On Monotone Strategy Equilibria in Simultaneous Auctions for Complementary Goods *

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Abstract

We explore existence and properties of equilibrium when $N \geq 2$ bidders compete for $L \geq 2$ objects via simultaneous but separate auctions. Bidders have private combinatorial valuations over all sets of objects they could win, and objects are complements in the sense that these valuations are supermodular in the set of objects won. We provide a novel partial order on types under which best replies are monotone, and demonstrate that Bayesian Nash equilibria which are monotone with respect to this partial order exist on any finite bid lattice. We apply this result to show existence of monotone Bayesian Nash equilibria in continuous bid spaces when a single global bidder competes for L objects against many local bidders who bid for single objects only, highlighting the step in this extension which fails with multiple global bidders. We therefore instead consider an alternative *equilibrium with endogenous tiebreaking* building on Jackson, Simon, Swinkels and Zame (2002), and demonstrate that this exists in general. Finally, we explore efficiency in simultaneous auctions with symmetric bidders, establishing novel sufficient conditions under which inefficiency in expectation approaches zero as the number of bidders increases.

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1 Introduction

Simultaneous bidding for multiple objects is a commonly occurring phenomenon in many real-world auction markets, but surprisingly little is known about the properties of equilibria in games involving simultaneous auctions when bidder payoffs are non-additive.[1] For example, when auctioning drilling rights in the US Outer Continental Shelf, the US Minerals Management Service typically offers (and bidders typically bid on) a large number of drilling tracts simultaneously. Prior empirical work (e.g. Hendricks and Porter, 1988, Hendricks, Pinkse and Porter, 2003) suggests that economically important complementarities may exist between tracts in close proximity. Yet little is presently known – either theoretically or empirically – about how such synergies might affect equilibrium behavior in such markets.[2]

This paper analyzes equilibrium within a class of mechanisms we refer to as *simultaneous standard auctions* for complementary goods. In this setting, a collection of $L \geq 2$ objects are offered for sale to a set of $N \geq 2$ bidders. Bidders have independent private valuations over *combinations* of objects, where objects are *complements* in the sense that bidders' valuations are supermodular in sets of objects won. Auctions are *simultaneous* in the sense that bidders may bid on each object individually but may not submit contingent or combinatorial bids, and *standard* in the sense that each object l is allocated to a high bidder in auction l and payments in auction l depend only on bids in auction l. So long as all auctions are standard, auctions for different objects may have different formats. For simplicity, we frame discussion in terms of a

versity.

[1]Examples of markets involving simultaneous bidding include highway procurement in many US states (e.g. Krasnokutskaya [2011], Somaini [2013], Groeger [2014] among others), recycling services in Japan (Kawai, 2010), cleaning services in Sweden (Lunander and Lundberg, 2013), oil and drilling rights in the US Outer Continental Shelf (Hendricks and Porter, 1988, Hendricks, Pinkse and Porter, 2003), and to a lesser extent US Forest Service timber harvesting (Athey, Levin and Siera, 2011, among many others).

[2]Notable exceptions are Fox and Bajari (2013), who estimate the deterministic component of bidder valuations in FCC simultaneous ascending spectrum auctions, and Gentry, Komarova and Schiraldi (2017), who empirically study simultaneous bidding in Michigan Department of Transportation highway procurement auctions.

single auctioneer, although this is inessential for our results.

The simultaneous standard auction game raises a number of significant theoretical challenges. Even assuming independent private types, each bidder's preference structure could in principle be as complex as a complete $(2^L - 1)$-dimensional set of valuations assigned by that bidder to each of the $2^L - 1$ possible non-empty subsets of objects. Meanwhile, the simultaneous standard auction permits bidders to submit (at most) L individual bids on the L objects being sold. Furthermore, as usual in auctions, payoffs in the resulting game may be discontinuous in bids. The end result is a discontinuous Bayesian game with high-dimensional types for which even basic properties – such as existence of Bayesian Nash equilibrium – are challenging to establish in general.

Section 2 introduces the model and shows that even the (strong) assumption of supermodular valuations is insufficient to ensure monotonicity of best replies with respect to the usual coordinatewise order on types – in fact, a strict coordinatewise increase in type can induce a strict coordinatewise decrease in best-reply bids. This turns out to be because the usual coordinatewise order on types imposes insufficient structure on marginal valuations. Motivated by this observation, Section 3 introduces a partial order on bidder types characterized by a finite number of linear inequalities on marginal valuations. These inequalities define a cone (with nonempty interior) strictly contained in the first-orthant cone of the $(2^L - 1)$-dimensional type space – i.e., the cone describing the usual coordinatewise order. This stronger partial order turns out to be sufficient for monotonicity in the sense that each bidder i has an interim best reply such that an increase in i's type with respect to our partial order will imply an increase in i's bids with respect to the usual coordinatewise order.

Equipped with this preliminary result, Section 4 builds on the methodology of Athey (2001), McAdams (2003) and Reny (2011) to establish existence of pure strategy equilibria on finite bid spaces which are monotone in the sense above. This result turns on one additional condition, which clarifies the relationship between the partial order cone and the support of bidder's joint distribution of valuations. Namely, we show that this relationship is "sufficiently rich" if this distribution is absolutely continuous with respect to the Lebesgue measure in the $(2^L - 1)$-dimensional space and the support of this distribution is regular enough. While the existence of a pure strategy Bayes-Nash Equilibrium follows from Milgrom and Weber (1985), the monotone characterization of the equilibria under a suitable partial order on types is novel in this setting.

We then proceed to consider continuous bidding spaces. First, in Section 5.1, we consider a special case similar in spirit to Krishna and Rosenthal (1996), in which a single global bidder bids in simultaneous first-price auctions for L objects against a collection of local bidders who bid for single objects only.[3] Building on proof techniques in Reny (2011), we show existence of a pure strategy Bayes-Nash equilibrium which is monotone with respect to our partial order. To the best of our knowledge, both existence and monotonicity are novel in this setting. Moreover, monotonicity is here pivotal in establishing existence; the proof turns on passing from a sequence of monotone equilibria on discrete spaces to the limit of this sequence in a continuous space, which is feasible only because the space of monotone strategies is known to be compact in the pointwise convergence topology if the partial order is sufficiently rich (Reny, 2011).

While we believe that this finding is of interest in its own right, this example also serves to highlight a subtle challenge arising in settings with more than one global bidder. Specifically, the interaction between strategic overbidding by global bidder i and dependence across auctions of bids by i's global rivals leads to uncertainty regarding a key technical property – better-reply security of Reny (1999) – needed to complete the existence proof. In Section 5.2, we therefore turn to an alternative solution concept, *equilibrium with endogenous tiebreaking*, building on the work of Jackson, Simon, Swinkels and Zame (2002). Jackson, Simon, Swinkels and Zame

[3] In this section only, we restrict attention to simultaneous *first-price* auctions, as our primary purpose is illustration. We conjecture, however, that arguments similar to those in 5.1 could be used to establish existence of monotone equilibria in standard auctions more generally.

(2002, henceforth JSSZ) define the *communication extension* $\mathcal{G}^c$ to a given game $\mathcal{G}$ as the game arising when, in addition to their actions under $\mathcal{G}$, players also submit cheap-talk indications of their types which the auctioneer may use (only) to resolve ties. A *solution to* $\mathcal{G}^c$ is a strategy profile for bidders plus a tiebreaking rule such that strategies are a Bayesian Nash equilibrium given the tiebreaking rule. Starting from a class of discontinuous games $\mathcal{G}$ which includes ours, JSSZ (2002) establish existence of solutions to $\mathcal{G}^c$ in which bidders play *distributional strategies* as defined by Milgrom and Weber (1985) and communication is truthful.

In the context of simultaneous auctions for complementary goods, we show that these general conclusions can be sharpened in at least three respects. First, rather than permitting bidders to communicate their full $(2^L - 1) \times 1$-dimensional types, we allow bidder i to submit (in addition to her bid vector b_i) only an $L \times 1$ vector of cheap-talk signals s_i; we refer to the latter as a *signalling extension* to distinguish it from the *communication extension* of JSSZ (2002). Second, we show the existence of a solution to the signalling extension in which the auctioneer's tiebreaking rule can be characterized by a set of L weakly monotone *tiebreaking precedence functions* $(\rho_1, ..., \rho_l)$, where the auctioneer randomizes object l independently among the set of high bidders in auction l with the highest tiebreaking precedence: i.e. among the set of bidders i with $b_{il} = \max_j\{b_{jl}\}$ and $\rho_l(s_{il}) = \max_j\{\rho_l(s_{jl})\}$. This characterization of tiebreaking sharpens that in JSSZ (2002), implying in particular existence of a solution where allocations and payments in auction l depend only on bids and signals in auction l. Furthermore, whereas JSSZ (2002) consider only existence in distributional strategies, we obtain existence in pure strategies which are additionally monotone in a suitable partial order sense.

Finally, in Section 6, we relate our model to the important question of the performance of auctions when the number of bidders becomes large. We consider the case of ex-ante symmetric bidders playing a Bayes-Nash equilibrium in symmetric monotone strategies. We obtain a sufficient condition that guarantees that the expected inefficiency in a symmetric monotone equilibrium converges to zero. Intuitively, this condition requires the support of private types to contain one type which dominates all others in a partial order strictly more restrictive than the one introduced in Section 3. While admittedly a strong restriction, this is to our knowledge one of the first positive results on efficiency in simultaneous auctions for complementary goods.

Proofs of all propositions and lemmas are collected in the Appendix.

Related literature The effect of complementarities in simultaneous auctions has been studied, among others and in very specific setups, by Bikhchandani (1999), by Rosenthal and Wang (1996) and Szentes and Rosenthal (2003) in simultaneous first-price auctions, by Krishna and Rosenthal (1996) in simultaneous second-price auction, and by Brusco and Lopomo (2002, 2009) and Cramton (1997) in simultaneous ascending auctions.

Szentes and Rosenthal (2003) study the simultaneous first-price mechanism in a complete information setting with two identical players who compete via simultaneous first-price auctions for three identical objects. Their analysis highlights the challenges involved in study of the simultaneous first-price mechanism – even in relatively simple settings, equilibrium turns out to have subtle and surprising properties. Similar complexity arises in Krishna and Rosenthal (1996), who study a setting where many identical objects are auctioned via simultaneous second-price auctions to two types of bidders: global bidders, who bid in multiple auctions, and local bidders, who bid in one auction only. Global bidders' preferences are characterized by a one-dimensional private type describing their valuation for each (identical) single object, with a deterministic, common knowledge synergy realized in the event of a multiple win. In contrast, we allow a much richer type space in which global bidders to have private valuations for each of the possible $(2^L - 1) \times 1$ combinations of auctioned objects. We thereby take a significant step toward characterizing equilibrium in a broad class of simultaneous auction games.

Games of incomplete information with payoffs supermodular in actions have also been studied by, among others, Athey (2001), McAdams (2003) and Reny (2011). Vives (1990) established the existence of pure-strategy equilibria in Bayesian games when payoffs are supermodular and upper-semicontinuous in actions. This could provide an alternative path to establishing existence when the bid space is finite, but does not speak to *monotone* equilibria. Meanwhile, Van Zandt and Vives (2007) demonstrate existence of monotone pure strategy equilibria in games with supermodular utility assuming (among other conditions) that utility is continuous and exhibits increasing differences in own and rival actions. The latter condition does not hold even in finite bid spaces, and therefore cannot be applied in our setting.[4]

There is also a substantial literature analyzing properties of various combinatorial auction mechanisms. Notable studies in this literature include Cantillon and Pesendorfer (2006), Ausubel and Milgrom (2002), Ausubel and Cramton (2004), Cramton (2006), Krishna and Rosenthal (1996), Klemperer (2008, 2010), Milgrom (2000a, 2000b), to mention just a few. Detailed surveys of this literature are given in de Vreis and Vorha (2003) and Cramton et al. (2006). While these studies also consider settings where bidders have preferences over combinations, the theoretical problems generated by simultaneous bidding differ substantially from those encountered in true combinatorial mechanisms.[5]

[4]As usual in auctions, once one moves from finite to continuous bid spaces, utility is no longer either continuous or semi-continuous in actions. Hence results based on these no longer apply.

[5]Though only tangentially related to our problem, there is also a growing literature on multi-unit discriminatory auctions of homogeneous objects. Reny (1999, 2011), Athey (2001), and McAdams (2006) address existence and properties of equilibrium in such auctions. Meanwhile, Hortacsu and Puller (2008), Hortacsu and McAdams (2010), and Hortacsu (2011) provide more empirical perspectives on multi-unit auctions.

2 Simultaneous standard auctions with complementarities

Consider a setting in which N risk-neutral bidders compete for L prizes allocated via a class of mechanisms we call *simultaneous standard auctions*, defined as follows:

Definition 1 (Simultaneous standard auctions)**.** *We say that objects $l = 1, ..., L$ are allocated via* simultaneous standard auctions *if the bidding mechanism is such that:*

1. *Bidders may bid for each object $l = 1, ..., L$ individually, but may not submit combination or contingent bids;*
2. *Each object l is allocated to a high bidder in auction l, with payments conditional on allocation determined solely by bids in auction l.*

Note that while allocations are always to a high bidder, payment rules need not be the same across l. In what follows, we frame discussion in terms of a single seller, although this is inessential for our results.

For ease of exposition, in analyzing monotonicity and bidding we will initially assume that ties are broken randomly and independently across objects:

Assumption 1 (Independent tie-breaking)**.** *Ties are broken independently across auctions; tie-breaking does not depend on bidders' types.*

We will maintain this assumption through Section 5.1, which demonstrates existence of monotone equilibria in continuous bid spaces with one global bidder. It will be dropped in Section 5.2, when we consider monotone equilibria with endogenous tiebreaking in continuous bid spaces with many global bidders.

Let an *outcome* from the perspective of bidder i be an $L \times 1$ indicator vector ω with a 1 in the lth place if object l is allocated to bidder i and a 0 in the lth place otherwise. Similarly, let the *outcome matrix* Ω for bidder i be the $(2^L - 1) \times L$ matrix whose rows contain (transposes of) each possible outcome $\omega \neq 0$: e.g. if $L = 2$,

$$\Omega^T = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

In what follows, we use the squared Euclidean norm $\|\omega\|^2$ to denote the number of objects allocated to bidder i in outcome ω.

Bidder preferences Let Y_i^ω denote the *combinatorial valuation* bidder i assigns to outcome ω. We normalize the outcome "win nothing" to zero, and assume that valuations are non-decreasing in the set of objects won:

Assumption 2 (Values Normalized and Non-decreasing)**.** $Y_i^0 = 0$ *and* Y_i^ω *is non-decreasing in the vector of objects won:* $\omega' \geq \omega$ *implies* $Y_i^{\omega'} \geq Y_i^\omega$*.*

Let Y_i be the $(2^L - 1) \times 1$ vector describing the combinatorial valuations i assigns to all possible winning outcomes (normalizing $Y_i^0 = 0$ as above), with elements of Y_i corresponding to rows in Ω. In what follows, we interpret Y_i as bidder i's *private type* in the bidding game, known to bidder i but unknown to rivals at the time of bidding. We further assume that private types Y_i are i.i.d. across bidders:

Assumption 3 (Independent Private Values)**.** *Each bidder* i *draws private type* Y_i *from a continuous c.d.f.* $F_{Y,i}$ *with compact support* $\mathcal{Y}_i \subset R^{2^L-1}$*, with* $F_{Y,i}$ *common knowledge and types drawn independent across bidders:* $Y_i \perp Y_j$ *for all* i, j*.*

As our focus is on monotone equilibria, in the bulk of our analysis we will further assume that objects are complements in the sense that combinatorial valuations are supermodular in the set of objects won:

Definition 2. *We will say that bidders have supermodular valuations if for any outcomes* ω_1, ω_2,

$$Y_i^{\omega_1 \wedge \omega_2} + Y_i^{\omega_1 \vee \omega_2} \geq Y_i^{\omega_1} + Y_i^{\omega_2},$$

where $\omega_1 \wedge \omega_2$ *denotes the meet of* ω_1, ω_2 *and* $\omega_1 \vee \omega_2$ *denotes the join of* ω_1, ω_2.

Supermodularity implies that winning a larger set of objects increases the marginal valuation i assigns to any additional object.

Actions and strategies Let $\mathcal{B}_{il}$ be the set of feasible bids for bidder i in auction l. For each bidder i, we assume that $\mathcal{B}_{il}$ is a compact subset of $\mathbb{R}^+$.[6] The action space for bidder i is the set of $L \times 1$ bid vectors $b_i = (b_{i1}, \ldots, b_{iL})^T$, with $b_i \in \mathcal{B}_i = \times_l \mathcal{B}_{il}$ and $\mathcal{B}_i$ a lattice in $\mathbb{R}^L$. As usual, a *pure strategy* for bidder i is a function $\sigma_i : \mathcal{Y}_i \rightarrow \mathcal{B}_i$. Let $\sigma = (\sigma_1, \ldots, \sigma_N)$ denote a pure strategy profile for all bidders, and $\sigma_{-i} = (\sigma_1, ..., \sigma_{i-1}, \sigma_{i+1}, ..., \sigma_N)$ denote a strategy profile for all bidders except i.[7]

Joint and marginal winning probabilities Let $P_i(b; \sigma_{-i})$ be the $(2^L - 1) \times 1$ vector describing the probability distribution over outcomes arising when i submits bid $b \in \mathcal{B}_{il}$ against rival strategies σ_{-i}, with $P_i^{\omega}(b; \sigma_{-i})$ the element of $P_i(b; \sigma_{-i})$ describing the probability of outcome ω. Similarly, let $\Gamma_i(b; \sigma_{-i})$ be the $L \times 1$ vector describing *marginal* win probabilities arising

[6] While we do not explicitly model reserve prices, these can easily be accommodated in our framework by introducing a dummy bidder whose action space is a singleton including only the relevant reserve prices.

[7] Note that although we do not discuss reserve prices explicitly, our framing here in fact implicitly accommodates arbitrary reserve prices. One could, for instance, simply include a dummy bidder whose bid space in each auction is a singleton equal to the relevant reserve. All results developed below would then immediately extend.

when i submits bid vector $b \in \mathcal{B}_{il}$ against rival strategy profile σ_{-i}, with $\Gamma_{il}(b; \sigma_{-i})$ the marginal probability i wins auction l. Observe that $\Gamma_i(b; \sigma_{-i})$ is related to $P_i(b; \sigma_{-i})$ by

$$\Gamma_i(b; \sigma_{-i}) = \Omega^T P_i(b; \sigma_{-i}).$$

Under Assumption 1, $\Gamma_{il}(b; \sigma_{-i})$ depends only on bid b_l. Furthermore, if ties occur with probability zero, $\Gamma_{il}(b; \sigma_{-i})$ is the c.d.f. of the maximum rival bid in auction l.

Interim payoffs and expected payments Let $\pi_i(b_i; y_i, \sigma_{-i})$ denote the expected interim payoff of bidder i with type $y_i \in \mathcal{Y}_i$ submitting bid vector b_i against rival strategies σ_{-i}. Maintaining Assumptions 1-3, we may write $\pi_i(b_i; y_i, \sigma_{-i})$ as follows:

$$\pi_i(b_i; y_i, \sigma_{-i}) = y_i^T P_i(b; \sigma_{-i}) - \sum_{l=1}^{L} c_{il}(b_{il}; \sigma_{-i}), \tag{1}$$

where $c_{il}(b_{il}; \sigma_{-i})$ denotes i's expected mechanism-determined payment in auction l as a function of i's bid b_{il} in auction l given rival strategies σ_{-i}. For example, if auction l is a first-price auction, then we would have

$$c_{il}(b_{il}; \sigma_{-i}) = b_{il} \Gamma_{il}(b_{il}; \sigma_{-i}).$$

Note that the additively separable form for payments follows jointly from our hypotheses of standard auctions and independent tiebreaking; the former implies that payments in auction l depend only on allocations and bids in auction l, while the latter implies that allocations in auction l depend only on bids in auction l.

3 Monotone best responses in simultaneous standard auctions with complementarities

A natural first question in analysis of simultaneous auctions is whether bidding strategies are monotone in any natural economic sense. As we show in Sections 4-6, monotonicity is useful in analyzing both technical questions such as existence of equilibrium and economic questions such as expected inefficiency in large markets. Furthermore, insofar as we are focusing on a setting with complementarities between objects, it is natural to expect that "higher valuations" should in some sense translate into higher bids. In this section, we show that this is in fact the case: there is a partial order $\succeq$ on the space of types $\mathcal{Y}_i$ such that if $y_i' \succeq y_i$, then i has a best reply at type y_i' which is coordinatewise greater than any best reply at type y_i.

Importantly, however, this partial order $\succeq$ is *not* the usual coordinatewise partial order; indeed, even when objects are complements, a strict coordinatewise increase in i's type can lead to a strict coordinatewise decrease in all elements of i's best-response bid. Intuitively, this is because the usual coordinatewise order on $\mathcal{Y}_i$ imposes insufficient structure on the *marginal value added* by any additional object.

