_2)'(t)dt + \int_{b_1}^{b_1+b_2-a_2} f((\gamma_1+\gamma_2)(t))(\gamma_1+\gamma_2)'(t)dt \\
&= \int_{a_1}^{b_1} f(\gamma_1(t))\gamma_1'(t)dt + \int_{b_1}^{b_1+b_2-a_2} f(\gamma_2(t-b_1+a_2))\gamma_2'(t-b_1+a_2)dt \\
&= \int_{\gamma_1} f(z)\,dz + \int_{a_2}^{b_2} f(\gamma_2(s))\gamma_2'(s)dt \quad \text{(via the substitution } s=t-b_1+a_2\text{)} \\
&= \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz.
\end{aligned}$$

□

We now prove a useful inequality.

Proposition 3.11. *Let D be a domain in $\mathbb{C}$, $\gamma : [a,b] \to D$ be a piecewise smooth path and $f : D \to \mathbb{C}$ be a continuous function. Then*

$$\left|\int_\gamma f(z)\,dz\right| \leq \left(\max_{z\in\{\gamma(t):\, t\in[a,b]\}} |f(z)|\right)\cdot(\text{length of } \gamma). \tag{3.3}$$

Proof. Consider first a continuous complex-valued function $\varphi : [a,b] \to \mathbb{C}$, for which we prove

$$\left|\int_a^b \varphi(t)dt\right| \leq \int_a^b |\varphi(t)|dt.$$

To see this, let $\int_a^b \varphi(t)dt = re^{i\theta}$, where $r \geq 0$ and $\theta \in (-\pi,\pi]$. Then

$$\begin{aligned}
\left|\int_a^b \varphi(t)dt\right| &= r = e^{-i\theta}re^{i\theta} = e^{-i\theta}\int_a^b \varphi(t)dt = \int_a^b e^{-i\theta}\varphi(t)dt \\
&= \int_a^b \operatorname{Re}\left(e^{-i\theta}\varphi(t)\right)dt + i\int_a^b \operatorname{Im}\left(e^{-i\theta}\varphi(t)\right)dt.
\end{aligned}$$

But the left hand side is real, and so the integral of the imaginary part on the right hand side must be zero, and so

$$\left|\int_a^b \varphi(t)dt\right| = \int_a^b \operatorname{Re}\left(e^{-i\theta}\varphi(t)\right)dt \leq \int_a^b \left|\operatorname{Re}\left(e^{-i\theta}\varphi(t)\right)\right|dt \leq \int_a^b \left|e^{-i\theta}\varphi(t)\right|dt = \int_a^b |\varphi(t)|\,dt.$$

The claim in the proposition now follows, since with $\varphi(t) := f(\gamma(t))\cdot\gamma'(t)$, $t \in [a,b]$, we obtain

$$\begin{aligned}
\left|\int_\gamma f(z)\,dz\right| &= \left|\int_a^b f(\gamma(t))\gamma'(t)dt\right| \leq \int_a^b |f(\gamma(t))\gamma'(t)|dt \\
&= \int_a^b |f(\gamma(t))||\gamma'(t)|dt \leq \max_{t\in[a,b]} |f(\gamma(t))| \int_a^b |\gamma'(t)|dt.
\end{aligned}$$

If $\gamma(t) = x(t) + iy(t)$, where x, y are real-valued, then

$$\int_a^b |\gamma'(t)|dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,dt = \text{length of } \gamma.$$

This completes the proof. □

Exercise 3.12. Calculate the upper bound given by (3.3) on the absolute value of the integral

$$\int_\gamma z^2\,dz,$$

where γ is the straight line path from 0 to $1+i$. Also, compute the integral and find its absolute value.

Exercise 3.13. Using the calculation done in Exercise 3.9, deduce that $\binom{2n}{n} \leq 4^n$.

3.3. Fundamental Theorem of Contour Integration

If f is the derivative of a holomorphic function, then the calculation of $\displaystyle\int_\gamma f(z)\,dz$ is easy, since we have (analogous to the Fundamental Theorem of Integral Calculus in the real setting):

Theorem 3.14 (Fundamental[1] Theorem of Contour Integration)**.** *Let D be a domain in $\mathbb{C}$ and $\gamma:[a,b]\to D$ be a piecewise smooth path. Suppose that $f:D\to\mathbb{C}$ is a continuous function in D, such that there is a holomorphic function $F:D\to\mathbb{C}$ such that $F'=f$ in D. Then*

$$\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a)).$$

Proof. For $z=x+iy\in D$, where x,y are real, define the real-valued functions U,V,u,v by

$$\begin{array}{rcl} F(x+iy) &=& U(x,y)+iV(x,y),\\ f(x+iy) &=& u(x,y)+iv(x,y). \end{array}$$

Also, set $\gamma(t)=x(t)+iy(t)$ ($t\in[a,b]$), where x,y are real-valued. Then by the Cauchy-Riemann equations, we have

$$u(x,y)+iv(x,y) = f(x+iy) = F'(x+iy) = \frac{\partial U}{\partial x}(x,y)+i\frac{\partial V}{\partial x}(x,y) = \frac{\partial V}{\partial y}(x,y)-i\frac{\partial U}{\partial y}(x,y).$$

By the chain rule and the above, we have

$$\frac{d}{dt}U(x(t),y(t)) = \frac{\partial U}{\partial x}(x(t),y(t))\cdot x'(t)+\frac{\partial U}{\partial y}(x(t),y(t))\cdot y'(t) = u(x(t),y(t))\cdot x'(t)-v(x(t),y(t))\cdot y'(t).$$

Similarly,

$$\frac{d}{dt}V(x(t),y(t)) = \frac{\partial V}{\partial x}(x(t),y(t))\cdot x'(t)+\frac{\partial V}{\partial y}(x(t),y(t))\cdot y'(t) = v(x(t),y(t))\cdot x'(t)+u(x(t),y(t))\cdot y'(t).$$

Thus

$$\begin{array}{rcl} \displaystyle\int_\gamma f(z)\,dz &=& \displaystyle\int_a^b f(\gamma(t))\gamma'(t)dt = \int_a^b \Big(u(x(t),y(t))+iv(x(t),y(t))\Big)(x'(t)+iy'(t))dt\\ &=& \displaystyle\int_a^b \frac{d}{dt}U(x(t),y(t))dt + i\int_a^b \frac{d}{dt}V(x(t),y(t))dt\\ &=& U(x(b),y(b))-U(x(a),y(a))+i(V(x(b),y(b))-V(x(a),y(a)))\\ &=& F(\gamma(b))-F(\gamma(a)). \end{array}$$

This completes the proof. ∎

[1] The naming of this result is done just to highlight the similarity with the real analysis analogue. However, in complex analysis, this isn't all that "fundamental". We will soon learn about Cauchy's Integral Theorem, which is certainly more fundamental!

In particular, $\int_\gamma f(z)\,dz$ is independent of the connecting path when f has an "antiderivative" or "primitive".

Example 3.15. Since $\frac{d}{dz}\left(\frac{z^2}{2}\right) = z$ $(z \in \mathbb{C})$, it follows that for any path γ joining 0 to $1+i$,

$$\int_\gamma z\,dz = \frac{(1+i)^2}{2} - \frac{0^2}{2} = \frac{1+2i+i^2}{2} = \frac{1+2i-1}{2} = i,$$

and so in particular, we recover the answer obtained in Example 3.6. $\Diamond$

Example 3.16. There is no function $F : \mathbb{C} \to \mathbb{C}$ such that $F'(z) = \overline{z}$ for all $z \in \mathbb{C}$. Indeed, the calculation in Example 3.5 shows that the contour integral along paths joining 0 to $1+i$ does depend on the path chosen. $\Diamond$

Exercise 3.17. Show, using the Cauchy-Riemann equations, that $z \mapsto \overline{z}$ has no primitive in $\mathbb{C}$.

Exercise 3.18 (Integration by Parts Formula)**.** Let f, g be holomorphic functions defined in a domain D, such that f', g' are continuous in D, and let γ be a piecewise smooth path in D from $z \in D$ to $w \in D$. Show that

$$\int_\gamma f(\zeta)g'(\zeta)d\zeta = f(z)g(z) - f(w)g(w) - \int_\gamma f'(\zeta)g(\zeta)d\zeta.$$

Exercise 3.19. Evaluate $\int_\gamma \cos z\,dz$, where γ is any path joining $-i$ to i.

Definition 3.20. A path $\gamma : [a,b] \to \mathbb{C}$ is said to be *closed* if $\gamma(a) = \gamma(b)$.

Corollary 3.21. *Let D be a domain in $\mathbb{C}$ and $\gamma : [a,b] \to D$ be a closed piecewise smooth path. Suppose that $f : D \to \mathbb{C}$ is a continuous function in D, such that there is a holomorphic function $F : D \to \mathbb{C}$ such that $F' = f$ in D. Then*

$$\int_\gamma f(z)\,dz = 0.$$

Example 3.22. Since for $m \in \mathbb{Z} \setminus \{0\}$ and $z \in D := \mathbb{C} \setminus \{0\}$, we have

$$\frac{d}{dz}\left(\frac{z^m}{m}\right) = z^{m-1} \quad (z \in \mathbb{C}),$$

it follows that for any closed path γ in D, $\int_\gamma z^k\,dz = 0$ for $k \neq -1$. What if $k = -1$? Note that

$$\frac{d}{dz}\text{Log } z = \frac{1}{z} \quad \text{for} \quad z \in \widetilde{D} := \mathbb{C} \setminus (-\infty, 0),$$

and so for any path $\widetilde{\gamma}$ in $\widetilde{D}$, we do have

$$\int_{\widetilde{\gamma}} \frac{1}{z}\,dz = 0.$$

However, in D, $\frac{1}{z}$ doesn't have a primitive; see Exercise 3.24. $\Diamond$

Exercise 3.23. Use the Fundamental Theorem of Contour Integration to write down the value of

$$\int_\gamma e^z\,dz$$

where γ is a path joining 0 and $a+ib$. Equate the answer obtained with the parametric evaluation along the straight line from 0 to $a+ib$, and deduce that

$$\int_0^1 e^{ax}\cos(bx)dx = \frac{a(e^a\cos b - 1) + be^a \sin b}{a^2+b^2}.$$

Exercise 3.24. Show that $\frac{1}{z}$ has no primitive in the punctured complex plane $\mathbb{C} \setminus \{0\}$.

3.4. The Cauchy Integral Theorem

We will now show one of the main results in complex analysis, called the Cauchy Integral Theorem.

Theorem 3.25 (The Cauchy Integral Theorem). *Let D be a domain in $\mathbb{C}$ and let $f : D \to \mathbb{C}$ be holomorphic in D. If $\gamma_1, \gamma_2 : [0,1] \to D$ are two closed piecewise smooth paths such that γ_1 is D-homotopic to γ_2, then*

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_1} f(z)\,dz.$$

Before we go further, let us try to understand the statement. Notice, first of all, that the two paths in D are *closed*. Secondly, what do we mean by saying that the two closed paths are "D-homotopic"? We have the following definition.

Definition 3.26. Let D be a domain in $\mathbb{C}$. Let $\gamma_0, \gamma_1 : [0,1] \to D$ be closed paths. Then γ_0 is said to be *D-homotopic* to γ_1 if there is a continuous function $H : [0,1] \times [0,1] \to D$ such that the following hold:

(H1) For all $t \in [0,1]$, $H(t,0) = \gamma_0(t)$.

(H2) For all $t \in [0,1]$, $H(t,1) = \gamma_1(t)$.

(H3) For all $s \in [0,1]$, $H(0,s) = H(1,s)$.

We can think of the H as a family of closed paths from $[0,1]$ to D, parametrized by "time", the s-variable. Initially, when $s = 0$, $H(\cdot,0)$ is the path γ_0, while finally, when the time $s = 1$, we end up with $H(\cdot,1)$, which is the path γ_1. So far, this is what (H1) and (H2) say. The requirement (H3) just says that at each point of time s, the intermediate path $\gamma_s := H(\cdot,s)$ is closed too. Figure 4 illustrates this.

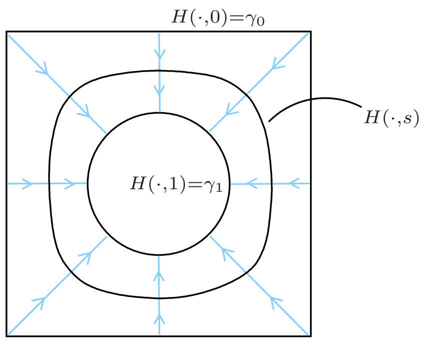


Figure 4. D-homotopic closed paths.

Exercise 3.27. Let D be a domain in $\mathbb{C}$. Show that D-homotopy is an equivalence relation on the set of all closed paths in D.

Proof of Theorem 3.25. We will make the simplifying assumption that the homotopy H is twice continuously differentiable. This smoothness condition can be omitted, but then the proof becomes technical. Moreover, the assumption of twice continuous differentiability is mild, and we will invoke this below when we will exchange the order of partial differentiation:

$$\frac{\partial^2 H}{\partial s \partial t} = \frac{\partial^2 H}{\partial t \partial s}.$$

Essentially, the proof proceeds by looking at the map $s \mapsto I(s)$ for $s \in [0,1]$, where $I(s)$ denotes the integral of f along $\gamma_s := H(\cdot,s)$. We will use differentiation under the integral sign with respect to s to show that $I'(s) \equiv 0$, showing that $s \mapsto I_s$ is constant, and in particular $I(0) = I(1)$, which is the desired conclusion.)

Set $\gamma_s := H(\cdot, s)$ to be the intermediate curve at time s, where $s \in [0,1]$, and define

$$I(s) = \int_{\gamma_s} f(z)\, dz,$$

for $s \in [0,1]$. We have

$$\begin{aligned}
\frac{dI}{ds}(s) &= \frac{d}{ds}\int_{\gamma_s} f(z)\,dz = \frac{d}{ds}\int_0^1 f(H(t,s))\frac{\partial H}{\partial t}(t,s)dt \\
&= \int_0^1 \frac{\partial}{\partial s}\left(f(H(t,s))\frac{\partial H}{\partial t}(t,s)\right) dt \\
&= \int_0^1 \left(f'(H(t,s))\frac{\partial H}{\partial s}(t,s)\frac{\partial H}{\partial t}(t,s) + f(H(t,s))\frac{\partial^2 H}{\partial s \partial t}(t,s)\right) dt \\
&= \int_0^1 \left(f'(H(t,s))\frac{\partial H}{\partial t}(t,s)\frac{\partial H}{\partial s}(t,s) + f(H(t,s))\frac{\partial^2 H}{\partial t \partial s}(t,s)\right) dt \\
&= \int_0^1 \frac{d}{dt}\left(f(H(t,s))\frac{\partial H}{\partial s}(t,s)\right) dt,
\end{aligned}$$

and so by the Fundamental Theorem of Integral Calculus,

$$\begin{aligned}
\frac{dI}{ds}(s) &= \int_0^1 \frac{d}{dt}\left(f(H(t,s))\frac{\partial H}{\partial s}(t,s)\right) dt = f(H(1,s))\frac{\partial H}{\partial s}(1,s) - f(H(0,s))\frac{\partial H}{\partial s}(0,s) \\
&= f(H(1,s))\lim_{\sigma\to s}\frac{H(1,\sigma)-H(1,s)}{\sigma - s} - f(H(1,s))\lim_{\sigma\to s}\frac{H(0,\sigma)-H(0,s)}{\sigma - s} \\
&= f(H(1,s))\lim_{\sigma\to s}\frac{H(1,\sigma)-H(1,s)}{\sigma - s} - f(H(1,s))\lim_{\sigma\to s}\frac{H(1,\sigma)-H(1,s)}{\sigma - s} = 0.
\end{aligned}$$

Hence the map $s \mapsto I(s) : [0,1] \to \mathbb{C}$ is constant. In particular,

$$\int_{\gamma_1} f(z)\,dz = I(1) = I(0) = \int_{\gamma_2} f(z)\,dz.$$

This completes the proof. □

Exercise 3.28. We have seen that if C is the circular path with center 0 and radius 1 traversed in the anticlockwise direction, then

$$\int_C \frac{1}{z}dz = 2\pi i.$$

Now consider the path S, comprising the four line segments which are the sides of the square with vertices $\pm 1 \pm i$, traversed anticlockwise. Draw a picture to convince yourself that S is $\mathbb{C}\setminus\{0\}$-homotopic to C. Evaluate parametrically the integral

$$\int_S \frac{1}{z}dz,$$

and confirm that the answer is indeed $2\pi i$.

Exercise 3.29. Let $\gamma : [0, 2\pi] \to \mathbb{C}$ be the elliptic path defined by $\gamma(t) = a\cos t + ib\sin t$, where a, b are positive. By considering

$$\int_\gamma \frac{1}{z}dz,$$

show that

$$\int_0^{2\pi} \frac{1}{a^2(\cos t)^2 + b^2(\sin t)^2}dt = \frac{2\pi}{ab}.$$

3.4.1. Special case: simply connected domains. An important special case is when a closed path γ is D-homotopic to a *point* (that is, the constant path $\widetilde{\gamma}(t) = w$ for all $t \in [0,1]$). In this case we say that γ is *D-contractible*. A domain in which every closed path is D-contractible is called *simply connected*.

For example, the domains $\mathbb{C}$, $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, $\mathbb{C} \setminus (-\infty, 0]$, are all simply connected, while the annulus $\{z \in \mathbb{C} : 1 < |z| < 1\}$ and the punctured complex plane $\mathbb{C} \setminus \{0\}$ aren't. See Figure 5. In these examples, we notice that the domains with "holes" aren't simply connected.

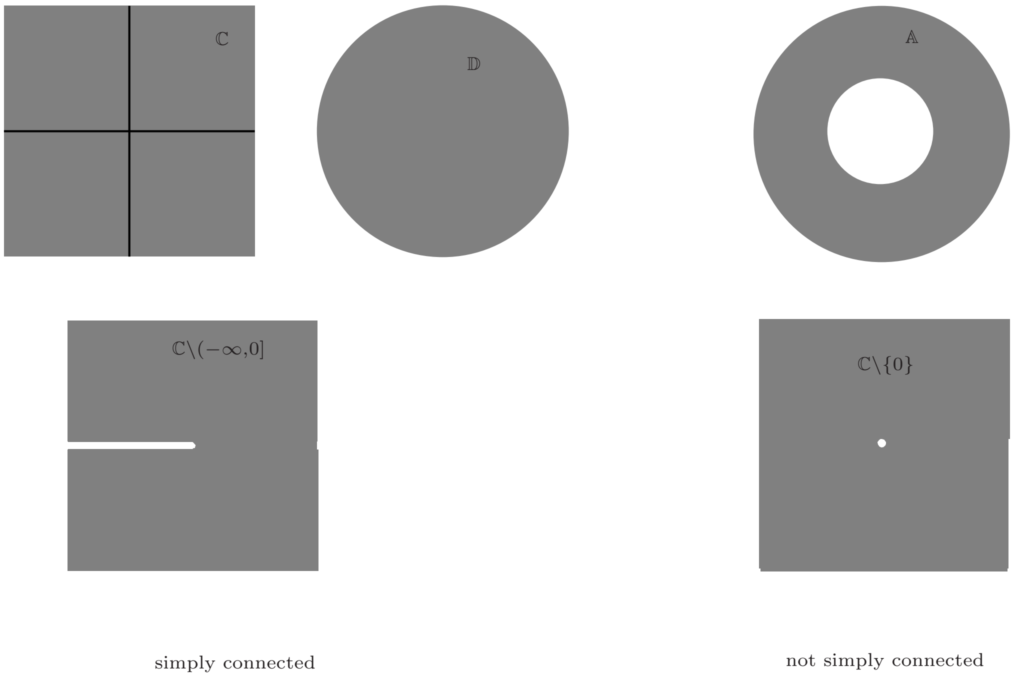


Figure 5. The domains $\mathbb{C}$, $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{C} \setminus (-\infty, 0]$ are simply connected, while the annulus $\mathbb{A} := \{z \in \mathbb{C} : 1 < |z| < 2\}$ and the punctured plane $\mathbb{C} \setminus \{0\}$ aren't.

Exercise 3.30. Show that any closed path is $\mathbb{C}$-contractible. Prove that any two closed paths are $\mathbb{C}$-homotopic.

We have the following corollary of the Cauchy Integral Theorem.

Corollary 3.31. *Let D be a simply connected domain, γ be a closed piecewise smooth path in D and $f : D \to \mathbb{C}$ be holomorphic in D. Then*

$$\int_\gamma f(z)\,dz = 0.$$

Proof. Let γ be D-contractible to a point, say, $w \in D$, and let $\widetilde{\gamma}$ be the constant path given by $\widetilde{\gamma}(t) = w$ for all $t \in [0,1]$. By the Cauchy Integral Theorem, we have

$$\int_\gamma f(z)\,dz = \int_{\widetilde{\gamma}} f(z)\,dz = \int_0^1 f(\widetilde{\gamma}(t))\widetilde{\gamma}'(t)\,dt = \int_0^1 f(w) \cdot 0\,dt = \int_0^1 0\,dt = 0.$$

This completes the proof. □

This corollary is itself also sometimes called the Cauchy Integral Theorem.

