MA319 Partial Differential Equations

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Contents

Introduction

What is a Partial Differential Equation?

A *Partial Differential Equation* (PDE) is an equation involving known and unknown functions of several real variables and their partial derivatives.

So a PDE is analogous to an ODE (Ordinary differential equation, which is an equation involving known and unknown functions of *one* variable t, usually thought of as time), except that now we have functions of *several* variables (say, $t, x_1, \cdots, x_n$).

Exercise 0.1 (ODE background). In our study of PDEs, we will need the knowledge of solutions to the following two ODEs, and the aim of this exercise is to quickly recall this.

(1) (First order linear ODE). Let $a : \mathbb{R} \to \mathbb{R}$ be a continuous function. Show that the initial value problem

$$\begin{cases} x'(t) = a(t)x(t), & t \in \mathbb{R}, \\ x(t_i) = x_i \in \mathbb{R}. \end{cases}$$

has a unique solution, given by $x(t) = \exp\left(\int_{t_i}^{t} a(\tau)d\tau\right) \cdot x_i$, $t \in \mathbb{R}$.

(2) (Second order linear ODE). Let $\lambda \in \mathbb{R}$. The aim of this exercise is to show that the equation

$$x''(t) + \lambda x(t) = 0, \quad x(0) = A \text{ and } x'(0) = B \tag{0.1}$$

has the unique solution given by

$$x(t) = \begin{cases} A\cos(kt) + \dfrac{B}{k}\sin(kt) & \text{if } \lambda = k^2,\ k > 0 \\ A + Bt & \text{if } \lambda = 0 \\ A\cosh(kt) + \dfrac{B}{k}\sinh(kt) & \text{if } \lambda = -k^2,\ k > 0. \end{cases}$$

To see this, first verify that the x given by the formula above is a solution.

One can show that if A, B are zero, then 0 is the only solution, as follows. First note that by virtue of the equation $x'' + \lambda x = 0$, it is clear that x is infinitely many times differentiable. Moreover, $x^{(2n)} = (-1)^n \lambda^n x$ and $x^{(2n-1)} = (-1)^{n-1}\lambda^n x$. Since

$x(0) = x'(0) = 0$, it follows that $x^{(k)}(0) = 0$ for all k. Using a Taylor polynomial of degree $2n-1$, Taylor's Formula gives the existence of a $\theta \in (0,1)$ such that

$$x(t) = \frac{x^{(2n)}(\theta t)}{(2n)!} t^{2n}.$$

Now suppose we choose any $c > 0$, and let $M > 0$ be such that $|x(t)| \leqslant M$ on $[-c, c]$. Then $|x^{(2n)}(t)| \leqslant |\lambda|^n M$ on $[-c, c]$. Thus

$$0 \leqslant |x(t)| \leqslant \frac{|\lambda|^n M}{(2n)!} c^{2n} = M \cdot \frac{(\sqrt{|\lambda|}c)^{2n}}{(2n)!} \overset{n\to\infty}{\longrightarrow} M \cdot 0 = 0.$$

Thus $x \equiv 0$ on $[-c, c]$. But as $c > 0$ was arbitrary, we conclude that $x \equiv 0$ on $\mathbb{R}$.

The uniqueness to (0.1) now follows immediately by just considering the difference of two possible solutions.

Here are a few examples of PDEs.

Example 0.1 (Transport Equation). Consider

$$\frac{\partial u}{\partial t}(x,t) + \frac{\partial u}{\partial x}(x,t) = 0.$$

The function $(x,t) \overset{u}{\mapsto} u(x,t)$ is the unknown function. In the sequel, it will be convenient to use the notation

$$\begin{aligned} u_t &:= \frac{\partial u}{\partial t}, \\ u_x &:= \frac{\partial u}{\partial x}. \end{aligned}$$

The transport equation can then be written simply as $u_t + u_x = 0$. ♢

In general, if w is a smooth enough function of the variables $x_1, x_2, \cdots, x_n$ (and possibly several others), then it will be convenient to use the following notation:

$$w_{x_1 \cdots x_n} := \frac{\partial^n w}{\partial x_1 \cdots \partial x_n}.$$

Example 0.2 (Mathematical physics). Most of the equations in Mathematical Physics are PDEs. So PDEs describe our physical universe! For example in (nonrelativistic) quantum mechanics, the fundamental equation is the *Schrödinger equation*. For the hydrogen atom, this equation is

$$-i\hbar\Psi_t = \frac{\hbar^2}{2m}(\Psi_{xx} + \Psi_{yy} + \Psi_{zz}) + \frac{e^2}{\sqrt{x^2+y^2+z^2}}\Psi,$$

where

m is the mass of the electron,

e is the charge of the electron,

$\hbar$ is $h/2\pi$, and h is Planck's constant,

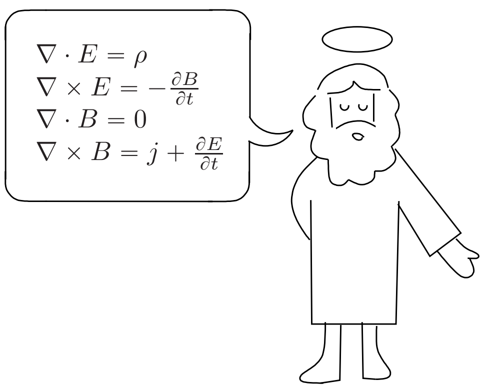
Ψ is the "wave function".

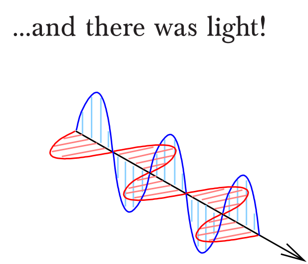
Note that this is a PDE. The quantity

$$\iiint_{\Omega} |\Psi|^2 dx dy dz$$

gives the probability of finding the electron in the region $\Omega \subset \mathbb{R}^3$ at time t. (This equation can be used for example to explain the energy levels of the electron in the hydrogen atom. Variations of the equation explain the structure of all atoms and molecules, and so all of Chemistry!)

As opposed to the above PDE example, with just one equation, one can also have a ***system*** of PDEs where there are $m > 1$ equations in $\ell > 1$ unknowns. A classical example is the Maxwell equations[1] of electromagnetism describing the evolution of the electric field E and the magnetic field B.





In Einstein's theory of General Relativity, describing spacetime, the "spacetime metric" is obtained as a solution to a PDE called the Einstein field equation. $\Diamond$

Exercise 0.2 (Schrödinger equation).

(1) (Probability is conserved). Consider the free one dimensional Schrödinger equation

$$i\hbar\Psi_t = -\frac{\hbar^2}{2m}\Psi_{xx}.$$

Show that for any C^∞ solution $\Psi : \mathbb{R} \times [0, \infty) \to \mathbb{C}$ such that $\Psi(\cdot, t)$ has compact support for all t,

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = \int_{-\infty}^{\infty} |\Psi(x,0)|^2 dx.$$

(2) The one dimensional Schrödinger equation is

$$i\hbar\Psi_t = -\frac{\hbar^2}{2m}\Psi_{xx} + V(x) \cdot \Psi,$$

[1] See Exercise 0.3 for an explanation of the notation $\nabla\times$ and $\nabla\cdot$.

where V is the potential energy. One method to find solutions is to assume that variables separate, that is, the solution has the form

$$\Psi(x,t) = X(x)T(t).$$

Substituting this in the Schrödinger equation gives

$$\frac{i\hbar T'(t)}{T(t)} = \frac{-\frac{\hbar^2}{2m}X''(x) + V(x)X(x)}{X(x)}.$$

As the left hand side depends only on t and the right hand side only on x, the only way equality can occur is if both sides are equal to the same constant, say E (for "energy"). So we obtain the equation following for T,

$$T'(t) = \frac{-iE}{\hbar}T(t),$$

which has the solution

$$T(t) = C\exp\Big(\frac{-iEt}{\hbar}\Big) = C\bigg(\cos\Big(\frac{Et}{\hbar}\Big) - i\sin\Big(\frac{Et}{\hbar}\Big)\bigg).$$

The equation for X is

$$-\frac{\hbar^2}{2m}X''(x) + (V(x) - E)X(x) = 0.$$

Consider a free particle of mass m confined to the interval $0 < x < \pi$ (so that $V \equiv 0$ in $(0,\pi)$), and suppose that $\Psi(0,t) = \Psi(\pi,t) = 0$ for all t. (Imagine the particle to be in an "infinite potential well".) Show that this problem has a nontrivial solution if and only if

$$E = \frac{n^2\hbar^2}{2m}, \quad n = 1, 2, 3, \cdots.$$

Sketch the probability density function $|\Psi|^2$ when $n = 1, 2$, and compute the probability that the particle is in the interval $[0, 1/4]$ in each case.

Exercise 0.3 (curl:=$\nabla\times$ and div:=$\nabla\cdot$). The aim of this exercise is to introduce the fundamental operators called curl and divergence from vector calculus. A *vector field* $v : \mathbb{R}^3 \to \mathbb{R}^3$ is just a map which assigns to every point $x \in \mathbb{R}^3$ a vector $v(x)$. For a vector field

$$v = (v_1, v_2, v_3) : \mathbb{R}^3 \to \mathbb{R}^3,$$

we define the *curl of* v, $\nabla \times v : \mathbb{R}^3 \to \mathbb{R}^3$, by

$$\nabla \times v := \det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{bmatrix} := \Big(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}, \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}, \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\Big).$$

The *divergence of* v, $\nabla \cdot v : \mathbb{R}^3 \to \mathbb{R}$, is defined by

$$\nabla \cdot v := \Big(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\Big) \cdot (v_1, v_2, v_3) = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}.$$

Roughly speaking, the curl measures the amount of "curling" taking place in the vector field, and points in a direction perpendicular to the plane in which the curling takes place—if the fingers of our right hand curl in the plane in which curling of v takes place around a point, then the thumb points in the direction of the curl $\nabla \times v$ at that point. The divergence measures the amount of "spreading out" taking place (that is, the extent to which a vector field acts like a source/sink at a point). To illustrate this, consider first the vector field

$$v(x, y, z) = (-y, x, 0).$$

Draw representative vectors of v in the xy-plane at the points

$$(1, 0, 0),\ (1, 1, 0),\ (0, 1, 0),\ (-1, 1, 0),\ (-1, 0, 0),\ (-1, -1, 0),\ (0, -1, 0),\ (1, -1, 0).$$

Calculate the curl and the divergence of v.

Next, consider the vector field

$$r(x, y, z) = (x, y, z).$$

Draw representative vectors of r in the xy-plane at the points

$$(1, 0, 0),\ (1, 1, 0),\ (0, 1, 0),\ (-1, 1, 0),\ (-1, 0, 0),\ (-1, -1, 0),\ (0, -1, 0),\ (1, -1, 0).$$

Calculate the curl and the divergence of r.

Exercise 0.4. Suppose that $f : \mathbb{R}^3 \to \mathbb{R}$ is a twice continuously differentiable function. Show that $\nabla \times (\nabla f) = 0$.

Exercise 0.5. Suppose that $v : \mathbb{R}^3 \to \mathbb{R}^3$ is has twice continuously differentiable components. Show that $\nabla \cdot (\nabla \times v) = 0$.

Exercise 0.6 (Finding potentials results in solutions to Maxwell's equations)**.** Let $\varphi : \mathbb{R}^4 \to \mathbb{R}$ be any twice continuously differentiable function, and $A : \mathbb{R}^4 \to \mathbb{R}^3$ be any function with twice continuously differentiable components. Set

$$\begin{aligned} E &:= -\nabla\varphi - \frac{\partial A}{\partial t}, \\ B &:= \nabla \times A. \end{aligned}$$

(φ is then called a ***scalar potential***, and A is called a ***vector potential*** for (E, B)). Also, set

$$\rho := \nabla \cdot E \text{ and } j = \nabla \times B - \frac{\partial E}{\partial t}.$$

Prove that these E, B, ρ, j satisfy Maxwell's equations of electromagnetism.

Example 0.3 (Economics). In Economics, the economic agents optimize, and in problems involving continuous-time optimization, for example

$$\begin{cases} \text{minimize} & \varphi(\mathbf{x}(t_f)) + \displaystyle\int_{t_i}^{t_f} F(t, \mathbf{x}(t), \mathbf{u}(t))dt \\ \text{subject to} & \mathbf{x}'(t) = f(t, \mathbf{x}(t), \mathbf{u}(t)), \quad t \in [t_i, t_f], \\ & \mathbf{x}(t_i) = x_i \in \mathbb{R}^n, \\ & \mathbf{u}(t) \in \mathbb{U} \subset \mathbb{R}^m, \quad t \in [t_i, t_f], \end{cases}$$

a sufficient condition for solvability is the existence of a solution to the ***Bellman equation***, which is the following PDE:

$$\frac{\partial V}{\partial t}(\boldsymbol{x}, t) + \min_{\boldsymbol{u} \in \mathbb{U}} \Big(\nabla_{\boldsymbol{x}} V(\boldsymbol{x}, t) \cdot f(t, \boldsymbol{x}, \boldsymbol{u}) + F(t, \boldsymbol{x}, \boldsymbol{u}) \Big) = 0, \quad \boldsymbol{x} \in \mathbb{R}^n,\ t \in [t_i, t_f]$$

with the boundary condition

$$V(\boldsymbol{x}, t_f) = \varphi(\boldsymbol{x}) \text{ for all } \boldsymbol{x} \in \mathbb{R}^n.$$

The unknown in this equation is $V : \mathbb{R}^n \times [t_i, t_f] \to \mathbb{R}$. $\Diamond$

Exercise 0.7 (Bellman's equation for an LQ[2] problem). Consider the optimal control problem

$$\begin{cases} \text{minimize} & \displaystyle\int_0^1 \Big((\mathbf{x}(t))^2 + (\mathbf{u}(t))^2\Big) dt \\ \text{subject to} & \mathbf{x}'(t) = \mathbf{u}(t), \quad t \in [0,1], \\ & \mathbf{x}(0) = x_i. \end{cases}$$

(1) Write down Bellman's equation for this problem with the appropriate boundary condition.

(2) It can be justified[3] that one expects the solution to be "separable": $V(x,t) = x^2 \cdot \mathbf{p}(t)$ for some function $\mathbf{p} : [0,1] \to \mathbb{R}$. Assuming this form of V, show that the *ordinary* differential equation that $\mathbf{p}$ must satisfy is

$$\mathbf{p}'(t) = (\mathbf{p}(t))^2 - 1, \quad t \in [0,1],$$

with $\mathbf{p}(1) = 0$. This is a well known ODE, called a *Riccati equation*, and it can be shown to have the unique solution

$$\mathbf{p}(t) = \frac{1 - e^{2(t-1)}}{1 + e^{2(t-1)}}, \quad t \in [0,1].$$

Thus $V(x,t) = x^2 \cdot \dfrac{1 - e^{2(t-1)}}{1 + e^{2(t-1)}}$, $x \in \mathbb{R}$, $t \in [0,1]$.

(Finally, using this V one can find the $\mathbf{u}$ that solves the given optimization problem following Bellman's method, again something we won't do here.)

Exercise 0.8 (Euler-Lagrange Equation in Calculus of Variations). Suppose that $\Omega \subset \mathbb{R}^d$ is a region, and that

$$\begin{array}{llll} \mathcal{L}: & \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d & \longrightarrow & \mathbb{R} \\ & (X_1, \cdots, X_d, U, V_1, \cdots, V_d) & \longmapsto & \mathcal{L}(X_1, \cdots, X_d, U, V_1, \cdots, V_d) \end{array}$$

is a given C^2 function (called the *Lagrangian density*). We are interested in finding $u \in C^1(\Omega)$ which minimize $I : C^1(\Omega) \to \mathbb{R}$ given by

$$I(u) = \int_\Omega \mathcal{L}(x_1, \cdots, x_d, u, u_{x_1}, \cdots, u_{x_d}) dx_1 \cdots dx_d = \int_\Omega \mathcal{L}\big(\boldsymbol{x}, u(\boldsymbol{x}), \nabla u(\boldsymbol{x})\big) d\boldsymbol{x}, \quad u \in C^1(\Omega).$$

In the subject of Calculus of Variations, using tools akin to ordinary Calculus, it can be shown that a necessary condition for a minimizer u of I is that it satisfies the following PDE, called the *Euler-Lagrange equation*:

$$\frac{\partial \mathcal{L}}{\partial U}\big(\boldsymbol{x}, u(\boldsymbol{x}), \nabla u(\boldsymbol{x})\big) - \sum_{i=1}^d \frac{\partial}{\partial x_i}\Big(\frac{\partial \mathcal{L}}{\partial V_i}\big(\boldsymbol{x}, u(\boldsymbol{x}), \nabla u(\boldsymbol{x})\big)\Big) = 0.$$

[2] LQ=**L**inear ODE, with a **Q**uadratic cost

[3] See for example [**S2**].

(This plays an important role in connection with the fundamental equations in the applied sciences, for example in Physics. The Einstein field equation, and the basic equations of electromagnetism are both the Euler-Lagrange equation for some Lagrangian density.) Here, we consider an example of deriving the Euler-Lagrange equation describing minimal area surfaces.

Consider a smooth surface in $\mathbb{R}^3$, representing the graph of a function $(x, y) \mapsto u(x, y)$ defined on an open set $\Omega \subset \mathbb{R}^2$. The area of the surface is given by

$$I(u) = \iint_{\Omega} \sqrt{1 + \|\nabla u\|_2^2}\, dxdy, \quad u \in C^2(\Omega).$$

Show that if u is a minimizer, then u must satisfy the PDE given by

$$(1 + u_x^2)u_{yy} - 2u_x u_y u_{xy} + (1 + u_y^2)u_{xx} = 0.$$

Verify that the following are solutions to this PDE: $u = Ax+By+C$ (plane) and $u = \tan^{-1}(y/x)$ (helicoid). In the case of the helicoid, show that the a parametric representation of the surface is given by

$$\begin{aligned} x(s,t) &= s \cdot \cos t, \\ y(s,t) &= s \cdot \sin t, \\ z(s,t) &= t, \end{aligned}$$

by setting $s = \sqrt{x^2 + y^2}$ and $t = \tan^{-1}(y/x)$. Use the command

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with(plots):  plot3d([s*cos(t), s*sin(t), t], s=-3..3, t=-3..3)
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to plot the surface (which is the minimimal area surface having a helix as its boundary).

Example 0.4 (Finance). In Finance, the ***Black-Scholes equation*** describes the price of the European option over time:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,$$

where

- V $(= V(S, t))$ is the price of the option,
- S is the stock price,
- t is the time,
- r is the risk-free interest rate,
- σ is the volatility of stock.

We will revisit this equation again, and obtain an explicit formula for V in terms of given data, by recasting it (via a change of variables) into a diffusion equation (which can be solved explicitly using the Fourier transform). We'll see this in Chapter 4. $\Diamond$

Example 0.5 (Navier-Stokes). A celebrated equation modeling incompressible fluid flow in hydrodynamics, which plays an important role in the design of airplanes, ships, flow of ink in a printer, the study of blood flow in arteries, etc., is following PDE, called the ***Navier-Stokes equation***:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \mathbf{u}.$$

(Here ∇^2 is the vector Laplacian, $\mathbf{u}$ is the velocity of the fluid, p is the pressure, ρ is the fluid density, ν is the kinematic viscosity.) Showing the existence and smoothness of solutions of the Navier-Stokes equation is a famous open problem in Mathematics, and is one of the Millennium Prize problems stated by the Clay Mathematics Institute in 2000—a correct solution to which results in a US $1,000,000 prize. ◊

Example 0.6 (Complex Analysis). In calculus, a function $f : (a, b) \to \mathbb{R}$ is said to be ***differentiable at*** $c \in (a, b)$ with a derivative $f'(c) \in \mathbb{R}$ if

$$\lim_{x\to c} \frac{f(x) - f(c)}{x - c} = f'(c).$$

Analogously, we can also define the complex derivative of a complex-valued function of a complex variable as follows. If $U \subset \mathbb{C}$ is a open subset of $\mathbb{C} = \mathbb{R}^2$, and $w \in U$, then $f : U \to \mathbb{C}$ is said to be ***complex differentiable at*** w with complex derivative $f'(w) \in \mathbb{C}$ if

$$\lim_{z\to w} \frac{f(z) - f(w)}{z - w} = f'(w).$$

This seemingly innocent generalization of the derivative from the real to complex case is anything but that! Indeed, the subject of complex analysis is radically different from real analysis. This big difference stems from the geometric meaning of complex multiplication in the plane. To illustrate the truly different nature of the two subjects, here is a remarkable fact: If a function $f : U \to \mathbb{C}$ is complex differentiable *once* in U, then it is *infinitely* many times complex differentiable in U. Clearly the real analogue of this statement is not true. There are functions that are real differentiable everywhere, but for which the derivative function is differentiable nowhere! Nevertheless, there is a deep link between complex analysis and real analysis via the path of PDEs! Indeed, one has the following result:

Theorem 0.1. *Let $U \subset \mathbb{C}$ be open and let $w \in U$. Then $f : U \to \mathbb{C}$ is complex differentiable at w if and only if $u := \mathrm{Re}(f)$ and $v := \mathrm{Im}(f)$ are real differentiable at w, and the* Cauchy-Riemann equations *hold at $w = (x_0, y_0)$:*

$$\begin{aligned} \frac{\partial u}{\partial x}(x_0, y_0) &= \frac{\partial v}{\partial y}(x_0, y_0), \\ \frac{\partial u}{\partial y}(x_0, y_0) &= -\frac{\partial v}{\partial x}(x_0, y_0). \end{aligned}$$

Moreover, then $f'(w) = \dfrac{\partial u}{\partial x}(x_0, y_0) + i\dfrac{\partial v}{\partial x}(x_0, y_0)$. ◊

Exercise 0.9. In the following, x, y will be real numbers.

(1) Using Theorem 0.1, show that $z = x + iy \mapsto \exp(z) := e^x(\cos y + i \sin y)$ is complex differentiable everywhere, and find its complex derivative.

(2) Using Theorem 0.1, show that $z = x + iy \mapsto \overline{z} := x - iy$ is complex differentiable nowhere.

As seen in the examples above, a (scalar) PDE for an unknown function u of the independent variables $x_1, \cdots, x_n$ is

$$F(x_1, \cdots, x_n, u, u_{x_1}, \cdots, u_{x_n}, u_{x_1 x_2}, \cdots) = 0,$$

where F is a known function. This equation is often supplemented with Initial Conditions and Boundary Conditions, as we will explain later. We will study only scalar equations in this course.

Why study PDEs?

We study PDEs because the need arises in applications, as suggested by the examples listed in the previous section. Traditionally, PDE models were common in Physics and Engineering, but recently they have been used in all applied sciences such as Biology, Chemistry, Economics and so on. This is understandable since in *any* area where there is an interaction between a number of variables, and where rates of changes are involved, one obtains PDE models.

What does “study” mean?

Our study of PDEs involves two interrelated aspects, qualitative and quantitative. Qualitative aspects concern deriving information about the solution to a given problem from the structure of the PDE and associated boundary and initial conditions. For example, we will learn later on that for a vibrating string with clamped ends, the total mechanical energy, defined as the sum of the kinetic energy (energy of motion) and the potential energy (energy from the string’s tension), is conserved. Hence from an energy perspective, the motion of the string is the exchange process between the kinetic and potential forms of energy. Fundamental qualitative questions for PDEs are:

(1) Existence: Does the PDE have a solution?
Clearly, in a description of reality leading to a PDE model, the absence of a solution would imply that we have an incorrect model.

(2) Uniqueness: Is the solution unique?
Again, in a realistic situation, we expect just one reality, and so if our PDE model has multiple solutions, this indicates that we have modelled incorrectly.

(3) Stability: Is it the case that a small change in the equation/ side conditions results in a small change in the solution?
In making any measurement, we incur experimental errors. Hence we may not know the precise values of the coefficients in our PDEs or the precise boundary and initial conditions. So the question of stability is a natural one: basically we are asking if the solution doesn’t change too much if we perturb the data slightly.

Collectively, if the answers to the above three questions are all “yes”, then we say that the PDE is *well-posed*, otherwise it is called *ill-posed*. In this first course on PDEs, we will learn the basics of well-posedness considerations for simple equations, prototypical of "big classes".

For a well-posed PDE, a natural question is also that of devising numerical methods to find an approximate solution. This is a part of the study of quantitative aspects of PDEs concerned with constructing explicit solutions or numerical approximations of solutions to PDEs. Almost always, determining exact solution formulas for PDEs involves a reduction to a bunch of ODEs. Another important question is that of *regularity of solutions*: that is, given regularity (such as smoothness) of the boundary or initial data, then how is this reflected in the solution? For example, the homogeneous heat equation always has smooth solutions, no matter how irregular the initial data, while the wave equation propagates the singular behaviour of the initial data.

Let us clarify what we mean by a solution to a PDE.

Function spaces, operators, solutions

If $k \in \mathbb{N}$ and $D \subset \mathbb{R}^n$, then we say that $u \in C^k(D)$ if $u : D \to \mathbb{R}$ is k times continuously differentiable on D. In other words, the partial derivatives

$$\frac{\partial^k}{\partial x_{i_1} \cdots \partial x_{i_k}}$$

exist in D for all $1 \leqslant i_1, \cdots, i_k \leqslant n$, and they are continuous on D. We take $C^0(D)$ to be the set of real valued continuous functions on D. Sometimes we simply write "$u \in C^k$" instead of "$u \in C^k(D)$".

A function $u \in C^k$ that satisfies a PDE of order k is called a *classical* or *strong* solution of the PDE.

However, in the modern theory of PDEs, we will often deal with solutions that are not classical, and will be referred to as *weak* solutions. These will satisfy the PDE too, albeit in a "generalized sense". We will see some examples of such solutions, and what they mean, later on in Chapter 5.

Exercise 0.10 (Transport equation). Show that if $f \in C^1(\mathbb{R})$, then $(x,t) \mapsto f(x - vt)$ satisfies $u_t + vu_x = 0$.

Exercise 0.11. Show that each of the following equations has a solution having the form $u(x,y) = e^{\alpha x + \beta y}$, with real α, β. Find constants α, β in each case.

(1) $u_x + 3u_y + u = 0$.
(2) $u_{xx} + u_{yy} = 5e^{x-2y}$.
(3) $u_{xxxx} + u_{yyyy} + 2u_{xxyy} = 0$.

Exercise 0.12 (An ill-posed problem). Show that the system

$$\begin{aligned} u_x &= ax^2y + y, \\ u_y &= x^3 + x \end{aligned}$$

with the condition $u(0,0) = 0$ has a unique solution when $a = 3$, but no solution when $a \neq 3$.

Order of a PDE

The *order* of a PDE is the order of the highest derivative that appears in the equation.

Example 0.7.

The Black-Scholes equation has order 2 (or "is a second order PDE").

The transport equation $u_x + u_t = 1$ has order 1 / is a first order PDE.

The diffusion equation $u_t - u_{xx} = 0$, the wave equation $u_{tt} - c^2 u_{xx} = 0$ and the Laplace equation $u_{xx} + u_{yy} = 0$ are second order PDEs.

$u_t + u_{xxxx} = 0$ is a fourth order PDE.

$(u_{tt})^2 - (u_{xx})^2 = \sin(xy)$ is a second order PDE. $\Diamond$

Linear versus nonlinear

Mappings between various function spaces are called ***operators***, typically denoted by L. The result of acting L on a function u is the function Lu. While studying PDEs, we will meet operators involving partial derivatives and will be called ***differential operators***. For example,

$$\begin{aligned} &u \xmapsto{L_1} x_1^2 u_{x_1} + \cdots + x_n^2 u_{x_n} &&: C^k \longrightarrow C^{k-1}, \\ &u \xmapsto{L_2} u_{x_1} \cdots u_{x_n} &&: C^k \longrightarrow C^{k-1}, \end{aligned}$$

are both differential operators.

A differential operator L satisfying

(L1) $L(u_1 + u_2) = L(u_1) + L(u_2)$ and

(L2) $L(\alpha \cdot u) = \alpha L(u)$

for all functions u_1, u_2, u in the domain of L and all scalars α will be called ***linear***. On the other hand, if (L1) or (L2) doesn't always hold for some L, then such a differential operator L is called ***nonlinear***. In the example above, L_1 is linear, but L_2 isn't. In general, it can be seen that a differential operator L of the form

$$Lu := P(x, \partial) := \sum_{0 \leqslant k_1, \cdots, k_n \leqslant N} c_{k_1, \cdots, k_n}(x_1, \cdots, x_n) \frac{\partial^{k_1 + \cdots + k_n}}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} u$$

is linear. If L is a linear differential operator, then the equation

$$Lu = f$$

is called a ***linear PDE***. Moreover, if all the coefficient functions $c_{k_1, \cdots, k_n}(x_1, \cdots, x_n)$ are constants, then we call $Lu = f$ a ***constant coefficient*** linear PDE.

Example 0.8.

The Black-Scholes equation is linear, but ***not*** constant coefficient. The diffusion equation $u_t - u_{xx} = 0$ is linear and constant coefficient. $x^9 u_x + e^{xy} u_y + (\sin(x^2 + y^2))^2 u = x^3$ is linear, but not constant coefficient.

$u_x^2 + u_y^2 = 1$ is nonlinear. $u_x x + u_y y = u^3$ is nonlinear. $\Diamond$

Homogeneous versus nonhomogeneous

If L is a linear differential operator, then the PDE

$$Lu = 0$$

is called *homogeneous*, while the equation

$$Lu = f$$

with a nonzero f is called *inhomogeneous* (or sometimes *nonhomogeneous*).

Example 0.9. The Laplace equation $u_{xx} + u_{yy} = 0$ is homogeneous.

The equation $u_{xx} + u_{yy} = f$ with a nonzero f is called *Poisson's equation.* For example, in electrostatics, one meets the Poisson's equation

$$u_{xx} + u_{yy} = -\rho,$$

where ρ is the charge density. $\Diamond$

In the context of linear PDEs, one has the following easy, but fundamental result.

Theorem 0.2 (Superposition Principle). *Let L be a linear differential operator on a region Ω of the independent variables. Then the following hold on Ω:*

(1) *A linear combination of solutions of the homogeonous PDE $Lu = 0$ is again a solution to $Lu = 0$.*

(2) *Let f be a function on Ω, and let u_p satisfy $Lu_p = f$. Then* every *solution u of $Lu = f$ has a decomposition*

$$u = u_p + u_h,$$

where u_h satisfies $Lu_h = 0$. (*In other words, every solution to $Lu = f$ is the sum of the particular solution u_p and a solution u_h of the homogeneous equation.*)

Proof.

(1) If $Lu_1 = 0$ and $Lu_2 = 0$, then for any scalars c_1, c_2, it follows, using the linearity of L that

$$L(c_1 \cdot u_1 + c_2 \cdot u_2) = c_1 \cdot L(u_1) + c_2 \cdot L(u_2) = c_1 \cdot 0 + c_2 \cdot 0 = 0.$$

Hence $c_1 \cdot u_1 + c_2 \cdot u_2$ is also a solution.

(2) Let u satisfy $Lu = f$. Define $u_h := u - u_p$. Then

$$Lu_h = L(u - u_p) = Lu - Lu_p = f - f = 0,$$

and so u_h satisfies the homogeneous equation $Lu_h = 0$. $\square$

Exercise 0.13. Classify the given PDEs by filling the table:

Order	Linear/ nonlinear	Constant/ nonconstant coefficient	Homogeneous/ inhomogeneous

(1) $u_t = u_{xx} + 2u_x + u$

(2) $u_t = xu_{xx} + e^{-t}$

(3) $u_{xx} + 3u_{xy} + u_{yy} = \sin x$

(4) $u_{tt} = uu_{xxxx}$.

The classical trinity

In this book, we will focus our attention on three important linear, second order PDEs: the diffusion equation, the wave equation and the Laplace equation in two independent variables.

- The *diffusion equation* is
$$u_t - u_{xx} = 0,$$
and models the physical phenomenon of the diffusion of heat or of matter. In the case of the diffusion of heat, imagine a hot rod cooling down as time passes, and the temperature $u(x, t)$ at place x along the rod and at time t satisfies the diffusion equation.
- The *wave equation* is
$$u_{tt} - c^2 u_{xx} = 0,$$
governs the motion of a vibrating string, and $u(x, t)$ denotes the displacement at place x and at time t.
- The *Laplace equation* is
$$u_{xx} + u_{yy} = 0,$$
arises in a variety of "steady-sate" problems and in complex analysis. The equation $u_{xx} + u_{yy} = f$ with a with a nonzero f is called the *Poisson equation.*

We focus on these since they are prototypical examples of whole classes of PDEs. We will elaborate on this in Chapter 2.

Exercise 0.14 (Laplacian). In $\mathbb{R}^n$, the operator

$$\Delta := \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

is called the *Laplace operator* or the *Laplacian.* The equation

$$\Delta u = 0$$

is called the *Laplace equation*, and the equation $\Delta u = f$ with a nonzero f is called the *Poisson equation.*

(1) (Harmonic functions). Twice continuously differentiable functions that satisfy the Laplace equation in two independent variables are called *Harmonic functions.* Show that the following functions are harmonic: $u(x, y) = x^2 - y^2$, $u(x, y) = e^x \cos y$, $u(x, y) = \log \sqrt{x^2 + y^2}$ (in $\mathbb{R}^2 \setminus \{(0, 0)\}$).

(2) (Link with complex analysis). Show that if $U \subset \mathbb{C}$ is open, and $f : U \to \mathbb{C}$ is complex differentiable in U, then $u := \mathrm{Re}(f)$ and $v := \mathrm{Im}(f)$ are harmonic in U.

Remark 0.1. In fact it can be shown that *every* harmonic function on a "simply connected" open set arises in this manner. So, in particular, every harmonic function is *locally* the real part of some complex differentiable function.

(3) (Harmonic functions give steady state behaviour). Consider the diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x} + \frac{\partial^2 u}{\partial y^2}$$

in two spatial variables. (Imagine a hot plate cooling down with time.) Show that if, after a long time, the temperature profile does not change with time ("steady-state"), then u satisfies the Laplace equation.

(4) (Link with Brownian motion). Consider a particle in a two dimensional domain D, say the unit square. Divide the square into N^2 identical little squares, and denote their vertices by (x_i, y_j). The size of each edge of a little square is h. A particle located at an internal vertex (x_i, y_j) jumps during a time interval τ to one of its nearest neighbours with equal probability (1/4). When the particle reaches the boundary ∂D, it dies. We ask:

What is the life expectancy $u(x, y)$ of a particle that starts its life at a point (x, y) in D as $h \to 0$, $\tau \to 0$, while maintaining $h^2/(2\tau) =: k$?

We shall arrive at an answer while relying on an intuitive notion of the life expectancy $u(x, y)$, that is, the "average time" it takes for a particle starting at (x, y) to reach ∂D. Obviously a particle starting its life at a boundary point dies at once, so that we have $u(x, y) = 0$ on ∂D. Now consider an internal point (x, y). A particle can go to one of its four neighbours with equal probability, and moreover it takes a time τ to get to one of these neighbours. So we obtain

$$u(x, y) = \tau + \frac{u(x + h, y) + u(x, y + h) + u(x - h, y) + u(x, y - h)}{4}.$$

Expand all the functions using a Taylor expansion, assuming $u \in C^3$, and take the limit as stipulated in the question to obtain the Poisson equation

$$\Delta u = -\frac{2}{k}, \quad (x, y) \in D,$$

with the boundary condition $u(x, y) = 0$ on ∂D.

(5) Let us consider an application of a one-dimensional analogue of the model considered in the previous item. Many models in the stock market are based on assuming that stock prices vary randomly. Assume that a broker buys a stock at a price m. The broker decides in advance to sell it if its price reaches an upper bound M (in order to cash the resulting profit) or a lower bound m (in order to minimise losses in case the

stock dives). How much time on an average will the broker hold the stock assuming that the stock price performs a Brownian motion? The relevant equation is

$$u''(x) = -\frac{1}{k}, \quad u(m) = u(M) = 0.$$

Compute the average time as a function of x.

(6) (Electrostatic/Gravitational potential). Show that

$$u(x, y, z) := \frac{1}{r} := \frac{1}{\sqrt{x^2 + y^2 + z^2}},$$

satisfies $\Delta u = 0$ on $\mathbb{R}^3 \backslash \{(0, 0, 0)\}$.

Exercise 0.15 (Fundamental solution of the diffusion equation). Show that

$$u(x, t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}, \quad x \in \mathbb{R}, t > 0$$

satisfies the diffusion equation $u_t = u_{xx}$ for $(x, t) \in \mathbb{R} \times (0, \infty)$. Use Maple to plot $u(\cdot, t)$ for $t = 1, 0.1, 0.001$.

Exercise 0.16 (Travelling wave solution to the wave equation). Show that if $f \in C^2$, then

$$u(x, t) = f(x + ct) + f(x - ct)$$

solves the wave equation $u_{tt} = c^2 u_{xx}$.

Initial Boundary Value Problems

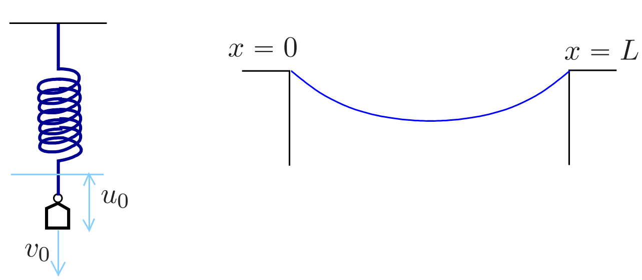
Recall that ODEs typically come with initial values or boundary values. For example, consider a spring mass system, where the displacement u satisfies

$$m\ddot{u} + ku = f$$

where m is the mass of the bob, k is the spring constant, and f is the externally applied force. See the left picture in the following figure. Typically, we would be interested in solving this equation given a pair of initial conditions of the form

$$u(0) = u_0 \text{ and } \dot{u}(0) = v_0,$$

which specify the *initial* position and velocity of the mass at the initial time $t = 0$. Taken together, the differential equation, with the pair of initial conditions is called an "initial value problem".



An example of a different sort is the following. Consider

$$u''(x) = -\frac{g}{c^2},$$

which models the displacement u of an elastic string along the position x due to gravity. See the picture on the right above. (Here g is the usual acceleration due to gravity and c^2 is a constant depending on the material properties of the string.) If the string is clamped between the two endpoints at $x = 0$ and $x = L$, then we have the boundary conditions

$$u(0) = 0 \text{ and } u(L) = 0.$$

Taken together, the differential equation, with the pair of boundary conditions is called a "boundary value problem".

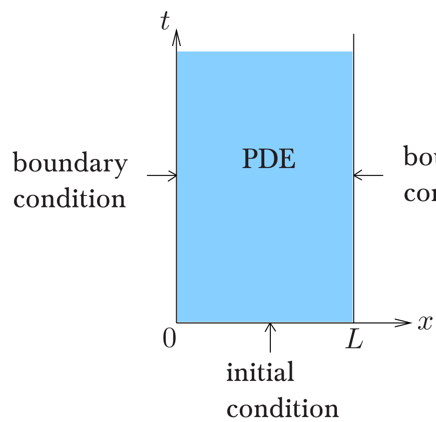
Generally, a PDE involves a time variable $t > 0$ and one or more spatial variables. Typically the spatial variables are restricted to some open set. For example, we may be interested in the temperature of a heated rod of length L in which case we consider the heat equation in the domain

$$\Omega := \{(t, x) : t > 0 \text{ and } 0 < x < L\}.$$

Physically it makes sense that information concerning the transfer of heat energy on the boundary is needed; in other words, we need boundary conditions at $x = 0$ and at $x = L$. There are three main types of boundary conditions that occur in most applications:

- Dirichlet conditions, specifying the unknown function on the boundary. (In the heat flow case, this means the temperature is specified at the two end points of the rod.)
- Neumann conditions, prescribing the normal derivative on the boundary. (In the heat flow case, this means specifying the rate of energy transfer, u_x at the two end points.)
- Robin conditions, where a linear combination of the function and the normal derivative is specified on the boundary.

In equations involving time, one or more initial conditions are usually needed to obtain a unique solution to the PDE. Typically this number equals the highest order of differentiation with respect to time occurring in the PDE. For example, in the diffusion equation, one initial condition is needed, while in the wave equation, two initial conditions are needed.



The combination of a PDE with associated boundary conditions and initial conditions is called an *initial boundary value problem.*

If time is not involved in the PDE (for example in the Laplace equation), the combination of the equation with associated boundary conditions is called a *boundary value problem.*

Some historical remarks

Many equations and results in PDEs bear the name of their originator. Here is a selective list of historical characters which gives a feeling of the span of time and an indication of the key milestones in the subject.

The one dimensional wave equation was introduced and analyzed by d'Alembert in 1752 as a model of a vibrating string. His work was extended by Euler (1759) and later by D. Bernoulli (1762) to 2 and 3 dimensions. The Laplace equation was first studied by Laplace in his work on gravitational potential fields around 1780. The heat equation was introduced by Fourier in his celebrated memoir *Théorie analytique de la chaleur* (1810). Thus, the three major examples of second-order PDEs—hyperbolic, elliptic and parabolic—had been introduced by the first decade of the 19th century, though their central role in the classification of PDEs, and related boundary value problems, were not clearly formulated until later in the century. Besides the three classical examples, a profusion of equations, associated with major physical phenomena, appeared in the period between 1750 and 1900:

- The Euler equation of incompressible fluid flows, 1755.
- The minimal surface equation by Lagrange in 1760 (the first application of the Euler-Lagrange principle in PDEs).
- The Monge-Ampére equation by Monge in 1775.
- The Laplace and Poisson equations, as applied to electric and magnetic problems, starting with Poisson in 1813, Green in 1828, and Gauss in 1839.