We begin by illustrating failure of coordinatewise monotonicity in the context of a two-object simultaneous first price example, then proceed to develop the partial order $\succeq$ and show that this is sufficient to restore monotonicity of best responses.

Example 1. *Consider two bidders competing for two objects via simultaneous first-price auctions. Suppose that bidder 2's fixed strategy is to bid either in auction 1 (with probability $\frac{1}{2}$) or in auction 2 (with probability $\frac{1}{2}$), drawing bids from the uniform $U[0;1]$ distribution in either case. Consider two types for bidder* 1*:*

$$y_1' = (0,0,0,2)^T, \quad y_1'' = (0,1,1,5/2)^T.$$

For $b_1, b_2 \in [0,1]$, these types correspond to the following profit functions:

$$\pi' = -b_1 \cdot \left(\frac{1}{2} - \frac{b_2}{2}\right) - b_2 \cdot \left(\frac{1}{2} - \frac{b_1}{2}\right) + (2 - b_1 - b_2) \cdot \frac{b_1 + b_2}{2},$$
$$\pi'' = -b_1 \cdot \left(\frac{1}{2} - \frac{b_2}{2}\right) + \left(\frac{5}{2} - b_2\right) \cdot \left(\frac{1}{2} - \frac{b_1}{2}\right) + \left(\frac{5}{2} - b_1 - b_2\right) \cdot \frac{b_1 + b_2}{2},$$

yielding best response bids $b' = \left(\frac{1}{2}, \frac{1}{2}\right)^T$ and $b'' = \left(\frac{1}{4}, \frac{1}{4}\right)^T$ respectively. Ignoring the first component, which corresponds to the case of winning no auctions, we see that y_1'' is strictly greater than y_1' in the coordinatewise sense.

Thus even when objects are complements in the (strong) sense of supermodular valuations, a strict coordinatewise increase in type (from y_1' to y_1'') can generate a strict coordinatewise decrease in i's best response bid (from b' to b''). As pointed out by Reny (2011) in a substantially different context (multi-unit auctions with risk-averse bidders), the fundamental problem is that the coordinatewise partial order on types imposes insufficient structure on *marginal value added*: for instance, when moving from y_1' to y_1'' in Example 1, the value added by object 2 (in events where i is already winning object 1) falls from 2 to 0.5 even as the value i assigns to winning objects 1 and 2 together increases from 2 to 2.5. The solution is to seek an alternative partial order on $\mathcal{Y}_i$ which imposes additional structure on these marginal valuations.

3.1 Partial order on types

We next turn to construction of an alternative partial order on $\mathcal{Y}_i$ sufficient to restore monotonicity of best replies. Specifically, building on the intuition in Example 1, we seek a partial order $\succeq$ on $\mathcal{Y}_i$ such that types which are "higher" with respect to $\succeq$ also have higher marginal valuations. Toward this end, we first make precise the notion of *marginal valuations* in this combinatorial context:

Definition 3 (Marginal valuations)**.** *Let ω and ω' be two outcomes such that $\omega' \geq \omega$ in the coordinatewise sense. For bidder i, the marginal valuation of objects corresponding to allocation*

ω' relative to those in allocation ω is defined as the difference

$$Y_i^{\omega'} - Y_i^{\omega}.$$

Recall that under Assumption 2 all marginal valuations are non-negative.

We seek a partial order $\succeq$ on the space of types $\mathcal{Y}_i$ such that $y_i' \succeq y_i$ implies that *every* marginal valuation is higher for type y_i' than for type y_i. Bearing in mind the combinatorial nature of marginal valuations, this leads to the following definition for the partial order $\succeq$:

Definition 4 (Partial order)**.** *We will say that*

$$\tilde{y}_i \succeq y_i$$

if and only if for any outcome ω and any object l such that $\omega_l = 0$ we have

$$\tilde{y}_i^{\omega \vee e_l} - \tilde{y}_i^{\omega} \geq y_i^{\omega \vee e_l} - y_i^{\omega}. \tag{PO}$$

Note that (by construction) the partial order (PO) is more restrictive than the usual coordinatewise order on $\mathcal{Y}_i$. In particular, choosing $\omega = 0$, we find that $\tilde{y}_i \succeq y_i$ implies $\tilde{y}_i^{e_l} \geq y_i^{e_l}$ for any object l, which in turn implies $\tilde{y}_i^{e_l \vee e_m} \geq y_i^{e_l \vee e_m}$ for any $l \neq m$ and so forth. Proceeding inductively in this way, we ultimately conclude that $\tilde{y}_i \succeq y_i$ implies $\tilde{y}_i \geq y_i$ coordinatewise, as desired in view of the discussion above.

While the economic motivation for the partial order (PO) is clear, it also has a useful geometric interpretation. Intuitively, this interpretation arises from the observation that there is a positive cone generating our partial order. Specifically, define the set $\mathcal{Z}_{2^L-1}$ as follows:

$$\mathcal{Z}_{2^L-1} = \left\{ z \in \mathbb{R}^{2^L-1} : \text{ taking } z^0 = 0, \quad \forall\, (\omega,\, l \text{ such that } \omega_l = 0) \quad z^{\omega \vee e_l} - z^{\omega} \geq 0 \right\}. \tag{2}$$

Then $\mathcal{Z}_{2^L-1}$ is a *solid cone* located in the first orthant of the $2^L - 1$-dimensional space $\mathbb{R}^{2^L-1}$.[8]

[8] To remind the readers, a nonempty subset Z of a vector space is said to be a cone if it satisfies the following three properties: (i) $Z + Z \subset Z$, (ii) $\alpha Z \subset Z$ for all $\alpha \geq 0$, (iii) $Z \cap (-Z) = \{0\}$. Any cone with a nonempty

Furthermore, for any $y_i \in \mathbb{R}^{2^L-1}$, the set $y_i + \mathcal{Z}_{2^L-1}$, which amounts to the translation of the cone $\mathcal{Z}_{2^L-1}$ in $\mathbb{R}^{2^L-1}$ to vertex y_i, represents all the realizations of i's type in $\mathbb{R}^{2^L-1}$ that dominate y_i in the sense of partial order (PO). Similarly, the set $y_i - \mathcal{Z}_{2^L-1}$, which amounts to the rotation of $\mathcal{Z}_{2^L-1}$ and then its translation in $\mathbb{R}^{2^L-1}$ to vertex y_i, represents all the realizations of i's type in $\mathbb{R}^{2^L-1}$ that are dominated by y_i under the partial order (PO).

3.2 Monotone best replies

To conclude this section, we demonstrate that the additional structure imposed by the partial order (PO) is in fact sufficient to restore an economically meaningful notion of monotonicity. Specifically, given any set of rival strategy profiles σ_{-i}, we show that if $\tilde{y}_i \succeq y_i$ in the sense of (PO), then at least one element of i's best reply bid correspondence at type $\tilde{y}_i$ is coordinatewise no smaller than every element of i's best reply bid correspondence at type y_i. We prove this proposition in three steps.

First, combining our hypothesis of simultaneous standard auctions with Assumptions 1–3 above, we show that supermodularity of valuations in the set of objects won implies supermodularity of interim payoffs $\pi_i(b_i; y_i, \sigma_{-i})$ as a function of b_i:

Lemma 1. *Suppose that Assumptions 1–3 hold. Fix a rival pure strategy profile σ_{-i}. Let y_i be a realization of bidder i's type. If valuations are supermodular in the sense of Definition 2, then the interim payoff function*

$$\pi_i(b_i; y_i, \sigma_{-i}) = y_i^T P(b_i; \sigma_{-i}) - \sum_{l=1}^{L} c_{il}(b_{il}; \sigma_{-i})$$

is supermodular in b_i.

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interior is a solid cone. In this case the interior of Z^{2^L-1} is clearly nonempty in $\mathbb{R}^{2^L-1}$; e.g. for $L = 2$ we have $(z^{e_1}, z^{e_2}, z^{e_1 \vee e_2})^T = (1,1,2)^T \in \mathcal{Z}_3^o$, for $L = 3$ we have $(z^{e_1}, z^{e_2}, z^{e_3}, z^{e_1\vee e_2}, z^{e_1 \vee e_3}, z^{e_2 \vee e_3}, z^{e_1 \vee e_2 \vee e_3})^T = (1,1,1,2,2,2,3)^T \in \mathcal{Z}_7^o$, and so forth for $L > 3$. Hence $\mathcal{Z}_{2^L-1}$ is a solid cone in $\mathbb{R}^{2^L-1}$.

Second, we establish that bidders' interim payoffs satisfy the following weak single crossing property in y_i:

Lemma 2. *Maintaining Assumptions 1–3, fix a pure strategy σ_{-i} for the rival bidders. Suppose that types $\tilde{y}_i$ and y_i are such that $\tilde{y}_i \succeq y_i$ in the sense of the partial order (PO), and suppose that $\tilde{b}_i \geq b_i$ in the coordinatewise sense. Then*

$$\pi_i(\tilde{b}_i; y_i, \sigma_{-i}) \geq \pi_i(b_i; y_i, \sigma_{-i}) \quad \Longrightarrow \quad \pi_i(\tilde{b}_i; \tilde{y}_i, \sigma_{-i}) \geq \pi_i(b_i; \tilde{y}_i, \sigma_{-i}).$$

Finally, we combine these results to establish the following weak monotonicity property on the set of i's best replies to σ_{-i}:

Proposition 1. *Maintaining Assumptions 1–3, suppose that valuations are supermodular in the sense of Definition 2. Fix a pure strategy profile σ_{-i} for the rival bidders. Let $b_i \in \mathcal{B}_i$ be a best response to σ_{-i} when bidder i's type is y_i, and $\tilde{b}_i \in \mathcal{B}_i$ be a best response to σ_{-i} when i's type is $\tilde{y}_i$, where $\tilde{y}_i$ and y_i are such that $\tilde{y}_i \succeq y_i$ in the sense of the partial order (PO). Then the bid vector $b_i \vee \tilde{b}_i$ is also a best response to σ_{-i} when i's type is $\tilde{y}_i$.*

Proposition 1 does not, of course, guarantee existence of a best reply strategy σ_i^* to σ_{-i}. It does, however, imply that if such a best reply exists, then there also exists a best reply strategy $\tilde{\sigma}_i^*$ which is monotone in that $y_i' \succeq y_i$ in the sense of (PO) implies $\tilde{\sigma}_i^*(y_i') \geq \tilde{\sigma}_i^*(y_i)$ in the usual coordinatewise order. This in turn provides a foundation for our analysis of monotone equilibrium below.

4 Monotone equilibrium in finite bid spaces

Building on the partial order (PO), we next turn to consider monotone equilibrium in finite bid spaces. Specifically, suppose that $\mathcal{B}$ is a finite lattice. Under an additional support condition on Y_i to be defined shortly, we show that there exists a Bayes-Nash equilibrium in pure strategies $(\sigma_1^*, ..., \sigma_N^*) : \mathcal{Y} \to \mathcal{B}$ with the property that $y_i' \succeq y_i$ in the sense of (PO) implies $\sigma_i^*(y_i') \geq \sigma_i^*(y_i)$ in the usual coordinatewise order.

Toward this end, we require one additional assumption, which guarantees that the support of Y_i is "sufficiently rich" to permit meaningful comparisons with respect to the partial order (PO):

Assumption 4. *There is a countable subset $\mathcal{Y}_i^\star$ of $\mathcal{Y}_i$ such that every set in F_{Y_i}- sigma-algebra assigned positive probability by F_{Y_i} contains two points between which (here "between" is understood in the partial order sense) lies a point in $\mathcal{Y}_i^\star$.*

For this assumption to hold jointly with atomlessness of the distribution of Y_i, it is necessary that there exists a positive F_{Y_i}-measure of points in the support $\mathcal{Y}_i$ that can be compared to each other by means of the partial order (PO). In particular, (PO) should not reduce to the trivial partial order on $\mathcal{Y}_i$. Fortunately, Assumption 4 turns out to follow from natural regularity conditions on the distribution of Y_i:

Proposition 2. *Suppose that $\mathcal{Y}_i$ is compact and has non-empty interior with respect to $\mathbb{R}^{2^L-1}$, and that the boundary of $\mathcal{Y}_i$ has zero Lebesgue measure in $\mathbb{R}^{2^L-1}$. Further suppose that the joint distribution of the (2^L-1)-dimensional vector Y_i is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^{2^L-1}$. Then Assumption 4 holds.*

We now turn to this section's main result: under Assumptions 1–4, at least one Bayes-Nash equilibrium monotone in the sense of (PO) exists on any finite bid lattice. In view of Proposition 1, the proof of this statement is relatively straightforward. Restricting bids to a finite lattice guarantees the continuity of the interim payoff function, which ensures that each player's interim best reply correspondence is non-empty. Hence by Proposition 1, we conclude that each player has a best reply which is monotone and join closed with respect to the partial order (PO) on types and the usual coordinatewise order on bids. To guarantee existence of an equilibrium in monotone pure strategies, we therefore need only verify conditions G.1-G.6 of Reny (2011), after which Theorem 4.1 of Reny (2011) delivers the result. Of these, only condition G.3 is potentially problematic, leading to our Assumption 4 and sufficient conditions provided in Proposition 2. We thereby conclude:

Proposition 3. *Maintaining Assumptions 1–4, suppose that valuations are supermodular in the sense of Definition 2. for each bidder i the bid space $\mathcal{B}_i \subset \mathbb{R}^L$ is a finite lattice.*[9] *Then there is an equilibrium in pure strategies which are monotone with respect to the partial order (PO) on types and the coordinatewise partial order on bids: i.e. such that $y_i' \succeq y_i$ in the sense of (PO) implies $b_i' \geq b_i$ in the usual coordinatewise sense.*

As in Reny (2011), Proposition 3 immediately extends to existence of symmetric monotone equilibria when bidders are symmetric:

Corrolary 1. *In addition to the hypotheses of Proposition 3, suppose that bidders are* symmetric *in the sense that $\mathcal{Y}_i = \mathcal{Y}_j$, $F_i = F_j$, and $\mathcal{B}_i = \mathcal{B}_j$ for all bidders i, j. Then there is an equilibrium in* symmetric *pure strategies monotone with respect to the partial order (PO) on types and the coordinatewise partial order on bids.*

The proof of this corollary follows immediately from the proof of Proposition 3 given in the Appenidix, but invoking Theorem 4.5 rather than Theorem 4.1 of Reny (2011). We therefore do not provide a separate proof.

Note that the main contribution of Proposition 3 is not existence per se,[10] but rather existence in strategies which are *monotone* with respect to a suitably defined partial order. Since our analysis in Sections 5 and 6 pivots on monotonicity, this additional structure turns out to be essential.

In Example 2 below we consider a setting with two objects and two ex-ante symmetric bidders whose valuation for the bundle of both objects is the sum of standalone valuations and a non-negative deterministic complementarity denoted as k. The bidding set consists of two points and, thus, is discrete. At the lower bid no loss will be incurred ex-post. However, a bidder may potentially incur a loss when submitting the higher bid, and whether such a loss happens depends on bidder's type as well as what that bidder won.

[9] E.g., $\mathcal{B}_i = \bigtimes_{l=1}^{L} \mathcal{B}_{il}$ where each $\mathcal{B}_{il}$ consists of the finite number of points.

[10] Existence of equilibrium in pure strategies could be obtained by, for instance, applying results in Milgrom and Weber (1985).

We consider three different auction formats (first-price, second-price and all-pay) and illustrate symmetric BNE in pure monotone strategies. In this example we find that for moderate and high levels of complementarity in first-price and all-pay auctions exposure happens with a positive probability and the probability of exposure as well as the maximal degree of exposure is increasing in k up to some level, after which it becomes constant. An analogous result holds for the second-price auction with the difference that exposure happens for any $k \geq 0$. We generally find that the exposure problem (looking at the maximum degree of exposure) in the all-pay auction is not more severe than that in the first-price auction due to bidders bidding more cautiously in the all-pay auction, and the exposure problem in the first-price auction is not more severe than that in the second-price auction.

Example 2. *Consider the case of two symmetric global bidders competing for two objects. The valuation for having two objects is*

$$Y_i^{(1,1)} = Y_i^{e_1} + Y_i^{e_2} + k, \quad i = 1, 2,$$

where $k \geq 0$ is constant, $Y_i^{e_1}$ and $Y_i^{e_2}$ are two independent random variables distributed uniformly on $[0, 1]$.

Suppose the the bidders are only allowed to bid either $b_1 = 0$ or $b_2 = \frac{1}{2}$ for each object, and ties for objects are broken independently with a fair lottery. We characterize equilibrium bidding strategies in a BNE in symmetric pure monotone strategies. We illustrate how BNE changes when complementarity k increases and discuss occurrences of exposure.

First-price auction

If $k = 0$ and, thus, we are in the case of case of no complementarities from winning two objects. Each biddes always bidding 0 for each object is a BNE. For $k \in (0, 1)$, the equilibrium bidding strategies are

$$\beta_i\left(Y_i^{(1,0)}, Y_i^{(0,1)}\right) = \begin{cases} \left(\frac{1}{2}, \frac{1}{2}\right), & \text{if} \quad (Y_i^{e_1}, Y_i^{e_2}) \in A, \\ \left(\frac{1}{2}, 0\right), & \text{if} \quad (Y_i^{e_1}, Y_i^{e_2}) \in B, \\ \left(0, \frac{1}{2}\right), & \text{if} \quad (Y_i^{e_1}, Y_i^{e_2}) \in C, \\ (0, 0), & \text{if} \quad (Y_i^{e_1}, Y_i^{e_2}) \in D, \end{cases} \tag{3}$$

where regions A,B,C and D are defined as follows:

$$A = \left\{ (y^{e_1}, y^{e_2}) \in [0,1]^2 : \, y^{(1,0)} + y^{(0,1)} \geq 2\gamma(k) - \frac{k}{2}, \; y^{e_1} \geq \gamma(k) - \frac{k}{2}, \; y^{e_2} \geq \gamma(k) - \frac{k}{2} \right\}, \quad (4)$$

$$C = \left\{ (y^{(e_1}, y^{e_2}) \in [0,1]^2 : \, y^{e_1} < \gamma(k) - \frac{k}{2}, \; y^{e_2} \geq \gamma(k) \right\},$$

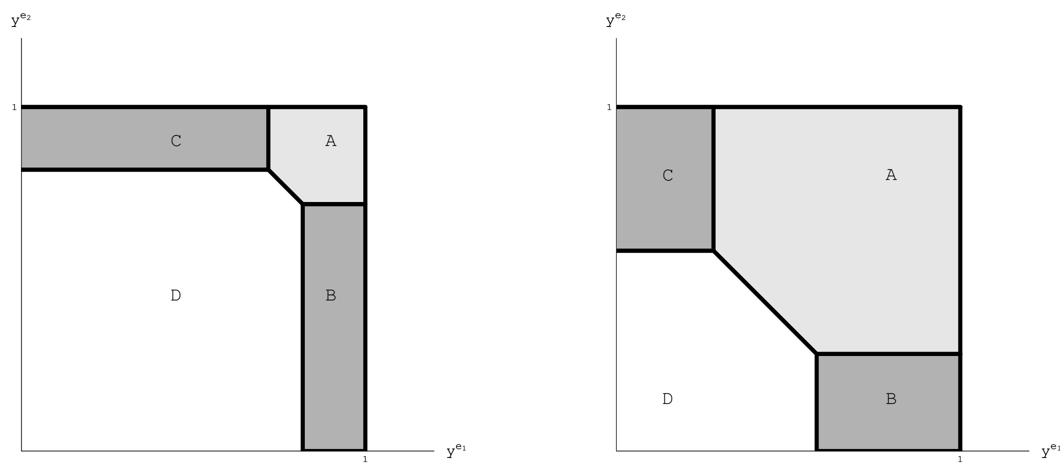
$$B = \left\{ (y^{e_1}, y^{(e_2}) \in [0,1]^2 : \, y^{e_1} \geq \gamma(k), \; y^{e_2} < \gamma(k) - \frac{k}{2} \right\},$$

$$D = [0,1]^2 \backslash (A \cup B \cup C),$$

and

$$\gamma(k) = \frac{\frac{k^3}{8} + \frac{3k^2}{8} - \frac{k}{2} + 1}{\frac{k^2}{2} + \frac{k}{2} + 1}. \quad k \in (0,1].$$

Figure 1 illustrates the BNE in symmetric monotone pure strategies when $k = 0.2$ and $k = 0.6$. For $k > 0$ small enough, there no exposure for any of the bidders.



$k = 0.2$ $\qquad$ $k = 0.6$

Figure 1: Illustration to Example 2 (first-price auction).

However, for $k \in (k^*_{fp,exp}, 1]$ with $k^*_{fp,exp} \approx 0.4171$ some bidders with valuations in region A become exposed when winning one object only. The maximum possible exposure (ex-post loss) in situation is $\frac{\frac{k^3}{4} + \frac{k^2}{4} + \frac{5k}{2} - 1}{k^2 + k + 2}$, which increases in k from 0 (at $k = k^*_{fp,exp}$) to 0.5 (at $k = 1$).