Example 3.32. For any closed path γ, we have

$$\int_\gamma e^z \, dz = 0,$$

since e^z is entire, and $\mathbb{C}$ is simply connected. In fact for any entire function f,

$$\int_\gamma f(z)\, dz = 0,$$

for any closed path γ. $\Diamond$

Exercise 3.33. Consider

$$I := \int_\gamma f(z)dz,$$

where f is a continuous function in a domain D containing the closed smooth path γ. True or false?

(1) If f is holomorphic in D and D is a disc, then $I = 0$.

(2) If $I \neq 0$, and f is holomorphic in D, then not all the points enclosed by γ are contained in D.

(3) If f is holomorphic in D, then $I = 0$.

Exercise 3.34. Applying Cauchy's Integral Theorem to e^z and integrating round a circular path, show that for all $r > 0$,

$$\int_0^{2\pi} e^{r\cos t} \cos(r \sin t + t) dt = 0.$$

Exercise 3.35 (Winding number of a curve)**.** Suppose that $\gamma : [0, 1] \to \mathbb{C}$ is a smooth closed path that does not pass through 0. We define the *winding number of* γ (about 0) to be

$$w(\gamma) := \frac{1}{2\pi i} \int_\gamma \frac{1}{z} dz = \frac{1}{2\pi i} \int_0^1 \frac{\gamma'(t)}{\gamma(t)} dt.$$

(1) Using the observation that $e^{2\pi i a} = 1$ if and only if $a \in \mathbb{Z}$, show that $w(\gamma) \in \mathbb{Z}$ by proceeding as follows. Define $\varphi : [0, 1] \to \mathbb{C}$ by

$$\varphi(t) = e^{\int_0^t \frac{\gamma'(s)}{\gamma(s)} ds}, \quad t \in [0, 1].$$

To show that $w(\gamma) \in \mathbb{Z}$, it suffices to show that $\varphi(1) = 1$. To this end, calculate $\varphi'(t)$, and use this expression to show that φ/γ is constant in $[0, 1]$. Use this fact to conclude that $\varphi(1) = 1$.

(2) Calculate the winding number of the curve $\Gamma_1 : [0, 1] \to \mathbb{C}$ given by $\Gamma_1(t) = e^{2\pi i t}$ ($t \in [0, 1]$).

(3) Prove that if $\gamma_1, \gamma_2 : [0, 1] \to \mathbb{C}$ are two smooth closed paths that do not pass through 0, and $\gamma_1 \cdot \gamma_2$ is their pointwise product, then

$$w(\gamma_1 \cdot \gamma_2) = w(\gamma_1) + w(\gamma_2).$$

(4) Let $m \in \mathbb{N}$. Calculate the winding number of the curve $\Gamma_m : [0, 1] \to \mathbb{C}$ given by $\Gamma_m(t) = e^{2\pi i m t}$ ($t \in [0, 1]$).

(5) Show that the winding number function $\gamma \mapsto w(\gamma)$ is "locally constant", by which we mean that if $\gamma_0 : [0, 1] \to \mathbb{C} \setminus \{0\}$ is a smooth closed path, then there is a $\delta > 0$ such that for every smooth closed path $\gamma : [0, 1] \to \mathbb{C} \setminus \{0\}$ such that such that

$$\|\gamma - \gamma_0\|_\infty := \max_{t\in[0,1]} |\gamma(t) - \gamma_0(t)| < \delta,$$

we have $w(\gamma) = w(\gamma_0)$. (In other words, if we equip the set of curves with the uniform topology, and equip $\mathbb{Z}$ with the discrete topology, then $\gamma \mapsto w(\gamma)$ is continuous.)

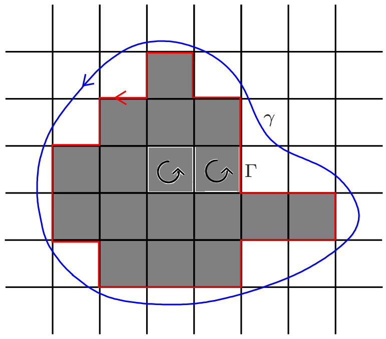


Figure 6. Cauchy Integral Theorem.

3.4.2. Pictorial justification of the Cauchy Integral Theorem. We now give a pictorial argument for the Cauchy Integral Theorem, based on the fact that a holomorphic function is locally an amplitwist.

Imagine a simple closed path γ whose interior is filled with a grid of small squares, each having a side length ϵ, aligned with the real and imaginary axes. We have shaded the squares that are entirely within γ, and we denote by Γ be the boundary of this shaded region. See Figure 6.

Because we have drawn relatively large squares, Γ is only a crude approximation to γ. However, as ϵ gets smaller, the shaded region fills the interior of γ better, and Γ follows γ more closely. We will later justify that the difference in integral of f around Γ and around γ is of order ϵ.

But first we will show that the integral of f around Γ is zero as ϵ goes to 0. Consider the sum of all the integrals of f taken anticlockwise around each of the shaded squares. When we add the integrals from these two squares, their common edge is traversed twice, but in opposite directions, and hence the integrals along it cancel. But this is true of every edge that lies *inside* the shaded region, so that when we add the integrals over all the shaded squares, the only integrals which do not cancel each other are the ones along edges that make up Γ. Thus

$$\int_\Gamma f(z)\,dz = \sum_{\text{shaded squares}} \int_\square f(z)\,dz. \tag{3.4}$$

The investigation of the integral of f around γ has thus been reduced to the study of the local effect of f on infinitesimal squares in the interior of γ. But on each infinitesimal square, the action of f is an amplitwist, and the image under f is thus again a square. Let us see what this says about each term in the above sum. Figure 7 shows a magnified picture of an infinitesimal square and the amplitwisted image of this square under f. The points a, b, c, d are mapped to A, B, C, D.

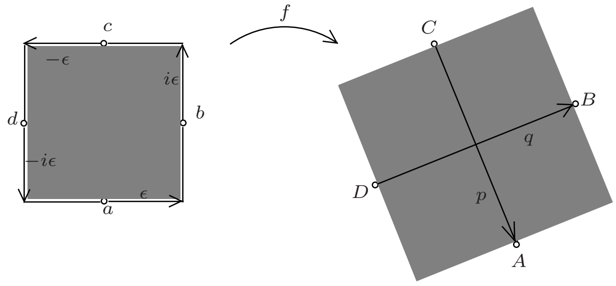


Figure 7. Amplitwist on an infinitesimal shaded square.

Then

$$\int_{\square} f(z)\,dz = A\cdot\epsilon + B\cdot(i\epsilon) + C\cdot(-\epsilon) + D\cdot(-i\epsilon) = (A-C)\epsilon + (B-D)i\epsilon = (p+iq)\epsilon = 0,$$

where the last equality follows because

$$q = \boxed{\text{p rotated through a right angle}} = ip,$$

using the fact that f, being a local amplitwist, maps the infinitesimal square to a square. Thus the vanishing of the loop integrals for holomorphic functions, that is, the Cauchy Integral Theorem is just a manifestation of their local amplitwist property!

Finally, we show the little detail we ignored, namely that the difference in integral of f around Γ and around γ is of order ϵ. Look at Figure 6 again, and let us look at the parts of white squares which lie between γ and the Γ. The number of such fragments of squares is approximately

$$\frac{\text{length of } \gamma}{\epsilon}.$$

On the other hand, the difference in the integrals of f along the portions of γ and Γ along a particular white fragment of an infinitesimal square can be seen to be of order ϵ^2 (since it is of the form $v \cdot dz$, where v is a vector of length of order ϵ, obtained as a sum of amplitwisted versions of vectors of length of order ϵ, and dz is of order ϵ). So the difference between the integral of f around Γ and γ is of order

$$\frac{\text{length of } \gamma}{\epsilon}\cdot\epsilon^2 = \epsilon\cdot(\text{length of } \gamma),$$

which goes to zero as ϵ goes to zero.

This completes the pictorial argument for the Cauchy Integral Theorem.

Finally, we highlight that for maps that do not possess the local amplitwist property the Cauchy Integral Theorem may fail, by looking at the nonholomorphic map $z \mapsto \overline{z}$. We will show that rather than the integral around the closed loop γ being 0, the contour integral of $\overline{z}$ around γ yields the area enclosed by γ!

First note that each infinitesimal square in our grid of squares in Figure 6 is still mapped to a square by the map $z \mapsto \overline{z}$, but now instead of the new squares being an amplitwisted version of the old squares, we have that the new squares are obtained simply by reflection in the real axis of the old squares. See Figure 8.

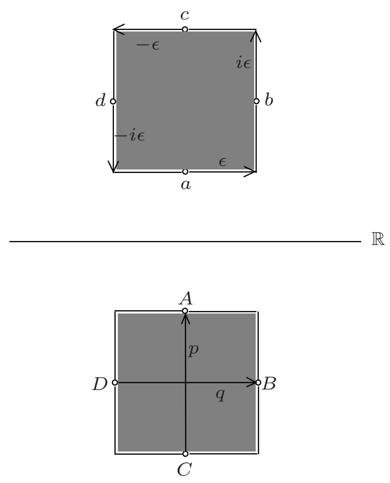


Figure 8. Action of the nonholomorphic map $z \mapsto \overline{z}$ on an infinitesimal shaded square.

This makes each term in the sum (3.4) nonzero:

$$\int_{\square} \overline{z}\, dz = A\cdot\epsilon + B\cdot(i\epsilon) + C\cdot(-\epsilon) + D\cdot(-i\epsilon) = (A-C)\epsilon + (B-D)i\epsilon = (p+iq)\epsilon = (i\epsilon+i\epsilon)\epsilon = 2i\epsilon^2,$$

If we add all these contributions, then we get

$$\int_{\Gamma} \overline{z}\, dz = \sum_{\text{shaded squares}} \int_{\square} \overline{z}\, dz = 2i\epsilon^2 \cdot (\text{number of infinitesimal squares inside } \gamma).$$

But

$$(\text{number of infinitesimal squares inside } \gamma) = \frac{\text{area enclosed by } \gamma}{\text{area of each infinitesimal square}},$$

and so

$$\int_{\Gamma} \overline{z}\, dz = 2i \cdot (\text{area enclosed by } \gamma).$$

So in contrast to the Cauchy Integral Theorem for holomorphic functions, we see that for this nonholomorphic function, the integral along a closed contour is not zero, but yields the area of the contour!

Exercise 3.36. Hold a coin of radius R down on a flat surface and roll another coin of radius r round it. The path traced by a point on the rim of the rolling coin is called an *epicycloid*, and it is a closed curve if $R = nr$, for some $n \in \mathbb{N}$. See Figure 9.

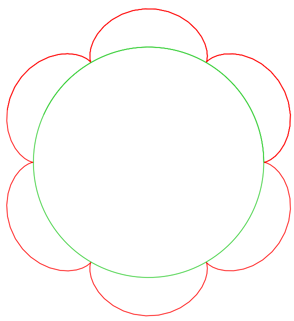


Figure 9. The epicycloid with $n = 6$.

(1) With the center of the fixed coin at the origin, show that the epicycloid can be represented parametrically as $z(t) = r((n+1)e^{it} - e^{i(n+1)t})$, $t \in [0, 2\pi]$.

(2) By evaluating the integral of $\overline{z}$ along the epicycloid, show that the area enclosed by the epicycloid is equal to $\pi r^2(n+1)(n+2)$.

In the rest of this chapter we will learn about several consequences of the Cauchy Integral Theorem. In particular, we will learn

(1) that in simply connected domains every holomorphic function possesses a primitive;

(2) that holomorphic functions are infinitely many times differentiable;

(3) that bounded entire functions are constants (Liouville's Theorem) (and also use this to prove the Fundamental Theorem of Algebra);

(4) a result called Morera's theorem, which is a sort of a converse to the Cauchy Integral Theorem.

3.5. Existence of a primitive

We will show that on a simply connected domain, every holomorphic function is the derivative of some holomorphic function.

Theorem 3.37. *If D is a simply connected domain and $f : D \to \mathbb{C}$ is holomorphic, then there is a holomorphic function $F : D \to \mathbb{C}$ such that $F'(z) = f(z)$ for all $z \in D$.*

Proof. Fix $z_0 \in D$. For any $z \in D$, let γ_z be any path in D joining z_0 to z. Define $F : D \to \mathbb{C}$ by

$$F(z) = \int_{\gamma_z} f(\zeta)\, d\zeta.$$

This gives a well-defined F, since if $\widetilde{\gamma}$ is another path joining z_0 to z, then the concatenation of γ_z with $-\widetilde{\gamma}$ is a closed path, and so by the Cauchy Integral Theorem,

$$0 = \int_{\gamma_z - \widetilde{\gamma}} f(\zeta)\, dz = \int_{\gamma_z} f(\zeta)\, d\zeta - \int_{\widetilde{\gamma}} f(\zeta)\, d\zeta,$$

and so $\displaystyle\int_{\gamma_z} f(\zeta)\, d\zeta = \int_{\widetilde{\gamma}} f(\zeta)\, d\zeta$.

Next we show the holomorphicity of F and that $F' = f$. Since f is holomorphic in D, it is also continuous there, and so given an $\epsilon > 0$, there is a $\delta > 0$ such that whenever $\zeta \in \mathbb{C}$ is such that $|\zeta - z| < \delta$, we have $\zeta \in D$ and $|f(\zeta) - f(z)| < \epsilon$. Thus if we take a w such that $0 < |w - z| < \delta$, then we have:

$$\frac{F(w) - F(z)}{w - z} = \frac{1}{w - z}\left(\int_{\gamma_w} f(\zeta)\, d\zeta - \int_{\gamma_z} f(\zeta)\, d\zeta\right).$$

If γ_{zw} is a straight line path joining z to w, then the concatenation of γ_w with the concatenation of $-\gamma_{wz}$ with $-\gamma_z$ is a closed path, and so by the Cauchy Integral Theorem, we obtain

$$\int_{\gamma_w} f(\zeta)\, d\zeta - \int_{\gamma_z} f(\zeta)\, d\zeta = \int_{\gamma_{zw}} f(\zeta)\, d\zeta.$$

See Figure 10.

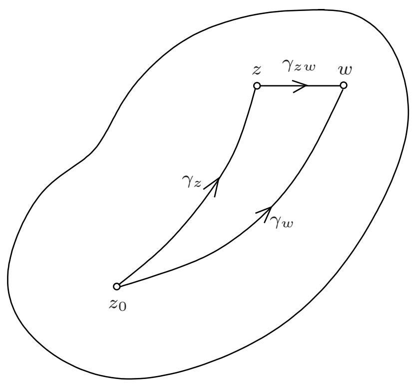


Figure 10. Existence of a primitive.

Thus,

$$\frac{F(w) - F(z)}{w - z} - f(z) = \frac{1}{w - z}\int_{\gamma_{zw}} f(\zeta)\, d\zeta - \frac{1}{w - z}\int_{\gamma_{zw}} f(z)\, d\zeta = \frac{1}{w - z}\int_{\gamma_{zw}} (f(\zeta) - f(z))\, d\zeta,$$

and so

$$\begin{aligned}\left|\frac{F(w)-F(z)}{w-z}-f(z)\right| &= \left|\frac{1}{w-z}\int_{\gamma_{zw}}(f(\zeta)-f(z))\,d\zeta\right| = \frac{1}{|w-z|}\left|\int_{\gamma_{zw}}(f(\zeta)-f(z))\,d\zeta\right| \\ &\leq \frac{1}{|w-z|}\,\epsilon|w-z| = \epsilon.\end{aligned}$$

Thus $F'(z)=f(z)$, and F is holomorphic. □

Exercise 3.38. Suppose that D is a domain. If f is holomorphic in D, and there is no F holomorphic in D such that $F'=f$ in D, then we know that D cannot be simply connected. Give a concrete example of such a D and f.

3.6. Cauchy Integral Formula

We will now learn about a result, called the Cauchy Integral Formula, which says, roughly speaking that if we have a closed path γ without self-intersections, and f is function holomorphic inside γ, then the value of f at any point inside γ is determined by the values of the function on γ! This illustrates the "rigidity" of holomorphic functions. Later on, in the next chapter, we will study a more general Cauchy Integral Formula, which will allow us to even express all the derivatives of f at any point inside γ in terms of the values of the function on γ. So we can consider the basic result in this section as the "$n=0$ case" of the more general result to follow. We begin with the following.

Proposition 3.39. *Let C_r be the circular path with center z_0 and radius $r>0$ traversed in the anticlockwise direction. Let $R>r$, $D_R=\{z\in\mathbb{C}:|z-z_0|<R\}$, and $f:D_R\to\mathbb{C}$ be such that f is continuous on D_R and holomorphic in the punctured disc $D_R\setminus\{z_0\}$. Then*

$$f(z_0)=\frac{1}{2\pi i}\int_{C_r}\frac{f(z)}{z-z_0}\,dz.$$

Proof. Let $\epsilon>0$. Then there is a $\delta>0$ (which we can arrange to be smaller than r) such that whenever $0<|z-z_0|\leq\delta$, we have $|f(z)-f(z_0)|<\epsilon$. Consider the circular path C_δ, with center z_0 and radius δ traversed in the anticlockwise direction. But C_δ and C_r are easily seen to be $D_R\setminus\{z_0\}$-homotopic. Indeed the homotopy H can be obtained by just taking the convex combination of the points on C_r and C_δ: $H(\cdot,s):=(1-s)C_r(\cdot)+sC_\delta(\cdot)$, $s\in[0,1]$. Thus by the Cauchy Integral Theorem, we have

$$\int_{C_r}\frac{f(z)}{z-z_0}\,dz=\int_{C_\delta}\frac{f(z)}{z-z_0}\,dz.$$

Hence,

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_{C_r}\frac{f(z)}{z-z_0}\,dz-f(z_0)\right| &= \left|\frac{1}{2\pi i}\int_{C_\delta}\frac{f(z)}{z-z_0}\,dz-f(z_0)\frac{1}{2\pi i}\int_{C_\delta}\frac{1}{z-z_0}\,dz\right| \\ &= \left|\frac{1}{2\pi i}\int_{C_\delta}\frac{f(z)-f(z_0)}{z-z_0}\,dz\right| \\ &\leq \left(\max_{z\in C_\delta}\frac{|f(z)-f(z_0)|}{2\pi|z-z_0|}\right)\cdot 2\pi\delta \\ &< \frac{\epsilon}{2\pi\delta}\cdot 2\pi\delta=\epsilon.\end{aligned}$$

Since $\epsilon>0$ was arbitrary, the claim follows. □

Corollary 3.40. *Let D be a domain, and let $f : D \to \mathbb{C}$ be holomorphic in D. Let $z_0 \in D$ and $r > 0$ be such that the circular path C_r with center z_0 and radius r, and its interior, is contained in D. Then we have*

$$f(z_0) = \frac{1}{2\pi i}\int_{C_r} \frac{f(z)}{z - z_0}dz.$$

Here is a basic version of the Cauchy Integral Formula. Note that as opposed to the previous Corollary 3.40, now the w can be *any* point inside the circle with center z_0 and radius r, and not necessarily the center z_0 as in the corollary above.

Corollary 3.41 (Cauchy's Integral Formula for circular paths)**.** *Let D be a domain, and let $f : D \to \mathbb{C}$ be holomorphic in D. Let $z_0 \in D$ and $r > 0$ be such that the circular path C_r with center z_0 and radius r, and its interior, is contained in D. Then we have*

$$f(w) = \frac{1}{2\pi i}\int_{C_r} \frac{f(z)}{z - w}dz, \quad |w - z_0| < r.$$

Proof. Let w be such that $|w - z_0| < r$. Choose a $\delta > 0$ small enough so that the circular path C_δ with center w and radius δ is contained in the interior of C_r. But now C_r and C_δ are D-homotopic; see Figure 11.

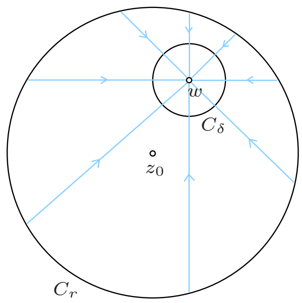


Figure 11. Homotopy between circles.

Thus it follows that $f(w) = \dfrac{1}{2\pi i}\displaystyle\int_{C_\delta} \frac{f(z)}{z - w}dz = \frac{1}{2\pi i}\int_{C_r} \frac{f(z)}{z - w}dz.$ $\square$

Exercise 3.42. Let $0 < a < 1$, and let γ be the unit circle with center 0 traversed anticlockwise. Show that

$$\int_\gamma \frac{i}{(z-a)(az-1)}dz = \int_0^{2\pi} \frac{1}{1 + a^2 - 2a\cos t}dt.$$

Use Cauchy's Integral Formula to deduce that $\displaystyle\int_0^{2\pi} \frac{1}{1 + a^2 - 2a\cos t}dt = \frac{2\pi}{1 - a^2}.$

Exercise 3.43. Fill in the blanks.

(1) $\displaystyle\int_\gamma \frac{e^z}{z-1}dz =$ ______, where γ is the circle $|z| = 2$ traversed in the anticlockwise direction.