- The Navier-Stokes equations for fluid flows in 1822 by Navier, followed by Poisson (1831) and Stokes (1845).
- Linear elasticity, Navier (1821) and Cauchy (1822).
- Maxwell's equations in electromagnetic theory in 1864.
- The Helmholtz equation and the eigenvalue problem for the Laplace operator in connection with acoustics in 1860.
- The Plateau problem (in the 1840s) as a model for soap bubbles.
- The Korteweg-de Vries equation (1896) as a model for solitary water waves.

A central connection between PDE and the mainstream of mathematical development in the 19th century arose from the role of PDE in the theory of analytic functions of a complex variable. Cauchy had observed in 1827 the link between the Cauchy-Riemann PDEs and complex differentiable functions. From the later point of view of Riemann (1851), this became the central defining feature of complex differentiable functions.

The modern theory of PDEs began with the work of Poincaré, when in 1890, he gave the first complete proof, in rather general domains, of the existence and uniqueness of a solution of the Laplace equation for any continuous Dirichlet boundary condition.

In his celebrated address to the International Mathematical Congress (ICM) in Paris in 1900, Hilbert presented 23 problems (the so-called Hilbert problems), two of which are concerned with the theory of nonlinear elliptic PDEs. In connection with one of these (Problem 20), Hilbert revived the interest in Riemann's approach to the Dirichlet principle. In connection with these problems extensive analysis by numerous mathematicians, for example, Lebesgue, Fubini, Courant, Fredholm, gave rise to new tools for the analysis of PDEs.

Another important machinery to carry through the study of solutions of PDEs was introduced by S.L. Sobolev in the mid 1930s: the definition of new classes of function spaces, now called the Sobolev spaces. Besides Sobolev, many advances were made in PDE theory in the 20th centry by the application of functional analysis, by Banach, Friedrichs, Browder, Gårding, Lax, J.L. Lions, Hille, Yosida, and others.

Laurent Schwartz, in his celebrated book *La théorie des distributions* (1950) presented the generalized solutions of PDEs in a new perspective. He created a calculus, based on extending the class of ordinary functions to a new class of objects, the distributions, while preserving many of the basic operations of analysis, including addition, multiplication by C^∞ functions, differentiation, as well as, under certain restrictions, convolution and Fourier transform. These tools proved to be fundamental to the study of PDEs, and has been the subject of intensive investigation beginning in the mid-1950s in the work of Ehrenpreis, Malgrange and Hörmander, revolutionizing the subject. PDE theory continues to be an active and a rich area of research even today.

Chapter 1

First Order PDEs

A first order PDE for an unknown function $(x_1, \cdots, x_n) \mapsto u(x_1, \cdots, x_n)$ has the general form

$$F(x_1, \cdots, x_n, u, u_{x_1}, \cdots, u_{x_n}) = 0,$$

where F is a given function of $2n+1$ variables. First order equations arise in a number of models appearing in the various applied sciences, and we give one simple example below, which also illustrates how a PDE can arise from basic considerations like a conservation law.

1.1. An example of deriving a PDE: traffic flow

Consider car traffic along a single lane of a highway only in one direction, and without any entrances or exits. Let $u(x,t)$ be the density of cars at position x and at time t. If there are a lot of cars, the it is reasonable to ignore the "granularity" of cars, and think of u as being a continuously differentiable real-valued function (rather than an integer-valued function).

Now imagine counting the number of cars which pass a lamppost at place x per unit time. This is called the ***flux*** of cars at place x and at time t, and is denoted by $\varphi(x,t)$.

If we imagine any stretch of highway between the points $x = a$ and $x = b$, it is clear that the total number of cars at time t in this stretch is

$$N(t) = \int_a^b u(x,t)dx.$$

Thus the rate of change of the number of cars at time t within this stretch is

$$\frac{dN}{dt}(t) = \frac{d}{dt}\int_a^b u(x,t)dx = \int_a^b \frac{\partial u}{\partial t}dx.$$

On the other hand, we know that there are no entrances or exits on this highway, and so by the "law of conservation of cars", it must be the case that this rate of change in the number of cars at time t over this stretch of highway equals the difference in the fluxes of cars at the endpoints $x = a$ and $x = b$, that is,

$$\int_a^b \frac{\partial u}{\partial t} dx = \frac{dN}{dt}(t) = \varphi(a,t) - \varphi(b,t).$$

But we can convert the right hand side into an integral too:

$$\varphi(a,t) - \varphi(b,t) = -\int_a^b \frac{\partial \varphi}{\partial x}(x,t)dx.$$

Consequently, we obtain *for all* a, b that

$$\int_a^b \Big(\frac{\partial u}{\partial t}(x,t) + \frac{\partial \varphi}{\partial x}(x,t)\Big) dx = 0.$$

We claim that this implies that

$$\frac{\partial u}{\partial t} + \frac{\partial \varphi}{\partial x} = 0$$

for all x and t. Suppose that there was some point (x_0, t_0) where

$$\frac{\partial u}{\partial t}(x_0,t_0) + \frac{\partial \varphi}{\partial x}(x_0,t_0) \neq 0,$$

and there is no loss of generality in assuming that this is a positive number ϵ. Assuming that u is C^1, there then exists a small neighbourhood (a, b) of x_0 such that

$$\frac{\partial u}{\partial t}(x,t_0) + \frac{\partial \varphi}{\partial x}(x,t_0) > \frac{\epsilon}{2} > 0$$

for $x \in (a, b)$. So the integral

$$\int_a^b \Big(\frac{\partial u}{\partial t}(x,t) + \frac{\partial \varphi}{\partial x}(x,t)\Big) dx > (b-a)\frac{\epsilon}{2} > 0,$$

a contradiction, completing the proof of our claim. So we arrive at the following PDE:

$$\frac{\partial u}{\partial t} + \frac{\partial \varphi}{\partial x} = 0.$$

The number φ of cars per hour passing a place equals the density u of cars times the velocity v of cars:

$$\varphi \,\frac{\text{cars}}{\text{hour}} = u\,\frac{\text{cars}}{\text{km}} \cdot v\,\frac{\text{km}}{\text{hour}}.$$

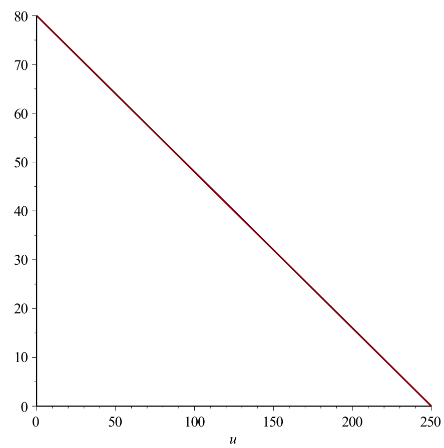
We make the simplifying assumption that the speed v a car moves is a known function f of the traffic density. This makes sense, since if the traffic density is high, cars move slowly, while if it is low, it is like driving on a free highway with its maximum speed limit. Here is an example of such a function f:

$$v = f(u) := V\Big(1 - \frac{u}{U}\Big), \quad 0 \leqslant u \leqslant U. \tag{1.1}$$

(Here V (freeway maximum speed) and U (maximum density of cars) are constants.) Hence our PDE for traffic flow becomes

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(uf(u)) = 0.$$

See the picture below, where $U = 250$ cars/km and $V = 80$ km/hr.



If we assume there is always light traffic, then we can take $v \equiv V$, so that the traffic flow PDE becomes

$$\frac{\partial u}{\partial t} + V\frac{\partial u}{\partial x} = 0,$$

which is the ***transport equation*** (which we had mentioned earlier).

Exercise 1.1. Show that the PDE for traffic flow, if f is given by (1.1) is

$$\frac{\partial u}{\partial t} + V\left(1 - \frac{2u}{U}\right)\frac{\partial u}{\partial x} = 0.$$

By taking the average car length as 4 meters, determine a value for U, the maximum possible density of cars (cars per km) along a stretch of single lane road.

For simplicity, we will only consider PDEs in two independent variables. One reason is that the geometric ideas will become very transparent, since the solution $u(x, y)$ to an equation

$$F(x, y, u, u_x, u_y) = 0,$$

will have a graph which can be visualized as a surface in $\mathbb{R}^3$. Also, we will focus on quasilinear equations.

Definition 1.1 (Quasilinear PDE). A PDE which is linear in the highest order of derivatives, with coefficients that can depend on lower order derivatives as well as on the independent variables, is called ***quasilinear***.

Example 1.1 (A first order quasilinear equation). The general first order quasilinear equation in two independent variables is

$$a\big(x,y,u(x,y)\big)\frac{\partial u}{\partial x}(x,y)+b\big(x,y,u(x,y)\big)\frac{\partial u}{\partial y}(x,y)=c\big(x,y,u(x,y)\big).$$

For example the ***inviscid Burgers's equation***, $u_t + uu_x = 0$ is an example of a quasilinear first order equation. $\Diamond$

Exercise 1.2. Write the general form of a second order quasilinear PDE.

In this chapter, we will learn a method for solving first order quasilinear equations in two independent variables:

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u),$$

where $a, b, c : \mathbb{R}^3 \to \mathbb{R}$ are given continuously differentiable (C^1) functions. The key idea behind the approach is a geometric one:

> The PDE says that the normal $(u_x, u_y, -1)$ to the graph of u is perpendicular to the vector (a, b, c) at each point on the graph of the solution u.

This information will be used to "knit" together a solution of the PDE, given data (values of the solution u along a curve in the independent variable plane). All this is a mystery right now, but will become clearer as we proceed.

1.2. The method of characteristics

The method of characteristics was developed in the 19th century by Hamilton in connection with his investigation of the propagation of light. Although one can stipulate precise assumptions which are ***sufficient*** for the existence of a unique solution to a quasilinear PDE which can be found using the method of characteristics, in our treatment, we will ***assume*** that there exists a function u whose graph is a surface which is the union of "characteristic curves" (constructed from the quasilinear equation and initial data), and show that this u then solves our quasilinear equation with the given initial data. This will give a concrete procedure to solve first order quasilinear equations in two independent variables by reducing it to solving the ODEs for the characteristic curves.

As we had mentioned earlier, the method of characteristics is based on "knitting" the solution surface with a one-parameter family of curves that intersect a given curve in space (where we have initial data for the PDE).

Suppose that we are given the quasilinear first order PDE in two independent variables

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u), \tag{Q}$$

where a, b, c are assumed to be continuously differentiable (C^1). Suppose that we are given the "initial condition" which is given parametrically by the curve $\gamma : \mathbb{R} \to \mathbb{R}^3$:

$$\mathbb{R} \ni s \overset{\gamma}{\longmapsto} \big(x_0(s), y_0(s), u_0(s)\big) \in \mathbb{R}^3.$$

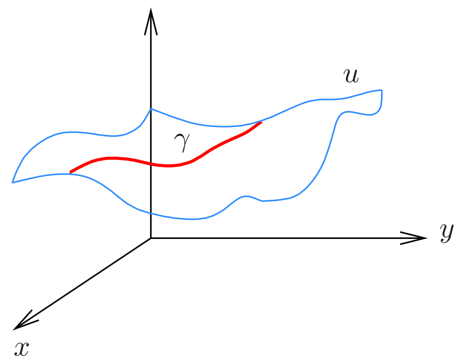
In other words, our solution u should satisfy

$$u\big(x_0(s), y_0(s)\big) = u_0(s) \qquad (s \in \mathbb{R}).$$

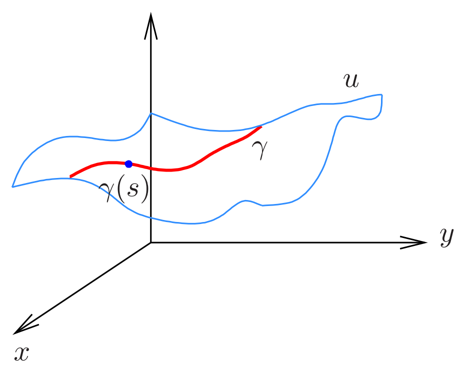
Pictorially, the surface (graph of u) in $\mathbb{R}^3$ we want to construct must pass through the curve γ, as shown below. Thus

(Q) and γ

constitute the data for our problem, which we refer to as the "initial value problem" or "Cauchy problem".



Let us fix $s \in \mathbb{R}$. This in turn fixes the point $\gamma(s) = \big(x_0(s), y_0(s), u_0(s)\big)$ in $\mathbb{R}^3$ lying on the image of the curve γ. See this point indicated as blue dot on the red curve in the picture below.



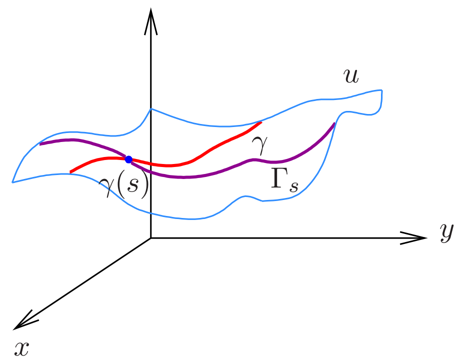
The idea behind the method of characteristics is that if we imagine the graph of u as a sheet of cloth, then we construct this sheet one "thread" at a time. To this end, consider, for this fixed s, the system of ODEs

$$\begin{aligned} \dot{X}_s(t) &= a\big(X_s(t), Y_s(t), U_s(t)\big) & X_s(0) &= x_0(s), \\ \dot{Y}_s(t) &= b\big(X_s(t), Y_s(t), U_s(t)\big) & Y_s(0) &= y_0(s), \\ \dot{U}_s(t) &= c\big(X_s(t), Y_s(t), U_s(t)\big) & U_s(0) &= u_0(s), \end{aligned}$$

where $t \in \mathbb{R}$. From the theory of ODEs, since $a, b, c \in C^1$, we know that these ODEs admit (at least a local) solution

$$t \xmapsto{\Gamma_s} \big(X_s(t), Y_s(t), U_s(t)\big).$$

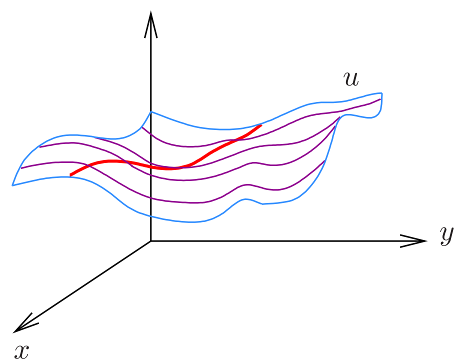
This curve Γ_s is called a *characteristic curve* for the pair (Q), γ. Note that Γ_s passes through the point $\gamma(s)$ at time $t = 0$. This is "one of the threads of the cloth" u. See the picture below.



Now suppose that the union of these characteristic curves Γ_s for the various s give rise to a surface which is the graph of a C^1 function $u : \mathbb{R}^2 \to \mathbb{R}$, that is, for each $(x, y) \in \mathbb{R}^2$ there is a unique $(s, t) \in \mathbb{R}^2$ such that $x = X_s(t)$, $y = Y_s(t)$, and moreover,

$$u\big(X_s(t), Y_s(t)\big) = U_s(t) \quad (s, t \in \mathbb{R}). \tag{1.2}$$

See the picture below.



u is made from the characteristic curves Γ_s

We claim that this u solves our PDE (Q). To see this we just differentiate both sides of (1.2) with respect to t and use the chain rule:

$$\frac{\partial u}{\partial x}\big(X_s(t), Y_s(t)\big) \cdot \dot{X}_s(t) + \frac{\partial u}{\partial y}\big(X_s(t), Y_s(t)\big) \cdot \dot{Y}_s(t) = \dot{U}_s(t),$$

and appealing to the definition of the characteristic curve Γ_s, we obtain

$$\begin{aligned}
&\frac{\partial u}{\partial x}\big(X_s(t), Y_s(t)\big) \cdot a\Big(X_s(t), Y_s(t), u\big(X_s(t), Y_s(t)\big)\Big) \\
&+ \frac{\partial u}{\partial y}\big(X_s(t), Y_s(t)\big) \cdot b\Big(X_s(t), Y_s(t), u(X_s(t), Y_s(t))\Big) \\
&\qquad\qquad\qquad\qquad = c\Big(X_s(t), Y_s(t), u\big(X_s(t), Y_s(t)\big)\Big).
\end{aligned}$$

With $(x, y) = \big(X_s(t), Y_s(t)\big)$, this becomes

$$\frac{\partial u}{\partial x}(x, y) \cdot a\big(x, y, u(x, y)\big) + \frac{\partial u}{\partial y}(x, y) \cdot b\big(x, y, u(x, y)\big) = c\big(x, y, u(x, y)\big).$$

Moreover, $u\big(x_0(s), y_0(s)\big) = u\big(X_s(0), Y_s(0)\big) = U_s(0) = u_0(s)$, and so u has the appropriate initial data as well. This completes the proof of the validity of the method of characteristics for solving the quasilinear equation (Q) with initial data γ.

Summarizing, in the method of characteristics, we construct u from the characteristic curves Γ_s (which amounts to solving a system of ODEs). The characteristic curves take with them a initial piece of the information from the initial data γ, and propagate it with them, by evolving along an ODE.

Let us demonstrate this method in the case of the transport equation.

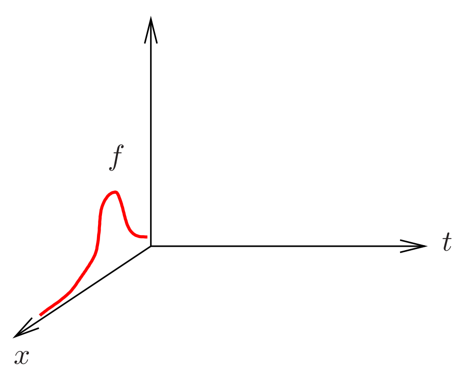
Example 1.2 (Transport equation). Consider the (linear) equation

$$u_t + cu_x = 0,$$

where (the speed) c is a constant. We are given the initial data

$$u(0, x) = f(x), \quad x \in \mathbb{R}.$$

Thus the initial data curve γ is given by $\gamma(s) = (s, 0, f(s))$, $s \in \mathbb{R}$. See the picture below.



The characteristic equations are

$$\begin{array}{ll} \dot{X}_s(\tau) = c & X_s(0) = s, \\ \dot{T}_s(\tau) = 1 & T_s(0) = 0, \\ \dot{U}_s(\tau) = 0 & U_s(0) = f(s), \end{array}$$

This has the unique solution given by

$$\begin{aligned} X_s(\tau) &= c\tau + s, \\ T_s(\tau) &= \tau, \\ U_s(\tau) &= f(s). \end{aligned}$$

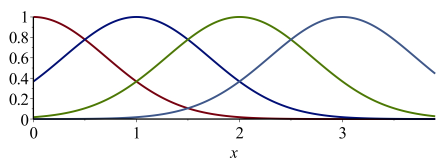
Thus with

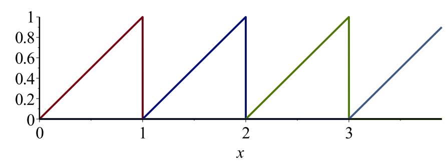
$$\begin{aligned} x &:= X_s(\tau) = c\tau + s, \\ t &:= T_s(\tau) = \tau, \end{aligned}$$

we have

$$u(x,t) = U_s(\tau) = f(s) = f(x - ct).$$

For example, if $f = e^{-x^2}$ and $c = 1$, then the graph of $u(\cdot, t)$ is shown on the left below below at the time instances $t = 0, 1, 2, 3$.





Note that the method of characteristics also seems to work *formally* for initial data f which may not be smooth. For example, if we take

$$f(x) = \begin{cases} x & \text{if } 0 < x < 1, \\ 0 & \text{otherwise}, \end{cases}$$

then the solution $u(\cdot, t) = f(\cdot - t)$ is displayed on the right above at the time instances $t = 0, 1, 2, 3$. However, this is not a *classical* solution, but solves the transport equation in a "weak" sense. We will revisit this in Chapter 5 when we learn about distributions, and make this precise. $\Diamond$

Of course, it may happen that the projections $\big(X_s(\cdot), Y_s(\cdot)\big)$ and $\big(X_{s'}(\cdot), Y_{s'}(\cdot)\big)$ of two characteristic curves Γ_s and $\Gamma_{s'}$ intersect each other. At such points, there are two available values for u: $U_s(t)$ and $U_{s'}(t')$, and we don't know what the value of u should be at $\big(X_s(t), Y_s(t)\big) = \big(X_{s'}(t'), Y_{s'}(t')\big)$. We give an example below.

Example 1.3 (Shock wave). Consider the inviscid Burgers's equation

$$u_t + uu_x = 0,$$

with the initial condition

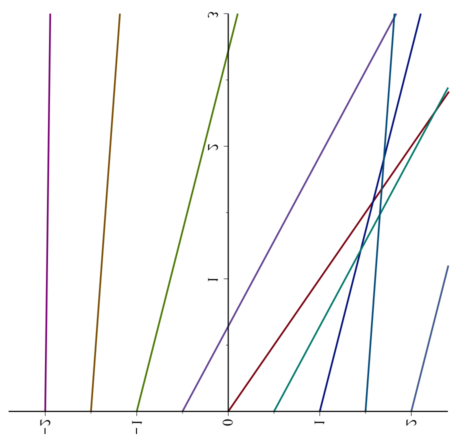
$$u(x, 0) = e^{-x^2}.$$

The characteristic equations are

$$\begin{array}{ll} \dot{X}_s(\tau) = U_s(\tau) & X_s(0) = s, \\ \dot{T}_s(\tau) = 1 & T_s(0) = 0, \\ \dot{U}_s(\tau) = 0 & U_s(0) = e^{-s^2}. \end{array}$$

This has the unique solution given by

$$\begin{array}{rcl} U_s(\tau) & = & e^{-s^2}, \\ T_s(\tau) & = & \tau, \\ X_s(\tau) & = & e^{-s^2}\tau + s. \end{array}$$



The projections of the characteristic curves are plotted above, and we see that they intersect. (Nevertheless, a "shock wave solution" can be given beyond the time where the characteristics intersect. For more details, we refer the interested student to the book [**K**].) ◊

Exercise 1.3. Solve $u_x = 1$ subject to the initial condition $u(0, y) = f(y)$.

Exercise 1.4. Solve $u_x + (\log u)u_y = 0$ subject to the initial condition $u(0, y) = e^y$.

Exercise 1.5. Solve $u_x + u^2 u_y = 0$ subject to the initial condition $u(0, y) = \sqrt{y}$, $x, y > 0$.

Exercise 1.6. Solve $u_x - x^2 u_y = -u$ subject to the initial condition $u(0, y) = f(y)$.

Exercise 1.7. Solve $u_x + yu_y = x$ subject to the initial condition $u(0, y) = f(y)$.

Exercise 1.8. Solve $u_x = -3yu$ subject to the initial condition $u(0, y) = f(y)$.

Exercise 1.9 (Advection equation). The equation

$$u_t + cu_x = k(x, t)$$

is used to model AIDS epidemics, fluid dynamics, and other situations involving the transport of matter with flow. As a concrete application, suppose that a factory spills out a pollutant in the air, which is carried by the wind blowing in the x direction at a speed c meters per second. Let

$u(x,t)$ be the density (number of particles per meter) at time t, and suppose that the particles are falling out of the air at a constant rate proportional to $u(x,t)$, with constant of proportionality $r > 0$. Then u satisfies the equation $u_t + cu_x = k(x,t)$, with $k(x,t) = -ru(x,t)$.

(1) If the initial condition is $u(x,0) = f(x)$, then show that $u(x,t) = e^{-rt}f(x-ct)$.

(2) Let M denote the number of particles in the air at time $t = 0$. Show that the number of particles at time $t > 0$ is $e^{-rt}M$.

Exercise 1.10 (Pollution assessment). A river flows in a planar region

$$D = \{(x,y) : |y| < 1, x \in \mathbb{R}\}.$$

A factory spills a contaminant into the river. The contaminant is further spread and convected by the flow in the river. The velocity field of the river water is only in the x-direction. The concentration of the contaminant at a point (x,y) in the river at time t is denoted by $u(x,y,t)$. Conservation of matter and momentum leads to the following first order PDE for u:

$$u_t - (y^2 - 1)u_x = 0.$$

Suppose that the initial condition is $u(x,y,0) = e^y e^{-x^2}$.

(1) Note that y appears in the PDE just as a parameter. So considering the PDE in the two independent variables (x,t), use the method of characteristics to find its solution $u_{(y)}(x,t)$, and hence find an expression for $u(x,y,t) = u_{(y)}(x,t)$.

(2) Next suppose that a fish lives near the point $(x,y) = (2,0)$ in the river. The fish can tolerate a contaminant concentration level of 0.5. If the concentration exceeds this level, the fish dies at once. Will the fish survive? If yes, explain why. If no, then find the time at which the fish will die.

Exercise 1.11 (Utility Theory). In Utility Theory, one encounters the following problem: Find all functions $u = u(x,y)$ with the property that the ratio between the marginal utilities with respect to x and y depends on (say) x only. Thus we must solve the equation

$$u_x - f(x)u_y = 0,$$

where f is a given function. Assuming that $u(0,y) = \varphi(y)$, where φ is given, show that

$$u(x,y) = \varphi\big(y + F(x)\big),$$

where F is given by

$$F(x) := \int_0^x f(\xi)d\xi.$$

Chapter 2

The classical trinity

In this chapter we will consider general second order PDEs and introduce the three main examples of second order PDEs we will study in this course.

2.1. Classification of second order linear PDEs

Consider a general second order linear PDE in two variables:

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G,$$

where A, B, C, D, E, F, G are known functions of x, y. Recall that this equation is called homogeneous if $G = 0$. The ***principal part*** of a linear PDE collects the terms of highest order, and in the case of our second order PDE, this is

$$p_N = Au_{xx} + 2Bu_{xy} + Cu_{yy}.$$

It turns out that this term has a lot to do with the qualitative behaviour of the solutions. The quadratic form

$$q(s,t) := As^2 + 2Bst + Ct^2$$

associated with the principal part is used to classify the second order PDE.

Definition 2.1 (Parabolic/Elliptic/Hyperbolic). We call the second order PDE

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G,$$

(1) ***elliptic*** if $AC - B^2 > 0$ everywhere,

(2) ***parabolic*** if $AC - B^2 = 0$ everywhere,

(3) ***hyperbolic*** if $AC - B^2 < 0$ everywhere.

Here “everywhere” means “everywhere in the set of interest”. The set of interest may be a strict subset of $\mathbb{R}^2$.

Here are a few examples.

Example 2.1 (Classical trinity classification!).

The Laplace-Poisson equation $u_{xx} + u_{yy} = g$ is elliptic. Indeed, here $A = C \equiv 1$, $B \equiv 0$, and so $AC - B^2 \equiv 1 > 0$.

The diffusion equation $u_t - u_{xx} = g$ is parabolic. Indeed, here $A = B \equiv 0$, $C \equiv -1$, and so $AC - B^2 \equiv 0$.

The wave equation $u_{tt} - c^2 u_{xx} = g$ is hyperbolic. Indeed, here $A \equiv 1$, $B \equiv 0$, $C \equiv -c^2$, and so $AC - B^2 \equiv -c^2 < 0$. $\diamondsuit$

In the above example, the coefficients were all constants, and so a global classification was possible. However, in the case of variable coefficients, the nature of the PDE might change depending on what subset of $\mathbb{R}^2$ is considered.

Example 2.2. Consider the equation $yu_{xx} + u_{yy} = 0$. Then $A = y$, $B = 0$, and $C = 1$. So $AC - B^2 = y$. Hence the PDE is elliptic if $y > 0$, parabolic if $y = 0$, and hyperbolic if $y < 0$. $\diamondsuit$

Why make this fuss about this (seemingly weird) classification? It turns out that after a change of variables, essentially any second order linear PDE can be brought to a canonical form resembling the Poisson/Diffusion/Wave equation (but possibly with first order terms) depending on whether the PDE was elliptic/parabolic/hyperbolic, respectively, to begin with. (This is analogous to what one does in Linear Algebra, where by a process involving diagonalization, one observes that the zero set in $\mathbb{R}^2$ of any quadratic polynomial in two variables is an ellipse, a parabola or a hyperbola.) Moreover, the respective PDE solutions share common features characteristic to each of these three classes. For example:

(1) the solutions to the elliptic PDE arise as the steady state energy minimal functions,
(2) the solutions to the parabolic PDE even out fluctuations,
(3) the solutions to hyperbolic PDEs behave like disturbances that persist and propagate.

Remark 2.1. Even for more than two variables, a similar classification is possible for second order PDEs. With the coefficients in the principal part of the equation, one builds a quadratic form. And depending on whether this quadratic form is definite, semidefinite or indefinite, one calls the equation elliptic, parabolic or hyperbolic, respectively. A classification for equations of order bigger than 2 and for *systems* of PDEs is also available (although it is not as complete as for order 2).

Exercise 2.1. Consider the *telegraph equation* important in electrical engineering,

$$v_{xx} = KLv_{tt} + (KR + LS)v_t + RSv,$$

where $v(x, t)$ is the electrostatic potential at time t at a point x units from one end of a transmission line that has an electrostatic capacity K, self-inductance L, resistance R, and leakage conductance S, all per unit length. Show that the equation is hyperbolic if $KL > 0$, and that it is parabolic if either K or L is zero.

2.2. Uniqueness and stability

A physical system must evolve in a determined way under given conditions, and experiments should be reproducible. In mathematical models, this corresponds to existence and uniqueness of solutions. We won't go into general existence questions here, since we will be constructing explicit solutions for our PDEs. But in order to justify that there aren't any solutions besides what our constructive procedures deliver, we will sometimes need to use the following uniqueness results.

2.2.1. Wave equation with Dirichlet conditions. Consider the wave equation with Dirichlet conditions:

$$\begin{array}{ll} \dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} = v & (0 < x < 1,\ t > 0), \\ u(0,t) = \alpha(t) \text{ and } u(1,t) = \beta(t) & (t > 0), \\ u(x,0) = f(x) \text{ and } u_t(x,0) = g(x) & (0 < x < 1). \end{array}$$

Let u_1, u_2 be two solutions to the problem. Then their difference $w := u_1 - u_2$ satisfies

$$\begin{array}{ll} \dfrac{\partial^2 w}{\partial t^2} - \dfrac{\partial^2 w}{\partial x^2} = 0 & (0 < x < 1,\ t > 0), \\ w(0,t) = 0 \text{ and } w(1,t) = 0 & (t > 0), \\ w(x,0) = 0 \text{ and } w_t(x,0) = 0 & (0 < x < 1). \end{array}$$

Consider the "energy integral"

$$E_w(t) := \frac{1}{2}\int_0^1 \left(\left(\frac{\partial w}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2 \right) dx.$$

Then we obtain

$$\begin{aligned} \frac{dE_w}{dt} &= \int_0^1 w_t w_{tt}\, dx + \int_0^1 w_x w_{xt}\, dx \\ &= \int_0^1 w_t w_{tt}\, dx + \left(w_x w_t \Big|_0^1 - \int_0^1 w_{xx} w_t\, dx \right) \\ &= \int_0^1 w_t \underbrace{(w_{tt} - w_{xx})}_{=0} dx + w_x(1,t) \underbrace{w_t(1,t)}_{=0} - w_x(0,t) \underbrace{w_t(0,t)}_{=0} = 0. \end{aligned}$$

So E_w must be constant[1], and so $E_w(t) = E_w(0) = 0$, where the latter equality follows from the initial conditions $w(x,0) = 0$ (implying $w_x(x,0) = 0$) and $w_t(x,0) = 0$. Hence we obtain for all $t > 0$ that

$$0 = E_w(t) = \frac{1}{2}\int_0^1 \left(\left(\frac{\partial w}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2 \right) dx,$$

[1] Energy is converved in the free vibrations of a string.

giving

$$w_t = 0 \text{ and } w_x = 0 \quad (0 < x < 1,\ t > 0).$$

So w is constant along vertical lines and along horizontal lines in $(0,1) \times (0,\infty)$. From here it also follows that w must be constant. From the initial condition $w(x,0) = 0$, we also see that the constant value must be zero. Thus $w \equiv 0$, and so $u_1 = u_2$!

Exercise 2.2. Consider a flexible beam with clamped ends at $x = 0$ and $x = 1$. Small wave motion in the beam satisfies

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} + \gamma^2 \frac{\partial^4 u}{\partial x^4} = 0 && (0 < x < 1,\ t > 0), \\ &u(0,t) = 0 = u(1,t) \text{ and } u_x(0,t) = 0 = u_x(1,t) = 0 && (t > 0), \end{aligned}$$

where γ is a a constant depending on the shape and the material of the beam. Show that the energy

$$E(t) = \frac{1}{2}\int_0^1 \Big((u_t)^2 + \gamma^2 (u_{xx})^2\Big) dx$$

is conserved along a solution u.

2.2.2. Diffusion equation with Dirichlet conditions. Consider the diffusion equation with Dirichlet conditions:

$$\begin{aligned} &\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = v && (0 < x < 1,\ t > 0), \\ &u(0,t) = \alpha(t) \text{ and } u(1,t) = \beta(t) && (t > 0), \\ &u(x,0) = f(x) && (0 < x < 1). \end{aligned}$$

Let u_1, u_2 be two solutions to the problem. Then their difference $w := u_1 - u_2$ satisfies

$$\begin{aligned} &\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} = 0 && (0 < x < 1,\ t > 0), \\ &w(0,t) = 0 \text{ and } w(1,t) = 0 && (t > 0), \\ &w(x,0) = 0 && (0 < x < 1). \end{aligned}$$

The PDE and integration by parts give

$$\begin{aligned} 0 &= \int_0^1 w(\underbrace{w_t - w_{xx}}_{=0}) dx = \int_0^1 w w_t dx - \int_0^1 w w_{xx} dx \\ &= \int_0^1 w w_t dx - (w w_x)\Big|_0^1 + \int_0^1 w_x^2 dx. \end{aligned}$$

The second term on the right hand side is 0 thanks to the boundary conditions. Since the third term is nonnegative, the first one must be $\leqslant 0$. Thus

$$\frac{1}{2}\frac{d}{dt}\int_0^1 w^2 dx = \int_0^1 w w_t dx \leqslant 0.$$

So we conclude that the map

$$t \mapsto \int_0^1 w^2 dx$$

is decreasing. Since its initial value is

$$\int_0^1 (w(x,0))^2 dx = \int_0^1 0^2 dx = 0,$$

it follows that

$$\int_0^1 w^2 dx \leqslant 0 \quad (t > 0).$$

But this implies that $w \equiv 0$ for all $t > 0$, that is, $u_1 \equiv u_2$.

Exercise 2.3 (Maximum Principle). Consider the heat equation

$$\begin{aligned} &\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 && (0 \leqslant x \leqslant 1,\ t \geqslant 0), \\ &u(0,t) = \alpha(t) \text{ and } u(1,t) = \beta(t) && (t \geqslant 0), \\ &u(x,0) = f(x) && (0 \leqslant x \leqslant 1). \end{aligned}$$

Suppose that f, α, β are bounded. Thus there are constants m, M such that

$$m \leqslant f(x) \leqslant M, \quad m \leqslant \alpha(t) \leqslant M, \quad m \leqslant \beta(t) \leqslant M \quad (0 \leqslant x \leqslant 1,\ t \geqslant 0).$$

Then the maximum principle states that if u is a solution to the above initial boundary value problem, then

$$m \leqslant u(x,t) \leqslant M \quad (0 \leqslant x \leqslant 1,\ t \geqslant 0).$$

We won't give a mathematical proof of this, but the result is surely expected physically. Indeed, the physical meaning of the last right-most inequality is if the initial temperature distribution and the temperature at the endpoints do not exceed a certain value M, then the temperature distibution inside the rod at any subsequent time will remain smaller than M. Similarly, if the initial temperature distribution and the temperature of the endpoints do not fall below a certain value m, then we expect that the temperature distribution inside the rod at any later time will stay bigger than m.

The aim of this exercise is three-fold. We would like to give a justification of the unqiueness using the Maximum Principle, to justify a Comparison Principle, and to justify stability of the solution under initial condition perturbations.

(1) (Uniqueness). Show that if u_1, u_2 are two solutions to the initial boundary value problem above, then the Maximum Principle implies that $u_1 = u_2$.

(2) (Comparison Principle). Suppose that (α_1, β_1, f_1) and (α_2, β_2, f_2) are two sets of initial/boundary condition data, such that the each function in the second set of data dominate the respective function from the first set of data: for all $t \geqslant 0$ and all $0 \leqslant x \leqslant 1$,

$$\alpha_1(t) \leqslant \alpha_2(t), \quad \beta_1(t) \leqslant \beta_2(t), \quad f_1(x) \leqslant f_2(x).$$

If u_1 is a solution to the initial boundary value problem with the first set of data, and u_2 is a solution to the initial boundary value problem with the second set of data, then show that

$$u_1(x,t) \leqslant u_2(x,t) \quad (0 \leqslant x \leqslant 1,\ t \geqslant 0).$$

(3) (Stability). Show the stability of solutions for perturbations of the initial/boundary values in the L^∞ norm, that is, if u_1 and u_2 are solutions to the initial boundary value problem with the initial/boundary data (α_1, β_1, f_1) and (α_2, β_2, f_2), respectively, then

$$\sup_{\substack{0 \leqslant x \leqslant 1 \\ t \geqslant 0}} |u_1(x,t) - u_2(x,t)| \leqslant \max \Big\{ \sup_{t \geqslant 0} |\alpha_1(t) - \alpha_2(t)|, \ \sup_{t \geqslant 0} |\beta_1(t) - \beta_2(t)|, \\ \sup_{0 \leqslant x \leqslant 1} |f_1(x) - f_2(x)| \Big\}.$$

Exercise 2.4. The aim of this exercise is to show that the maximum principle does not hold when one has an internal source of heat. Based on our physical intuition, we expect this, but let us see a concrete example. The problem is the following:

$$\begin{aligned} &\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 2(t+1) + x(1-x) && (0 \leqslant x \leqslant 1,\ t \geqslant 0), \\ &u(0,t) = 0 \text{ and } u(1,t) = 0 && (t \geqslant 0), \\ &u(x,0) = x(1-x) && (0 \leqslant x \leqslant 1). \end{aligned}$$

(1) Verify that u given by $u(x,t) = (t+1)x(1-x)$ is a solution.

(2) What are the maximum values of the initial and boundary data?

(3) Show that the maximum principle conclusion does not hold.

2.2.3. The Laplace equation with Dirichlet conditions. In order to prove the uniqueness of solutions for the Laplace equation with Dirichlet boundary conditions, we will need some preliminaries from vector calculus in the plane, notably Green's Identity, which we establish now. These results are useful in their own right and various versions are used extensively in PDE theory.

Definition 2.2.

A ***smooth curve*** (in $\mathbb{R}^2$) is a map $t \overset{\gamma}{\longmapsto} \big(x(t), y(t)\big) : [a,b] \to \mathbb{R}^2$, where x, y are continuously differentiable.

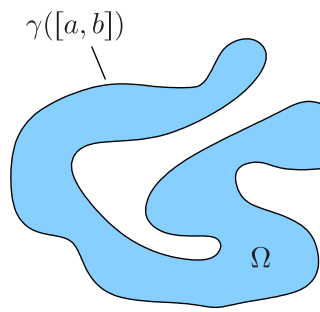
A smooth curve is ***closed*** if $\gamma(a) = \gamma(b)$.

A closed curve is ***simple*** if whenever $t_1, t_2 \in [a,b]$ with $t_1 \neq t_2$ and $\gamma(t_1) = \gamma(t_2)$, then $t_1, t_2 \in \{a, b\}$. (That is, γ doesn't intersect itself except at the endpoints.)

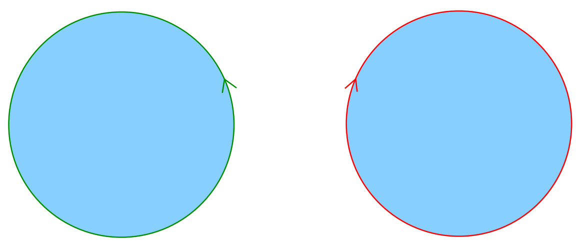
Example 2.3. The curve $\gamma : [0, 2\pi] \to \mathbb{R}^2$ given by $\gamma(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$, is a simple curve. The image of the curve γ is just a circle with center $(0,0)$ and radius 1. $\Diamond$

In our example, we note that the simple curve, namely the circle, divides the plane into two regions: one bounded, which is the interior of the circle, that is, the disk $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$, and another region, which is the unbounded and is the exterior of the circle: $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 > 1\}$. This is no coincidence. Jordan proved that a simple curve γ *always* divides the plane into two regions: one bounded and interior to γ, and one unbounded and exterior to γ. This result is known as the

Jordan Curve Theorem. In order to prove our uniqueness result for Laplace's equation, we will assume that we are working in a region Ω which is the interior of a simple curve γ.



A simple curve $\gamma : [a,b] \to \mathbb{R}^2$ is ***positively oriented*** if, while thinking of $\gamma(t)$ as the position at time t, the motion is such that the interior of γ always lies on the left. See the pictures below.



γ given by
$\gamma(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$,
is positively oriented.

γ given by
$\gamma(t) = (\cos(-t), \sin(-t))$, $t \in [0, 2\pi]$,
is negatively oriented.