For $k \in (1, 2]$, the equilibrium bidding strategies are

$$\beta_i \left(Y_i^{(1,0)}, Y_i^{(0,1)} \right) = \begin{cases} \left(\frac{1}{2}, \frac{1}{2} \right), & \textit{if} \quad (Y_i^{e_1}, Y_i^{e_2}) \in A, \\ (0,0), & \textit{if} \quad (Y_i^{e_1}, Y_i^{e_2}) \in D, \end{cases} \quad (5)$$

where regions A and D are described as follows:

$$D = \left\{ (y^{e_1}, y^{e_2}) \in [0,1]^2 : \, y^{e_1} + y^{e_2} \leq \gamma(k) \right\}, \qquad (6)$$
$$A = [0,1]^2 \backslash D,$$

with

$$\gamma(k) = \frac{2-k}{\sqrt{1+(2-k)(k-1)}+1}.$$

Some bidders with valuations in region A are exposed when winning one object only. The maximum possible exposure (ex-post loss) in situation is 0.5.

When $k \geq 2$, bidders always submit $b_2 = \frac{1}{2}$ for each object. Again, some bidders are exposed when winning one object only and the maximum possible exposure (ex-post loss) in situation is 0.5.

Second-price auction

In a second-price auction, with $k = 0$, the equilibrium bidding strategies are such that each bidder submits b_1 in auction ℓ if $y^{e_\ell} < \frac{1}{3}$ and submits b_2 otherwise. Due to the second-price nature of the auction, some bidders may be exposed even in the situation of $k = 0$ when winning one object and having to pay b_2 for that object. The maximum possible degree of exposure in this situation is equal to $\frac{1}{3}$.

When the complementarity becomes strictly positive, the equilibrium bidding strategies change. When $k \in (0, k^*_{sp})$, where $k^*_{sp} \approx 0.8385$ (to be exact, k^*_{sp} satisfies $\frac{\left(k^*_{sp}\right)^3}{8} + \frac{\left(k^*_{sp}\right)^2}{8} + k^*_{sp} - 1 = 0$), then the monotone bidding strategies have the form (3)-(4) with

$$\gamma(k) = \frac{\frac{k^3}{8} + \frac{5k^2}{8} + \frac{k}{2} + 1}{3 + \frac{k^2}{2} + \frac{3k}{2}}.$$

Some bidders with values in A are exposed when winning one object and paying b_2 for that object. The maximum possible degree of such exposure in this situation is equal to $\frac{0.5+\frac{k}{4}-\frac{k^3}{8}-\frac{k^2}{8}}{3+\frac{k^2}{2}+\frac{3k}{2}}$. Moreover, when $k \in [0, k^*_{sp,exp1})$, where $k^*_{sp,exp1} \approx 0.5214$, some bidders with values in A may be exposed even when winning both objects and having to pay b_2 for each object. The maximum possible degree of such exposure is equal to $\frac{1-k-\frac{3k^2}{2}-\frac{k^3}{2}}{3+\frac{3k}{2}+\frac{k^2}{2}}$ – it decreases in k from $\frac{1}{3}$ (at $k = 0$) to 0 (at $k = k^*_{sp,exp1}$).

For $k \in [k^*_{sp}, 2)$, the equilibrium bidding strategies are in the form (5)-(6) with

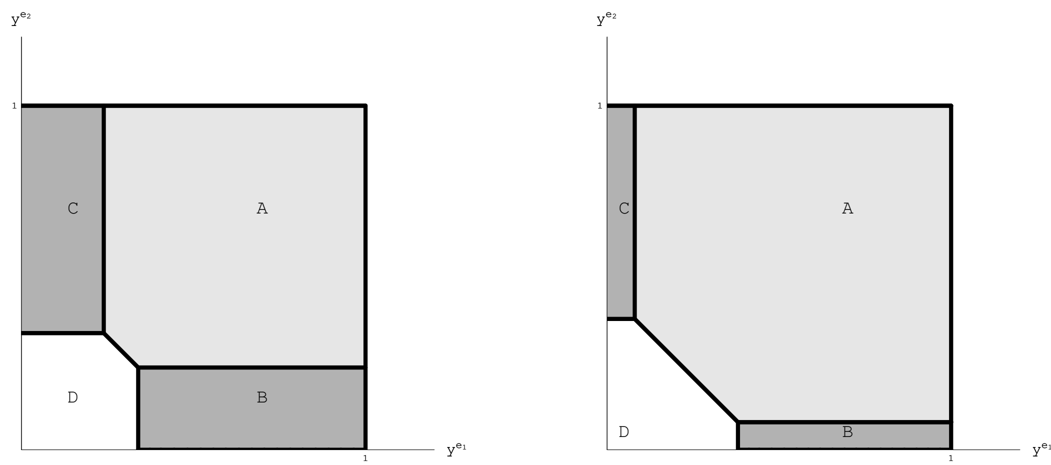
$$\gamma(k) = \frac{2-k}{\sqrt{1+(2-k)(k+1)}+1}.$$

In this case some bidders with values in A are exposed when winning one object and paying

b_2 *for that object. The maximum possible degree of such exposure in this situation is equal to* 0.5. *Moreover, some bidders with values in A may be exposed even when winning both objects and having to pay* b_2 *for each object. The maximum possible degree of such exposure is equal to* $\dfrac{2-k}{\sqrt{1+(2-k)(k+1)}+1}-1.$

For $k \geq 2$, *bidders always submit* $b_2 = \frac{1}{2}$ *for each object. The maximum possible degree of such exposure is equal to 1.*

The graphical illustration of these regions and equilibrium bidding behavior for $k = 0.2$ *and* $k = 0.6$ *is given in Figure 2.*



$k = 0.2$ $\qquad$ $k = 0.6$

Figure 2: Illustration to Example 2 (second-price auction).

All-pay auction

In an all-pay auction, with $k = 0$, *the equilibrium bidding strategies are such that each type submits* $b_1 = 0$ *for each object.*

When $k \in (0, k^*_{ap})$, *where* $k^*_{ap} \approx 1.5418$ *(to be exact,* k^*_{ap} *satisfies* $\frac{(k^*_{ap})^3}{16} + \frac{k^*_{ap}}{2} - 1 = 0$*), the monotone bidding strategies have the form (3)-(4) with*

$$\gamma(k) = \frac{\frac{k^3}{16} + \frac{k^2}{4} + 1}{\frac{k^2}{4} + \frac{k}{2} + 1}.$$

For $k > 0$ *small enough, there no exposure for any of the bidders.*

However, for $k \in (k^*_{ap,exp}, 4]$ *with* $k^*_{ap,exp} \approx 0.5911$ *some bidders with valuations in region A become exposed when winning one object only. The maximum possible exposure (ex-post loss)* *in situation is* $\dfrac{\frac{k^3}{16}+\frac{k^2}{4}+1}{\frac{k^2}{4}+\frac{k}{2}+1}-\frac{1}{2}(k+1)$.

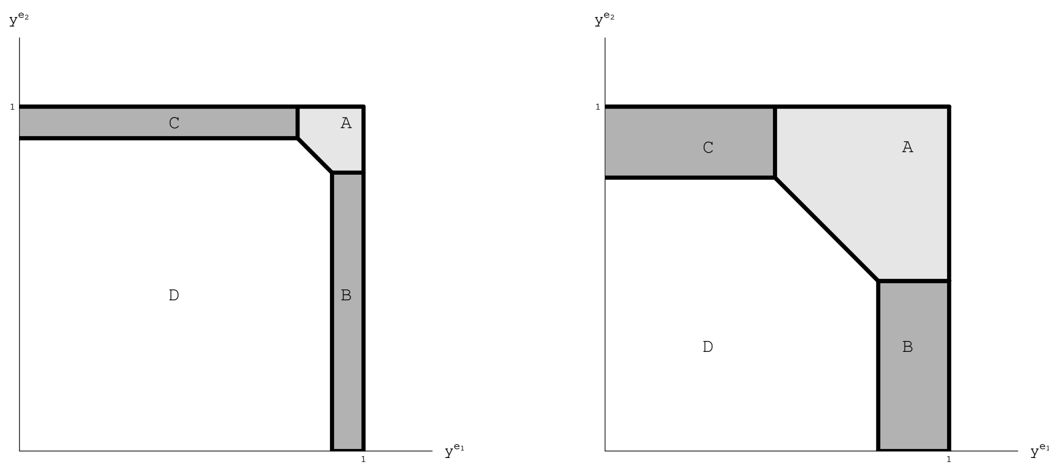
For $k \in [k^*_{ap}, 4)$, *the equilibrium bidding strategies are in the form (5)-(6) with*

$$\gamma(k) = \frac{4-k}{\sqrt{1+k(4-k)}+1}.$$

In this case some bidders with values in A *are exposed when winning one object and paying* b_2 *for that object. The maximum possible degree of such exposure in this situation is equal to* 0.5.

For $k \geq 4$, *bidders always submit* $b_2 = \frac{1}{2}$ *for each object. Again, some bidders are exposed when winning one object only and the maximum possible exposure (ex-post loss) in situation is 0.5.*

The graphical illustration of the equilibrium bidding strategies for $k = 0.2$ *and* $k = 0.6$ *is given in Figure 3.*



$k = 0.2$ $\qquad$ $k = 0.6$

Figure 3: Illustration to Example 2 (all-pay auction).

Similar to the case of single-object auction, for $k > 0$ *(up to a threshold), bidders' equilibrium behavior is more aggressive in the second-price auction than in the first price auction, and is more cautious in the all-pay auction than in the first-price auction.*

5 Monotone equilibria in continuous bid spaces

We now turn from finite to continuous bid spaces, applying Proposition 3 to establish two new results. We begin with a special case inspired by Krishna and Rosenthal (1996), in which a single *global bidder* bids for multiple objects against many *local bidders* who bid for single objects only. We demonstrate existence of a monotone pure strategy Bayes-Nash equilibrium in this context, building on limiting techniques in Reny (2011) to extend from discrete to continuous bid spaces. While instructive in its own right, this example also highlights a subtle technical challenge: with more than one global bidder, interaction between strategic overbidding by global bidders and strategic dependence of bids across auctions renders uncertain a key technical condition – *better-reply security* of Reny (1999) – needed to complete the extension proof. Returning to the full model, we therefore consider instead a more general solution concept inspired by the work of JSSZ (2002): *equilibrium with endogenous tiebreaking.* We define this solution concept in detail in Section 5.2, and demonstrate that equilibria with endogenous tiebreaking exist for any number of global bidders.

5.1 One global bidder and many local bidders

First consider the following special case of our general model: suppose that one global bidder competes for L objects against many local bidders, with each local bidder competing in exactly one auction. Let bidder 0 denote the global bidder and let $l_1, \ldots, l_N$ denote the auctions in which local bidders $1, \ldots, N$ are competing respectively. We specialize our assumptions to this environment as follows:

Assumption 5. *The global bidder is risk-neutral and draws private type Y_0 (which contains combinatorial valuations for all $\omega \neq 0$) from a continuous (2^L-1)-variate c.d.f. F_0 with compact support $\mathcal{Y}_0 \subset \mathbb{R}^{2^L-1}$. The global bidder's valuations are monotone ($Y^{\omega'} \geq Y^{\omega}$ for $\omega' \geq \omega$) and supermodular in the sense of Definition 2.*

Assumption 6. *Each local bidder i is risk-neutral and draws private type Y_i for object l_i from univariate continuous c.d.f. F_i with compact support $\mathcal{Y}_i \subset \mathbb{R}$, $i = 1, \ldots, N$. Types are independently distributed across all bidders.*

For simplicity, and for this section only, further suppose that each auction $l = 1, ..., L$ is a *first-price* auction. We conjecture that similar results could be shown for other standard auctions, but do not pursue this further here as our goal is illustration.

For each $l = 1, \ldots, L$, let $\mathcal{B}_l \subset \mathbb{R}$ be a compact interval describing feasible bids in auction l. Then a pure a strategy for the global bidder 0 is a map $\sigma_0 : \mathcal{Y}_0 \to \mathcal{B}_0$ with $\mathcal{B}_0 \equiv \times_{l=1}^{L} \mathcal{B}_l$, while a pure strategy for local bidder i is a map $\sigma_i : \mathcal{Y}_i \to \mathcal{B}_{l_i}$.

It is straightforward to construct a sequence of finite lattices $\{\ddot{\mathcal{B}}_0^k; \ddot{\mathcal{B}}_1^k, \ldots, \ddot{\mathcal{B}}_L^k\}_{k=1}^{\infty}$ such that as $k \to \infty$ each $\ddot{\mathcal{B}}_j^k$ becomes increasingly dense in $\mathcal{B}_j$, $j = 0, \ldots, L$. By Proposition 3, for each k the collection of such finite lattices will induce a monotone equilibrium strategy profile $\ddot{\sigma}^k$, where monotonicity of the strategy of the global bidder is understood in the sense of the partial order (PO) on the type space and the coordinatewise order for bid vectors, and monotonicity of the strategies of the local bidders are understood in the usual univariate sense. By Lemma A.13 in Reny (2011), the space of strategies monotone with respect to our partial order is compact in the pointwise convergence topology, hence the sequence of strategies $\{\ddot{\sigma}^k\}_{k=1}^{\infty}$ will have a subsequence $\{\ddot{\sigma}^{k_j}\}_{j=1}^{\infty}$ which converges pointwise a.e. to a limit σ^*. This limit σ^* is a monotone pure strategy profile by construction; we seek to show that it also defines an equilibrium on the continuous bid space $\mathcal{B} \equiv \{\mathcal{B}_0; \mathcal{B}_1, \ldots, \mathcal{B}_L\}$.

To achieve this, we apply the concept of *better-reply security* introduced in Reny (1999), defined formally as follows:

Definition 5 (Secure a payoff)**.** *Player i can secure a payoff of $\alpha \in \mathbb{R}$ at strategy profile $\sigma \in \mathcal{S}$ if there exists $\bar{\sigma}_i \in \mathcal{S}_i$ such that $\Pi_i(\bar{\sigma}_i; \sigma'_{-i}) \geq \alpha$ for all σ'_{-i} in some open neighborhood of σ_{-i}.*

Definition 6 (Better-Reply Secure)**.** *A game $G = (\mathcal{S}_i, \Pi_i)_{i=1}^{N}$ is better-reply secure if whenever*

$(\tilde{\sigma}, \tilde{\Pi})$ *is in the closure of the graph of the vector payoff function* $\mathbf{\Pi}(\cdot)$ *and* $\tilde{\sigma}$ *is not an equilibrium, some player* i *can secure a payoff strictly above* $\tilde{\Pi}_i$ *at* $\tilde{\sigma}$.

By Remark 3.1 in Reny (1999) (p. 1038), if a game is better-reply secure, then the limit of a convergent sequence of ϵ-equilibria, as ϵ tends to zero, are pure strategy equilibria. To establish existence of an equilibrium in monotone pure strategies on the continuous space $\mathcal{B}$, it is therefore sufficient to demonstrate the following:

(ii) There exists a sequence of finite lattices $\{\ddot{\mathcal{B}}_0^k; \ddot{\mathcal{B}}_1^k, \ldots, \ddot{\mathcal{B}}_L^k\}_{k=1}^{\infty}$ such that if $\ddot{\sigma}^k$ is a monotone pure strategy equilibrium on $\{\ddot{\mathcal{B}}_0^k; \ddot{\mathcal{B}}_1^k, \ldots, \ddot{\mathcal{B}}_L^k\}$ for each k, then $\{\ddot{\sigma}^k\}_{k=1}^{\infty}$ is a sequence of ϵ-equilibria on $\mathcal{B}$ for which $\epsilon \to 0$;

(ii) The bidding game is better-reply secure when bids may be submitted on $\mathcal{B}$.

We now establish each of these in turn.

Lemma 3. *Suppose that Assumptions 1, 4, 5, and 6 hold, and that each auction* $l = 1, ..., L$ *is a first-price auction.*

Let $\{\ddot{\mathcal{B}}_0^k; \ddot{\mathcal{B}}_1^k, \ldots, \ddot{\mathcal{B}}_L^k\}_{k=1}^{\infty}$ *be any sequence of finite lattices such that:*

1. $\ddot{\mathcal{B}}_0^k \subset \mathcal{B}_0$, $\ddot{\mathcal{B}}_l^k \subset \mathcal{B}_l$ *for all* $l = 1, \ldots, L$, *and*
$$\mathcal{H}(\ddot{\mathcal{B}}_j^k, \mathcal{B}_j) \to 0 \text{ as } k \to \infty,$$
for $j = 0, \ldots, N$, *where* $\mathcal{H}(\cdot, \cdot)$ *stands for the Hausdorff distance.*
2. *For each* $l = 1, \ldots, L$, $\ddot{\mathcal{B}}_l^k$ *is a subset of* $\ddot{\mathcal{B}}_{0l}^k$ *such that* $\min \ddot{\mathcal{B}}_{0l}^k < \min \ddot{\mathcal{B}}_l^k$, $\max \ddot{\mathcal{B}}_{0l}^k > \max \ddot{\mathcal{B}}_l^k$, *and for any* $b_l', b_l'' \in \ddot{\mathcal{B}}_l^k$ *there exists a point* $b_l''' \in \ddot{\mathcal{B}}_{0l}^k$ *such that* $b_l' < b_l''' < b_l''$.

Let $\ddot{\sigma}^k$ *be a monotone pure strategy equilibrium for bid space* $\ddot{\mathcal{B}}^k$. *Then for any sequence* $\{\epsilon^m\}$ *such that* $\epsilon^m > 0$ *and* $\epsilon^m \to 0$, *there exists a subsequence* $\{k_m\}_{m=1}^{\infty}$ *of* $k = 1, 2, ...$ *such that strategy profile* $\ddot{\sigma}^{k_m}$ *is an* ϵ^m*-equilibrium on the unrestricted* $\mathcal{B}$.

Note that the proof of Lemma 3 turns on choosing lattices $\{\ddot{\mathcal{B}}_0^k; \ddot{\mathcal{B}}_1^k, \ldots, \ddot{\mathcal{B}}_1^k\}$ such that the bid lattice of the global bidder is always finer than the product of the bid lattices of the local bidders. This guarantees that at each point along the sequence of finite grids considered the global bidder can resolve ties in the direction she most prefers. This in turn allows us to use the global bidder's revealed preference on finite lattices to bound the her potential gains from deviation in continuous bid space.

Lemma 4. *Suppose that Assumptions 1, 4, 5, and 6 hold, and that each auction* $l = 1, ..., L$ *is a first-price auction. Then the simultaneous first-price auction game with one global and many local bidders is better-reply secure when considering monotone strategies played by the bidders.*

While we relegate details to the Appendix, we emphasize that the proof of Lemma 4 is more complicated here than in standard single- or multi-unit auctions. In standard auctions, better-reply security follows almost automatically from the fact that bidders almost surely bid below their marginal valuations. One can therefore construct payoff-securing deviations by slightly increasing bids at any point involving potential ties.[11] Here, in contrast, the global bidder may *strategically overbid* for a given object – i.e. to submit a bid strictly above her marginal valuation – in the hope of winning higher-order combinations. The proof of Lemma 4 therefore in fact turns on *independence of rival bids* faced by the global bidder. This allows us to assert that any increase in the marginal probability of winning object l proportionally increases the probability of winning all combinations involving object l, and hence to conclude that the global bidder will always want to break relevant ties in her favor.

Finally, combining Lemmas 3 and 4, Remark 3.1 in Reny (1999), and Lemma A.13 of Reny (2011) as described above, we obtain this subsection's main result:

Proposition 4. *Suppose that Assumptions 1, 4, 5, and 6 hold, and that each auction* $l = 1, ..., L$ *is a first-price auction. In the simultaneous first-price auction game with one global bidder and many local bidders, a monotone pure strategy equilibrium exists on the compact convex bid space* $\mathcal{B}$.

Now consider what may go wrong with more than one global bidder. Recall that, due to the possibility of strategic overbidding by the global bidder, our proof of Lemma 4 (better-reply

[11] see, e.g., Reny (1999, 2011) for examples of this argument.

security) turns crucially on *independence* of bids by local rivals across auctions. Unfortunately, however, if multiple global bidders are present, bids by global rivals may in principle exhibit arbitrary dependence across auctions through strategies σ. When combined with the possibility of strategic overbidding, this in turn may be problematic for better-reply security. For example, consider a setting in which two global bidders compete for two objects. Imagine a sequence of strategies along which these bidders converge to a tie in auction 1, such that at each strategy profile in the sequence, any type of bidder 1 bidding "just above" the tie point in auction 1 also submits a bid for object 2 which wins auction 2 with certainty. Then any deviation by bidder 2 to a point "just above" the tie will produce a strict increase in the probability that bidder 2 wins auction 1, without increasing the probability that bidder 2 wins objects 1 and 2 together. If bidder 2 also engages in strategic overbidding for object 1, this could in turn imply a strict *decrease* in bidder 2's expected payoff. Hence even if the limit profile (with ties) is not an equilibrium, bidder 2 may not have a profitable deviation. This in turn undermines any proof of better-reply security paralleling Lemma 4, and thereby any proof of existence paralleling Proposition 4.