(2) $\displaystyle\int_\gamma \frac{z^2+1}{z^2-1}dz =$ ______, where γ is the circle $|z - 1| = 1$ traversed in the anticlockwise direction.

(3) $\displaystyle\int_\gamma \frac{z^2+1}{z^2-1}dz =$ ______, where γ is the circle $|z - i| = 1$ traversed in the anticlockwise direction.

(4) $\displaystyle\int_\gamma \frac{z^2+1}{z^2-1}dz =$ ______, where γ is the circle $|z + 1| = 1$ traversed in the anticlockwise direction.

(5) $\displaystyle\int_\gamma \frac{z^2+1}{z^2-1}dz =$ ______, where γ is the circle $|z| = 3$ traversed in the anticlockwise direction.

Exercise 3.44. Does $z \mapsto \dfrac{1}{z(1-z^2)}$ have a primitive in $\{z \in \mathbb{C} : 0 < |z| < 1\}$?

Corollary 3.45 (Cauchy's Integral Formula for general paths)**.** *Let*

(1) *D be a domain,*

(2) *$f : D \to \mathbb{C}$ be holomorphic in D,*

(3) *$z_0 \in D$, and*

(4) *γ be a closed path in D which is $D \setminus \{z_0\}$-homotopic to a circular path C centered at z_0, such that C and its interior is contained in D.*

Then we have

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0} dz.$$

Proof. By the Cauchy Integral Formula for circular paths, it follows that

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz.$$

But since γ is $D \setminus \{z_0\}$-homotopic to C, by the Cauchy Integral Theorem we have

$$\frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0} dz.$$

This completes the proof. $\square$

This result highlights the "rigidity" associated with holomorphic functions mentioned earlier. By this we mean that their highly structured nature (everywhere locally an amplitwist) enables one to pin down their precise behaviour from very limited information. That is, even if we know the effect of a holomorphic function in a small portion of the plane, its values can be inferred at other far away points in a unique manner. Figure 12 illustrates this in the case of the Cauchy Integral Formula, where knowing the values of f on the curve γ enables one to determine the values at all points in the shaded region!

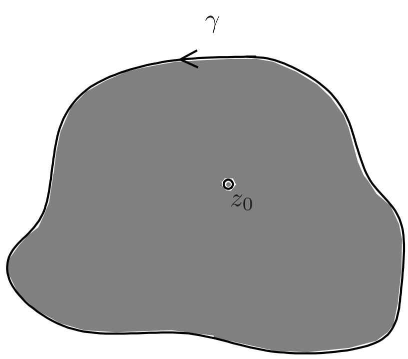


Figure 12. Rigidity of holomorphic functions.

Exercise 3.46. Integrate the following functions over the circular path given by $|z| = 3$ traversed in the anticlockwise direction:

(1) $\text{Log}(z - 4i)$.

(2) $\dfrac{1}{z-1}$.

(3) Principal value of i^{z-3}.

Exercise 3.47. Let F be defined by $F(z) = \dfrac{e^{iz}}{z^2+1}$ and let $R > 1$.

(1) Let σ be the closed semicircular path formed by the segment S of the real axis from $-R$ to R, followed by the circular arc T of radius R in the upper half plane from R to $-R$. Show that
$$\int_\sigma F(z)dz = \frac{\pi}{e}.$$

(2) Prove that $|e^{iz}| \le 1$ for z in the upper half plane, and conclude that for large enough $|z|$,
$$|F(z)| \le \frac{2}{|z|^2}.$$

(3) Show that $\lim\limits_{R\to\infty} \displaystyle\int_T F(z)\,dz = 0$, and so $\lim\limits_{R\to\infty} \displaystyle\int_S F(z)\,dz = \frac{\pi}{e}$.

(4) Conclude, by parameterizing the integral over S in terms of x and just considering the real part, that
$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}dx := \lim_{R\to\infty} \int_{-R}^{R} \frac{\cos x}{1+x^2}dx = \frac{\pi}{e}.$$

Exercise 3.48. Evaluate $\displaystyle\int_0^{2\pi} e^{e^{i\theta}}\,d\theta$.

3.7. Holomorphic functions are infinitely many times differentiable

In this section we prove the fundamental property of holomorphic functions in a domain, namely that they are infinitely many times complex differentiable.

Corollary 3.49. *Let D be a domain, and let $f : D \to \mathbb{C}$ be holomorphic in D. Then for all $z_0 \in D$, $f''(z_0), f'''(z_0), \ldots$ all exist.*

Proof. Let g be defined by
$$g(z) = \begin{cases} \dfrac{f(z)-f(z_0)}{z-z_0} & \text{if } z \neq z_0, \\ f'(z_0) & \text{if } z = z_0. \end{cases}$$
Clearly g is holomorphic in $D \setminus \{z_0\}$ and continuous in D. Choose a $r > 0$ small enough so that the disc with center z_0 and radius $2r$ is contained in D. Then the circular path C_r with center z_0 and radius r is contained in D. Then by Proposition 3.39, it follows that
$$\begin{aligned} f'(z_0) &= g(z_0) = \frac{1}{2\pi i}\int_{C_r} \frac{f(z)-f(z_0)}{(z-z_0)^2}dz \\ &= \frac{1}{2\pi i}\int_{C_r} \frac{f(z)}{(z-z_0)^2}dz - \frac{f(z_0)}{2\pi i}\int_{C_r} \frac{1}{(z-z_0)^2}dz \\ &= \frac{1}{2\pi i}\int_{C_r} \frac{f(z)}{(z-z_0)^2}dz. \end{aligned} \tag{3.5}$$
Thus for w near z_0 inside C_r, but with $w \neq z_0$, we have that
$$\begin{aligned} \frac{f'(w)-f'(z_0)}{w-z_0} &= \frac{1}{w-z_0}\left(\frac{1}{2\pi i}\int_{C_r} \frac{f(z)}{(z-w)^2}dz - \frac{1}{2\pi i}\int_{C_r} \frac{f(z)}{(z-z_0)^2}dz\right) \\ &= \frac{1}{2\pi i}\int_{C_r} \frac{f(z)(2z-z_0-w)}{(z-w)^2(z-z_0)^2}dz, \end{aligned}$$
and so we guess that
$$\lim_{w\to z_0} \frac{f'(w)-f'(z_0)}{w-z_0} = \frac{2}{2\pi i}\int_{C_r} \frac{f(z)}{(z-z_0)^3}dz.$$

We prove this claim below. We have

$$\frac{f'(w)-f'(z_0)}{w-z_0}-\frac{2}{2\pi i}\int_{C_r}\frac{f(z)}{(z-z_0)^3}dz=(w-z_0)\frac{1}{2\pi i}\int_{C_r}\frac{(3z-z_0-2w)f(z)}{(z-w)^2(z-z_0)^3}dz$$

But the continuous map

$$(z,w)\mapsto\left|\frac{(3z-z_0-2w)f(z)}{(z-w)^2(z-z_0)^3}\right|$$

on the compact set $C_r\times\left\{w:|w-z_0|\le\frac{r}{2}\right\}$ has a maximum value $M\ge 0$. Hence,

$$\left|\frac{f'(w)-f'(z_0)}{w-z_0}-\frac{2}{2\pi i}\int_{C_r}\frac{f(z)}{(z-z_0)^3}dz\right|\le|w-z_0|Mr.$$

This shows that f' is differentiable at z_0. As the choice of z_0 was arbitrary, f' is holomorphic in D. But now replacing f by f', we see that f'' is holomorphic in D. Continuing in this manner, we get the desired conclusion. □

Exercise 3.50. Suppose f is holomorphic in a domain D. Is it clear that if $n\in\mathbb{N}$, then $f^{(n)}$ has a continuous derivative?

3.8. Liouville's Theorem and the Fundamental Theorem of Algebra

Here is one more instance of the rigidity associated with holomorphicity.

Theorem 3.51 (Liouville's Theorem)**.** *Every bounded entire function is constant.*

Proof. Let $M\ge 0$ be such that for all $z\in\mathbb{C}$, $|f(z)|\le M$. Suppose that $w\in\mathbb{C}$, and let γ be the circular path with center w and radius R, where R is any positive number. Then (from the proof of Corollary 3.49, see in particular (3.5))

$$|f'(w)|=\left|\frac{1}{2\pi i}\int_\gamma\frac{f(z)}{(z-w)^2}dz\right|\le\frac{M}{2\pi R^2}\cdot 2\pi R=\frac{M}{R}.$$

But since $R>0$ was arbitrary, it follows that $f'(w)=0$. So $f'(w)=0$ for all $w\in\mathbb{C}$, and hence f is constant. One way to see this is the following. If $z\in\mathbb{C}$, by considering the straight line path γ_z joining 0 to z, we have

$$f(z)-f(0)=\int_{\gamma_z}f'(\zeta)\,d\zeta=0.$$

This completes the proof. □

This result can be used to give a short proof of the Fundamental Theorem of Algebra[2].

Corollary 3.52 (Fundamental Theorem of Algebra)**.** *Every polynomial of degree ≥ 1 has a root in $\mathbb{C}$.*

Proof. Suppose $p(z)=c_0+c_1z+\cdots+c_dz^d$ is a polynomial with $d\ge 1$, and such that it has no root in $\mathbb{C}$. That is, for all $z\in\mathbb{C}$, $p(z)\neq 0$. But then the function f defined by

$$f(z)=\frac{1}{p(z)}\quad(z\in\mathbb{C})$$

[2]Despite its name, there is no purely algebraic proof of the theorem, since any proof must use the completeness of the reals, which is not an algebraic concept. Additionally, it is not really fundamental for modern algebra; its name was given at a time when the study of algebra was mainly concerned with the solutions of polynomial equations with real or complex coefficients.

is entire. (See Exercise 2.12.) By Exercise 1.26, there exist $M, R > 0$ such that $|p(z)| \geq M|z|^d$ whenever $|z| > R$. In the compact set $\{z \in \mathbb{C} : |z| \leq R\}$, the continuous function $z \mapsto |p(z)|$ has a positive minimum m. Thus

$$|f(z)| \leq \min \left\{ \frac{1}{MR^d}, \frac{1}{m} \right\}.$$

By Liouville's Theorem, f must be constant, and so p must be a constant, a contradiction to the fact that $d \geq 1$. □

Exercise 3.53. Let f be an entire function such that f is bounded below, that is, there is a $\delta > 0$ such that for all $z \in \mathbb{C}$, $|f(z)| \geq \delta$. Show that f is a constant.

Exercise 3.54. Show that an entire function whose range of values avoids a disc $\{w \in \mathbb{C} : |w - w_0| < r\}$ must be a constant.

Exercise 3.55. Assume that f is an entire function that is periodic in both the real and in the imaginary direction, that is, there exist T_1, T_2 in $\mathbb{R}$ such that $f(z) = f(z + T_1) = f(z + iT_2)$ for all $z \in \mathbb{C}$. Prove that f is constant.

Exercise 3.56. A classical theme in the theory of entire functions is to ask:

"If f is entire and $|f(z)|$ behaves such-and-such as $|z|$ gets large, then is f actually equal to so-and-so?".

Here is one instance of this.

(1) Show that if f is entire and $|f(z)| \leq |e^z|$, for all $z \in \mathbb{C}$, then in fact f is equal to ce^z for some complex constant c with $|c| \leq 1$. (Thus if a nonconstant entire function "grows" no faster than the exponential function, it *is* an exponential function.)

(2) A naive student may argue that the assertion is false because he has heard that "polynomials grow more slowly than the exponential function", but surely $p \neq e^z$. Dispel his confusion by showing that if p is a polynomial satisfying $|p(z)| \leq |e^z|$ for all $z \in \mathbb{C}$, then $p \equiv 0$.
Hint: Look at $z = x < 0$.

3.9. Morera's Theorem: converse to Cauchy's Integral Theorem

Theorem 3.57 (Morera's Theorem)**.** *Let D be a domain and let $f : D \to \mathbb{C}$ be a continuous function such that for every closed rectangular path γ in every disc contained in D,*

$$\int_\gamma f(z)dz = 0.$$

Then f is holomorphic in D.

Proof. Let $z_0 \in D$, and let $R > 0$ be such that the disc Δ with center z_0 and radius R is contained in D. Define $F : \Delta \to \mathbb{C}$ by

$$F(z) = \int_{\gamma_{z_0,z}} f(\zeta)\, d\zeta,$$

where $\gamma_{z_0,z}$ is the path joining z_0 to z by first moving horizontally and then moving vertically. We will first show that F is holomorphic in Δ, and its derivative is f. From this it follows that f (being the derivative of a holomorphic function) is itself holomorphic in Δ, and the proof will be finished.

Let $z \in \Delta$. Suppose $\epsilon > 0$. Since f is continuous, there exists $\delta > 0$ be such that whenever $|w - z| < \delta$, $w \in \Delta$ and $|f(w) - f(z)| < \epsilon$. We have

$$F(w) - F(z) = \int_{\gamma_{z_0,w}} f(\zeta)\, d\zeta - \int_{\gamma_{z_0,z}} f(\zeta)\, d\zeta.$$

Using the fact that the integral of f on closed rectangular paths is zero, it follows that

$$F(w) - F(z) = \int_{\gamma_{z,w}} f(\zeta)\, d\zeta,$$

where $\gamma_{z,w}$ is the path joining z to w by again first moving horizontally and then moving vertically. See Figure 13 which shows one particular case.

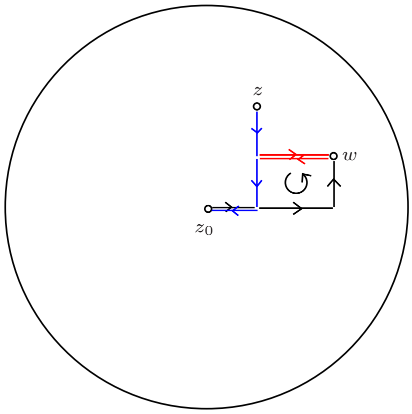


Figure 13. Morera's Theorem.

Thus

$$\frac{F(w)-F(z)}{w-z} - f(z) = \frac{1}{w-z}\int_{\gamma_{z,w}} f(\zeta)\,d\zeta - f(z)\frac{1}{w-z}\int_{\gamma_{z,w}} 1\,d\zeta = \frac{1}{w-z}\int_{\gamma_{z,w}} (f(\zeta)-f(z))\,d\zeta,$$

where we have used the Fundamental Theorem of Contour Integration for the holomorphic function 1 to obtain

$$\int_{\gamma_{z,w}} 1\,d\zeta = w-z.$$

Consequently,

$$\left|\frac{F(w)-F(z)}{w-z} - f(z)\right| = \left|\frac{1}{w-z}\int_{\gamma_{z,w}} (f(\zeta)-f(z))\,d\zeta\right| \leq \frac{\epsilon}{|w-z|}(|\mathrm{Re}(w-z)|+|\mathrm{Im}(w-z)|) < 2\epsilon.$$

This completes the proof. □

Exercise 3.58. Suppose f is continuous in a domain D and also holomorphic on all of D except possibly at one point $z_0 \in D$. Then f is actually holomorphic on all of D.

Remark 3.59 (Outside the scope of this course)**.** Morera's Theorem can be used to show several very useful results. We mention one such:

Theorem 3.60. *Suppose $(f_n)_{n\in\mathbb{N}}$ is a sequence of holomorphic functions defined in a domain D, and $(f_n)_{n\in\mathbb{N}}$ converges to f uniformly on every compact subset of D. (That is, on any compact $K \subset D$, the functions f_n restricted to K converge uniformly to f.) Then f is holomorphic on D.*

Proof. Let z be an arbitrary point of D, and let Δ be an open disc whose closure is contained in D. Then by Cauchy's Theorem,

$$\int_R f_n(z)dz = 0$$

for any rectangle R contained in Δ. On the other hand, we know that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f in the closure of Δ, and so

$$0 = \lim_{n\to\infty}\int_R f_n(z)dz = \int_R f(z)dz.$$

Since the f_n are continuous, f will also be continuous on D. By Morera's Theorem, f is holomorphic on Δ. Since z was arbitrary, this means f is holomorphic on all of D. □

Chapter 4

Taylor and Laurent Series

In this chapter we will first learn about the fundamental result which says that a holomorphic function has a power series expansion around any point in the domain where it lives, and vice versa, every power series gives rise to a holomorphic function. En route we will also prove further fundamental results on holomorphic functions:

(1) The (general) Cauchy Integral Formula and the Cauchy inequality.

(2) The classification of zeros and the Identity Theorem.

(3) The Maximum Modulus Theorem.

In the second part of the chapter, we will learn about Laurent series, which are like power series, except that negative integer powers of the variable also occur in the expansion. This will be useful to study functions that are holomorphic in annuli (and in particular punctured discs). They are also useful to classify "singularities", and to evaluate some real integrals, as we will see at the end of this chapter.

4.1. Series

Just like with real series, given a sequence $(a_n)_{n\in\mathbb{N}}$ of complex numbers, one can form a new sequence $(s_n)_{n\in\mathbb{N}}$ of its *partial sums*:

$$\begin{aligned} s_1 &:= a_1, \\ s_2 &:= a_1 + a_2, \\ s_3 &:= a_1 + a_2 + a_3, \\ &\vdots \end{aligned}$$

If $(s_n)_{\mathbb{N}}$ converges, we say that the series $\displaystyle\sum_{n=1}^{\infty} a_n$ *converges*, and we write $\displaystyle\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$.

If the sequence $(s_n)_{\mathbb{N}}$ does not converge we say that the series $\displaystyle\sum_{n=1}^{\infty} a_n$ *diverges*.

We say that the series $\displaystyle\sum_{n=1}^{\infty} a_n$ *converges absolutely* if the real series $\displaystyle\sum_{n=1}^{\infty} |a_n|$ converges.

From the result in Exercise 1.27, which says that a complex sequence converges if and only if the sequences of its real and imaginary parts converge, it follows that

$$\sum_{n=1}^{\infty} a_n$$

converges if and only if the two real series

$$\sum_{n=1}^{\infty} \mathrm{Re}(a_n) \quad \text{and} \quad \sum_{n=1}^{\infty} \mathrm{Im}(a_n)$$

converge. Thus the results from real analysis lend themselves for use in testing the convergence of complex series. For example, it is easy to prove the following two facts, which we leave as exercises.

Exercise 4.1. If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Exercise 4.2. If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n$ converges.

Exercise 4.3. Show that if $|z| < 1$, then $\sum_{n=0}^{\infty} z^n$ converges and that $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$.

Exercise 4.4. Show that if $|z| < 1$, then $\sum_{n=1}^{\infty} nz^{n-1} = \frac{1}{(1-z)^2}$.

Exercise 4.5. Show that the series $\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$ converges for all $s \in \mathbb{C}$ satisfying $\mathrm{Re}(s) > 1$. Thus

$$s \mapsto \zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$$

is a well-defined map in the half-plane given by $\mathrm{Re}(s) > 1$, and is called the *Riemann ζ-function*. The link of the Riemann zeta function with the number theoretic world of primes is brought out by the *Euler Product Formula*, which says that if $p_1 := 2 < p_2 := 3 < p_3 := 5 < \cdots$ is the infinite list of primes in increasing order, then

$$\zeta(s) = \lim_{K\to\infty} \prod_{k=1}^{K} \frac{1}{1 - \frac{1}{p_k^s}}, \quad \mathrm{Re}(s) > 1.$$

Bernhard Riemann (1826-1866) showed that the ζ-function can be extended holomorphically to $\mathbb{C}\setminus\{1\}$. It can be shown that the ζ-function has zeros at $-2, -4, -6, \ldots$, called “trivial zeros”, but it also has other zeros. All the non-trivial zeros Riemann computed turned out to lie on the line $\mathrm{Re}(s) = \frac{1}{2}$. This led him to formulate the following conjecture, which is a famous unsolved problem in Mathematics.

Conjecture 4.6 (Riemann Hypothesis). *All non-trivial zeros of the ζ-function lie on the line* $\mathrm{Re}(s) = \frac{1}{2}$.

4.2. Power series are holomorphic

Theorem 4.7. *For a power series $\sum_{n=0}^{\infty} c_n z^n$, exactly one of the following hold:*

(1) *Either it is absolutely convergent for all $z \in \mathbb{C}$.*

(2) *Or there is a unique nonnegative real number R such that*

   (a) $\sum_{n=0}^{\infty} c_n z^n$ *is absolutely convergent for all $z \in \mathbb{C}$ with $|z| < R$, and*

   (b) $\sum_{n=0}^{\infty} c_n z^n$ *is divergent for all $z \in \mathbb{C}$ with $|z| > R$.*

(The unique $R > 0$ in the above theorem is called the *radius of convergence* of the power series, and if the power series converges for all $z \in \mathbb{C}$, we say that the power series has an infinite radius of convergence, and write "$R = \infty$".)

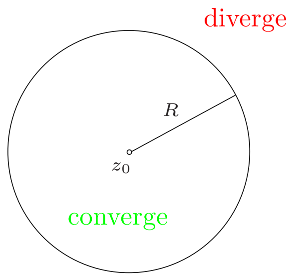


Figure 1. Convergence region of a power series in $\mathbb{C}$.