Definition 2.3 (Line integral). Let $t \xmapsto{\gamma} \big(x(t), y(t)\big) : [a,b] \to \mathbb{R}^2$ be a smooth curve and $f : \gamma([a,b]) \to \mathbb{R}$ be a continuous function. The ***line integral of*** f ***over*** γ is defined by

$$\int_\gamma f(x,y)ds := \int_a^b f\big(x(t), y(t)\big)\sqrt{(x'(t))^2 + (y'(t))^2}dt.$$

We also introduce two other integrals which will be of interest to us:

$$\begin{aligned}\int_\gamma f(x,y)dx &:= \int_a^b f\big(x(t), y(t)\big)x'(t)dt,\\ \int_\gamma f(x,y)dy &:= \int_a^b f\big(x(t), y(t)\big)y'(t)dt.\end{aligned}$$

The line integral has many properties akin to the Riemann integral, and are readily verified. We state some of these properties for the integral with ds. Similar identities also hold with dx or dy.

(1) (Linearity). For scalars α, β and continuous f, g we have

$$\int_\gamma \big(\alpha f(x,y) + \beta g(x,y)\big) ds = \alpha \int_\gamma f(x,y)ds + \beta \int_\gamma g(x,y)ds.$$

(2) (Concatenation). Let

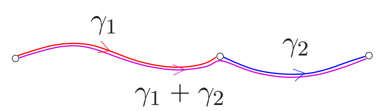
$$\begin{aligned} \gamma_1 &: [a_1, b_1] \to \mathbb{R}^2 \text{ and} \\ \gamma_2 &: [a_2, b_2] \to \mathbb{R}^2 \end{aligned}$$

be two paths such that

$$\gamma_1(b_1) = \gamma_2(a_2)$$

(so that γ_2 starts where γ_1 ends). Define $\gamma_1 + \gamma_2 : [a_1, b_1 + b_2 - a_2] \to \mathbb{R}^2$ to be their "concatenation", given by:

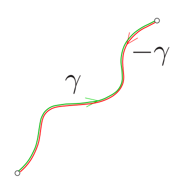
$$(\gamma_1 + \gamma_2)(t) = \begin{cases} \gamma_1(t) & \text{for } a_1 \leqslant t \leqslant b_1, \\ \gamma_2(t - b_1 + a_2) & \text{for } b_1 \leqslant t \leqslant b_1 + b_2 - a_2. \end{cases}$$



If f is continuous on the image of $\gamma_1 + \gamma_2$, then

$$\int_{\gamma_1+\gamma_2} f(x,y)ds = \int_{\gamma_1} f(x,y)ds + \int_{\gamma_2} f(x,y)ds.$$

(3) (Opposite curve).
Given a smooth curve $\gamma : [a,b] \to \mathbb{R}^2$, its ***opposite path***, $-\gamma : [a,b] \to \mathbb{R}^2$, is defined by $(-\gamma)(t) = \gamma(a+b-t)$, $t \in [a,b]$. Then $(-\gamma)(a) = \gamma(b)$ and $(-\gamma)(b) = \gamma(a)$, and so $-\gamma$ starts where γ ends, and ends at the starting point of γ, while traversing the same path of γ, but in the opposite direction.



But why do we denote the opposite path by $-\gamma$? Here's why.

$$\int_{-\gamma} f(x,y)ds = -\int_\gamma f(x,y)ds.$$

Some more notation: If $N, M : \gamma([a, b]) \to \mathbb{R}$ are two continuous functions, then

$$\int_\gamma \big(M(x, y)dx + N(x, y)dy\big) := \int_\gamma M(x, y)dx + \int_\gamma N(x, y)dy.$$

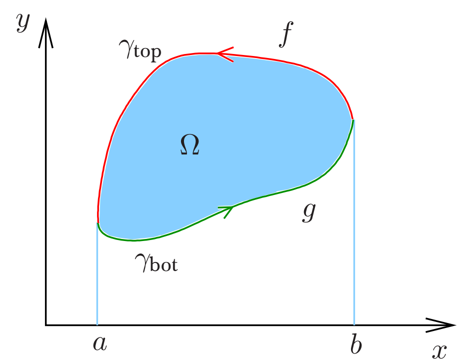
With these preliminaries out of the way, we are now ready to state Green's[2] Theorem. This result relates the line integral around a simple curve to a double integral over the region bounded by the curve.

Theorem 2.1 (Green's Theorem). *Let γ be a positively oriented simple curve with interior Ω, and let M, N be continuous functions with continuous partial derivatives on the image of γ and in Ω. Then*

$$\int_\gamma \big(M(x, y)dx + N(x, y)dy\big) = \iint_\Omega \Big(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\Big)dxdy.$$

Proof. We will first prove this result in the case when γ is a *standard* curve, by which we mean that no vertical or horizontal line can intersect γ in more than two points. Once this is done, we will see how to derive the result also in the case of non-standard curves.

As illustrated in the picture below, given a standard curve γ, we can find an interval $[a, b]$ and two differentiable functions f and g on $[a, b]$, such that γ is composed of a top portion γ_{top}, which is the graph of f, and a bottom portion γ_{bot}, which is the graph of g. Since γ is positively oriented, the reverse of γ_{top} is parametrized by $\big(x, f(x)\big)$, as x runs from a to b, while γ_{bot} is parametrized by $\big(x, g(x)\big)$, as x runs from a to b.



So

$$\begin{aligned} -\int_{\gamma_{\text{top}}} M(x, y)dx &= \int_a^b M\big(x, f(x)\big)dx \text{ and} \\ \int_{\gamma_{\text{bot}}} M(x, y)dx &= \int_a^b M\big(x, g(x)\big)dx. \end{aligned}$$

[2]Named after the British mathematical physicist Green (1793-1841).

Also,

$$
\begin{aligned}
\iint\limits_{\Omega} \frac{\partial M}{\partial y} dxdy &= \iint\limits_{\Omega} \frac{\partial M}{\partial y} dydx = \int_a^b \Big(\int_{g(x)}^{f(x)} \frac{\partial M}{\partial y} dy \Big) dx \\
&= \int_a^b \Big(M\big(x, f(x)\big) - M\big(x, g(x)\big)\Big) dx \\
&= -\int_{\gamma_{\text{top}}} M(x,y)dx - \int_{\gamma_{\text{bot}}} M(x,y)dx \\
&= -\int_{\gamma} M(x,y)dx.
\end{aligned}
$$

In a similar manner, one can show that

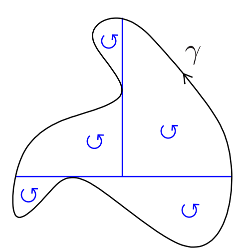
$$\iint\limits_{\Omega} \frac{\partial N}{\partial x} dxdy = \int_{\gamma} N(x,y)dy.$$

Upon subtraction, we obtain

$$\iint\limits_{\Omega} \Big(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \Big) dxdy = \int_{\gamma} \big(M(x,y)dx + N(x,y)dy \big),$$

establishing Green's Theorem for standard curves.

What about non-standard curves? Here is the sketch of how it works. The result follows simply by subdividing the region into regions with positively oriented standard curve boundaries. Let γ_k, $k = 1, 2, 3, \cdots, n$ be the resulting boundary curves, and Ω_k the region inside γ_k. The construction is illustrated in the picture below with $n = 5$.



Each curve consists of portions of the curve γ and portions not on γ. The portions on γ are traversed once in the positive direction, while the ones not on γ are traversed twice, in opposite ways. As a result, the sum of the integrals over all γ_k add up to the integral over γ, since the integrals over the portions not on γ cancel out. Applying

Green's Theorem for the standard curves γ_k, and adding the integrals, we obtain

$$\begin{aligned}\int_\gamma \big(M(x,y)dx+N(x,y)dy\big) &= \sum_{k=1}^{n}\int_{\gamma_k}\big(M(x,y)dx+N(x,y)dy\big)\\ &= \sum_{k=1}^{n}\iint\limits_{\Omega_k}\Big(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\Big)dxdy\\ &= \iint\limits_{\Omega}\Big(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\Big)dxdy.\end{aligned}$$

This completes the proof. □

Example 2.4 (Area as a line integral). Let γ be a positively oriented simple curve with interior Ω. Then the area of Ω is given by

$$\int_\gamma -ydx.$$

Take $M(x,y) = -y$ and $N(x,y) = 0$. Then

$$\frac{\partial N}{\partial x} = 0 \text{ and } \frac{\partial M}{\partial y} = -1,$$

and so by Green's Theorem,

$$\begin{aligned}\text{area of } \Omega &= \iint\limits_{\Omega} dxdy = \iint\limits_{\Omega}\Big(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\Big)dxdy\\ &= \int_\gamma\big(M(x,y)dx+N(x,y)dy\big) = \int_\gamma -ydx.\end{aligned}$$ ◊

Exercise 2.5. Let γ be a positively oriented simple curve with interior Ω. Show that the area enclosed by γ is also given by any of the following two expressions:

$$\int_\gamma xdy, \quad \frac{1}{2}\int_\gamma(-ydx+xdy).$$

Exercise 2.6. Let C be the positively oriented circle given by $C(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$. Verify Green's Theorem with $M(x,y) = y^2$ and $N(x,y) = -x$.

Exercise 2.7. Find the area enclosed by the ellipse given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$$

by expressing it as a line integral. What happens when $a = b$?

Exercise 2.8 (Divergence Theorem in the plane). Let γ be a positively oriented simple curve with interior Ω, and u be a twice continuously differentiable function on Ω and on γ. Then

$$\Big(\iint\limits_{\Omega}\nabla\cdot(\nabla u)\,dxdy\Big) = \iint\limits_{\Omega}\Delta u\,dxdy = \int_\gamma\Big(-\frac{\partial u}{\partial y}dx+\frac{\partial u}{\partial x}dy\Big).$$

Using Green's Theorem, one can derive Green's Identity, which will be used to show the uniqueness of solutions to the Laplace equation.

Theorem 2.2 (Green's Identity). *Let γ be a positively oriented simple curve with interior Ω. Let u, v have continuous second order partial derivatives in Ω and on γ. Then*

$$\iint_{\Omega} \big(u\Delta v + \nabla u \cdot \nabla v\big)dxdy = \int_{\gamma} u\frac{\partial v}{\partial n}ds.$$

Before we go on to prove this, let us clarify what we mean by the symbol

$$\frac{\partial v}{\partial n},$$

called the *normal derivative along the curve* γ:

$$\frac{\partial v}{\partial n} := \nabla u \cdot n = (u_x, u_y) \cdot \frac{\big(y'(t), -x'(t)\big)}{\sqrt{(x'(t))^2 + (y'(t))^2}}.$$

In the above expression, we recognize $(y'(t), -x'(t))$ as the normal vector to the curve γ (at the point $\gamma(t)$), and the length of this normal vector is $\sqrt{(x'(t))^2 + (y'(t))^2}$. So

$$\frac{(y'(t), -x'(t))}{\sqrt{(x'(t))^2 + (y'(t))^2}}$$

is the unit normal vector to the curve at the point $\gamma(t)$. Hence the normal derivative of v along the curve at a point $\gamma(t)$ is just the directional derivative of v in the direction given by the normal vector to the curve at the point $\gamma(t)$.

Proof of Green's Identity. First note that

$$\begin{aligned}
\int_{\gamma} u\frac{\partial v}{\partial n}ds &= \int_a^b u\frac{v_x y' - v_y x'}{\sqrt{(x')^2 + (y')^2}}\sqrt{(x')^2 + (y')^2}dt \\
&= \int_a^b u(v_x y' - v_y x')dt = \int_a^b uv_x y' dt - \int_a^b uv_y x' dt \\
&= \int_{\gamma} uv_x dy - \int_{\gamma} uv_y dx = \int_{\gamma} u(-v_y dx + v_x dy).
\end{aligned}$$

Let $M = -uv_y$ and $N = uv_x$. Then

$$\begin{aligned}
M_y &= -u_y v_y - uv_{yy}, \\
N_x &= u_x v_x + uv_{xx}.
\end{aligned}$$

Applying Green's Theorem, we obtain

$$\begin{aligned}
\int_\gamma u\frac{\partial v}{\partial n}ds &= \int_\gamma u(-v_y dx + v_x dy) = \int_\gamma (Mdx + Ndy) = \iint\limits_\Omega \Big(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\Big)dxdy \\
&= \iint\limits_\Omega \big(u_x v_x + uv_{xx} + u_y v_y + uv_{yy}\big)dxdy \\
&= \iint\limits_\Omega (u\Delta v + \nabla u \cdot \nabla v)dxdy.
\end{aligned}$$

This completes the proof. □

Exercise 2.9 (Green's Second Identity). Let γ be a positively oriented simple curve with interior Ω. Let u, v have continuous second order partial derivatives in Ω and on γ. Then show that

$$\iint\limits_\Omega (u\Delta v - v\Delta u)dxdy = \int_\gamma \Big(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\Big)ds.$$

Theorem 2.3 (Uniqueness). *Let γ be a simple curve with interior Ω and f be a continuous function on γ. Then the boundary value problem*

$$\begin{cases} \Delta u = \rho \text{ in } \Omega, \\ u|_\gamma = f, \end{cases}$$

has at most one continuous solution.

Proof. Suppose that u_1, u_2 satisfy

$$\begin{aligned}
&\Delta u_1 = \rho = \Delta u_2 \text{ in } \Omega, \text{ and} \\
&u_1|_\gamma = f = u_2|_\gamma.
\end{aligned}$$

Set $w := u_1 - u_2$. Then

$$\begin{aligned}
&\Delta w = 0 \text{ in } \Omega, \text{ and} \\
&w|_\gamma = f - f = 0.
\end{aligned}$$

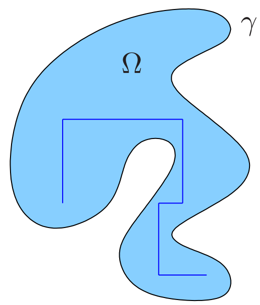
We want to show that $w \equiv 0$ in Ω. Applying Green's Identity with $u := w$ and $v := w$ gives

$$\iint\limits_\Omega \nabla w \cdot \nabla w \, dxdy = \int_\gamma w\frac{\partial w}{\partial n}ds = 0.$$

Here we have used the fact that $\Delta w = 0$ in Ω, and also that w vanishes on γ. So

$$\iint\limits_\Omega (w_x^2 + w_y^2)dxdy = 0,$$

and since the integrand is nonnegative and continuous, it follows that $w_x = w_y \equiv 0$ on Ω. This means that w is constant along vertical segments and along horizontal segments. But then (since any two points in Ω can be joined by a "stepwise" path comprising horizontal and vertical segments) w must be constant in Ω.



As w is zero on the boundary of Ω, and since w is continuous, it follows that the value of this constant must be 0. So $w \equiv 0$ on Ω and the range of γ, as wanted. $\square$

Exercise 2.10 (Uniqueness up to constants in the Neumann problem for the Poisson equation). Let γ be a simple curve with interior Ω and f be a continuous function on γ. Suppose that u_1, u_2 are two solutions to the boundary value problem

$$\left\{ \begin{array}{l} \Delta u = \rho \text{ in } \Omega, \\ \left.\dfrac{\partial u}{\partial n}\right|_{\gamma} = f. \end{array} \right.$$

Show that there exists a constant C such that $u_2 \equiv u_1 + C$ on Ω.

Exercise 2.11 (Compatibility condition in Neumann problems). Let γ be a simple curve with interior Ω.

(1) If v has continuous second order partial derivatives in Ω and on γ, then show that

$$\iint\limits_{\Omega} \Delta v \, dxdy = \int_{\gamma} \frac{\partial v}{\partial n} ds.$$

(2) Now suppose that u has continuous second order partial derivatives in Ω and on γ and that $\Delta u = 0$ in Ω. Show that the normal derivative of u must integrate to 0 along the boundary, that is,

$$\int_{\gamma} \frac{\partial u}{\partial n} ds = 0.$$

This means that the boundary values of the normal derivative of a harmonic function u cannot be arbitrary; they must satisfy the compatibility condition

$$\int_{\gamma} \frac{\partial u}{\partial n} ds = 0.$$

(3) Consider the Neumann problem for the interior $\mathbb{D}$ of the unit circle C with center $(0, 0)$ and radius 1:

$$\left\{ \begin{array}{l} \Delta u = 0 \text{ in } \mathbb{D}, \\ \left.\dfrac{\partial u}{\partial n}\right|_{C} (\cos t, \sin t) = f(t), \;\; t \in [0, 2\pi], \end{array} \right.$$

where f is the continuous function given by

$$f(t) = \left\{ \begin{array}{ll} \sin t & \text{if } t \in [0, \pi], \\ 0 & \text{if } t \in [\pi, 2\pi]. \end{array} \right.$$

Does this problem have a solution?

Exercise 2.12. Let γ be a simple curve with interior Ω. Suppose that u has continuous second order partial derivatives in Ω and on the range of γ, and also suppose that $\Delta u = 0$ in Ω.

(1) Show that $\displaystyle\int_\gamma \left(\frac{\partial u}{\partial y}dx - \frac{\partial u}{\partial x}dy\right) = 0.$

(2) Moreover , if v has continuous second order partial derivatives in Ω and on γ, and if $v = 0$ on γ, then show that

$$\iint\limits_{\Omega} \nabla u \cdot \nabla v \, dxdy = 0.$$

Exercise 2.13 (Dirichlet's Principle). The aim of this exercise is to show that harmonic functions are energy minimizing functions.

Let γ be a simple curve with interior Ω. Let u be the unique solution to the Dirichlet Problem for the Laplacian:

$$\begin{cases} \Delta u = 0 \text{ in } \Omega, \\ u|_\gamma = f. \end{cases}$$

The ***energy*** of a function φ defined on Ω is

$$E(\varphi) = \frac{1}{2}\iint\limits_{\Omega} \|\nabla\varphi\|_2^2 \, dxdy.$$

Dirichlet's Principle states that among all the functions v on $\Omega \cup \gamma$ that satisfy the Dirichlet boundary condition $v|_\gamma = f$, the one that minimizes the energy integral is the harmonic function u! That is, if $v|_\gamma = f$, then $E(v) \geqslant E(u)$. Follow the outline below to prove the principle.

(1) Write $v = u + (v - u) =: u + w$, and note that $w|_\gamma = 0$. Show that

$$\|\nabla v\|_2^2 = \|\nabla u\|_2^2 + 2\nabla v \cdot \nabla w + \|\nabla w\|_2^2.$$

(2) Show that $E(v) = E(u) + E(w)$. Conclue that $E(v) \geqslant E(u)$.

Exercise 2.14 (Mean Value Property).

(1) (For the case of a disc). Consider the Dirichlet problem for the Laplace equation in the disc $D_R := \{(x, y) \in \mathbb{R}^2 : (x - x_0)^2 + (y - y_0)^2 < R^2\}$, with smooth data f on the boundary ∂D_R:

$$\begin{cases} \Delta u = 0 \text{ in } D_R, \\ u|_{\partial D_R} = f \end{cases}$$

Show that $\displaystyle u(x_0, y_0) = \frac{1}{2\pi}\int_0^{2\pi} f(x_0 + R\cos\theta, y_0 + R\sin\theta)d\theta.$

(In other words the value at the center of the disc D_R is the average/mean of the values of u on the boundary of the disc D_R).

Hint: Differentiate $\displaystyle r \mapsto \frac{1}{2\pi}\int_0^{2\pi} u(x_0 + r\cos\theta, y_0 + r\sin\theta)d\theta$, $r \in [0, R]$, and use Exercise 2.8.

(2) (General domains). Suppose that Ω is any region in $\mathbb{R}^2$, and u is harmonic in Ω. Then for any closed disc $\overline{D}_R := \{(x, y) \in \mathbb{R}^2 : (x - x_0)^2 + (y - y_0)^2 \leqslant R^2\} \subset \Omega$, we have

$$u(x_0, y_0) = \frac{1}{2\pi}\int_0^{2\pi} u(x_0 + R\cos\theta, y_0 + R\sin\theta)d\theta.$$

Exercise 2.15 (Maximum Principle). The aim of this exercise is twofold: first of all to prove the Maximum Principle for the Dirichlet problem for the Laplace equation, and secondly, to apply the Maximum Principle for obtaining uniquenss of solutions to the Dirichlet problem for the Laplace equation.

Maximum Principle. Let γ be a simple curve with interior Ω. Let u be a twice continuously differentiable function on Ω and on γ which satisfies the Dirichlet problem for the Laplacian with continuous boundary data f:

$$\begin{cases} \Delta u = 0 \text{ in } \Omega, \\ u|_\gamma = f \end{cases}$$

Then $\max\limits_{\Omega\cup\gamma} u = \max\limits_{\gamma} f$ and $\min\limits_{\Omega\cup\gamma} u = \min\limits_{\gamma} f$.

Note that the continuous function u has a maximizer and a minimizer on the compact set $\Omega \cup \gamma$. Similarly, the continuous function f has a maximizer and a minimizer on the compact set γ (or rather the image of the curve γ).

(1) For $\epsilon > 0$, define $v(x, y) = u(x, y) + \epsilon(x^2 + y^2)$. Show that $\Delta v > 0$ in Ω.

(2) Show that the maximizer of v on $\Omega \cup \gamma$ exists, but it can't lie in Ω.

(3) If $(x_*, y_*) \in \gamma$ is a maximizer of v, then for all $(x, y) \in \Omega \cup \gamma$,

$$\begin{aligned} u(x, y) \leqslant v(x, y) \leqslant v(x_*, y_*) &= u(x_*, y_*) + \epsilon(x_*^2 + y_*^2) \\ &\leqslant \max_\gamma f + \epsilon \cdot d^2, \end{aligned}$$

where d is the largest distance of the points of γ to the origin $(0, 0)$. Conclude that $\max\limits_{\Omega\cup\gamma} u = \max\limits_{\gamma} f$.

(A similar proof can be given to show that also $\min\limits_{\Omega\cup\gamma} u = \min\limits_{\gamma} f$.)

(Uniqueness). Show, using the Maximum Principle, the uniqueness of solutions to the Dirichlet problem for the Laplacian:

$$\begin{cases} \Delta u = 0 \text{ in } \Omega, \\ u|_\gamma = f \end{cases}$$

Remark 2.2. In the initial and boundary value problems considered so far, we have only looked at ***bounded*** spatial regions. However, we will also consider ***unbounded*** domains in Chapter 4, where one has some natural constraints on the solutions, for instance boundedness or convergence to 0 at ∞. Even in these cases one has uniqueness. We satisfy ourselves by describing three situations where one has a unique solution provided the initial and boundary conditions are sufficiently regular. For the wave equation and the diffusion equation, the proofs are simply a modification of the proofs given above. But the proof in the case of the Laplace equation is more involved.

(1) Laplace's equation $u_{xx} + u_{yy} = \rho$ in the half plane $x \in \mathbb{R}$, $y > 0$, with a given boundary condition $u(\cdot, 0) = f$ has at most one bounded solution.

(2) The Diffusion equation $u_t - u_{xx} = v$ in the half plane $x \in \mathbb{R}$, $t > 0$ with a given initial condition $u(\cdot, 0) = f$ has at most one solution u with the property that for each x, $u(x, t) \to 0$ as $t \to \infty$.

(3) The wave equation $u_{tt} - u_{xx} = v$ in the half plane $x \in \mathbb{R}$, $t > 0$ with given initial conditions $u(\cdot, 0) = f$, $u_t(\cdot, 0) = g$ has at most one solution u with the property that for each x, $u(x,t) \to 0$ as $t \to \infty$.

As far as the question of stability is concerned (that is, whether small changes in the initial and boundary conditions result in small changes of the solution), for our second order PDEs considered above, this desired property of stability does hold. In other words, our 2nd order linear PDE problems are well-posed.

2.3. Discretization and the finite difference method

In this section we give a quick overview of an important class of numerical method for PDEs, called the *finite difference method*[3]. The idea is to replace the derivatives appearing in the differential equation by their difference quotients.

We recall that if f is a three times continuously differentiable function, then Taylor's formula gives

$$f(x \pm h) = f(x) \pm h f'(x) + \frac{h^2}{2} f''(x) + o(h^2) \text{ as } h \to 0.$$

Here "$\varphi(x,h) = \psi(x,h) + o(g(h))$ as $h \to 0$" means that

$$\lim_{h \to 0} \frac{\varphi(x,h) - \psi(x,h)}{g(h)} = 0.$$

Using this Taylor expansion for f, it follows that

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= f'(x) + o(1), \\ \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} &= f''(x) + o(1) \end{aligned}$$

as $h \to 0$.

By replacing the derivatives in our second order PDEs with such difference quotients, we get difference equations. We elaborate on this method by considering the case of the diffusion equation as an illustrative example.

Example 2.5. Consider the initial and boundary value problem for the diffusion equation given by

$$\begin{aligned} &\frac{\partial u}{\partial t} = a \frac{\partial^2 u}{\partial x^2} && (0 < x < 1,\ t > 0), \\ &u(0,t) = \alpha(t) \text{ and } u(1,t) = \beta(t) && (t > 0), \\ &u(x,0) = f(x) && (0 < x < 1). \end{aligned}$$

[3] There is another important numerical method in connection with PDEs, called the ***finite element method***, where the idea is to decompose the domain of the PDE into a finite number of simple polygonal pieces (called elements) and approximate the solution in these pieces by very simple functions. However, we will not treat this here. The interested reader is referred to [**St**, §8.5].

For $j = 0, 1, \cdots, n$, we set

$$\Delta x = \frac{1}{n} \text{ and } x_j = \frac{j}{n}.$$

Also, let

$$t_k = k\Delta t,$$

where $\Delta t > 0$, and we will choose it appropriately below. With

$$u[j,k] := u(x_j, t_k) \quad (j = 0, 1, \cdots, n, \quad k = 0, 1, 2, 3, \cdots),$$

and by replacing the derivatives in the diffusion equation by their respective difference quotient approximations, we obtain

$$\frac{1}{\Delta t}(u[j,k+1] - u[j,k]) = \frac{a}{(\Delta x)^2}(u[j-1,k] - 2u[j,k] + u[j+1,k]),$$

where $j = 1, \cdots, n-1$ and $k = 0, 1, 2, 3, \cdots$. Now we choose Δt so that

$$\alpha = a\frac{\Delta t}{(\Delta x)^2},$$

where α will be determined later! Then we obtain

$$u[j,k+1] = \alpha u[j-1,k] + (1-2\alpha)u[j,k] + \alpha u[j+1,k], \tag{2.1}$$

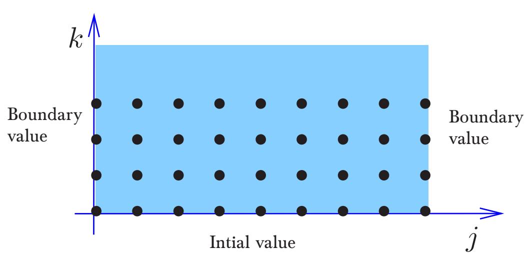
that is,

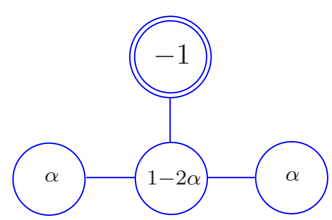
$$-u[j,k+1] + \alpha u[j-1,k] + (1-2\alpha)u[j,k] + \alpha u[j+1,k] = 0. \tag{2.2}$$

This is a linear equation system with infinitely many unknowns, namely the

$$u[j,k], \quad j = 1, \cdots, n-1, \quad k = 1, 2, 3, \cdots .$$

In the picture on the left below, we have sketched the grid of points on which $u[j,k]$ is defined.





The system has a special structure, which makes it easy to solve. From the equation (2.1), we see that the values

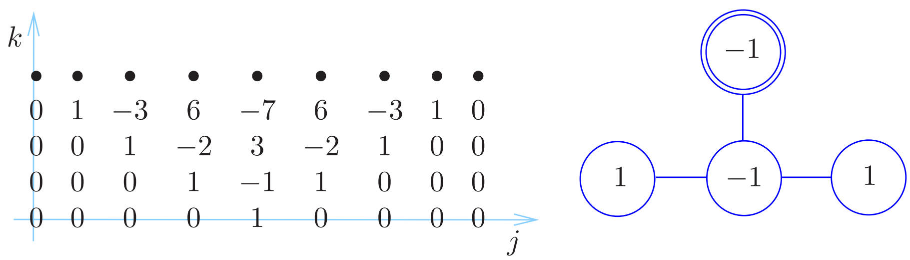
$$u[j,1], \quad j = 1, \cdots, n-1,$$

can be determined from the initial values

$$u[j,0], \quad j = 0, 1, \cdots, n,$$

and thereafter the values of u on row $k+1$ can be determined from the values of u on row k. Here one also uses the boundary values corresponding to $j=0$ and $j=n$. The solution procedure can be described by the schematic "molecule" shown on the right in the previous picture, where inside the rings, we have shown the coefficients appearing in the equation (2.2). The molecule moves over the grid, row-wise, from bottom upwards, and in each position, the value of u in the marked "atom" with double rings is determined with the help of the values of u in the others.

Now for the choice of α. It can happen that the values obtained for u grow without bound. Then the difference method is said to be ***unstable***. For example, if $\alpha = 1$, the molecule is shown on the right, and for the initial boundary value data shown on the left, we get the values of u as shown in the interior of the grid on the left.



The reason behind the growing values is that one of the coefficients in the molecule (the central atom) is negative. On the other hand, if we choose the value of α so that

$$0 \leqslant \alpha \leqslant \frac{1}{2}, \tag{2.3}$$

then all the coefficients in (2.1) are nonnegative. In this case the formula is telling us that $u[j,k+1]$ is a ***convex combination*** of the three values of the function on row k. Indeed, $\alpha + (1-2\alpha) + \alpha = 1$. So the new function value $u[j,k+1]$ lies between the largest and the smallest value determined previously. Thus if the initial and boundary values are bounded, then it follows that the values of u everywhere in the grid will stay bounded! So the difference method is now ***stable***. (When the difference method is stable, the rounding and other errors stay under control, while they grow unboundedly when the difference method is unstable.) $\Diamond$

Exercise 2.16 (Finite difference method for the Laplace equation). Consider the Dirichlet Problem for the Laplacian:

$$\begin{cases} \Delta u = 0 \text{ in } R, \\ u|_{\partial R} = f \end{cases}$$

where R is the rectangle $\{(x,y) \in \mathbb{R}^2 : 0 < x < a,\ 0 < y < b\}$ with boundary ∂R. Set

$$\Delta x := \frac{a}{m}, \quad \Delta y := \frac{b}{n},$$

and $u[i,j] = u(i\Delta x, j\Delta y)$, $i = 0,1,\cdots,m$, $j = 0,1,\cdots,n$.

(1) Obtain a recurrence relation for the $u[i,j]$ by using difference quotient approximations.

(2) Consider the case of a square. Take $m = n$. Show that $u[i, j]$ is the average of the values of u at the four neighbouring points of the grid.

(3) Let $a = 1$, $m = n = 3$, and the boundary condition be given by $f(x, 0) = \sin(\pi x)$ for $0 \leqslant x \leqslant 1$ and 0 elsewhere on the boundary of the square. Determine $u[i, j]$ for all $1 \leqslant i, j \leqslant 2$.

Chapter 3

Separation of variables

In this chapter we will learn a method of solving linear PDEs in certain bounded domains based on Fourier series theory, the superposition principle, and "separating variables". By separating variables in a solution, we mean that the function of several variables arises as a product of functions of each variable separately. We have mentioned the superposition principle earlier, although in this chapter, we will need to superimpose infinitely many functions. We will begin our discussion with Fourier Series of periodic functions.

3.1. Fourier series

We are familiar with a Taylor series of an analytic[1] function $f : (-a, a) \to \mathbb{R}$, where one expresses f as a combination of the simple functions $1, x, x^2, x^3, \cdots$: for example,

$$\begin{aligned} \frac{1}{1-x} &= 1 + x + x^2 + x^3 + \cdots \quad (-1 < x < 1), \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + - \cdots \quad (x \in \mathbb{R}). \end{aligned}$$

With Fourier series, we are interested in expanding a "T-periodic" function in terms of the special set of functions

$$1,\ \cos(\omega_0 x),\ \cos(2\omega_0 x),\ \cos(3\omega_0 x), \cdots \quad ,\ \sin(\omega_0 x),\ \sin(2\omega_0 x),\ \sin(3\omega_0 x), \cdots .$$

Here $\omega_0 := 2\pi/T$. Note that the constant function $1 = \cos(0 \cdot \omega_0 x)$. Thus a Fourier series expansion[2] of a function f is an expression of the type

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \big(a_n \cos(n\omega_0 x) + b_n \sin(n\omega_0 x)\big).$$

[1] See for example [**S**, Remark 6.3, p. 335].

[2] The somewhat strange "2" appearing in the first term $a_0/2$ is related to wanting a nicer unifying formula for the a_n, which works for *all* n, as we shall see later — if we did not have this 2, then we would have a formula for the a_n for $n \geqslant 1$, and a different formula (containing an extra 2!) when $n = 0$.

Definition 3.1 (Periodic function). A function $f : \mathbb{R} \to \mathbb{R}$ is said to be ***periodic*** if there exists a $T > 0$, called a ***period of*** f, such that

$$\text{for all } x \in \mathbb{R},\ f(x+T) = f(x).$$

f is then said to be T***-periodic***.

Example 3.1. Let $T > 0$ and $\omega_0 := 2\pi/T$. Let $n \in \mathbb{N}$. The functions $\sin(n\omega_0 x)$ and $\cos(n\omega_0 x)$ are T-periodic, since for all $x \in \mathbb{R}$,

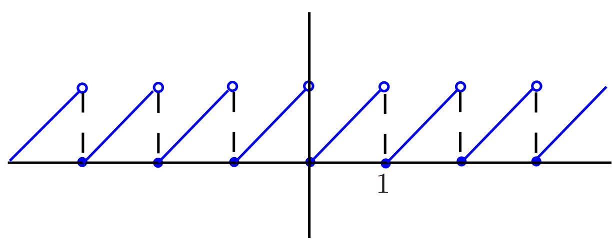
$$\begin{aligned}\sin\big(n\omega_0(x+T)\big) &= \sin(n\omega_0 x + 2\pi \cdot n) = \sin(n\omega_0 x),\\ \cos\big(n\omega_0(x+T)\big) &= \cos(n\omega_0 x + 2\pi \cdot n) = \cos(n\omega_0 x).\end{aligned} \qquad \Diamond$$

Example 3.2. The fractional part function $\{\cdot\} : \mathbb{R} \to \mathbb{R}$ is defined by

$$\{x\} = x - \lfloor x \rfloor, \quad x \in \mathbb{R}.$$

Here $\lfloor x \rfloor$ denotes the greatest integer $\leqslant x$. It is clear that if $x \in \mathbb{Z}$, then $\lfloor x \rfloor = x$. Also, for all real x, $\lfloor x+1 \rfloor = \lfloor x \rfloor + 1$. So if $x \in \mathbb{Z}$, then $\{x+1\} = 0 = \{x\}$, and for noninteger x,

$$\{x+1\} = x + 1 - \lfloor x+1 \rfloor = x + 1 - (\lfloor x \rfloor + 1) = x - \lfloor x \rfloor = \{x\}.$$



Hence $\{\cdot\}$ is 1-periodic. $\Diamond$

Note that the function $\{\cdot\}$ is not continuous at the integer points, but *is* smooth everywhere else, and moreover, it is also well behaved in the sense that the limits on either side of the discontinuity of both the function and its derivative exist. It will be convenient to develop the Fourier expansion theory for such "piecewise smooth" functions. So we give the following definition.

Definition 3.2 (Piecewise smooth). A function $f : [a, b] \to \mathbb{R}$ is called ***piecewise smooth*** if there exist finitely many points $c_0 := a < c_1 < \cdots < c_n < b =: c_{n+1}$ such that f is continuous and continuously differentiable on each subinterval (c_i, c_{i+1}), and the limits

$$\lim_{x\to a+} f(x),\quad \lim_{x\to c_1-} f(x),\ \lim_{x\to c_1+} f(x),\ \cdots,\ \lim_{x\to c_n-} f(x),\ \lim_{x\to c_n+} f(x),\ \lim_{x\to b-} f(x),$$
$$\lim_{x\to a+} f'(x),\ \lim_{x\to c_1-} f'(x),\ \lim_{x\to c_1+} f'(x),\ \cdots,\ \lim_{x\to c_n-} f'(x),\ \lim_{x\to c_n+} f'(x),\ \lim_{x\to b-} f'(x),$$

all exist. That is, at each point of discontinuity, the left and right hand limits of the function and also of its derivative exist.

A T-periodic function $f : \mathbb{R} \to \mathbb{R}$ is ***piecewise smooth*** if f is piecewise smooth on $[0, T]$.

Exercise 3.1. Which of the following functions is piecewise smooth on $\mathbb{R}$? In each case $f(0) = 0$, and for $x \neq 0$, f is given by:

- ☐ (A) $f(x) = \sin(1/x)$
- ☐ (B) $f(x) = x\sin(1/x)$
- ☐ (C) $f(x) = x^2\sin(1/x)$
- ☐ (D) $f(x) = x^3\sin(1/x)$.

Exercise 3.2. Let $f, g : \mathbb{R} \to \mathbb{R}$ be both T-periodic. Show that their pointwise product $f \cdot g$ is also T-periodic.

Exercise 3.3. Show that if $f_1, \cdots f_k$ are all T-periodic, and $c_1, \cdots, c_k$ are real numbers, then their linear combination $c_1 \cdot f_1 + \cdots + c_k \cdot f_k$ (defined pointwise) is also T-periodic.

Exercise 3.4. Show that the function $\cos x + \cos(\sqrt{2}x)$ is not periodic. Does this contradict the previous exercise? (This function is an example of an almost periodic function.)

Exercise 3.5. Show that piecewise smooth periodic functions on $\mathbb{R}$ are bounded.

Exercise 3.6. Let f be a continuous T-periodic function on $\mathbb{R}$. Show that for any $a \in \mathbb{R}$,

$$\int_0^T f(x)dx = \int_a^{a+T} f(x)dx.$$

Hint: Consider $F(x) := \displaystyle\int_x^{x+T} f(\xi)dx$.

In our study of Fourier series, the questions we ask are:

(1) For which functions is a Fourier expansion guaranteed?

(2) If we know that a function has a Fourier expansion, then how are the "Fourier coefficients" (the a_n and b_n) computed?

These questions are answered by the following result:

Theorem 3.1. *Let f be a T-periodic piecewise smooth function. Set $\omega_0 = 2\pi/T$. Then there exist real sequences $(a_n)_{n\geqslant 0}$ and $(b_n)_{n\geqslant 1}$ such that for all $x \in \mathbb{R}$,*

$$\frac{f(x+) + f(x-)}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty} \big(a_n \cos(n\omega_0 x) + b_n \sin(n\omega_0 x)\big).$$

Moreover, the Fourier coefficients a_n, b_n are given by

$$\begin{aligned} a_n &= \frac{2}{T}\int_{\text{period}} f(x)\cos(n\omega_0 x)dx, \quad n = 0, 1, 2, 3, \cdots \\ b_n &= \frac{2}{T}\int_{\text{period}} f(x)\sin(n\omega_0 x)dx, \quad n = 1, 2, 3, \cdots. \end{aligned}$$

(In the above, the subscript "period" for the integral means that the integral is from $x = a$ to $x = a + T$, with any[3] real a.)

[3] In practice, while doing computations for a concrete periodic function, we will take some convenient interval for integration, such as $[-T/2, T/2]$ or $[0, T]$.

Proof. See the appendix to this chapter, Section 3.8. $\square$

Remark 3.1. In particular, if the T-periodic piecewise smooth function f is continuous at x, then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \big(a_n \cos(n\omega_0 x) + b_n \sin(n\omega_0 x)\big).$$

The smoother f is, the faster the Fourier coefficients converge to 0, and the faster the Fourier series converges to f. For a function with discontinuities, the Fourier series does ***not*** converge to the function uniformly. This means that no partial sum of the Fourier series approximates the function well on the ***whole*** interval. Irrespective of how many terms one takes in the partial sum, the approximation error to the function does not become small near the discontinuity, and this is called ***Gibb's phenomenon.***

Exercise 3.7 (Triangular wave). The 2π periodic triangular wave is given on $[-\pi, \pi]$ by

$$f(x) = \begin{cases} \pi + x & \text{if } x \in [-\pi, 0], \\ \pi - x & \text{if } x \in [0, \pi]. \end{cases}$$

(1) Sketch the graph of f.
(2) Find the Fourier series of f.
(3) Plot partial sums with 3, 33 and 333 terms using Maple.
(4) Show that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}.$$

Exercise 3.8 (Gibbs Phenomenon).

(1) (Square wave) The 2π periodic square wave is given on $[-\pi, \pi]$ by

$$f(x) = \begin{cases} -1 & \text{if } x \in (-\pi, 0), \\ 1 & \text{if } x \in [0, \pi]. \end{cases}$$

Show that the Fourier series coefficients are given by $a_n := 0$ for all n, $b_n = 0$ for all even n, and

$$b_n := \frac{4}{n\pi} \quad \text{if } n \text{ is odd.}$$

Let s_N is the partial sum given by

$$s_N(x) = 4 \sum_{n=0}^{N} \frac{\sin((2n+1)x)}{(2n+1)\pi}.$$

It can be shown that s_N takes its maximal value at

$$x = \frac{\pi}{2(N+1)}.$$

Plot s_N for $N = 3, 33, 333$, and calculate the maximal value in each case. Note that the "overshoot" does not converge to 0, but rather to a value of about 0.17898. This is an instance of Gibbs Phenomenon.