5.2 Monotone equilibrium with endogenous tie-breaking

Recall that the technical challenge in establishing better-reply security with many global bidders is not overbidding per se; rather, it is the fact that by slightly increasing her bid in auction l, bidder i may win auction l only against types of global rivals against which i is likely to lose in other auctions. There may therefore in principle exist a sequence of strategies $\{\sigma^k\}_{k=1}^{\infty}$, converging to a limit $\tilde{\sigma}$ involving ties, such that no type tying at the limit wishes to deviate at any point along the sequence, but a positive measure of tying types wish to deviate at the limiting profile $\tilde{\sigma}$. The fundamental problem in such a case is that independent tiebreaking at the limiting profile $\tilde{\sigma}$ may *lose information* regarding the order in which near-ties are broken along the sequence – types tying at the limit could submit different bids in auction l (and hence have near-ties resolved with differing precedences) at every strategy profile σ^k in the sequence. If this tie-breaking order could be preserved in the limit, then $\tilde{\sigma}$ would in fact represent a monotone pure strategy equilibrium.

Motivated by this observation, in analyzing simultaneous auctions with many global bidders, we focus on a solution concept which generalizes Bayes-Nash equilibrium along the lines proposed by Jackson, Simon, Swinkels and Zame (2002, henceforth JSSZ). In what follows, we refer to this solution concept as an *equilibrium with endogenous tiebreaking* in the *signalling extension* of the simultaneous bidding game. We define this solution concept formally as follows.

Let $\mathcal{G}$ be a simultaneous auction game satisfying Assumptions 2–4 on primitives above. In what follows, we no longer presume that ties are resolved independently across actions according to some pre-specified tiebreaking rule – i.e. we no longer maintain Assumption 1 above. Rather, from this point forward, we interpret $\mathcal{G}$ as a *game of indeterminate outcomes* in the language of JSSZ (2002): that is, a game requiring only that each object l is awarded to a high bidder in auction l, without specifying which high bidder will receive the object in the event of a tie.

We define the *signalling extension* $\mathcal{G}^s$ to the game of indeterminate outcomes $\mathcal{G}$ by augmenting each bidder's strategy space as follows. For each auction l, we allow bidder i to submit, in addition to her bid vector b_i, a vector of cheap-talk signal $s_i \in [0,1]^L$ indicating her desired *tiebreaking precedence* in each auction l. These signals are irrelevant for allocations and payoffs except in case of ties, in which case the auctioneer may consider $(s_1, ..., s_N)$ in determining how to break ties.

A *pure strategy* for bidder i in the signalling extension $\mathcal{G}^s$ is therefore a function $\sigma_i \times s_i : \mathcal{Y}_i \to \mathcal{B}_i \times S_i$, where $\sigma_i : \mathcal{Y}_i \to \mathcal{B}_i$ denotes i's bidding strategy as above and $\tau_i : \mathcal{Y}_i \to \mathcal{S}_i$ denotes i's *tiebreaking strategy* in the signalling extension.[12] As above, let $\mathcal{Y} = \times_i \mathcal{Y}_i$, $\mathcal{B} = \times_i \mathcal{B}_i$, and

[12]Since we focus on pure strategies, writing bidder i's strategy as the cross-product of her bidding and tiebreaking strategies involves no loss of generality.

$\mathcal{S} = \times_i \mathcal{S}_i$ collect type, bid, and signal spaces across bidders respectively, with $\sigma = (\sigma_1, ..., \sigma_N)$ and $\tau = (\tau_1, ..., \tau_N)$ denoting profiles of bidding and tiebreaking strategies across bidders.

For each auction $l \in \{1, \ldots, L\}$, let Δ_l^N be the N-dimensional simplex describing all possible probabilities with which object l could be allocated to each of the N competing bidders. Let the correspondence $\Theta : \mathcal{B} \times \mathcal{S} \to \Delta_1^N \times \cdots \times \Delta_L^N$ describe the set of all allocation rules feasible in the signalling extension $\mathcal{G}^s$ to $\mathcal{G}$: that is, the set of rules such that object l is allocated to bidder i only if $b_{il} = \max_j b_{jl}$. Following JSSZ (2002), we define an *allocation rule* $\theta : \mathcal{B} \times \mathcal{S} \to \Delta_1^N \times \cdots \times \Delta_L^N$ as a selection from the correspondence Θ.

As in JSSZ (2002), we define a *pure strategy solution* to the signalling extension $\mathcal{G}^s$ as an allocation rule θ^* plus a profile of pure bidding and tiebreaking strategies $\sigma^* \times \tau^*$ such that $\sigma^* \times \tau^*$ constitute a Bayesian Nash equilibrium given the allocation rule θ^*. We refer to any such solution as an *equilibrium with endogenous tiebreaking.*

We are now in position to state this section's main result: under our assumptions on primitives, there exists an equilibrium with endogenous tiebreaking in which (i) bidders play monotone pure strategies, (ii) allocation of object l depends only on bids and signals in auction l, and (iii) tiebreaking is weakly monotone in signals.

Proposition 5. *Let $\mathcal{G}$ be an L-object N-bidder simultaneous auction game. Suppose that assumptions 2–4 hold, that $\mathcal{B}_i \subset \mathbb{R}^L$ is a compact lattice for each $i = 1, \ldots, N$, and that valuations are supermodular in the sense of Definition 2. Then the signalling extension $\mathcal{G}^s$ to $\mathcal{G}$ admits a pure strategy solution $\theta^*; \sigma^* \times \tau^*$ such that:*

1. *Both bidding strategies σ^* and signalling strategies τ^* are monotone with respect to the partial order (PO) on types and the coordinatewise order on actions;*

2. *The auctioneer's allocation rule θ^* can be characterized by a $L \times 1$ vector of anonymous* tiebreaking precedence rules $(\rho_1, ..., \rho_L)$ *such that:*

   (a) *For each $l = 1, ..., L$, ρ_l is a weakly monotone function from $[0, 1]$ to $[0, 1]$;*

   (b) *Bidder $i = 1, ..., N$ submitting signal $s_{il} \in [0, 1]$ is assigned tiebreaking precedence $\rho_l(s_{il}) \in [0, 1]$ in auction l;*

(c) *Object l is allocated randomly and uniformly among high bidders in auction l with the highest tiebreaking precedence.*

While we defer the full proof of Proposition 5 to the Appenidix, the intuition is worth sketching briefly here. As noted above, the fundamental problem in passing from the sequence of finite equilibria $\{\sigma^k\}_{k=1}^{\infty}$ to the limit σ^* is that the relative tiebreaking precedence along the sequence need not be preserved in the limit. But since this relative tiebreaking precedence is determined (along the sequence) by the $L \times 1$ vectors of bids submitted by each bidder, it must be encodable in (at most) an additional $L \times 1$ of signals for each bidder. It only remains to find an encoding such that any bidder types separated by a non-vanishing measure of bids along the sequence also submit strictly different signals in the limit; we accomplish this by linking τ_l^* to the measure of the maximum bid in auction l. This construction implies that (almost) no type of bidder i can strictly gain from submitting any (b_i, s_i) in the range of equilibrium strategies $\times_j(\sigma_j^* \times \tau_j^*)$; otherwise, bidder i could also eventually gain along the sequence $\{\sigma^k\}_{k=1}^{\infty}$ by bidding like the relevant type y_j of bidder j. The role of the weakly monotone tiebreaking precendence rule ρ_l is simply to rule out profitable off-equilibrium deviations by mapping every signal $s \in [0, 1]$ to the same tiebreaking precedence as some on-equilibrium signal s_i in the range of $\times_i \tau_i^*$. Hence, under ρ_l, bidder i's set of deviations in auction l is payoff-equivalent to i's set of on-equilibrium deviations in auction l. In view of the observations above, this guarantees that $\sigma^* \times \tau^*$ is an equilibrium profile under ρ.

As with Proposition 3 above, Proposition 5 extends immediately to existence of symmetric equilibria with endogenous tiebreaking when bidders are symmetric:

Corrolary 2. *In addition to the hypotheses of Proposition 5, suppose that bidders are symmetric in the sense that $\mathcal{Y}_i = \mathcal{Y}_j$, $F_i = F_j$, and $\mathcal{B}_i = \mathcal{B}_j$ for all bidders i, j. Then the signalling extension $\mathcal{G}^s$ to $\mathcal{G}$ admits a pure strategy solution $\theta^*; \sigma^* \times \tau^*$ satisfying the conclusions of Proposition 5 plus the property that strategies are symmetric: i.e. that $\sigma_i^* \times \tau_i^* = \sigma_j^* \times \tau_j^*$ for all bidders i, j.*

The proof of this fact follows immediately from the proof of Proposition 5, replacing the generic sequence $\{\sigma^k\}$ of finite equilibria derived from Proposition 5 with a sequence of symmetric equilibria derived from Corollary 1.

Although obviously related to JSSZ (2002)'s general result on existence of equilibria with endogenous tiebreaking, Proposition 5 and Corollary 2 strengthen JSSZ (2002)'s conclusions in several important respects. First, whereas JSSZ (2002) generically guarantee existence of a solution to the communication extension $\tilde{\mathcal{G}}^c$ only in *distributional strategies* as defined by Milgrom and Weber (1985), we obtain existence of such a solution in *pure strategies* which are additionally *monotone* with respect to the (PO). Second, we provide much more structure on tiebreaking than is available using only results in JSSZ (2002). For instance, Proposition 5 yields a solution in which auctions are truly both simultaneous and separable: i.e. in which allocation of object l depends only on bids and signals in auction l. Similarly, weak monotonicity of tiebreaking rules under Proposition 5 implies that allocation probabilities are weakly monotone in types. These sharper monotonicity conclusions turn out to be crucial in our analysis of limiting inefficiency in Section 6, for which existence in distributional strategies is insufficient.

Example 3 below is numerical and illustrates monotone bidding strategies in a symmetric BNE equilibrium when two ex-ante symmetric bidders are competing for two objects and their valuations for the bundle of both objects is characterized by a positive complementarity of 0.3. This visualisation helps to build intuition of what these monotone strategies look like in some simultaneous auctions settings. We also briefly discuss the effect of such a positive complementarity on revenue and efficiency in the aforementioned BNE.

Example 3. *Consider the case of two symmetric global bidders competing for two objects. The valuation for having two objects is*

$$Y_i^{(1,1)} = Y_i^{e_1} + Y_i^{e_2} + 0.3, \quad i = 1, 2,$$

where $Y_i^{e_1}$ *and* $Y_i^{e_2}$ *are two independent random variables distributed uniformly on* $[1, 2]$*. Bidders can bid any non-negative price on each object.*

Due to the constancy of the complementarity, the monotonicity of strategies β_ℓ, $\ell = 1, 2$, in the partial order sense described in this paper is equivalent to monotonicity in $Y_i^{e_1}$ (standalone valuation for object 1) and in $Y_i^{e_2}$ (standalone valuation for object 2). Because if it very difficult to derive a closed form for the equilibrium bidding strategies even in this framework, we approximate these strategies by employing numerical simulations. We consider a dense enough discrete bidding grid on $[0, M]$ (for $M > 0$ large enough) and iterate best responses on this grid until we attain the convergence of these best responses under a pre-determined convergence criterion. After such equilibrium best responses are obtained, we fit them on a continuous bidding by employing tensor product polynomials (in $Y_i^{e_1}, Y_i^{e_2}$) on $[1, 2]^2$.

The illustration of equilibrium bidding strategies as functions of $Y_i^{e_1}$ (for different fixed values of $Y_i^{e_2}$) is given in Figure 4 . The monotonicity of strategies in each Y^{e_ℓ}, $\ell = 1, 2$, can be clearly seen in that graphical illustration – first of all, for a fixed $Y_i^{e_2}$,the equilibrium bidding function on object 1 is strictly increasing in $Y_i^{e_1}$; second, for a higher fixed $Y_i^{e_2}$ the equilibrium bidding strategy on object 1 is also higher. Also, as expected, in the equilibrium we have approximated we have that for a given Y^{e_ℓ}, each bidder submits a bid higher that in the case of additive valuations (zero complementarity).

expected seller's revenue	1.7642
expected bidder's profit	0.2777
prob of i winning both objects	0.3456
prob of i winning only one object	0.3079
prob of inefficiency when both objects are won by the same bidder	0
prob of inefficiency in an auction when each bidder wins only one object	0.0480
prob of inefficiency in both auctions when each bidder wins only one object	0
prob of exposure when both objects are won by the same bidder	0
prob of having exposure in an auction when each bidder wins only one object	0.1241
prob of having exposure in both auctions when each bidder wins only one object	0
average degree exposure in an auction (conditional on having exposure)	0.0775
average percentage exposure in an auction (conditional on having exposure)	0.0673
average MLT in an auction	0.1639
average MLT in an auction when winning only that auction	0.1216
average total MLT when winning both auctions	0.3655

Table 1: Numerical results in Example 3 . MLT stands for money left on the table.

As we can see, the presence of a strictly positive complementarity in the valuation for both objects results in some exposure as well as revenue gains for the seller and lower expected profit for a bidder relative to the benchmark case of zero complementarity ($k = 0$ in $Y_i^{(1,1)} = Y_i^{e_1} + Y_i^{e_2} + k$). Indeed, in the case of zero complementarity the expected seller's revenue (for simplicity suppose that it is the same seller auctioning off both objects) is equal to 1.3333 and

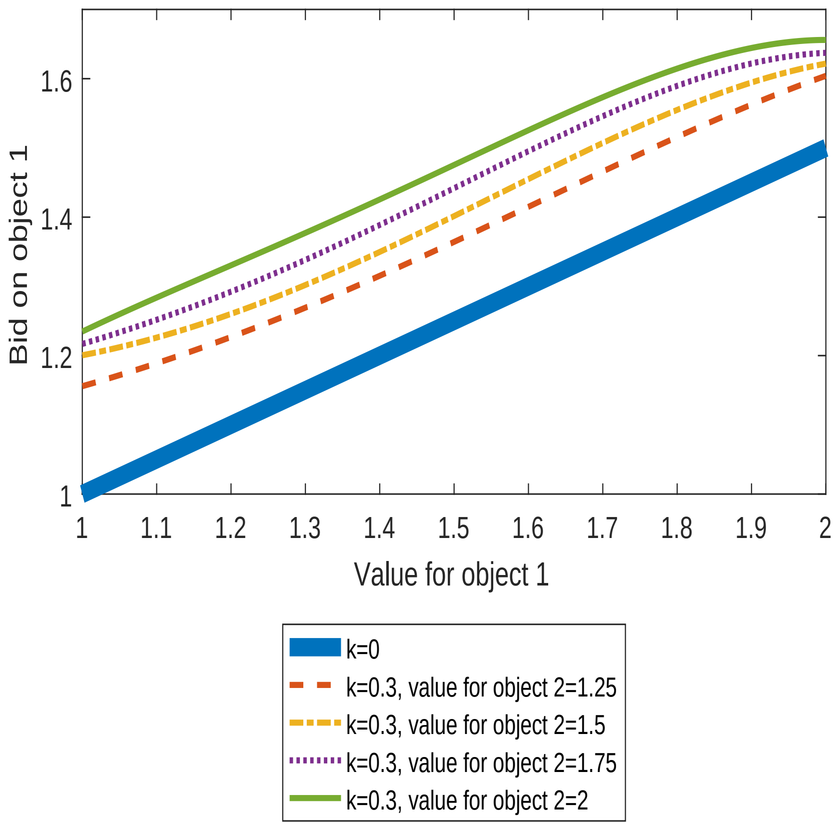


Figure 4: Illustration of equilibrium bidding strategies in Example 3. k denotes the complementarity in $Y_i^{(1,1)} = Y_i^{e_1} + Y_i^{e_2} + k$.

each bidder's expected profit is equal to 0.3333. As we can see, the exposure can happen only when a bidder wins one object only and the average loss by a bidder conditional on a bidder being exposed is 0.0775 (or 0.0673 in percentage terms). As is expected, the simultaneity of two auctions taking place results in a positive probability of an inefficient allocation of an object.

6 Many-bidder limit

In this section we relate our model to the important question of the performance of simultaneous auctions when the number of bidders N becomes large. Throughout this subsection we focus on the case when all bidders have supermodular valuations and are ex-ante symmetric (both in terms on distributions of valuations and facing the same bidding set). As in Section 5.2, we assume that the bid space $\mathcal{B}$ is continuous and that bidders play monotone equilibria with endogenous tiebreaking at each finite N; note that by Proposition 5 at least one such equilibrium exists at each N.[13] We explore the following question: under what conditions can we guarantee that expected inefficiency approaches zero as the number of bidders N become large?

It turns out that there are conditions under which we can guarantee that expected efficiency loss approaches zero as $N \to \infty$. Specifically, suppose that there exists a point $y_0 \in \mathcal{Y}$ that dominates any other point in $\mathcal{Y}$ in a partial order that is *strictly more restrictive* than partial order (PO): that is, suppose that there exists a closed convex cone $\widetilde{\mathcal{Z}}_{2^L-1}$ such that

$$\widetilde{\mathcal{Z}}_{2^L-1} \backslash \{0\} \subset \mathcal{Z}^o_{2^L-1} \tag{7}$$

(that is, all the rays defining $\widetilde{\mathcal{Z}}_{2^L-1}$ lie strictly away from the border of cone $\mathcal{Z}_{2^L-1}$) and there exists $y_0 \in \mathcal{Y}$ such that

$$\mathcal{Y} \subset y_0 - \widetilde{\mathcal{Z}}_{2^L-1}. \tag{8}$$

We show that under this condition expected inefficiency approaches zero as $N \to \infty$. Note that

[13] Throughout this section we will implicitly be exploiting the property of the communications extension equilibria that bidders submit their true values as their signals. This fact implies that if two bidders tie for an object and one bidder has strictly stronger type in the partial order (PO) sense, then this bidder will win the object with probability 1.

(7)-(8) can be equivalently formulated as follows: for any $y \in \mathcal{Y} \backslash \{y_0\}$

$$y_0^{\omega \vee e_l} - y_0^{\omega} > y^{\omega \vee e_l} - y^{\omega} \quad \text{if } \omega_l = 0. \tag{9}$$

We first illustrate this property in a simple special case with $L = 2$ objects. We then formulate the general result in Proposition 6 below.

Example 4. *Consider $L = 2$ and suppose that the support of the vector $(Y_i^{e_1}, Y_i^{e_2})$ is described as the following polyhedron in $[0,1]^2$:*

$$\mathcal{W} = \left\{ (y^{e_1}, y^{e_2})^T : 0 \le y^{e_1} \le 1, \quad 0 \le y^{e_2} \le 1,\ y^{e_2} \ge 5y^{e_1} - 4,\ 3y^{e_2} \le y^{e_1} + 2 \right\}.$$

This set is illustrated in Figure 5(a).

Suppose that for each i,

$$Y_i^{e_1 \vee e_2} = Y_i^{e_1} + Y_i^{e_2}.$$

Then the support of each bidder's vector of valuations is

$$\mathcal{Y} = \left\{ (y^{e_1}, y^{e_2}, y^{e_1 \vee e_2})^T :\ 0 \le y^{e_1} \le 1, \quad 0 \le y^{e_2} \le 1, \quad y^{e_2} \ge 5y^{e_1} - 4, \right. \\ \left. 3y^{e_2} \le y^{e_1} + 2, \quad y^{e_1 \vee e_2} = y^{e_1} + y^{e_2} \right\}.$$

Point $y_0 = (1,1,2)^T$ dominates any other point in $\mathcal{Y}$ not just in the partial order (PO) but in a strictly more restrictive partial order given by the following positive cone $\widetilde{\mathcal{Z}}_3$:

$$\widetilde{\mathcal{Z}}_3 = \left\{ (z^{e_1}, z^{e_2}, z^{e_1 \vee e_2})^T :\ -z^{e_1} + 3z^{e_2} \ge 0,\ 5z^{e_1} - z^{e_2} \ge 0,\ z^{e_1 \vee e_2} - z^{e_1} - z^{e_2} \ge 0 \right\}.$$

In other words, conditions (7) and (8) are satisfied.

Let N be large enough so we expect the bidder with the closest value to y_0 to be very close with a high probability. In other words, we can easily establish that

$$r_N = \min_{i=1} \|Y_i - y_0\| \xrightarrow{p} 0 \quad \text{as } N \to \infty,$$

and, hence, for large enough N, r_N is small enough with a very high probability.

Since the valuation $Y^{e_1 \vee e_2}$ is fully determined by Y^{e_1} and Y^{e_2}, it is enough to consider the projection of $(Y^{e_1}, Y^{e_2}, Y^{e_1 \vee e_2})^T$ on the two-dimensional space of standalone valuations $(Y^{e_1}, Y^{e_2})^T$ to evaluate the losses in efficiency.

