Preferences and Performance in Simultaneous First-Price Auctions: A Structural Analysis*

Matthew Gentry[†] Tatiana Komarova[‡] Pasquale Schiraldi[§]

February 2020

Abstract

Motivated by the empirical prevalence of simultaneous bidding across a wide range of auction markets, we develop and estimate a structural model of strategic interaction in simultaneous first-price auctions when objects are heterogeneous and bidders have preferences over combinations. We establish non-parametric identification of primitives in this model under standard exclusion restrictions, providing a basis for both estimation and testing of preferences over combinations. We then apply our model to data on Michigan Department of Transportation (MDOT) highway procurement auctions, quantifying the magnitude of cost synergies and evaluating the performance of the simultaneous first-price mechanism in the MDOT marketplace.

*We thank editor Aureo de Paula and three anonymous referees for many helpful comments and suggestions which have greatly improved the paper. We are also grateful to Philip Haile, Ali Hortacsu, Ken Hendricks, Paul Klemperer, Paulo Somaini and Balazs Szentes for their comments, suggestions and insights, as well as many seminar and conference participants for helpful discussion. We thank Joachim Groeger and Paulo Somaini for graciously sharing their data. Financial support from ESRC Grant #ES/N000056/1 is gratefully acknowledged. An earlier draft of this paper was circulated under the title "Simultaneous First-Price Auctions with Preferences over Combinations: Identification, Estimation and Application."

[†]Florida State University, mgentry@fsu.edu

[‡]London School of Economics, t.komarova@lse.ac.uk

[§]London School of Economics and CEPR, p.schiraldi@lse.ac.uk

1 Introduction

Simultaneous bidding in multiple first-price auctions is a commonly occurring but rarely discussed phenomenon in many real-world auction markets.[1] In environments where values over combinations are non-additive in the set of objects won, bidders must account for the possibility of winning multiple auctions at the time of bidding. This in turn substantially alters the strategic bidding problem compared to the standard first price auction, with ambiguous welfare implications depending on the importance of synergies (either positive or negative) among objects. We develop a structural model of bidding in simultaneous first-price auctions and study identification and estimation in this framework. We then apply our methodology to estimate cost synergies arising in Michigan Department of Transportation (MDOT) highway procurement auctions, using the resulting estimates to analyze revenue and efficiency performance of the simultaneous first-price mechanism in this application.[2]

To illustrate the policy questions arising in simultaneous multi-object auctions, note that given a set of L heterogeneous objects for sale, bidders i's preference structure could in principle be as complex as a complete 2^L-dimensional set of signals describing the valuations i assigns to each of the 2^L possible subsets of objects. Meanwhile, the simultaneous first-price mechanism allows bidders to submit (at most) L individual bids on the L objects being sold. Consequently, the simultaneous first-price auction format is necessarily inefficient—the "message space" (standalone bids) is insufficiently rich to allow bidders to express their true preferences. The auctioneer could alleviate this "message space" problem by, for instance, allowing combination bids, but these still need not produce efficient allocations, and may

[1]To underscore the prevalence of simultaneous bidding in applications, note that many widely studied first-price marketplaces in fact exhibit simultaneous bids. Concrete examples include markets for highway procurement in many US states (Krasnokutskaya (2011), Somaini (2015), Li and Zheng (2009), Groeger (2014), many others), snow-clearing in Montreal (Flambard and Perrigne (2006)), recycling services in Japan (Kawai (2011)), cleaning services in Sweden (Lunander and Lundberg (2013)), oil and drilling rights in the US Outer Continental Shelf (Hendricks et al. (2003)), and to a lesser extent US Forest Service timber harvesting (Lu and Perrigne (2008), Li and Zhang (2010), Athey et al. (2011), many others).

[2]This paper focuses on complementarities arising when auctions are run simultaneously. This complements the literature on potential linkages in valuations over time, e.g. Balat (2015), De Silva (2005), De Silva et al. (2005), Groeger (2014) and Jofre-Bonet and Pesendorfer (2003) among others.

involve substantial practical costs; see, e.g., Cramton et al. (2006) for a review. Hence in evaluating the simultaneous first-price format, it is first necessary to assess potential welfare and revenue effects of simultaneous bidding, about which little is presently known.

We develop a structural empirical model of bidding in simultaneous first-price auctions when objects are heterogeneous and bidders have non-additive preferences over combinations, to our knowledge the first in the literature. We represent the total value bidder i assigns to each combination as the sum of two components: the sum of i's *standalone valuations* for each object in the combination individually, plus a combination-specific *complementarity* (either positive or negative) capturing the incremental change in value i associates with winning the combination as a whole. We interpret standalone valuations as private information drawn independently across bidders conditional on observables, and complementarities as deterministic functions of observables.[3] We find this framework natural in a variety of procurement contexts—when, for instance, non-additivity in preferences can be represented as the expectation over a cost shock realized following a multiple win. Furthermore, and crucially, our framework collapses immediately to the standard separable model when complementarities are zero, supporting formal testing of this hypothesis.

Building on this framework, we make four main contributions to the literature on structural analysis of auction markets. First, we establish a new set of identification results applicable even when complementarities are non-zero. We start by showing that optimal behavior in this environment yields an inverse bidding system non-parametrically identified up to the unknown function describing complementarities, which in turn collapses to the standard inverse bidding function of Guerre et al. (2000) when complementarities are zero. Under natural exclusion restrictions—namely, that marginal distributions of standalone valuations are invariant either to characteristics of rival bidders or characteristics of other objects—we then translate this inverse bidding system into a system of linear equations in unknown bid-

[3]Note that this structure does not restrict dependence between i's standalone valuations for different objects in the market. We view this flexibility as critical, as in practice we expect i's standalone valuations to be positively correlated among each other even if independent from other bidders valuations.

der complementarities, with excludable variation in competition and other characteristics yielding non-parametric identification of these.

Second, we develop a two-step procedure by which to estimate primitives in our structural model. First, in Step 1, we estimate the multivariate joint distribution of bids as a function of bidder- and auction-level characteristics. In Step 2, we use pairwise differencing and GMM estimation to estimate complementarities using the inverse bidding system. Once complementarities are estimated, it is straightforward to use the inverse bidding system to estimate standalone project completion costs for each bidder.

Third, we apply our structural framework to analyze simultaneous bidding in Michigan Department of Transportation (MDOT) highway procurement markets. We view this market as prototypical of our target application: large numbers of projects are auctioned simultaneously (an average of 45 per letting round in our 2005-2015 sample period), more than half of bidders bid on at least two projects simultaneously (with an average of 2.7 bids per round across all bidders in the sample), and combination and contingent bidding are explicitly forbidden. Within this marketplace, we show that factors such as size of other projects, number of bidders in other auctions, and the relative distance between projects have substantial reduced-form impacts on i's bid in auction l, a finding hard to rationalize in standard separable models. We then apply the estimation algorithm described above to recover structural estimates of primitives. We find substantial complementarities in this application: comparing the 10th to the 90th percentile of estimated complementarities, we find that a combination win may generate anything from approximately 23.8 percent cost savings to approximately 11 percent cost increases depending on bidder and project characteristics, with large, heterogeneous, and overlapping projects more likely to be substitutes.