(2) Suppose that for a piecewise smooth T-periodic function, the Fourier coefficients are absolutely convergent, that is,

$$\sum_{n=1}^{\infty} |a_n| < \infty \text{ and } \sum_{n=1}^{\infty} |b_n| < \infty.$$

Prove that the convergence in Theorem 3.1 is uniform[4], that is, Gibbs Phenomenon does not occur.

Exercise 3.9 (Even and odd functions). Let f be a $2L$-period function and let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right),$$

be its Fourier series.

(1) Show that f is even if and only if $b_n = 0$ for all $n = 1, 2, 3, \cdots$. In this case,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right),$$

where $a_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$, $n = 0, 1, 2, 3, \cdots$.

(2) Show that f is odd if and only if $a_n = 0$ for all $n = 0, 1, 2, 3, \cdots$. In this case,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

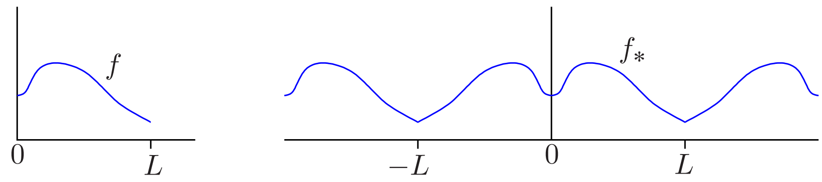
where $b_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$, $n = 1, 2, 3, \cdots$.

3.1.1. Half period expansions.

In the sequel, we will often need to represent by a Fourier series, a function f that is only defined in a finite interval, say $[0, L]$. (This may appear as the initial condition in our PDE problem.) Since f is not periodic, the result above is not applicable. However, in this section we will see that we can ***extend*** f in a periodic manner to the whole of $\mathbb{R}$ to obtain a $2L$-periodic function f_*, and then use the results from the previous section. This can be done in one of two ways: if we make f_* an even function, then we will get a cosine expansion for f in $[0, L]$, while if f_* is an odd function, then we will get a sine expansion of f.

1° Even half period extension. We first extend f on the interval $(-L, L)$ by setting $f_*(-x) := f(x)$, $-L < x < 0$, and then extend this periodically with period $2L$ to get f_* on the whole of $\mathbb{R}$. The following picture illustrates this.

[4] Recall that if $f, f_n : \mathbb{R} \to \mathbb{R}$ ($n \in \mathbb{N}$) are functions, then the sequence $(f_n)_{n\in\mathbb{N}}$ is said to ***converge uniformly to*** f if for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n > N$, and for all $x \in \mathbb{R}$, $|f_n(x) - f(x)| < \epsilon$.

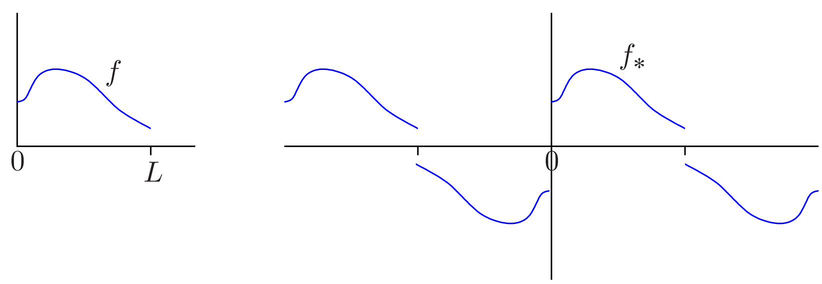


Now we expand f_* in its Fourier series, and note that the b_ns are all zeros. Thus for $x \in (0, L)$ where f is continuous we have

$$f(x) = f_*(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L},$$

where $a_n = \dfrac{2}{L} \displaystyle\int_0^L f(x) \cos \frac{n\pi x}{L} dx$ for all $n = 0, 1, 2, 3, \cdots$.

2° Odd half-period extension. We first extend f on the interval $(-L, L)$ by setting $f_*(-x) := -f(x)$, $-L < x < 0$, and then extend this periodically with period $2L$ to get f_* on the whole of $\mathbb{R}$. The following picture illustrates this.



Now we expand f_* in its Fourier series, and note that the a_ns are all zeros. Thus for $x \in (0, L)$ where f is continuous, we have

$$f(x) = f_*(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L},$$

where $b_n = \dfrac{2}{L} \displaystyle\int_0^L f(x) \sin \frac{n\pi x}{L} dx$ for all $n = 1, 2, 3, \cdots$.

3.2. Dimensionless form

We will now consider the classical trinity of PDEs, and often we will assume them to live on the special interval $[0, \pi]$, and also set physical constants to 1. We do this in order to simplify formulae; so that we can focus on the essential things, rather than get distracted by other superficial stuff. Let us show that there is no loss in generality in doing this, since it is only a matter of scaling space/time variables.

Consider as a specific example, the diffusion equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2}, \quad (0 < x < L,\ t > 0),$$

where a is a constant. By defining

$$x_1 = \frac{\pi x}{L},$$

we see that when x ranges over $(0, L)$, x_1 ranges over $(0, \pi)$. Then our PDE leads to the equation[5]

$$\frac{\partial u}{\partial t} = \frac{a\pi^2}{L^2}\frac{\partial^2 u}{\partial x_1^2}, \quad (0 < x_1 < \pi,\ t > 0).$$

Now the ugly constant can also be removed by defining

$$t_1 = \frac{a\pi^2}{L^2}t,$$

so that

$$\frac{\partial u}{\partial t_1} = \frac{\partial^2 u}{\partial x_1^2}, \quad (0 < x_1 < \pi,\ t_1 > 0).$$

We say then that our original equation has been converted into a "dimensionless form". (The terminology arises from "dimensional analysis" in mechanics, where all physical quantities can be expressed in terms of length, mass and time.)

With these preliminaries out of the way, we are now ready to describe the method of separation of variables for finding PDE solutions.

3.3. Diffusion equation with Dirichlet conditions

We begin by considering the one dimensional homogeneous diffusion equation with Dirichlet boundary conditions, namely:

$$\begin{aligned}
&\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 && (0 < x < \pi,\ t > 0),\\
&u(0,t) = u(\pi,t) = 0 && (t > 0),\\
&u(x,0) = f(x) && (0 < x < \pi).
\end{aligned}$$

One can interpret this equation as a model for heat diffusion in a wall of thickness π, with much bigger other dimensions, and such that the temperature at either side is 0. Note that both the PDE and the boundary conditions are homogeneous. That the PDE is homogeneous means that there are not heat sources or sinks inside the region. We will assume that the initial value function f is piecewise smooth. We will demonstrate the method of separation of variables in the following steps.

[5] Here there is a slight abuse of notation, where instead of u, we should have, strictly speaking, used the function u_1 defined by $u_1(x_1, t) := u(\frac{Lx_1}{\pi}, t)$.

Step 1. Determine all nonzero functions of the form $u(x,t) = X(x)T(t)$ such that

$$\begin{aligned} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \qquad & (0 < x < \pi,\ t > 0), \\ u(0,t) = u(\pi,t) = 0 \quad & (t > 0). \end{aligned}$$

Inserting such a u into the PDE gives

$$X(x)T'(t) - X''(x)T(t) = 0.$$

If $X(x) \neq 0$ and $T(t) \neq 0$, we obtain

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)}.$$

The left hand side is a function of t, and the right hand side is a function of x. As they are equal, it follows that their common value must be a constant. (After all, changing t doesn't change the left hand side, while changing x doesn't change the right hand side!) Denoting the common constant value by $-\lambda$, we obtain

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda,$$

that is,

$$\begin{aligned} X''(x) + \lambda X(x) &= 0, \\ T'(t) + \lambda T(t) &= 0. \end{aligned}$$

The variables x and t are now "separated" in the sense that they satisfy two separate (ordinary) differential equations. The boundary conditions $u(0,t) = u(\pi,t) = 0$ imply that $X(0)T(t) = 0 = X(\pi)T(t)$. If $X(0) \neq 0$ or $X(\pi) \neq 0$, then we would obtain $T \equiv 0$, and so also $u \equiv 0$. As we are interested in nonzero solutions, we assume that

$$X(0) = X(\pi) = 0.$$

Step 2. As the above ODE for X comes equipped with boundary conditions, we will treat this equation first (and it will help us to narrow down the possible values of λ, as we shall see). We know (see Exercise 0.1) that the solution to $X'' + \lambda X = 0$ has different character depending on whether λ is positive, negative or 0, and we will show that nontrivial solutions for X arise only when λ is positive.

1° $\lambda < 0$. Then $X'' + \lambda X = 0$ has the general solution given by

$$X(x) = A\cosh(\sqrt{|\lambda|}x) + B\sinh(\sqrt{|\lambda|}x),$$

where A, B are constants. Now $X(0) = 0$ gives $A \cdot 1 + B \cdot 0 = 0$, and so $A = 0$. Thus $X(x) = B\sinh(\sqrt{|\lambda|}x)$. Next, the boundary condition $X(\pi) = 0$ gives $B\sinh(\sqrt{|\lambda|}\pi) = 0$. But B is not zero (for otherwise X, being $B\sinh(\sqrt{|\lambda|}x)$ would then be identically zero, and so would u be). So $\sinh(\sqrt{|\lambda|}\pi) = 0$, giving $\lambda = 0$, a contradiction.

2° $\lambda = 0$. Now $X'' = 0$, and so $X(x) = Ax + B$. Then $X(0) = 0 = X(\pi)$ give $B = 0$ and $A\pi + B = 0$. Thus $A = 0 = B$, resulting in the trivial solution again.

3° $\lambda > 0$. Then

$$X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$$

where A, B are constants. The boundary conditions $X(0) = 0 = X(\pi)$ give $A = 0$ and

$$0 = A\cos(\sqrt{\lambda}\pi) + B\sin(\sqrt{\lambda}\pi) = 0 + B\sin(\sqrt{\lambda}\pi) = B\sin(\sqrt{\lambda}\pi).$$

If $B = 0$, we would get $X \equiv 0$, again resulting in the trivial solution. So $B \neq 0$, and hence $\sin(\sqrt{\lambda}\pi) = 0$. As $\sqrt{\lambda}\pi > 0$, we conclude from $\sin(\sqrt{\lambda}\pi) = 0$ that $\sqrt{\lambda}\pi \in \pi\mathbb{N}$, and so $\lambda = n^2$, where $n \in \mathbb{N}$. So with

$$\lambda = \lambda_n = n^2, \quad n = 1, 2, 3, \cdots,$$

the nontrivial solutions are

$$X_n(x) = \sin(nx) \quad (0 < x < \pi,\ n = 1, 2, 3, \cdots).$$

(In linear algebraic terminology, we say that the differential operator

$$-\frac{d^2}{dx^2}$$

with homogeneous Dirichlet boundary conditions on the interval $[0, \pi]$ has the ***eigenfunctions*** X_n, with ***eigenvalues*** $\lambda_n = n^2$, $n = 1, 2, 3, \cdots$.)

Next we solve the equation for T, namely $T'(t) + \lambda T(t) = 0$, where $\lambda = -n^2$, giving

$$T_n(t) = c_n e^{-\lambda_n t} = c_n e^{-n^2 t}.$$

Thus the PDE and the boundary conditions (but not yet the initial condition!) are satisfied by

$$u_n(x, t) = T_n(t)X_n(x) = c_n e^{-n^2 t}\sin(nx), \quad n = 1, 2, 3, \cdots.$$

In order that the intial condition is satisfied too, the plan is to take a combination of these u_ns, with appropriate c_ns, and determine these using a Fourier expansion of f. We will see this in the next step.

Step 3. Build formally the infinite sums

$$u(x, t) = \sum_{n=1}^{\infty} u_n(x, t) = \sum_{n=1}^{\infty} T_n(t)X_n(x) = \sum_{n=1}^{\infty} c_n e^{-n^2 t}\sin(nx).$$

It remains to find c_n so that the initial condition is satisfied. In order to start with the given initial value, we put $t = 0$ and obtain

$$f(x) = u(x, 0) = \sum_{n=1}^{\infty} c_n \sin(nx) \quad (0 < x < \pi).$$

Note that each $\sin(nx)$, $n \in \mathbb{N}$, is 2π-periodic, and so the right-hand side is the Fourier series of a 2π-periodic function. Comparing this with the sine series for the odd half-period 2π-periodic extension of f, we obtain

$$c_n = \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)\,dx.$$

Thus with these values of c_ns, u given by

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2 t}\sin(nx),$$

does satisfy the initial condition.

If we had a ***finite*** sum, then we would have been done, thanks to the superposition principle. Indeed, we could have just said that u satisfies the PDE and the homogeneous boundary conditions because each of the u_ns do. But as we have an infinite sum / series at hand, we must consider convergence issues, and we do so in the next step.

Step 4. From Step 3, we know that for $t = 0$, and with piecewise smooth and continuous data f, the series does converge to $f(x)$ for all $0 < x < \pi$. In order to study the convergence for $t > 0$, we investigate the size of the terms. We know that f is bounded, that is, there is an M such that $|f(x)| \leqslant M$ for $0 < x < \pi$. Thus

$$|c_n| = \left|\frac{2}{\pi}\int_0^\pi f(x)\sin(nx)\,dx\right| \leqslant \frac{2}{\pi}\int_0^\pi |f(x)||\sin(nx)|dx \leqslant \frac{2}{\pi}\int_0^\pi M\cdot 1dx = 2M.$$

Hence $|c_n e^{-n^2 t}\sin(nx)| \leqslant 2Me^{-n^2 t}$. Now $e^{-n^2 t} < \dfrac{1}{n^2 t}$, thanks to the inequality

$$e^{n^2 t} = 1 + \frac{n^2 t}{1!} + \cdots > n^2 t.$$

So it follows by the Comparison Test that the series for u converges (absolutely).

Since each u_n satisfies the PDE, it follows that u also satisfies the PDE because it can be shown that termwise differentiation is allowed in this case, as sketched below.

Proposition 3.2 (Termwise differentiation). *If on an interval I*

(1) $\displaystyle\sum_{n=1}^{\infty} f_n$ ***is pointwise convergent, and***

(2) $\displaystyle\sum_{n=1}^{\infty} f_n'$ ***is uniformly convergent,***

then $\displaystyle\left(\sum_{n=1}^{\infty} f_n\right)' = \sum_{n=1}^{\infty} f_n'$ ***on*** I.

This result from Real Analysis and its proof can be found, for example, in the notes[6] on MA203 or [**R**, Theorem 7.17, p. 152].

[6] available at `http://personal.lse.ac.uk/sasane/ma203.pdf`

We will show using this that we can differentiate our series termwise twice with respect to x and once with respect to t. We will show that the series obtained by termwise differentiation,

$$\begin{aligned} \frac{\partial}{\partial x}: &\qquad \sum_{n=1}^{\infty} n c_n e^{-n^2 t}\cos(nx) \\ \frac{\partial^2}{\partial x^2}, \frac{\partial}{\partial t}: &\qquad \sum_{n=1}^{\infty} -n^2 c_n e^{-n^2 t}\sin(nx) \end{aligned}$$

converge uniformly in the strips $S_\delta := \{(x,t) : 0 < x < \pi,\ t \geqslant \delta\}, \quad \delta > 0$.

To do this, one can use the "Weierstrass M-test" [**R**, Theorem 7.10, p.148].

Proposition 3.3. *If on an interval I, $|f_n(x)| \leqslant m_n$, $x \in I$, $n \in \mathbb{N}$, and $\sum\limits_{n=1}^{\infty} m_n$ converges, then $\sum\limits_{n=1}^{\infty} f_n$ converges uniformly on I.*

Since in our case the coefficients c_n are bounded, the terms satisfy

$$|n^2 c_n e^{-n^2 t}\sin(nx)| \leqslant M n^2 e^{-n^2\delta} \quad (t \geqslant \delta).$$

Using the estimate $e^{n^2\delta} > \frac{n^4\delta^2}{2!}$, we obtain $n^2 e^{-n^2\delta} < \frac{2}{\delta^2 n^2}$, and so from the Weierstrass M-test, the termwise differentiated series converge uniformly in the strips S_δ, justifying the termwise differentiation in S_δ. As $\delta > 0$ was arbitrary, we see that the PDE is satisfied by u everywhere in the region $t > 0$ and $0 < x < \pi$:

$$\Big(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\Big)u = \Big(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\Big)\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty}\Big(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\Big)u_n = \sum_{n=1}^{\infty} 0 = 0.$$

Remark 3.2 (Regularity of the solution: $u \in C^\infty$!). As a curiosity, note also that along the same lines as the above proof, we can in fact see that u may be differentiated arbitrarily many times with respect to x and with respect to t, so that u is a C^∞ function! This is an example of a "regularity result" mentioned in the preface.

Remark 3.3 (The factor $e^{-n^2 t}$). Also we note that the presence of the factor $e^{-n^2 t}$ was crucial in the justification of termwise differentiation. Such exponential factors appear with all types of diffusion problems. In the sequel, we proceed with the knowledge that the above treatment can be carried out, without actually doing so in each case—so we will be skipping "Step 4".

The factor $e^{-n^2 t}$ also helps the convergence speed. For a fixed $t > 0$, the terms go quickly to 0 as $n \to \infty$, and so in numerical calculations one can often get away with using just the first few terms of the series for u.

Thus we have solved the initial boundary value problem. Are there any other solutions? The answer is no, thanks to the unqiueness shown in §2.2.2.

The method that we have used is called the *Fourier method* or the *method of separation of variables.* From the calculations above, we remember that the solution to the homogeneous diffusion equation with homogeneous Dirichlet conditions can be expressed as a sine series. This can be used to solve concrete problems, and instead of doing separation of variables from scratch, we *assume* a solution in the form of a sine series. The following example illustrates this.

Example 3.3. A rod of length L is insulated, and heated uniformly to $100°$C. At time $t = 0$, the endpoints are cooled down to $0°$C, and are kept at this temperature (for example by touching the ends by ice cubes). We wish to determine how the temperature evolves in time. Thus we have

$$\begin{aligned} &\frac{\partial u}{\partial t} - a\frac{\partial^2 u}{\partial x^2} = 0 && (0 < x < L,\ t > 0), \\ &u(0,t) = u(L,t) = 0 && (t > 0), \\ &u(x,0) = 100 && (0 < x < L), \end{aligned}$$

where a is a constant. As we have homogeneous Dirichlet conditions, the solution has the form

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin\frac{n\pi x}{L}.$$

Putting this in the PDE gives $\displaystyle\sum_{n=1}^{\infty}\Big(u_n'(t) + \frac{an^2\pi^2}{L^2}u_n(t)\Big)\sin\frac{n\pi x}{L} = 0.$

Thus uniqueness of coefficients in the Fourier expansion gives

$$u_n'(t) + \frac{an^2\pi^2}{L^2}u_n(t) = 0, \quad n = 1, 2, 3, \cdots .$$

These have the solutions $u_n(t) = c_n e^{-\frac{an^2\pi^2 t}{L^2}}$, $n = 1, 2, 3, \cdots$. Thus

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\frac{an^2\pi^2 t}{L^2}} \sin\frac{n\pi x}{L}.$$

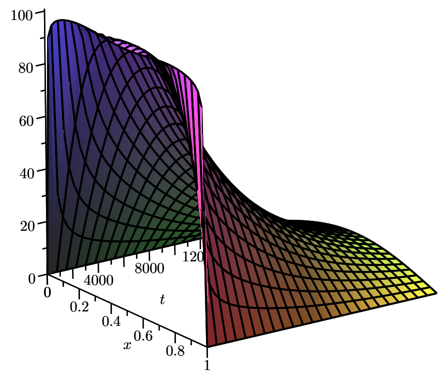
Putting $t = 0$ gives $100 = u(x,0) = \displaystyle\sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L}$ $(0 < x < L)$. Hence

$$c_n = \frac{2}{L}\int_0^L 100 \sin\frac{n\pi x}{L}dx = \frac{200\big(1-\cos(n\pi)\big)}{n\pi} = \frac{200\big(1-(-1)^n\big)}{n\pi}.$$

Consequently, the temperature of the rod is given by

$$u(x,t) = \sum_{n=1}^{\infty} \frac{200\big(1-(-1)^n\big)}{n\pi} e^{-\frac{an^2\pi^2 t}{L^2}} \sin\frac{n\pi x}{L}.$$

Here is a plot of the graph of u. The calculations have been done for iron, for which $a = 1.5 \cdot 10^{-5}\ m^2/s$, and with $L = 1\ m$.



We have also sketched the temperature profiles $u(\cdot, t)$ as functions of x for various different fixed values of t.

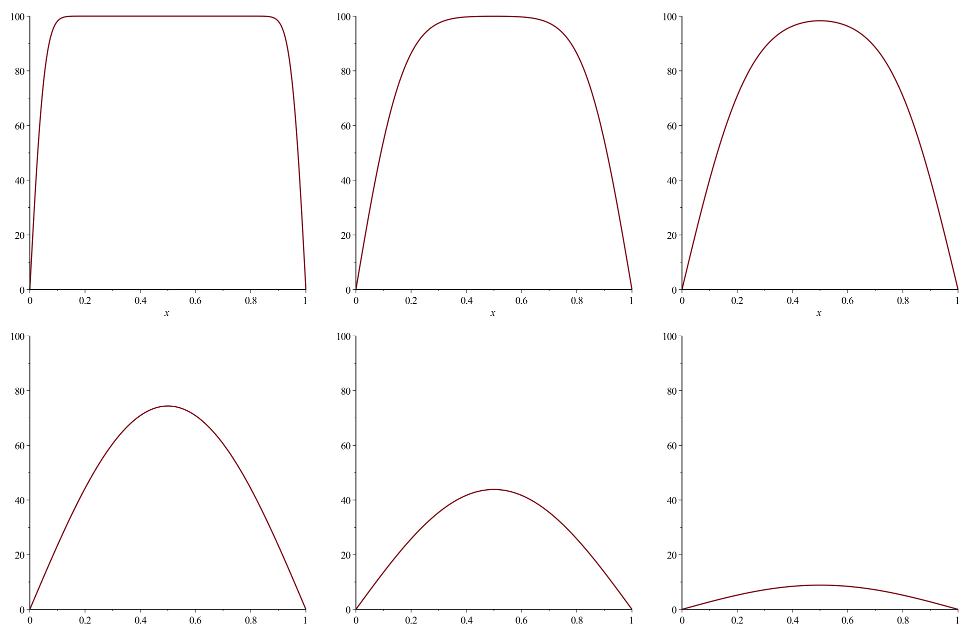


Figure 1. The temperature profiles at times t equal to 1 minute, 10 minutes, 20 minutes, 1 hour, 2 hours and 5 hours.

We note that the temperature u converges to 0 as time increases, as expected physically (since the endpoints are maintained at temperature 0). Also, the bigger the constant a is, the faster this happens. $\Diamond$

Exercise 3.10. Solve

$$\begin{aligned}
&\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 && (0 < x < 1,\ t > 0),\\
&u(0,t) = u(1,t) = 0 && (t > 0),\\
&u(x,0) = x && (0 < x < 1).
\end{aligned}$$

You may use the Fourier series expansion for an odd half-period 2-periodic extension of $u(x,0)$:

$$u(x,0) = x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{\pi n} \sin(n\pi x) \quad (0 < x < 1).$$

Exercise 3.11. Solve

$$\begin{aligned}
&\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 && (0 < x < 1,\ t > 0),\\
&u(0,t) = u(1,t) = 0 && (t > 0),\\
&u(x,0) = \sin(\pi x) + 2\sin(3\pi x) && (0 < x < 1).
\end{aligned}$$

Exercise 3.12 (Separation of variables). Find solutions u of the form $u(x,y) = X(x)Y(y)$ for the following:

(1) $u_x + u_y = 0$
(2) $u_x + u_y = 2(x+y)u$
(3) $u_{xy} - u = 0$.

3.4. Diffusion equation with Neumann conditions

We shall now change the boundary conditions from Dirichlet to Neumann and do the corresponding calculations for the initial boundary value problem

$$\begin{aligned}
&\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 && (0 < x < \pi,\ t > 0),\\
&u_x(0,t) = u_x(\pi,t) = 0 && (t > 0),\\
&u(x,0) = f(x) && (0 < x < \pi).
\end{aligned}$$

For the case of the diffusion of heat in a rod, these boundary conditions have the physical meaning that the endpoints are completely insulated (so that the “heat flow/transfer” u_x is 0 at these points). Assuming that $u(x,t) = X(x)T(t)$ gives

$$\begin{aligned}
X''(x) + \lambda X(x) &= 0,\\
T'(t) + \lambda T(t) &= 0.
\end{aligned}$$

The boundary conditions give

$$X'(0) = X'(\pi) = 0.$$

We consider the three cases:

1° $\lambda < 0$. Then $X(x) = A\cosh(\sqrt{|\lambda|}x) + B\sinh(\sqrt{|\lambda|}x)$ for some constants A, B. So $X'(x) = A\sqrt{|\lambda|}\sinh(\sqrt{|\lambda|}x) + B\sqrt{|\lambda|}\cosh(\sqrt{|\lambda|}x)$. The boundary condition $X'(0) = 0$ gives $B\sqrt{|\lambda|} = 0$ and so $B = 0$. Hence we have $X'(x) = A\sqrt{|\lambda|}\sinh(\sqrt{|\lambda|}x)$. The boundary condition $X'(\pi) = 0$ now gives $A\sqrt{|\lambda|}\sinh(\sqrt{|\lambda|}\pi) = 0$. If $A = 0$, then $X \equiv 0$, and $u \equiv 0$. So $A \neq 0$ and hence $\lambda = 0$, which is a contradiction.

2° $\lambda = 0$. Then $X(x) = Ax + B$ for some A, B. Then $X'(x) = A$. The boundary conditions give $A = 0$. So B can be arbitrary. Thus for $\lambda = 0$, we obtain a nontrivial solution,

$$X_0(x) = 1,$$

and its multiples.

3° $\lambda > 0$. Then $X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$ for some constants A, B. We have $X'(x) = -A\sqrt{\lambda}\sin(\sqrt{\lambda}x) + \sqrt{\lambda}B\cos(\sqrt{\lambda}x)$. The boundary conditions give

$$\begin{aligned}
&0 = X'(0) = \sqrt{\lambda}B \implies B = 0, \text{ and using this,}\\
&0 = X'(\pi) = -\sqrt{\lambda}A\sin(\sqrt{\lambda}\pi) + 0 \implies (A = 0 \text{ or } \lambda = n^2,\ n \in \mathbb{N}).
\end{aligned}$$

So the nontrivial solutions are obtained when $\lambda = n^2$, $n = 1, 2, 3, \cdots$, and then

$$X_n(x) = \cos(nx), \quad n = 1, 2, 3, \cdots .$$

Summarizing: the differential operator $-\frac{d^2}{dx^2}$ with homogeneous Neumann conditions on $[0, \pi]$ have eigenfunctions

$$X_n(x) = \cos(nx),$$

corresponding to the eigenvalues $\lambda_n = n^2$, $n = 0, 1, 2, 3, \cdots$. Note that the case $\lambda = 0$ has been included in the above by starting from $n = 0$.

Now let us look at the t variable. With $\lambda = n^2$, we have

$$T_n(t) = c_n e^{-n^2 t}, \quad n = 0, 1, 2, 3, \cdots .$$

Thus the PDE and the boundary conditions are satisfied by

$$u_n(x, t) = X_n(x)T_n(t) = c_n e^{-n^2 t}\cos(nx), \quad n = 0, 1, 2, 3, \cdots .$$

Now we build a formal sum

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t) = c_0 + \sum_{n=1}^{\infty} c_n e^{-n^2 t}\cos(nx).$$

Using the initial condition gives

$$f(x) = c_0 + \sum_{n=0}^{\infty} c_n \cos(nx).$$

Using an even half period extension of f, we have that

$$\begin{aligned} c_0 &= \frac{1}{\pi}\int_0^\pi f(x)dx, \\ c_n &= \frac{2}{\pi}\int_0^\pi f(x)\cos(nx)\, dx, \quad n = 1, 2, 3, \cdots . \end{aligned}$$

In the same way as the Dirichlet case, one can show that the series converges and gives the unique solution to the problem.

In this case, we note that the solution consists of a ***stationary***, time independent part ($n = 0$), and ***transient*** parts ($n \geqslant 1$). The transient parts contain the factor $e^{-n^2 t}$, $n = 1, 2, 3, \cdots$, and converge to 0 as $t \to \infty$. The stationary part is constant, equal to

$$c_0 = \frac{1}{\pi}\int_0^\pi f(x)dx,$$

which is the "average value" of the initial condition! For an insulated rod this means that the temperature after a long time is approximately constant along the rod (and the constant value is the average value of the initial temperature). This is what one expects to happen physically.

An important conclusion of the above calculations is that in the case of homogeneous Neumann condition at both ends, one can write the solution to the diffusion equation as a cosine series.

Exercise 3.13. Solve

$$\begin{aligned} &\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 && (0 < x < 1,\ t > 0), \\ &u_x(0,t) = u_x(1,t) = 0 && (t > 0), \\ &u(x,0) = x && (0 < x < 1). \end{aligned}$$

You may use the Fourier series expansion for an even half-period 2-periodic extension of $u(x, 0)$:

$$u(x,0) = x = \frac{1}{2} - \frac{4}{\pi^2}\sum_{n=0}^{\infty}\frac{\cos\big((2n+1)\pi x\big)}{(2n+1)^2} \qquad (0 < x < 1).$$

Exercise 3.14 (Average temperature). Consider the Neumann problem for the diffusion equation,

$$\begin{aligned} &\frac{\partial u}{\partial t} - a\frac{\partial^2 u}{\partial x^2} = 0 && (0 < x < L,\ t > 0), \\ &u_x(0,t) = u_x(L,t) = 0 && (t > 0), \\ &u(x,0) = f(x) && (0 < x < L). \end{aligned}$$

Define the average temperature by $T_{\text{av}}(t) := \dfrac{1}{L}\displaystyle\int_0^L u(x,t)dx$.

Show that the average temperature does not change with time. What is the constant value in terms of the initial condition f?

Exercise 3.15 (Uniqueness). Consider the Neumann problem for the diffusion equation,

$$\begin{aligned}
&\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = v && (0 < x < 1,\ t > 0),\\
&u_x(0,t) = \alpha(t),\ u_x(1,t) = \beta(t) && (t > 0),\\
&u(x,0) = f(x) && (0 < x < 1).
\end{aligned}$$

Define the energy by $E(t) := \dfrac{1}{2}\displaystyle\int_0^1 \big(u(x,t)\big)^2 dx$.

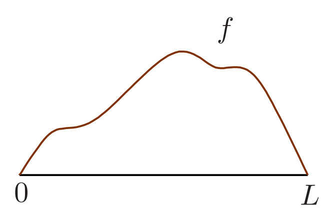
Show by energy considerations that the above initial boundary value problem has a unique solution.

3.5. Wave equation with Dirichlet conditions

Let us now consider the following problem, describing the wave equation in one dimension:

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0 && (0 < x < L,\ t > 0),\\
&u(0,t) = u(L,t) = 0 && (t > 0),\\
&u(x,0) = f(x) \text{ and } u_t(x,0) = g(x) && (0 < x < L).
\end{aligned}$$

This may describe the motion of a thin elastic string which is not acted upon by any force. The boundary conditions mean that the ends of the string are kept fixed. Among the initial conditions, the function f gives the initial shape of the string, while g gives the initial speed. We had shown the uniqueness of the solution in Subsection 2.2.1.



Once again, we use the method of separation of variables. Set

$$u(x,t) = X(x)T(t).$$

By putting this into the PDE, we get $XT'' = c^2 X''T$. So

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = -\lambda,$$

where λ is a constant. Thus we arrive at

$$\begin{aligned}
X'' + \lambda X &= 0,\\
T'' + c^2\lambda T &= 0.
\end{aligned}$$

The boundary conditions give $X(0) = X(L) = 0$. So we arrive at exactly the same equation for X with the same boundary conditions as in Section 3.3. Proceeding in the same manner, we see that the nontrivial solutions exist only in the case when $\lambda > 0$, and then

$$X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x).$$

The boundary conditions give

$$\begin{aligned} 0 &= X(0) = A, \\ 0 &= X(L) = B\sin(\sqrt{\lambda}L) \quad \Longrightarrow \quad (B = 0 \text{ or } \sqrt{\lambda}L = n\pi,\ n \in \mathbb{N}) \end{aligned}$$

So we see that the nontrivial solutions to $X'' + \lambda X = 0$ with the boundary conditions $X(0) = X(L) = 0$ exist when

$$\lambda = \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \cdots,$$

and then the solutions are

$$X_n(x) = \sin\frac{n\pi x}{L}, \quad n = 1, 2, 3, \cdots.$$

Next we solve the ODE $T'' + c^2\lambda T = 0$, where $\lambda = \lambda_n = \left(\frac{n\pi}{L}\right)^2$, giving

$$T_n(t) = a_n\cos\frac{n\pi ct}{L} + b_n\sin\frac{n\pi ct}{L}.$$

Analogous to the diffusion equation, we form the formal sum

$$u(x,t) = \sum_{n=1}^{\infty} X_n(x)T_n(t) = \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi ct}{L} + b_n\sin\frac{n\pi ct}{L}\right)\sin\frac{n\pi x}{L}. \tag{3.1}$$

Here we must choose the constants a_n and b_n so that the initial conditions are satisfied. Putting $t = 0$ in the expressions for u and u_t gives

$$\begin{aligned} f(x) = u(x,0) &= \sum_{n=1}^{\infty} a_n\sin\frac{n\pi x}{L}, \\ g(x) = u_t(x,0) &= \sum_{n=1}^{\infty} b_n\frac{n\pi c}{L}\sin\frac{n\pi x}{L}. \end{aligned}$$

The second line above is obtained by formally differentiating the series for u termwise with respect to t. We get the coefficients a_n, b_n by expanding f, g in their respective (odd half period extension) sine series. Thus

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx, \\ b_n &= \frac{L}{n\pi c}\cdot\frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}dx. \end{aligned}$$

With these coefficients, (3.1) is the unique solution to the initial boundary value problem.

We observe that just as in the diffusion equation, the homogeneous Dirichlet conditions at both ends give rise to a solution in the form of a sine series. With this knowledge, we can directly assume a solution to the problem in the form of a sine series, and perform calculations in specific examples, as we shall see below in the case of a plucked string.

Remark 3.4. The careful reader is probably awaiting an explanation for the validity of the termwise differentiation in (3.1) so that one can check that the wave equation is solved by this u. In this series, the coefficients are of the same order of size as the coefficients for the initial condition functions. There is no reason to expect that the series in (3.1) would be termwise differentiable unless such a thing happens with the series for the functions in the initial conditions. So the function in (3.1) is not a solution to the problem in the *classical* sense. Despite this, (3.1) *is* the physical solution to the wave equation. The mathematical explanation of this is that the function satisfies the wave equation with derivatives *in the sense of distributions*, and we will elaborate on this in Chapter 5.

Remark 3.5. The various steps in the solution of the wave equation can be interpreted physically. The separated solution

$$T_n(t)X_n(x) = T_n(t)\sin\frac{n\pi}{L}x$$

represents a vibration where the "form" or "shape" of the string in the x direction stays the same over time, while its amplitude varies periodically in time. Such a vibration is called a *standing wave* or a *normal mode*. The frequency of T_n,

$$\omega_n = \frac{n\pi c}{L},$$

is called the *natural frequency*. In the pictures below, the standing waves have been plotted for $n = 1, 2, 3$. The points where $X_n = 0$ are called *nodes*.

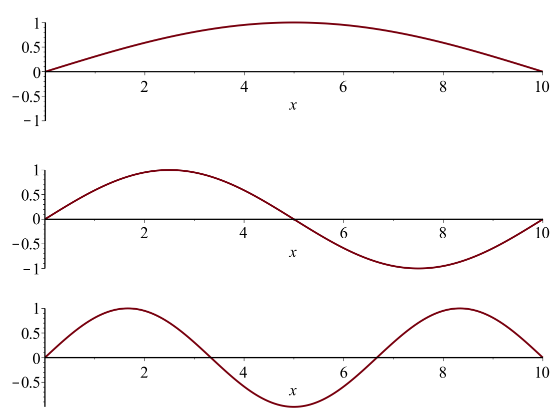


Figure 2. Standing wave profiles for $n = 1, 2, 3$.

The idea behind the solution can be expressed physically by saying that the wave motion of the string is the superposition of standing waves.

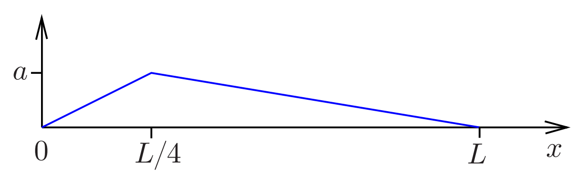
As is well-known, a vibrating string, for example in a guitar, often produces a sound. How we perceive it depends on what simple harmonic vibrations are involved. The *fundamental frequency* of the string corresponds to the standing wave with $n = 1$:

$$\omega_1 = \frac{\pi c}{L}.$$

The other n values are called *overtones*. In what degree these appear depends on the Fourier coefficients in (3.1). The size of the coefficients determines the intensity of the corresponding tone. As we have seen above, this depends on the initial conditions.

Let us do some computations in some musical examples.

Example 3.4 (Guitar). Consider a plucked guitar string of length L, which is lifted up to a height a at a point having distance $L/4$ from one end point, and let go from rest (so that the initial speed is 0). At the initial moment, the string profile looks like this:



Let us determine the evolution of the displacement u of the string. u is the solution to

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 && (0 < x < L,\ t > 0), \\ &u(0,t) = u(L,t) = 0 && (t > 0), \\ &u(x,0) = f(x) \text{ and } u_t(x,0) = 0 && (0 < x < L), \end{aligned}$$

where

$$f(x) = \begin{cases} \dfrac{4a}{L} x & \text{if } 0 \leqslant x \leqslant \dfrac{L}{4}, \\ \dfrac{4a}{3L}(L - x) & \text{if } \dfrac{L}{4} \leqslant x \leqslant L. \end{cases}$$

Since we have homogeneous Dirichlet boundary conditions, we expect a solution in the form of a sine series:

$$u(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi ct}{L} + b_n \sin \frac{n\pi ct}{L} \right) \sin \frac{n\pi x}{L}.$$

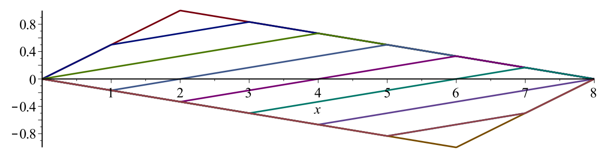
The coefficients a_n, b_n are determined from the initial conditions. As $g \equiv 0$, it follows that $b_n = 0$, $n = 1, 2, 3, \cdots$. On the other hand, one can calculate

$$a_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx = \frac{32a}{3n^2\pi^2} \sin \frac{n\pi}{4}, \quad n = 1, 2, 3, \cdots .$$

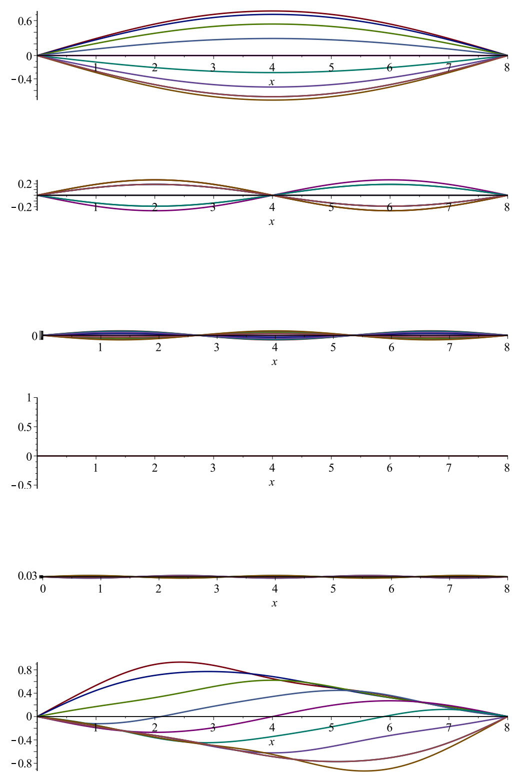
Summarizing, the solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} \frac{32a}{3n^2\pi^2} \sin\frac{n\pi}{4} \cos\frac{n\pi ct}{L} \sin\frac{n\pi x}{L}.$$

The figure below shows the shape of the string at uniformly spaced time instances.



In the following picture , we have shown the standing waves for $n = 1, 2, 3, 4, 5$, together with their sum (bottom picture) at various time instances. $\diamondsuit$



Remark 3.6. In the case of the guitar string, there are no additional forces, which give a homogeneous wave equation. In this case, one talks of *free vibrations*. On the other hand, if the string is made to vibrate with a bow, as is the case in a violin, one has *forced vibrations*, and one has an *in*homogeneous wave equation at hand. In this case, there appear additional terms (due to the particular solution) besides the terms in (3.1).