Let $q_N = \min\limits_{i=1,\ldots,N} \|(Y_i^{e_1}, Y_i^{e_2})^T - (y_0^{e_1}, y_0^{e_2})^T\|$. We can easily show that $q_N \le r_N \le 2\sqrt{2} q_N$. Therefore, $q_N \xrightarrow{p} 0$ as $N \to \infty$. Again, we conclude that for large enough N, q_N is very small *with a very high probability.*

For a given sample $\overline{\mathbf{y}} = \left\{(y_i^{e_1}, y_i^{e_2})^T\right\}_{i=1}^N$ *of standalone valuations, let* $y(N)$ *be the vector of standalone valuations in the sample closest to* $(y_0^{e_1}, y_0^{e_2})^T$*: that is,* $y(N) = \arg \min_{i=1,\ldots,N} \|(y_i^{e_1}, y_i^{e_2})^T - (y_0^{e_1}, y_0^{e_2})^T\|$.

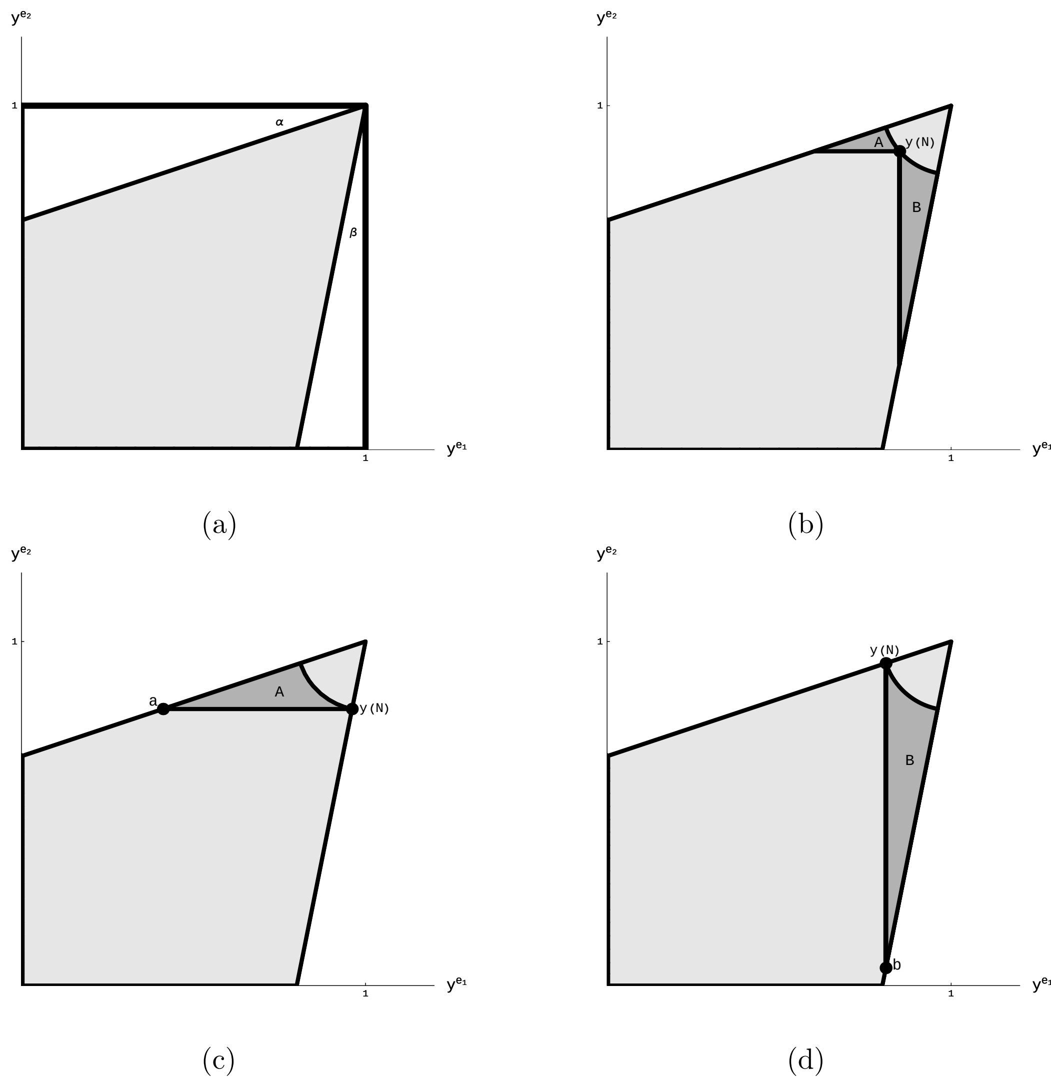


Figure 5: Illustration to Example 4.

Figure 5(b) shows an example of such $y(N)$ *on the boundary of the two-dimensional ball* $B_{q_N}\left((y_0^{e_1}, y_0^{e_2})^T\right)$ *and ares* A *and* B *such the the valuations* $(y_i^{e_1}, y_i^{e_2}, y_i^{e_1 \vee e_2})^T$*, where* $(y_i^{e_1}, y_i^{e_2}) \in A \cup B$ *are not comparable in the partial order (PO) to the valuation vector with standalone* *valuations $y(N)$.*

Figure 5(c) shows the case when the area of type A has the largest possible mass. Notice that point $a \in A$ has the largest distant (in the Euclidean norm in $\mathbb{R}^2$) in A to $(y_0^{e_1}, y_0^{e_2})^T$. We can show that the Euclidean distance of a to $(y_0^{e_1}, y_0^{e_2})^T$ is equal to $\sqrt{\frac{250}{26}} q_N$. Indeed, from the sine law for triangles

$$\frac{q_N}{\sin\alpha} = \frac{dist\big(a, \big(y_0^{e_1}, y_0^{e_2})^T\big)}{\sin(\frac{\pi}{2} + \beta)}.$$

Figure 5(d) shows the case when the set of type B has the largest possible mass. Notice that point $b \in B$ has the largest distant (in the Euclidean norm in $\mathbb{R}^2$) in B to $(y_0^{e_1}, y_0^{e_2})^T$. We can show that the Euclidean distance of b to $(y_0^{e_1}, y_0^{e_2})^T$ is equal to $\sqrt{\frac{234}{10}} q_N$. Once again, this can be easily shown from the sine law for triangles:

$$\frac{q_N}{\sin\beta} = \frac{dist\big(b, \big(y_0^{e_1}, y_0^{e_2})^T\big)}{\sin(\frac{\pi}{2} + \alpha)}.$$

For simplicity, we will just use the fact that both these distances $dist\big(a, \big(y_0^{e_1}, y_0^{e_2})^T\big)$ and $dist\big(b, \big(y_0^{e_1}, y_0^{e_2})^T\big)$ are bounded by $5q_N$.

Let $\omega_i^*(\overline{\mathbf{y}})$ denote a solution to the efficient allocation problem: that is, it is a solution to

$$\max_{\omega_i: \sum_i \omega_i = (1,1)} \sum_{i=1}^N y_i^{\omega_i},$$

and let $\tilde{\omega}_i(\overline{\mathbf{y}})$ denote a bundle won by bidder i in a symmetric monotone equilibrium in pure strategies. For notational simplicity, below these two sets of bundles simply as ω_i^* and $\tilde{\omega}_i$.

We want to show that

$$E\left[\sum_{i=1}^N Y_i^{\omega_i^*} - \sum_{i=1}^N Y_i^{\tilde{\omega}_i}\right] \to 0 \quad \text{as } N \to \infty.$$

Let us analyze different occurrences of inefficiencies in a symmetric monotone equilibrium. Bidders with standalone valuations in region C will not win anything because the bidder with standalone valuations $y(N)$ will be submitting higher bids for all the objects. Thus, exploiting that $Y^{e_1 \vee e_2} = Y^{e_1} + Y^{e_2}$, we obtain that all the possible inefficiencies in the allocation of objects will be bounded from above by the sum of $\mathcal{L}_1$ and $\mathcal{L}_2$, where

$$\mathcal{L}_1 = \max\Bigg\{ \max_{(y_i^{e_1}, y_i^{e_2})^T \in A,\, (y_j^{e_1}, y_j^{e_2})^T \in A} |y_i^{e_1} - y_j^{e_1}|, \max_{(y_i^{e_1}, y_i^{e_2})^T \in B,\, (y_j^{e_1}, y_j^{e_2})^T \in B} |y_i^{e_1} - y_j^{e_1}|, \\ \max_{(y_i^{e_1}, y_i^{e_2})^T \in A,\, (y_j^{e_1}, y_j^{e_2})^T \in B} |y_i^{e_1} - y_j^{e_1}| \Bigg\}$$

and

$$\mathcal{L}_2 = \max\Bigg\{ \max_{(y_i^{e1},y_i^{e2})^T\in A,\,(y_j^{e1},y_j^{e2})^T\in A} |y_i^{e_2} - y_j^{e_2}|, \max_{(y_i^{e1},y_i^{e2})^T\in B,\,(y_j^{e1},y_j^{e2})^T\in B} |y_i^{e_2} - y_j^{e_2}|, \max_{(y_i^{e1},y_i^{e2})^T\in A,\,(y_j^{e1},y_j^{e2})^T\in B} |y_i^{e_2} - y_j^{e_2}| \Bigg\}.$$

Obviously,

$$\mathcal{L}_1 \le 2 \max_{(y_i^{e1},y_i^{e2})^T\in A} \|(y_i^{e_1}, y_i^{e_2})^T - (y_0^{e_1}, y_i^{e_2})^T\| + 2 \max_{(y_i^{e1},y_i^{e2})^T\in B} \|(y_i^{e_1}, y_i^{e_2})^T - (y_0^{e_1}, y_i^{e_2})^T\| \le 20q_N \le 20r_N$$

Analogously, $\mathcal{L}_2 \le 20r_N$.

Let $\mathcal{L}_N$ *denote the efficiency loss in a symmetric monotone equilibrium. It is a random variable as it depends on realizations of* $Y_1, \dots, Y_N$. *Then*

$$\begin{aligned} E[\mathcal{L}_N] &= E\left[\mathcal{L}_N \,\middle|\, \min_{i=1,\dots,N} \|Y_i - y_0\| \le r_N\right] \cdot P\left(\min_{i=1,\dots,N} \|Y_i - y_0\| \le r_N\right) \\ &\quad + E\left[\mathcal{L}_N \,\middle|\, \min_{i=1,\dots,N} \|Y_i - y_0\| > r_N\right] \cdot \left(1 - P\left(\min_{i=1,\dots,N} \|Y_i - y_0\| \le r_N\right)\right) \\ &\quad \le 40r_N \cdot P\left(\min_{i=1,\dots,N} \|Y_i - y_0\| \le r_N\right) + M \cdot \left(1 - P\left(\min_{i=1,\dots,N} \|Y_i - y_0\| \le r_N\right)\right), \end{aligned}$$

where $M < \infty$ *is some constant (the losses of efficiency are always bounded as the support* $\mathcal{Y}$ *of bidders' values is bounded in* $\mathbb{R}^3$*). Now, due to the condition* $y_0 \in \mathcal{Y}$, *we can always choose the rate of* $r_N \to 0$ *such that*

$$P\left(\min_{i=1,\dots,N} \|Y_i - y_0\| \le r_N\right) \to 1 \quad \textit{as } N \to \infty.$$

Then we can see that

$$E[\mathcal{L}_N] \to 0 \quad \textit{as } N \to \infty.$$

□

The argument in Example 4 relies on the fact that both angles α and β were strictly positive and therefore we were able to bound losses of efficiency by a random variable converging in probability to zero. We now build on this intuition to formulate a result applicable to the general case:

Proposition 6. *Suppose Assumptions 2–4 hold and that all N bidders participating in the auctions have supermodular valuations, are ex-ante symmetric, have the same bidding sets and play an equilibrium in symmetric monotone strategies. In addition, assume that there is a point y_0 in the support $\mathcal{Y}$ of bidders' values and a closed convex cone $\widetilde{\mathcal{Z}}_{2^L-1}$ such that conditions (7) and (8) hold.*

Let $\mathcal{L}_N$ denote the efficiency loss in a symmetric monotone equilibrium. Then

$$E[\mathcal{L}_N] \to 0 \text{ as } N \to \infty.$$

Note that under the conditions of Proposition 6 we cannot guarantee that it is *the same bidder* who wins all the objects (it is relatively easy to construct examples illustrating that) even in the limit, the result of Proposition 6 stems from the fact in the limit it is only bidders of types increasingly closer to y_0 (possibly many of those) who win at least one object.

Many-bidder and many-object limit We want to finish this section with a discussion of what happens in the limit when both the number of bidders and the size of the market are growing to infinity. Namely, in this scenario we would like to formulate a result analogous to the result of Proposition 6.

We will suppose that as the market grows, all bidders have supermodular preferences over all objects in the market. Thus, in the limit the support $\mathcal{Y}^\infty$ of is a subset of $\mathbb{R}^\infty$. Each vector of valuations is an infinite sequence $\{y^\omega\}_{\omega\in\{0,1\}^\infty}$, with the order of $\omega \in \{0,1\}^\infty$ chosen in some way. Analogously to the case above, we assume that there is $y_0 \in \mathcal{Y}^\infty \backslash \{y_0\}$ that dominates any other point in $\mathcal{Y}^\infty$ in the partial order strictly more restrictive than (PO):

$$\forall\, (\omega \in \{0,1\}^\infty) \text{ such that } \omega_l = 0, \quad y_0^{\omega\vee e_l} - y_0^\omega > y^{\omega\vee e_l} - y^\omega. \tag{10}$$

We show that under this conditions, as the number of bidders and the number of objects increase to infinity, all the objects are increasingly won by types increasingly close to type

y_0 and, thus, in the limit the expected inefficiency is equal to zero. This is formalized in Proposition 7 below.

Proposition 7. *Suppose Assumptions 2–4 hold and that for any L all N bidders participating in the auctions have supermodular valuations, are ex-ante symmetric, have the same bidding sets and play an equilibrium in symmetric monotone strategies. In addition, suppose that $L \to \infty$ as $N \to \infty$ and*

(i) there is a point y_0 in the support $\mathcal{Y}^\infty$ of bidders' values over an infinite sequence of objects such that condition (10) holds;

(ii) a bound on the individual components of 2^{L-1}-dimensional elements in the projection of $\mathcal{Y}^\infty$ on $\mathbb{R}^{2^L-1}$ grows linearly in L.

Let $\mathcal{L}_{N,L}$ denote the efficiency loss in a symmetric monotone equilibrium with L objects. Then if $L \to \infty$ slowly enough (as $N \to \infty$), then

$$E\left[\mathcal{L}_{N,L}\right] \to 0 \text{ as } N \to \infty.$$

As in the case of fixed L, the asymptotic efficiency result stems from the fact that objects are increasingly won by bidders whose types are increasingly close to type y_0 (or its projection on the $\mathbb{R}^{2^L-1}$). The slow enough growth of L is required to guarantee that the effect of winners clustering around y_0 dominates the maximum cumulative size of inefficiency that grows proportionally with L. Regarding condition (ii) in Proposition 7, note that the upper bound in (ii) has to be growing to infinity with L (due to the supermodularity condition of preferences for any L). The condition on the linear rate means that when we add an extra object to existing L objects the valuations for subsets of objects grow in a bounded way.

7 Conclusion

Building on techniques in Athey (2001), McAdams (2003), Reny (2011) and JSSZ (2002), we establish the existence of pure strategy monotone equilibria in a class of standard simultaneous auction mechanisms for complementary goods. All the analysis is conducted under the assumption of supermodular preferences. In general bidders' valuations across different bundles of objects are not additive across individual objects in these bundles. Monotonicity on the space of types is understood in the partial order sense given by a cone described by a finite number of linear inequalities on increasing differences of bidders valuations between a larger and a smaller sets of objects, while monotonicity on the space of bids is understood in the coordinatewise order sense. Finally, we establish novel sufficient conditions under which expected inefficiency approaches zero as the number of bidders increases.

An alternative approach to existence could be the differential equations approach. Because bidders' preferences are over many bundles of objects, the existence problem would be characterized by a system of partial differential equations with some boundary conditions. Issues of existence of solutions of systems of partial differential equations are infamously much more difficult than those in the systems of ordinary differential equations and, therefore, the approach did not seem realistic to us.

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Appendix: Proofs

Proof of Lemma 1

Payments $\sum_{l=1}^{L} c_{il}(b_{il};\sigma_{-i})$ are additively separable across auctions, hence modular in $b_i = (b_{il},...,b_{il})$ by construction. To establish supermodularity of interim payoffs

$$\pi_i(b_i; y_i, \sigma_{-i}) = y_i^T P(b_i; \sigma_{i-}) - \sum_{l=1}^{L} c_{il}(b_{il};\sigma_{-i})$$

in b_i, it is therefore sufficient to establish that expected valuations $y_i^T P(b_i;\sigma_{i-})$ are supermodular in b_i for any $y_i \in \mathcal{Y}_i$.

Toward this end, for any realization y_{-i} of the rivals' types, let the $(\sum N_l - l) \times 1$ vector $B_{-i} = \sigma_{-i}(Y_{-i})$ be the vector of all rival bids across all auctions. With slight abuse of notation, let $y_i(b; B_{-i})$ denote player i's expected valuation given type y_i, own bid vector b and the complete rival bid vector B_{-i}:

$$y_i(b; B_{-i}) = E_\omega[y_i^\omega | y_i, b, B_{-i}],$$

where the expectation is taken over ties. Note that, in the absence of ties, bidder i will win each auction in which his bid is the highest, $\omega = \omega(b, B_{-i})$ will be deterministic, and $y_i(b; B_{-i}) = y_i^{\omega(b,B_{-i})}$.

The function $y_i^T P_i(b,\sigma_{-i})$ can be written as the expectation of $y_i(b; B_{-i})$ with respect to the distribution of the rival bids:

$$y_i^T P_i(b,\sigma_{-i}) = \int y_i(b; \sigma_{-i}(Y_{-i}))\ F_{-i}(dY_{-i}).$$

Let $b', b'' \in \mathcal{B}_i$. We want to show that

$$y_i^T P_i(b'' \vee b'; \sigma_{-i}) + {y_i}^T P_i(b'' \wedge b'; \sigma_{-i}) \geq {y_i}^T P_i(b''; \sigma_{-i}) + {y_i}^T P_i(b'; \sigma_{-i}),$$

or equivalently, that

$$\int [y_i(b'' \vee b'; \sigma_{-i}(Y_{-i})) + y_i(b'' \wedge b'; \sigma_{-i}(Y_{-i})) - y_i(b''; \sigma_{-i}(Y_{-i})) - y_i(b'; \sigma_{-i}(Y_{-i}))]\ F_{Y_{-i}}(dY_{-i}) \geq 0. \quad (11)$$

For a specific realization y_{-i} of rivals' types, if there are no ties for $(b''; \sigma_{-i}(y_{-i}))$ and $(b'; \sigma_{-i}(y_{-i}))$, then

$$\omega(b'' \wedge b'; \sigma_{-i}(y_{-i})) = \omega(b'; \sigma_{-i}(y_{-i})) \wedge \omega(b''; \sigma_{-i}(y_{-i})),$$
$$\omega(b'' \vee b'; \sigma_{-i}(y_{-i})) = \omega(b'; \sigma_{-i}(y_{-i})) \vee \omega(b''; \sigma_{-i}(y_{-i})),$$

and

$$
\begin{aligned}
y_i(b'' \vee b'; \sigma_{-i}(y_{-i})) + y_i(b'' \wedge b'; \sigma_{-i}(y_{-i})) - y_i(b''; \sigma_{-i}(y_{-i})) - y_i(b'; \sigma_{-i}(y_{-i}) \\
= y_i^{\omega(b'' \vee b'; \sigma_{-i}(y_{-i}))} + y_i^{\omega(b'' \wedge b'; \sigma_{-i}(y_{-i}))} - y_i^{\omega(b'; \sigma_{-i}(y_{-i}))} - y_i^{\omega(b''; \sigma_{-i}(y_{-i}))} \\
= y_i^{\omega(b'; \sigma_{-i}(y_{-i})) \vee \omega(b''; \sigma_{-i}(y_{-i}))} + y_i^{\omega(b'; \sigma_{-i}(y_{-i})) \wedge \omega(b''; \sigma_{-i}(y_{-i}))} - y_i^{\omega(b'; \sigma_{-i}(y_{-i}))} - y_i^{\omega(b''; \sigma_{-i}(y_{-i}))} \geq 0.
\end{aligned}
$$

Alternatively, if there are ties at either $(b''; \sigma_{-i}(y_{-i}))$ or $(b'; \sigma_{-i}(y_{-i}))$, then some of the outcome vectors $\omega(b'' \wedge b'; \sigma_{-i}(y_{-i}))$, $\omega(b'' \vee b'; \sigma_{-i}(y_{-i}))$, $\omega(b'; \sigma_{-i}(y_{-i}))$, $\omega(b''; \sigma_{-i}(y_{-i}))$ will be stochastic. Let $\lambda_2(b'; b''; \sigma_{-i}(y_{-i}))$ denote the joint distribution of $\omega(b'; \sigma_{-i}(y_{-i}))$ and $\omega(b''; \sigma_{-i}(y_{-i}))$. Note that the distribution of $\omega(b'' \vee b'; \sigma_{-i}(y_{-i}))$ coincides with the distribution of $\omega(b'; \sigma_{-i}(y_{-i})) \vee \omega(b''; \sigma_{-i}(y_{-i}))$, and the distribution of $\omega(b'' \wedge b'; \sigma_{-i}(y_{-i}))$ coincides with the distribution of $\omega(b'; \sigma_{-i}(y_{-i})) \wedge \omega(b''; \sigma_{-i}(y_{-i}))$. Therefore,

$$
\begin{aligned}
y_i(b'' \vee b'; \sigma_{-i}(y_{-i})) &= E_{(\omega_1, \omega_2) \sim \lambda_2(b'; b''; \sigma_{-i}(y_{-i}))} \left[y_i^{\omega_1 \vee \omega_2} | y_i, b, \sigma_{-i}(y_{-i}) \right], \\
y_i(b'' \wedge b'; \sigma_{-i}(y_{-i})) &= E_{(\omega_1, \omega_2) \sim \lambda_2(b'; b''; \sigma_{-i}(y_{-i}))} \left[y_i^{\omega_1 \wedge \omega_2} | y_i, b, \sigma_{-i}(y_{-i}) \right].
\end{aligned}
$$

Since we can write

$$
\begin{aligned}
y_i(b'; \sigma_{-i}(y_{-i})) &= E_{(\omega_1, \omega_2) \sim \lambda_2(b'; b''; \sigma_{-i}(y_{-i}))} \left[y_i^{\omega_1} | y_i, b, \sigma_{-i}(y_{-i}) \right], \\
y_i(b''; \sigma_{-i}(y_{-i})) &= E_{(\omega_1, \omega_2) \sim \lambda_2(b'; b''; \sigma_{-i}(y_{-i}))} \left[y_i^{\omega_2} | y_i, b, \sigma_{-i}(y_{-i}) \right],
\end{aligned}
$$

then the integrand in (11) can be written as

$$
E_{(\omega_1, \omega_2) \sim \lambda_2(b'; b''; \sigma_{-i}(y_{-i}))} \left[y_i^{\omega_1 \vee \omega_2} + y_i^{\omega_1 \wedge \omega_2} - y_i^{\omega_1} - y_i^{\omega_2} | y_i, b, \sigma_{-i}(y_{-i}) \right]. \tag{12}
$$

Because for every realization of (ω_1, ω_2) from the distribution $\lambda_2(b'; b''; \sigma_{-i}(y_{-i}))$ we have

$$
y_i^{\omega_1 \vee \omega_2} + y_i^{\omega_1 \wedge \omega_2} - y_i^{\omega_1} - y_i^{\omega_2} \geq 0,
$$

the expectation in (12) is also non-negative. Thus, for every y_{-i}, the integrand in (11) is non-negative. Therefore, (11) holds.