Proof. Let

$$S := \left\{ y \in [0, \infty) : \text{ there exists a } z \in \mathbb{C} \text{ such that } y = |z| \text{ and } \sum_{n=0}^{\infty} c_n z^n \text{ converges} \right\}.$$

Clearly $0 \in S$. We consider the following two possible cases.

$\underline{1}^\circ$ S is not bounded above. Then given $z \in \mathbb{C}$, there exists a $y \in S$ such that $|z| < y$. But since $y \in S$, there exists a $z_0 \in \mathbb{C}$ such that $y = |z_0|$ and

$$\sum_{n=0} c_n z_0^n$$

converges. It follows that its nth term goes to 0 as $n \to \infty$, and in particular, it is bounded: $|c_n z_0^n| \leq M$. Then with $r := \dfrac{|z|}{|z_0|}$ (< 1), we have

$$|c_n z^n| = |c_n z_0^n| \left(\frac{|z|}{|z_0|} \right)^n \leq M r^n \quad (n \in \mathbb{N}).$$

But $\displaystyle\sum_{n=0}^{\infty} M r^n$ converges ($r < 1$!), and so by the comparison test, $\displaystyle\sum_{n=0}^{\infty} c_n z^n$ is absolutely convergent.

$\underline{2}^\circ$ S is bounded above. Let $R := \sup S$. If $z \in \mathbb{C}$ and $|z| < R$, then by the definition of supremum, it follows that there exists a $y \in S$ such that $|z| < y$. Then we repeat the proof in 1° as follows. Since $y \in S$, there exists a $z_0 \in \mathbb{C}$ such that $y = |z_0|$ and

$$\sum_{n=0}^{\infty} c_n z_0^n$$

converges. It follows that its nth term goes to 0 as $n \to \infty$, and in particular, it is bounded: $|c_n z_0^n| \leq M$. Then with $r := \dfrac{|z|}{|z_0|}$ (< 1), we have

$$|c_n z^n| = |c_n z_0^n| \left(\frac{|z|}{|z_0|} \right)^n \leq M r^n \quad (n \in \mathbb{N}).$$

But $\displaystyle\sum_{n=0}^{\infty} M r^n$ converges ($r < 1$!), and so by the comparison test, $\displaystyle\sum_{n=0}^{\infty} c_n z^n$ is absolutely convergent.

Finally, if $z \in \mathbb{C}$ and $|z| > R$, then $|z| \notin S$, and by the definition of S, $\displaystyle\sum_{n=0}^{\infty} c_n z^n$ diverges.

The uniqueness of R can be seen as follows. If $R, \widetilde{R}$ have the property described in the theorem and $R < \widetilde{R}$, then

$$R < r := \frac{R + \widetilde{R}}{2} < \widetilde{R}.$$

Because $0 < r < \widetilde{R}$, $\displaystyle\sum_{n=1}^{\infty} c_n r^n$ converges, while as $R < r$, $\displaystyle\sum_{n=0}^{\infty} c_n r^n$ diverges, a contradiction. $\square$

Remark 4.8. Just as with real power series, complex power series may diverge at every point on the boundary (given by $|z| = R$), or diverge on some points of the boundary and converge at other points of the boundary, or converge at all the points on the boundary.

Exercise 4.9. Show that $\displaystyle\sum_{n=1}^{\infty} n^n z^n$ converges only when $z = 0$.

Exercise 4.10. Show that $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n^n}$ converges for all $z \in \mathbb{C}$.

Exercise 4.11. Let $(c_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers. Show that the radius of convergence of the real power series

$$\sum_{n=0}^{\infty} |c_n| x^n$$

coincides with that of the complex power series $\displaystyle\sum_{n=0}^{\infty} c_n z^n$.

Exercise 4.12. Find the radius of convergence of the following complex power series:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^n, \quad \sum_{n=1}^{\infty} \frac{1}{n^2} z^n, \quad \sum_{n=1}^{\infty} n^{2012} z^n, \quad \sum_{n=1}^{\infty} \frac{1}{n^n} z^n, \quad \sum_{n=0}^{\infty} \frac{1}{n!} z^n.$$

Consider a polynomial $p : \mathbb{C} \to \mathbb{C}$ given by

$$p(z) := c_0 + c_1 z + c_2 z^2 + \cdots + c_d z^d, \quad z \in \mathbb{C},$$

for some complex numbers $c_0, \dots, c_d$ and some integer $d \in \{0, 1, 2, \dots\}$. For such a polynomial p, it is immediately seen (by successive differentiation, and subsequent evaluation at $z = 0$) that the coefficients $c_0, c_1, \dots, c_d$ are related to the derivatives of p at 0:

$$\begin{array}{rcl} p(0) &=& c_0, \\ p'(0) &=& c_1, \\ p''(0) &=& 2c_2, \\ & \cdots & \\ p^{(k)}(0) &=& k! c_k, \\ & \cdots & \end{array}$$

so that

$$c_k = \frac{p^{(k)}(0)}{k!}, \quad k = 0, 1, 2, 3, \dots.$$

Hence one obtains the following expansion for p:

$$p(z) = p(0) + p'(0) z + \frac{p''(0)}{2!} z^2 + \cdots + \frac{p^{(d)}(0)}{d!} z^d.$$

We will see in Corollary 4.15 below, that for power series, which are similar to polynomials (except that we have possibly infinitely many nonzero coefficients), a similar formula holds in the region of convergence.

We first prove the result which says that, just like polynomials, power series are holomorphic inside the disc where they converge.

Theorem 4.13. *Let $R > 0$ and let the power series $f(z) := \sum_{n=0}^{\infty} c_n z^n$ converge for $|z| < R$. Then*

$$f'(z) = \sum_{n=1}^{\infty} nc_n z^{n-1} \quad \text{for } |z| < R.$$

Proof. First we show that the power series

$$g(z) := \sum_{n=1}^{\infty} nc_n z^{n-1} = c_1 + 2c_2 z + \cdots + nc_n z^{n-1} + \ldots$$

is absolutely convergent for $|z| < R$. Fix z and let r be such that $|z| < r < R$. By hypothesis

$$\sum_{n=0}^{\infty} c_n r^n$$

converges, and so there is some positive number M such that $|c_n r^n| < M$ for all n. Let $\rho := \frac{|z|}{r}$. Then $0 \le \rho < 1$, and

$$|nc_n z^{n-1}| \le \frac{nM|z|^{n-1}}{r^n} = \frac{Mn\rho^{n-1}}{r}.$$

But from Exercise 4.4, we know that $\sum_{n=1}^{\infty} n\rho^{n-1}$ converges[1]. Hence by the Comparison Test,

$$\sum_{n=1}^{\infty} nc_n z^{n-1}$$

converges absolutely.

Now we show that $f'(z_0) = g(z_0)$ for $|z_0| < R$, that is,

$$\lim_{z \to z_0} \left(\frac{f(z) - f(z_0)}{z - z_0} - g(z_0) \right) = 0.$$

As before, let r be such that $|z_0| < r < R$ and since $z \to z_0$, we may also restrict z so that $|z| < r$.

Suppose $\epsilon > 0$. As $\sum_{n=1}^{\infty} nc_n r^{n-1}$ converges absolutely, there is an index N such that

$$\sum_{n=N}^{\infty} |nc_n r^{n-1}| < \frac{\epsilon}{4}.$$

Now keep N fixed. We can write

$$\frac{f(z) - f(z_0)}{z - z_0} - g(z_0) = \sum_{n=1}^{\infty} c_n \left(z^{n-1} + z^{n-2} z_0 + \cdots + z z_0^{n-2} + z_0^{n-1} - n z_0^{n-1} \right).$$

We let S_1 be the sum of the first $N-1$ terms of this series (that is, from $n = 1$ to $n = N-1$) and S_2 be the sum of the remaining terms. Then

$$|S_2| \le \sum_{n=N}^{\infty} |c_n| \left(r^{n-1} + r^{n-1} + \cdots + r^{n-1} + nr^{n-1} \right) = \sum_{n=N}^{\infty} 2n|c_n| r^{n-1} < \frac{\epsilon}{2}.$$

Also,

$$S_1 = \sum_{n=1}^{N} c_n \left(z^{n-1} + z^{n-2} z_0 + \cdots + z z_0^{n-2} + z_0^{n-1} - n z_0^{n-1} \right)$$

[1] to $\frac{1}{(1-\rho)^2}$

is a polynomial in z and by the algebra of limits, $\lim_{z \to z_0} S_1 = 0$. Hence there is a $\delta > 0$ such that whenever $|z - z_0| < \delta$, we have $|S_1| < \frac{\epsilon}{2}$. Thus for $|z| < r$ and $0 < |z - z_0| < \delta$, we have

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - g(z_0) \right| \leq |S_1| + |S_2| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This means that $f'(z_0) = g(z_0)$, as claimed. □

Remark 4.14. The above two results imply that if we replace the real variable in a real power series which converges in an interval $(-R, R)$ by a complex variable, then we can "extend/continue" the real power series to a holomorphic function in the disc given by $|z| < R$ in the complex plane. So we can view *real analytic* functions (namely functions of a real variable having a local power series expansion) as restrictions of holomorphic functions. This again highlights the interplay between the world of real analysis and complex analysis. (We have seen a previous instance of this interaction when we studied the Cauchy-Riemann equations.)

By a repeated application of the previous result, we have the following.

Corollary 4.15. *Let $R > 0$ and let the power series $f(z) := \sum_{n=0}^{\infty} c_n z^n$ converge for $|z| < R$. Then*

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)(n-2)\cdots(n-k+1)c_n z^{n-k} \quad \text{for } |z| < R, \quad k \geq 1. \tag{4.1}$$

In particular, for $n \geq 0$, $c_n = \frac{1}{n!} f^{(n)}(0)$.

Proof. This is straightforward, and the last claim follows by setting $z = 0$ in (4.1):

$$f^{(k)}(0) = k(k-1)(k-2)\cdots 1 c_k + \left(z \cdot \sum_{n=k+1}^{\infty} n(n-1)(n-2)\cdots(n-k+1)c_n z^{n-k-1} \right)\Bigg|_{z=0} = k! c_k.$$

Also, $f(0) = c_0$. □

More generally, we can consider power series which are centered at a point z_0, and the following result follows easily from the above.

Corollary 4.16. *Let $z_0 \in \mathbb{C}$, $R > 0$ and $f(z) := \sum_{n=0}^{\infty} c_n (z - z_0)^n$ converge for $|z - z_0| < R$. Then*

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)(n-2)\cdots(n-k+1)c_n (z - z_0)^{n-k} \quad \text{for } |z - z_0| < R, \quad k \geq 1.$$

In particular, for $n \geq 0$, $c_n = \frac{1}{n!} f^{(n)}(z_0)$.

Remark 4.17 (Uniqueness of coefficients)**.** Suppose that

$$\sum_{n=0}^{\infty} c_n (z - z_0)^n \text{ and } \sum_{n=0}^{\infty} \widetilde{c}_n (z - z_0)^n$$

are two power series which both converge to the same function f in an open disk with center z_0 and radius $R > 0$. Then from the above, for $n \geq 0$, we have

$$c_n = \frac{f^{(n)}(z_0)}{n!} = \widetilde{c}_n.$$

Exercise 4.18. For $|z| < 1$, what is $1^2 + 2^2 z + 3^2 z^2 + 4^2 z^3 + \cdots$?

Exercise 4.19. True or false? All statements refer to power series of the form $\sum_{n=0}^{\infty} c_n z^n$.

(1) The set of points z for which the power series converges equals either the singleton set $\{0\}$ or some open disc of finite positive radius or the entire complex plane, but no other type of set.
(2) If the power series converges for $z = 1$, then it converges for all z with $|z| < 1$.
(3) If the power series converges for $z = 1$, then it converges for all z with $|z| = 1$.
(4) If the power series converges for $z = 1$, then it converges for $z = -1$.
(5) Some power series converge at all points of an open disc with center 0 of some positive radius, and also at certain points on the boundary of the disc (that is the circle bounding the disc), and at no other points.
(6) There are power series that converge on a set of points which is exactly equal to the closed disc given by $|z| \le 1$.
(7) If the power series diverges at $z = i$, then it diverges at $z = 1 + i$ as well.

4.3. Taylor series

We have seen in the last section that complex power series define holomorphic functions in their disc of convergence. In this section, we will show that conversely, every holomorphic function f defined in a domain D possesses a power series expansion in a disc around any point $z_0 \in D$.

Theorem 4.20. *If f is holomorphic in the open disc $D_R := \{z \in \mathbb{C} : |z - z_0| < R\}$, then*

$$f(z) = c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + c_3(z - z_0)^3 + \cdots, \quad z \in D_R,$$

where for $n \ge 0$,

$$c_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz,$$

and C is the circular path with center z_0 and radius r, where $0 < r < R$ traversed in the anticlockwise direction.

Proof. Let $z \in D_R$. Initially, let r be such that $|z - z_0| < r < R$. Then by Cauchy's Integral Formula,

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z_0 + z_0 - z} d\zeta = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)\left(1 - \dfrac{z - z_0}{\zeta - z_0}\right)} d\zeta$$

Set $w := \dfrac{z - z_0}{\zeta - z_0}$. Then $|w| = \dfrac{|z - z_0|}{r} < 1$. Thus

$$\frac{1}{1 - \dfrac{z - z_0}{\zeta - z_0}} = \frac{1}{1 - w} = 1 + w + w^2 + w^3 + \cdots + w^{n-1} + \frac{w^n}{1 - w},$$

and so

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_C f(\zeta) \left(\frac{1}{\zeta - z_0} + \frac{z - z_0}{(\zeta - z_0)^2} + \cdots + \frac{(z - z_0)^{n-1}}{(\zeta - z_0)^n} + \frac{(z - z_0)^n}{(\zeta - z_0)^n (\zeta - z)} \right) d\zeta \\ &= c_0 + c_1(z - z_0) + \cdots + c_{n-1}(z - z_0)^{n-1} + R_n(z), \end{aligned}$$

where

$$R_n(z) := \frac{1}{2\pi i} \int_C \frac{f(\zeta)(z - z_0)^n}{(\zeta - z_0)^n (\zeta - z)} d\zeta.$$

There is a $M > 0$ such that for all $\zeta \in C$, $|f(\zeta)| < M$. Moreover, $|\zeta - z_0| = r$ and

$$|\zeta - z| = |\zeta - z_0 - (z - z_0)| \ge |\zeta - z_0| - |z - z_0| = r - |z - z_0|,$$

and so

$$|R_n(z)| \le \left(\frac{|z-z_0|}{r}\right)^n \frac{M}{r-|z-z_0|} \stackrel{n\to\infty}{\longrightarrow} 0.$$

Thus the series $c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + c_3(z-z_0)^3 + \cdots$ converges to $f(z)$. Note that we have only shown the expression

$$c_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz$$

where r is such that $|z-z_0| < r < R$. But by the Cauchy Integral Theorem, we see that this integral is independent of r, and any value of $r \in (0, R)$ can be chosen here. $\square$

Corollary 4.21 (Taylor[2] Series)**.** *Let D be a domain and let $f : D \to \mathbb{C}$ be holomorphic. If $z_0 \in D$, then*

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \frac{f'''(z_0)}{3!}(z-z_0)^3 + \ldots, \quad |z-z_0| < R,$$

where R is the radius of the largest open disk with center z_0 contained in D. Also,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz, \tag{4.2}$$

where C is the circular path with center z_0 and radius r, where $0 < r < R$ traversed in the anticlockwise direction.

(4.2) is called the (general) *Cauchy Integral formula.*

Proof. From Theorem 4.20, we have

$$f(z) = c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + c_3(z-z_0)^3 + \ldots, \quad z \in D_R, \tag{4.3}$$

where $D_R := \{z \in \mathbb{C} : |z-z_0| < R\}$ and R is the radius of the largest open disk with center z_0 contained in D. Also, for $n \ge 0$,

$$c_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz,$$

and C is the circular path with center z_0 and radius r, where $0 < r < R$ traversed in the anticlockwise direction. But from Corollary 4.16, we know that the power series above is infinitely many times differentiable, and also, for $n \ge 0$,

$$\frac{1}{n!}f^{(n)}(z_0) = c_n.$$

Thus the result follows. $\square$

Example 4.22. The exponential function f, $z \mapsto f(z) := e^z$, is entire. Since

$$\frac{d}{dz}e^z = e^z,$$

it follows that $f^{(n)}(0) = 1$, and so

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(z-0)^n = \sum_{n=0}^{\infty} \frac{1}{n!}z^n,$$

for all $z \in \mathbb{C}$. $\Diamond$

---

[2]Named after Brook Taylor (1685-1731) who, among others, studied this expansion in the context of real analytic functions

Example 4.23. The function f defined by $f(z) = \text{Log}(z)$ is holomorphic in $\mathbb{C} \setminus (-\infty, 0]$. The largest open disc with center $z_0 = 1$ in this cut plane is $D = \{z \in \mathbb{C} : |z-1| < 1\}$. Since

$$f^{(n)}(z_0) = \frac{(-1)^n (n-1)!}{z_0^n} = (-1)^n (n-1)!,$$

we have $\text{Log}(1+w) = w - \dfrac{w^2}{2} + \cdots + \dfrac{(-1)^n w^n}{n} + \cdots$ for $|w| < 1$. ◊

Exercise 4.24. Show that for $z \in \mathbb{C}$: $\sin z = z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - + \cdots$ and $\cos z = 1 - \dfrac{z^2}{2!} + \dfrac{z^4}{4!} - + \cdots$.

Exercise 4.25. Find the Taylor series of the polynomial $z^6 - z^4 + z^2 - 1$ with the point $z_0 = 1$ taken as the center.

Corollary 4.26 (Cauchy's inequality)**.** *Suppose f is holomorphic in $D_R := \{z \in \mathbb{C} : |z - z_0| < R\}$ and $|f(z)| \le M$ for all $z \in D_R$. Then for $n \ge 0$,*

$$|f^{(n)}(z_0)| \le \frac{n! M}{R^n}.$$

Proof. Let C be the circle with center z_0 and radius $r < R$. Then

$$|f^{(n)}(z_0)| = \left| \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz \right| \le \frac{n!}{2\pi} \max_{z \in C} \left| \frac{f(z)}{(z-z_0)^{n+1}} \right| \cdot 2\pi r = \frac{n!}{2\pi} \frac{M}{r^{n+1}} 2\pi r = \frac{n! M}{r^n}.$$

The claim now follows by passing the limit $r \nearrow R$. □

Exercise 4.27. Suppose that f is an entire function for which there is an $M > 0$ and an $n \in \mathbb{N}$ such that for all $z \in \mathbb{C}$, $|f(z)| \le M|z|^n$. Use Cauchy's inequality to prove that $f^{(n+1)}(z) = 0$ for all z and show that f is a polynomial of degree at most n.

Exercise 4.28. Evaluate

$$\int_C \frac{\sin z}{z^{2012}} dz,$$

where C is the circular path with center 0 and radius 1 traversed in the anticlockwise direction.

4.4. Classification of zeros

Suppose p is a nonzero polynomial function. Then by the Division Algorithm, given any $z_0 \in \mathbb{C}$, we can divide p by $z - z_0$, and find the quotient polynomial q and the remainder $r \in \mathbb{C}$:

$$p(z) = (z - z_0) q(z) + r.$$

If the z_0 was a *zero* of p, that is, if $p(z_0) = 0$, then we must have $r = 0$, and so $p(z) = (z - z_0)q(z)$. Note that if p is of degree d, then q has degree $d - 1$. We can now ask if $q(z_0) = 0$, and if so, repeat the above with p replaced by q to get $p(z) = (z - z_0)q(z) = (z - z_0)^2 q_1(z)$, where q_1 is a polynomial with degree $d - 2$. Proceeding in this manner, we will eventually obtain

$$p(z) = (z - z_0)^m q_{m-1}(z),$$

where q_{m-1} is a polynomial of degree $d - m$, and $q_{m-1}(z_0) \neq 0$.

We will see in this section that a similar thing holds when instead of the polynomial, we have a holomorphic function. But let us first define what we mean by a zero of a holomorphic function.

Definition 4.29. Let D be a domain and $f : D \to \mathbb{C}$ be holomorphic in D.

A point $z_0 \in D$ is called a *zero of f* if $f(z_0) = 0$.

If there is a least $m \in \mathbb{N}$ such that

(1) $f^{(m)}(z_0) \neq 0$ and

(2) $f(z_0) = \cdots = f^{(m-1)}(z_0) = 0$,

then z_0 is said to be *a zero of f of order m*. (We adopt the convention that $f^{(0)} := f$.)

We have the following result on the classification of zeros of a holomorphic function.