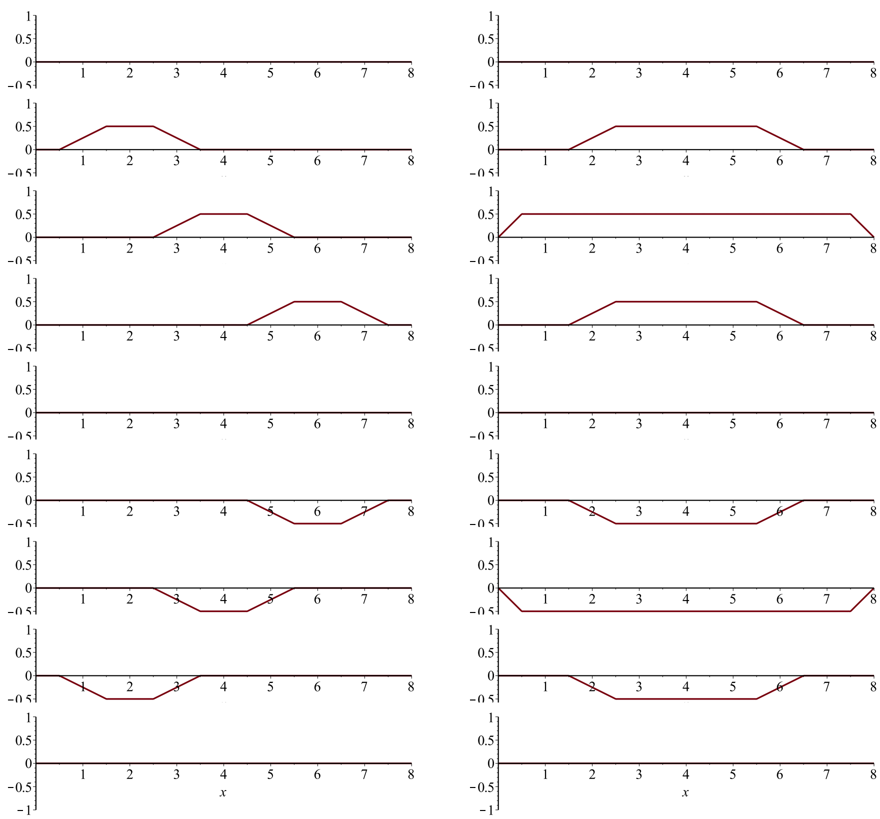
Example 3.5 (Piano). Consider a taut piano string of length L and density per unit length ρ, getting an impulse P from a hammer blow, where the width of the hammer is b, and the midpoint of the hammer is at $x = a$ along the string. In this case, the initial conditions become

$$\begin{aligned} u(x,0) &= 0, \\ u_t(x,0) &= \begin{cases} \dfrac{P}{\rho b} & \text{if } |x-a| < \dfrac{b}{2}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The solution is given by (3.1), where the coefficients are

$$\begin{aligned} a_n &= 0, \\ b_n &= -\frac{2LP}{b\rho c}\frac{1}{n^2\pi^2}\left(\cos\frac{n\pi}{L}\left(a+\frac{b}{2}\right) - \cos\frac{n\pi}{L}\left(a-\frac{b}{2}\right)\right) \\ &= \frac{4LP}{b\rho c}\frac{1}{n^2\pi^2}\left(\sin\frac{n\pi a}{L}\right)\left(\sin\frac{n\pi b}{2L}\right). \end{aligned}$$

The pictures below show the profile of the string at various time instances. In the left pictures, a is small, while in the right pictures, $a = L/2$.



In the left pictures, the pulse seems to travel left to right, gets reflected from the right end, propagates to the left end, gets reflected there, and continues in this manner. In the right pictures, the wave spreads to both ends, gets reflected at both ends, and this process continues.

One can see how the sound depends on a (where the hammer hits), and on b (the width of the hammer). For example, when one plays a^1 (440 Hz), one also gets overtones:

$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$
440 Hz	880 Hz	1320 Hz	1760 Hz	2200 Hz	2640 Hz	3080 Hz
a^1	a^2	e^1	a^3	c^4	e^4	g^4

To a musical ear, among these overtones, the overtone $n = 7$ sounds dissonant with the $n = 1$ note, while the others are perceived to form a pleasing combination. So it is desirable to suppress this $n = 7$ overtone. This can be achieved by simply making the coefficient $b_7 = 0$! To this end, we could take $a = L/7$, and then we see that one of the factors in the term b_7, namely

$$\sin \frac{n\pi a}{L} = \sin \frac{n\pi(L/7)}{L} = \sin \frac{\pi n}{7} \overset{(n=7)}{=} 0,$$

so that $b_7 = 0$, as desired. Note that when we take $a = L/7$, we are hitting the string at a node of the 7th overtone. $\Diamond$

In all of our examples of the vibrating string, we only considered fixed endpoints, that is homogeneous Dirichlet boundary conditions. In the case of acoustic vibrations, for example in an organ pipe, one also encounters Neumann boundary conditions. Just as with the homogeneous Neumann conditions for the diffusion equation, one gets a solution as a cosine series also for the wave equation with Neumann boundary conditions.

Exercise 3.16. Solve

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 && (0 < x < \pi/2,\ t > 0), \\ &u(0,t) = u(\pi/2,t) = 0 && (t > 0), \\ &u(x,0) = 0 \text{ and } u_t(x,0) = x\cos x && (0 < x < \pi/2). \end{aligned}$$

You may use the Fourier series expansion for an odd half-period π-periodic extension of $u(x,0)$:

$$u_t(x,0) = x\cos x = \sum_{n=1}^{\infty} \frac{16}{\pi}(-1)^{n+1}\frac{n}{(4n^2-1)^2}\sin(2nx) \quad (0 < x < \pi/2).$$

Use Maple to plot the graphs of $u(\cdot,t)$ at time instances

$$t = 0, \frac{\pi}{4}, \frac{\pi}{3}, \frac{3\pi}{4}, \pi$$

first with just one term of the solution series, and next with 333 terms. Can you explain mathematically why are the plots almost the same by noting the Fourier coefficients of the initial condition?

3.5.1. D'Alembert's solution. One can express the solution to the wave equation with Dirichlet boundary conditions in a somewhat simpler form, without going through the Fourier series malarchy. Consider the problem

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 && (0 < x < L,\ t > 0), \\ &u(0,t) = u(L,t) = 0 && (t > 0), \\ &u(x,0) = f(x) \text{ and } u_t(x,0) = g(x) && (0 < x < L). \end{aligned}$$

Let the odd half period $2L$-periodic extension of f, g be denoted by f_* and g_*, respectively. Then the solution to the above is given by ***d'Alembert's Formula***,

$$u(x,t) = \frac{f_*(x-ct) + f_*(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g_*(\xi)d\xi.$$

We can check this by direct differentiation. We have

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial}{\partial t}\Big(\frac{f_*(x-ct) + f_*(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g_*(\xi)d\xi\Big) \\ &= \frac{-cf_*'(x-ct) + cf_*'(x+ct)}{2} + \frac{1}{2c}\cdot\big(g_*(x+ct)\cdot c - g_*(x-ct)\cdot(-c)\big) \\ &= \frac{c}{2}\big(f_*'(x+ct) - f_*'(x-ct)\big) + \frac{1}{2}\big(g_*(x+ct) + g_*(x-ct)\big). \end{aligned}$$

Differentiating again with respect to t, we obtain

$$\frac{\partial^2 u}{\partial t^2} = \frac{c^2}{2}\big(f_*''(x+ct) + f_*''(x-ct)\big) + \frac{c}{2}\big(g_*'(x+ct) - g_*'(x-ct)\big). \tag{3.2}$$

Similarly, by differentiating u with respect to x we obtain

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial}{\partial x}\Big(\frac{f_*(x-ct) + f_*(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g_*(\xi)d\xi\Big) \\ &= \frac{f_*'(x-ct) + f_*'(x+ct)}{2} + \frac{1}{2c}\cdot\big(g_*(x+ct) - g_*(x-ct)\big). \end{aligned}$$

Differentiating again with respect to x, we obtain

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2}\big(f_*''(x+ct) + f_*''(x-ct)\big) + \frac{1}{2c}\big(g_*'(x+ct) - g_*'(x-ct)\big). \tag{3.3}$$

It follows from (3.2) and (3.3) that $\dfrac{\partial^2 u}{\partial t^2} - c^2\dfrac{\partial^2 u}{\partial x^2} = 0$.

Let us check that the boundary conditions are satisfied. Note that

$$u(0,t) = \frac{f_*(-ct) + f_*(ct)}{2} + \frac{1}{2c}\int_{-ct}^{ct} g_*(\xi)d\xi = 0 + 0 = 0$$

since f_*, g_* are odd. Now we would like to check $u(L,t) = 0$ too. To this end, we make the following observations:

(1) Using the oddness and $2L$-periodicity of f_*, we have

$$f_*(L+ct) = f_*(L+ct-2L) = f_*(ct-L) = f_*(-(L-ct)) = -f_*(L-ct).$$

(2) The map $\eta \mapsto g_*(\eta+L)$ is odd since g_* is odd and $2L$-periodic:

$$g_*(-\eta+L) = g_*(-\eta+L-2L) = g_*(-\eta-L) = g_*(-(\eta+L)) = -g_*(\eta+L).$$

So we have $\displaystyle\int_{L-ct}^{L+ct} g_*(\xi)d\xi \overset{(\eta=\xi-L)}{=} \int_{-ct}^{ct} g_*(\eta+L)d\eta = 0.$

Hence $u(L,t) = \dfrac{1}{2}\big(\underbrace{f_*(L-ct)+f_*(L+ct)}_{=0 \text{ by } (1)}\big) + \underbrace{\dfrac{1}{2c}\displaystyle\int_{L-ct}^{L+ct} g_*(\xi)d\xi}_{=0 \text{ by } (2)} = 0+0 = 0.$

Finally, we can check if the initial conditions is satisfied. We have

$$u(x,0) = \frac{f_*(x)+f_*(x)}{2} + \frac{1}{2c}\int_x^x g_*(\xi)d\xi = f_*(x)+0 = f_*(x) = f(x) \quad (0<x<L).$$

Also, from our previous calculation, we have

$$\begin{aligned} \frac{\partial u}{\partial t}(x,0) &= \frac{c}{2}\big(f_*'(x+0) - f_*'(x-0)\big) + \frac{1}{2}\big(g_*(x+0)+g_*(x-0)\big) \\ &= 0 + g_*(x) = g(x) \qquad\qquad (0<x<L). \end{aligned}$$

When the initial velocity g is zero, d'Alembert's solution takes on the simpler form

$$u(x,t) = \frac{f_*(x-ct)+f_*(x+ct)}{2},$$

which has an interesting geometric interpretation. For a fixed t, the graph of $f_*(\cdot - ct)$ is just a shifted version of the graph of f_* by ct units to the right. As t increases, the graph travels to the right, representing a travelling wave, moving to the right with a speed c. Similarly the graph of $f_*(\cdot + ct)$ with increasing t represents a travelling wave moving to the left with speed c. And the solution of the wave equation is an average of these two travelling waves moving in opposite directions, and the shape of the wave is determined by the initial shape of the string. Revisit the pictures shown in Example 3.4.

Exercise 3.17. Consider the wave equation with Dirichlet boundary conditions and with initial speed 0:

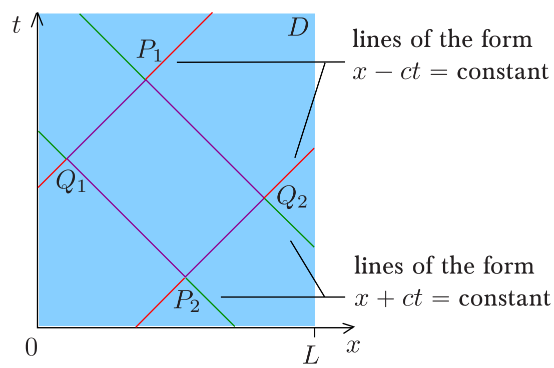
$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0 && (0<x<L,\ t>0), \\ &u(0,t) = u(L,t) = 0 && (t>0), \\ &u(x,0) = f(x) \text{ and } u_t(x,0) = 0 && (0<x<L). \end{aligned}$$

Use the trigonometric identity $2(\sin A)(\cos B) = \sin(A-B)+\sin(A+B)$ to derive d'Alembert's Formula from the solution given by the separation of variables method.

Exercise 3.18 (Characteristic Parallelogram). The lines given by the equations $x - ct$ = constant and $x + ct$ = constant are called ***characteristic lines*** in the domain $D := (0, L) \times (0, \infty) \subset \mathbb{R}^2$ for the wave equation with Dirichlet boundary conditions:

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 && (0 < x < L,\ t > 0),\\
&u(0,t) = u(L,t) = 0 && (t > 0),\\
&u(x,0) = f(x) \text{ and } u_t(x,0) = g(x) && (0 < x < L).
\end{aligned}$$

A parallelogram in the domain D which has sides along characteristic lines is called a ***characteristic parallelogram.***



Show that if (P_1, P_2) and (Q_1, Q_2) are pairs of opposite vertices of a characteristic parallelogram, then the solution u satisfies

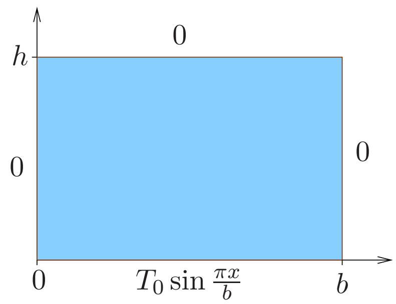
$$u(P_1) + u(P_2) = u(Q_1) + u(Q_2).$$

Hint: With $G(x) := \int_0^x g_*(\xi)d\xi$, we have $\int_{x-ct}^{x+ct} g_*(\xi)d\xi = G(x+ct) - G(x-ct)$.

3.6. Laplace equation

Our next example uses the separation of variables method for Laplace's equation in a rectangle with homogeneous Dirichlet boundary conditions on two opposite sides. Once we know this case, we can handle the general case by superposition.

Example 3.6. Consider the problem of finding the ***steady state / equilibrium*** temperature $u(x, y)$ in a non-insulated building where the temperature outside is $0°$C and the temperature in the ground under the house is given by a sine function, as shown in the cross section of the building:



This is described by

$$\begin{aligned} &\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 && (0 < x < b,\ 0 < y < h), \\ &u(0,y) = u(b,y) = 0 && (0 < y < h), \\ &u(x,0) = T_0 \sin \tfrac{\pi x}{b} \text{ and } u(x,h) = 0 && (0 < x < b). \end{aligned}$$

With

$$u(x,y) = X(x)Y(y),$$

we get from the PDE that $X''Y + XY'' = 0$, and so

$$\frac{X''}{X} = -\frac{Y''}{Y}$$

from which we conclude that their common value must be a constant, say $-\lambda$. Thus we arrive at the equations

$$\begin{aligned} X'' + \lambda X &= 0, \\ Y'' - \lambda Y &= 0. \end{aligned}$$

The boundary conditions for X are $X(0) = X(b) = 0$. In the x-direction, we thus have the same situation as the diffusion equation with homogeneous Dirichlet boundary conditions. We know then that for nontrivial solutions to exist, we must have

$$\lambda = \lambda_n := \left(\frac{n\pi}{b}\right)^2, \quad n = 1, 2, 3, \cdots .$$

The solutions are then given by

$$X_n(x) = \sin \frac{n\pi x}{b}.$$

Next we solve the equation in the y-direction, $Y'' - \lambda Y = 0$, for $\lambda = (n\pi/b)^2$. The general solution is[7]

$$Y_n(y) = a_n e^{\frac{n\pi y}{b}} + b_n e^{-\frac{n\pi y}{b}}.$$

[7] One could equivalently use hyperbolic functions here.

The the temperature distribution can be written in the form

$$u(x,y) = \sum_{n=1}^{\infty} Y_n(y)X_n(x) = \sum_{n=1}^{\infty} \left(a_n e^{\frac{n\pi y}{b}} + b_n e^{-\frac{n\pi y}{b}}\right) \sin \frac{n\pi x}{b}.$$

It remains to choose a_n and b_n so that the conditions at $y = 0$ and $y = h$ are satisfied. These give

$$\begin{aligned} u(x,0) &= T_0 \sin \frac{\pi x}{b} = \sum_{n=1}^{\infty} (a_n + b_n) \sin \frac{n\pi x}{b}, \\ u(x,h) &= 0 = \sum_{n=1}^{\infty} \left(a_n e^{\frac{n\pi h}{b}} + b_n e^{-\frac{n\pi h}{b}}\right) \sin \frac{n\pi x}{b}. \end{aligned}$$

By the uniqueness of coefficients in Fourier expansions, we obtain

$$\begin{aligned} a_1 + b_1 &= T_0, \\ a_n + b_n &= 0, \quad n = 2, 3, 4, \cdots, \\ a_n e^{\frac{n\pi h}{b}} + b_n e^{-\frac{n\pi h}{b}} &= 0, \quad n = 1, 2, 3, \cdots. \end{aligned}$$

From the equations

$$\begin{aligned} a_n + b_n &= 0, \\ a_n e^{\frac{n\pi h}{b}} + b_n e^{-\frac{n\pi h}{b}} &= 0, \end{aligned}$$

for $n = 2, 3, 4, \cdots$, we conclude that

$$a_n = b_n = 0, \quad n = 2, 3, 4, \cdots.$$

On the other hand, from the equations

$$\begin{aligned} a_1 + b_1 &= T_0, \\ a_1 e^{\frac{\pi h}{b}} + b_1 e^{-\frac{\pi h}{b}} &= 0, \end{aligned}$$

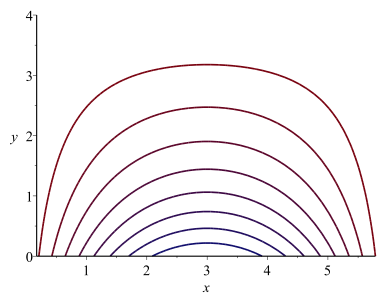
we obtain

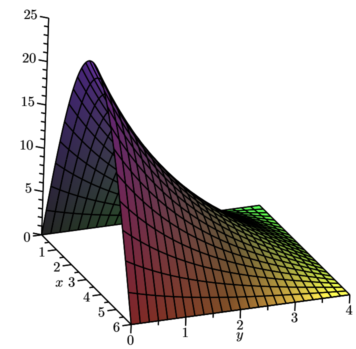
$$a_1 = -T_0 \frac{e^{-\frac{\pi h}{b}}}{e^{\frac{\pi h}{b}} - e^{-\frac{\pi h}{b}}} \quad \text{and} \quad b_1 = T_0 \frac{e^{\frac{\pi h}{b}}}{e^{\frac{\pi h}{b}} - e^{-\frac{\pi h}{b}}}.$$

Substituting these in the series for u, we obtain

$$u(x,y) = T_0 \frac{e^{\frac{\pi(h-y)}{b}} - e^{-\frac{\pi(h-y)}{b}}}{e^{\frac{\pi h}{b}} - e^{-\frac{\pi h}{b}}} \sin \frac{\pi x}{b} = T_0 \frac{\sinh \dfrac{\pi(h-y)}{b}}{\sinh \dfrac{\pi h}{b}} \sin \frac{\pi x}{b}.$$

With the help of this expression, we can determine the "isotherms" (which are level curves of the temperature u). One can also plot the graph of u.





We observe that in homogeneous Dirichlet problems in the x-direction have a solution in the form of a sine series. So one can also solve the problem alternatively by assuming this form for the solution, and calculating the y-dependent coefficients. $\Diamond$

Exercise 3.19. Solve

$$\begin{aligned}
&\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 && (0<x<1,\ 0<y<1),\\
&u(0,y)=u(1,y)=0 && (0<y<1),\\
&u(x,0)=2\sin(\pi x) \text{ and } u(x,1)=-\sin(2\pi x) && (0<x<1).
\end{aligned}$$

3.7. (∗) Inhomogeneous Dirichlet and Neumann problems

So far, we have looked at the separation of variables method for

- homogeneous PDEs, and
- homogeneous boundary conditions.

We shall now make a few remarks to show how one can also handle the cases when we don't have such a situation, and we will do so by just considering a few examples.

3.7.1. Inhomogeneous boundary value problems. As a typical case, we consider the diffusion equation

$$\begin{aligned}
&\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}=0 && (0<x<\pi,\ t>0),\\
&u(0,t)=A \text{ and } u(\pi,t)=B && (t>0),\\
&u(x,0)=f(x) && (0<x<\pi).
\end{aligned}$$

If $A\neq 0$ or $B\neq 0$, then the (Dirichlet) boundary conditions are not homogeneous, and we can't directly use the Fourier method. To handle this, we will bring this to the homogeneous boundary value case by subtracting a time-independent "stationary /

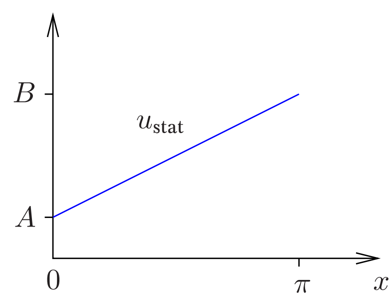
steady-state / equilibrium" temperature distribution, $x \mapsto u_{\text{stat}}(x)$, such that it satisfies the differential equation and the boundary conditions:

$$\begin{aligned} &u''_{\text{stat}}(x) = 0 \quad (0 < x < \pi), \\ &u_{\text{stat}}(0) = A \text{ and } u_{\text{stat}}(\pi) = B. \end{aligned}$$

Thus

$$u_{\text{stat}}(x) = A + \frac{B-A}{\pi} x,$$

as shown in the picture below.



Now set

$$v(x,t) = u(x,t) - u_{\text{stat}}(x).$$

Then

$$\begin{aligned} &\frac{\partial v}{\partial t} = \frac{\partial u}{\partial t}, \quad \frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 u}{\partial x^2}, \\ &v(0,t) = u(0,t) - u_{\text{stat}}(0) = A - A = 0, \\ &v(\pi,t) = u(\pi,t) - u_{\text{stat}}(\pi) = B - B = 0, \\ &v(x,0) = u(x,0) - u_{\text{stat}}(x) = f(x) - u_{\text{stat}}(x) =: \widetilde{f}(x), \end{aligned}$$

and so v satisfies

$$\begin{aligned} &\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = 0 && (0 < x < \pi, \ t > 0), \\ &v(0,t) = 0 \text{ and } v(\pi,t) = 0 && (t > 0), \\ &v(x,0) = \widetilde{f}(x) && (0 < x < \pi). \end{aligned}$$

But this new problem for v has ***homogeneous*** boundary conditions (albeit with a different, but ***known*** initial $\widetilde{f}$). This can be solved as before, and so we can also find u:

$$u(x,t) = u_{\text{stat}}(x) + v(x,t) = u_{\text{stat}}(x) + \sum_{n=1}^{\infty} c_n e^{-n^2 t} \sin(nx).$$

The term $v(x,t)$ depends on the initial condition (as opposed to u_{stat}, which doesn't), and is called the ***transient solution***. For the diffusion equation, the transient solution contains the factors $e^{-n^2 t}$ with $n \geqslant 1$, and goes to 0 as $t \to \infty$. The temperature distribution thus approaches the stationary solution u_{stat} as $t \to \infty$.

Even more generally, one could have boundary conditions that are time dependent, that is,

$$u(0,t) = A(t) \text{ and } u(\pi,t) = B(t) \quad (t > 0).$$

Then there does not exist a time independent solution to the differential equation satisfying the boundary conditions. Nevertheless, the problem can be handled in a similar manner as the above treatment. Define

$$\begin{aligned} \widetilde{u}(x,t) &= A(t) + \frac{B(t)-A(t)}{\pi} x, \\ v(x,t) &= u(x,t) - \widetilde{u}(x,t). \end{aligned}$$

Then the boundary conditions for v ***do*** become homogeneous:

$$\begin{aligned} v(0,t) &= u(0,t) - \widetilde{u}(0,t) = A(t) - A(t) = 0, \\ v(\pi,t) &= u(\pi,t) - \widetilde{u}(\pi,t) = B(t) - B(t) = 0. \end{aligned}$$

However, now the PDE becomes inhomogeneous:

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - \left(\frac{\partial \widetilde{u}}{\partial t} - \frac{\partial^2 \widetilde{u}}{\partial x^2}\right) = 0 - \left(\frac{\partial \widetilde{u}}{\partial t} - 0\right) = -\frac{\partial \widetilde{u}}{\partial t}.$$

How one can solve inhomogeneous equations is discussed in the next subsection.

3.7.2. Inhomogeneous PDEs. Consider again as an illustrative example, the diffusion equation problem

$$\begin{aligned} &\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = w(x,t) \quad (0 < x < \pi,\ t > 0), \\ &u(0,t) = u(\pi,t) = 0 \quad (t > 0), \\ &u(x,0) = f(x) \qquad\qquad (0 < x < \pi). \end{aligned}$$

Owing to the boundary conditions, we expect a solution $u(x,t)$ which for fixed t can be expanded as a sine series. So let us write

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin(nx).$$

The functions w and f can also be expanded into sine series:

$$\begin{aligned} w(x,t) &= \sum_{n=1}^{\infty} w_n(t) \sin(nx), \\ f(x) &= \sum_{n=1}^{\infty} f_n \sin(nx). \end{aligned}$$

Substituting this into the PDE gives us formally that

$$\sum_{n=1}^{\infty} \big(u_n'(t) + n^2 u_n(t)\big) \sin(nx) = \sum_{n=1}^{\infty} w_n(t) \sin(nx).$$

Uniqueness of the Fourier series coefficients gives

$$u_n'(t) + n^2 u_n(t) = w_n(t), \quad n = 1, 2, 3, \cdots .$$

The initial condition gives

$$u_n(0) = f_n, \quad n = 1, 2, 3, \cdots .$$

From the above system of ODEs and with the initial conditions, we can find out $u_n(\cdot)$ for $n = 1, 2, 3, \cdots$. This determines the u as well.

In the case when one has a ***time independent*** right hand side $w = w(x)$, then one can also solve the equation alternatively by subtracting a time independent ***particular solution***

$$u = u_{\text{stat}}(x)$$

which satisfies the differential equation and the boundary conditions. One first find u_{stat} by solving

$$\begin{aligned} -u_{\text{stat}}''(x) &= w(x), \\ u_{\text{stat}}(0) = u_{\text{stat}}(\pi) &= 0. \end{aligned}$$

Next, set

$$v(x,t) := u(x,t) - u_{\text{stat}}(x).$$

Then v satisfies

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - \Big(\frac{\partial u_{\text{stat}}}{\partial t} - \frac{\partial^2 u_{\text{stat}}}{\partial x^2}\Big) = w - (0 + w) = 0,$$

that is, v satisfies the homogeneous equation

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = 0$$

with homogeneous boundary conditions. So using our separation of variables method, we can determine v, and hence also u. Just as in the previous subsection, again the transient part $v(x,t)$ dies out as t increases, and the solution $u(x,t)$ approaches the stationary solution $u_{\text{stat}}(x)$.

This solution method is an application of the well-known principle

$$u = u_{\text{hom}} + u_{\text{part}},$$

with the homogeneous solution $u_{\text{hom}} = v$ and the particular solution $u_{\text{part}} = u_{\text{stat}}$, for solving linear equations.

Exercise 3.20 (Duhamel's Principle). For $s \geqslant 0$, let $(x,t,s) \mapsto v(x,t,s)$ be the solution of the following initial boundary value problem (which depends on the parameter s):

$$\begin{aligned} &v_t - v_{xx} = 0 && (0 < x < L,\ t > s),\\ &v(0,t,s) = 0 = v(L,t,s) && (t \geqslant s),\\ &v(x,s,s) = F(x,s) && (0 \leqslant x \leqslant L). \end{aligned}$$

Prove that u given by $u(x,t) = \displaystyle\int_0^t v(x,t,s)ds$ is a solution of the inhomogeneous problem

$$\begin{aligned} &u_t - u_{xx} = F(x,t) && (0 < x < L,\ t > 0),\\ &u(0,t) = 0 = u(L,t) && (t \geqslant 0),\\ &u(x,0) = 0 && (0 \leqslant x \leqslant L). \end{aligned}$$

Hint: Use Leibniz's Integral Rule saying that if $(x,t) \mapsto \varphi(x,t)$ is a function such that φ_t exists and is continuous, and $t \mapsto a(t), b(t)$ are continuously differentiable, then

$$\frac{d}{dt}\Big(\int_{a(t)}^{b(t)} \varphi(x,t)\,dx\Big) = \int_{a(t)}^{b(t)} \varphi_t(x,t)\,dx + \varphi\big(b(t),t\big)\cdot b'(t) - \varphi\big(a(t),t\big)\cdot a'(t).$$

For a proof of this, see for example [**F**].

3.7.3. Selected additional examples.

Example 3.7 (Nuclear fission reactor). If a nucleus of U^{235} is hit by a neutron with enough speed, a nuclear fission reaction takes place. Besides the energy produced, 2 or 3 new neutrons are also produced. If the conditions for collision are sufficiently good, then an avalanche of neutrons is produced. This is exploited in a nuclear reactor where the chain reaction, once it has started, is controlled by control rods. For starting the chain reaction, the reactor must be larger than a certain minimum size. This can be determined by solving an eigenvalue problem, which can be illustrated by the following simplified, one dimensional model.

Consider a slab of Uranium of thickness L. The concentration of the neutrons we are interested in is described by a diffusion equation model:

$$\frac{\partial u}{\partial t} - a\frac{\partial^2 u}{\partial x^2} = cu,$$

where a, c are positive constants. The function u is the density of neutrons. We assume that the density of neutrons outside the slab of Uranium is 0, and so we arrive at the Dirichlet homogeneous boundary conditions

$$u(0,t) = u(L,t) = 0 \quad (t > 0).$$

As usual, we assume that

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin\frac{n\pi x}{L},$$

which gives $u_n'(t) + a\lambda_n u_n(t) = cu_n(t)$, where $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, that is

$$u_n'(t) = (c - a\lambda_n)u_n(t).$$

Thus

$$u_n(t) = c_n e^{(c-a\lambda_n)t},$$

where the coefficients c_n can be determined from the initial condition. Hence the solution has the form

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{(c-a\lambda_n)t} \sin\frac{n\pi x}{L}.$$

The sign of $c - a\lambda_n$ is significant for the growth of the exponential function for large t. Since λ_1 is the smallest eigenvalue, we have

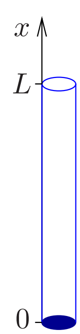
1° (undercritical) if $c < a\lambda_1$, then $u(x,t) \to 0$ as $t \to \infty$,

2° (critical) if $c = a\lambda_1$, then $u(x,t) \to c_1 \sin\frac{\pi x}{L}$ as $t \to \infty$,

3° (overcritical) if $c > a\lambda_1$, then we expect $u(x,t) \to \infty$ as $t \to \infty$.

Since $\lambda_1 = \pi^2/L^2$, we obtain the critical thickness of the slab for starting the chain reaction is $L = \pi\sqrt{a/c}$. $\diamondsuit$

Example 3.8 (Organ pipe). Consider an organ pipe which is closed at one end ($x = 0$) and open at the other ($x = L$) as shown.



The air pressure u fluctuations satisfies the following wave equation and boundary conditions:

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0 \qquad (0 < x < L,\ t > 0),$$

$$u_x(0,t) = 0 \text{ and } u(L,t) = 0 \quad (t > 0).$$

The eigenvalue problem

$$X''(x) + \lambda X(x) = 0, \quad X'(0) = 0, \quad X(L) = 0$$

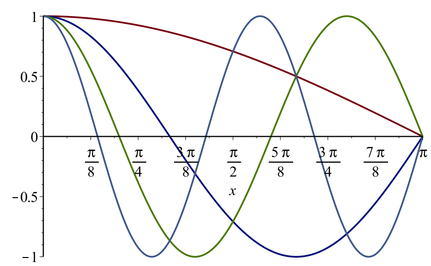
has the solutions

$$X_n(x) = \cos(\beta_n x),$$

where

$$\beta_n = \left(n - \frac{1}{2}\right)\frac{\pi}{L}, \quad n = 1, 2, 3, \cdots,$$

and $\lambda = \lambda_n = \beta_n^2$. The modes / standing waves are shown in the picture below:



Assuming that $u(x,t) = \sum_{n=1}^{\infty} u_n(t) X_n(x)$ gives

$$u_n''(t) + c^2 \beta_n^2 u_n(t) = 0,$$

and so the solution has the form

$$u(x,t) = \sum_{n=1}^{\infty} \big(a_n \cos(c\beta_n t) + b_n \sin(c\beta_n t)\big) \cos(\beta_n x). \qquad \Diamond$$

Exercise 3.21. Solve

$$\begin{aligned} &u_{tt} - u_{xx} = 0 && (0 < x < \pi,\ t > 0), \\ &u_x(0,t) = 0 = u_x(\pi, t) && (t > 0), \\ &u(x,0) = 0 \text{ and } u_t(x,0) = 0 && (0 < x < \pi). \end{aligned}$$

You may use the following Fourier expansion of the 2π-periodic function given by $x \mapsto |x|$ on the interval $(-\pi, \pi)$:

$$|x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\cos\big((2n+1)x\big)}{(2n+1)^2}, \quad x \in (-\pi, \pi).$$

3.8. Proof of the Fourier Series Theorem

We will just prove the following result:

Theorem 3.4 (The Fourier Series Theorem). *Let $f : \mathbb{R} \to \mathbb{R}$ be T-periodic and piecewise smooth, and set $\omega_0 := 2\pi/T$. Then*

$$\frac{f(x+) + f(x-)}{2} = \sum_{n=-\infty}^{\infty} f_n e^{in\omega_0 x} := \lim_{N\to\infty} \sum_{n=-N}^{N} f_n e^{in\omega_0 x} \quad (x \in \mathbb{R}),$$

where f_n are the Fourier coefficients of f, defined by

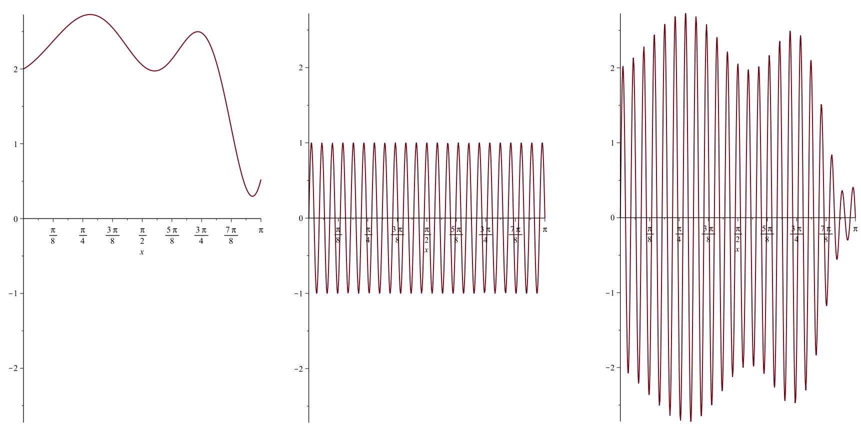
$$f_n = \frac{1}{T}\int_{-T/2}^{T/2} f(x)e^{-in\omega_0 x}dx \quad (n \in \mathbb{Z}).$$

The proof of this result is based on two auxiliary results, which we prove first. To begin with, we observe that the Fourier coefficients tend to 0 as $n \to \pm\infty$, and this is based on the following more general result.

Lemma 3.5 (Riemann-Lebesgue Lemma). *If $f : [a, b] \to \mathbb{R}$ is piecewise smooth, then*

$$\lim_{\omega \to \pm\infty} \int_a^b f(x)e^{i\omega x}dx = 0.$$

The result is intuitively clear, since if our function f is as shown in the leftmost picture, then by multiplying it with a high frequency harmonic $e^{i\omega x} = \cos(\omega x) + i\sin(\omega x)$, we get a function as depicted in the rightmost picture, with a small magnitude of the integral (since the area above the x-axis cancels the parts below the x-axis).



Proof. Suppose first that f is continuously differentiable on $[a, b]$. Then we can use integration by parts to obtain

$$\int_a^b f(x)e^{i\omega x}dx = \frac{1}{i\omega}f(x)e^{i\omega x}\Big|_a^b - \frac{1}{i\omega}\int_a^b f'(x)e^{i\omega x}dx.$$

Since $|e^{i\omega x}| = 1$, we obtain the estimate:

$$\Big|\int_a^b f(x)e^{i\omega x}dx\Big| \leqslant \frac{|f(b)| + |f(a)|}{|\omega|} + \frac{1}{|\omega|}\int_a^b |f'(x)|dx.$$

It is then immediate that the right hand side goes to 0 as $\omega \to \pm\infty$.

If f is just piecewise smooth, we can split $[a, b]$ into subintervals where f is continuously differentiable and repeat the above on the subintervals. □

Here is the next result we will need to prove our Fourier Series Theorem.

Lemma 3.6. *If* $f : [-T/2, T/2] \to \mathbb{R}$ *is piecewise smooth, then*

$$\lim_{a\to\infty} \int_{-T/2}^{T/2} f(x) \frac{\sin(ax)}{x} dx = \pi \frac{f(0+) + f(0-)}{2}.$$

Proof. We will just prove

$$\lim_{a\to\infty} \int_{0}^{T/2} f(x) \frac{\sin(ax)}{x} dx = \pi \frac{f(0+)}{2}.$$

Then by replacing x by $-x$ gives

$$\lim_{a\to\infty} \int_{-T/2}^{0} f(x) \frac{\sin(ax)}{x} dx = \lim_{a\to\infty} \int_{0}^{T/2} f(-x) \frac{\sin(ax)}{x} dx = \pi \frac{f(0-)}{2}.$$

Define

$$\begin{aligned} I(a) &:= \int_0^{T/2} f(x) \frac{\sin(ax)}{x} dx \\ &= \underbrace{\int_0^{T/2} \frac{f(x) - f(0+)}{x} \sin(ax) dx}_{=: I_1(a)} + f(0+) \underbrace{\int_0^{T/2} \frac{\sin(ax)}{x} dx}_{=: I_2(a)}. \end{aligned}$$

We will show that $\lim_{a\to\infty} I_1(a) = 0$ and $\lim_{a\to\infty} I_2(a) = \frac{\pi}{2}$.
To calculate the limit of $I_2(a)$, we make use of the known integral

$$\int_0^\infty \frac{\sin x}{x} = \frac{\pi}{2}.$$

This gives

$$\lim_{a\to\infty} I_2(a) = \lim_{a\to\infty} \int_0^{T/2} \frac{\sin(ax)}{x} dx \overset{(\xi := ax)}{=} \lim_{a\to\infty} \int_0^{aT/2} \frac{\sin \xi}{\xi} d\xi = \frac{\pi}{2}.$$

Now let $\epsilon > 0$. We will show that $|I_1(a)| < \epsilon$ for all large enough a. Since

$$f'(0+) = \lim_{x \searrow 0} \frac{f(x) - f(0+)}{x}$$

exists, the function $\big(f(x) - f(0+)\big)/x$ is bounded on $(0, T/2]$, that is,

$$|f(x) - f(0+)| \leqslant Mx \quad (0 < x \leqslant T/2)$$

for some $M > 0$. Let $\delta > 0$ be such that $\delta < \epsilon/(2M)$ and $\delta < T/2$. Then

$$\Big| \int_0^\delta \frac{f(x) - f(0+)}{x} \sin(ax) dx \Big| \leqslant M \int_0^\delta |\sin(ax)| dx \leqslant M\delta < \frac{\epsilon}{2}.$$

Thus we have

$$\begin{aligned}
|I_1(a)| &= \Big|\int_0^{\delta} \frac{f(x)-f(0+)}{x}\sin(ax)dx + \int_{\delta}^{T/2} \frac{f(x)-f(0+)}{x}\sin(ax)dx\Big| \\
&< \frac{\epsilon}{2} + \Big|\int_{\delta}^{T/2} \frac{f(x)-f(0+)}{x}\sin(ax)dx\Big|.
\end{aligned}$$

But on the interval $[\delta, T/2]$, the function

$$x \mapsto \frac{f(x)-f(0+)}{x}$$

is piecewise smooth (since the only possible singularity is at $x = 0$, and we are away from 0!). The Riemann-Lebesgue Lemma therefore applies, giving

$$\lim_{a\to\infty} \int_{\delta}^{T/2} \frac{f(x)-f(0+)}{x}\sin(ax)dx = 0.$$

So for all sufficiently large a, $|I_1(a)| < \epsilon/2 + \epsilon/2 = \epsilon$, and this completes the proof. □

These two lemmas now allow us to prove the Fourier Series Theorem, and before we give this proof, we make one important observation, which will use. If $z := e^{i\varphi} \neq 1$, where φ is a real number (which is not an integral multiple of 2π), then

$$\begin{aligned}
\sum_{n=-N}^{N} e^{in\varphi} &= \sum_{n=-N}^{N} z^n = z^{-N}\sum_{n=0}^{2N} z^n = z^{-N}\frac{1-z^{2N+1}}{1-z} = \frac{z^{-N}-z^{N+1}}{1-z} \\
&= \frac{z^{-N-\frac{1}{2}} - z^{N+\frac{1}{2}}}{z^{-\frac{1}{2}} - z^{\frac{1}{2}}} = \frac{e^{-i\varphi(N+\frac{1}{2})} - e^{i\varphi(N+\frac{1}{2})}}{e^{-i\frac{\varphi}{2}} - e^{i\frac{\varphi}{2}}} \\
&= \frac{\sin\left((N+\frac{1}{2})\varphi\right)}{\sin\frac{\varphi}{2}}. \qquad (3.4)
\end{aligned}$$

Proof of the Fourier Series Theorem. Let

$$s_N(x) := \sum_{n=-N}^{N} f_n e^{in\omega_0 x}.$$

We need to show that $\lim\limits_{N\to\infty} s_N(x) = \dfrac{f(x+)+f(x-)}{2}$ for all x.