□

Proof of Lemma 2

Denote

$$
\begin{aligned}
b_i^{(0)} &= (b_{i1}, b_{i2}, b_{i3}, \ldots, b_{i,L-1}, b_{iL})^T, \\
b_i^{(1)} &= (\tilde{b}_{i1}, b_{i2}, b_{i3}, \ldots, b_{i,L-1}, b_{iL})^T, \\
b_i^{(2)} &= (\tilde{b}_{i1}, \tilde{b}_{i2}, b_{i3}, \ldots, b_{i,L-1}, b_{iL})^T, \\
&\ldots \\
b_i^{(L)} &= (\tilde{b}_{i1}, \tilde{b}_{i2}, \tilde{b}_{i3}, \ldots, \tilde{b}_{i,L-1}, \tilde{b}_{iL})^T.
\end{aligned}
$$

Represent the difference $\pi_i(\tilde{b}_i; y_i, \sigma_{-i}) - \pi_i(b_i; y_i, \sigma_{-i})$ in the following way:

$$\pi_i(\tilde{b}_i; y_i, \sigma_{-i}) - \pi_i(b_i; y_i, \sigma_{-i}) = \sum_{l=1}^{L} \left(\pi_i(b_i^{(l)}; y_i, \sigma_{-i}) - \pi_i(b_i^{(l-1)}; y_i, \sigma_{-i}) \right)$$

Now represent the interim payoff function $\pi_i(b, y_i, \sigma_{-i})$ by applying what we refer to throughout this paper as a marginalization technique:

$$\pi_i(b_i, y_i, \sigma_{-i}) = \sum_{\omega:\omega_l=0} y_i^{\omega}(P_i^{\omega}(b_i, \sigma_{-i}) + P_i^{\omega\vee e_l}(b_i, \sigma_{-i})) + \sum_{\omega:\omega_l=0} (y_i^{\omega\vee e_l} - y_i^{\omega}) P_i^{\omega\vee e_l}(b_i, \sigma_{-i}) - \sum_{m=1}^{L} c_{im}(b_{im}, \sigma_{-i}).$$

Note that if $\omega_l = 0$, then $P_i^{\omega}(b; \sigma_{-i}) + P_i^{\omega\vee e_l}(b; \sigma_{-i})$ is the probability of winning all objects $m \neq l$ for which $\omega_m = 1$ and losing all objects $m \neq l$ for which $\omega_m = 0$. Thus, this is the probability of the event that does not depend on the allocation of object l (also using Assumption 1). Therefore, $P_i^{\omega}(b; \sigma_{-i}) + P_i^{\omega\vee e_l}(b; \sigma_{-i})$ does not depend on the bid for object l. Therefore, if $\omega_l = 0$, then

$$P_i^{\omega}(b_i^{(l)}; \sigma_{-i}) + P_i^{\omega\vee e_l}(b_i^{(l)}; \sigma_{-i}) - P_i^{\omega}(b_i^{(l-1)}; \sigma_{-i}) - P_i^{\omega\vee e_l}(b_i^{(l-1)}; \sigma_{-i}) = 0$$

because bid vectors $b_i^{(l)}$ and $b_i^{(l-1)}$ differ only in the bid submitted for object l. Hence,

$$\pi_i\left(b_i^{(l)}; y_i, \sigma_{-i}\right) - \pi_i\left(b_i^{(l-1)}; y_i, \sigma_{-i}\right) = \sum_{\omega:\omega_l=0} (y_i^{\omega\vee e_l} - y_i^{\omega}) \left(P_i^{\omega\vee e_l}(b_i^{(l)}, \sigma_{-i}) - P_i^{\omega\vee e_l}(b_i^{(l-1)}, \sigma_{-i}) \right) - c_{il}(\tilde{b}_{il}, \sigma_{-i}) + c_{il}(b_{il}, \sigma_{-i}).$$

Note that because the component corresponding to the bid submitted for object l in the vector $b_i^{(l)}$ is greater than the one in the vector $b_i^{(l-1)}$, and all the other components in the two vectors are the same, then

$$\forall\, \omega : \omega_l = 0 \qquad P_i^{\omega\vee e_l}(b_i^{(l)}, \sigma_{-i}) - P_i^{\omega\vee e_l}(b_i^{(l-1)}, \sigma_{-i}) \geq 0.$$

Combining this across all objects l, we have

$$\pi_i(\tilde{b}_i; y_i, \sigma_{-i}) - \pi_i(b_i; y_i, \sigma_{-i}) = \sum_{l=1}^{L} \sum_{\omega:\omega_l=0} (y_i^{\omega\vee e_l} - y_i^{\omega}) \underbrace{\left(P_i^{\omega\vee e_l}(b_i^{(l)}, \sigma_{-i}) - P_i^{\omega\vee e_l}(b_i^{(l-1)}, \sigma_{-i}) \right)}_{\geq 0} - \sum_{l=1}^{L} c_{il}(\tilde{b}_{il}, \sigma_{-i}) + \sum_{l=1}^{L} c_{il}(b_{il}, \sigma_{-i}).$$

The analogous representation for type $\tilde{y}_i$ is

$$\pi_i(\tilde{b}_i; \tilde{y}_i, \sigma_{-i}) - \pi_i(b_i; \tilde{y}_i, \sigma_{-i}) = \sum_{l=1}^{L} \sum_{\omega:\omega_l=0} (\tilde{y}_i^{\omega \vee e_l} - \tilde{y}_i^{\omega}) \underbrace{\left(P_i^{\omega \vee e_l}(b_i^{(l)}, \sigma_{-i}) - P_i^{\omega \vee e_l}(b_i^{(l-1)}, \sigma_{-i})\right)}_{\geq 0}$$
$$- \sum_{l=1}^{L} c_{il}(\tilde{b}_{il}, \sigma_{-i}) + \sum_{l=1}^{L} c_{il}(b_{il}, \sigma_{-i}).$$

From the definition of the partial order we then conclude that

$$\pi_i(\tilde{b}_i; \tilde{y}_i, \sigma_{-i}) - \pi_i(b_i; \tilde{y}_i, \sigma_{-i}) \geq \pi_i(\tilde{b}_i; y_i, \sigma_{-i}) - \pi_i(b_i; y_i, \sigma_{-i}),$$

which implies the weak single crossing property. □

Proof of Proposition 2

Let μ_{2^L-1} denote the Lebesgue measure in $\mathbb{R}^{2^L-1}$. Fix any $S \subset \mathcal{Y}_i$ with a positive F_{Y_i} measure. By the absolute continuity assumption then, $\mu_{2^L-1}(S) > 0$. We start by showing that we can find two points in S – denote them as $\bar{y}_i$ and $\tilde{y}_i$, – such that all the inequalities in the definition of the partial order will be satisfied strictly. In other words, we want to find $\bar{y}_i, \tilde{y}_i \in S$ such that $\tilde{y}_i \in \bar{y}_i + \mathcal{Z}^o_{2^L-1}$. Set $\mathcal{Z}^o_{2^L-1}$ is the interior in $\mathbb{R}^{2^L-1}$ of cone $\mathcal{Z}_{2^L-1}$ defined in (2) (it is discussed in the main text of the paper that $\mathcal{Z}^o_{2^L-1} \neq \emptyset$).

We claim that we can take any $\bar{y}_i \in S$ that is a point of (Lebesgue) density of S. The property of being a point of density of S means that

$$\lim_{r \downarrow 0} \frac{\mu_{2^L-1}\left((\bar{y}_i + \mathbf{B}_r) \cap S\right)}{\mu_{2^L-1}\left(\bar{y}_i + \mathbf{B}_r\right)} = 1,$$

where $\mathbf{B}_r$ denotes the open ball in $\mathbb{R}^{2^L-1}$ centered at 0 of radius $r > 0$.

Note that if $\bar{y}_i \in S$ is a point of density of S and the set $(y_i + \mathcal{Z}) \cap S$ is empty, then

$$\begin{aligned}
\frac{\mu_{2^L-1}\left((\bar{y}_i + \mathbf{B}_r) \cap S\right)}{\mu_{2^L-1}\left(\bar{y}_i + \mathbf{B}_r\right)} &= \frac{\mu_{2^L-1}\left((\bar{y}_i + \mathbf{B}_r) \cap S \cap (\bar{y}_i + \mathcal{Z}^o_{2^L-1})^c\right)}{\mu_{2^L-1}\left(\bar{y}_i + \mathbf{B}_r\right)} \\
&\leq \frac{\mu_{2^L-1}\left((\bar{y}_i + \mathbf{B}_r) \cap (\bar{y}_i + \mathcal{Z}^o_{2^L-1})^c\right)}{\mu_{2^L-1}\left(\bar{y}_i + \mathbf{B}_r\right)}
\end{aligned}$$

for all $r > 0$. Note that the ratio

$$\frac{\mu_{2^L-1}\left(\mathcal{Z}^o_{2^L-1} \cap \mathbf{B}_r\right)}{\mu_{2^L-1}\left(\mathbf{B}_r\right)}$$

is strictly between zero and one does not depend on r.[14] Denote this ratio as $c_L \in (0, 1)$. Using the

[14]To see this, consider the scaling transform $y \to \frac{y}{r}$. It maps the $\mathcal{Z}^o_{2^L-1} \cap \mathbf{B}_r$ to $\mathcal{Z}^o_{2^L-1} \cap \mathbf{B}_1$. Since the Jacobian

translation invariance property of the Lebesgue measure we obtain then that

$$\frac{\mu_{2^L-1}\left((\bar{y}_i+\mathbf{B}_r)\cap S\right)}{\mu_{2^L-1}\left(\bar{y}_i+\mathbf{B}_r\right)} \le \frac{\mu_{2^L-1}\left((\bar{y}_i+\mathbf{B}_r)\cap(\bar{y}_i+\mathcal{Z}^o_{2^L-1})^c\right)}{\mu_{2^L-1}\left(\bar{y}_i+\mathbf{B}_r\right)} = 1-c_L$$

for all $r>0$, but this contradicts the property of $\bar{y}_i$ being a point of density of S. This contradiction means that for any point $\bar{y}_i$ of Lebesgue density of S, we can find $\tilde{y}_i \in \bar{y}_i + \mathcal{Z}^o_{2^L-1}$ to give us the conclusion. By the Lebesgue property a.e. $\bar{y}_i \in S$ is a point of density.

By convexity, the boundary of each set in the finite union of sets composing $\mathcal{Y}_i$ has Lebesgue measure zero in $\mathbb{R}^{2^L-1}$. Since the boundary of a finite union of sets is contained in the finite union of the respective boundaries, we conclude that the boundary of $\mathcal{Y}_i$ has Lebesgue measure zero in $\mathbb{R}^{2^L-1}$. Therefore, without loss of generality, we can select both $y_i \in S$ and $\tilde{y}_i \in S\cap(y_i+\mathcal{Z}^o_{2^L-1})$ to be in the interior of $\mathcal{Y}_i$ in $\mathbb{R}^{2^L-1}$. Therefore, we can find a small $r>0$ such that the ball $\tilde{y}_i+\mathbf{B}_r$ is contained in $(y_i+\mathcal{Z}^o_{2^L-1})\cap\mathcal{Y}_i$. Consider the intersection $(\tilde{y}_i+\mathbf{B}_r)\cap(\tilde{y}_i+\mathcal{Z}^o_{2^L-1})$. This intersection is non-empty and is an open set in $\mathbb{R}^{2^L-1}$. Every point in this intersection dominates $\bar{y}_i$ but is dominated by $\tilde{y}_i$ in the sense of partial order (PO). As a union of a finite number of compact sets, $\mathcal{Y}_i$ is compact in $\mathbb{R}^{2^L-1}$ and, hence, has a dense countable subset. Denote it as $\mathcal{Y}_i^\star$. Set $(\tilde{y}_i+\mathbf{B}_r)\cap(\tilde{y}_i+\mathcal{Z}^o_{2^L-1})$ being a non-empty open set in $\mathbb{R}^{2^L-1}$ clearly has to contain a point from $\mathcal{Y}_i^\star$. $\square$

Proof of Proposition 3

We define the partial order $\succeq$ on the space of types as in (PO). Let us define the partial order in each bid (= action) space as the coordinatewise order:

$$\tilde{b}_i \succeq b_i \quad \Longleftrightarrow \quad \tilde{b}_i \ge b_i,$$

where $\tilde{b}_i \ge b_i$ means that each component on the L-dimensional vector $\tilde{b}_i$ is greater or equal than the corresponding component of b_i.

We verify conditions of Theorem 4.1 and Proposition 4.4 in Reny (2011). Let us start with conditions G1-G6 in Theorem 4.1.

G1 Obviously, the partial order given in Definition 4 is transitive:

$$\tilde{\tilde{y}}_i \succeq \tilde{y}_i,\ \tilde{y}_i \succeq y_i \quad \Rightarrow \quad \tilde{\tilde{y}}_i \succeq y_i;$$

it is also reflexive:

$$y_i \succeq y_i;$$

and it is also antisymmetric:

$$\tilde{y}_i \succeq y_i \text{ and } y_i \succeq \tilde{y}_i \quad \Rightarrow \quad \tilde{y}_i = y_i.$$

of this sort of transform is independent of y, the ratio of the volume of the intersection of shape $\mathcal{Z}^o_{2^L-1}$ with he ball $\mathbf{B}_r$ to the volume of $\mathbf{B}_r$ is invariant under such a transform.

Let $\mathcal{A}(\Re^{2^L} \cap \mathcal{Y}_i)$ denote the Borel sigma-field on $\Re^{2^L} \cap \mathcal{Y}_i$. Let us show that

$$G_i \equiv \{(\tilde{y}_i, y_i) \in \mathcal{Y}_i \times \mathcal{Y}_i \ : \tilde{y}_i \succeq y_i\} \in \mathcal{A}(\Re^{2^L} \cap \mathcal{Y}_i) \times \mathcal{A}(\Re^{2^L} \cap \mathcal{Y}_i).$$

Notice that the partial order between $\tilde{y}_i$ and y_i can be fully expresses as comparison of certain linear inequalities for $\tilde{y}_i$ and y_i. In other words, the order $\tilde{y}_i \succeq y_i$ can be fully described as a system of linear inequalities

$$M_L \tilde{y}_i - M_L y_i \geq 0,$$

where matrix M_L is defined in a certain way. Therefore, G_i is the intersection of a closed convex polyhedron in $\mathbb{R}^{2 \cdot 2^L}$ with $\mathcal{Y}_i \times \mathcal{Y}_i$, and thus, $G_i \in \mathcal{A}(\Re^{2 \cdot 2^L} \cap \mathcal{Y}_i \times \mathcal{Y}_i)$. Under the Euclidean metric $\Re^{2^L} \cap \mathcal{Y}_i$ is a locally compact Hausdorff space. Therefore, by e.g. Proposition 7.6.2 on page 220 in Cohn (2013),

$$\mathcal{A}(\Re^{2 \cdot 2^L} \cap \mathcal{Y}_i \times \mathcal{Y}_i) = \mathcal{A}(\Re^{2^L} \cap \mathcal{Y}_i) \times \mathcal{A}(\Re^{2^L} \cap \mathcal{Y}_i),$$

and, hence, $G_i \in \mathcal{A}(\Re^{2^L} \cap \mathcal{Y}_i) \times \mathcal{A}(\Re^{2^L} \cap \mathcal{Y}_i)$.

G2 The atomlessness of the measure on the sigma-algebra for $\mathcal{Y}_i$ follows from the continuity of c.d.f. $F_{Y,i}$ in Assumption 3.

G3 This is assumed explicitly.

G4 $\mathcal{B}_i$ is assumed to be a lattice. Also,

$$\left\{(\tilde{b}_i, b_i) \in \mathcal{B}_i \times \mathcal{B}_i \ : \ \tilde{b}_i \succeq b_i\right\}$$

is closed in the product topology on $\mathbb{R}^L \times \mathbb{R}^L$ because the partial order on $\mathcal{B}_i$ is the coordinate-wise partial order.

G5 Each $\mathcal{B}_i$ is a locally-complete Euclidean metric lattice.

G6 The function

$$u_i(b_i, b_{-i}; y_i) = \sum_{\omega} (y_i^{\omega} - \omega' b_i) \cdot \prod_{l:\ \omega^l = 1} \left(1(b_{il} > \max_{j \neq i} b_{jl}) + T_{il}(S_l, b_{il}) 1(b_{il} = \max_{j \neq i} b_{jl}) \right) \times \prod_{l:\ \omega^l = 0} 1(b_{il} < \max_{j \neq i} b_{jl}) \quad (13)$$

is jointly measurable. It is bounded since both $\mathcal{Y}_i$ and $\mathcal{B}_i$ are bounded. Note that $T_{il}(S_l, b_{il})$, where S_l denotes the set of bidders tying for the highest bid in auction l, stands for the probability of obtaining object l in case of ties. This formulation accords with Assumption 1.

The function $u_i(b_i; y_i)$ is also continuous in $b_i \in \mathcal{B}_i$ for every $y_i \in \mathcal{Y}_i$ since by assumption $\mathcal{B}_i$ consists of a finite number of points.

Thus, G1-G6 in Theorem 4.1 in Reny (2011) hold. Clearly, Lemmas 1 and 2 of this paper imply (i) in Proposition 4.4 in Reny (2011). $\square$

Proof of Lemma 3

Fix $\epsilon > 0$. Choose k such that $\delta_k \equiv \max_{j=0,\ldots,N} \mathcal{H}(\ddot{\mathcal{B}}_j^k, \mathcal{B}_j) < \epsilon/L$.

For local bidder $j = 1, \ldots, N$ with type realization $y_j \in \mathcal{Y}_j$, the argument is entirely standard; we reproduce it here only for completeness. Let $\pi_j^* = \max_{b_j \in \mathcal{B}_j} \pi_j(y_j, b_j; \tilde{\sigma}_{-j}^k)$ and $b_j^* = \arg\max_{b_j \in \mathcal{B}_j} \pi_j(y_j, b; \tilde{\sigma}_{-j}^k)$, and let $\ddot{b}_j$ be the smallest element of $\ddot{\mathcal{B}}_j$ such that $\ddot{b}_j \geq b_j^*$. Clearly, we must have $b_j^* \leq y_j$, hence the loss associated with bidding $\ddot{b}_j$ (rather than b_j^*) can be no greater than δ_k. But $\ddot{b}_j$ was feasible for bidder j under grid space $\ddot{\mathcal{B}}^k$, whence by equilibrium $\pi_j(y_j, \ddot{\sigma}_j^k(y_j); \ddot{\sigma}_{-j}^k) \geq \pi_j(y_j, \bar{b}_j; \ddot{\sigma}_{-j}^k) \geq \pi_k^* - \delta_k > \pi_k^* - \epsilon$. Since y_j was arbitrary, this bound also applies in expectation. Thus local bidder j can gain no more than ϵ by deviating from strategy profile $\ddot{\sigma}_j^k$ when rivals play strategy profile $\ddot{\sigma}_{-j}^k$; i.e. $\ddot{\sigma}_{-j}^k$ satisfies the conditions of an ϵ-equilibrium for local bidder j.