Proposition 4.30 (Classification of zeros)**.** *Let D be a domain and $f : D \to \mathbb{C}$ be holomorphic in D such that $z_0 \in D$ is a zero of f. Then there are exactly two possibilities:*

$\underline{1}^\circ$ *There is a positive R such that $f(z) = 0$ for all z satisfying $|z - z_0| < R$.*

$\underline{2}^\circ$ *z_0 is a zero of order m and there exists a holomorphic function $g : D \to \mathbb{C}$ such that $g(z_0) \neq 0$ and $f(z) = (z - z_0)^m g(z)$ for all $z \in D$.*

Proof. We have a power series expansion for f in a disc with some radius $R > 0$ and center z_0:

$$f(z) = c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + c_3(z - z_0)^3 + \cdots \quad \text{for } |z - z_0| < R.$$

Since $f(z_0) = 0$, we know that $c_0 = 0$. Now there are exactly two possibilities:

$\underline{1}^\circ$ All the c_n are zero.

$\underline{2}^\circ$ There is a smallest $m \geq 1$ such that $c_m \neq 0$.

The first case gives $f(z) = 0$ whenever $|z - z_0| < R$.

In the latter case, we have $c_0 = c_1 = \cdots = c_{m-1} = 0$, and so using the fact that

$$c_n = \frac{f^{(n)}(z_0)}{n!}, \quad n \geq 0,$$

it follows that z_0 is a zero of order m. Moreover, from the power series expansion, we have

$$f(z) = c_m(z - z_0)^m + c_{m+1}(z - z_0)^{m+1} + \cdots = (z - z_0)^m \sum_{k=0}^{\infty} c_{m+k}(z - z_0)^k \tag{4.4}$$

for $|z - z_0| < R$. Thus, if we define $g : D \to \mathbb{C}$ by

$$g(z) = \begin{cases} \dfrac{f(z)}{(z - z_0)^m} & \text{for } z \neq z_0, \\ \displaystyle\sum_{k=0}^{\infty} c_{m+k}(z - z_0)^k & \text{for } |z - z_0| < R, \end{cases}$$

From (4.4), the two definitions give the same value whenever both are applicable. The function g is seen to be holomorphic near z_0 by the power series expansion for g, while at any z in D that is different from z_0, it is holomorphic by the first definition. Finally, $g(z_0) = c_m \neq 0$. This completes the proof. $\square$

Exercise 4.31. Let D be a domain, $m \in \mathbb{N}$, $R > 0$ and $z_0 \in D$. Let $f, g : D \to \mathbb{C}$ be holomorphic functions such that $g(z_0) \neq 0$ and whenever $|z - z_0| < R$, $f(z) = (z - z_0)^m g(z_0)$. Prove that z_0 is a zero of f of order m.

Exercise 4.32. Find the order of the zero z_0 for the function f in each case:

(1) $z_0 = i$ and $f(z) = (1 + z^2)^4$.

(2) $z_0 = 2n\pi i$, where n is an integer, and $f(z) = e^z - 1$.

(3) $z_0 = 0$ and $f(z) = \cos z - 1 + \frac{1}{2}(\sin z)^2$.

Exercise 4.33. Let f be holomorphic in a disc that contains a circle γ in its interior. Suppose there is exactly one zero z_0 of order 1 of f, which lies in the interior of γ. Prove that

$$z_0 = \frac{1}{2\pi i} \int_\gamma \frac{z f'(z)}{f(z)} dz.$$

Exercise 4.34. Let D be a domain and f be holomorphic in D such that f has a zero of order $m > 1$ at $z_0 \in D$. Prove that the function $z \mapsto (f(z))^2$ has a zero of order $2m$ at z_0, and that f' has a zero of order $m - 1$ at z_0.

4.5. The Identity Theorem

In this section, we will learn the Identity Theorem, which once again highlights the rigidity of holomorphic functions.

Lemma 4.35. *Let D be a domain and $f : D \to \mathbb{C}$ be a holomorphic function in D. Suppose that $(z_n)_{n\in\mathbb{N}}$ is a sequence of distinct zeros of f which converges to a point z_* in D. Then f is identically zero in a disc with some radius $R > 0$ centered at z_*.*

Proof. By the continuity of f,

$$f(z_*) = \lim_{n\to\infty} f(z_n) = \lim_{n\to\infty} 0 = 0.$$

From Proposition 4.30, if f is not identically zero in a disc centered at z_* with some positive radius, then z_* is a zero of order m of f, and there exists a holomorphic function $g : D \to \mathbb{C}$ such that

$$f(z) = (z - z_*)^m g(z) \quad (z \in D)$$

and $g(z_*) \neq 0$. Since g is continuous, $g(z) \neq 0$ in a disc with some radius $R > 0$ centered at z_*. But then for large n ensuring that z_n satisfies $0 < |z_n - z_*| < R$, we have the contradiction that

$$0 = f(z_n) = (z_n - z_*)^m g(z_n) \neq 0.$$

Hence f is identically zero in a disc centered at z_* with some positive radius. □

Theorem 4.36. *Let D be a domain and $f : D \to \mathbb{C}$ be holomorphic in D. Suppose that $(z_n)_{n\in\mathbb{N}}$ is a sequence of distinct zeros of f which converges to a point z_* in D. Then f is identically zero in D.*

Proof. Let $w \in D$. Then there is a path $\gamma : [0, 1] \to D$ such that $\gamma(0) = z_*$ and $\gamma(1) = w$. We know that there is a $R > 0$ such that whenever $|z - z_*| < R$, we have $f(z) = 0$. In particular, for small positive ts, $|\gamma(t) - z_*| < R$, and so $f(\gamma(t)) = 0$. We will show that in fact $f(\gamma(t)) = 0$ for *all* $t \in [0, 1]$, and in particular $f(w) = f(\gamma(1)) = 0$ too. To this end, define

$$T = \sup\{\tau \in [0, 1] : \forall t \in [0, \tau],\ f(\gamma(t)) = 0\}.$$

This T exists by the least upper bound property of $\mathbb{R}$, since $\{\tau \in [0, 1] : \forall t \in [0, \tau],\ f(\gamma(t)) = 0\}$ is bounded above by 1 and is nonempty because 0 belongs to it. By continuity of $f \circ \gamma$, $f(\gamma(T)) = 0$. By Lemma 4.35, there is a disc of some positive radius r centered at $\gamma(T)$ such that f is identically zero in that disc. But if $T < 1$, then for ts in $(T, 1]$ that are close enough to T, we will have $|\gamma(t) - \gamma(T)| < r$, and so $f(\gamma(t)) = 0$ for such ts. This contradicts the definition of T. Hence $T = 1$, and so $f(w) = f(\gamma(1)) = 0$. This completes the proof. □

The following is an immediate consequence of this result.

Corollary 4.37 (Identity Theorem)**.** *Let D be a domain and $f, g : D \to \mathbb{C}$ be holomorphic in D. Suppose that $(z_n)_{n\in\mathbb{N}}$ is a sequence of distinct points in D which converges to a point z_* in D, and such that for all $n \in \mathbb{N}$, $f(z_n) = g(z_n)$. Then $f(z) = g(z)$ for all $z \in D$.*

Proof. Define $\varphi : D \to \mathbb{C}$ by $\varphi(z) = f(z) - g(z)$, and note that the z_ns are zeros of the holomorphic function φ. By the result above, φ must be identically zero in D, and so the claim follows. □

Exercise 4.38. Show that there is only one entire function f such that $f(x) = e^x$ for all $x \in \mathbb{R}$.

Exercise 4.39. Show, using the Identity Theorem, that for all $z_1, z_2 \in \mathbb{C}$,

$$\cos(z_1 + z_2) = (\cos z_1)(\cos z_2) - (\sin z_1)(\sin z_2),$$

by appealing to the corresponding identity when z_1, z_2 are real numbers.

Exercise 4.40. Let D be a domain and let $H(D)$ be the set of all functions holomorphic in D. Then it is easy to check that $H(D)$ is a commutative ring with the pointwise operations

$$\begin{aligned}(f+g)(z) &= f(z)+g(z),\\ (f\cdot g)(z) &= f(z)g(z),\end{aligned}$$

for $z \in D$ and $f, g \in H(D)$. (By a *commutative ring* R, we mean a set R with two laws of composition $+$ and $\cdot$ such that $(R, +)$ is an Abelian group, $\cdot$ is associative, commutative and has an identity, and the distributive law holds: for $a, b, c \in R$, $(a+b)\cdot c = a\cdot c + b \cdot c$.) Check that $H(D)$ is an *integral domain*, that is, a nonzero ring having no zero divisors. In other words, if $f \cdot g = 0$ for $f, g \in H(D)$, then either $f = 0$ or $g = 0$.

If instead of $H(D)$, we consider the set $C(D)$ of all complex-valued *continuous* functions on D, then $C(D)$ is again a commutative ring with pointwise operations. Is $C(D)$ an integral domain? (This shows that continuous functions are not as "rigid" as holomorphic functions.)

Exercise 4.41. Let f, g be holomorphic functions in a domain D. Which of the following conditions imply $f = g$ identically in D?

(1) There is a sequence $(z_n)_{n\in\mathbb{N}}$ of distinct points in D such that $f(z_n) = g(z_n)$ for all $n \in \mathbb{N}$.
(2) There is a convergent sequence $(z_n)_{n\in\mathbb{N}}$ of distinct points in D with $\lim\limits_{n\to\infty} z_n \in D$ such that $f(z_n) = g(z_n)$ for all $n \in \mathbb{N}$.
(3) γ is a smooth path in D joining distinct points $a, b \in D$ and $f = g$ on γ.
(4) $w \in D$ is such that $f^{(n)}(w) = g^{(n)}(w)$ for all $n \geq 0$.

Exercise 4.42. Suppose that f is an entire function, and that in every power series

$$f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n,$$

at least one coefficient is 0. Prove that f is a polynomial.

4.6. The Maximum Modulus Theorem

In this section, we prove an important result, known as the Maximum Modulus Theorem.

Theorem 4.43 (Maximum Modulus Theorem)**.** *Let D be a domain and let $f : D \to \mathbb{C}$ be holomorphic in D. Suppose that there is a $z_0 \in D$ such that for all $z \in D$, $|f(z_0)| \geq |f(z)|$. Then f is constant on D.*

Proof. Let $r > 0$ be such that the disc with center z_0 and radius $2r$ is contained in D. Let γ be the circular path defined by $\gamma(t) = z_0 + re^{it}$, $t \in [0, 2\pi]$. Then by the Cauchy Integral Formula,

$$f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_0}dz = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})dt,$$

and since $|f(z_0 + re^{it})| \leq |f(z_0)|$ for all t, the above yields

$$|f(z_0)| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{it})|dt \leq |f(z_0)|.$$

By rearranging, this implies that

$$\frac{1}{2\pi}\int_0^{2\pi} \left(|f(z_0)| - |f(z_0 + re^{it})|\right) dt = 0.$$

But the integrand is pointwise nonnegative, and so $|f(z_0 + re^{it})| = |f(z_0)|$ for all t. But by replacing r be any smaller number, the same conclusion would hold. Thus we have that f maps the disc Δ with center z_0 and radius r into the circle $\{w \in \mathbb{C} : |w| = |f(z_0)|\}$. This implies by Example 2.25 that f is constant on Δ. The Identity Theorem now implies that f must be constant on the whole of D. $\square$

Exercise 4.44. Let D be a domain and let $f : D \to \mathbb{C}$ be a nonconstant holomorphic map. Prove that there is no maximizer for the map $z \mapsto |f(z)|$ on D.

Exercise 4.45 (Minimum Modulus Theorem)**.** Let D be a domain and let $f : D \to \mathbb{C}$ be holomorphic in D. Suppose that there is a $z_0 \in D$ such that for all $z \in D$, $|f(z_0)| \leq |f(z)|$. Then prove that either $f(z_0) = 0$ or f is constant on D.

Exercise 4.46. Consider the function f defined by $f(z) = z^2 - 2$. Find the maximum and minimum value of $|f(z)|$ on $\{z \in \mathbb{C} : |z| \leq 1\}$.

4.7. Laurent series

Laurent series generalize Taylor series. Indeed, while a Taylor series

$$\sum_{n=0}^{\infty} c_n (z - z_0)^n$$

has nongenative powers of the term $z - z_0$, and converges in a disc, a *Laurent series* is an expression of the type

$$\sum_{n \in \mathbb{Z}} c_n (z - z_0)^n,$$

which has negative powers of $z - z_0$ too, and we will see that it "converges" in an annulus and gives a holomorphic function there. Conversely, we will also learn that if we have a holomorphic function in an annulus and has singularities that lie in the "hole" inside the annulus, then the function has a Laurent series expansion in the annulus. For example, we know that for $z \neq 0$, the following holds:

$$e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{z}\right)^n = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}.$$

Let us first define precisely what we mean by the convergence of the Laurent series $\sum_{n \in \mathbb{Z}} c_n (z - z_0)^n$.

Definition 4.47. The Laurent series $\sum_{n \in \mathbb{Z}} c_n (z - z_0)^n$ is said to *converge* if the two series

$$\sum_{n=1}^{\infty} c_{-n} (z - z_0)^{-n} \quad \text{and} \quad \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

converge. In this case, we write

$$\sum_{n \in \mathbb{Z}} c_n (z - z_0)^n = \sum_{n=1}^{\infty} c_{-n} (z - z_0)^{-n} + \sum_{n=0}^{\infty} c_n (z - z_0)^n,$$

and call it the *sum* of the Laurent series.

Let us first find out for what z the Laurent series *could* possibly converge. From Theorem 4.7. the power series

$$\sum_{n=0}^{\infty} c_n (z - z_0)^n$$

converges inside a disc $\{z \in \mathbb{C} : |z| < R_2\}$ and diverges outside it. The series

$$\sum_{n=1}^{\infty} c_{-n} w^n$$

also converges inside a disc $\{w \in \mathbb{C} : |w| < r_1\}$ and diverges outside it. Thus

$$\sum_{n=1}^{\infty} c_{-n} (z - z_0)^{-n}$$

converges for

$$\frac{1}{|z-z_0|} < r_1,$$

that is for $|z-z_0| > \frac{1}{r_1} =: R_1$, and diverges for $|z-z_0| < R_1$. Hence the Laurent series converges in the annulus $\{z \in \mathbb{C} : R_1 < |z-z_0| < R_2\}$ and diverges if either $|z-z_0| < R_1$ or $|z-z_0| > R_2$. Also since both

$$z \mapsto \sum_{n=0}^{\infty} c_n(z-z_0)^n \quad \text{and} \quad z \mapsto \sum_{n=1}^{\infty} c_{-n}(z-z_0)^{-n}$$

are holomorphic in $\{z \in \mathbb{C} : R_1 < |z-z_0| < R_2\}$, so is their sum.

That conversely, a function holomorphic in an annulus has a Laurent series expansion is the content of the following theorem.

Theorem 4.48. *If f is holomorphic in the annulus $\mathbb{A} := \{z \in \mathbb{C} : R_1 < |z-z_0| < R_2\}$, where $R_1 \geq 0$, then*

$$f(z) = \sum_{n\in\mathbb{Z}} c_n(z-z_0)^n \quad \textit{for } z \in \mathbb{A},$$

where

$$c_n = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-z_0)^{n+1}}d\zeta,$$

and C is the circular path given by $C(t) = z_0 + re^{it}$ $(t \in [0, 2\pi])$, and r is any number such that $R_1 < r < R_2$. Moreover the series is unique.

Proof. (Existence.) Let $z \in \mathbb{A}$, and fix r_1 and r_2 such that $R_1 < r_1 < |z-z_0| < r_2 < R_2$. Let γ_1 and γ_2 be the circular paths given by

$$\begin{aligned} \gamma_1(t) &= z_0 + r_1 e^{it}, \\ \gamma_2(t) &= z_0 + r_2 e^{it}, \end{aligned}$$

for $t \in [0, 2\pi]$. Let $\gamma_3 : [r_1, r_2] \to \mathbb{A}$ be the path

$$\gamma_3(t) = ti\frac{z}{|z|}.$$

Thus γ_3 is a straight line path that avoids z and joins γ_1 to γ_2. See Figure 2.

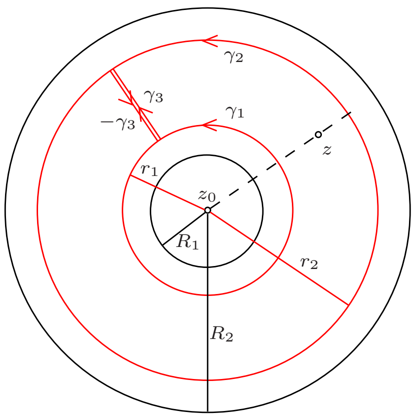


Figure 2. Laurent series.

Clearly the path $\gamma := \gamma_2 - \gamma_3 - \gamma_1 + \gamma_3$ is $\mathbb{A}$-contractible to z, and so by the Cauchy Integral Formula,

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}d\zeta = \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\zeta)}{\zeta - z}d\zeta - \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(\zeta)}{\zeta - z}d\zeta,$$

since the contour integral along γ_3 cancels with that along $-\gamma_3$. We have

$$\frac{1}{2\pi i}\int_{\gamma_2}\frac{f(\zeta)}{\zeta-z}d\zeta=\frac{1}{2\pi i}\int_{\gamma_2}\frac{f(\zeta)}{\zeta-z_0+z_0-z}d\zeta=\frac{1}{2\pi i}\int_{\gamma_2}\frac{f(\zeta)}{(\zeta-z_0)\left(1-\dfrac{z-z_0}{\zeta-z_0}\right)}d\zeta$$

Set $w:=\dfrac{z-z_0}{\zeta-z_0}$. Then $|w|=\dfrac{|z-z_0|}{r_2}<1$. Thus

$$\frac{1}{1-\dfrac{z-z_0}{\zeta-z_0}}=\frac{1}{1-w}=1+w+w^2+w^3+\cdots+w^{n-1}+\frac{w^n}{1-w},$$

and so

$$\begin{aligned}\frac{1}{2\pi i}\int_{\gamma_2}\frac{f(\zeta)}{\zeta-z}d\zeta &= \frac{1}{2\pi i}\int_{\gamma_2}f(\zeta)\left(\frac{1}{\zeta-z_0}+\cdots+\frac{(z-z_0)^{n-1}}{(\zeta-z_0)^n}+\frac{(z-z_0)^n}{(\zeta-z_0)^n(\zeta-z)}\right)d\zeta\\ &= c_0+c_1(z-z_0)+\cdots+c_{n-1}(z-z_0)^{n-1}+R_n(z),\end{aligned}$$

where

$$R_n(z):=\frac{1}{2\pi i}\int_{\gamma_2}\frac{f(\zeta)(z-z_0)^n}{(\zeta-z_0)^n(\zeta-z)}d\zeta.$$

Here we have used the fact that γ_2 is $\mathbb{A}$-homotopic to any circle C with center z_0 and radius r where $R_1<r<R_2$, and so by Theorem 3.25,

$$\int_{\gamma_2}\frac{f(\zeta)}{(\zeta-z_0)^k}d\zeta=\int_C\frac{f(\zeta)}{(\zeta-z_0)^k}d\zeta=c_k$$

for $k=1,\dots,n-1$. There is a $M>0$ such that for all $\zeta\in\gamma_2$, $|f(\zeta)|<M$. Moreover, $|\zeta-z_0|=r_2$ and

$$|\zeta-z|=|\zeta-z_0-(z-z_0)|\geq|\zeta-z_0|-|z-z_0|=r_2-|z-z_0|,$$

and so

$$|R_n(z)|\leq\left(\frac{|z-z_0|}{r_2}\right)^n\frac{M}{r_2-|z-z_0|}\overset{n\to\infty}{\longrightarrow}0.$$

Thus

$$c_0+c_1(z-z_0)+c_2(z-z_0)^2+c_3(z-z_0)^3+\cdots=\frac{1}{2\pi i}\int_{\gamma_2}\frac{f(\zeta)}{\zeta-z}d\zeta.$$

Similarly, we have

$$-\frac{1}{2\pi i}\int_{\gamma_1}\frac{f(\zeta)}{\zeta-z}d\zeta=\frac{1}{2\pi i}\int_{\gamma_1}\frac{f(\zeta)}{(z-z_0)-(\zeta-z_0)}d\zeta=\frac{1}{2\pi i}\int_{\gamma_1}\frac{f(\zeta)}{(z-z_0)\left(1-\dfrac{\zeta-z_0}{z-z_0}\right)}d\zeta$$

Set $w:=\dfrac{\zeta-z_0}{z-z_0}$. Then $|w|=\dfrac{r_1}{|z-z_0|}<1$. Thus

$$\frac{1}{1-\dfrac{\zeta-z_0}{z-z_0}}=\frac{1}{1-w}=1+w+w^2+w^3+\cdots+w^{n-1}+\frac{w^n}{1-w},$$

and so

$$\begin{aligned}-\frac{1}{2\pi i}\int_{\gamma_1}\frac{f(\zeta)}{\zeta-z}d\zeta &= \frac{1}{2\pi i}\int_{\gamma_1}f(\zeta)\left(\frac{1}{z-z_0}+\cdots+\frac{(\zeta-z_0)^{n-1}}{(z-z_0)^n}+\frac{(\zeta-z_0)^n}{(z-z_0)^n(z-\zeta)}\right)d\zeta\\ &= c_{-1}(z-z_0)^{-1}+c_{-2}(z-z_0)^{-2}+\cdots+c_{-(n-1)}(z-z_0)^{-(n-1)}+\widetilde{R}_n(z),\end{aligned}$$

where

$$\widetilde{R}_n(z):=\frac{1}{2\pi i}\int_{\gamma_1}\frac{f(\zeta)(\zeta-z_0)^n}{(z-z_0)^n(z-\zeta)}d\zeta.$$

Here we have used the fact that γ_1 is $\mathbb{A}$-homotopic to any circle C with center z_0 and radius r where $R_1 < r < R_2$, and so by Theorem 3.25,

$$\int_{\gamma_1} \frac{f(\zeta)}{(\zeta - z_0)^k} d\zeta = \int_C \frac{f(\zeta)}{(\zeta - z_0)^k} d\zeta = c_k$$

for $k = -1, \ldots, -(n-1)$. There is a $M > 0$ such that for all $\zeta \in \gamma_1$, $|f(\zeta)| < M$. Moreover, $|\zeta - z_0| = r_1$ and

$$|z - \zeta| = |(z - z_0) - (\zeta - z_0)| \geq |z - z_0| - |\zeta - z_0| = |z - z_0| - r_1,$$

and so

$$|\widetilde{R}_n(z)| \leq \left(\frac{r_1}{|z - z_0|}\right)^n \frac{M}{|z - z_0| - r_1} \overset{n\to\infty}{\longrightarrow} 0.$$

Thus

$$c_{-1}(z - z_0)^{-1} + c_{-2}(z - z_0)^{-2} + \cdots + c_{-(n-1)}(z - z_0)^{-(n-1)} + \cdots = -\frac{1}{2\pi i} \int_{\gamma_1} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

This completes the proof of the part on the existence of the Laurent expansion.