First we will derive an integral representation for s_N by substituting the integral expression for f_n, as follows.

$$\begin{aligned}
s_N(x) &= \sum_{n=-N}^{N} f_n e^{in\omega_0 x} = \sum_{n=-N}^{N} \frac{1}{T}\Big(\int_{-T/2}^{T/2} f(\xi)e^{-in\omega_0\xi}d\xi\Big)e^{in\omega_0 x} \\
&= \frac{1}{T}\int_{-T/2}^{T/2}\Big(f(\xi)\sum_{n=-N}^{N} e^{in\omega_0(x-\xi)}\Big)d\xi \\
&= \frac{1}{T}\int_{-T/2}^{T/2} f(\xi)\frac{\sin\big((N+\frac{1}{2})\omega_0(x-\xi)\big)}{\sin\frac{\omega_0(x-\xi)}{2}}d\xi \quad \text{(using (3.4))} \\
&= \frac{1}{T}\int_{x-T/2}^{x+T/2} f(x-y)\frac{\sin\big((N+\frac{1}{2})\omega_0 y\big)}{\sin\frac{\omega_0 y}{2}}dy \quad \text{(using } y = x-\xi\text{)} \\
&= \frac{1}{T}\int_{-T/2}^{T/2} f(x-y)\frac{\sin((N+\frac{1}{2})\omega_0 y)}{\sin\frac{\omega_0 y}{2}}dy,
\end{aligned}$$

where to obtain the last equality, we have used the T-periodicity of the integrand (check this!). This establishes the integral representation of s_N, namely

$$\begin{aligned}
s_N(x) &= \frac{1}{T}\int_{-T/2}^{T/2} f(x-y)\frac{\sin\big((N+\frac{1}{2})\omega_0 y\big)}{\sin\frac{\omega_0 y}{2}}dy \\
&= \frac{1}{T}\int_{-T/2}^{T/2} \underbrace{f(x-y)\frac{y}{\sin\frac{\omega_0 y}{2}}}_{=:g(y)} \frac{\sin\big((N+\frac{1}{2})\omega_0 y\big)}{y}dy.
\end{aligned}$$

Now we will apply Lemma 3.6. Note that since $\lim\limits_{y\to 0}\dfrac{y}{\sin\frac{\omega_0 y}{2}} = \dfrac{2}{\omega_0}$, we obtain

$$g(0+) = 2\cdot\frac{f(x-)}{\omega_0} \quad \text{and} \quad g(0-) = 2\cdot\frac{f(x+)}{\omega_0}.$$

Thus, using Lemma 3.6, we have

$$\begin{aligned}
\lim_{N\to\infty} s_N(x) &= \lim_{N\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} g(y)\frac{\sin\big((N+\frac{1}{2})\omega_0 y\big)}{y}dy \\
&= \frac{1}{T}\cdot\pi\cdot\frac{g(0+)+g(0-)}{2} = \frac{2\pi}{\omega_0 T}\cdot\frac{f(x+)+f(x-)}{2} \\
&= \frac{f(x+)+f(x-)}{2}.
\end{aligned}$$

This completes the proof. □

Chapter 4

Integral transform methods

We continue our study of initial and boundary value problems for the various types of PDEs we considered in Chapter 3. But while we have so far considered *bounded* domains of space, we now consider ***unbounded domains.***

The method of separation of variables we used in the previous chapter to solve the diffusion equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \quad (x \in I,\ t > 0)$$

with given boundary conditions relied on the fact that I is a bounded interval. We found the solution to be of the form

$$u(x,t) = \sum_k T_k(t) X_k(x),$$

where X_k was a linear combination of sine and cosine functions. These functions X_k appeared as eigenfunctions of a certain differential operator which took into account the boundary conditions.

When we now drop the boundedness of I, for example, taking $I = \mathbb{R}$ in the problem above, then we can no longer expect the solution to be expressed as a series arising from an eigenfunction expansion. We will see, perhaps not totally unexpectedly, that the solution is given by an analogous expression, where the summation index k is replaced by a continuous variable ξ, and the summation sign by an integral:

$$u(x,t) = \int_{\mathbb{R}} T(\xi, t) e^{ix\xi} d\xi.$$

In order to obtain this, we will use the Fourier transform, which is developed in the first section, and subsequently used to solve our PDEs in infinite spatial domains.

4.1. Fourier transform

Just like we used Fourier series to represent a periodic function, there is an analogue of the Fourier series for *non*-periodic functions, called the "Fourier transform". When we had a (nice) periodic function f with a period T, then we could express

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{2\pi n x}{T} + b_n \sin \frac{2\pi n x}{T} \right) = \sum_{n \in \mathbb{Z}} f_n e^{in\frac{2\pi}{T}x},$$

that is, f could be thought of as a 'discrete superposition" of harmonic functions used in the Fourier series, where the harmonic functions all have frequencies which are integral multiples of a fundamental frequency $2\pi/T$. For a (nice) non-periodic function, we will find

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(\xi) e^{i\xi x} d\xi,$$

where $\widehat{f}$ denotes the *Fourier transform* of the function f, defined by

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx \quad (\xi \in \mathbb{R}).$$

So now the function can be thought of as a "continuous superposition" of harmonic functions (as opposed to the discrete superposition met earlier in the context of Fourier series of a periodic function). Note that below, as in our Fourier series results, f is piecewise smooth, and we make the standing assumption that

$$f(x) = \frac{f(x+) + f(x-)}{2}$$

everywhere in order to simplify matters. Sometimes we will also denote Fourier transformation $\widehat{\cdot}$ by $\mathcal{F}$.

Theorem 4.1 (Fourier Integral Theorem). *Let $f : \mathbb{R} \to \mathbb{C}$ be an absolutely integrable function. Then*

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(\xi) e^{i\xi x} d\xi \quad (x \in \mathbb{R}),$$

where $\widehat{f} : \mathbb{R} \to \mathbb{C}$ is the Fourier transform of f, defined by

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx \quad (\xi \in \mathbb{R}).$$

Proof. See the appendix to this chapter, Section 4.7. □

Note that in the above, a function $f : \mathbb{R} \to \mathbb{C}$ is called *absolutely integrable* if

$$\int_{-\infty}^{\infty} |f(x)| dx < \infty.$$

For example, $x \mapsto e^{-x^2}$ is absolutely integrable, while $x \mapsto e^{iax}$ (a real) isn't.

Example 4.1. Let $f : \mathbb{R} \to \mathbb{R}$ be the absolutely integrable function given by

$$f(x) = \mathbf{1}_{[-a,a]}(x) \cdot \cos(\xi_0 x),$$

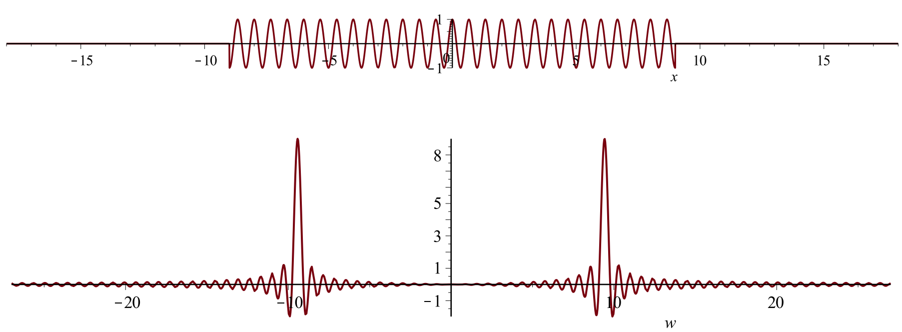
where $\mathbf{1}_{[-a,a]}$ is the indicator function / characteristic function of the interval $[-a, a]$:

$$\mathbf{1}_{[-a,a]}(x) := \begin{cases} 1 & \text{if } x \in [-a, a], \\ 0 & \text{if } x \notin [-a, a]. \end{cases}$$

Then the Fourier transform of f is given by (for $\xi \neq \pm\xi_0$)

$$\begin{aligned}
\widehat{f}(\xi) &= \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx = \int_{-a}^{a} \cos(\xi_0 x) e^{-i\xi x} dx \\
&= \int_{-a}^{a} \frac{e^{i\xi_0 x} + e^{-i\xi_0 x}}{2} e^{-i\xi x} dx = \frac{1}{2}\Big(\int_{-a}^{a} e^{i(\xi_0 - \xi)x} dx + \int_{-a}^{a} e^{-i(\xi_0 + \xi)x} dx \Big) \\
&= \frac{e^{i(\xi_0 - \xi)a} - e^{-i(\xi_0 - \xi)a}}{2i(\xi_0 - \xi)} + \frac{e^{-i(\xi_0 + \xi)a} - e^{i(\xi_0 + \xi)a}}{-2i(\xi_0 + \xi)} \\
&= \frac{\sin\big((\xi_0 - \xi)a\big)}{\xi_0 - \xi} + \frac{\sin\big((\xi_0 + \xi)a\big)}{\xi_0 + \xi}.
\end{aligned}$$

The picture below shows the graphs of f (top) when $a = 9$ and $\xi_0 = 3\pi$, and its Fourier transform $\widehat{f}$ (bottom).



Note the peaks near $\pm 3\pi$, suggesting that the harmonics are concentrated near these frequencies, as intuitively expected (since for a "large" interval, f is just $\cos(3\pi x)$). $\Diamond$

Exercise 4.1. Let $a > 0$. Find the Fourier transform of $\mathbf{1}_{[-a,a]}$.

An important Fourier transform which we will need in the sequel is that of the Gaussian function.

Example 4.2 (Fourier transform of the Gaussian). Let $a > 0$, and consider the Gaussian function

$$f(x) = e^{-ax^2}.$$

We'd like to find its Fourier transform.

Recall that $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$. It'll help in our computation of the Fourier transform.

(With $I := \displaystyle\int_{-\infty}^{\infty} e^{-x^2}dx$, we have

$$\begin{aligned} I^2 &= \Big(\int_{-\infty}^{\infty} e^{-x^2}dx\Big)\Big(\int_{-\infty}^{\infty} e^{-y^2}dy\Big) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}dxdy \\ &= \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\,drd\theta = \pi. \end{aligned}$$

So $I = \displaystyle\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$.)

As f is an even function, we have

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx = 2\int_0^{\infty} e^{-ax^2}\cos(\xi x)dx.$$

Differentiating under the integral sign and using integration by parts yields

$$\begin{aligned} \widehat{f}'(\xi) &= -2\int_0^{\infty} xe^{-ax^2}\sin(\xi x)dx \\ &= \frac{e^{-ax^2}}{a}\sin(\xi x)\Big|_0^{\infty} - \int_0^{\infty}\xi\cos(\xi x)\frac{e^{-ax^2}}{a}dx \\ &= 0 - \frac{\xi}{2a}\widehat{f}(\xi). \end{aligned}$$

So we have obtained a differential equation for $\widehat{f}$! Note that

$$\frac{d}{d\xi}\Big(e^{\frac{\xi^2}{4a}}\widehat{f}(\xi)\Big) = e^{\frac{\xi^2}{4a}}\cdot\frac{2\xi}{4a}\cdot\widehat{f}(\xi) + e^{\frac{\xi^2}{4a}}\cdot\widehat{f}'(\xi) = 0.$$

Hence for all ξ,

$$e^{\frac{\xi^2}{4a}}\widehat{f}(\xi) = 1\widehat{f}(0) = \int_{-\infty}^{\infty} e^{-ax^2}dx \overset{(\xi=\sqrt{a}x)}{=} \int_{-\infty}^{\infty} e^{-\xi^2}\frac{1}{\sqrt{a}}d\xi = \sqrt{\frac{\pi}{a}}.$$

Consequently, $\widehat{f}(\xi) = \sqrt{\dfrac{\pi}{a}}e^{-\frac{\xi^2}{4a}}$. ◊

Remark 4.1.

The Fourier transform of the Gaussian $x \mapsto e^{-ax^2}$ is again a Gaussian, $\xi \mapsto \sqrt{\dfrac{\pi}{a}}e^{-\frac{\xi^2}{4a}}$.

Exercise 4.2. What is the Fourier transform of $e^{-a|x|}$ where $a > 0$? What about $\dfrac{1}{a^2+x^2}$?

The map $f \mapsto \widehat{f}$ is called *Fourier transformation*. It is easily seen to be a linear transformation (for example from the vector space of all absolutely integrable functions on $\mathbb{R}$ to the vector space of all bounded and continuous functions on $\mathbb{R}$, where both vector spaces are equipped with pointwise operations). In our use of Fourier transformation to solve PDEs, we will also need some further properties of $\widehat{\cdot} = \mathcal{F}$, given below.

Lemma 4.2 (Translation). *If $f : \mathbb{R} \to \mathbb{C}$ is absolutely integrable, then for every $a \in \mathbb{R}$, $\mathcal{F}\big(f(\cdot + a)\big) = e^{i\xi a}\mathcal{F}(f)$.*

Proof. We have

$$\begin{aligned} \widehat{f(\cdot + a)}(\xi) &= \int_{-\infty}^{\infty} f(x+a)e^{-i\xi x}dx \\ &= \int_{-\infty}^{\infty} f(u)e^{-i\xi(u-a)}du \quad (x + a = u) \\ &= e^{i\xi a}\int_{-\infty}^{\infty} f(u)e^{-i\xi u}du \\ &= e^{i\xi a} \cdot \widehat{f}(\xi). \end{aligned}$$

□

Exercise 4.3. Show that if $x \mapsto f(x) : \mathbb{R} \to \mathbb{C}$ is absolutely integrable, then for fixed $t \in \mathbb{R}$ and $c \in \mathbb{R}$,

$$\mathcal{F}_x\Big(\frac{f(\cdot + ct) + f(\cdot - ct)}{2}\Big)(\xi) = \widehat{f}(\xi) \cdot \cos(c\xi t) \quad (\xi \in \mathbb{R}).$$

Lemma 4.3 (Differentiation). *If $f, f' : \mathbb{R} \to \mathbb{C}$ are absolutely integrable, then*

$$\widehat{f'} = \big(\xi \mapsto i\xi \widehat{f}(\xi)\big).$$

Proof. Note that by the Fundamental Theorem of Calculus,

$$f(x) = \int_0^x f'(\xi)d\xi + f(0).$$

From here it follows, by using the fact that f' is absolutely integrable, that

$$\lim_{x\to\pm\infty} f(x)$$

exist. Moreover, since f is absolutely integrable on $\mathbb{R}$, it also follows that these limits must equal 0. We have

$$\widehat{f'}(\xi) = \int_{-\infty}^{\infty} f'(x)e^{-i\xi x}dx = f(x)e^{-i\xi x}\Big|_{-\infty}^{\infty} + i\xi\int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx = 0 + i\xi\widehat{f}(\xi).$$

This completes the proof. □

Convolution. A natural question associated with Fourier transformation is this: If f, g are absolutely integrable, then what is the "inverse Fourier transform" of the pointwise product of the Fourier transforms $\widehat{f}$ and $\widehat{g}$? Thus:

$$\begin{array}{ccc} f & \longleftrightarrow & \widehat{f}, \\ g & \longleftrightarrow & \widehat{g}, \\ \boxed{\ ?\ } & \longleftrightarrow & \widehat{f} \cdot \widehat{g}. \end{array}$$

We will learn below that the answer to the question is:

$$\boxed{\text{The convolution } f * g \text{ of } f \text{ and } g.}$$

First, let us define what we mean by the convolution of two functions.

Definition 4.1 (Convolution). If $f, g : \mathbb{R} \to \mathbb{C}$ are two functions, then their ***convolution*** $f * g$ is defined by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x-y)dy \quad (x \in \mathbb{R}).$$

Exercise 4.4. Show that convolution products commute, that is, $f * g = g * f$.

Exercise 4.5. Let $a > 0$ and g be absolutely integrable. Show that

$$\big(\mathbf{1}_{[-a,a]} * g\big)(x) = \int_{x-a}^{x+a} g(y)dy, \quad x \in \mathbb{R}.$$

Sufficient for the existence of of $f*g$ is that f is bounded ("$f \in L^\infty$") and g is absolutely integrable ("$g \in L^1$"), and then $f * g$ is bounded too:

$$\begin{aligned} |(f*g)(x)| &= \Big|\int_{-\infty}^{\infty} f(y)g(x-y)dy\Big| \leqslant \int_{-\infty}^{\infty} |f(y)||g(x-y)|dy \\ &\leqslant \Big(\sup_{y\in\mathbb{R}}|f(y)|\Big)\Big(\int_{-\infty}^{\infty}|g(x-y)|dy\Big) = \underbrace{\Big(\sup_{y\in\mathbb{R}}|f(y)|\Big)}_{=:\|f\|_\infty}\underbrace{\Big(\int_{-\infty}^{\infty}|g(\eta)|d\eta\Big)}_{\|g\|_1}, \end{aligned}$$

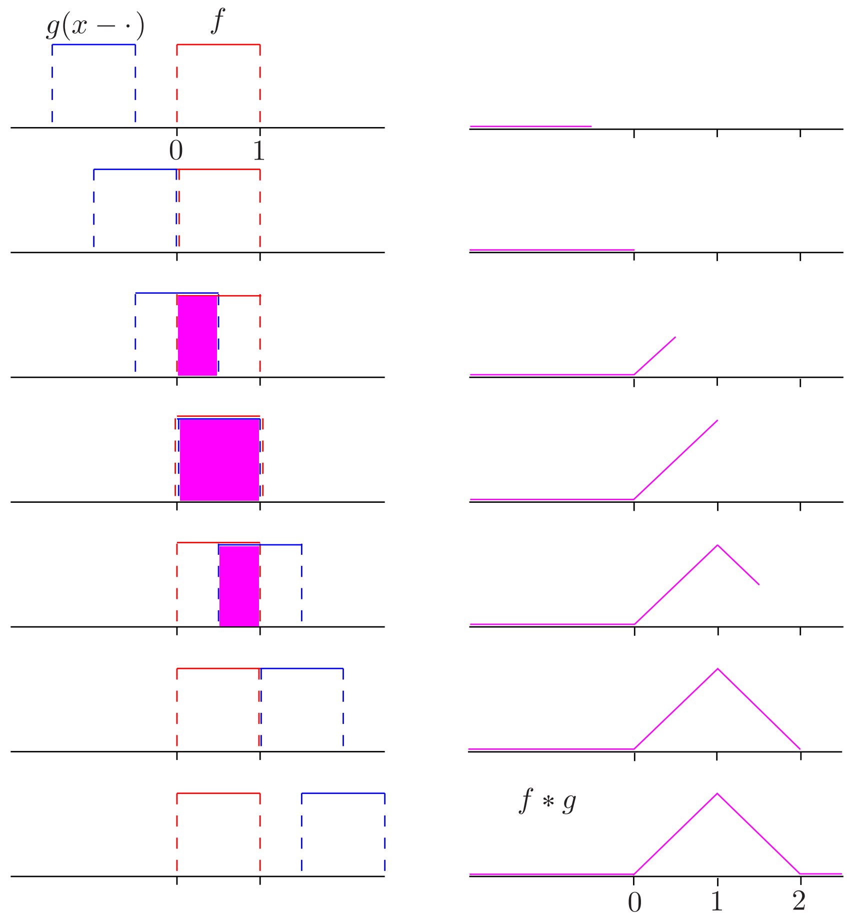
and so $\|f*g\|_\infty \leqslant \|f\|_\infty\|g\|_1$.

Why is the operation $*$ called "convolution"? In ordinary language, one of the meanings of convolution is "folding", and this is the origin of the mathematical terminology. Indeed, g is first reflected about the y-axis, then translated (by x), and then this resulting function is multiplied pointwise by f, and finally the result is integrated in order to obtain $f * g$ at x. (The "folding" part is when g is reflected, and the reflected function is "overlapped" with the graph of f.) The best way to understand this convoluted explanation is to see an example and the associated pictures.

Example 4.3. Let us find out the convolution $\mathbf{1}_{[0,1]} * \mathbf{1}_{[0,1]}$, where $\mathbf{1}_{[0,1]}$ is the indicator function of the interval $[0,1]$:

$$\mathbf{1}_{[0,1]}(x) = \begin{cases} 1 & \text{if } x \in [0,1], \\ 0 & \text{if } x \in \mathbb{R}\backslash[0,1]. \end{cases}$$

We have $(\mathbf{1}_{[0,1]} * \mathbf{1}_{[0,1]})(x) = \begin{cases} 0 & \text{if } x \leqslant 0, \\ x & \text{if } 0 \leqslant x \leqslant 1, \\ 2-x & \text{if } 1 \leqslant x \leqslant 2, \\ 0 & \text{if } x \geqslant 2. \end{cases}$



$\diamond$

Theorem 4.4 (Convolution Theorem). *If $f, g : \mathbb{R} \to \mathbb{C}$ are absolutely integrable, then $f * g$ is also absolutely integrable, and*

$$\widehat{f * g} = \widehat{f} \cdot \widehat{g}.$$

Proof. We have

$$\begin{aligned}
\|f * g\|_1 &= \int_{-\infty}^{\infty} |(f*g)(x)| dx = \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} f(y) g(x-y) dy \right| dx \\
&\leqslant \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(y) g(x-y)| dy dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(y)||g(x-y)| dx dy \\
&= \int_{-\infty}^{\infty} |f(y)| \int_{-\infty}^{\infty} |g(x-y)| dx dy = \int_{-\infty}^{\infty} |f(y)| \int_{-\infty}^{\infty} |g(u)| du dy \\
&= \left(\int_{-\infty}^{\infty} |f(y)| dy \right) \left(\int_{-\infty}^{\infty} |g(u)| du \right) = \|f\|_1 \|g\|_1.
\end{aligned}$$

So $f * g$ is absolutely integrable too. Moreover,

$$\begin{aligned}
\widehat{f * g}(\xi) &= \int_{-\infty}^{\infty} (f * g)(x) e^{-i\xi x} dx = \int_{-\infty}^{\infty} \Big(\int_{-\infty}^{\infty} f(y) g(x-y) dy \Big) e^{-i\xi x} dx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y) g(x-y) e^{-i\xi x} dy dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y) g(x-y) e^{-i\xi x} dx dy \\
&= \int_{-\infty}^{\infty} f(y) \Big(\int_{-\infty}^{\infty} g(x-y) e^{-i\xi x} dx \Big) dy \\
&= \int_{-\infty}^{\infty} f(y) \Big(\int_{-\infty}^{\infty} g(u) e^{-i\xi(u+y)} du \Big) dy \\
&= \int_{-\infty}^{\infty} f(y) e^{-i\xi y} \widehat{g}(\xi) dy = \widehat{g}(\xi) \int_{-\infty}^{\infty} f(y) e^{-i\xi y} dy = \widehat{g}(\xi) \widehat{f}(\xi).
\end{aligned}$$

This completes the proof. □

Exercise 4.6. Give a slicker proof of the commutativity of the convolution product for absolutely integrable functions (as compared with the solution to Exercise 4.4), using the Convolution Theorem.

Exercise 4.7. Let $a > 0$ and g be absolutely integrable. Show that

$$\Big(\mathcal{F}\Big(\int_{x-a}^{x+a} g(y) dy \Big) \Big)(\xi) = \begin{cases} 2\widehat{g}(\xi) \cdot \dfrac{\sin(\xi a)}{\xi} & \text{if } \xi \neq 0, \\ 2a\widehat{g}(\xi) & \text{if } \xi = 0. \end{cases}$$

Hint: Use Exercises 4.1 and 4.5.

4.2. The Diffusion equation

Consider the diffusion equation describing the temperature in an infinite rod:

$$\begin{aligned}
&\frac{\partial u}{\partial t} - a \frac{\partial^2 u}{\partial x^2} = 0 && (x \in \mathbb{R},\ t > 0), \\
&u(x,t) \to 0 \text{ as } t \to \infty && (x \in \mathbb{R}), \\
&u(x,0) = f(x) && (x \in \mathbb{R}).
\end{aligned}$$

To begin with we will calculate formally, and obtain a candidate expression for the solution involving the initial condition f. Then we will impose reasonable conditions on f in order to convince ourselves that the expression we have obtained is meaningful, and does solve the problem. Knowing uniqueness, we can then conclude that this is *the* solution.

A partial Fourier transformation in the x-direction in the PDE gives

$$\int_{-\infty}^{\infty} e^{-ix\xi} \frac{\partial u}{\partial t} dx - a \int_{-\infty}^{\infty} e^{-ix\xi} \frac{\partial^2 u}{\partial x^2} dx = 0. \tag{4.1}$$

In our formal calculations, we assume that it is allowed to interchange the order of differentiation in the first term, that is,

$$\int_{-\infty}^{\infty} e^{-ix\xi}\frac{\partial u}{\partial t}dx = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} e^{-ix\xi}u(x,t)dx = \frac{\partial}{\partial t}\widehat{u}(\xi,t),$$

where

$$\widehat{u}(\xi,t) := (\mathcal{F}_x u)(\xi,t) := \int_{-\infty}^{\infty} e^{-ix\xi}u(x,t)dx.$$

For the other term in (4.1), we use the derivative rule for the Fourier transform, giving us

$$\int_{-\infty}^{\infty} e^{-ix\xi}\frac{\partial^2 u}{\partial x^2}dx = \big(\mathcal{F}_x(u_{xx}(\cdot,t))\big)(\xi) = (i\xi)^2\widehat{u}(\xi,t) = -\xi^2\widehat{u}(\xi,t).$$

Substituting all this in (4.1), yields the following ODE for $\widehat{u}(\xi,\cdot)$:

$$\frac{\partial\widehat{u}}{\partial t}(\xi,t) + a\xi^2\widehat{u}(\xi,t) = 0.$$

The general solution is

$$\widehat{u}(\xi,t) = c(\xi)e^{-a\xi^2 t}.$$

Note that the "constant" c depends on ξ, that is, $c = c(\xi)$. To determine $c(\xi)$, we put $t = 0$, and obtain

$$c(\xi) = \widehat{u}(\xi,0) = \int_{-\infty}^{\infty} e^{-ix\xi}u(x,0)dx = \int_{-\infty}^{\infty} e^{-ix\xi}f(x)dx = \widehat{f}(x).$$

Thus

$$\widehat{u}(\xi,t) = \widehat{f}(x)e^{-a\xi^2 t}.$$

Now we know from Example 4.2 that

$$e^{-\alpha x^2} \overset{\mathcal{F}_x}{\longmapsto} \sqrt{\frac{\pi}{\alpha}}e^{-\frac{\xi^2}{4\alpha}}$$

and so it follows that

$$e^{-at\xi^2} \overset{\mathcal{F}_x^{-1}}{\longmapsto} \frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{4at}}e^{-x^2/4at} =: G(x,t).$$

Since $\widehat{u}$ is the product of $\widehat{f}$ and $\widehat{G(\cdot,t)}$, it follows that u is the convolution of f and $G(\cdot,t)$:

$$u(x,t) = \big(G(\cdot,t)*f\big)(x) = \frac{1}{\sqrt{4\pi at}}\int_{-\infty}^{\infty} e^{-(x-y)^2/4at}f(y)dy.$$

Remark 4.2. So far, we did not say anything about the initial condition f. We now make some remarks on how properties of f affect the u, and ensure that u solves our problem.

Since G is rapidly decreasing as $x \to \pm\infty$, the convolution integral converges if for example f is absolutely integrable. One can then retrace the steps above, including the exchange of the derivative with the integral, and find that u satisfies the initial value problem. It can also be shown that $u(x,t) \to 0$ as $t \to \infty$. The uniqueness theorem then ensures that this is the only solution.

Another consequence of the rapid decay of the G and its regularity is that the convolution integral above can be differentiated under the integral sign as many times as one wishes, either with respect to x or with respect to t. Thus without demanding any regularity on the initial condition function f (not even continuity!), we nevertheless get an ***infinitely differentiable*** u for all $t > 0$!

Even more generally, one may assume that f is an element "of the Schwartz class of tempered distributions" $\mathcal{S}'(\mathbb{R})$. We will mention these distributions again at the end of the next chapter on Distributions. Also, one can consider initial conditions that are not in $\mathcal{S}'(\mathbb{R})$, but for some $k > 0$, $e^{-kx}f(x)$ is absolutely integrable, and then one may use the Laplace transform to solve the initial value problem. We will see this later.

Before we consider a concrete example of an initial condition f, we mention the following standard notation, which will be useful in our calculations:

$$\begin{aligned} \operatorname{erf} x &:= \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy \quad &&\text{(error function)}, \\ \operatorname{erfc} x &:= 1 - \operatorname{erf} x &&\text{(complementary error function)}. \end{aligned}$$

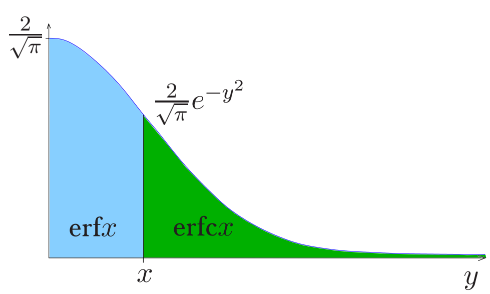
Since

$$\int_0^\infty e^{-y^2} dy = \frac{\sqrt{\pi}}{2},$$

we have erf $x \to 1$ as $x \to \infty$, and

$$\operatorname{erfc} x = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-y^2} dy.$$

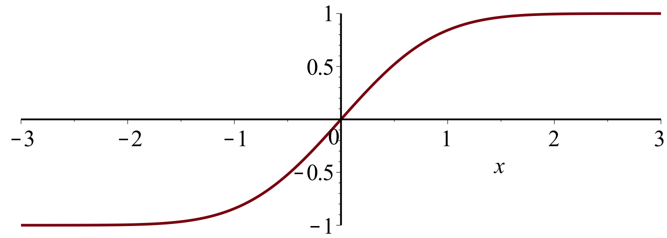
The following picture illustrates this:



By definition, erf x is a primitive function of $\frac{2}{\sqrt{\pi}}e^{-x^2}$. Moreover, erf is odd, and

$$\lim_{x \to -\infty} \operatorname{erf}(x) = -1.$$

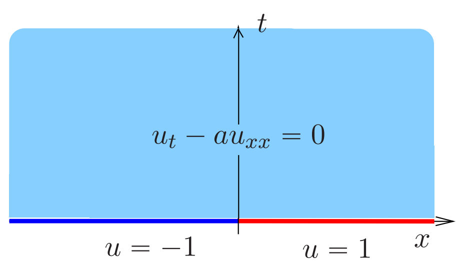
The picture below shows the graph of erf.



Example 4.4. Let the initial condition function f be given by

$$f(x) = \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x < 0. \end{cases}$$

(Imagine two infinite rods, at uniform temperatures of $1°$C and $-1°$C joined at one end at time $t = 0$.)



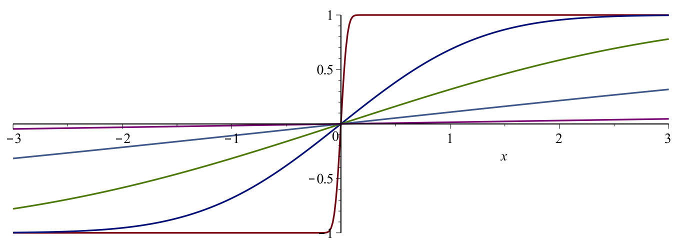
Then we have the initial boundary value problem

$$\begin{aligned} \frac{\partial u}{\partial t} - a\frac{\partial^2 u}{\partial x^2} &= 0 && (x \in \mathbb{R},\ t > 0), \\ u(x,t) &\to 0 \text{ as } t \to \infty && (x \in \mathbb{R}), \\ u(x,0) &= f(x) && (x \in \mathbb{R}). \end{aligned}$$

The solution is given by

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{4\pi at}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4at} f(y)dy. \\ &= \frac{1}{\sqrt{4\pi at}} \Big(-\int_{-\infty}^{0} e^{-(x-y)^2/4at} dy + \int_{0}^{\infty} e^{-(x-y)^2/4at} dy \Big) \\ &= -\frac{1}{\sqrt{\pi}} \int_{x/\sqrt{4at}}^{\infty} e^{-\xi^2} d\xi + \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x/\sqrt{4at}} e^{-\xi^2} d\xi \quad (\text{with } \xi = \frac{x-y}{\sqrt{4at}}) \\ &= -\Big(\frac{1}{2}\text{erf}(\xi)\Big)\Big|_{x/\sqrt{4at}}^{\infty} + \Big(\frac{1}{2}\text{erf}(\xi)\Big)\Big|_{-\infty}^{x/\sqrt{4at}} \\ &= \text{erf}\Big(\frac{x}{\sqrt{4at}}\Big). \end{aligned}$$

The solution spatial profiles at various instances of t are depicted below:



Note that despite the discontinuity at $x = 0$ in the initial condition f, the solution looks very smooth for $t > 0$. $\Diamond$

Remark 4.3 (Infinite speed of propagation). We remark that a consequence of the convolution formula is that the initial condition at $x = 0$ affects u at *every* x for *every* $t > 0$. Thus the diffusion equation model implies an infinite speed of propagation from the initial condition. (An extreme illustration of this in the diffusion of matter case is this: if you put a drop of ink in a lake, then it immediately spreads to the opposite shore!) Clearly, an infinite propagation is not physically realistic, and this suggests that the model is a bit too simplified. So one should bear in mind that the model won't give a good description of physical reality for very tiny values of t.

Exercise 4.8 (Fourier transform method with mixed derivatives). Solve the initial value problem

$$\begin{aligned} \frac{\partial u}{\partial t \partial x} &= \frac{\partial^2 u}{\partial x^2} \qquad (x \in \mathbb{R},\ t > 0), \\ u(x,0) &= \frac{1}{1+x^2} \qquad (x \in \mathbb{R}). \end{aligned}$$

Plot the graphs of $u(\cdot, t)$ at $t = 0, 1, 2, 3$.

Exercise 4.9 (An equation with nonconstant coefficients). Solve

$$\begin{aligned} t\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} &= 0 \quad (x \in \mathbb{R},\ t > 0), \\ u(x,0) &= f(x) \quad (x \in \mathbb{R}) \end{aligned}$$

formally using the Fourier transform method. Assuming that $f \in C^1$, check that the obtained solution does satisfy the initial value problem.

Exercise 4.10 (Tikhonov's example, 1935). Let f be defined by

$$f(t) = \begin{cases} e^{-1/t^2} & \text{if } t > 0, \\ 0 & \text{if } t \geqslant 0. \end{cases}$$

Then it can be shown[1] that $f \in C^\infty$. New set

$$u(x,t) := \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} f^{(k)}(t) \quad (x \in \mathbb{R}, t \in \mathbb{R}).$$

[1]See for example, [**S**, p.335 and p. 490-492].

Check formally that this u satisfies the diffusion equation, that is, $u_t = u_{xx}$ ($x \in \mathbb{R}, t \in \mathbb{R}$).

(In fact it can be shown rigorously that $f \in C^\infty(\mathbb{R}^2)$ and that it satisfies the diffusion equation; see for example, [**H**, Example 2, p.50].)

Remark: The function $u \equiv 0$ is clearly a solution, to the initial value problem with $u(x, 0) = 0$ ($x \in \mathbb{R}$), and the above series gives a new nonzero solution with the same initial condition. So the uniqueness is violated! However, it can be shown that it is not the case that

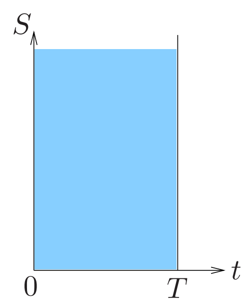
$$\text{for all } x \in \mathbb{R},\ u(x, t) \to 0 \text{ as } t \to \infty.$$

So this example shows the relevance of this condition; see Remark 2.2.(2) on page 44.

4.3. Application: European option pricing

In the sub-world of Finance in Economics, there are "financial instruments" called options. An ***option*** is the right, but not the obligation, to buy or sell a stock for an agreed upon price at some time in the future. The agreed upon price is called the ***strike price*** K. Options come with a time limit at which they must be exercised, called the ***expiry date*** T. An option to buy is called a ***call option***, while one to sell is called a ***put option***. A fundamental question in Finance is then:

What is a (fair) price V of the option?



It is assumed that the reader has met (or accepts on faith) the considerations behind obtaining the Black-Scholes equation for the pricing of European options:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \quad S \geqslant 0,\ t \in [0, T],$$

where S, t are independent variables, and

$V(S, t)$ is the value of the option at stock price S and time t,
T is the expiry date of the option,
σ is the volatility of the asset,
r is the risk-free interest rate.

T, σ, r are known, and the unknown is $V(S, t)$ for $S \geqslant 0$ and for $t \in [0, T]$.

We note that the Black-Scholes equation is a second order, linear, parabolic, homogeneous, non-constant coefficient PDE, and we will solve it by reducing it to a diffusion equation, and using the Fourier transform method.