Finally, building on our structural estimates, we measure potential inefficiencies associated with the simultaneous first price auction design. Toward this end, we compare the simultaneous first-price auction used in the MDOT marketplace with a simple efficient combinatorial benchmark: the Vickrey-Clarke-Groves (VCG) mechanism. As expected, the VCG mechanism yields lower social costs: our estimates suggest total social gains of approximately 6 percent, with relatively larger gains in lettings with larger complementarities. Interestingly, however, the counterfactual VCG auction also slightly increases MDOT's procurement costs by about 1 percent. In other words, even in the presence of substantial complementarities, the simultaneous first-price auction format appears to perform very well.[4] This is, to our knowledge, the first formal comparison of the simultaneous first-price and VCG mechanisms, and may help to explain the popularity of the simultaneous first-price format in practice.[5]

While this is to our knowledge the first structural analysis of bidding in simultaneous first-price auctions, our work builds on a small but growing structure literature analyzing combinatorial auctions.[6] Cantillon and Pesendorfer (2006) analyze combinatorial first-price sealed-bid auctions for London bus routes, using the possibility of package bidding to identify bidder preferences over combinations. More recently, Kim et al. (2014) employ a related methodology to analyze the large-scale combinatorial auctions used in procurement of Chilean school meals. A key source of identification in these combinatorial settings is observation of package bids, which are directly informative regarding relative preferences for specific combinations. Since, by construction, we observe only standalone bids, identification in our simultaneous first-price setting is a substantially different (and more challenging) problem, for which we develop a novel solution.

We are also aware of two recent studies structurally analyzing synergies across auctions when package bids are not available. In the context of FCC simultaneous ascending auctions,

[4]We also explored other leading combinatorial mechanisms, such as the descending proxy auction of Ausubel and Milgrom (2002). By construction, these lead to the same efficient allocation as the VCG auction, and in preliminary tests they also led to very similar expected revenue. For this reason, we chose to focus for simplicity on the VCG auction.

[5]More generally, this analysis contributes to the growing literature that aims to understand the performance of different auction formats in procurement auctions. Other studies in this area includes among others: Athey et al. (2011) who compare open versus sealed bid auctions for timber harvesting contracts, Lewis and Bajari (2011) who compare first price versus scoring rules with time incentives for the procurement of roadwork contracts, and Decarolis (2017) who studies first price auctions with ex-post screening and average-bid auctions in the Italian procurement.

[6]Although only tangentially related to our problem, there is also a growing empirical literature on *multi-unit auctions*, which focus on markets for homogeneous, divisible goods like electricity and treasury bills. See e.g. Fevrier et al. (2004); Chapman et al. (2007); Kastl (2011); Hortacsu and Puller (2008); Hortacsu and McAdams (2010) and Hortacsu (2011); Wolak (2007); and Reguant (2014).

Bajari and Fox (2013) estimate the deterministic component of bidder valuations under the assumption that the allocation of licenses is pairwise stable in matches, a condition which need not hold in the simultaneous first-price setting we consider here. Meanwhile, Kong (2018) studies identification and estimation in sequential auctions allowing for synergies and affiliation in each bidder's private valuations across auctions, complementing our analysis allowing for both features in simultaneous auctions.[7] Although the economic considerations motivating our analysis closely parallel those of Kong (2018), simultaneous and sequential bidding introduce fundamentally different empirical challenges which render our respective methodological contributions entirely distinct.

Paralleling these structural studies, there is also a small reduced-form literature seeking to quantify the role of preferences over combinations in multi-object auctions. Ausubel et al. (1997) and Moreton and Spiller (1998) measure synergy effects in FCC spectrum auctions. Lunander and Lundberg (2013) empirically compare combinatorial and simultaneous first-price auctions in a Swedish market for internal cleaning services, finding that bidders inflate their standalone bids in combinatorial auctions relative to first-price auctions but that this does not significantly affect the procurer's final costs. De Silva (2005) and De Silva et al. (2005) analyze spatial synergies in Oklahoma Department of Transportation highway procurement auctions, finding that previous winners participate more often and bid more aggressively in subsequent nearby projects. These findings are consistent with the hypothesis of spatial synergies in procurement, motivating the structural model we consider here.

Finally, from a more theoretical perspective, there have been several studies analyzing strategic interaction in stylized models involving simultaneous first-price auctions; see, for example, Szentes and Rosenthal (2003) and Ghosh (2012) among others. Gentry et al. (2019) study existence and proprieties of equilibrium in a setting closely paralleling that studied here. There is also a substantial literature analyzing properties of various combinatorial

[7] We emphasize that both we and Kong (2018) allow for affiliation in valuations across objects for each bidder, but not affiliation in valuations across bidders. Formally, therefore, both studies fall within the independent private values paradigm.

auction mechanisms: Ausubel and Milgrom (2002), Ausubel and Cramton (2004), Cramton (1997, 2006), Krishna and Rosenthal (1996), Klemperer (2008, 2010), Milgrom (2000b, 2000a), and Rosenthal and Wang (1996), to mention just a few. Detailed surveys of this literature are given in De Vreis and Vorha (2003) and Cramton et al. (2006).[8]

The rest of this paper is organized as follows. Section 2 outlines the model of simultaneous first-price auctions on which our structural analysis is built. Section 3 studies identification in this model. Section 4 describes the Michigan Department of Transportation (MDOT) highway procurement marketplace, and Section 5 presents our structural estimation strategy and the estimation results. Section 6 compares MDOT's simultaneous first-price format with a combinatorial VCG mechanism. Finally, Section 7 concludes. Additional results are collected in a set of technical appendices: Appendix A collects technical proofs, Appendix B extends our framework to incorporate entry, and Appendices C-H present extended identification, testing, and Monte Carlo simulation results.

2 Empirical framework

Consider a population of simultaneous first-price lettings. In each letting t, a set $\mathcal{N}_t = \{1, ..., N_t\}$ of risk-neutral bidders compete for (subsets of) a set $\mathcal{L}_t = \{1, ..., L_t\}$ of objects allocated via separate but simultaneous first-price auctions. Each bidder $i \in \mathcal{N}_t$ participates in a set of auctions, $\mathcal{L}_{it} \subset \mathcal{L}_t$, submitting a scalar bid b_{itl} in each auction l in which she participates. Bidding is simultaneous and objects are awarded auction by auction: the high bidder in auction l wins object l and pays her bid, with ties broken independently across bidders and auctions. Let L_{it} denote the number of auctions in which bidder i is participating, and $b_{it} \equiv (b_{itl})_{l \in \mathcal{L}_{it}}$ denote the $L_{it} \times 1$ vector of bids submitted by i in letting t.

[8]There is also a growing theoretical literature on simultaneous first-price auctions in computer science; see Feldman et al. (2012) and Syrgkanis (2012) among others. This literature focuses primarily on deriving bounds on the "Bayesian price of anarchy," or fractional efficiency loss, in simultaneous first-price auction markets. Results in this literature are largely restricted to settings with negative complementarities, and even in these settings bounds tend to be wide (e.g. Feldman et al. (2012) show that Bayesian Nash equilibrium captures at least half of total social surplus).

For each letting t, the econometrician observes the following data. First, for each object $l = 1, ..., L_t$ auctioned in letting t, the econometrician observes a vector of characteristics X_{lt} describing this object. Second, for each bidder i active in letting t, the econometrician observes bidder i's bid vector b_{it}, participation set $\mathcal{L}_{it}$, and a vector of generic bidder characteristics Z_{it}. In what follows, let $X_t \equiv (X_{1,t}, ..., X_{L_t,t})$ describe characteristics of all objects auctioned in letting t, and $Z_t \equiv (Z_{1t}, ..., Z_{N_t,t})$ describe characteristics of all active bidders.