For the global bidder 0, the argument is somewhat more subtle, turning on the fact that action spaces for bidder 0 are always such that bidder 0 can choose (at cost no greater than the bid increment) whether and how to resolve potential ties. Fixing type $y_0 \in \mathcal{Y}_0$ and holding strategies of local bidders fixed at $\ddot{\sigma}_{-0}^k$, let π_0^* denote bidder 0's supremum payoff over the unrestricted bid space $\mathcal{B}_0$. Let $\{b_0^m\}_{m=1}^\infty$ be any sequence of $L \times 1$ bid vectors for bidder 0 such that $\pi_0(y_0, b_0^m; \ddot{\sigma}_{-0}^k) \to \pi_0^*$ as $m \to 0$. Let $\{b_{0l}^m\}_{m=1}^\infty$ be the scalar sequence such that for each $m = 1, 2, ...$ the scalar b_{0l}^m is the lth element of the $L \times 1$ bid vector b_0^m; i.e. the element describing bidder 0's bid in auction l. Construct a sequence $\{\ddot{b}_{0l}^m\}_{m=1}^\infty$ from $\{b_{0l}^m\}_{m=1}^\infty$ as follows:

$$\ddot{b}_{0l}^m = \min\{b_l' \in \ddot{\mathcal{B}}_{0l}^k : b_l' \geq \max\{b_l'' \in \ddot{\mathcal{B}}_l | b_l'' \leq b_{0l}^m\}\}.$$

Recall that $\ddot{\mathcal{B}}_l^k$ is a selection from $\ddot{\mathcal{B}}_{0l}^k$ such that for any two points b_l', b_l'' in $\ddot{\mathcal{B}}_l^k$ there exists a third point $b_l''' \in \ddot{\mathcal{B}}_{0l}^k \backslash \ddot{\mathcal{B}}_l^k$ such that $b_l' < b_l''' < b_l''$. Furthermore, from the perspective of bidder 0, allocation probabilities for each l are flat outside the support $\ddot{\mathcal{B}}_l^k$ of rival strategies $\ddot{\sigma}_{-0}^k$. Hence by construction the sequence $\ddot{b}_{0l}^m$ will yield the same allocation probabilities as the sequence b_{0l}^m for any $m = 1, 2, ...$ Thus bidder 0's expected payoffs along bid sequence $\ddot{b}_{0l}^m$ can differ from bidder 0's expected payoffs along bid sequence b_{0l}^m by at most the bid increment δ_k. Repeating this construction separately for each $l = 1, ..., L$, we obtain a sequence $\{\ddot{b}_0^m\}_{m=1}^\infty$ contained in $\ddot{\mathcal{B}}_0^k$, yielding the same allocations as $\{b_0^m\}_{m=1}^\infty$, and for which payoffs differ (relative to $\{b_0^m\}_{m=1}^\infty$) by at most $L\delta_k$. But (by equilibrium) bidder 0's payoffs under $\ddot{\sigma}_0^k$ must be weakly greater than bidder 0's payoffs at each $\ddot{b}_0^m$, and the sequence $\ddot{b}_0^m$ attains a payoff within $L\delta_k < \epsilon$ of bidder 0's unrestricted supremum at type y_0. Since y_0 was arbitrary, this bound on bidder 0's interim gains also implies a bound on bidder 0's ex ante gains, whence it follows that $\ddot{\sigma}^k$ is an ϵ-equilibrium on the unrestricted bid space $\mathcal{B}$. □

Proof of Lemma 4

Let $(\tilde{\sigma}, \tilde{\Pi})$ be any point in the closure of the vector payoff function

$$\mathbf{\Pi}(\sigma) = (\Pi_0(\sigma); \Pi_1(\sigma), \ldots, \Pi_N(\sigma)),$$

i.e. any $\tilde{\Pi}$ for which there exists a sequence of monotone strategies $\{\sigma^k\}_{k=1}^\infty$ such that $\sigma^k \to \tilde{\sigma}$ (weak*) and $\mathbf{\Pi}(\sigma^k) \to \tilde{\Pi}$. Suppose that $\tilde{\sigma}$ is not an equilibrium. We need to show that some player $i \in \{0, 1, \ldots, N\}$ can secure a payoff strictly above $\tilde{\Pi}_i$ at $\tilde{\sigma}$.

First suppose that under the limiting strategy $\tilde{\sigma}$ relevant ties occur with probability zero. The vector ex ante payoff function $\mathbf{\Pi}(\cdot)$ is then continuous in a neighborhood of $\tilde{\sigma}$, from which it follows that $\tilde{\mathbf{\Pi}} = \mathbf{\Pi}(\tilde{\sigma})$. By hypothesis $\tilde{\sigma}$ is not an equilibrium, hence there exists a player i and a strategy σ_i^0 such that

$$\Pi_i(\sigma_i^0, \tilde{\sigma}_{-i}) > \Pi_i(\tilde{\sigma}_i; \sigma_{-i})$$

Let Π_i^* be i's supremum payoff against rival strategies $\tilde{\sigma}_{-i}$, and let $\{\sigma_i^n\}_{n=1}^{\infty}$ be any sequence of strategies for i such that $\Pi_i(\sigma_i^n; \tilde{\sigma}_{-i}) \to \Pi_i^*$. Recall that ties are broken randomly across bidders and auctions. Hence if ties occur with probability bounded away from zero along the sequence $\{\sigma^n\}_{n=1}^{\infty}$, it must be that i is indifferent as to how these are resolved. This in turn implies that i's supremum payoff Π_i^* can also be approached by a strategy sequence $\{\sigma_i^n\}_{n=1}^{\infty}$ such that for each n the random vector $B_i = \sigma_i^n(Y_i)$ has a joint distribution with atomless marginals.[15] Choose any such sequence. Since $\Pi_i^* \geq \Pi_i^0 > \tilde{\Pi}_i$ there will exist an n such that

$$\Pi_i(\sigma_i^n; \tilde{\sigma}_{-i}) > \tilde{\Pi}_i.$$

But by construction σ_i^n generates a joint bid distribution with atomless marginals, hence there is no rival strategy profile under which i ties with positive probability under σ_i^n. Thus $\Pi_i(\sigma_i^n; \cdot)$ is also continuous in σ_{-i} at σ_i^n for each n. Hence for large enough n, we will have

$$\Pi_i(\sigma_i^n; \sigma_{-i}) > \tilde{\Pi}_i$$

for all σ_{-i} in a neighborhood of $\tilde{\sigma}_{-i}$. Thus bidder i can secure a payoff strictly above $\tilde{\Pi}_i$.

Now suppose that under the limiting strategy $\tilde{\sigma}$, a relevant tie occurs with positive probability for at least one auction $l = 1, \ldots, L$. In particular, suppose that such a tie occurs at bid $\bar{b}_l$. By construction, this tie must involve at least one local bidder i. Let $\tilde{Y}_i(\bar{b}_l)$ be the set of types such that i bids $\bar{b}_l$ under the limiting strategy $\tilde{\sigma}$; note that $\tilde{Y}_i(\bar{b}_l)$ has positive measure by construction. Clearly we must have $v_i \geq \bar{b}_l$ almost everywhere in $\tilde{Y}_i(\bar{b}_l)$, since otherwise i could strictly gain by deviating to a bid strictly below $\bar{b}_l$ on a relevant subset of types. Furthermore, we can have $v_i = \bar{b}_l$ on at most a measure-0 subset of $\tilde{Y}_i(\bar{b}_l)$. Hence we must have $v_i > \bar{b}_l$ almost everywhere in $\tilde{Y}_i(\bar{b}_l)$. But in this case either i wins with probability 1 against any other rival type submitting limiting bid $\bar{b}_l$ along the sequence σ^k, or i can secure a payoff strictly above $\tilde{\Pi}_i$ by deviating on $\tilde{Y}_i(\bar{b}_l)$ to a bid infinitesimally above $\bar{b}_l$.

Thus for any sequence of strategies converging to a tie between two local bidders, or for any sequence of strategies converging to a tie between the global bidder and one local bidder such that tying types of the local bidder lose with strictly positive probability along the sequence, at least one player i can secure a payoff strictly greater than $\tilde{\Pi}_i$. But one potential problem case still remains: a sequence of strategies σ^k converging to a relevant tie between the global bidder 0 and a single local bidder i, in which almost every type of bidder 0 tying at $\bar{b}_l$ in the limit loses against almost every type of bidder i tying at $\bar{b}_l$ in the limit with probability approaching one along the sequence. We will show that the global bidder can secure a payoff strictly above $\tilde{\Pi}_0$ in this case.

Toward this end, suppose that σ^k is a sequence satisfying the description above. Let $V(y)$ and

[15] For instance, one could add a small continuously distributed error ϵ to each bid, with the support of ϵ tending to a unit mass at 0 and the sign of ϵ adjusted as needed to ensure i's preferred tie-breaking resolution.

$K(y)$ denote bidder 0's standalone valuation and complementarity vectors at type $y \in \mathcal{Y}_0$ respectively, and let $\bar{Y}_{0l}$ be the set of $y \in \mathcal{Y}_0$ such that $\tilde{\sigma}_{0l} = \bar{b}_l$. For each $l = 1, ..., L$, define $\Gamma_l(y; \sigma^k)$ as the marginal probability that bidder 0 with type $y \in \mathcal{Y}_0$ wins auction $l = 1$ under strategy profile σ^k, and let $\tilde{\Gamma}_l(y) = \lim_{k\to\infty} \Gamma_l(y; \sigma^k)$.[16] Since independence of types across local rivals implies independence of bids across both rivals and auctions, we may then write the limiting interim payoff of bidder 0 at type y as follows:

$$\begin{aligned}
\tilde{\pi}(y) &= \tilde{\Gamma}_l(y)[V_l(y) - \bar{b}_l] + \sum_{\ell \neq l} \tilde{\Gamma}_\ell(y)[V_\ell(y) - \tilde{\sigma}_{0l}(y)] + \sum_{\omega} K^\omega(y) \prod_{\ell=1}^{L} \tilde{\Gamma}_\ell(y)^{\omega_\ell}[1 - \tilde{\Gamma}_\ell(y)]^{1-\omega_\ell} \\
&= \sum_{\ell \neq l} \tilde{\Gamma}_\ell(y)[V_\ell(y) - \tilde{\sigma}_{0l}(y)] + \sum_{\omega:\omega_l=0} K^\omega \prod_{\ell \neq l} \tilde{\Gamma}_\ell(y)^{\omega_\ell}[1 - \tilde{\Gamma}_\ell(y)]^{1-\omega_\ell} \\
&+ \tilde{\Gamma}_l(y) \cdot \left\{ [V_l(y) - \bar{b}_l] + \sum_{\omega:\omega_l=0} [K^{\omega \vee e_l}(y) - K^\omega(y)] \prod_{\ell \neq l} \tilde{\Gamma}_\ell(y)^{\omega_\ell}[1 - \tilde{\Gamma}_\ell(y)]^{1-\omega_\ell} \right\}.
\end{aligned} \tag{14}$$

Now restrict attention to the subset of y in $\bar{\mathcal{Y}}_{0l}$ such that $V_l(y) > 0$; since $V_l(y) = 0$ for at most a set of F_0-measure 0 in $\mathcal{Y}_0$, this subset has the same F_0-measure as $\bar{\mathcal{Y}}_{0l}$. Suppressing y in notation, we may write the final term in braces as follows:

$$(V_l - \bar{b}_l) + \sum_{\omega:\omega_l=0} [K^{\omega \vee e_l} - K^\omega] \prod_{\ell \neq l} \tilde{\Gamma}_\ell^{\omega_\ell}[1 - \tilde{\Gamma}_\ell]^{1-\omega_\ell}. \tag{15}$$

First suppose that the term (15) above is strictly positive for F_0-a.e. $y \in \bar{\mathcal{Y}}_{0l}$. Then bidder 0 can secure a payoff strictly above $\tilde{\Pi}_0$ by deviating at each $y \in \bar{\mathcal{Y}}_{0l}$ to a bid b_l' slightly greater than $\bar{b}_l$.

Now suppose instead that the term (15) is nonpositive on a subset of $\bar{\mathcal{Y}}_{0l}$ with positive F_0-measure. The supermodularity of complementarities implies $K^{\omega \vee e_l} \geq K^\omega$ for all ω, whence

$$V_l + \sum_{\omega:\omega_l=0} [K^{\omega \vee e_l} - K^\omega] \prod_{\ell \neq l} \tilde{\Gamma}_\ell^{\omega_\ell}[1 - \tilde{\Gamma}_\ell]^{1-\omega_\ell} > 0.$$

But when local rivals employ undominated strategies, any $b_l > 0$ implies a strictly positive probability of winning auction l. Hence there exists a $b_l' \in (0, V_l)$ for which both $\tilde{\Gamma}_l' > 0$ and

$$V_l - b_l' + \sum_{\omega:\omega_l=0} [K^{\omega \vee e_l} - K^\omega] \prod_{\ell \neq l} \tilde{\Gamma}_\ell^{\omega_\ell}[1 - \tilde{\Gamma}_\ell]^{1-\omega_\ell} > 0.$$

The first line of (14) does not depend on $\tilde{\Gamma}_l$, so the only effect of deviating to bid b_l' at type y is to replace the nonpositive final term in the limiting payoff (14) with a strictly positive limiting term. By construction, this represents a strict increase in limiting payoffs, which bidder 0 may secure against any sequence of rival strategies converging to $\tilde{\sigma}_{-0}$ by infinitesimally increasing the deviating bid b_l'. Furthermore, by hypothesis, such securing deviations exist on a set of types $y \in \bar{\mathcal{Y}}_0$ of positive F_0-measure. Hence in this case bidder 0 can secure an ex ante payoff strictly greater than $\tilde{\Pi}_0$.

[16] Note that by construction we have $\tilde{\Gamma}_l(y) \neq \Gamma_l(y; \tilde{\sigma})$ for $y \in \bar{\mathcal{Y}}_{0l}$.

Taken together, the cases above establish that for any point $(\tilde{\sigma}, \tilde{\mathbf{\Pi}})$ in the closure of the graph of the vector ex ante payoff function $\mathbf{\Pi}(\cdot)$, if $\tilde{\sigma}$ is not an equilibrium then at least one player $i \in \{0, 1, \ldots, N\}$ can secure a payoff strictly above $\tilde{\Pi}_i$. This is what was to be shown. $\square$

Proof of Proposition 5

The proof of Proposition 5 is by construction.

Consider any sequence $\{\ddot{\mathcal{B}}_1^k, ..., \ddot{\mathcal{B}}_N^k\}_{k=1}^{\infty}$ of *finite* bid lattices for bidders $i = 1, ..., N$ such that $\mathcal{H}(\ddot{\mathcal{B}}_i^k, \mathcal{B}_i) \to 0$ for all i, where $\mathcal{H}$ denotes Hausdorff distance. Letting $\ddot{\mathcal{B}}^k = \times_{i=1}^N \ddot{\mathcal{B}}_i^k$ denote the Cartesian product of action spaces for all bidders, Proposition 3 implies that for each k there exists a monotone pure strategy bidding equilibrium σ^k on $\ddot{\mathcal{B}}^k$, where as usual monotonicity is understood in the sense of the partial order (PO) on types and the coordinatewise order on bids. As in Section 5.1, we may invoke Lemma A.13 in Reny (2011) to conclude that the space of strategies monotone with respect to (PO) is compact, and hence that the sequence $\{\ddot{\sigma}^k\}_{k=1}^{\infty}$ has a subsequence $\{\ddot{\sigma}^{k_j}\}_{j=1}^{\infty}$ which converges pointwise a.e.-F to a monotone pure strategy limit σ^*. With more than one global bidder, however, we may *no longer* leverage better-reply security to conclude that σ^* does not contain ties.

Now augment the limit strategy profile σ^* with a profile of tiebreaking strategies $\tau^* = (\tau_1^*, ..., \tau_N^*)$ constructed as follows. For each strategy profile σ^k along the sequence $\{\sigma^k\}_{k=1}^{\infty}$, let $\Gamma_{il}^k(b_l) \equiv \Pr(\sigma_{il}^k(Y_i) \leq b_l)$ denote the c.d.f. of bidder i's bid in auction l, $\bar{\Gamma}_l^k(b_l) \equiv \prod_{i=1}^N \Gamma_{il}^k(b_l)$ denote the c.d.f. of the maximum bid among all bidders in auction l, and $\bar{\gamma}_l^k(b_l)$ denote the (discrete) p.d.f. associated with the c.d.f $\bar{\Gamma}$. For each bidder i and auction l, let $\tau_{il}^k(y_i) \equiv \bar{\Gamma}_l^k(\sigma_{il}^k(y_i)) - \frac{1}{2}\bar{\gamma}_l^k(\sigma_{il}^k(y_i))$ denote the c.d.f. of the maximum bid among all bidders in auction l, evaluated at the point $b_l = \sigma_{il}^k(y_i)$, less 1/2 the p.d.f. of the maximum bid among all bidders in auction l, also evaluated at the point $b_l = \sigma_{il}^k(y_i)$. Note that $\bar{\Gamma}_l^k(b_l) - \frac{1}{2}\bar{\gamma}_l^k(b_l)$ is monotone in b_l (since each step of the c.d.f is equal to the p.d.f. $\bar{\gamma}_l^k(b_l)$). Furthermore, for any two sequences of bids $\{b_l^k\}_{k=1}^k$, $\{\tilde{b}_l^k\}_{k=1}^k$ such that both $b_l^k \in \ddot{\mathcal{B}}_l^k$ and $\tilde{b}_l^k \in \ddot{\mathcal{B}}_l^k$, if $\bar{\Gamma}_l^k(b_l^k) - \frac{1}{2}\bar{\gamma}_l^k(b_l) \to \bar{\Gamma}_l^k(\tilde{b}_l^k) - \frac{1}{2}\bar{\gamma}_l^k(\tilde{b}_l^k)$ then we must have either (i) both $\bar{\gamma}_l^k(b_l^k) \to 0$ and $\bar{\gamma}_l^k(\tilde{b}_l^k) \to 0$, or (ii) eventually $b_l^k = \tilde{b}_l^k$. If $b_l^k \neq \tilde{b}_l^k$, then $\bar{\Gamma}_l^k(b_l^k) - \frac{1}{2}\bar{\gamma}_l^k(b_l)$, $\bar{\Gamma}_l^k(\tilde{b}_l^k) - \frac{1}{2}\bar{\gamma}_l^k(\tilde{b}_l^k)$ must differ by at least $\frac{1}{2}\max\{\bar{\gamma}_l^k(b_l^k), \bar{\gamma}_l^k(\tilde{b}_l^k)\}$.

Finally, for each bidder i, define an $L \times 1$ vector $\tau_i^k(y_i) = (\tau_{i1}^k(y_i), ..., \tau_{iL}^k(y_i))$ stacking up the functions τ_{il}^k just defined. Observe that, by construction, $\tau_{1l}^k, ..., \tau_{Nl}^k$ *preserve order* of bids across bidders: i.e. for any bidders i, j, we have $\tau_{il}^k(y_i) \gtreqless \tau_{jl}^k(y_j)$ as $\sigma_{il}^k(y_i) \gtreqless \sigma_{jl}^k(y_j)$. Furthermore, the vector $\tau_i^k(y_i)$ inherits monotonicity of σ_i^k: $y_i' \succeq y_i$ in the sense of (PO) implies $\tau_i^k(y_i') \geq \tau_i^k(y_i)$ in the usual coordinatewise sense. Hence defining $\tau^k = (\tau_1^k, ..., \tau_1^k)$ and focusing on the subsequence such that $\sigma^k \to \sigma^*$, it follows that there exists a further subsequence $\{k_j\}_{j=1}^{\infty}$ such that τ^k also converges pointwise a.e.-F to a monotone limit τ^*.

What do we gain from this construction? The *cardinal* limit σ^* cannot preserve ordinal information on bids along the sequence – a tie under σ^* implies only that a positive measure of types have the same limit bid. But the *ordinal* limit τ^* does: if $\tau_{il}^*(y_i) < \tau_{jl}^*(y_j)$, then we must eventually have $\tau_{il}^k(y_i) < \tau_{jl}^k(y_)$ and hence $\sigma_{il}^k(y_i) < \sigma_{jl}^k(y_j)$ almost surely *even if* also $\sigma_{il}^k(y_i) \to \sigma_{jl}^k(y_j)$ (and hence types y_i, y_j tie under σ^*). Consequently, the set of ties arising with respect to τ^* (as opposed to those arising with respect to σ^*) will correspond, up to a set of measure zero, with the set of types eventually tying along the sequence σ^k. This latter fact is central to our construction, so we state and prove it separately as a lemma.