4.3.1. Reduction to the diffusion equation. If K is the strike price of the option, then we define

$$\begin{aligned} &S =: Ke^x \text{ or } x := \log \frac{S}{K}, \\ &\tau := \frac{(T-t)\sigma^2}{2}, \\ &V(S,t) =: Kv(x,\tau) \text{ or } v(x,\tau) := \frac{V(S,t)}{K}. \end{aligned}$$

Then

$$\begin{aligned} \frac{\partial V}{\partial t} &= K\frac{\partial v}{\partial \tau}\frac{\partial \tau}{\partial t} = K\frac{\partial v}{\partial \tau} \cdot \Big(-\frac{\sigma^2}{2}\Big) = -\frac{K\sigma^2}{2}\frac{\partial v}{\partial \tau}, \\ \frac{\partial V}{\partial S} &= K\frac{\partial v}{\partial x}\frac{\partial x}{\partial S} = K\frac{\partial v}{\partial x} \cdot \frac{1}{S/K} \cdot \frac{1}{K} = \frac{K}{S}\frac{\partial v}{\partial x}, \\ \frac{\partial^2 V}{\partial S^2} &= -\frac{K}{S^2}\frac{\partial v}{\partial x} + \frac{K}{S}\frac{\partial^2 v}{\partial x^2}\frac{\partial x}{\partial S} = -\frac{K}{S^2}\frac{\partial v}{\partial x} + \frac{K}{S}\frac{\partial^2 v}{\partial x^2}\frac{1}{S/K} \cdot \frac{1}{K} \\ &= -\frac{K}{S^2}\frac{\partial v}{\partial x} + \frac{K}{S^2}\frac{\partial^2 v}{\partial x^2}. \end{aligned}$$

By substituting these in the Black-Scholes equation, we obtain

$$-\frac{K\sigma^2}{2}\frac{\partial v}{\partial \tau} + \frac{\sigma^2 S^2}{2} \cdot \frac{K}{S^2}\Big(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}\Big) + rS\Big(\frac{K}{S}\frac{\partial v}{\partial x}\Big) - rKv = 0,$$

that is,

$$\begin{aligned} 0 &= -\frac{\sigma^2}{2}\frac{\partial v}{\partial \tau} + \frac{\sigma^2}{2}\Big(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}\Big) + r\frac{\partial v}{\partial x} - rv \\ &= -\frac{\sigma^2}{2}\bigg(\frac{\partial v}{\partial \tau} - \frac{\partial^2 v}{\partial x^2} + \Big(1 - \frac{2r}{\sigma^2}\Big)\frac{\partial v}{\partial x} + \frac{2r}{\sigma^2}v\bigg). \end{aligned}$$

Consequently, with $a := \dfrac{2r}{\sigma^2} - 1$, we obtain

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + a\frac{\partial v}{\partial x} - (a+1)v.$$

Now set

$$u(x,t) := e^{(\frac{a^2}{4}+a+1)\tau} e^{\frac{a}{2}x} v(x,\tau).$$

Then

$$\begin{aligned} \frac{\partial u}{\partial \tau} &= \Big(\frac{a^2}{4} + a + 1\Big)u + e^{(\frac{a^2}{4}+a+1)\tau} e^{\frac{a}{2}x}\frac{\partial v}{\partial \tau}, \\ \frac{\partial v}{\partial x} &= \frac{a}{2}u + e^{(\frac{a^2}{4}+a+1)\tau} e^{\frac{a}{2}x}\frac{\partial v}{\partial x}, \end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2 v}{\partial x^2} &= \frac{a}{2}\Big(\frac{a}{2}u + e^{(\frac{a^2}{4}+a+1)\tau} e^{\frac{a}{2}x}\frac{\partial v}{\partial x}\Big) + \frac{a}{2}e^{(\frac{a^2}{4}+a+1)\tau} e^{\frac{a}{2}x}\frac{\partial v}{\partial x} + e^{(\frac{a^2}{4}+a+1)\tau} e^{\frac{a}{2}x}\frac{\partial^2 v}{\partial x^2} \\
&= \frac{a^2}{4}u + ae^{(\frac{a^2}{4}+a+1)\tau} e^{\frac{a}{2}x}\frac{\partial v}{\partial x} + e^{(\frac{a^2}{4}+a+1)\tau} e^{\frac{a}{2}x}\frac{\partial^2 v}{\partial x^2} \\
&= \frac{a^2}{4}u + e^{(\frac{a^2}{4}+a+1)\tau} e^{\frac{a}{2}x}\Big(a\frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2}\Big) \\
&= \frac{a^2}{4}u + e^{(\frac{a^2}{4}+a+1)\tau} e^{\frac{a}{2}x}\Big(\frac{\partial v}{\partial \tau} + (1+a)v\Big) \\
&= \Big(\frac{a^2}{4} + a + 1\Big)u + e^{(\frac{a^2}{4}+a+1)\tau} e^{\frac{a}{2}x}\frac{\partial v}{\partial \tau} \\
&= \frac{\partial u}{\partial \tau}.
\end{aligned}$$

Thus

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2},$$

where

$$\begin{aligned}
S &= Ke^x, \\
\tau &= \frac{(T-t)\sigma^2}{2}, \\
V(S,t) &= Ke^{-(\frac{a^2}{4}+a+1)\tau} e^{-\frac{a}{2}x}u(x,\tau), \\
a &= \frac{2r}{\sigma^2} - 1.
\end{aligned}$$

4.3.2. Solution to the Black-Scholes equation. The payoff at expiry for European options is given by

$$V(S,T) = \begin{cases} \max\{S-K, 0\} & \text{for a call option,} \\ \max\{K-S, 0\} & \text{for a put option.} \end{cases}$$

Let us solve the Black-Scholes equation for a call option. (The computation is analogous for a put option.) Note that when $t = T$, $\tau = 0$. Thus

$$\begin{aligned}
u(x,0) &= \frac{1}{K}e^{\frac{a}{2}x}V(S,T) \\
&= \frac{1}{K}e^{\frac{a}{2}x}\max\{S-K,0\} \\
&= \frac{1}{K}e^{\frac{a}{2}x}\max\{Ke^x-K,0\} \\
&= \max\{e^{(\frac{a}{2}+1)x} - e^{\frac{a}{2}x}, 0\}.
\end{aligned}$$

By substituting this in the solution to the diffusion equation for u, we obtain

$$\begin{aligned} u(x,\tau) &= \frac{1}{\sqrt{4\pi\tau}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4\tau}} u(y,0)dy \\ &= \frac{1}{\sqrt{4\pi\tau}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4\tau}} \max\{e^{(\frac{a}{2}+1)y} - e^{\frac{a}{2}y}, 0\}dy. \end{aligned}$$

We also note that $e^{(\frac{a}{2}+1)y} - e^{\frac{a}{2}y} \begin{cases} \geqslant 0 & \text{if } y \geqslant 0, \\ < 0 & \text{if } y < 0. \end{cases}$

Thus

$$u(x,\tau) = \frac{1}{\sqrt{4\pi\tau}}\int_{0}^{\infty} e^{-\frac{(x-y)^2}{4\tau}} \big(e^{(\frac{a}{2}+1)y} - e^{\frac{a}{2}y}\big)dy.$$

Let Φ denote the normal cumulative distribution function, defined by

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}dy.$$

Then using the change of variable $\eta = \dfrac{x+2\tau\alpha - y}{\sqrt{2\tau}}$, it can be seen that

$$\frac{1}{\sqrt{4\pi\tau}}\int_{0}^{\infty} e^{-\frac{(x-y)^2}{4\tau}} e^{\alpha y}dy = e^{\alpha(x+\alpha\tau)}\Phi\Big(\frac{x+2\tau\alpha}{\sqrt{2\tau}}\Big).$$

Hence $u(x,\tau) = e^{(\frac{a}{2}+1)(x+\frac{a\tau}{2}+\tau)}\Phi\Big(\dfrac{x+\tau a+2\tau}{\sqrt{2\tau}}\Big) - e^{\frac{a}{2}(x+\frac{a\tau}{2})}\Phi\Big(\dfrac{x+\tau a}{\sqrt{2\tau}}\Big)$. Finally,

$$\begin{aligned} V(S,t) &= Ke^{-(\frac{a^2}{4}+a+1)\tau}e^{-\frac{a}{2}x}u(x,\tau) \\ &= Ke^{x}\Phi\Big(\frac{x+\tau a+2\tau}{\sqrt{2\tau}}\Big) - Ke^{-(a+1)\tau}\Phi\Big(\frac{x+\tau a}{\sqrt{2\tau}}\Big) \\ &= K\frac{S}{K}\Phi\Big(\underbrace{\frac{\log(S/K)+(T-t)(r+\frac{\sigma^2}{2})}{\sigma\sqrt{T-t}}}_{=:d_1}\Big) \\ &\quad -Ke^{-r(T-t)}\Phi\Big(\underbrace{\frac{\log(S/K)+(T-t)(r-\frac{\sigma^2}{2})}{\sigma\sqrt{T-t}}}_{=:d_2}\Big), \end{aligned}$$

that is,

$$V(S,t) = S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2), \tag{4.2}$$

where

$$\begin{aligned} d_1 &= \frac{\log \frac{S}{K} + (T-t)(r + \frac{\sigma^2}{2})}{\sigma\sqrt{T-t}}, \\ d_2 &= \frac{\log \frac{S}{K} + (T-t)(r - \frac{\sigma^2}{2})}{\sigma\sqrt{T-t}}, \\ \Phi(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy. \end{aligned}$$

Example 4.5 (A numerical example). Suppose that the current price of a security is £62 per share. The continuously compounded interest rate is 10% per year. The volatility of the price of the security is $\sigma = 20\%$ per year. The cost of a five-month European call option with a strike price of £60 per share can be found using the formula (4.2). We have

$$S = 62, \quad t = 0, \quad r = 0.1, \quad \sigma = 0.2, \quad T = \frac{5}{12}, \quad K = 60.$$

Then

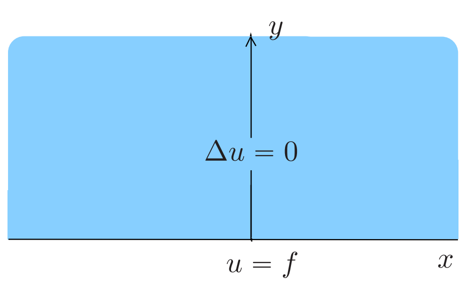
$$\begin{aligned} d_1 &= \frac{\log \frac{62}{60} + (\frac{5}{12} - 0)(0.1 + \frac{0.1^2}{2})}{0.2\sqrt{\frac{5}{12} - 0}} \approx 0.6413, \\ d_2 &= \frac{\log \frac{62}{60} + (\frac{5}{12} - 0)(0.1 - \frac{0.1^2}{2})}{0.2\sqrt{\frac{5}{12} - 0}} \approx 0.5122. \end{aligned}$$

Using $\Phi(0.6413) \approx 0.7393$ and $\Phi(0.5122) \approx 0.6957$, we obtain $V \approx 5.7981$, that is, the price of the call option is £5.7981. ◊

4.4. Dirichlet's problem for a half plane

We determine *bounded* solutions to

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 \quad (y > 0, \ x \in \mathbb{R}), \\ u(x, 0) = f(x). \end{cases}$$



In the same manner as with the Diffusion Equation, we begin with formal calculations, and postpone the discussion of the assumptions on f. A Fourier transformation in the x-direction in the Laplace equation gives the differential equation

$$(i\xi)^2 \widehat{u}(\xi, y) + \frac{\partial^2 \widehat{u}}{\partial y^2}(\xi, y) = 0,$$

where $\widehat{u} = \mathcal{F}_x u(\cdot, y)$ denotes the Fourier transform of $x \mapsto u(x, y)$. This differential equation has the general solution

$$\widehat{u}(\xi, y) = a(\xi)e^{\xi y} + b(\xi)e^{-\xi y},$$

where the "constants" a, b depend on ξ. The boundary condition gives

$$\widehat{u}(\xi, 0) = \widehat{f}(\xi).$$

Setting $y = 0$ in the expression for $\widehat{u}(\xi, y)$ gives

$$a(\xi) + b(\xi) = \widehat{f}(\xi).$$

So we get a constraint on a, b, but this is not enough to determine these functions. We now use the condition that u is bounded. To do this, note that if there is $\xi > 0$ for which $a(\xi) \neq 0$, then $|a(\xi)e^{\xi y}| \to \infty$ as $y \to \infty$. Similarly, we see that if there is $\xi < 0$ for which $b(\xi) \neq 0$, then $|b(\xi)e^{-\xi y}| \to \infty$ as $y \to \infty$. We cannot expect u to be bounded if $\widehat{u}$ is unbounded in this manner. So we take a and b to be such that

$$\begin{cases} a(\xi) = 0 & \text{if } y > 0, \\ b(\xi) = 0 & \text{if } y < 0 \end{cases}$$

Together with the constraint $a(\xi) + b(\xi) = \widehat{f}(\xi)$, we now obtain that

$$a(\xi) = \begin{cases} 0 & \text{if } y > 0, \\ \widehat{f}(\xi) & \text{if } y < 0. \end{cases} \quad \text{and} \quad b(\xi) = \begin{cases} \widehat{f}(\xi) & \text{if } y > 0, \\ 0 & \text{if } y < 0. \end{cases}$$

Thus

$$\widehat{u}(\xi, y) = \widehat{f}(\xi)e^{-|\xi| y}.$$

From Exercise 4.2 it follows that $e^{-|\xi| y} \overset{\mathcal{F}_x^{-1}}{\longmapsto} \dfrac{1}{\pi} \dfrac{y}{x^2 + y^2} =: P(x, y)$.

Using the Convolution Theorem we now obtain that

$$u(x, y) = (P * f)(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(x - \xi)^2 + y^2} f(\xi) d\xi.$$

Remark 4.4. Just as with the diffusion equation, one can check that the integral is convergent if f is for example absolutely integrable or bounded, and that the calculations can be traced backwards, so that we do have a legitimate solution to our problem. Moreover, with the uniqueness theorem, it follows that this is the only possible solution. The function P above is called the *Poisson kernel*.

We remark that without the boundedness assumption, the solution is not unique, for example, we may simply add y to u to obtain a new solution with the same Dirichlet boundary condition!

Example 4.6. Suppose that the initial condition for the Dirichlet problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 \quad (y > 0,\ x \in \mathbb{R}), \\ u(x,0) = f(x). \end{cases}$$

is given by

$$f(x) = \begin{cases} T_0 & \text{if } -a \leqslant x \leqslant a, \\ 0 & \text{otherwise.} \end{cases}$$

Then the bounded solution u is given by

$$\begin{aligned} u(x,y) &= \frac{1}{\pi}\int_{-a}^{a} \frac{y}{(x-\xi)^2+y^2} T_0 d\xi = \frac{T_0 y}{\pi}\int_{-a}^{a} \frac{1}{(x-\xi)^2+y^2} d\xi \\ &= \frac{T_0}{\pi y}\int_{-a}^{a} \frac{1}{(\frac{\xi-x}{y})^2+1} d\xi \overset{(u=\frac{\xi-x}{y})}{=} \frac{T_0}{\pi y}\int_{\frac{-a-x}{y}}^{\frac{a-x}{y}} \frac{1}{u^2+1} y du \\ &= \frac{T_0}{\pi}\left(\tan^{-1}\left(\frac{a-x}{y}\right) - \tan^{-1}\left(\frac{-a-x}{y}\right)\right) \\ &= \frac{T_0}{\pi}\left(\tan^{-1}\left(\frac{a-x}{y}\right) + \tan^{-1}\left(\frac{a+x}{y}\right)\right). \end{aligned}$$

If $C = (x,y)$ is any point in the upper half plane, and $A = (-a,0)$ and $B = (0,a)$, then by looking at the three cases below, it is easy to see that

$$\angle ACB = \tan^{-1}\left(\frac{a-x}{y}\right) + \tan^{-1}\left(\frac{a+x}{y}\right).$$

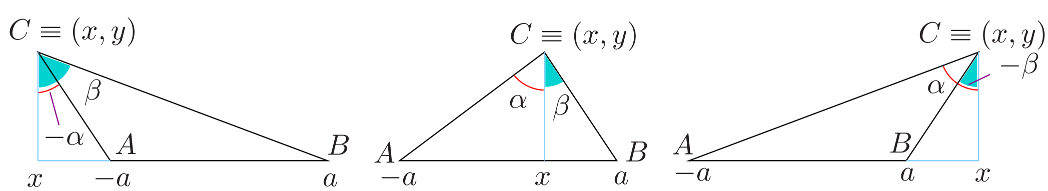


Figure 1. Here $\alpha := \tan^{-1}\left(\frac{a+x}{y}\right)$ and $\beta := \tan^{-1}\left(\frac{a-x}{y}\right)$.

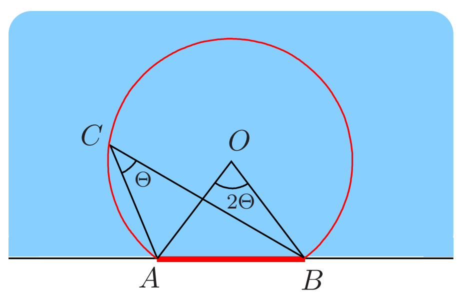
Thus, if the position of C is at (x,y), and if we denote $\angle ACB$ by $\Theta(x,y)$, then our PDE solution can be rewritten as

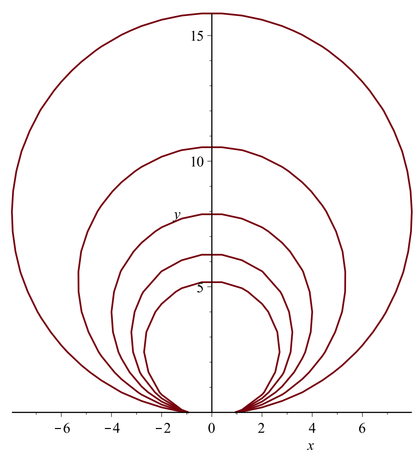
$$u(x,y) = \frac{\Theta(x,y)}{\pi} T_0.$$

In order to understand this function, we can look at its level curves. Imagining this to be a temperature distribution, the level curves of u are called *isotherms* (iso=same, therm=temperature). Note that the largest Θ can be is π, and so $0 \leqslant u \leqslant T_0$. If $T \in [0, T_0]$, then the level curve for this fixed temperature is

$$\{(x,y) : x \in \mathbb{R},\ y \geqslant 0,\ u(x,y) = T\} = \left\{(x,y) : x \in \mathbb{R},\ y \geqslant 0,\ \Theta(x,y) = \frac{T}{T_0}\pi\right\}.$$

In other words, we are looking at all points C which subtend a fixed angle $(\frac{T}{T_0}\pi)$ at the line segment AB. But from elementary geometry (see the picture on the left below), we know that this is a circular arc that passes through A, B and has center O such that $\angle AOB = 2\Theta$.





For example if $T_0 = 100°$C, and $a = 1$, then the isotherms are depicted in the picture on the right above. $\Diamond$

Exercise 4.11. Show analytically that the isotherms from Example 4.6 are circular arcs.

Hint: Take $\tan$ of both sides of equation $u = T$ and use the angle addition trigonometric formula for $\tan$.

Exercise 4.12.

(1) Using the Convolution Theorem, prove the semigroup property of the Poisson kernel, that is, if

$$P_y(x) := P(x,y) = \frac{1}{\pi}\frac{y}{x^2 + y^2},$$

then $P_{y_1} * P_{y_2} = P_{y_1+y_2}$.

(2) Solve the Laplace equation in the upper half plane with the Dirichlet boundary condition

$$u(x,0) = f(x) = \frac{1}{4 + x^2}.$$

What are the isotherms in this case? Using Maple, plot a few of these, when T takes the values $0.06, 0.1, 0.2$. Also plot a the graph of u using Maple, for example in the region $x \in [-6, 6]$ and $y \in [0, 7]$.

Exercise 4.13. Let D be the interior of a simple curve in $\mathbb{R}^2$. Let

$$\varphi : D \to \mathbb{U} := \{z \in \mathbb{C} : \text{Im}(z) > 0\}$$

be complex differentiable in D. Show that if $u : \mathbb{U} \to \mathbb{R}$ is harmonic, then $u \circ \varphi : D \to \mathbb{R}$ is harmonic too.

Now suppose that $\varphi : D \to \mathbb{U}$ is a bijection, and also that $\varphi^{-1} : \mathbb{U} \to D$ is complex differentiable. We call such a map φ a ***conformal map***. Based on the calculation done for the previous part of the exercise, we conclude that a function $u : D \to \mathbb{U}$ is harmonic if and only if $u \circ \varphi : D \to \mathbb{R}$ is harmonic. Thus the existence of a conformal map taking D to $\mathbb{U}$ allows one to transplant harmonic functions from the (possibly complicated) domain D to the (geometrically simple) domain $\mathbb{U}$. This mobility has the advantage that the Dirichlet Problem in D can be solved by first moving over to $\mathbb{U}$, solving it there, and then transplanting the solution back to D.

A first natural question is then the following: Given a domain D which is the interior of a simple curve, is there a conformal map taking D to $\mathbb{U}$? The answer is "yes"!

Theorem 4.5 (Riemann Mapping Theorem). *Let D be the interior of a simple curve in $\mathbb{R}^2$. Then there exists a conformal map $\varphi : D \to \mathbb{U}$.*

Thus the above result guarantees a conformal map, but unfortunately the proof does not give a practical algorithm for finding it.

Show that the "Möbius transformation" $\varphi : \mathbb{D} \to \mathbb{U}$, given by

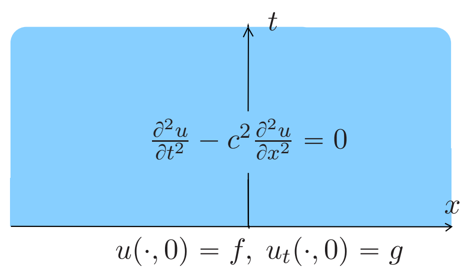
$$\varphi(s) = i\frac{1+z}{1-z}, \quad s \in \mathbb{D},$$

is a conformal map from the disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ to the upper half plane $\mathbb{U}$.

4.5. Oscillations of an infinite string

Let us now consider the boundary value problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2 \dfrac{\partial^2 u}{\partial x^2} = 0 & (t > 0,\ x \in \mathbb{R}), \\ u(x, 0) = f(x) & (x \in \mathbb{R}), \\ \dfrac{\partial u}{\partial t}(x, 0) = g(x) & (x \in \mathbb{R}). \end{cases}$$



Taking Fourier transform in the x-direction gives

$$\begin{aligned} \frac{d}{dt^2}\widehat{u}(\xi,t)+c^2\xi^2\widehat{u}(\xi,t) &= 0, \\ \widehat{u}(\xi,0) &= \widehat{f}(\xi), \\ \frac{d}{dt}\widehat{u}(\xi,0) &= \widehat{g}(\xi). \end{aligned}$$

The general solution of the ODE (in t) above is

$$\widehat{u}(\xi,t)=A(\xi)\cos(c\xi t)+B(\xi)\sin(c\xi t),$$

where A, B are constant in t (but are functions of ξ). We can determine A, B using the initial conditions:

$$\widehat{f}(\xi)=\widehat{u}(\xi,0)=A(\xi)$$

$$\widehat{g}(\xi)=\frac{d}{dt}\widehat{u}(\xi,0)=c\xi B(\xi).$$

Thus

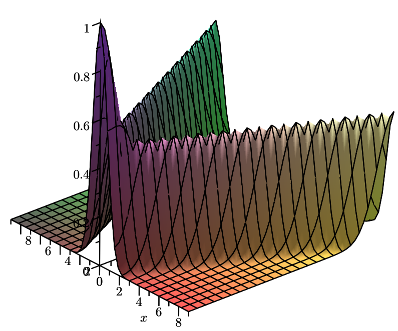
$$\widehat{u}(\xi,t)=\widehat{f}(\xi)\cos(c\xi t)+\frac{1}{c\xi}\widehat{g}(\xi)\cdot\sin(c\xi t), \quad \xi\neq 0.$$

Using the result from Exercise 4.7 (and under the assumption that for each t, $\mathcal{F}_x(u(\cdot,t))$ is continuous), we obtain

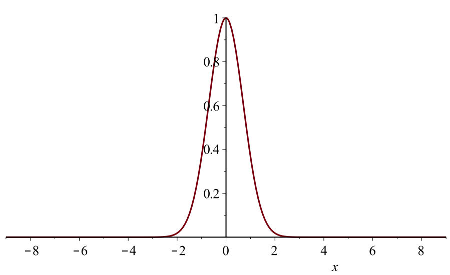
$$u(x,t):=\frac{f(x-ct)+f(x+ct)}{2}+\frac{1}{2c}\int_{x-ct}^{x+ct}g(y)dy,$$

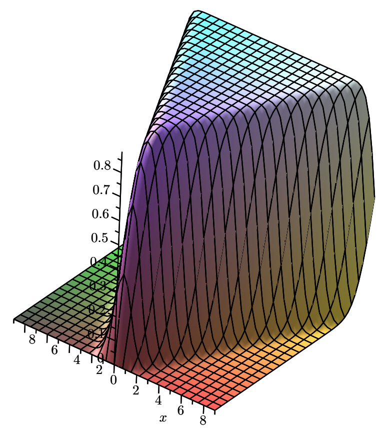
This is known as ***D'Alembert's Formula***.

We remark that the solution comprises two parts: one which depends on the string's initial wave form, and the second, which depends on the initial speed. These two parts are illustrated below for some given f, g. The first term in u is shown below, where the evolution can be thought of as the propagation of the initial shape in both directions with a speed c.



On the other hand, the plots of the initial speed and the second term in the solution are shown below:





Exercise 4.14. A pressure wave generated as a result of an explosion satisfies

$$P_{tt} - 16P_{xx} = 0$$

in the domain $\{(x,t) : x \in \mathbb{R}, t > 0\}$, where $P(x,t)$ is the pressure at the point x and time t. The initial conditions at the explosion time $t = 0$ are

$$\begin{aligned} P(x,0) &= \begin{cases} 10 & \text{if } |x| \leqslant 1, \\ 0 & \text{if } |x| > 1, \end{cases} \\ P_t(x,0) &= \begin{cases} 1 & \text{if } |x| \leqslant 1, \\ 0 & \text{if } |x| > 1. \end{cases} \end{aligned}$$

A building is located at $x_0 = 10$. The engineer who designed the building determined that it will sustain a pressure up to $P = 6$. Will the building collapse?

Exercise 4.15 (Stability). Consider the Cauchy Initial Value Problem for the wave equation:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2 \dfrac{\partial^2 u}{\partial x^2} = 0 & (t > 0,\ x \in \mathbb{R}), \\ u(x,0) = f(x) & (x \in \mathbb{R}), \\ \dfrac{\partial u}{\partial t}(x,0) = g(x) & (x \in \mathbb{R}), \end{cases}$$

where $f \in C^2$ and $g \in C^1$. For $i = 1, 2$, let u_i be the solution to the initial value problem with initial data (f_i, g_i). Fix $T > 0$. Given any $\epsilon > 0$, show that there is a $\delta > 0$ such that if

$$\|f_1 - f_2\|_\infty := \sup_{x \in \mathbb{R}} |f_1(x) - f_2(x)| < \delta \text{ and } \|g_1 - g_2\|_\infty := \sup_{x \in \mathbb{R}} |g_1(x) - g_2(x)| < \delta,$$

then $\sup_{\substack{x \in \mathbb{R}, \\ t \in [0,T]}} |u_1(x,t) - u_2(x,t)| < \epsilon$.

(This shows stability of classical solutions on finite time intervals for the wave equation in the L^∞ norm.)

4.6. The Laplace transform method

In this section, we will use the Laplace transform for solving our second order PDEs where the spatial variable x lives on the half line $[0, \infty)$.

Definition 4.2. If $f : \mathbb{R} \to \mathbb{C}$ is a piecewise smooth function, then its Laplace transform F is defined by

$$F(s) = \int_0^\infty f(t)e^{-st}dt,$$

for those $s \in \mathbb{C}$ for which the integral exists.

So in general the Laplace transform for a given f is defined only for a subset of the complex plane. Typically we will only consider f which are exponentially bounded, that is, there exist M, a such that

$$|f(t)| \leqslant Me^{at} \quad (t > 0).$$

Then the Laplace transform of F exists for all $\text{Re}(s) > a$. Indeed,

$$\begin{aligned}\int_0^\infty |f(t)e^{-st}|dt &= \int_0^\infty |f(t)|e^{-\text{Re}(s)t}dt \\ &\leqslant \int_0^\infty Me^{at}e^{-\text{Re}(s)t}dt = M\int_0^\infty e^{(a-\text{Re}(s))t}dt = \frac{M}{\text{Re}(s) - a} < \infty.\end{aligned}$$

(It can be shown that for any piecewise smooth function, whenever the Laplace transform converges for some $s_0 \in \mathbb{C}$, it converges also for all s in the half plane in $\mathbb{C}$ given by $\text{Re}(s) > \text{Re}(s_0)$.)

It can be seen that Laplace transformation is linear. We will denote the Laplace transformation operator by $\mathcal{L}$.

Example 4.7. Let us find out the Laplace transform of the constant function 1. For $\text{Re}(s) > 0$, we have

$$F(s) = \int_0^\infty 1e^{-st}dt = \frac{e^{-st}}{-s}\Big|_0^\infty = \frac{1}{s}.$$

Thus $\big(\mathcal{L}(1)\big)(s) = 1/s$ for $\text{Re}(s) > 0$. $\Diamond$

The following further properties will be useful to us. (We use capital letters below to denote the Laplace transform: thus the Laplace transform of f is denoted by F, etc.)

(1) (Injectivity).

If f is continuous on $(0, \infty)$, and if F is zero on the real interval (a, ∞) for some real a, then $f(t) = 0$ for all $t > 0$. (See for example [**A**, Exercise 11.38].)

(2) (Differentiation). $\mathcal{L}(f')(s) = sF(s) - f(0-)$.

Reason: We have for an exponentially bounded f satisfying $|f(t)| \leqslant Me^{at}$ that for $\mathrm{Re}(s) > a$, $f(t)e^{-st} \to 0$ as $t \to \infty$. Then for $\epsilon > 0$, by integrating by parts, we have

$$\begin{aligned} \int_{-\epsilon}^{\infty} f'(t)e^{-st}dt &= f(t)e^{-st}\Big|_{-\epsilon}^{\infty} + s\int_{-\epsilon}^{\infty} f(t)e^{-st}dt \\ &= -f(-\epsilon)e^{s\epsilon} + s\int_{-\epsilon}^{\infty} f(t)e^{-st}dt. \end{aligned}$$

Passing the limit as $\epsilon \searrow 0$, we obtain $\big(\mathcal{L}(f)\big)(s) = -f(0-) + sF(s)$.

(3) The *causal convolution* of two functions f, g is defined by

$$(f \circledast g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau \quad (t > 0).$$

Then $\mathcal{L}(f \circledast g) = F \cdot G$.

Reason: We have

$$\begin{aligned} \mathcal{L}(f \circledast g)(s) &= \int_0^{\infty} (f * g)(t)e^{-st}dt = \int_0^{\infty} \Big(\int_0^t f(\tau)g(t-\tau)d\tau\Big)e^{-st}dt \\ &= \int_0^{\infty}\int_0^t f(\tau)g(t-\tau)e^{-st}d\tau dt = \int_0^{\infty}\int_{\tau}^{\infty} f(\tau)g(t-\tau)e^{-st}dtd\tau \\ &= \int_0^{\infty} f(\tau)\Big(\int_{\tau}^{\infty} g(t-\tau)e^{-st}dt\Big)d\tau \\ &= \int_0^{\infty} f(\tau)\Big(\int_0^{\infty} g(u)e^{-s(u+\tau)}du\Big)d\tau \quad (u = t-\tau) \\ &= \int_0^{\infty} f(\tau)e^{-s\tau}\Big(\int_0^{\infty} g(u)e^{-su}du\Big)d\tau = \int_0^{\infty} f(\tau)e^{-s\tau}G(s)d\tau \\ &= G(s)\int_0^{\infty} f(\tau)e^{-s\tau}d\tau = G(s)F(s). \end{aligned}$$

(4) Shifting rules:

(a) (Shift in the "s-domain"). $\mathcal{L}\big(e^{s_0 t}f(t)\big)(s) = \big(\mathcal{L}(f)\big)(s-s_0)$ for $s - s_0$ in the domain of $\mathcal{L}f$.

Reason:

$$\Big(\mathcal{L}\big(e^{s_0 t}f(t)\big)\Big)(s) = \int_0^{\infty} e^{s_0 t}f(t)e^{-st}dt = \int_0^{\infty} f(t)e^{-(s-s_0)t}dt = \big(\mathcal{L}(f)\big)(s-s_0).$$

A similar result holds for a shift in the time-domain.

(b) (Shift in the time-domain). For $t_0 \geqslant 0$, $\big(\mathcal{L}\big(f(t-t_0)1_{[t_0,\infty)}\big)\big)(s) = e^{-st_0}\big(\mathcal{L}(f)\big)(s)$.
Reason:

$$\begin{aligned}
\mathcal{L}\big(f(t-t_0)\mathbf{1}_{[t_0,\infty)}(t)\big)(s) &= \int_0^\infty f(t-t_0)\mathbf{1}_{[t_0,\infty)}(t)e^{-st}dt = \int_{t_0}^\infty f(t-t_0)e^{-st}dt \\
&= \int_0^\infty f(\tau)e^{-s(\tau+t_0)}d\tau \quad (\tau = t-t_0) \\
&= \int_0^\infty f(\tau)e^{-s\tau}e^{-st_0}d\tau = e^{-st_0}\int_0^\infty f(\tau)e^{-s\tau}d\tau \\
&= e^{-st_0}\big(\mathcal{L}(f)\big)(s).
\end{aligned}$$

Let us now see how we can use the Laplace transform to solve PDEs. The idea is similar to what we did with the Fourier transform, but usually one takes the Laplace transform with respect to t. Here are couple of examples.

Example 4.8 (A first order equation). Consider the problem

$$\begin{cases} \dfrac{\partial u}{\partial x} + x\dfrac{\partial u}{\partial t} = 0 & (t > 0,\ x \in \mathbb{R}), \\ u(x,0) = 0 & (x \in \mathbb{R}), \\ u(0,t) = t & (t > 0). \end{cases}$$

Taking Laplace transform with respect to t, we obtain

$$\mathcal{L}\Big(\frac{\partial u}{\partial x}\Big) + x\big(s\mathcal{L}(u) - u(x,0)\big) = 0.$$

Here $u(x,0) = 0$. In the first term, we assume that we may interchange integration and differentiation:

$$\mathcal{L}\Big(\frac{\partial u}{\partial x}\Big) = \int_0^\infty e^{-st}\frac{\partial u}{\partial x}dt = \frac{\partial}{\partial x}\int_0^\infty e^{-st}u(x,t)dt = \frac{\partial}{\partial x}\mathcal{L}u.$$

Writing $U(x,s) := \big(\mathcal{L}(u(x,\cdot))\big)(s)$, we obtain

$$\frac{\partial U}{\partial x} + xsU = 0.$$

This is an ODE with x as the independent variable (and parameter s). The general solution is

$$U(x,s) = U(0,s)e^{\int_0^x -\xi s d\xi} = U(0,s)e^{-sx^2/2}.$$

Since $\big(\mathcal{L}(t)\big)(s) = 1/s^2$, the condition $u(0,t) = t$ yields $U(0,s) = 1/s^2$, and so

$$U(x,s) = \frac{1}{s^2}e^{-sx^2/2}.$$

By the shifting rule, with $t_0 = x^2/2 \geqslant 0$, we obtain

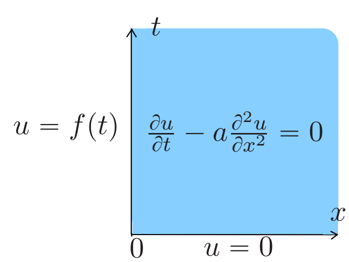
$$u(x,t) = \left(t - \frac{x^2}{2}\right)\mathbf{1}_{[x^2/2,\infty)}(t) = \begin{cases} 0 & \text{if } 0 \leqslant t \leqslant \dfrac{x^2}{2}, \\ t - \dfrac{x^2}{2} & \text{if } t > \dfrac{x^2}{2}. \end{cases}$$

Since we proceeded formally, we need to check that this solution does satisfy the PDE, and this can be done. ◊

Example 4.9 (Heat equation for a semi-infinite rod). Consider the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - a\dfrac{\partial^2 u}{\partial x^2} = 0 & (t, x > 0), \\ u(0,t) = f(t) & (t > 0), \\ u(x,0) = 0 & (x > 0). \end{cases}$$

The problem is illustrated in the picture below.



Using Laplace transformation with respect to t, we obtain

$$sU(x,s) - \underbrace{u(x,0)}_{=0} = a\frac{\partial^2 U}{\partial x^2}(x,s).$$

This ODE in x (with parameter s) has the general solution (assuming s is positive and large)

$$U(x,s) = A(s)e^{\sqrt{s}x/\sqrt{a}} + B(s)e^{-\sqrt{s}x/\sqrt{a}}.$$

Assuming that U stays bounded as $s \to \infty$, we set $A \equiv 0$. To determine B, we obtain from the boundary condition $u(0,t) = f(t)$ that

$$U(0,s) = F(s) = B(s) \cdot 1.$$

Hence

$$U(x,s) = F(s)e^{-\sqrt{s}x/\sqrt{a}}.$$

By the convolution rule, it follows that $u(x,t)$ must be the convolution of f with the function whose Laplace transform is $e^{-\sqrt{s}x/\sqrt{a}}$. Suppose we are given[2] that

$$\mathcal{L}\left(\frac{k}{2\sqrt{\pi t^3}}\exp\left(-\frac{k^2}{4t}\right)\right)(s) = e^{-k\sqrt{s}} \quad (k>0).$$

Thus we obtain (with $k = x/\sqrt{a} > 0$)

$$\mathcal{L}\left(\frac{x}{2\sqrt{a\pi t^3}}\exp\left(-\frac{x^2}{4at}\right)\right)(s) = e^{-\sqrt{s}x/\sqrt{a}} \quad (k>0).$$

Hence

$$u(x,t) = f \circledast \frac{x}{2\sqrt{a\pi t^3}}\exp\left(-\frac{x^2}{4at}\right) = \frac{x}{2\sqrt{a\pi}}\int_0^t f(t-\tau)\frac{1}{\sqrt{\tau^3}}\exp\left(-\frac{x^2}{4a\tau}\right)d\tau.$$

Note that the solution at time t depends on the value of f on the interval $[0,t]$. This is expected since the temperature of the heat reservoir in the ***future*** ($\tau > t$) cannot possibly affect the temperature of the rod ***now***. Let us find an explicit u in the simple case when $f \equiv T_0$ (constant heat source), when the convolution above is doable:

$$u(x,t) = \frac{xT_0}{2\sqrt{a\pi}}\int_0^t \frac{1}{\sqrt{\tau^3}}\exp\left(-\frac{x^2}{4a\tau}\right)d\tau.$$

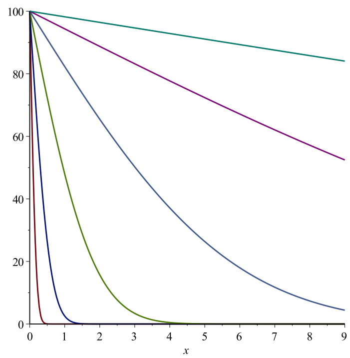
Substituting $z = \dfrac{x}{2\sqrt{a\tau}}$, we have $dz = \dfrac{x}{2\sqrt{a}\sqrt{\tau^3}}\left(-\dfrac{1}{2}\right)d\tau$, and so

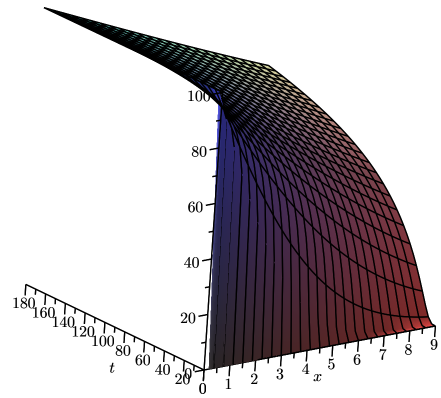
$$u(x,t) = \frac{2T_0}{\sqrt{\pi}}\int_{\frac{x}{2\sqrt{at}}}^{\infty} e^{-z^2}dz = T_0 \cdot \operatorname{erfc}\left(\frac{x}{2\sqrt{at}}\right),$$

where erfc is the ***complementary error function***, defined by

$$\operatorname{erfc}(x) := 1 - \operatorname{erf}(x) = 1 - \frac{2}{\sqrt{\pi}}\int_0^x e^{-z^2}dz = \frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-z^2}dz.$$

[2]This can be found out using Maple or looked up for example in the mathematical reference work called *Abramowitz and Stegun: Handbook of Mathematical Functions*, published in 1964. Thus 1046 page treatise has been one of the most comprehensive sources of information on special functions, containing definitions, identities, approximations, plots, and tables of values of many functions used in applied mathematics. The notation used in the Handbook is the standard for applied mathematics even today. The Laplace transform we have looked up is actually entry 29.3.82 on page 1026 of the Handbook, available on the web at: `http://people.math.sfu.ca/~cbm/aands/`





Taking $T_0 = 100$, we can plot the temperature profiles at various time instances, say $t = 0.01, 0.1, 1, 10, 100, 1000$ (picture on the left above), and also the graph of u for $x \in [0, 9]$ and $t \in [0, 180]$ (picture on the right above). $\Diamond$

Example 4.10 (Semi-infinite string). Consider the oscillations of an infinite string, subject to the following conditions:

(1) The string is initially at rest on the x-axis from $x = 0$ to ∞ (semi-infinite).

(2) For time $t > 0$, the left end of the string is displaced according to

$$u(0,t) = f(t) := \begin{cases} \sin t & \text{if } 0 \leqslant t \leqslant 2\pi, \\ 0 & \text{otherwise.} \end{cases}$$

(3) Furthermore, $\lim_{x\to\infty} u(x,t) = 0$ for $t \geqslant 0$.

Thus we have

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2 \dfrac{\partial^2 u}{\partial x^2} = 0 & (t, x > 0), \\ u(0,t) = f(t) \text{ and } \lim\limits_{x\to\infty} u(x,t) = 0 & (t > 0), \\ u(x,0) = \dfrac{\partial u}{\partial t}(x,0) = 0 & (x > 0). \end{cases}$$

(Of course, there is no infinite string, but our model describes a long string or rope of negligible weight with its right end fixed far out on the x-axis.) Taking Laplace transform with respect to t, we have for $x > 0$ that

$$c^2 \mathcal{L}\Big(\frac{\partial^2 u}{\partial x^2}\Big) = \mathcal{L}\Big(\frac{\partial^2 u}{\partial t^2}\Big) = s\big(\mathcal{L}(u_t)\big) - \underbrace{u_t(x,0)}_{=0} = s\big(sU - \underbrace{u(x,0)}_{=0}\big) = s^2 U.$$

Interchanging differentiation and integration in the leftmost term, we obtain

$$\frac{\partial^2 U}{\partial x^2} - \frac{s^2}{c^2}U = 0.$$

The general solution to this (assuming s is positive and large) is

$$U(x,s) = A(s)e^{sx/c} + B(s)e^{-sx/c}.$$

We have $U(0,s) = \mathcal{L}(u(0,\cdot)) = \mathcal{L}(f) = F(s)$. On the other hand, assuming that the order of integrating with respect to t and taking the limit as $x \to \infty$ can be interchanged, we obtain

$$\lim_{x\to\infty} U(x,s) = \lim_{x\to\infty} \int_0^\infty e^{-st}u(x,t)dt = \int_0^\infty e^{-st} \lim_{x\to\infty} u(x,t)dt = 0.$$

This implies that $A \equiv 0$, because $c > 0$ and for all (large) $s > 0$, the function $x \mapsto e^{sx/c}$ goes to infinity as $x \to \infty$. So we now obtain

$$F(s) = U(0,s) = B(s),$$

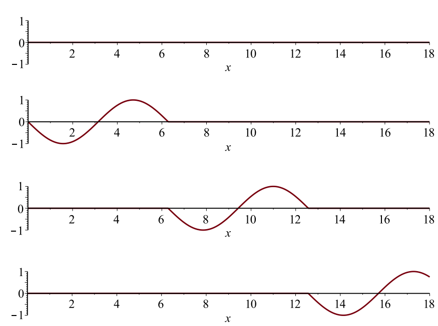
and so $U(x,s) = F(s)e^{-sx/c}$. By the Shifting Rule, with $t_0 = x/c \geqslant 0$, we obtain

$$u(x,t) = f\left(t - \frac{x}{c}\right) \cdot \mathbf{1}_{[\frac{x}{c},\infty)}(t),$$

that is,

$$u(x,t) = \begin{cases} \sin\left(t - \frac{x}{c}\right) & \text{if } \frac{x}{c} < t < \frac{x}{c} + 2\pi, \\ 0 & \text{otherwise.} \end{cases}$$

The pictures below show the plots of $u(\cdot,t)$ for various time instances ($t = 0, 2\pi, 4\pi, 6\pi$):



Thus the solution u describes a single-period sine waveform travelling to the right with speed c. Note that a point x stays at rest until $t = x/c$, the time needed to reach that x if one starts at $t = 0$ (start of the motion at the left end) and travels with speed c. The result agrees with our physical intuition. Since we have worked formally we need to check that our solution satisfies the wave equation. This can be done, except that it does so in the "sense of distributions", and it is a weak solution. We will revisit this in Chapter 5. $\diamondsuit$

Exercise 4.16 (Transport revisited). Solve

$$\begin{cases} \dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial x} = 0 & (t > 0,\ x > 0), \\ u(0,t) = f(t) & (t \geqslant 0), \\ u(x,0) = 0 & (x \geqslant 0), \end{cases}$$

using the Laplace transform method.

Exercise 4.17 (Falling string). Consider a semi-infinite string fixed at one end, falling under gravity:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2\dfrac{\partial^2 u}{\partial x^2} = -g & (t > 0,\ x > 0), \\ u(0,t) = 0 & (t > 0), \\ u(x,0) = 0 = u_t(x,0) & (x > 0). \end{cases}$$

We will solve this using the Laplace transform method by following the steps below.

(1) Show that $U(x,s) := \big(\mathcal{L}\big(u(x,\cdot)\big)\big)(s)$ satisfies $s^2 U - c^2 \dfrac{\partial^2 U}{\partial x^2} = -\dfrac{g}{s}$.

(2) Show that $U_{\text{particular}}(x,s) := -\dfrac{g}{s^3}$ satisfies the ODE in part (1).