Following Cantillon and Pesendorfer (2006) and Bajari and Fox (2013), we analyze bidding in the simultaneous first-price auction taking participation as given. That is, we take the endogenous outcome of interest to be the bid vectors $(b_{1t})_{i=1}^{N_t}$ submitted by each bidder, conditional on auction characteristics X_t, bidder characteristics Z_t, and participation sets $(\mathcal{L}_{it})_{i=1}^{N_t}$. We view this as a natural, and arguably necessary, first step toward understanding simultaneous first-price auction markets: here, as elsewhere, one cannot analyze participation without understanding bidding. Importantly, however, one can also view our analysis as applying to bidding within a two-stage entry and bidding model in which entry is interpreted as a process of value discovery; the key hypothesis in this case is that bidders discover private information about valuations only following costly entry. We return to this point in detail in Section 3.4 below, with a completely specified entry and bidding model in Appendix B.

For concreteness, we follow many prior studies on highway procurement auctions, e.g. Bajari and Ye (2003), Krasnokutskaya (2011), and Krasnokutskaya and Seim (2011) among many others, in assuming that bidders observe the participation structure $(\mathcal{L}_{it})_{i=1}^{N_t}$ at the time of bidding. We note, however, that our identification analysis applies equally when bidders observe only the set of *potential* participants in each auction; e.g., the set of planholders as in Li and Zheng (2009). In this case, one would simply reinterpret $\mathcal{L}_{it}$ as the set of auctions in which i is a potential participant, then proceed as we describe below.

In either case, to streamline notation, we adopt the convention that bidder i's characteristics Z_{it} include her participation set $\mathcal{L}_{it}$. From the perspective of both bidders and the econometrician, the common-knowledge observables (X_t, Z_t) fully characterize letting t.

Our baseline model turns on two sets of structural assumptions: the first regarding bidder preferences, the second regarding equilibrium behavior. In this section, we describe each of these in turn. We discuss extensions of this baseline model to settings with endogenous participation, unobserved heterogeneity, and richer preferences in Section 3.4.

2.1 Bidder preferences

Since our identification analysis applies at the population level conditional on specific realizations of common-knowledge observables (X_t, Z_t), for the next two sections we suppress the letting subscript t for notational compactness. We reintroduce the letting subscript t when discussing estimation in Section 5.

By construction, if bidder $i = 1, ..., N$ participates in $L_i \geq 1$ auctions, then she may win any of 2^{L_i} possible combinations of objects. We index these possible combinations with an $L_i \times 1$ outcome vector ω_i, where $\omega_{il} = 1$ if object l is allocated to bidder i and $\omega_{il} = 0$ otherwise. We represent the set of all 2^{L_i} combinations possible for bidder i with a $2^{L_i} \times L_i$ outcome matrix Ω_i, where each row of Ω_i corresponds to a distinct outcome ω_i. For example, if $L_i = 2$, then Ω_i would be given by

$$\Omega_i = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Equivalently, one may view each outcome ω_i as the binary representation of some integer in the set $\{1, ..., 2^{L_i}\}$ indexing i's possible combinations, with Ω_i collecting all such binary representations. With slight abuse of notation, we therefore use the shorthand "$\omega \in \Omega_i$" to indicate that outcome ω is possible for bidder i.

Bidders have preferences over combinations of objects, which need not be additive over elements in the combination. Specifically, to each potential outcome $\omega \in \Omega_i$, bidder i

associates a combinatorial valuation Y_i^ω, which she receives in the event that this outcome is realized. Let $Y_i \equiv [Y_i^\omega]_{\omega\in\Omega_i}$, a $2^{L_i} \times 1$ vector, collect i's combinatorial valuations Y_i^ω for all possible outcomes $\omega \in \Omega_i$. For simplicity, and without loss of generality, we normalize the value of winning nothing to zero: $Y_i^0 = 0$.

Let bidder i's *standalone valuation* for object l, denoted V_{il}, be the valuation i assigns to the outcome "i wins object l alone". Accordingly, let i's *standalone valuation vector*, denoted V_i, be the $L_i \times 1$ vector describing i's standalone valuations for each object in her participation set: $V_i \equiv [V_{il}]_{l=1}^{L_i}$. Finally, let K_i^ω denote i's *complementarity* between objects in combination $\omega \in \Omega_i$, defined as the difference between i's combinatorial valuation Y_i^ω and the sum of i's standalone valuations for objects won under ω:

$$K_i^\omega = Y_i^\omega - \omega^T V_i.$$

Let $K_i \equiv [K_i^\omega]_{\omega\in\Omega_i}$ be the $2^{L_i} \times 1$ vector containing the complementarities associated by i with each possible outcome $\omega \in \Omega_i$. Note that, by construction, we have

$$Y_i \equiv \Omega_i V_i + K_i.$$

We may thus equivalently represent bidder i's preferences in terms of the pair (V_i, K_i), where V_i describes i's valuations for each object individually, while K_i reflects departures from additivity in i's preferences over combinations. In particular, our model reduces to the canonical additively separable case if and only if $K_i = 0$ for all i.

As usual, we interpret standalone valuation vectors V_i as stochastic and private information for each bidder i. We further assume that standalone valuation vectors $(V_1, ..., V_N)$ are distributed independently across bidders conditional on observables:[9]

Assumption 1 (Independent private standalone valuations)**.** *For each bidder* $i = 1, ..., N$,

[9]In Appendix C, we allow for the presence of auction-level unobserved heterogeneity A_l, which is common knowledge among bidders but unobserved to the econometrician.

standalone valuations V_i are distributed according to a joint c.d.f. $F_i(\cdot|Z,X)$, with V_i independent from V_j for all $j \neq i$, and $F_i(\cdot|Z,X)$ common knowledge.

While standalone valuations are stochastic private information, we model complementarities K_i as determined by observables. We view this structure as natural in applications such as highway contracting, snow cleaning (Flambard and Perrigne (2006)), recycling (Kawai (2011)), and cleaning (Lunander and Lundberg (2013)), where factors such as capacity constraints, distance between projects, timing of projects, or types of work are the main considerations motivating analysis of complementarities. We emphasize, however, that insofar as our model interprets all correlation across bidders and complementarities across objects as arising through observables, the suitability of the model will inherently depend heavily on what observables are available.

Assumption 2 (Deterministic complementarities)**.** *For all $i = 1, ..., N$ and each combination $\omega \in \Omega_i$, $K_i^\omega = \kappa_i^\omega(Z, W_i^\omega)$, where W_i^ω is an observed vector of characteristics of the combination $\omega \in \Omega_i$, and $\kappa_i^\omega(Z, W_i^\omega)$ is common knowledge to bidders.*

One may also interpret the function $\kappa_i^\omega(Z, W_i)$ as reflecting bidders' expectations, at the time of bidding, over ex ante unknown synergy effects associated with winning combination ω. The crucial hypothesis is that, at the time of bidding, this expectation depends only on common-knowledge observables. For example, if value discovery is costly, bidders may invest in learning standalone valuations prior to bidding, but invest in discovering idiosyncratic synergy effects only following a multiple win. In Appendix D, we generalize our identification analysis to settings where complementarities additionally incorporates an ex ante unknown affine transformation of standalone valuations. This extension further allows compementarities to be stochastic private information, so long as the private information component of bidders' expected complementarities can be fully explained by standalone valuations.

In what follows, let $W_i \equiv [W_i^\omega]_{\omega \in \Omega_i}$ collect characteristics of all combinations of auctions in which bidder i participates, and $W \equiv (W_1, ..., W_N)$ collect characteristics of all combinations of auctions in which all bidders participate. With slight abuse of notation, we write $K_i = \kappa_i(Z, W_i)$ to indicate the vectorization of Assumption 2 across all potential combinations $\omega \in \Omega_i$ of auctions in which bidder i participates.