Lemma 5. *Consider any auction $l = 1, ..., L$ and any $(\bar{b}_l, \bar{t}_l)$ such that there exists some bidder $i = 1, ..., N$ and type y_i such that $\sigma^k_{il}(y_i) \to \bar{b}_l$ and $\bar{\Gamma}^k_l(\sigma^k_{il}(y_i)) \to \bar{t}_l$ as $k \to 0$. Then for any bidder $j = 1, ..., N$ and for a.e.-F_j type y_j of bidder j, the following statements hold:*

- *If $\tau^*_{jl}(y_j) < \bar{t}_l$, then eventually $\sigma^k_{jl}(y_j) < \sigma^k_{il}(y_i)$ as $k \to \infty$;*
- *If $\tau^*_{jl}(y_j) > \bar{t}_l$, then eventually $\sigma^k_{jl}(y_j) > \sigma^k_{il}(y_i)$ as $k \to \infty$.*
- *If $\tau^*_{jl}(y_j) = \bar{t}_l$, then either (i) eventually $\sigma^k_{jl}(y_j) = \sigma^k_{il}(y_i)$, or (ii) the $\bar{\Gamma}^k$-measure of bids in the closed interval $[\min\{\sigma^k_{il}(y_i), \sigma^k_{jl}(y_j)\}, \max\{\sigma^k_{il}(y_i), \sigma^k_{jl}(y_j)\}]$ eventually approaches zero.*

Proof of Lemma 5. First suppose that $\tau^*_{jl}(y_j) < \bar{t}_l$. Consider the subset of types y_j such that both σ^k and $\bar{\Gamma}_j(\sigma^k_{jl}(y_j))$ are pointwise convergent; recall that this set has full measure with respect to F_j. If $\tau^*_{jl}(y_j) < \bar{t}_l$, then by definition $\lim \bar{\Gamma}^k_l(\sigma^k_{il}(y_i)) < \bar{t}_l = \lim \bar{\Gamma}^k_l(\sigma^k_{il}(y_i))$. Hence we must eventually have $\bar{\Gamma}^k_l(\sigma^k_{il}(y_i)) < \bar{\Gamma}^k_l(\sigma^k_{il}(y_i))$. But bearing in mind that $\bar{\Gamma}^k_l$ is the c.d.f. of the maximum bid in auction l, this can only hold if there exists a positive measure of types submitting bids strictly greater than $\sigma^k_{jl}(y_j)$ and weakly less than $\sigma^k_{il}(y_i)$. Hence in particular we must have $\sigma^k_{jl}(y_j) < \sigma^k_{il}(y_i)$. Next suppose that that $\tau^*_{jl}(y_j) > \bar{t}_l$. In this case, reversing the arguments in the last paragraph show that we must eventually have $\sigma^k_{jl}(y_j) > \sigma^k_{il}(y_i)$. Finally, suppose that $\tau^*_{jl}(y_j) = \bar{t}_l$. Further suppose that the $\bar{\Gamma}^k$-measure of bids in the closed interval $[\min\{\sigma^k_{il}(y_i), \sigma^k_{jl}(y_j)\}, \max\{\sigma^k_{il}(y_i), \sigma^k_{jl}(y_j)\}]$ does *not* approach zero. Recall that by hypothesis $\lim_{k\to\infty} \bar{\Gamma}^k_l(\sigma^k_{il}(y_i)) \to \bar{t}_l = \lim_{k\to\infty} \bar{\Gamma}^k_l(\sigma^k_{il}(y_i))$. Given the construction of $\bar{\Gamma}^k_l(\cdot)$, this implies both that the mass of bids strictly less than $\sigma^k_{il}(y_i))$ must eventually approach the mass of bids strictly less than $\sigma^k_{jl}(y_j))$, and that the mass of bids strictly greater than $\sigma^k_{il}(y_i))$ must eventually approach the mass of bids strictly greater than $\sigma^k_{jl}(y_j))$. But also (by hypothesis) there is always a positive mass of bids weakly between $\sigma^k_{il}(y_i)$ and $\sigma^k_{jl}(y_j)$. These statements can hold simultaneously only if $\sigma^k_{il}(y_i) = \sigma^k_{jl}(y_j)$. □

Lemma 5 immediately implies that bidder i with type y_i cannot strictly gain from submitting any $(\bar{b}_l, \bar{t}_l)$ which is on the equilibrium path in the sense that there exists some bidder $j = 1, ..., N$ and type y_j with $\sigma^k_{il}(y_i) \to \bar{b}_l$ and $\bar{\Gamma}^k_l(\sigma^k_{il}(y_i)) \to \bar{t}_l$. For by Lemma 1, the outcomes which would obtain by submitting such a $(\bar{b}_l, \bar{t}_l)$ can differ by at most a set of measure zero from the outcomes which would have eventually obtained had bidder i imitated type y_j of bidder j along the sequence of equilibria σ^k. Hence if deviating to $(\bar{b}_l, \bar{t}_l)$ is strictly profitable under $\sigma^* \times \tau^*$, i must also have had a strictly profitable deviation at some point along the sequence σ^k. This contradicts the hypothesis that σ^k is a sequence of equilibria.

So i cannot have a profitable deviation on the equilibrium path. To ensure that the strategies $\sigma^* \times \tau^*$ are an equilibrium, it is therefore sufficient to construct a collection of tiebreaking precendence rules $\rho^*_1, ..., \rho^*_L$ such that the set of deviations available to i in the limit are effectively equivalent to those on the equilibrium path.

Toward this end, consider the tiebreaking rule $\rho^*_l(t)$ equal to the c.d.f. of the random variable $T = \max\{\tau^*_{1l}(Y_1), \tau^*_{Nl}(Y_N)\}$ evaluated at t. Then ρ^*_l is a weakly monotone function from $[0, 1]$ to $[0, 1]$. Furthermore, if the auctioneer evaluates ties according to the tiebreaking rule $\rho^*_l(t)$, then by construction submitting any tiebreaking signal $t^* \in [0, 1]$ is effectively equivalent (up to an irrelevant set of measure zero) to submitting the next lowest tiebreaking signal on the equilibrium path. Hence under tiebreaking rule ρ_l^*, bidder i's set of feasible deviations in auction l is effectively equivalent to her set of feasible deviations on the equilibrium path. In view of the arguments above, it follows that $\sigma^* \times \tau^*$ is an equilibrium under the $L \times 1$ vector of tiebreaking rules $\rho^* = (\rho_1^*, ..., \rho_L^*)$.

Proof of Proposition 6

Conditions (7) and (8) imply that $\mathcal{Y} \subset y_0 - \mathcal{Z}_{2^L-1}$, that is, y_0 dominates any other point in $\mathcal{Y}$ in the partial order (PO) sense. Clearly, $\min_{i=1,...,N} \|Y_i - y_0\| \xrightarrow{p} 0$ as $N \to \infty$. Denote $r_N = \min_{i=1,...,N} \|Y_i - y_0\|$. Suppose that N is large enough, so r_N can be considered small enough with high probability. Let $y_{2^L-1}(N)$ denote the valuation vector among $\{Y_i\}_{i=1}^N$ closest to y_0 in the Euclidean norm.

Thus, $y_{2^L-1}(N)$ lies on the boundary of the (2^L-1)-dimensional ball $B_{r_N}(y_0)$. By definition, there are no bidders with valuations that dominate $y_{2^L-1}(N)$ in the partial order (PO) sense. All the bidders with valuations in $(y_{2^L-1}(N) - \mathcal{Z}_{2^L-1}) \cap \mathcal{Y}$ that are different from y_{2^L-1} will not win any objects as the bidder with the valuation vector $y_{2^L-1}(N)$ will submit bids not smaller than theirs and will beat them in the tie-breaking rule.

Thus, the only bidders who can win objects are the bidder with the valuation at $y_{2^L-1}(N)$ and bidders with valuations in the set

$$\mathcal{Y} \backslash \left(B_{r_N}(y_0) \cup (y_{2^L-1}(N) - \mathcal{Z}_{2^L-1})\right) = \left(\mathcal{Y} \backslash (y_{2^L-1}(N) - \mathcal{Z}_{2^L-1})\right) \backslash B_{r_N}(y_0). \tag{16}$$

(this set is analogous to the union of regions A and B in Example 4). Thus, when considering the loss of efficiency we only have to consider the extent of misallocation within this set.

Step 1. We start by finding a bound on the largest distance between y_0 and a point in the set $\mathcal{Y} \backslash \left(B_{r_N}(y_0) \cup (y_{2^L-1}(N) - \mathcal{Z}_{2^L-1})\right)$. Since $\mathcal{Y} \subset y_0 - \widetilde{\mathcal{Z}}_{2^L-1}$, this distance will not exceed the largest distance between y_0 and a point in the set

$$\left((y_0 - \widetilde{\mathcal{Z}}_{2^L-1}) \backslash (y_{2^L-1}(N) - \mathcal{Z}_{2^L-1})\right) \backslash B_{r_N}(y_0). \tag{17}$$

The set $y_0 - \widetilde{\mathcal{Z}}_{2^L-1}$ is a cone with the vertex at y_0, and the set $y_{2^L-1}(N) - \mathcal{Z}_{2^L-1}$is a cone with the vertex at $y_{2^L-1}(N)$.

In order to find the largest distance between y_0 and a point in set (17), take an extreme ray (let us call it R_1) of the cone $y_{2^L-1}(N) - \mathcal{Z}_{2^L-1}$ (that is, a ray that cannot be be represented as a convex combination of other rays in that cone) and find its intersection with the boundary of $y_0 - \widetilde{\mathcal{Z}}_{2^L-1}$ (they will intersect because of condition (7)). The intersection point (let us call it e.g. a) is the intersection of ray R_1 and some extreme ray (let us call it R_2) of the cone $y_0 - \widetilde{\mathcal{Z}}_{2^L-1}$. In fact, both R_1 and R_2 can be written as

$$R_1 = \left\{y_{2^L-1}(N) + \lambda_1(a - y_{2^L-1}(N)) : \lambda_1 \geq 0\right\}, \quad R_2 = \{y_0 + \lambda_2(a - y_0) : \lambda_2 \geq 0\}.$$

Let $\alpha(R_1, R_2, a)$ denote the angle between rays R_1 and R_2 at point a. Clearly, this angle coincides with the angle $\alpha(y_0 - y_{2^L-1}(N) + R_1, R_2, y_0)$ between rays $y_0 - y_{2^L-1}(N) + R_1$ (this is an extreme ray of $y_0 - \mathcal{Z}_{2^L-1}$, it originates at y_0 and is parallel to R_1) and R_2 at y_0. Condition (7) allows us to conclude that

$$\underline{\alpha} = \min_{\substack{R_1 \text{ is extreme ray of } y_{2^L-1}(N)-\mathcal{Z}_{2^L-1} \\ R_2 \text{ is extreme ray of } y_0-\widetilde{\mathcal{Z}}_{2^L-1}}} \alpha(R_1, R_2, a) = \min_{\substack{R_1^\dagger \text{ is extreme ray of } y_0-\mathcal{Z}_{2^L-1} \\ R_2 \text{ is extreme ray of } y_0-\widetilde{\mathcal{Z}}_{2^L-1}}} \alpha(R_1^\dagger, R_2, y_0) > 0.$$

Let R_3 denote the ray originating from y_0 and passing through $y_{2^L-1}(N)$:

$$R_3 = \left\{ y_0 + \lambda_3(y_{2^L-1}(N) - y_0) : \lambda_3 \geq 0 \right\}.$$

Clearly, ray R_3 lies in the cone $y_0 - \widetilde{\mathcal{Z}}_{2^L-1}$.

Since $\mathcal{Z}_{2^L-1}$ lies in the positive orthant $(\mathbb{R}^+)^{2^L-1}$, then the angle between any two extreme rays of $y_0 - \mathcal{Z}_{2^L-1}$ cannot be greater than $\frac{\pi}{2}$. As a consequence, the angle between any two extreme rays of $y_0 - \widetilde{\mathcal{Z}}_{2^L-1}$ cannot be greater than $\frac{\pi}{2} - 2\underline{\alpha}$. Because of condition (7)), we can also conclude that

$$\overline{\alpha} = \max_{\substack{R_1 \text{ is extreme ray of } y_{2^L-1}(N)-\mathcal{Z}_{2^L-1} \\ R_2 \text{ is extreme ray of } y_0-\widetilde{\mathcal{Z}}_{2^L-1}}} \alpha(R_1, R_2, a) = \max_{\substack{R_1^\dagger \text{ is extreme ray of } y_0-\mathcal{Z}_{2^L-1} \\ R_2 \text{ is extreme ray of } y_0-\widetilde{\mathcal{Z}}_{2^L-1}}} \alpha(R_1^\dagger, R_2, y_0) < \frac{\pi}{2}.$$

Now taking all this into account, we can conclude that the sum of the angle $\alpha(R_1, R_2, a)$ and the angle $\alpha(R_2, R_3, y_0)$ (between rays R_2 and R_3 at y_0) is not greater than

$$\frac{\pi}{2} - 2\underline{\alpha} + \overline{\alpha} < \pi.$$

Therefore, the points y_0, $y_{2^L-1}(N)$ and a form a usual two-dimensional triangle in the $(2^L - 1)$-dimensional space.

By the sine law of triangle,

$$\frac{r_N}{\sin \alpha(R_1, R_2, a)} = \frac{dist(a, y_0)}{\sin(\pi - \alpha(R_1, R_2, a) - \alpha(R_2, R_3, y_0))}.$$

Taking it from here, obtain that

$$dist(a, y_0) = \frac{r_N}{\sin \alpha(R_1, R_2, a)} \cdot \sin(\alpha(R_1, R_2, a) + \alpha(R_2, R_3, y_0)) \leq \frac{r_N}{\sin \underline{\alpha}},$$

and $\sin \underline{\alpha} > 0$ due to $0 < \underline{\alpha} < \frac{\pi}{4}$.

Step 2.

Let us analyze different occurrences of inefficiencies in a symmetric monotone equilibrium.

Bidders with valuations in set $y_{2^L-1}(N) - \mathcal{Z}_{2^L-1}$ will not win anything because the bidder with the valuation vector $y_{2^L-1}(N)$ will be submitting higher bids for all the objects. Thus, the only bidders winning objects are in the set (16).

Let $\mathcal{L}_N$ denote the efficiency loss in a symmetric monotone equilibrium. It is a random variable as it depends on realizations of $Y_1, \ldots, Y_N$. Let $\mathcal{P}_k$, $k = 1, \ldots, L$, denote the set of all possible partitions of the set of N into k subsets. Any partition in $\mathcal{P}_k$ can be described by k 01 L-dimensional vectors

$\eta_{k,1}, \ldots, \eta_{k,k}$ that capture a membership of an object by assigning 1. Then

$$\mathcal{L}_N \leq \max_{k=1,\ldots,L} \max_{\eta_{k,1},\ldots,\eta_{k,k} \in \mathcal{P}_k} \sum_{q=1}^{k} \max_{y_i, y_j \in \mathcal{Y} \setminus \left(B_{r_N}(y_0) \cup (y_{2^L-1}(N) - \mathcal{Z}_{2^L-1})\right)} |y_i^{\eta_{k,q}} - y_j^{\eta_{k,q}}|.$$

We can then bound it as follows:

$$\begin{aligned}
\mathcal{L}_N &\leq L \cdot \max_{y_i, y_j \in \mathcal{Y} \setminus \left(B_{r_N}(y_0) \cup (y_{2^L-1}(N) - \mathcal{Z}_{2^L-1})\right)} \|y_i - y_j\| \\
&\leq L \cdot \max_{y_i, y_j \in \left((y_0 - \widetilde{\mathcal{Z}}_{2^L-1}) \setminus (y_{2^L-1}(N) - \mathcal{Z}_{2^L-1})\right) \setminus B_{r_N}(y_0)} \|y_i - y_j\| \\
&\leq 2L \cdot \max_{y_i \in \left((y_0 - \widetilde{\mathcal{Z}}_{2^L-1}) \setminus (y_{2^L-1}(N) - \mathcal{Z}_{2^L-1})\right) \setminus B_{r_N}(y_0)} \|y_i - y_0\| \\
&\leq \frac{2L r_N}{\sin \underline{\alpha}}.
\end{aligned}$$

The final step is to look at the expectation

$$\begin{aligned}
E\left[\mathcal{L}_N\right] &= E\left[\mathcal{L}_N \mid \min_{i=1,\ldots,N} \|Y_i - y_0\| \leq r_N^*\right] \cdot P\left(\min_{i=1,\ldots,N} \|Y_i - y_0\| \leq r_N^*\right) \\
&\quad + E\left[\mathcal{L}_N \mid \min_{i=1,\ldots,N} \|Y_i - y_0\| > r_N^*\right] \cdot \left(1 - P\left(\min_{i=1,\ldots,N} \|Y_i - y_0\| \leq r_N^*\right)\right) \\
&\quad \leq \frac{2L r_N^*}{\sin \underline{\alpha}} \cdot P\left(\min_{i=1,\ldots,N} \|Y_i - y_0\| \leq r_N^*\right) + M \cdot \left(1 - P\left(\min_{i=1,\ldots,N} \|Y_i - y_0\| \leq r_N^*\right)\right),
\end{aligned}$$

where $M < \infty$ is some constant (the losses of efficiency are always bounded as the support $\mathcal{Y}$ of bidders' values is bounded in $\mathbb{R}^{2^L-1}$).

Now, due to the condition $y_0 \in \mathcal{Y}$, we can always choose the rate of $r_N^* \to 0$ such that

$$P\left(\min_{i=1,\ldots,N} \|Y_i - y_0\| \leq r_N^*\right) \to 1 \quad \text{as } N \to \infty$$

(that is, $r_N^* \to 0$ slowly enough). Then we can see that

$$E\left[\mathcal{L}_N\right] \to 0 \quad \text{as } N \to \infty.$$

□

Proof of Proposition 7

Let $\mathcal{Y}^{2^L-1}$ denote the projection of $\mathcal{Y}^\infty$ onto the space $\mathbb{R}^{2^L-1}$, and let $y_{i,2^L-1} \in \mathcal{Y}^{2^L-1}$ be the projection of type $y_i \mathbb{R}^\infty$ onto the space $\mathbb{R}^{2^L-1}$. Then $y_{0,2^L-1}$ dominates any other type in $\mathcal{Y}^{2^L-1}$ in the partial order strictly more restrictive than the partial order (PO). Let $\|\cdot\|_{2^L-1}$ denote the Euclidean norm in $\mathbb{R}^{2^L-1}$.

Then, analogously to the proof of Proposition 6, we can show that

$$E\left[\mathcal{L}_{N,L}\right] \leq \frac{2Lr_N^*}{\sin \underline{\alpha}_{2^L-1}} \cdot P\left(\min_{i=1,\ldots,N} \left\|Y_{i,2^L-1} - y_{0,2^L-1}\right\|_{2^L-1} \leq r_N^*\right)$$
$$+ ML \cdot \left(1 - P\left(\min_{i=1,\ldots,N} \left\|Y_{i,2^L-1} - y_{0,2^L-1}\right\|_{2^L-1} \leq r_N^*\right)\right),$$

where $\underline{\alpha}_{2^L-1}$ is the minimum angle between extreme rays of cone $\mathcal{K}_{2^L-1}$ and the the extreme rays of the cone that describes the partial order strictly more restrictive than (PO) and such that $y_{0,2^L-1}$ dominates any other point in $\mathcal{Y}^{2^L-1}$ in this strictly more restrictive order. The condition (10) permits us to take $\lim_{L\to\infty} \underline{\alpha}_{2^L-1} = \underline{\alpha} > 0$. Constant ML bounds each component in $\mathcal{Y}^{2^L-1}$ by condition (ii). Rewrite the expression above as

$$E\left[\mathcal{L}_{N,L}\right] \leq L\Bigg(\frac{2r_N^*}{\sin \underline{\alpha}} \cdot P\left(\min_{i=1,\ldots,N} \left\|Y_{i,2^L-1} - y_{0,2^L-1}\right\|_{2^L-1} \leq r_N^*\right)$$
$$+ M \cdot \left(1 - P\left(\min_{i=1,\ldots,N} \left\|Y_{i,2^L-1} - y_{0,2^L-1}\right\|_{2^L-1} \leq r_N^*\right)\right)\Bigg).$$

Note that

$$\|Y_i - y_0\| \leq r_N^* \quad \Rightarrow \quad \left\|Y_{i,2^L-1} - y_{0,2^L-1}\right\|_{2^L-1} \leq r_N^*,$$

where $\|\cdot\|$ stands for the l^2 norm in the space of infinite sequence. Therefore, choosing $r_N^* \to 0$ slow enough such that

$$P\left(\min_{i=1,\ldots,N} \|Y_i - y_0\| \leq r_N^*\right) \to 1 \quad \text{as } N \to \infty$$

and choosing the rate with which L goes to infinity slow enough to guarantee that $Lr_N^* \to 0$ will give us that $E\left[\mathcal{L}_{N,L}\right] \to 0$.