(Uniqueness.) Cauchy's Integral Formula allows us to show that the Laurent expansion is unique, that is, if

$$f(z) = \sum_{n\in\mathbb{Z}} \widetilde{c}_n (z - z_0)^n \quad \text{for } R_1 < |z - z_0| < R_2,$$

then for all n, $\widetilde{c}_n = c_n$.

First we note that if $n \neq 1$, then $(z - z_0)^n = \dfrac{d}{dz}\left(\dfrac{(z - z_0)^{n+1}}{n+1}\right)$, and so

$$\int_C (z - z_0)^n dz = 0 \quad (n \neq 1),$$

where C is given by $C(t) = z_0 + re^{it}$ $(t \in [0, 2\pi])$. By a direct calculation,

$$\int_C \frac{1}{z - z_0} dz = \int_0^{2\pi} \frac{1}{re^{it}} ire^{it} dt = 2\pi i.$$

Hence, if term-by-term integration is justified in the annulus, we would have

$$\int_C (z - z_0)^{-m-1} \sum_{n\in\mathbb{Z}} \widetilde{c}_n (z - z_0)^n dz = \sum_{n\in\mathbb{Z}} \widetilde{c}_n \int_C (z - z_0)^{n-m-1} dz = 2\pi i \widetilde{c}_m,$$

and the claim about the uniqueness of coefficients would be proved. This term-by-term integration can be justified as follows. We have

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \widetilde{c}_n (z - z_0)^{n-m-1} &= \left(\cdots + \frac{\widetilde{c}_{m-2}}{(z - z_0)^3} + \frac{\widetilde{c}_{m-1}}{(z - z_0)^2}\right) + \frac{\widetilde{c}_m}{z - z_0} + (\widetilde{c}_{m+1} + \widetilde{c}_{m+2}(z - z_0) + \cdots) \\
&= f_1(z) + \frac{\widetilde{c}_m}{z - z_0} + f_2(z).
\end{aligned}$$

We need only show that f_1, f_2 each have a primitive in the annulus and then

$$\int_C \sum_{n\in\mathbb{Z}} \widetilde{c}_n (z - z_0)^{n-m-1} dz = \int_C \left(f_1(z) + \frac{\widetilde{c}_m}{z - z_0} + f_2(z)\right) dz = 0 + 2\pi i \widetilde{c}_m + 0 = 2\pi i \widetilde{c}_m,$$

as required. But $f_2(z) = \displaystyle\sum_{n=1}^{\infty} \widetilde{c}_{m+n} (z - z_0)^{n-1}$ for $|z - z_0| < R_2$, and so if

$$F_2(z) := \sum_{n=1}^{\infty} \frac{\widetilde{c}_{m+n}}{n} (z - z_0)^n \quad \text{for } |z - z_0| < R_2,$$

then $\dfrac{d}{dz} F_2(z) = f_2(z)$, and so F_2 is a primitive of f_2.

For f_1, we have

$$f_1(z) = \sum_{n=1}^{\infty} \frac{\widetilde{c}_{m-n}}{(z-z_0)^{n+1}} = \sum_{n=1}^{\infty} \widetilde{c}_{m-n} w^{n+1},$$

where $w = \dfrac{1}{z-z_0}$, and this is valid for $R_2 > |z-z_0| > R_1$, and so

$$\sum_{n=1}^{\infty} \widetilde{c}_{m-n} w^{n+1}$$

converges for $|w| < \dfrac{1}{R_1}$. If we set

$$G(w) = -\sum_{n=1}^{\infty} \frac{\widetilde{c}_{m-n}}{n} w^n \quad \text{for } |w| < \frac{1}{R_1},$$

then $\dfrac{d}{dz} G(w) = -\displaystyle\sum_{n=1}^{\infty} \widetilde{c}_{m-n} w^{n-1}$. Hence if define F_1 by

$$F_1(z) = G\left(\frac{1}{z-z_0}\right) = \sum_{n=1}^{\infty} \frac{\widetilde{c}_{m-n}}{n} (z-z_0)^{-n} \quad \text{for } z \in \mathbb{A},$$

then $\dfrac{d}{dz} F_1(z) = \left(G'\left(\dfrac{1}{z-z_0}\right)\right)\left(-\dfrac{1}{(z-z_0)^2}\right) = (z-z_0)^{-2} \displaystyle\sum_{n=1}^{\infty} \widetilde{c}_{m-n}(z-z_0)^{-n+1} = f_1(z)$. $\square$

Note that the uniqueness of coefficients is valid only if we consider a particular fixed annulus. It can happen that the same function has different Laurent expansions, but valid in *different* annuli, as shown in the following example.

Example 4.49. Consider the function f defined by $f(z) = \dfrac{1}{z(z-1)}$.

Then if we consider the annulus $|z| > 1$, then the Laurent series expansion of f is given by

$$f(z) = \frac{1}{z(z-1)} = \frac{1}{z^2\left(1-\frac{1}{z}\right)} = \frac{1}{z^2}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right) = \sum_{n=2}^{\infty} z^{-n}.$$

On the other hand, if we consider the annulus $0 < |z| < 1$, then the Laurent series expansion of f is

$$f(z) = \frac{1}{z(z-1)} = -\frac{1}{z}\left(1 + z + z^2 + z^3 + \cdots\right) = -\frac{1}{z} - \sum_{n=0}^{\infty} z^n.$$

$\diamond$

Example 4.50. Consider the function f given by $f(z) = z^3 e^{\frac{1}{z}}$. Then f is holomorphic in $\mathbb{C} \setminus \{0\}$, and its Laurent expansion in the annulus $|z| > 0$ is given by

$$f(z) = z^3 e^{\frac{1}{z}} = z^3\left(1 + \frac{1}{1!}\frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \cdots\right) = z^3 + z^2 + \frac{z}{2} + \frac{1}{6} + \sum_{n=1}^{\infty} \frac{1}{(n+3)!} z^{-n}.$$

$\diamond$

4.8. Classification of singularities

If we look at the three functions

$$\frac{\sin z}{z}, \quad \frac{1}{z^3}, \quad e^{\frac{1}{z}},$$

then we notice that each of them is not defined at 0, but we will see that the "singularity" at 0 in each case has a very different nature. We give the following definition.

Definition 4.51. Let f be holomorphic in the punctured disc $\{z \in \mathbb{C} : 0 < |z - z_0| < R\}$ for some $R > 0$. Then we call z_0 an *isolated singularity* of f. The singularity is called

(1) *removable* if there is a function F, holomorphic for $|z - z_0| < R$ such that $F = f$ for $0 < |z - z_0| < R$.

(2) *a pole* if $\lim\limits_{z \to z_0} |f(z)| = +\infty$, that is, for every $M > 0$ there is a $\delta > 0$ such that whenever $0 < |z - z_0| < \delta$, $|f(z)| > M$.

(3) *essential* if z_0 is neither removable nor a pole.

Example 4.52.

(1) The function $\dfrac{\sin z}{z}$ has a removable singularity at 0, since for $z \neq 0$, we have

$$\frac{\sin z}{z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - + \cdots\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n},$$

and the right hand side, being a power series with an infinite radius of convergence (why?) defines an entire function.

(2) The function $\dfrac{1}{z^3}$ has a pole at 0, since $\lim\limits_{z \to 0} \dfrac{1}{|z|^3} = +\infty$.

(3) The function $e^{\frac{1}{z}}$ has an essential singularity at 0. Indeed, 0 is not a removable singularity, because for example

$$\lim_{x \searrow 0} e^{\frac{1}{x}} = +\infty.$$

0 is also not a pole, since $\lim\limits_{x \nearrow 0} e^{\frac{1}{x}} = 0$, showing that it isn't true that $\lim\limits_{z \to 0} |f(z)| = +\infty$. $\Diamond$

The following result gives a characterization of the types of singularities in terms of limiting behaviour.

Proposition 4.53. *Suppose z_0 is an isolated singularity of f. Then*

(1) *z_0 is removable if and only if* $\lim\limits_{z \to z_0} (z - z_0) f(z) = 0$.

(2) *z_0 is a pole if and only if there exists an $n \in \mathbb{N}$ such that*
   (a) $\lim\limits_{z \to z_0} (z - z_0)^{n+1} f(z) = 0$ *and*
   (b) *it is not the case that* $\lim\limits_{z \to z_0} (z - z_0) f(z) = 0$.

   We then call the smallest such n the order *of the pole z_0.*

Proof. (1) Let z_0 be removable, and F be holomorphic in $D_R := \{z \in \mathbb{C} : |z - z_0| < R\}$ such that $F = f$ for $0 < |z - z_0| < R$. Then using the fact that F is continuous at z_0, we obtain

$$\lim_{z \to z_0} (z - z_0) f(z) = \lim_{z \to z_0} (z - z_0) F(z) = 0 \cdot F(z_0) = 0.$$

Conversely, suppose that

$$\lim_{z \to z_0} (z - z_0) f(z) = 0,$$

and that f is holomorphic in the punctured disc $\{z \in \mathbb{C} : 0 < |z - z_0| < R\}$. Then f has a Laurent expansion

$$f(z) = \sum_{n \in \mathbb{Z}} c_n (z - z_0)^n,$$

where

$$c_n = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{(z - z_0)^{n+1}} dz,$$

and C_r is a circular path of any positive radius $r < R$ with center z_0. We show that $c_{-n} = 0$ for $n \in \mathbb{N}$. Indeed, given $\epsilon > 0$, first choose r small enough so that whenever $|z - z_0| = r$, we have $|(z - z_0)f(z)| < \epsilon$. Then we have for $n \in \mathbb{N}$ that

$$|c_{-n}| = \left| \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{(z - z_0)^{-n+1}} dz \right| \leq \frac{1}{2\pi} \frac{\epsilon}{r^{-n+2}} 2\pi r \leq \epsilon R^{n-1}.$$

Since the choice of $\epsilon > 0$ was arbitrary, it follows that $c_{-n} = 0$ for all $n \in \mathbb{N}$. Consequently, with

$$F(z) := \sum_{n=0}^{\infty} c_n (z - z_0)^n,$$

we see that F is holomorphic in $\{z \in \mathbb{C} : |z - z_0| < R\}$, and $F = f$ in the punctured disc $\{z \in \mathbb{C} : 0 < |z - z_0| < R\}$.

(2) Suppose that z_0 is a pole of f. Then there is some $R > 0$ such that $|f(z)| > 1$ in the punctured disc $D := \{z \in \mathbb{C} : 0 < |z - z_0| < R\}$. Define g in this punctured disc D by

$$g(z) = \frac{1}{f(z)}.$$

Since z_0 is a pole of f, it follows that $\lim\limits_{z \to z_0} g(z) = 0$. In particular, also

$$\lim_{z \to z_0} (z - z_0) g(z) = 0,$$

and so by the first part above, g has a holomorphic extension G to $\{z \in \mathbb{C} : |z - z_0| < R\}$. Also,

$$G(z_0) = \lim_{z \to z_0} g(z) = 0.$$

So z_0 is a zero of G, and since G is not identically zero in a neighbourhood of z_0, it follows from the result on the classification of zeros that z_0 has some order n, and there is a holomorphic function φ defined for $|z - z_0| < R$ such that $\varphi(z_0) \neq 0$ and $G(z) = (z - z_0)^n \varphi(z)$. In particular, in D we have

$$f(z) = \frac{1}{g(z)} = \frac{1}{G(z)} = \frac{1}{(z - z_0)^n \varphi(z)}.$$

Hence $(z - z_0)^{n+1} f(z) = \dfrac{(z - z_0)}{\varphi(z)} \longrightarrow \dfrac{0}{\varphi(z_0)} = 0$ as $z \to z_0$. Also, it can't be the case that

$$\lim_{z \to z_0} (z - z_0) f(z) = 0,$$

since otherwise, by the result in the previous part, z_0 would be a removable singularity.

Conversely, suppose that there exists an $n \in \mathbb{N}$ such that

$$\lim_{z \to z_0} (z - z_0)^{n+1} f(z) = 0$$

and that f is holomorphic in the punctured disc $\{z \in \mathbb{C} : 0 < |z - z_0| < R\}$. Suppose that we have chosen the smallest such n, that is,

$$\neg \left(\lim_{z \to z_0} (z - z_0)^n f(z) = 0 \right). \tag{4.5}$$

Then $\lim_{z \to z_0} (z - z_0)\left((z - z_0)^n f(z)\right) = 0$, and so by the previous part, there is a holomorphic extension h of $(z - z_0)^n f(z)$ to the disc $\{z \in \mathbb{C} : |z - z_0| < R\}$. In particular, for $0 < |z - z_0| < R$,

$$f(z) = \frac{h(z)}{(z - z_0)^n}.$$

From (4.5), $h(z_0) = \lim_{z \to z_0} h(z) = \lim_{z \to z_0} (z - z_0)^n f(z) \neq 0$. Since $h(z) \to h(z_0) \neq 0$ as $z \to z_0$,

$$|f(z)| = \frac{|h(z)|}{|z - z_0|^n} \to +\infty.$$

So z_0 is a pole of f. $\square$

We now show a result which illustrates the "wild" behaviour of a function f at its essential singularity z_0. It says that given any complex number w, any $\epsilon > 0$, and any arbitrary small punctured disc Δ with center z_0, there is a point z in Δ such that $f(z)$ lies within a distance ϵ from w. So the image of any punctured disc centered at the essential singularity is dense in $\mathbb{C}$. Or in even more descriptive terms, f comes arbitrarily close to any complex value in every neighbourhood of z_0.

Theorem 4.54 ("Casorati-Weierstrass"[3]). *Suppose z_0 is an essential singularity of f. Then for every complex number w, every $\delta > 0$ and every $\epsilon > 0$, there exists a $z \in \mathbb{C}$ such that $|z - z_0| < \delta$ and $|f(z) - w| < \epsilon$.*

Proof. Suppose that there is a w and an $\epsilon > 0$ such that for all z in some punctured disc D with center z_0, we have $|f(z) - w| \geq \epsilon$. Then the function g defined by

$$g(z) = \frac{1}{f(z) - w}$$

for $z \in D$ is holomorphic there, and

$$\lim_{z \to z_0} (z - z_0) g(z) = 0,$$

since g is bounded in D (by $1/\epsilon$). So g has a removable singularity at z_0. Let m be the order of the zero of g at z_0. (Set $m = 0$ if $g(z_0) \neq 0$.) Write $g(z) = (z - z_0)^m \varphi(z)$, where φ is holomorphic in D and $\varphi(z_0) \neq 0$. Then

$$(z - z_0)^{m+1} f(z) = (z - z_0)^{m+1} w + \frac{(z - z_0)}{\varphi(z)} \longrightarrow 0w + \frac{0}{\varphi(z_0)} = 0.$$

Thus we arrive at the contradiction that either f has a removable singularity at z_0 (when $m = 0$) or a pole at z_0 (when $m \in \mathbb{N}$). $\square$

Example 4.55. The function $e^{\frac{1}{z}}$ has an essential singularity at $z = 0$. It has no limit as z approaches 0 along the imaginary axis. (Why?) It becomes unboundedly large as z approaches 0 through positive real values, and it approaches 0 as z approaches 0 through negative real values.

We show that it takes on any given nonzero value $w = |w|e^{i\theta}$ in any arbitrarily small neighbourhood of $z = 0$. Setting $z = re^{it}$, we need to solve

$$e^{\frac{1}{z}} = e^{\frac{\cos t}{r} - i\frac{\sin t}{r}} = |w|e^{i\theta},$$

and so by equating the absolute value and arguments, we obtain

$$\cos t = r \log |w| \quad \text{and} \quad \sin t = -r\theta.$$

[3] This result was published by Weierstrass in 1876 (in German) and by the Sokhotski in 1873 (in Russian). So it was called Sokhotski's theorem in the Russian literature and Weierstrass's theorem in the Western literature. The same theorem was published by Casorati in 1868, and by Briot and Bouquet in the first edition of their book (1859), called *Theorie des fonctions doublement periodiques, et en particulier, des fonctions elliptiques.* However, Briot and Bouquet removed this theorem from the second edition (1875).

Using $(\cos t)^2 + (\sin t)^2 = 1$, we have

$$r = \frac{1}{\sqrt{(\log |w|)^2 + \theta^2}} \quad \text{and} \quad \tan t = -\frac{\theta}{\log |w|}.$$

But we are allowed to increase θ by integral multiples of 2π, without changing w. Bearing this in mind, it is clear from the above expression for r that we can make r as small as we please. $\Diamond$

The above example illustrates a much stronger theorem than the Casorati-Weierstrass Theorem, due to Picard, which says that the image of any punctured disc centered at an essential singularity misses at most one point of $\mathbb{C}$! In our example above, the exceptional value is $w = 0$. A proof of Picard's Theorem is beyond the scope of these notes, but can be found in Conway's book [**C**].

Proposition 4.56. *Suppose that z_0 is an isolated singularity of f with the Laurent series expansion*

$$f(z) = \sum_{n\in\mathbb{Z}} c_n (z - z_0)^n \quad \textit{for } 0 < |z - z_0| < R,$$

where R is some positive number. Then

(1) *z_0 is a removable singularity if and only if $c_n = 0$ for all $n < 0$.*

(2) *z_0 is a pole if and only if there is some $m \in \mathbb{N}$ such that $c_{-m} \neq 0$ and $c_n = 0$ for all $n < -m$. Moreover, the order of the pole coincides with m.*

(3) *z_0 is an essential singularity if and only if there are infinitely many indices $n < 0$ such that $c_n \neq 0$.*

Proof. (1) Suppose that z_0 is a removable singularity. Then f has a holomorphic extension F defined for $|z - z_0| < R$. But then this holomorphic F has a Taylor series expansion

$$F(z) = \sum_{n=0}^{\infty} \widetilde{c}_n (z - z_0)^n \text{ for } |z - z_0| < R.$$

In particular, for $0 < |z - z_0| < R$, we obtain

$$f(z) = \sum_{n=0}^{\infty} \widetilde{c}_n (z - z_0)^n = \sum_{n\in\mathbb{Z}} c_n (z - z_0)^n,$$

and by the uniqueness of the Laurent series expansion in an annulus, it follows that $c_n = \widetilde{c}_n$ for $n \geq 0$, and $c_n = 0$ for all $n < 0$.

Conversely, suppose that $c_n = 0$ for all $n < 0$. Then defining F for $|z - z_0| < R$ by

$$F(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n,$$

we see that F is a holomorphic extension of f to $\{z \in \mathbb{C} : |z - z_0| < R\}$.

(2) Suppose z_0 is a pole of order m. Then the function $(z - z_0)^m f(z)$ has a removable singularity at z_0. Hence by the previous part,

$$(z - z_0)^m f(z) = (z - z_0)^m \sum_{n\in\mathbb{Z}} c_n (z - z_0)^n = \sum_{n\in\mathbb{Z}} c_n (z - z_0)^{n+m}$$

has all coefficients of negative powers of $z - z_0$ equal to 0. Hence

$$(z - z_0)^m f(z) = c_{-m} + c_{-m+1}(z - z_0) + c_{-m+2}(z - z_0)^2 + \cdots \tag{4.6}$$

and so

$$f(z) = \frac{c_{-m}}{(z - z_0)^m} + \frac{c_{-m+1}}{(z - z_0)^{m-1}} + \cdots + \frac{c_{-1}}{(z - z_0)} + c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \cdots .$$

Moreover, the function $c_{-m}+c_{-m+1}(z-z_0)+c_{-m+2}(z-z_0)^2+\cdots$ is holomorphic, and so it has the limit c_{-m} as $z \to z_0$. On the other hand, since z_0 is a pole of order m, we know that it is not the case that $\lim_{z\to z_0}(z-z_0)^m f(z)=0$. Consequently, (4.6) implies that $c_{-m}\neq 0$.