We denote combination characteristics W and auction characteristics X distinctly for conceptual clarity, although in practice W will typically be derived through transformation of X. For example, in our highway procurement application, X includes factors such as project size, project location, and type of work in each project, whereas W includes factors such as distance between projects, sum of project sizes, and the degree of schedule overlap among projects in each combination. We view these as the main factors inducing potential nonseparability in this market, motivating Assumption 2 as noted above.

Taken together, Assumptions 1 and 2 embed the canonical separable IPV model within a substantially richer framework allowing both flexible nonparametric complementarities and arbitrary dependence among elements of V_i for each bidder i. We view the latter as an essential empirical complement to the former, since "reduced form" correlation in i's bids could be driven either by complementarities in i's preferences or by dependence in i's valuations. By leaving such dependence unrestricted, we focus cleanly on identification of nonadditivities per se, even when bidder i's own valuations are correlated across auctions.

2.2 Equilibrium behavior

Let $M \equiv (Z, X, W)$ denote the set of market characteristics observed to both bidders and the econometrician. Let $\mathcal{V}_i \subset \mathbb{R}_+^{L_i}$ denote the support of the standalone valuation vector V_i for bidder $i = 1, ..., N$, and let $\mathcal{B}_l \subset \mathbb{R}_+$ denotes the set of feasible bids in auction $l = 1, ..., L$. Generically, one would define a pure strategy for bidder i as a mapping from the space of combinatorial valuation vectors Y_i to the space of feasible bids. Under Assumptions 1 and 2, however, each bidder's private information is fully described by their vector of standalone valuations V_i. To emphasize this point, in what follows we focus on $\mathcal{V}_i$ as the type space for bidder i in market M, and define a pure strategy for bidder i in market M as a mapping

$\sigma_i^M : \mathcal{V}_i \to \mathcal{B}_i$, where $\mathcal{B}_i \equiv \times_{l \in \mathcal{L}_i} \mathcal{B}_l$ denotes i's action space in the simultaneous bidding game.[10] Let $\sigma^M = (\sigma_1^M, ..., \sigma_N^M)$ denote a strategy profile for all bidders in market M, and σ_{-i}^M denote a strategy profile for all rivals of bidder i.

Building on the first-order approach of Guerre et al. (2000), we base identification on necessary conditions for best-response behavior in simultaneous first-price auctions. For this analysis to proceed, we require the following assumptions on bidder behavior:

Assumption 3. *For each market structure M, the distribution of bids observed at M are generated by a strategy profile σ^M which is a Bayesian Nash equilibrium of the simultaneous bidding game. Furthermore, for each M, only one strategy profile σ^M is played.*

When complementarities are zero, existence of a pure strategy equilibrium is immediate and uniqueness follows under regularity conditions (Lebrun (1999)). More generally, with nonzero complementarities, existence of a pure strategy equilibrium in any discrete bid space follows from results in Milgrom and Weber (1985). Analysis of equilibrium with arbitrary complementarities in continuous bid spaces would be a fundamental breakthrough in its own right, and as such is well beyond the scope of this paper.[11] In this respect, our setting parallels many other studies on complex auction games, in which either existence (Bajari and Fox (2013) on spectrum auctions, Ausubel and Milgrom (2002) on proxy auctions) or uniqueness (Jofre-Bonet and Pesendorfer (2003), Roberts and Sweeting (2013), Somaini (2015) and references therein) is assumed as it cannot be guaranteed in general.[12]

[10]We focus on pure strategies for expositional simplicity, but this is without essential loss of generality; all results below apply equally when bidders play mixed strategies.

[11]"Fundamental" in the sense that existing theoretical tools appear inadequate to study existence in settings with complementarities. As in multi-unit auctions, the presence of both multidimensional bids and multidimensional types leads to failure of classical differential-equations approaches to Bayes-Nash equilibrium. Monotonicity-based methods widely used in multi-unit auctions—e.g. Athey (2004), McAdams (2006), and Reny (2011))—can be applied in special cases, but do not apply at the level of generality we consider here. Other approaches—e.g. that of Jackson et al. (2002) applied in Cantillon and Pesendorfer (2006)—deliver generalizations of Bayes-Nash equilibrium, but not Bayes-Nash equilibrium itself. See Gentry et al. (2019) for a detailed discussion of these issues, plus results on equilibrium existence in some special cases.

[12]We note, however, that almost every real-world bid space is ultimately discrete. For instance, if bidders must bid in pennies, then existence is guaranteed as noted above. In this sense, we see existence as of more theoretical than practical concern. In the main text, we follow the literature's convention of interpreting bid spaces as continuous, and proceed to analyze identification. Appendix F provides a more general partial identification analysis applicable in settings where discreteness is viewed as empirically important.

To leverage necessary conditions for optimal behavior, we require only the hypotheses on equilibrium behavior stated in Assumption 3. For such an analysis to yield point (rather than partial) identification of model primitives, we further require equilibrium behavior to satisfy the following additional conditions:

Assumption 4. *For each observed market structure M, the equilibrium strategy profile σ^M is such that (i) the joint cumulative distribution function of bids is absolutely continuous, and (ii) for any auction $l = 1, ..., L$ and any bidders i, j active in auction l, the marginal distributions of bids b_{il}, b_{jl} have common lower support.*

As above, under the null of separability ($K_i = 0$), these properties follow immediately from standard regularity conditions; when $K_i \neq 0$, we require them as assumptions, although key implications of these assumptions can be verified directly.[13] In practice, absolute continuity implies that marginal bid distributions are atomless, which in turn permits extension of the Guerre et al. (2000) first-order approach to settings with simultaneous auctions. If equilibrium bid distributions instead involve atoms, model primitives will typically be partially identified, although identified sets may be quite informative as we show in Appendix F. Meanwhile, common lower support ensures that bidders do not submit never-winning (or null) bids with positive probability. This may fail if, for example, bidders draw standalone valuations from distributions with asymmetric supports, or in the presence of binding public reserve prices. If so, the data will yield only an upper bound on standalone valuations consistent with such bids, although complementarities will in many cases remain point identified. We analyze this case in detail in Appendix F.4.

3 Nonparametric identification

We analyze identification based on a large number of simultaneous first-price auction markets. By hypothesis, for each market, the econometrician observes the vector of common-

[13]Specifically, one can verify whether the data involve ties and test whether bids exhibit common support, although testing for absolute continuity is infeasible in finite samples.

knowledge covariates M characterizing bidders, objects, and combinations in the market, together with the bid vectors $(b_i)_{i=1}^N$ submitted by each bidder active in the market. The identification problem is to recover the nonparametric primitives $F_i(\cdot|Z,X)$ and $\kappa_i(Z,W_i)$ for each bidder i active in market M.