(3) Show that the general solution to the ODE from part (1) is given by

$$U(x,s) = A(s)e^{\frac{s}{c}x} + B(s)e^{-\frac{s}{c}x} - \frac{g}{s^3}.$$

(4) Now we make the assumption that for each s, $x \mapsto U(x,s)$ is bounded. Find u.

(5) Take $c = 1$ and $g = 9.8\,\text{ms}^{-2}$. Plot u for $t = 1, 2, 3, 4, 5, 6$.

4.7. Proof of the Fourier Transform Theorem

The proof of the Fourier Series Theorem relied on the two technical results we proved in Lemma 3.5 and Lemma 3.6. Analogously, one can show the following two results, which we will use in order to prove the Fourier Integral Theorem.

Lemma 4.6. *If $f : \mathbb{R} \to \mathbb{C}$ is absolutely integrable, then* $\displaystyle\lim_{\xi \to \pm\infty} \int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx = 0.$

Lemma 4.7. *If $f : \mathbb{R} \to \mathbb{C}$ is absolutely integrable, then*

$$\lim_{a \to \infty} \int_{-\infty}^{\infty} f(x)\frac{\sin(ax)}{x}dx = \pi f(0).$$

As the proofs of these auxiliary results are almost identical to those of Lemmas 3.5 and 3.6, we will not give them here.

Proof of the Fourier Transform Theorem. We have

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{f}(\xi)e^{i\xi x}d\xi &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty} f(\omega)e^{-i\xi\omega}d\omega\Big)e^{i\xi x}d\xi \\
&= \lim_{a\to\infty}\frac{1}{2\pi}\int_{-a}^{a}\int_{-\infty}^{\infty} f(\omega)e^{i\xi(x-\omega)}d\omega d\xi \\
&= \lim_{a\to\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-a}^{a} f(\omega)e^{i\xi(x-\omega)}d\xi d\omega \\
&= \lim_{a\to\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\omega)\Big(\int_{-a}^{a} e^{i\xi(x-\omega)}d\xi\Big)d\omega \\
&= \lim_{a\to\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\omega)\frac{e^{ia(x-\omega)}-e^{-ia(x-\omega)}}{i(x-\omega)}d\omega \\
&= \lim_{a\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty} f(\omega)\frac{\sin\big(a(x-\omega)\big)}{x-\omega}d\omega.
\end{aligned}$$

In the above we changed the order of integration to obtain the third equality, and this is allowed thanks to the fact that f is absolutely integrable. Now, the final right hand side in the above equation array is

$$\int_{-\infty}^{\infty} f(\omega)\frac{\sin\big(a(x-\omega)\big)}{x-\omega}d\omega \overset{(y=x-\omega)}{=} \int_{-\infty}^{\infty} f(x-y)\frac{\sin(ay)}{y}dy,$$

and so, in light of the above,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{f}(\xi)e^{i\xi x}d\xi = \lim_{a\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty} f(x-y)\frac{\sin(ay)}{y}dy = f(x),$$

using Lemma 4.7. □

Chapter 5

Distributions and weak solutions

There are three reasons to study "distributions" or "generalized functions":

(1) To mathematically model the situation when one has an impulsive force (imagine a blow to an object which changes its momentum, but the force itself is supposed to act "impulsively", that is the time interval when the force is applied is 0 !). Similar situations arise in other instances in mathematics and the applied sciences.

(2) To develop a calculus which captures more general situations than the classical case. For example, what is the derivative of $|x|$ at $x = 0$? It will turn out that this is also useful to talk about weaker notions of solutions of PDEs.

(3) To extend the Fourier transform theory to functions that may not be absolutely integrable. For example, what[1] is the Fourier transform of the constant function 1?

It turns out that the theory of distributions solves all of these three problems in one go. This seems like a miracle, and naturally there is a price to pay. The price is that everything classical is now replaced by a weaker notion, but nevertheless this is useful since it is often sufficient for what one wants to do. An example is that as opposed to functions on $\mathbb{R}$, which have a well-defined value at every point $x \in \mathbb{R}$, we can no longer talk about the value of a distribution at a point of $\mathbb{R}$. Another instance in the context of PDEs is an example we met earlier, where we had a plucked guitar string, and we discovered that the solution we obtained formally doesn't solve the PDE in a classical sense. Notwithstanding this, it turns out that the PDE *is* satisfied in the sense of distributions. So with this motivation, we will learn the very basics of the theory of distributions in this chapter and see a glimpse of its applications to PDEs.

[1] Although the classical Fourier transform does not exist, it can be shown that in the sense of distributions, the Fourier transform of the constant function 1 is the Dirac delta distribution δ.

We make a brief historical remark about the story of the development of distributions. The prime example of a distribution, the "delta function" δ_a, was introduced[2] by the English physicist P.A.M. Dirac in the 1930s in order to do quantum mechanical computations (as eigenstates of the position operator). However, a firm mathematical foundation for this and other generalized functions had to wait till the 1950s when the French mathematician Laurent Schwartz introduced the concept of distributions and developed its theory. For this, he was awarded the Fields medal.

5.1. Test functions, distributions, and examples

Let us first quickly recall a few definitions from the topology of $\mathbb{R}^d$:

Definition 5.1.

(1) (The Euclidean 2-norm $\|\cdot\|_2$ on $\mathbb{R}^d$).
For $\mathbf{x} = (x_1, \cdots, x_d) \in \mathbb{R}^d$, $\|\mathbf{x}\|_2 := \sqrt{x_1^2 + \cdots + x_d^2}$.

(2) (Open ball). An *open ball with center* $\mathbf{a} \in \mathbb{R}^d$ *and radius* $r > 0$ is the set

$$B(\mathbf{a}, r) := \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x} - \mathbf{a}\|_2 < r\}.$$

(3) (Open set). A set $U \subset \mathbb{R}^d$ is *open* if for every $\mathbf{x} \in U$, there exists an $r > 0$ such that $B(\mathbf{x}, r) \subset U$.

(4) (Bounded set). A set $S \subset \mathbb{R}^d$ is *bounded* if there exists an $R > 0$ such that $S \subset B(\mathbf{0}, R)$.

(5) (Closed set). A set $F \subset \mathbb{R}^d$ is *closed* if $\mathbb{R}^d \backslash F$ is open.

(6) (Compact set). A set $K \subset \mathbb{R}^d$ is *compact* if it is closed and bounded.

Definition 5.2 (Test function). A *test function* $\varphi : \mathbb{R}^d \to \mathbb{R}$ is an infinitely differentiable function for which there exists a compact set outside which φ vanishes. The set of all test functions is denoted by $\mathcal{D}(\mathbb{R}^d)$. Equipped with pointwise operations, $\mathcal{D}(\mathbb{R}^d)$ is a real vector space.

Example 5.1. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be given by

$$\varphi(x) = \begin{cases} e^{-\frac{1}{1-x^2}} & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geqslant 1, \end{cases}$$

is an element of $\mathcal{D}(\mathbb{R})$. It is clear that φ vanishes outside the compact interval $[-1, 1]$. Moreover, it is also infinitely many times differentiable. Indeed, it can be seen that the function $f : \mathbb{R} \to \mathbb{R}$ given by

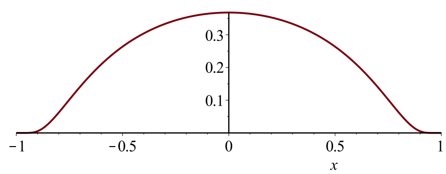
$$f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x > 0, \\ 0 & \text{if } x \leqslant 0, \end{cases}$$

[2]There were, however, earlier usages of such an object; for example an infinitely tall, unit impulse function was used by Cauchy in the early 19th century. The Dirac delta function as such was introduced as a "convenient notation" by Dirac in his book, *The Principles of Quantum Mechanics*, where he called it the "delta function", as a continuous analogue of the discrete Kronecker delta.

is clearly infinitely many times differentiable outside 0, and also

$$\forall n \in \mathbb{N}, \quad \lim_{x \to 0} f^{(n)}(x) = 0,$$

showing that f is infinitely many times differentiable everywhere on $\mathbb{R}$. (The details will be given in Exercise 5.1.) And our function φ is just the composition of f with the polynomial $x \mapsto 1 - x^2$. The picture below shows the graph of φ.



Similarly, we could have composed f with the function

$$\mathbf{x} \mapsto 1 - \|\mathbf{x}\|_2^2 = 1 - (x_1^2 + \cdots + x_d^2) : \mathbb{R}^d \to \mathbb{R}$$

and obtained a function in $\mathcal{D}(\mathbb{R}^d)$ that is C^∞ and is zero outside the closed unit ball $B(\mathbf{0}, 1) := \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\|_2 \leqslant 1\}$. $\Diamond$

Whenever $\varphi \in \mathcal{D}(\mathbb{R}^d)$, we have that for every $\epsilon > 0$ and every $\mathbf{a} \in \mathbb{R}^d$, also the function

$$\mathbf{x} \mapsto \varphi\Big(\frac{\mathbf{x} - \mathbf{a}}{\epsilon}\Big)$$

belongs to $\mathcal{D}(\mathbb{R}^d)$. By taking linear combinations, we see that we get a huge abundance of functions in $\mathcal{D}(\mathbb{R}^d)$. It is also easy to see that $\mathcal{D}(\mathbb{R}^d)$ is closed under partial differentiation. In the following, it will be convenient to introduce the following notation: if $\mathbf{k} = (k_1, \cdots, k_d)$ is a multi-index of nonnegative integers, then

$$D^{\mathbf{k}} := \frac{\partial^{k_1 + \cdots + k_d}}{\partial x_1^{k_1} \cdots \partial x_d^{k_d}}.$$

In this notation, we have: $\varphi \in \mathcal{D}(\mathbb{R}^d) \Rightarrow D^{\mathbf{k}}\varphi \in \mathcal{D}(\mathbb{R}^d)$ for all $\mathbf{k}$.

Exercise 5.1 (A C^∞ function which is not analytic). $(*)$

(1) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$, continuously differentiable on $\mathbb{R}_* := \mathbb{R} \backslash \{0\}$, and such that

$$\lim_{x \to 0} f'(x)$$

exists. Show that f is continuously differentiable on $\mathbb{R}$.

(2) Let $f : \mathbb{R} \to \mathbb{R}$ be $n - 1$ times continuously differentiable, n times continuously differentiable on $\mathbb{R}_*$, and such that

$$\lim_{x \to 0} f^{(n)}(x)$$

exists. Show that f is n times continuously differentiable on $\mathbb{R}$.

(3) Let $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x > 0, \\ 0 & \text{if } x \leqslant 1. \end{cases}$$

Show that f is infinitely many times differentiable.
Hint: Using induction on n, show that for $x > 0$, $f^{(n)}(x)$ is of the form $R_n(x)f(x)$, with R_n a rational function. Prove that $\lim_{x\to 0} x^{-n}f(x) = 0$ for all $n \in \mathbb{N}$.

Exercise 5.2. Solve $u_x = 0$ in $\mathcal{D}(\mathbb{R}^2)$.

Definition 5.3 (Convergence in $\mathcal{D}(\mathbb{R}^d)$). We say that a sequence $(\varphi_n)_{n\in\mathbb{N}}$ converges to φ in $\mathcal{D}(\mathbb{R}^d)$ if

(1) there exists a compact set $K \subset \mathbb{R}^d$ such that all the φ_n vanish outside K, and
(2) φ_n converges uniformly[3] to φ and for each multi-index $\mathbf{k}$, $D^{\mathbf{k}}\varphi_n$ converges uniformly to $D^{\mathbf{k}}\varphi$.

We then simply write $\varphi_n \xrightarrow{\mathcal{D}} \varphi$.

Definition 5.4 (Distribution). A ***distribution*** T on $\mathbb{R}^d$ is a map $T : \mathcal{D}(\mathbb{R}^d) \to \mathbb{R}$ such that

(1) (Linearity) For all $\varphi, \psi \in \mathcal{D}(\mathbb{R}^d)$ and all $\alpha \in \mathbb{R}$, $T(\varphi + \psi) = T(\varphi) + T(\psi)$ and $T(\alpha \cdot \varphi) = \alpha \cdot T(\varphi)$.
(2) (Continuity) $\varphi_n \xrightarrow{\mathcal{D}} \varphi \;\Rightarrow\; T(\varphi_n) \to T(\varphi)$.

The set of all distributions is denoted by $\mathcal{D}'(\mathbb{R}^d)$. With pointwise operations, $\mathcal{D}'(\mathbb{R}^d)$ is a vector space. We will usually denote $T(\varphi)$ for $\varphi \in \mathcal{D}(\mathbb{R}^d)$, by $\langle T, \varphi\rangle$.

Remark 5.1. It is sufficient to check the continuity requirement with $\varphi = 0$, since from the linearity of T, it follows that

$$T(\varphi_n) - T(\varphi) = T(\varphi - \varphi_n),$$

and it is clear that $(\varphi_n \xrightarrow{\mathcal{D}} \varphi) \;\Leftrightarrow\; (\varphi_n - \varphi \xrightarrow{\mathcal{D}} 0)$.

Example 5.2 ($L^1_{\text{loc}}(\mathbb{R}^d)$ functions are distributions). Let $f : \mathbb{R}^d \to \mathbb{R}$ be a ***locally integrable*** function (written $f \in L^1_{\text{loc}}(\mathbb{R}^d)$), that is, for every compact set K,

$$\int_K |f(\mathbf{x})|d\mathbf{x} < \infty.$$

Then f defines a distribution T_f as follows:

$$\langle T_f, \varphi\rangle := \int_{\mathbb{R}^d} f(\mathbf{x})\varphi(\mathbf{x})d\mathbf{x}, \quad \varphi \in \mathcal{D}(\mathbb{R}^d).$$

[3] That is, for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n > N$ and all $\mathbf{x} \in K$, $|\varphi_n(\mathbf{x}) - \varphi(\mathbf{x})| < \epsilon$.

The integral exists since φ is bounded and zero outside a compact set, and so we are actually integrating over a compact set. It is also easy to see that T_f is linear. Moreover, if $\varphi_n \xrightarrow{\mathcal{D}} 0$, with φ_n all vanishing outside a compact set K, then

$$|\langle T_f, \varphi_n\rangle| = \Big| \int_{\mathbb{R}^d} f(\mathbf{x})\varphi_n(\mathbf{x})d\mathbf{x}\Big| \leqslant \int_K |f(\mathbf{x})||\varphi_n(\mathbf{x})|d\mathbf{x} \leqslant \|\varphi_n\|_\infty \int_K |f(\mathbf{x})|d\mathbf{x} \to 0,$$

since φ_n converges to 0 uniformly on K (the derivatives play no role here).

The distributions of the type T_f, where f is locally integrable, are called *regular distributions.*

For example, the Heaviside function H

$$H(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x < 0, \end{cases}$$

is an example of a locally integrable function. We denote the corresponding distribution by the same symbol. ◊

It can be shown that the inclusion map

$$L^1_{\text{loc}}(\mathbb{R}^d) \subset \mathcal{D}'(\mathbb{R}^d)$$

is injective. So just like we identify integers as rationals, we may think of all locally integrable functions as distributions. But distributions are more general, as shown by the following example.

Example 5.3 (Dirac delta distribution). The distribution $\delta \in \mathcal{D}'(\mathbb{R}^d)$ is defined by

$$\langle \delta, \varphi\rangle = \varphi(\mathbf{0}), \quad \varphi \in \mathcal{D}(\mathbb{R}^d).$$

More generally, one defines, for $\mathbf{a} \in \mathbb{R}^d$, a distribution $\delta_{\mathbf{a}}$ by

$$\langle \delta_{\mathbf{a}}, \varphi\rangle = \varphi(\mathbf{a}), \quad \varphi \in \mathcal{D}(\mathbb{R}^d).$$

It is evident that $\delta_{\mathbf{a}}$ is linear and continuous on $\mathcal{D}(\mathbb{R}^d)$, that is, it is a distribution.

The delta distribution is not regular: there is no function f such that $\delta_{\mathbf{a}} = T_f$. Nevertheless, in a huge amount of literature, one encounters a manner of writing that suggests that $\delta_{\mathbf{a}}$ is a regular distribution. In place of $\langle \delta, \varphi\rangle$, one writes

$$\int_{\mathbb{R}^d} \delta(\mathbf{x})\varphi(\mathbf{x})d\mathbf{x} = \varphi(\mathbf{0}).$$

Similarly sometimes one writes $\displaystyle\int_{\mathbb{R}^d} \delta_{\mathbf{a}}(\mathbf{x})\varphi(\mathbf{x})d\mathbf{x} = \varphi(\mathbf{a})$.

One then talks about delta "functions" instead of delta *distributions.* This is of course incorrect (see the exercise below), but in some sense useful if one wants to do formal manipulations in order to guess answers, or in order to get physical insights etc. With this fallcious understanding, one often depicts the "graph of $\delta \in \mathcal{D}'(\mathbb{R})$" as a spike, with the intuitive feeling that the "δ function is everywhere 0, but is infinity at $x = 0$, and has integral over $\mathbb{R}$ equal to 1"! ◊

Exercise 5.3. Show that there is no function $\delta : \mathbb{R} \to \mathbb{R}$ which has the property that for all $a > 0$,

(1) δ is Riemann integrable on $[-a, a]$,

(2) for every C^∞ function φ vanishing outside $[-a, a]$, $\displaystyle\int_{-a}^{a} \delta(x)\varphi(x)dx = \varphi(0)$.

5.2. Derivatives in the distributional sense

Let us first consider the case when $d = 1$.

Definition 5.5 (Distributional derivative). If $T \in \mathcal{D}'(\mathbb{R})$, then $T' \in \mathcal{D}'(\mathbb{R})$ is defined by

$$\langle T', \varphi \rangle = -\langle T, \varphi' \rangle, \quad \varphi \in \mathcal{D}(\mathbb{R}).$$

Note that if $\varphi \in \mathcal{D}(\mathbb{R})$, then clearly $\varphi' \in \mathcal{D}(\mathbb{R})$. So the right hand side above is well defined. Moreover, the map

$$\varphi \mapsto -\langle T, \varphi' \rangle$$

is linear. This map is also continuous on $\mathcal{D}(\mathbb{R})$. Indeed, if $\varphi_n \xrightarrow{\mathcal{D}} 0$, then also $\varphi_n' \xrightarrow{\mathcal{D}} 0$, and so

$$-\langle T, \varphi_n' \rangle \to 0.$$

Thus $T' \in \mathcal{D}'(\mathbb{R})$.

Lemma 5.1. *If $f \in C^1(\mathbb{R})$, then $(T_f)' = T_{f'}$.*

Proof. Let $\varphi \in \mathcal{D}(\mathbb{R})$ be such that it vanishes outside $[a, b]$. Then using integration by parts,

$$\begin{aligned}\langle (T_f)', \varphi \rangle &= -\langle T_f, \varphi' \rangle = -\int_{\mathbb{R}} f(x)\varphi'(x)dx = -\int_a^b f(x)\varphi'(x)dx \\ &= 0 - \Big(-\int_a^b f'(x)\varphi(x)dx \Big) = \langle T_{f'}, \varphi \rangle.\end{aligned}$$

(Here we have used the facts that φ, φ' are zero outside $[a, b]$, and $\varphi(a) = \varphi(b) = 0$.) This completes the proof. $\square$

Remark 5.2. The above result means that whenever one identifies the function f with the distribution T_f, then the two possible interpretations of the derivative which arise—the classical sense versus the new distributional sense—coincide. However, the next example shows that now we can differentiate functions which we couldn't earlier, albeit we can do so only in the distributional sense.

Example 5.4 ($H' = \delta$). For any test function $\varphi \in \mathcal{D}(\mathbb{R})$, we know that $\varphi(x) = 0$ for all sufficiently large x, and so

$$\langle H', \varphi \rangle = -\langle H, \varphi' \rangle = -\int_0^\infty \varphi' dx = -\varphi(x)\Big|_0^\infty = \varphi(0) = \langle \delta, \varphi \rangle,$$

and so $H' = \delta$. $\Diamond$

Example 5.5 (Dipole). The derivative δ' of δ, is called the ***dipole***, and is given by

$$\langle \delta', \varphi \rangle = -\langle \delta, \varphi' \rangle = -\varphi'(0),$$

for all $\varphi \in \mathcal{D}(\mathbb{R})$. $\diamondsuit$

Proposition 5.2 (Jump Rule). *Let f be continuously differentiable on $\mathbb{R}$ except at the point $a \in \mathbb{R}$, where the limits $f(a+), f(a-), f'(a+), f'(a-)$ exist. Then f, f' are locally integrable, and*

$$(T_f)' = T_{f'} + \big(f(a+) - f(a-)\big)\delta_a.$$

We think of $f(a+) - f(a-)$ as the jump in f at the point a. One can formulate this result by saying:

> The derivative of f in the sense of distributions is
> the classical derivative plus δ_a times the jump in f at a.

Proof. Let $\varphi \in \mathcal{D}(\mathbb{R})$, and suppose that φ is 0 outside $[\alpha, \beta]$, and that $a \in [\alpha, \beta]$. Then

$$\begin{aligned}
\langle (T_f)', \varphi \rangle &= -\langle T_f, \varphi' \rangle = -\int_\alpha^\beta f(x)\varphi'(x)dx \\
&= -\int_\alpha^a f(x)\varphi'(x)dx - \int_a^\beta f(x)\varphi'(x)dx \\
&= \int_\alpha^a f'(x)\varphi(x)dx - f(a-)\varphi(a) + \int_a^\beta f'(x)\varphi(x)dx + f(a+)\varphi(a) \\
&= \int_\alpha^\beta f'(x)\varphi(x)dx + \big(f(a+) - f(a-)\big)\varphi(a) \\
&= \langle T_{f'}, \varphi \rangle + \big(f(a+) - f(a-)\big)\langle \delta_a, \varphi \rangle \\
&= \big\langle T_{f'} + \big(f(a+) - f(a-)\big)\delta_a, \varphi \big\rangle. \qquad \square
\end{aligned}$$

Remark 5.3. This result can be extended to the case when f is continuously differentiable everywhere except for a finite number of points a_k, and at these points a_k, the function satisfies the same assumptions as stipulated above. This then leads to

$$(T_f)' = T_{f'} + \sum_k \big(\underbrace{f(a_k+) - f(a_k-)}_{=:\sigma_k}\big)\delta_{a_k}.$$

The proof is analogous. In fact the result even extends to the case when f has infinitely many jump discontinuities provided that in any compact interval, one finds only finitely many discontinuities. The sum on the right hand side is the distribution defined by

$$\Big\langle \sum_k \sigma_k \delta_{a_k}, \varphi \Big\rangle = \sum_k \sigma_k \varphi(a_k),$$

where, for a given test function φ, only finitely many terms on the right hand side are nonzero.

Exercise 5.4. Show that

$$\begin{aligned}\frac{d}{dx}H(x)\cos x &= -H(x)\sin x+\delta,\\ \frac{d}{dx}H(x)\sin x &= H(x)\cos x.\end{aligned}$$

Exercise 5.5 (Fundamental solution to the 1D Laplace equation). Show that the equation

$$\frac{d^2}{dx^2}E=\delta$$

is satisfied by $E := \frac{1}{2}|x|$.

When $d > 1$, the definition of the distributional derivative is analogous.

Definition 5.6. Let $T \in \mathcal{D}'(\mathbb{R}^d)$. Then the ith-partial derivative of T, $1 \leqslant i \leqslant d$, is defined by

$$\Big\langle \frac{\partial T}{\partial x_i}, \varphi \Big\rangle = -\Big\langle T, \frac{\partial \varphi}{\partial x_i}\Big\rangle.$$

Exercise 5.6. Show that for all $T \in \mathcal{D}'(\mathbb{R}^d)$ and all i, j, $\dfrac{\partial^2 T}{\partial x_i \partial x_j} = \dfrac{\partial^2 T}{\partial x_j \partial x_i}$.

Exercise 5.7. The Heaviside function in two variables, $H : \mathbb{R}^2 \to \mathbb{R}$, is defined by

$$H(x,y)=\begin{cases}1 & \text{if } x\geqslant 0 \text{ and } y\geqslant 0,\\ 0 & \text{if } x<0 \text{ or } y<0.\end{cases}$$

(That is, H is the indicator function $1_{[0,\infty)^2}$ of the "first quadrant".) Show that $\dfrac{\partial^2 H}{\partial x \partial y} = \delta_{\mathbf{0}}$.

5.3. Weak solutions

A *weak solution* to a PDE will be one which is not a classical solution, but satisfies the PDE in the sense of distributions.

5.3.1. Weak solution to the transport equation. We had seen that the transport equation,

$$\begin{cases}\dfrac{\partial u}{\partial t} - c\dfrac{\partial u}{\partial x} = 0,\\ u(x,0) = f(x)\end{cases}$$

has the solution

$$u(x,t) = f(x+ct)$$

provided f is continuously differentiable. Relaxing this latter condition, we claim that this u is a weak solution provided f is locally integrable. To see this, let $\varphi \in \mathcal{D}(\mathbb{R}^2)$. Then we have

$$\langle u_t - cu_x, \varphi\rangle = \langle u, -\varphi_t + c\varphi_x\rangle = -\iint\limits_{\mathbb{R}^2} f(x+ct)\big(\varphi_t(x,t) - c\varphi_x(x,t)\big)dxdt.$$

Hence to prove this claim, it must be shown that the above integral is zero. To do this, we will make the following change of variables:

$$\boxed{\begin{array}{l}\xi = x + ct\\ \eta = t\end{array}} \quad \longleftrightarrow \quad \boxed{\begin{array}{l}x = \xi - c\eta,\\ t = \eta.\end{array}}$$

Recall that for a double integral, one has the following "change of variables" formula under the change of variables given by the map $(\xi, \eta) \overset{\Psi}{\mapsto} (x, t)$:

$$\iint\limits_{\mathbb{R}^2} F(x,y)dxdy = \iint\limits_{\mathbb{R}^2} (F \circ \Psi)(\xi, \eta) \cdot |J(\xi, \eta)| d\xi d\eta,$$

where $J(\xi, \eta)$ is the Jacobian determinant given by

$$J(\xi, \eta) = \det \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial \eta} \\ \dfrac{\partial t}{\partial \xi} & \dfrac{\partial t}{\partial \eta} \end{bmatrix}.$$

In our case, the derivative of the map $(\xi, \eta) \overset{\Psi}{\mapsto} (x, t)$ is

$$\begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial \eta} \\ \dfrac{\partial t}{\partial \xi} & \dfrac{\partial t}{\partial \eta} \end{bmatrix} = \begin{bmatrix} 1 & -c \\ 0 & 1 \end{bmatrix},$$

whose determinant is 1. Furthermore,

$$\begin{aligned} \varphi_t &= (\varphi \circ \Psi)_\xi \cdot \xi_t + (\varphi \circ \Psi)_\eta \cdot \eta_t = c(\varphi \circ \Psi)_\xi + (\varphi \circ \Psi)_\eta, \\ \varphi_x &= (\varphi \circ \Psi)_\xi \cdot \xi_x + (\varphi \circ \Psi)_\eta \cdot \eta_x = (\varphi \circ \Psi)_\xi. \end{aligned}$$

Thus $\varphi_t - c\varphi_x = c(\varphi \circ \Psi)_\xi + (\varphi \circ \Psi)_\eta - c(\varphi \circ \Psi)_\xi = (\varphi \circ \Psi)_\eta$. So

$$\begin{aligned} &\iint\limits_{\mathbb{R}^2} f(x + ct)\big(\varphi_t(x,t) - c\varphi_x(x,t)\big) dxdt \\ &\qquad = \iint\limits_{\mathbb{R}^2} f(\xi) \cdot (\varphi \circ \Psi)_\eta(\xi, \eta) \cdot |1| d\xi d\eta \\ &\qquad = \int_{\mathbb{R}} f(\xi) \int_{\mathbb{R}} (\varphi \circ \Psi)_\eta(\xi, \eta) d\eta d\xi \\ &\qquad = \int_{\mathbb{R}} f(\xi) \Big((\varphi \circ \Psi)(\xi, \eta) \Big|_{\eta=-\infty}^{\eta=+\infty} \Big) d\xi \\ &\qquad = \int_{\mathbb{R}} f(\xi) \cdot 0 d\xi = 0, \end{aligned}$$

where we have used the Fundamental Theorem of Calculus to simplify the inner integral, and used the fact that φ has compact support to obtain

$$\begin{aligned}(\varphi\circ\Psi)(\xi,\eta)\Big|_{\eta=-\infty}^{\eta=+\infty} &= \lim_{\eta\to\infty}(\varphi\circ\Psi)(\xi,\eta)-\lim_{\eta\to-\infty}(\varphi\circ\Psi)(\xi,\eta)\\ &= \lim_{\eta\to\infty}\varphi(\xi-c\eta,\eta)-\lim_{\eta\to-\infty}\varphi(\xi-c\eta,\eta)=0-0=0.\end{aligned}$$

This proves the claim.

5.3.2. Weak solution to the wave equation. Recall that if $f\in C^2(\mathbb{R})$, then

$$u(x,t):=\frac{f(x+ct)+f(x-ct)}{2} \tag{5.1}$$

is a classical solution to

$$\frac{\partial^2 u}{\partial t^2}-c^2\frac{\partial^2 u}{\partial x^2}=0$$

with the initial condition $u(x,0)=f(x)$ and with zero intial speed $u_t(x,0)=0$. Let us now show that even when f is locally integrable, u given by (5.1) satisfies the wave equation, but in the sense of distributions. In order to do this, we will use our result from the previous section, where we considered the transport equation. We have just seen that u_+ given by

$$u_+(x,t):=f(x+ct),$$

where $f\in L^1_{\text{loc}}$ is a dstributional solution of the transport equation

$$\frac{\partial u}{\partial t}-c\frac{\partial u}{\partial x}=0.$$

By replacing c by $-c$, we also see that u_- given by

$$u_-(x,t):=f(x-ct)$$

is a distributional solution to

$$\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}=0.$$

But

$$\frac{\partial^2}{\partial t^2}-c^2\frac{\partial^2}{\partial x^2}=\Big(\frac{\partial}{\partial t}-c\frac{\partial}{\partial x}\Big)\Big(\frac{\partial}{\partial t}+c\frac{\partial}{\partial x}\Big),$$

and using this observation, we will find a weak solution to the wave equation too. Let u be given by (5.1), and $\varphi\in\mathcal{D}(\mathbb{R}^2)$. Then

$$\begin{aligned}\Big\langle\frac{\partial^2 u}{\partial t^2}-c^2\frac{\partial^2 u}{\partial x^2},\varphi\Big\rangle &= \Big\langle\Big(\frac{\partial}{\partial t}-c\frac{\partial}{\partial x}\Big)\Big(\frac{\partial}{\partial t}+c\frac{\partial}{\partial x}\Big)u,\varphi\Big\rangle\\ &= -\Big\langle\Big(\frac{\partial}{\partial t}-c\frac{\partial}{\partial x}\Big)f(x+ct),\Big(\frac{\partial}{\partial t}+c\frac{\partial}{\partial x}\Big)\varphi\Big\rangle\\ &\quad -\Big\langle\Big(\frac{\partial}{\partial t}+c\frac{\partial}{\partial x}\Big)f(x-ct),\Big(\frac{\partial}{\partial t}-c\frac{\partial}{\partial x}\Big)\varphi\Big\rangle\\ &= -0-0=0.\end{aligned}$$

Exercise 5.8 (Weak solution exists, but no classical solution). Show that

$$u(x) = \begin{cases} c & \text{if } x < 0, \\ x + c & \text{if } x > 0, \end{cases}$$

is a weak solution of the ODE $u' = H$, where H is the Heaviside function.

5.3.3. Multiplication by C^∞ functions. In general, it is not possible to define the product of two distributions. For example, the product of two locally integrable functions is not in general locally integrable. ($f := \sqrt{|x|}^{-1}$ is locally integrable, but f^2 isn't!) So the product of two regular distributions in general may not define a distribution.

However, one *can* define the product of a function $\alpha \in C^\infty(\mathbb{R}^d)$ with a distribution $T \in \mathcal{D}'(\mathbb{R}^d)$ by setting

$$\langle \alpha T, \varphi \rangle = \langle T, \alpha\varphi \rangle, \quad \varphi \in \mathcal{D}(\mathbb{R}^d).$$

Note that if $\varphi \in \mathcal{D}(\mathbb{R}^d)$, then it is in particular in $C^\infty(\mathbb{R}^d)$, and so it is clear that $\alpha\varphi$ is infinitely many times differentiable. Moreover, as φ vanishes outside a compact set, so does $\alpha\varphi$. Hence $\alpha\varphi \in \mathcal{D}(\mathbb{R}^d)$, and the right hand side makes sense. It is also easy to see that

$$\varphi \mapsto \langle T, \alpha\varphi \rangle : \mathcal{D}(\mathbb{R}^d) \to \mathbb{R}$$

is linear, thanks to the linearity of T. Finally, it can be shown (using the Leibniz Rule) that if $\varphi_n \xrightarrow{\mathcal{D}} 0$, then also $\alpha\varphi_n \xrightarrow{\mathcal{D}} 0$. Consequently, $\alpha T \in \mathcal{D}'(\mathbb{R}^d)$.

Proposition 5.3. *If $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $\alpha \in C^\infty(\mathbb{R}^d)$, then*

$$\alpha T_f = T_{\alpha f}.$$

Proof. α is bounded on every compact set, and so it follows that αf is locally integrable. For $\varphi \in \mathcal{D}(\mathbb{R}^d)$, we have

$$\langle \alpha T_f, \varphi \rangle = \langle T_f, \alpha\varphi \rangle = \int_{\mathbb{R}^d} f(\mathbf{x})\alpha(\mathbf{x})\varphi(\mathbf{x}) d\mathbf{x} = \langle T_{\alpha f}, \varphi \rangle.$$

This completes the proof. □

The above result means that whenever we identify as usual the elements of $L^1_{\text{loc}}(\mathbb{R}^d)$ with distributions, then the two a priori different manners of forming the product with α lead to the same result.

Example 5.6. One can think of the distribution $H(x)\cos x$ as the product of the C^∞ function $\cos x$ with the distribution $H(x)$. ◊

Proposition 5.4. *The following calculation rules hold.*
For $T, T_1, T_2 \in \mathcal{D}'(\mathbb{R}^d)$, $\alpha_1, \alpha_2, \alpha, \beta \in C^\infty(\mathbb{R}^d)$, we have

(1) $\alpha(T_1 + T_2) = \alpha T_1 + \alpha T_2$
(2) $(\alpha_1 + \alpha_2)T = \alpha_1 T + \alpha_2 T$
(3) $(\alpha\beta)T = \alpha(\beta T)$

(4) $1T = T$.

(Thus $\mathcal{D}'(\mathbb{R}^d)$ is a $C^\infty(\mathbb{R}^d)$-module[4].)

Proof. All of these follow from the definition of multiplication of distributions by C^∞ functions. For example, to check (3), note that for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$, we have

$$\langle(\alpha\beta)T, \varphi\rangle = \langle T, (\alpha\beta)\varphi\rangle = \langle T, \beta(\alpha\varphi)\rangle = \langle \beta T, \alpha\varphi\rangle = \Big\langle \big(\alpha(\beta T)\big), \varphi\Big\rangle,$$

proving the claim. □

The product rule for differentiation is valid in the same manner as for functions.

Theorem 5.5 (Product Rule). *For $T \in \mathcal{D}'(\mathbb{R}^d)$ and $\alpha \in C^\infty(\mathbb{R}^d)$,*

$$\begin{aligned} (d = 1): \qquad & (\alpha T)' = \alpha' T + \alpha T' \\ (d > 1): \qquad & \frac{\partial}{\partial x_i}(\alpha T) = \Big(\frac{\partial \alpha}{\partial x_i}\Big)T + \alpha\Big(\frac{\partial T}{\partial x_i}\Big). \end{aligned}$$

Proof. When $d = 1$ and $\varphi \in \mathcal{D}(\mathbb{R})$, we have $(\alpha\varphi)' = \alpha'\varphi + \alpha\varphi'$, and so

$$\begin{aligned} \langle(\alpha T)', \varphi\rangle &= -\langle \alpha T, \varphi'\rangle = -\langle T, \alpha\varphi'\rangle \\ &= -\langle T, (\alpha\varphi)'\rangle + \langle T, \alpha'\varphi\rangle \\ &= \langle T', \alpha\varphi\rangle + \langle \alpha' T, \varphi\rangle \\ &= \langle \alpha T', \varphi\rangle + \langle \alpha' T, \varphi\rangle \\ &= \langle \alpha T' + \alpha' T, \varphi\rangle. \end{aligned}$$

The proof is analogous when $d > 1$. □

Theorem 5.6. *If $\mathbf{a} \in \mathbb{R}^d$ and $\alpha \in C^\infty(\mathbb{R}^d)$, then*

$$\alpha\delta_{\mathbf{a}} = \alpha(a)\delta_{\mathbf{a}}.$$

Proof. For $\varphi \in \mathcal{D}(\mathbb{R}^d)$, we have

$$\langle \alpha\delta_{\mathbf{a}}, \varphi\rangle = \langle \delta_{\mathbf{a}}, \alpha\varphi\rangle = (\alpha\varphi)(\mathbf{a}) = \alpha(\mathbf{a})\varphi(\mathbf{a}) = \alpha(\mathbf{a})\langle \delta_{\mathbf{a}}, \varphi\rangle = \langle \alpha(\mathbf{a})\delta_{\mathbf{a}}, \varphi\rangle. \qquad \square$$

Example 5.7. We have $x\delta = 0$, $(\cos x)\delta = \delta$, $(\sin x)\delta = 0$. ◊

Exercise 5.9. Redo Exercise 5.4 using the Product Rule.

Exercise 5.10 (Fundamental solutions). Show that if $\lambda \in \mathbb{R}$, $n \in \mathbb{N}$ and $\omega \in \mathbb{R}\backslash\{0\}$, then

(1) $\displaystyle\Big(\frac{d}{dx} - \lambda\Big)H(x)e^{\lambda x} = \delta$

(2) $\displaystyle\frac{d^n}{dx^n}\Big(H(x)\frac{x^{n-1}}{(n-1)!}\Big) = \delta$

(3) $\displaystyle\Big(\frac{d^2}{dx^2} + \omega^2\Big)H(x)\frac{\sin(\omega x)}{\omega} = \delta.$

[4] A module is just like a vector space, except that the underlying field is replaced by a ring—here the ring is $C^\infty(\mathbb{R}^d)$.

Exercise 5.11. Show that if $\alpha \in C^\infty(\mathbb{R})$, then $\alpha\delta' = \alpha(0)\delta' - \alpha'(0)\delta$. Conclude that $x\delta' = -\delta$.

Exercise 5.12. Show that for all $T \in \mathcal{D}'(\mathbb{R})$, we have

$$\Big[x, \frac{d}{dx}\Big]T := x\frac{dT}{dx} - \frac{d}{dx}(xT) = -T.$$

(Thus the *commutant* $\Big[x, \dfrac{d}{dx}\Big] = -1$.)

5.4. Fourier transform

We make a few final parting remarks, which are not really a part of the course, but which give a glimpse of what lies ahead.

The idea is that we would like to define the Fourier transform of a nice distribution T by setting

$$\langle \widehat{T}, \varphi \rangle = \langle T, \widehat{\varphi} \rangle.$$

But $\widehat{\varphi}$ may not have compact support[5], and so we need to enlarge our set of test functions. This leads to the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ of test functions turn out to be appropriate. We will not define these here, but simply remark that this allows us to consider the dual space $\mathcal{S}'(\mathbb{R}^d)$ of the Schwartz class of test functions, giving the space of tempered distributions

$$\mathcal{S}'(\mathbb{R}^d) \subset \mathcal{D}'(\mathbb{R}^d).$$

Much of the classical Fourier transform theory can be extended appropriately for the class of tempered distributions. This allows one to rigorously justify some of the formal calculations done in the previous chapters. It also gives rise to some important auxiliary concepts which are quite useful in the theory of PDEs. One such notion is the concept of a fundamental solution for a linear PDE.

Definition 5.7 (Fundamental Solution). Given a linear partial differential operator with constant coefficients,

$$D = \sum_{|\mathbf{k}| \leqslant K} a_{\mathbf{k}} D^{\mathbf{k}},$$

a ***fundamental solution*** is a distribution $E \in \mathcal{D}'(\mathbb{R}^d)$ such that

$$DE = \delta.$$

(In the above, $|\mathbf{k}| := k_1 + \cdots + k_d$ for $\mathbf{k} = (k_1, \cdots, k_d)$.)

These are useful since they allow one to solve the inhomogeneous equation

$$Du = g.$$

For a suitable g (for example a distribution with compact support), it can be shown that

$$u := E * g$$

[5] In fact, it can be shown that $\widehat{\varphi}$ belongs to $\mathcal{D}$ if and only if $\varphi = 0$.

does the job: $Du = D(E * g) = (DE) * g = \delta * g = g$. There is also a deep theorem of Malgrange and Ehrenpreiss which says that every nonzero operator with constant coefficients has a fundamental solution. Fundamental solutions with appropriate boundary conditions specific to a PDE problem are sometimes referred to as *Green's functions.*

Example 5.8. It follows from Exercise 5.5 that a fundamental solution for the one-dimensional Laplacian operator

$$\frac{d^2}{dx^2}$$

is $E_{\mathrm{p}} := |x|/2$. In fact if we add to E_{p} any solution to the homogeneous equation $u'' = 0$, then it will also be a fundamental solution. So $ax + b + |x|/2$ with arbitrary a, b are all fundamental solutions of the one-dimensional Laplacian operator. $\lozenge$

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