Conversely, suppose that there is some $m \in \mathbb{N}$ such that $c_{-m}\neq 0$ and $c_n=0$ for all $n<-m$. Then we have $(z-z_0)^m f(z)=c_{-m}+c_{-m+1}(z-z_0)+c_{-m+2}(z-z_0)^2+\cdots$, and since the right hand side defines a holomorphic function in $\{z\in\mathbb{C} : |z-z_0|<R\}$, it follows that

$$\lim_{z\to z_0}(z-z_0)^m f(z)=c_{-m}\neq 0, \quad \text{and} \quad \lim_{z\to z_0}(z-z_0)^{m+1}f(z)=0.$$

Thus z_0 is a pole of order m of f.

(3) This is immediate from the previous two parts and the fact that an essential singularity is neither a removable singularity nor a pole. □

Exercise 4.57. Let D be a domain and f be holomorphic in D such that f has only one zero z_0 in D, and the order of z_0 is $m>1$. Show that the function $z\mapsto \frac{1}{f(z)}$ has a pole of order m at z_0.

Exercise 4.58. Let D be a disc with center z_0. Suppose that f is nonzero and holomorphic in $D\setminus\{0\}$, and that f has a pole of order m at z_0. Show that the function $z\mapsto\frac{1}{f(z)}$ has a holomorphic extension g to D, and that g has a zero of order m at z_0.

Exercise 4.59. Let D be a domain and $z_0\in D$. Suppose that f has a pole of order m at z_0 and that f has the Laurent series expansion

$$f(z)=\sum_{n\in\mathbb{Z}} c_n(z-z_0)^n \quad \text{for } 0<|z-z_0|<R,$$

where R is some positive number. Show that $c_{-1}=\dfrac{1}{(m-1)!}\lim_{z\to z_0}\dfrac{d^{m-1}}{dz^{m-1}}\left((z-z_0)^m f(z)\right)$.

Exercise 4.60. True or false?

(1) If f has a Laurent expansion $z^{-1}+c_0+c_1z+\cdots$, convergent in some punctured disc about the origin, then f has a pole at 0.

(2) A function may have different Laurent series centered at z_0, depending on the annulus of convergence selected.

(3) If f has an isolated singularity at z_0, then it may be expanded in a Laurent series centered at z_0 and is convergent in some punctured disc given by $0<|z-z_0|<R$.

(4) If a Laurent series for f convergent in some annulus given by $R_1<|z-z_0|<R_2$ is actually a Taylor series (no negative powers of $z-z_0$), then this series actually converges in the full disc given by $|z-z_0|<R_2$ (at least).

(5) If the last conclusion holds, then f has at worst removable singularities in the full disc given by $|z-z_0|<R_2$ and may be considered holomorphic throughout this disc.

Exercise 4.61. Decide the nature of the singularity, if any, at 0 for the following functions. If the function is holomorphic or the singularity is isolated, expand the function in appropriate powers of z convergent in a punctured disc given by $0<|z|<R$.

$$\sin z, \quad \sin\frac{1}{z}, \quad \frac{\sin z}{z}, \quad \frac{\sin z}{z^2}, \quad \frac{1}{\sin\frac{1}{z}}, \quad z\sin\frac{1}{z}$$

Exercise 4.62. True or false?

(1) $\lim_{z\to 0}|e^{\frac{1}{z}}|=+\infty$.

(2) If f has a pole of order m at z_0, then there exists a polynomial p such that $f-\dfrac{p}{(z-z_0)^m}$ has holomorphic extension to a disc around z_0.

(3) If f is holomorphic in a neighbourhood of 0, then there is an integer m such that $\dfrac{f}{z^n}$ has a pole at 0 whenever $n>m$.

(4) If f,g have poles of order m_f, m_g respectively at z_0, then their pointwise product $f\cdot g$ has a pole of order m_f+m_g at z_0.

Exercise 4.63. Give an example of a function holomorphic in all of $\mathbb{C}$ except for essential singularities at the two points 0 and 1.

Exercise 4.64. The function f given by $f(z) = (z-1)^{-1}$ has the Laurent series $z^{-1} + z^{-2} + z^{-3} + \cdots$ for $|z| > 1$. A naive student, observing that this series has infinitely many negative powers of z, concludes that the point 0 is an essential singularity of f. Point out the flaw in his argument.

Exercise 4.65. Prove or disprove: If f and g have a pole and an essential singularity respectively at the point z_0, then fg has an essential singularity at z_0.

Exercise 4.66. Prove, using the Casorati-Weierstrass Theorem, that if f has an essential singularity at z_0, and if w is any complex value whatever, then there exists a sequence $z_1, z_2, z_3, \cdots$ such that

$$\lim_{n\to\infty} z_n = z_0 \quad \text{and} \quad \lim_{n\to\infty} f(z_n) = w.$$

4.9. Residue Theorem

Suppose that D is a domain and $z_0 \in D$ is an isolated singularity of $f : D \setminus \{z_0\} \to \mathbb{C}$. Then we know that f has a Laurent series expansion

$$f(z) = \sum_{n\in\mathbb{Z}} c_n (z-z_0)^n \text{ for } 0 < |z-z_0| < R,$$

where R is some positive number. We then know that

$$2\pi i c_{-1} = \int_C f(z)dz,$$

where C is a circular path with center z_0 and any radius $r < R$. If γ is a closed path in D which is $D \setminus \{z_0\}$-homotopic to C, then we have, using the Cauchy Integral Theorem, that

$$\int_\gamma f(z)dz = \int_C f(z)dz = 2\pi i c_{-1}. \tag{4.7}$$

We call c_{-1} the *residue of f at z_0*, since we can imagine integrating termwise formally in the Laurent series expansion of f, and this is what is "left over" (=residue):

$$\int_\gamma f(z)dz = \int_\gamma \sum_{n\in\mathbb{Z}} c_n (z-z_0)^n dz = \sum_{n\in\mathbb{Z}} c_n \int_\gamma (z-z_0)^n dz = 2\pi i c_{-1}.$$

(Because the integral $\displaystyle\int_\gamma (z-z_0)^n dz = 0$ for $n \geq 0$ thanks to the fact that $(z-z_0)^n$ is entire. Also

$$\frac{d}{dz}\frac{(z-z_0)^{n+1}}{n+1} = (z-z_0)^n,$$

for $n \leq -2$, showing that $(z-z_0)^n$ has a primitive.) We will denote the residue of f at z_0 by $\operatorname{res}(f, z_0)$.

(4.7) gives a way of computing contour integrals via calculating the residue of f at z_0 (which amounts to finding the value of the coefficient c_{-1} in the Laurent expansion of f). We know that there is a way to compute this if z_0 is a pole; see Exercise 4.59. Some real integrals can be calculated in this manner. Here is an example.

Example 4.67. Consider the real integral $\displaystyle\int_0^{2\pi} \frac{1}{5+3\cos\theta} d\theta$.

We view this as the contour integral along a circular path as follows. First we write

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+\frac{1}{z}}{2} = \frac{z^2+1}{2z},$$

where $z := e^{i\theta}$. So if γ is the circular path with radius 1 and center at 0 defined by $\gamma(\theta) = e^{i\theta}$ ($\theta \in [0, 2\pi]$), then $\gamma'(\theta)d\theta = ie^{i\theta}d\theta = izd\theta$, and so

$$\int_0^{2\pi} \frac{1}{5+3\cos\theta}d\theta = \int_\gamma \frac{1}{5+3\cdot\left(\dfrac{z^2+1}{2z}\right)}\frac{1}{iz}dz = \int_\gamma -\frac{2i}{(3z+1)(z+3)}dz.$$

Let f be the function defined by

$$f(z) = -\frac{2i}{(3z+1)(z+3)}.$$

Then f has two poles, one at $-\frac{1}{3}$ and the other at -3, both of order 1. Of these, only the one at $-\frac{1}{3}$ lies inside γ. Thus,

$$\int_0^{2\pi} \frac{1}{5+3\cos\theta}d\theta = \int_\gamma f(z)dz = 2\pi i \cdot \operatorname{res}\left(f, -\frac{1}{3}\right).$$

Now

$$f(z) = \frac{c_{-1}}{z+\dfrac{1}{3}} + c_0 + c_1\left(z+\frac{1}{3}\right) + c_2\left(z+\frac{1}{3}\right)^2 + \cdots,$$

and so

$$\left(z+\frac{1}{3}\right)f(z) = c_{-1} + c_0\left(z+\frac{1}{3}\right) + c_1\left(z+\frac{1}{3}\right)^2 + \ldots$$

is holomorphic inside C. Thus

$$c_{-1} = \lim_{z\to-\frac{1}{3}}\left(z+\frac{1}{3}\right)f(z) = \lim_{z\to-\frac{1}{3}} -\frac{2i}{3(z+3)} = -\frac{i}{4}.$$

Consequently, $\displaystyle\int_0^{2\pi} \frac{1}{5+3\cos\theta}d\theta = \frac{\pi}{2}$. $\Diamond$

More generally, if f has a finite number of *poles* in D, a result similar to (4.7) holds, and this is the content of the following.

Theorem 4.68 (Residue Theorem)**.** *Let D be a domain, and suppose that f is a function that is holomorphic in $D \setminus \{p_1, \ldots, p_K\}$ and has poles of order $m_1, \ldots, m_K$, respectively at $p_1, \ldots, p_K$. Let γ be a closed path in D which avoids $p_1, \ldots, p_K$ and for each $k = 1, \ldots, K$, γ is $D \setminus \{p_k\}$-homotopic to a circle C_k centered at p_k such that the interior of C_k is contained in D and contains only the pole p_k. Then*

$$\int_\gamma f(z)dz = 2\pi i\sum_{k=1}^{K}\operatorname{res}(f, p_k).$$

Proof. For each $k = 1, \ldots, K$, we can write

$$f(z) = \sum_{n=1}^{m_k} c_{-n,k}(z-p_k)^{-n} + \sum_{n=0}^{\infty} c_{n,k}(z-p_k)^n = f_k(z) + h_k(z),$$

where the sum with negative powers of $z - p_k$ is denoted by f_k, and the sum with nonnegative powers of $z - p_k$ is denoted by h_k. Note that h_k is holomorphic, and that f_k is a rational function, with only one singularity in D at p_k. Thus $f - f_k$ is holomorphic in a small disk around p_k.

Set $g := f - (f_1 + \cdots + f_K)$. Since

$$g = (f - f_k) - \sum_{j\neq k} f_j,$$

and observing that both $f - f_k$ and each f_j for $j \neq k$ is holomorphic at p_k. This happens with each $k \in \{1, \ldots, K\}$. Thus, g is holomorphic in D. We note that as γ is $D \setminus \{p_1\}$-homotopic to a circle C_1 centered at p_1, which is in turn D-homotopic to the point p_1, it follows that γ is D-contractible to p_1. So by the Cauchy Integral Theorem,

$$\int_\gamma g(z)dz = 0,$$

that is,

$$\int_\gamma f(z)dz = \sum_{k=1}^{K} \int_\gamma f_k(z)dz = 2\pi i \sum_{k=1}^{K} c_{-1,k} = 2\pi i \sum_{k=1}^{K} \operatorname{res}(f, p_k).$$

This finishes the proof. □

Exercise 4.69. Evaluate $\displaystyle\int_\gamma \frac{\operatorname{Log}(z)}{1+e^z}dz$ along the path γ shown in Figure 3.

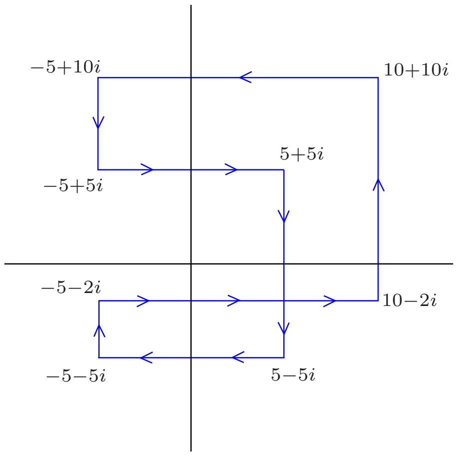


Figure 3. The curve γ.

As we mentioned earlier, the Residue Theorem can be used to calculate contour integrals, and sometimes gives an easy way to calculate some real integrals. Let us see how it can be used to calculate the improper integrals of rational functions.

Consider a real integral of the type

$$\int_{-\infty}^{\infty} f(x)dx.$$

Such an integral, for which the interval of integration is not finite, is called an *improper integral*, and it is defined as

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{a\to-\infty} \int_a^0 f(x)dx + \lim_{b\to+\infty} \int_0^b f(x)dx, \tag{4.8}$$

when both the limits on the right hand side exist. In this case, there also holds that

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{r\to+\infty} \int_{-r}^{r} f(x)dx. \tag{4.9}$$

(However, the expression on the right hand side in (4.9) may exist even if the limits in (4.8) may not exist. For example,

$$\lim_{r\to+\infty} \int_{-r}^{r} xdx = \lim_{r\to+\infty} \left(\frac{r^2}{2} - \frac{r^2}{2}\right) = 0,$$

but $\displaystyle\int_0^b xdx = \frac{b^2}{2}$, and so $\displaystyle\lim_{b\to+\infty} \int_0^b xdx$ does not exist.

We call the right hand side in (4.9), if it exists, the *Cauchy principal value* of the integral.

We will assume that the function f in (4.8) is a real rational function whose denominator is different from 0 for all real x, and whose degree is at least two units higher than the degree of the numerator. Then it can be seen that the limits on the right hand side in (4.8) exist, and so we can start from (4.9). We consider the corresponding contour integral

$$\int_\gamma f(z)dz,$$

around a path γ, as shown in Figure 4.

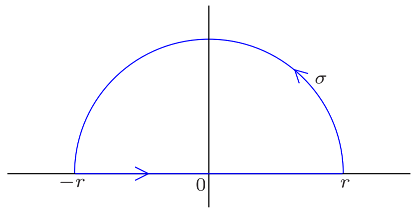


Figure 4. The path γ consisting of the semicircular arc σ and the straight line path joining $-r$ to r along the real axis.

Since f is rational, $z \mapsto f(z)$ has finitely many poles in the upper half plane, and if we choose r large enough, γ encloses all of these poles in its interior. By the Residue Theorem, we then obtain

$$\int_\gamma f(z)dz = \int_\sigma f(z)dz + \int_{-r}^{r} f(x)dx = 2\pi i \sum_{k:\ \mathrm{Im}(p_k)>0} \mathrm{res}(f, p_k),$$

where the sum consists of terms for all the poles that lie in the upper half-plane. From this, we obtain

$$\int_{-r}^{r} f(x)dx = 2\pi i \sum_{k:\ \mathrm{Im}(p_k)>0} \mathrm{res}(f, p_k) - \int_\sigma f(z)dz.$$

We show that as r increases, the value of the integral over the corresponding semicircular arc σ approaches 0. Indeed, from the fact that the degree of the denominator of f is at least two units higher than the degree of the numerator, it follows that there are M, r_0 large enough such that

$$|f(z)| < \frac{M}{|z|^2} \quad (|z| = r > r_0).$$

Hence

$$\left| \int_\sigma f(z)dz \right| \le \frac{M}{r^2}\pi r = \frac{M\pi}{r} \quad (r > r_0).$$

Consequently, $\displaystyle\lim_{r\to+\infty} \int_\sigma f(z)dz = 0$, and so

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{k:\ \mathrm{Im}(p_k)>0} \mathrm{res}(f, p_k).$$

Let us see an example of this method in action.

Example 4.70. We will show that $\displaystyle\int_0^\infty \frac{1}{1+x^4}dx = \frac{\pi}{2\sqrt{2}}$.

The function f given by $f(z) = \dfrac{1}{1+z^4}$ has four poles of order 1:

$$p_1 = e^{\frac{\pi i}{4}}, \quad p_2 = e^{\frac{3\pi i}{4}}, \quad p_3 = e^{\frac{5\pi i}{4}}, \quad p_4 = e^{\frac{7\pi i}{4}}.$$

These are depicted in Figure 5.

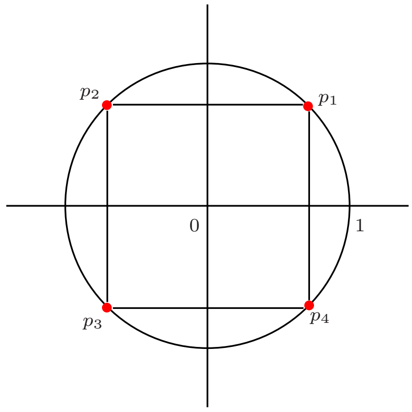


Figure 5. The four poles of f.

The first two of these poles lie in the upper half plane. We have

$$\begin{aligned}
\operatorname{res}(f, p_1) &= \lim_{z\to p_1} \frac{z-p_1}{1+z^4} = \lim_{z\to p_1} \frac{1}{\dfrac{(1+z^4)-(1+p_1^4)}{z-p_1}} = \frac{1}{\dfrac{d}{dz}(1+z^4)\Big|_{z=p_1}} = \frac{1}{4p_1^3} = -\frac{1}{4}e^{\frac{\pi i}{4}}, \\
\operatorname{res}(f, p_2) &= \lim_{z\to p_2} \frac{z-p_2}{1+z^4} = \lim_{z\to p_2} \frac{1}{\dfrac{(1+z^4)-(1+p_2^4)}{z-p_2}} = \frac{1}{\dfrac{d}{dz}(1+z^4)\Big|_{z=p_2}} = \frac{1}{4p_2^3} = \frac{1}{4}e^{-\frac{\pi i}{4}}.
\end{aligned}$$

Thus

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4} dx = 2\pi i \left(-\frac{1}{4}e^{\frac{\pi i}{4}} + \frac{1}{4}e^{-\frac{\pi i}{4}}\right) = \pi \sin\frac{\pi}{4} = \frac{\pi}{\sqrt{2}}.$$

Since f is even, that is, $f(x) = f(-x)$ for all $x \in \mathbb{R}$, it follows that

$$\int_0^{\infty} \frac{1}{1+x^4} dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{1+x^4} dx = \frac{\pi}{2\sqrt{2}}.$$

$\Diamond$

Here is one more example, but this time we integrate a nonrational function.

Example 4.71 (Fresnel Integrals[4])**.** We will show that

$$\int_0^{\infty} \cos(x^2) dx = \int_0^{\infty} \sin(x^2) dx = \frac{\sqrt{\pi}}{2\sqrt{2}}.$$

We consider $\displaystyle\int_\gamma e^{iz^2} dz$, where γ is shown in Figure 6.

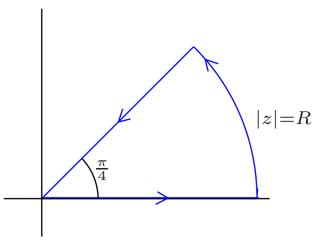


Figure 6. The path γ.

[4]These integrals arise in optics, in the description of diffraction phenomena.

Since e^{iz^2} is entire, we have

$$0=\int_\gamma e^{iz^2}dz=\int_0^R e^{ix^2}dx+\int_0^{\frac{\pi}{4}} e^{iR^2e^{2i\theta}}iRe^{i\theta}d\theta-\int_0^R e^{it^2e^{i\frac{\pi}{2}}}e^{i\frac{\pi}{4}}dt.$$

We will show that the middle integral goes to 0 as R increases. First note that

$$\left|e^{iR^2e^{2i\theta}}iRe^{i\theta}\right|=\left|Re^{R^2(i\cos(2\theta)-\sin(2\theta))}\right|=Re^{-R^2\sin(2\theta)}.$$

But Figure 7 shows that whenever the angle t is such that $0<t<\frac{\pi}{2}$, we have

$$\frac{2}{\pi}\le\frac{\sin t}{t}.$$

Indeed, the length of the arc PAQ, which is $2t$, is clearly less than the length of the semicircular arc PBQ, which is $\pi\sin t$, and so the inequality above follows. (Alternately, we could note that $t\mapsto\sin t$ is concave in $[0,\pi]$, because its second derivative is $-\sin t$, which is nonpositive in $[0,\pi]$, and so the graph of $\sin t$ is lies above that of the straight line graph of $\frac{2}{\pi}t$ joining the two points $(0,0)$ and $(\frac{\pi}{2},1)$.)

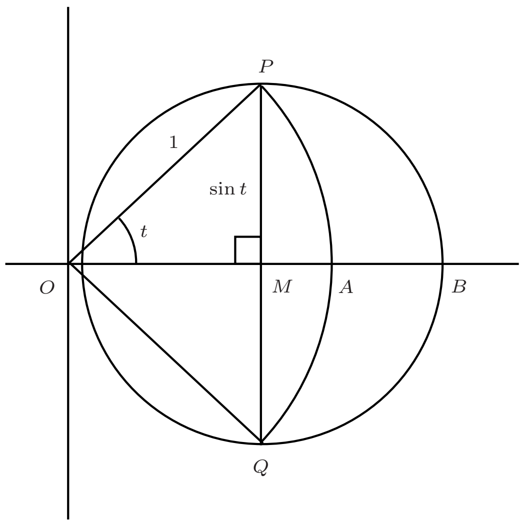


Figure 7. Here P is any point on the circle with center 0 and radius 1 such that OP makes an angle t with the positive real axis. We reflect P in the real axis to get the point Q, and let M be the intersection of PQ with the real axis. With M as center and radius PM, we draw a circle, meeting the real axis on the right of A at the point B.