In analyzing this problem, we adopt the following notation. For each bidder $i = 1, ..., N$, let $G_i(\cdot|M)$ be the joint cumulative distribution function of the $L_i \times 1$ bid vector b_i submitted by i conditional on market characteristics M, and $g_i(\cdot|M)$ be the corresponding conditional joint density. Taking equilibrium rival strategies σ_{-i}^M as given, suppose that bidder i submits bid $b_i \in \mathcal{B}_i$. For each auction $l \in \mathcal{L}_i$, let $\Gamma_{il}(b_i|M)$ denote this bidder's marginal probability of winning auction l, and for each combination $\omega \in \Omega_i$, let $P_i^\omega(b_i|M)$ denote the joint probability that she wins combination ω, both interpreted as functions of i's bid vector b_i taking rival strategies σ_{-i}^M as given. Finally, let $\Gamma_i(b_i|M) \equiv (\Gamma_{il}(b_i|M))_{l\in\mathcal{L}_i}$, an $L_i \times 1$ vector, collect marginal win probabilities $\Gamma_{il}(b_i|M)$ across auctions $l \in \mathcal{L}_i$, and let $P_i(b_i|M) \equiv [P_i^\omega(b_i|M)]_{\omega\in\Omega_i}$, a $2^{L_i} \times 1$ vector, collect combinatorial win probabilities $P_i^\omega(b_i|M)$ across combinations $\omega \in \Omega_i$. Note that, if there are no ties, then i's marginal probability of winning auction l, i.e. $\Gamma_{il}(b_i|M)$, is simply the c.d.f. of the maximum rival bid in auction l, evaluated at i's bid b_{il}. Furthermore, by construction, marginal win probabilities $\Gamma_i(b_i|M)$ are related to combinatorial win probabilities $P_i(b_i|M)$ by the identity $\Gamma_i(b_i|M) \equiv \Omega_i^T P_i(b_i|M)$.

Observe that, under Assumption 3, $G_i(\cdot|M)$ is identified directly for each $i = 1, ..., N$, with identification of $(G_i(\cdot|M))_{i=1}^N$ implying identification of $P_i(\cdot|M)$ and $\Gamma_i(\cdot|M)$ for all i. In what follows, we thus take bid distributions $G_i(\cdot|M)$, marginal win probabilities $\Gamma_i(\cdot|M)$, and combinatorial win probabilities $P_i(\cdot|M)$ as known. We aim to recover $F_i(\cdot|Z,X)$ and $\kappa_i(Z,W_i)$ given knowledge of $G_i(\cdot|M)$, $P_i(\cdot|M)$, and $\Gamma_i(\cdot|M)$ for all $i = 1, ..., N$.

We begin by showing that bidder i's primitives (F_i, κ_i) are nonparametrically identified up to κ_i. We then provide sufficient conditions for identification of κ_i based on excludable variation in either the set of competitors faced or the characteristics of other objects. Finally, we discuss identification under several extentions to the baseline model, including endogenous participation and additively separable auction-level heterogeneity.

3.1 Nonparametric identification of F_i up to κ_i

Consider the bidding problem faced by bidder $i = 1, .., N$ with preferences (V_i, K_i) in market M, where standalone valuations V_i are drawn privately from $F_i(\cdot|X, Z)$ and complementarities $K_i = \kappa_i(Z, W_i)$ are common knowledge as described above. By hypothesis, taking rival strategies σ_{-i}^M as given, this bidder optimally submits the $L_i \times 1$ bid vector $b_i \in \mathcal{B}_i$ which maximizes her expected interim profit function

$$\pi_i^M(b_i; V_i, K_i) = \Gamma_i(b_i|M)^T(V_i - b_i) + P_i(b_i|M)^T K_i, \tag{1}$$

where $\Gamma_i(b_i|M)^T(V_i - b_i)$ reflects the expected sum of bidder i's canonical standalone payoffs over each auction individually, and $P_i(b_i|M)^T K_i$ reflects the change in i's expected payoffs induced by non-additivities in her preferences over combinations.

Under Assumption 4, one can show that the interim function profit function (1) is differentiable in b_i almost surely with respect to the measure on $\mathcal{B}_i$ induced by $G_i(\cdot|M)$; we establish this formally in the proof of Proposition 1 below. Hence, under the hypothesis of equilibrium play, almost every bid $b_i \in \mathcal{B}_i$ submitted by i must satisfy the $L_i \times 1$ system of necessary first-order conditions a.e.:

$$\nabla_b \Gamma_i(b_i|M)(V_i - b_i) = \Gamma_i(b_i|M) - \nabla_b P_i(b_i|M)K_i, \tag{2}$$

where $\nabla_b \Gamma_i(b_i|M)$ is an $L_i \times L_i$ diagonal matrix and $\nabla_b P_i$ is an $2^{L_i} \times L_i$ matrix. Clearly, the system (2) is not solvable for (V_i, K_i) jointly; we have only L_i equations for $2^{L_i} - 1$ unknowns. But under Assumptions 3 and 4, this system is almost surely solvable for V_i given K_i. In other words, under the hypothesis $K_i = \kappa_i(Z, W_i)$, there almost surely exists a unique candidate for V_i at which b_i satisfies first order necessary conditions for a best response:

Proposition 1. *Suppose that Assumptions 1-4 hold. Let K_i be any candidate for bidder i's unknown complementarity vector $\kappa_i(Z, W_i)$: i.e. any vector in $\mathbb{R}^{2^{L_i}}$ whose first $L_i + 1$ components are zero.*[14] *Then for almost every b_i drawn from $G_i(\cdot|M)$, there exists a unique, identified vector $\xi_i(b_i|M; K_i)$ solving (2) under the hypothesis $\kappa_i(Z, W_i) = K_i$:*

$$\xi_i(b_i|M; K_i) \equiv \Upsilon_i(b_i|M) - \Psi_i(b_i|M) \cdot K_i, \tag{3}$$

where $\Upsilon_i(b_i|M)$ is an identified $L_i \times 1$ vector defined by

$$\Upsilon_i(b_i|M) \equiv b_i + \nabla_b \Gamma_i(b_i|M)^{-1} \Gamma_i(b_i|M), \tag{4}$$

and $\Psi_i(b_i|M)$ is an identified $L_i \times 2^{L_i}$ matrix defined by

$$\Psi_i(b_i|M) \equiv \nabla_b \Gamma_i(b_i|M)^{-1} \nabla_b P_i(b_i|M)^T. \tag{5}$$

Furthermore, if $K_i = \kappa_i(Z, W)$, then $V_i = \xi_i(b_i|M; K_i)$ almost surely.

Proof. See Appendix A. □

Note that, interpreted as a function of K_i, $\xi_i(b_i|M; K_i)$ is affine in K_i for all b_i and M. The additive term $\Upsilon_i(b_i|M)$ in this affine function is the canonical auction-by-auction inverse bidding function of Guerre et al. (2000), vectorized over the L_i auctions played by i.[15] The multiplicative term $\Psi_i(b_i|M) \cdot K_i$ adjusts this standard inverse bidding function for potential nonadditivities in i's preferences, reflected in the conjectured complementarity vector K_i. The weights $\Psi_i(b_i|M)$ on K_i correspond, intuitively, to the marginal effect of increasing each

[14] These zero components correspond to the outcomes in which bidder i wins either no objects ($\omega = (0, \dots, 0)$) or one object ($\omega'\omega = 1$), for which complementarities are zero by construction.

[15] To see this, recall that under Assumption 4 the lth element of $\Gamma_i(b_i|M)$ is simply the c.d.f. of the maximum bid among i's rivals in auction l, evaluated at b_{il}. Hence the lth element of $\Upsilon_i(b_i|M)$ reduces to

$$\Upsilon_{il}(b_i|M) = b_{il} + \frac{\Gamma_{i,l}(b_{il}|M)}{\gamma_{i,l}(b_{il}|M)} \quad \text{where} \quad \gamma_{i,l}(b_{il}|M) \equiv \frac{d}{db_{il}} \Gamma_{i,l}(b_{il}|M),$$

i.e., the usual standalone inverse bid function of Guerre et al. (2000) in auction l.

bid b_{il} on i's probability of winning each higher-order combination, relative to the marginal effect of b_{il} on i's overall probability of winning auction l.

Finally, observe that if the conjecture $\kappa_i(Z, W_i) = K_i$ is in fact correct, then we must have $V_i = \xi_i(b_i|M; K_i)$ almost surely. Hence to each candidate K_i for $\kappa_i(Z, W_i)$, there corresponds a unique, identified candidate $\hat{F}_i(\cdot|M; K_i)$ for the unknown c.d.f. $F_i(\cdot|Z, X)$:

$$\hat{F}_i(v|M; K_i) = \int_{\mathcal{B}_i} 1[\xi_i(B_i|M; K_i) \leq v]\, G_i(dB_i|M). \tag{6}$$

In other words, if $\kappa_i(Z, W_i)$ were known, then we could recover $F_i(\cdot|X, Z)$ immediately through the identity $F_i(\cdot|X, Z) \equiv \hat{F}_i(\cdot|M; \kappa_i(Z, W_i))$. Identification of bidder i's primitives therefore reduces to recovery of the unknown non-parametric function $\kappa_i(Z, W_i)$.

3.2 Nonparametric identification of complementarities based on variation in rival characteristics

In view of Proposition 1, it is also clear that further structure is necessary for identification: under Assumptions 1-4, we can identify valuations only up to complementarities. But suppose that, to these assumptions, we add the hypothesis that bidder i's primitives (F_i, κ_i) depend only on bidder i's characteristics Z_i, not on the characteristics of rival bidders Z_{-i}:

Assumption 5. *For all bidders i, $F_i(\cdot|Z, X) = F_i(\cdot|Z_i, X)$ and $\kappa_i(Z, W_i) = \kappa_i(Z_i, W_i)$.*

Similar assumptions have been widely employed in the empirical auction literature; see, e.g., Guerre et al. (2009), and Somaini (2015) among others. We will show that under Assumption 5, variation in competitor characteristics Z_{-i} induces a large (infinite) set of restrictions on the finite vector $\kappa_i(Z_i, W_i)$. [16] Under mild conditions on variation in Z_{-i} made precise below, these restrictions will have the unique solution $K_i = \kappa_i(Z, W_i)$, leading to nonparametric identification of $\kappa_i(Z, W_i)$ and hence the model as above.

[16]Notice that as Z_{-i} refers also to the set of participated auctions, variation in Z_{-i} could also imply variation in W_{-i} if there is a change in the auctions in which $-i$ participates.

Toward this end, consider any bidder $i = 1, ..., N$. Fix any realization of auction characteristics X, own characteristics Z_i, and combination characteristics W_i. Let $M \equiv (X, Z, W)$ and $M' \equiv (X', Z', W')$ be any two market structures such that the characteristics affecting i's primitives are held constant, but the characteristics of i's competitors vary: that is, such that $X = X'$, $Z_i = Z'_i$, and $W_i = W'_i$, but $Z_{-i} \neq Z'_{-i}$. Bids observed at market structures M and M' will of course typically correspond to different realizations of i's standalone valuations V_i. But, in view of Assumption 5, we know that V_i is drawn from the same distribution $F_i(\cdot|X, Z_i)$ under both M and M'. Furthermore, from Proposition 1, we know that for each market structure M and each candidate complementarity K_i, there exists a unique, identified candidate $\hat{F}_i(\cdot|M; K_i)$ for the unknown distribution $F_i(\cdot|X, Z_i)$. Hence, if $K_i = \kappa_i(Z_i, W_i)$, then for almost every $v \in \mathbb{R}^{L_i}$ we must have

$$\hat{F}_i(v|M; K_i) = F_i(v|X, Z_i) = \hat{F}_i(v|M'; K_i). \tag{7}$$

Clearly, if $\hat{F}_i(\cdot|M; K_i)$ and $\hat{F}_i(\cdot|M'; K_i)$ coincide almost everywhere, then the expectations of random vectors drawn from these distributions must also coincide. But recall that, by definition, $\hat{F}_i(\cdot|M; K_i)$ is the distribution of the random vector $\hat{V}_i \equiv \xi_i(B_i|M; K_i)$, where $B_i \sim G_i(\cdot|M)$. Hence, if $K_i = \kappa_i(Z_i, W_i)$, then in view of (7) we must also have

$$\int_{\mathcal{B}_i} \xi_i(B_i|M; K_i)\, G_i(dB_i|M) = \int_{\mathcal{B}_i} \xi_i(B_i|M'; K_i)\, G_i(dB_i|M'). \tag{8}$$

Finally, recall that $\xi_i(\cdot|M; K_i)$ is affine in K_i. Hence we may equivalently rewrite each integral in (8) as an identified affine function of K_i as follows:

$$\begin{aligned}\int_{\mathcal{B}_i} \xi_i(B_i|M; K_i)\, G_i(dB_i|M) &= \int_{\mathcal{B}_i} \left[\Upsilon_i(B_i|M) - \Psi_i(B_i|M) \cdot K_i\right]\, G_i(dB_i|M) \\ &\equiv \bar{\Upsilon}_i(M) - \bar{\Psi}_i(M) \cdot K_i, \end{aligned} \tag{9}$$

where $\bar{\Upsilon}_i(M)$, an identified $L_i \times 1$ vector, and $\bar{\Psi}_i(M)$, an identified $L_i \times 2^{L_i}$ matrix, denote the expectations of the functions $\Upsilon_i(\cdot|M)$ and $\Psi_i(\cdot|M)$ defined in Proposition 1 with respect to bids drawn from i's equilibrium bid distribution $G_i(\cdot|M)$:

$$\begin{aligned}\bar{\Upsilon}_i(M) &\equiv \int_{\mathcal{B}_i} \Upsilon_i(B_i|M)\, G_i(dB_i|M),\\ \bar{\Psi}_i(M) &\equiv \int_{\mathcal{B}_i} \Psi_i(B_i|M)\, G_i(dB_i|M).\end{aligned}$$

Substituting (9) into (8) under the hypothesis $K_i = \kappa_i(Z_i, W_i)$, we ultimately obtain a system of L_i linear restrictions on the unknown vector $\kappa_i(Z_i, W_i) \in \mathcal{K}_i$:

$$\left[\bar{\Upsilon}_i(M) - \bar{\Upsilon}_i(M')\right] - \left[\bar{\Psi}_i(M) - \bar{\Psi}_i(M')\right] \cdot \kappa_i(Z_i, W_i) = 0. \tag{10}$$

Recall that the first $L_i + 1$ elements of $\kappa_i(Z_i, W_i)$ are zero by construction. Hence (10) is effectively a system of L_i equations in $2^{L_i} - L_i - 1$ unknowns. When $L_i > 2$, we have $2^{L_i} - L_i - 1 > L_i$, hence the system (10) alone will be insufficient to identify $\kappa_i(Z_i, W_i)$. But recall that (10) must hold for *any* pair of markets M, M' such that $X = X'$, $Z_i = Z_i'$ and $W_i = W_i'$. In other words, for given (X, Z_i, W_i), every distinct realization of rival characteristics Z_{-i} generates an additional set of L_i linear restrictions parallelling (10), all of which must hold simultaneously at $K_i = \kappa_i(Z_i, W_i)$. Pooling such linear restrictions across many markets with varying rival characteristics Z_{-i}, we ultimately conclude:

Proposition 2. *Consider any bidder $i = 1, ..., N$ and any realization of i's bidder and combination characteristics (Z_i, W_i). Let $M^0, M^1, ..., M^J$ be any collection of market structures such that Z_i, W_i and X are constant for all $j = 1, ..., J$, and suppose that the submatrix formed by the last $(2^{L_i} - L_i - 1)$ columns of the $J(J-1)L_i \times 2^{L_i}$ matrix*

$$\boldsymbol{\Delta\bar{\Psi}} \equiv \left[\ \bar{\Psi}_i(M^j) - \bar{\Psi}_i(M^k)\ \right]_{j,k \in \{1,...,J\}}$$

has rank $2^{L_i} - L_i - 1$. Then $\kappa_i(Z_i, W_i)$ is identified.