Applying this inequality with $t=2\theta$ yields

$$\left|e^{iR^2e^{2i\theta}}iRe^{i\theta}\right|=Re^{-R^2\sin(2\theta)}\le Re^{-R^2\frac{4\theta}{\pi}},$$

and so

$$\left|\int_0^{\frac{\pi}{4}} e^{iR^2e^{2i\theta}}iRe^{i\theta}d\theta\right|\le R\int_0^{\frac{\pi}{4}}e^{-4R^2\frac{\theta}{\pi}}d\theta=\frac{\pi}{4R}(1-e^{-R^2}),$$

which tends to 0 as $R\to+\infty$. Hence we obtain

$$\int_0^\infty e^{ix^2}dx=\lim_{R\to+\infty}\int_0^R e^{it^2e^{i\frac{\pi}{2}}}e^{i\frac{\pi}{4}}dt=\frac{1+i}{\sqrt{2}}\int_0^\infty e^{-t^2}dt=\frac{(1+i)\sqrt{\pi}}{2\sqrt{2}}.$$

Here we have used the known[5] fact that

$$\int_0^\infty e^{-x^2}dx=\frac{\sqrt{\pi}}{2}.$$

[5] With $I:=\int_0^\infty e^{-x^2}dx$, $I^2=\left(\int_0^\infty e^{-x^2}dx\right)\left(\int_0^\infty e^{-y^2}dy\right)=\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dxdy=\int_0^{\frac{\pi}{2}}\int_0^\infty e^{-r^2}rdrd\theta=\frac{\pi}{4}$.

Consequently,

$$\int_0^\infty \cos(x^2)dx = \int_0^\infty \sin(x^2)dx = \frac{\sqrt{\pi}}{2\sqrt{2}}$$

by equating the real and imaginary parts ♢

Exercise 4.72. Evaluate $\displaystyle\int_0^{2\pi} \frac{\cos\theta}{5+4\cos\theta}d\theta$.

Exercise 4.73. Evaluate the following integrals:

(1) $\displaystyle\int_0^\infty \frac{1}{1+x^2}dx$.

(2) $\displaystyle\int_0^\infty \frac{1}{(a^2+x^2)(b^2+x^2)}dx$, where $a > b > 0$.

(3) $\displaystyle\int_0^\infty \frac{1}{(1+x^2)^2}dx$.

(4) $\displaystyle\int_0^\infty \frac{1+x^2}{1+x^4}dx$.

Exercise 4.74. Let $n \in \mathbb{N}$ and C be the circular path $C(t) = e^{it}$ ($t \in [0, 2\pi]$). Evaluate

$$\int_C \frac{e^z}{z^{n+1}}dz.$$

Deduce that $\displaystyle\int_0^{2\pi} e^{\cos\theta}\cos(n\theta - \sin\theta)d\theta = \frac{2\pi}{n!}$.

Exercise 4.75. Let f have a zero of order 1 at z_0, so that $\frac{1}{f}$ has a pole of order 1 at z_0. Prove that

$$\operatorname{res}\left(\frac{1}{f}, z_0\right) = \frac{1}{f'(z_0)}.$$

Exercise 4.76. Prove that $\operatorname{res}\left(\frac{1}{\sin z}, k\pi\right) = (-1)^k$.

Exercise 4.77. The nth Fibonacci number f_n, where $n \geq 0$, is defined by the following recurrence relation:

$$f_0 = 1, \quad f_1 = 1, \quad f_n = f_{n-1} + f_{n-2} \text{ for } n \geq 2.$$

Let $F(z) := \displaystyle\sum_{n=0}^\infty f_n z^n$.

(1) Prove by induction that $f_n \leq 2^n$ for all $n \in \mathbb{N}$.

(2) Using the estimate $f_n \leq 2^n$, deduce that the radius of convergence of F is at least $\frac{1}{2}$.

(3) Show that the recurrence relation among the f_n implies that $F(z) = \dfrac{1}{1-z-z^2}$.
Hint: Write down the Taylor series for $zF(z)$ and $z^2F(z)$ and add.

(4) Verify that $\operatorname{res}\left(\dfrac{1}{z^{n+1}(1-z-z^2)}, 0\right) = f_n$.

(5) Using the Residue Theorem, prove $f_n = \dfrac{1}{\sqrt{5}}\left(\left(\dfrac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\dfrac{1-\sqrt{5}}{2}\right)^{n+1}\right)$.

Hint: Integrate $\dfrac{1}{z^{n+1}(1-z-z^2)}$ around a circle with center 0 and radius R and show that this integral vanishes as $R \to +\infty$.

Chapter 5

Harmonic functions

In this last chapter, we study harmonic functions, namely functions $u : U \to \mathbb{R}$ defined in an open set U of $\mathbb{R}^2$, which satisfy an important PDE called the *Laplace equation*:

$$\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ in } U.$$

Besides being the prototype of an important class of PDEs ("elliptic equations"), this equation arises in applications (in connection with gravitational fields, electrostatic fields, steady-state heat conduction, incompressible fluid flow, Brownian motion, computer animation etc.).

What is the link between harmonic functions and complex analysis? We will see that the real and imaginary parts u, v of a holomorphic function $f = u + iv$ are harmonic functions, and using this link, will derive some important properties of harmonic functions.

Definition 5.1. Let U be an open subset of $\mathbb{R}^2$. A function $u : U \to \mathbb{R}$ is called *harmonic* if u has continuous partial derivatives of order 2, and

$$\frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) = 0 \text{ for all } (x, y) \in U.$$

Example 5.2. Let $U = \mathbb{R}^2$. Consider the function $u : U \to \mathbb{R}$ given by $u(x, y) = x^2 - y^2$ for $(x, y) \in \mathbb{R}^2$. Since

$$\frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) = 2 - 2 = 0,$$

it follows that u is harmonic. $\Diamond$

Exercise 5.3. Show that the following functions u are harmonic in the corresponding open set U.

(1) $u(x, y) = \log(x^2 + y^2)$, $U = \mathbb{R}^2 \setminus \{(0, 0)\}$.
(2) $u(x, y) = e^x \sin y$, $U = \mathbb{R}^2$.

We show below that the real and imaginary parts of a holomorphic function in an open set are harmonic functions.

Proposition 5.4. *Let U be an open subset of $\mathbb{C}$ and let $f : U \to \mathbb{C}$ be holomorphic in U. Then $u := \mathrm{Re}(f)$ and $v := \mathrm{Im}(f)$ are harmonic functions in U.*

Proof. We have $f(x + iy) = u(x, y) + iv(x, y)$ for $(x, y) \in U$. Since f is infinitely many times differentiable, we know that u, v have partial derivatives of all orders, and so by the Cauchy-Riemann equations, we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) = -\frac{\partial^2 u}{\partial y^2},$$

and so u is harmonic. Similarly,

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial x}\left(-\frac{\partial u}{\partial y}\right) = -\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = -\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right) = -\frac{\partial^2 v}{\partial y^2},$$

and so v is harmonic as well. (Alternately, we could have also noted that $v = \text{Re}(-if)$.) $\Box$

Example 5.5. Then the function u given by $u(x, y) = x^2 - y^2$ considered in Example 5.2 is in fact the real part of the holomorphic function $z \mapsto z^2$, and so is harmonic. Similarly the function $v(x, y) := xy$, which is half the imaginary part of $z \mapsto z^2$ is harmonic as well. $\Diamond$

In fact, a converse to the above result holds when the open set in question is a simply connected domain. (For more general domains, it can happen that there are harmonic functions which aren't the real part of a holomorphic function; see Exercise 5.8.)

Proposition 5.6. *Let D be a simply connected domain and $u : D \to \mathbb{R}$ be harmonic in D. Then there is another harmonic function v defined in D such that $f := u + iv$ is holomorphic in D.*

(The function v is then called a *harmonic conjugate* of u.)

Proof. We will explicitly construct the holomorphic f with real part u, and then $v := \text{Im}(f)$ will serve as the required harmonic function. First set

$$g = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}.$$

The plan is to prove that g is holomorphic, and then to construct a primitive of g, which will be the f we are after. To show that g is holomorphic, we will use Theorem 2.20. First we note that since u is harmonic, the functions

$$\text{Re}(g) = \frac{\partial u}{\partial x} \quad \text{and} \quad \text{Im}(g) = -\frac{\partial u}{\partial y}$$

have continuous partial derivatives, and moreover, they satisfy the Cauchy-Riemann equations:

$$\begin{aligned}
\frac{\partial}{\partial x}(\text{Re}(g)) &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}(\text{Im}(g)),\\
\frac{\partial}{\partial y}(\text{Re}(g)) &= \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = -\frac{\partial}{\partial x}\left(-\frac{\partial u}{\partial y}\right) = -\frac{\partial}{\partial x}(\text{Im}(g)).
\end{aligned}$$

Hence g is holomorphic in D, and by Theorem 3.37, it has a primitive G in D. Decompose $G = U + iV$ into its real and imaginary parts U, V. Then

$$\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = g = G' = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y},$$

where the last equation follows from the Cauchy-Riemann equations. From the above, we see that

$$\frac{\partial(u - U)}{\partial x} = 0,$$

and so it follows from the Fundamental Theorem of Integral Calculus, that $u - U$ is locally constant along horizontal lines. Similarly, since

$$\frac{\partial(u - U)}{\partial y} = 0,$$

$u - U$ is also locally constant along vertical lines. But any two points in U can be joined by a stepwise path, and so $u - U$ is constant in D, that is, there is a constant c such that $u = U + c$ in D. Hence $f := G - c$ is a holomorphic function in D whose real part is u. $\Box$

Exercise 5.7. Find harmonic conjugates for the following harmonic functions in $\mathbb{R}^2$:

$$e^x \sin y, \quad x^3 - 3xy^2 - 2y, \quad x(1 + 2y).$$

Exercise 5.8. Show that there is no holomorphic function f defined in $\mathbb{C} \setminus \{0\}$ whose real part is the harmonic function u defined by $u(x, y) = \log(x^2 + y^2)$, $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$.

Hint: If v is a harmonic conjugate of u, then also $h(z) := z^2 e^{-(u+iv)}$ is holomorphic. Find $|h|$, and conclude that $h' = 0$. Show that $h' = 0$ implies that $2/z$ has a primitive in $\mathbb{C} \setminus \{0\}$, which is impossible.

Exercise 5.9. Is it possible to find a $v : \mathbb{R}^2 \to \mathbb{R}$ so that f defined by $f(x + iy) = x^3 + y^3 + iv(x, y)$, $(x, y) \in \mathbb{R}^2$, is holomorphic in $\mathbb{C}$?

The above two results allow a fruitful interaction between harmonic and holomorphic functions. For instance we have the following.

Corollary 5.10. *Harmonic functions are infinitely many times differentiable.*

(Note that the definition of a harmonic function demands only *twice* continuous differentiability. The remarkable result here says that thanks to the fact that the Laplace equation is satisfied, in fact the function has got to be infinitely differentiable. A result of this type is called a *regularity result* in PDE theory.)

Proof. Suppose that u is a harmonic function in an open set U. Let $z_0 = (x_0, y_0) \in U$. Then there is a $r > 0$ such that the disc D with center z_0 and radius r is contained in U. But D is simply connected, and so there is a holomorphic function f defined in D, whose real part is u. But then u is infinitely many times differentiable in D, and in particular at $z_0 \in D$. As the choice of $z_0 \in D$ was abritrary, the result follows. $\square$

Exercise 5.11. Show that all partial derivatives of a harmonic function are harmonic.

Exercise 5.12. Show that the set $\text{Har}(U)$ of all harmonic functions on an open set U forms a real vector space with pointwise operations.

Exercise 5.13. Is the pointwise product of two harmonic functions also necessarily harmonic?

Using the Cauchy Integral Formula, we immediately obtain the following "mean value property" of harmonic functions, which says that the value of a harmonic function is the average (or mean) of the values on a circle with that point as the center.

Theorem 5.14 (Mean-value property of harmonic functions)**.** *Suppose u is harmonic in an open set U. Let $z_0 \in U$, and let $R > 0$ be such that $\{z \in \mathbb{C} : |z - z_0| < R\}$ is contained in U. Then*

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{it}) dt$$

for all r such that $0 < r < R$.

Proof. The disc $D := \{z \in \mathbb{C} : |z - z_0| < R\}$ is simply connected, and so there is a holomorphic function f defined in D, whose real part is u. But now by the Cauchy Integral Formula, if C is the circular path given by $C(t) = z_0 + re^{it}$ $(t \in [0, 2\pi])$, then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{it})}{re^{it}} ire^{it} dt = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it}) dt.$$

Equating real parts, the claim is proved. $\square$

From the Maximum Modulus Theorem (see page 64), we also obtain the following.

Theorem 5.15. *Suppose u is harmonic in a simply connected domain U. Suppose that $z_0 \in U$ is a point such that $u(z_0) \geq u(z)$ for all $z \in D$. Then u is constant in U.*

Proof. There is a holomorphic function f defined in D whose real part equals u. But then the function $g : D \to \mathbb{C}$ defined by $g(z) = e^{f(z)}$ $(z \in D)$ is holomorphic too. Its absolute value is e^u, and since

$$|g(z_0)| = e^{u(x_0,y_0)} \geq e^{u(x,y)} = |g(z)| \quad \text{for } z = x + iy \in D,$$

it follows from the Maximum Modulus Theorem, that g must be constant in D. This means that there is a number c such that

$$f(z) - 2k(z)\pi i = c$$

where $k : D \to \mathbb{Z}$ is an integer valued function in D. But from the above equation, we see that k is continuous in D, and so, being integer-valued, k must be constant in D. So f, and hence its real part u, is constant in D. □

In particular, this has an important consequence about the uniqueness of solutions to the Dirichlet problem, as explained below.

Let $\mathbb{D}$ be the open disc with center 0 and radius 1. The boundary of $\mathbb{D}$ is the circle $\mathbb{T}$ with center 0 and radius 1. Let $\varphi : \mathbb{T} \to \mathbb{R}$ be a continuous function. φ is called the *boundary data*. Then the *Dirchlet problem* is the following:

Find a continuous $u : \mathbb{D} \cup \mathbb{T} \to \mathbb{R}$ such that $\Delta u = 0$ in $\mathbb{D}$ and $u|_{\mathbb{T}} = \varphi$.

Proposition 5.16. *The solution to the Dirichlet problem is unique.*

Proof. Indeed, let u_1, u_2 be two distinct solutions corresponding to the boundary data φ, and suppose that there is a point $w \in \mathbb{D}$ where $u_1(w) > u_2(w)$. (Note that on the boundary u_1 and u_2 coincide.) Then $u := u_1 - u_2$ is a solution corresponding to the boundary data 0. Let $z_0 \in \mathbb{D} \cup \mathbb{T}$ be the maximizer for the real-valued continuous function u on the compact set $\mathbb{D} \cup \mathbb{T}$. Then z_0 can't be in $\mathbb{T}$, for otherwise, the maximum value must be 0, and in particular,

$$0 = u(z_0) \geq u(w) = u_1(w) - u_2(w) > 0,$$

a contradiction. Thus $z_0 \in \mathbb{D}$. Then we have in particular that $u(z_0) \geq u(z)$ for $z \in \mathbb{D}$, and so by Theorem 5.15, u must be constant in $\mathbb{D}$. But as u is continuous on $\mathbb{D} \cup \mathbb{T}$, and u is 0 on $\mathbb{T}$, it follows that the constant value of u must be 0 everywhere in $\mathbb{D} \cup \mathbb{T}$. Hence $u_1 = u_2$. □

Remark 5.17. It can be shown that the following expression, called the *Poisson Integral Formula*, gives the solution to the Dirichlet problem with boundary data φ:

$$u(re^{it}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2} \varphi(e^{i\theta}) d\theta \quad (0 \leq r < 1,\ 0 \leq t \leq 2\pi).$$

This can be derived using the Cauchy Integral Formula, but there are some technical subtleties, and so we will not prove this here.

Exercise 5.18 (Some half-plane Dirichlet problems)**.** Given the "boundary data" $b : \mathbb{R} \to \mathbb{R}$, we consider the problem of finding a continuous, real-valued function h defined in the closed upper half-plane $y \geq 0$, such that h is harmonic in the open upper half-plane $y > 0$ and moreover, $h(x, 0) = b(x)$.

(1) If b is just a polynomial p, then show that we can simply take $h(x, y) = \text{Re}(p(x + iy))$.

(2) Prove that if

$$b(x) = \frac{1}{1 + x^2},$$

then $(x, y) \mapsto \text{Re}(b(x + iy))$ is *not* a solution (because of the pole at $z = i$). Show that

$$h(x, y) := \text{Re}\left(\frac{i}{z + i}\right) = \frac{y + 1}{x^2 + (y + 1)^2}$$

gives a solution to the Dirichlet problem.

Exercise 5.19. Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a harmonic function such that $u(x, y) > 0$ for all $(x, y) \in \mathbb{R}^2$. Prove that u is constant.

Hint: Let f be an entire function whose real part is u. Consider e^{-f}.

Exercise 5.20. The regularity of functions satisfying the Laplace equation is not completely for free. Here is an example to show that a *discontinuous* function may satisfy the Laplace equation! Consider the function $u : \mathbb{R}^2 \to \mathbb{R}$ defined to be the real part of e^{-1/z^4} when $z \neq 0$ and 0 at the origin.

(1) Verify that u is discontinuous at 0.

(2) Check that $u(x, 0) = e^{-1/x^4}$, $u(0, y) = e^{-1/y^4}$.

(3) Being the real part of a holomorphic function in $\mathbb{C} \setminus \{0\}$, we know already that u satisfies the Laplace equation everywhere in $\mathbb{R}^2 \setminus \{(0, 0)\}$. Show that also
$$\frac{\partial^2 u}{\partial x^2}(0, 0) \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2}(0, 0)$$
exist, and that $\dfrac{\partial^2 u}{\partial x^2}(0, 0) + \dfrac{\partial^2 u}{\partial y^2}(0, 0) = 0$.

Exercise 5.21. Let D_1, D_2 be domains in $\mathbb{C}$. Let $\varphi : D_1 \to D_2$ be holomorphic. Show that if $h : D_2 \to \mathbb{R}$ is harmonic, then $h \circ \varphi : D_1 \to \mathbb{R}$ is harmonic as well.

Now suppose that $\varphi : D_1 \to D_2$ is holomorphic, a bjiection, and also $\varphi^{-1} : D_2 \to D_1$ is holomorphic. We call such a map φ a *biholomorphism.* Conclude that a function $h : D_2 \to \mathbb{R}$ is harmonic if and only if $h \circ \varphi$ is harmonic.

Thus the existence of a biholomorphism between two domains allows one to transplant harmonic (or even holomorphic) functions from one domain to the other. This mobility has the advantage that if D_1 is "nice" (like a half plane or a disc), while D_2 is complicated, then problems (like the Dirichlet Problem) in D_2 can be solved by first moving over to D_1, solving it there, and then transplanting the solution to D_2.

A first natural question is then the following: Given two domains D_1 and D_2, is there a biholomorphism between them? An answer is provided by the Riemann Mapping Theorem, a proof of which is beyond the scope of these notes, but can be found for example in [**C**].

Theorem 5.22 (Riemann Mapping Theorem)**.** *Let D be a proper (that is, $D \neq \mathbb{C}$) simply connected domain. Then there exists a biholomorphism $\varphi : D \to \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$.*

Thus the above result guarantees a biholomorphism between any two proper simply connected domains (by a passage through $\mathbb{D}$). Unfortunately, the proof does not give a practical algorithm for finding the biholomorphism. Show that the "Möbius transformation" $\varphi : \mathbb{H} \to \mathbb{D}$, where $\mathbb{H} := \{s \in \mathbb{C} : \mathrm{Re}(s) > 0\}$, given by
$$\varphi(s) = \frac{s - 1}{s + 1}, \quad s \in \mathbb{H},$$
is a biholomorphism between the right half plane $\mathbb{H}$ and the disc $\mathbb{D}$.

Bibliography

[C] John Conway. *Functions of One Complex Variable I*. Second Edition. Springer, 1978.

[N] Tristan Needham. *Visual Complex Analysis*. Oxford University Press, 1997.

[R] Reinhold Remmert. *Theory of complex functions*. Translated from the second German edition by Robert Burckel. Springer, 1991.
(Excellent for following the history of the subject.)

[S] William Shaw. *Complex Analysis with* MATHEMATICA. Cambridge University Press, 2006.
(Contains many beautiful pictures and many applications.)

There are many books on Complex Analysis at an undergraduate level. The above choice of four is based on the different takes on the subject adopted in them.

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