Recall that the identification criterion (10) exploits only invariance of *first moments* of $F_i(\cdot|Z_i, X)$, even though the underlying distributional invariance restriction (7) implies that relations analogous to (8) hold for the whole characteristic function. The system of equations in Proposition 2 merely provides a simple and testable sufficient condition under which the full characteristic system has a unique solution. Note also that variation in, e.g., number of rivals in each auction will produce exactly the kind of variation needed for full column rank of $\Delta\bar{\boldsymbol{\Psi}}$: nonlinear changes in the Jacobian Ψ_i of probabilities of winning different combinations, which map into bidding as weights on the unknown vector $\kappa_i(Z_i, W_i)$. Even discrete variation in Z_{-i} thus naturally gives rise to full column rank of $\Delta\bar{\boldsymbol{\Psi}}$, yielding nonparametric identification of $\kappa_i(Z_i, W_i)$ and hence the model as above.

3.3 Nonparametric identification of complementarities based on variation in characteristics of other objects

While the restriction that own primitives are invariant to competitor characteristics is both natural and widely employed, it could potentially be violated in environments with richer strategic interaction among players. For instance, if there is an upstream market for subcontractors, then capacity utilization by i's rivals could in principle affect i's costs. We therefore also consider nonparametric identification of $\kappa_i(\cdot)$ based on excludable variation in characteristics of other auctions. This key hypothesis underlying this approach, also widely maintained in the literature, is that standalone valuations in each auction l depend only on object l's characteristics X_l, not on the characteristics of other objects X_{-l}:

Assumption 6. *For each bidder i and object $l \in \mathcal{L}_i$, $F_{il}(\cdot|Z, X) = F_{il}(\cdot|Z, X_l)$.*

This assumption allows both $F_i(\cdot|Z, X)$ and $\kappa_i(Z, W_i)$ to depend on Z_{-i}, but requires each marginal distribution to be invariant to characteristics of other objects. The subsequent identification argument closely follows the steps above, with variation in X_{-l} replacing variation in Z_{-i}, and is therefore omitted for brevity.[17] Obviously, where appropriate, Assumptions 5 and 6 can also be maintained jointly, as we do in our application.

3.4 Extensions

While our analysis so far has focused on the baseline model defined in Assumptions 1-4, our identification insights extend to accommodate endogenous participation, unobserved heterogeneity, complementarities depending on standalone valuations, and positive probabilities of ties or never-winning bids. We develop these extensions in detail in Appendices B-F below. For completeness, however, we also describe the main ideas of each extension briefly here.

Appendix B formally embeds our bid-stage analysis within a two-stage entry and bidding game. In this extension, following Levin and Smith (1994), Krasnokutskaya and Seim (2011), Moreno and Wooders (2011), Athey et al. (2011), Groeger (2014), and Li and Zhang (2015) among others, we interpret entry as a process of value discovery. Bidders first simultaneously choose which combinations of auctions to enter on the basis of the common-knowledge primitives $(F_i, K_i)_{i=1}^N$ plus a vector of private, potentially combinatorial, entry costs. Conditional on entry, each bidder i then discovers their standalone valuations V_i for each auction in which they have entered. Finally, based on private valuations V_i plus the common-knowledge characteristics M, entering bidders submit bids as above.

How does such endogenous participation change our understanding of the underlying bidding game? Clearly, the sets of auctions which bidders actually enter will not be random; rather, bidders will endogenously select into both auctions for which they expect high valuations and combinations for which they anticipate positive complementarities. This implies, for instance, that the complementarities we actually observe will differ from those which would arise if bidders were randomly assigned to auctions, a point to which we return in interpreting our results. Crucially, however, so long as this selection is solely on the basis of the common-knowledge primitives $(F_i, K_i)_{i=1}^N$, entry in fact strengthens prospects for iden-

[17] Note that variation in X_{-l} may also vary W_{-i} without varying W_i if i does not bid in auction l and some rival does.

tification in two respects. First, this model of participation provides a formal equilibrium justification for Assumption 5 above.[18] Second, the combinations of auctions which bidders choose to enter will themselves convey additional information about complementarities. We develop both points more fully in Appendix B.

Next, in Appendix C, we extend our identification analysis to accommodate unobserved auction heterogeneity, modeled as an auction-level characteristic A_l such that $V_{il} = U_{il} + A_l$, with $U_i \equiv [U_{il}]_{l=1}^{L_i}$ independent private information for each bidder and A_l common knowledge to bidders but not the econometrician. The main complication induced by such unobserved heterogeneity is a first step in which, following Krasnokutskaya (2011), one nets out variation in bids driven by the separable unobservables $A \equiv [A_l]_{l=1}^{L}$. Non-parametric identification of bidder-level primitives then proceeds as described above.

In Appendix D, we generalize our non-parametric identification results to the case where compementarities are stochastic but their randomness can be fully explained by the standalone valuations. Such a case could arise if, for instance, winning two auctions together increases i's valuation for one or both objects by a fixed percentage.

In Appendix E, we discuss additional identifying restrictions induced by variation in combination characteristics W_i. In a parametric environment, variation in W_i alone will often be sufficient to identify the parameters governing κ_i. Nonparametrically, however, variation in W_i alone is typically insufficient to identify κ_i, although such variation does enrich the system of identifying restrictions based on variation in Z_{-i} and X_{-l} derived above.

Finally, in Appendix F, we extend our analysis to accomodate potential violations of Assumption 4, such as might arise if the bid distribution involves atoms or if bidders submit

[18] If bidders instead select into entry on the basis of private information about their valuations V_i, as in the "selective entry" models of Roberts and Sweeting (2013) and Gentry and Li (2014), then matters would be more complicated. In this case, for instance, the set of potential competitors bidder i faces will generally affect the information sets at which bidder i choose to enter, and hence the distribution of valuations F_i drawn upon entry. Even in this case, however, the identification insights below still apply so long as bidder i observes the set of rival entrants prior to bidding. In this case, one would include the set of potential competitors in each auction in X, and the set of actual entrants in Z. Furthermore, regardless of whether bidders observe actual rivals prior to bidding, the testing insights we develop below continue to apply: even when entry is selective, cross-auction spillovers can arise only if auctions are not additively separable.

bids which never win. We show that Assumption 4 can be viewed as a sufficient condition for point identification within a richer partial identification framework maintaining only Assumptions 1-3. More generally, if the bid distribution involves atoms, then both κ_i and F_i will typically be partially identified, although identified sets may be quite informative as we show in Appendix H.1. Meanwhile, if bidders submit never-winning bids, then the lower tail of standalone valuations corresponding to these bids will be endogenously truncated. So long as the support of standalone valuations is sufficiently rich, however, quantiles of standalone valuations above the truncation point will be identified up to κ_i and invariant to Z_{-i}. Focusing on these quantiles yields a continuum of identifying restrictions, a subset of the system (7), which will generically identify κ_i, although F_i will be identified only on the region of valuations corresponding to non-trivial bids.

4 Application: Michigan Highway Procurement

We now turn to our empirical application: Michigan Department of Transportation (MDOT) highway construction and maintenance contracts. MDOT allocates contracts for a wide range of highway construction and maintenance services via low-price sealed-bid auctions. The vast majority of MDOT projects are allocated via large simultaneous letting rounds, which take place on average every three weeks.[19] There are an average of 45 auctions per letting round and more than half (56 percent) of bidders submit bids on multiple contracts within a letting.[20] A bid is an itemized description of unit costs for each line item specified in contract plans; bids are submitted to MDOT project by project, with the winner of each project the bidder submitting the bid involving the lowest total project costs. Contracts are advertised up to ten weeks prior to letting, with the closing deadline for submitting, amending or withdrawing bids typically 10am on the letting date. MDOT then publicly opens bids and

[19] There are only two months without lettings.

[20] MDOT runs a pre-qualification process, which ensures quality of work. The process involves a check on the financial status of the firm and its backlogs from all construction activities. A bid submission includes a detailed break down of all costs involved in the contract. The winner is determined solely by the total cost of the project.

allocates contracts, with winning bidders held liable for completion of contracts won. In view of prior evidence on complementarities and capacity constraints in highway procurement, we expect factors such as capacity constraints, project proximity, project types, and scheduling overlap to induce substantial non-additivities in bidder payoffs across auctions. In this application, we focus on potential complementarities across auctions within a MDOT letting round, abstracting from potential complementarities across letting rounds.[21]

4.1 Data

MDOT provides detailed records on contracts auctioned, bids received, and letting outcomes on its letting website (http://www.michigan.gov/mdot). Drawing from these records, we observe data on (almost) all contracts auctioned by MDOT over the sample period January 2005 to March 2014.[22] Our sample includes a total of 8224 auctions, where for each auction the following information is observed: project description, project location, pre-qualification requirements, the internal MDOT engineer's estimate of total project cost, and the list of participating firms and their bids. Based on project descriptions, we classify projects into five project types: bridge work, major construction, paving (primarily hot-mix asphalt), safety (e.g. signing and signals), and miscellaneous, leading to a final distribution of projects across types summarized in Table 1. As evident from Table 1, roughly 80 percent of contracts are for road and bridge construction and maintenance broadly defined, with the remainder split between safety and other miscellaneous construction.

The data contains information on a total of 714 unique bidders active in the MDOT marketplace over our sample period, which we classify by size and scope of activity as follows. We define a bidder as "regular" if it submitted more than 100 bids in the sample period, and "fringe" otherwise. This yields a total of 36 regular bidders, with all other bidders classified

[21] A formal analysis of both static and dynamic complementarities is beyond the scope of the current paper, although it would be a very interesting avenue for future research.

[22] MDOT records for a small number of contracts are incomplete. Although we have data from October 2002 to March 2014, we have discarded the first few years (from October 2002 to December 2004) as we use lettings from these years to construct bidder backlog variables.

Table 1: Summary of Projects by Type

Contract Type	Frequency
Bridge	13.33
Major Construction	9.64
Paving	56.33
Safety	12.25
Miscellaneous	8.45

as "fringe." For the subsample of bidders who have submitted more than 50 bids, we also collect data on number and location of plants by firm. This data is derived from a variety of sources: OneSource North America Business Browser, Dun and Bradstreet, Hoover's, Yellowpages.com and firms' websites. We then further classify bidders as "large" or "small" based on this data, with "large" bidders those owning at least 6 plants in Michigan. We thus obtain a final classification of 8 large regular bidders, 28 small regular bidders, and 686 fringe bidders (of which 4 are large bidders) in the MDOT marketplace.

Table 2 surveys the auction side of the MDOT marketplace. The first key feature emerging from this table is the large number of contracts auctioned simultaneously in the market: a mean of 45 per letting, with a maximum of 133 on a single letting date.[23] On average about five bids are received per contract, which is small relative to the average number of bidders (approximately 84) active in any given letting. For each contract, MDOT prepares an internal "Engineer's Estimate" of expected procurement cost released to bidders before bidding; as evident from the dispersion in this estimate, projects vary substantially in size and complexity. The statistic "Money Left on the Table" measures the percent difference between lowest and second-lowest bids. On average, this is 7.4 percent, or roughly $112,000 per contract, suggesting the presence of substantial uncertainty over rival bids.

Table 3 summarizes bidder behavior in the MDOT marketplace. Consistent with Table 2, the average bidder competes in roughly 2.7 auctions per round, with large and regular bidders competing in substantially more. The variable "backlog" provides a bidder-specific measure

[23]Note that smaller supplemental lettings are occasionally held two or three weeks after the main letting in a given month.

Table 2: Auction Level Summary Statistics

	Mean	St. Dev.	Min	Max
Auctions per Round	45.19	35.67	1	133
Total Bids per Round	228.1	180.9	1	669
Distinct Bidders per Round	83.97	57.06	1	207
Number of Bidders per Auction	5.048	3.186	1	28
Large Regular Bidders per Auction	0.407	0.674	0	3
Regular Bidders per Auction	1.500	1.362	0	7
Fringe Bidders per Auction	3.149	2.926	0	23
Engineer's Estimate (in thousands)	1,514	4,689	4.412	165,313
Project Duration (in days)	175.8	205.1	2	1,838
Money Left on the Table	0.0744	0.0966	0	3.016

Table 3: Bidder Level Summary Statistics

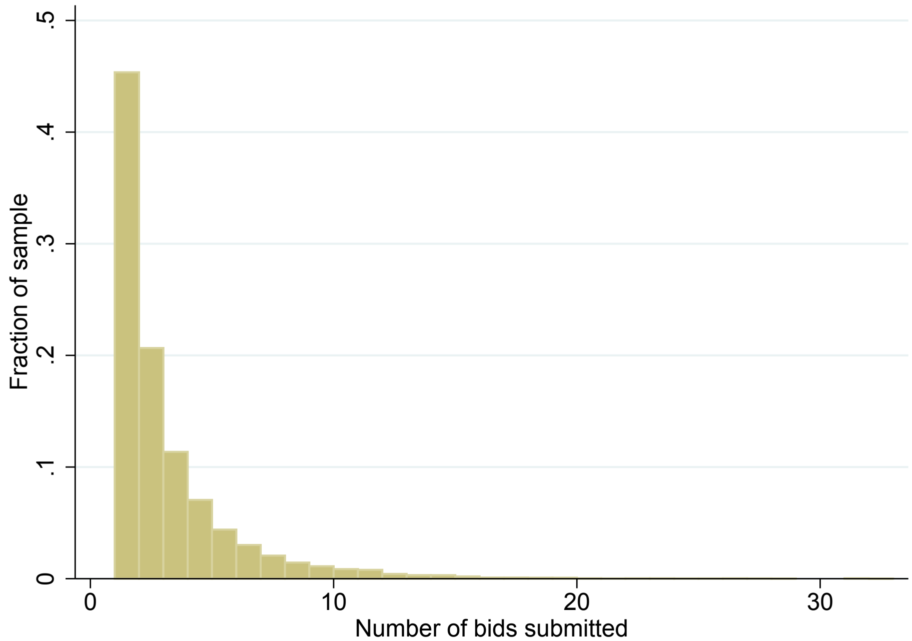
	Mean	St. Dev.	Min	Max
Bids by Round	2.716	2.785	1	33
Bids by Round if Large	6.545	6.237	1.000	33.000
Bids by Round if Regular	5.96	4.58	1.00	33.00
Backlog (in millions)	5.792	19.01	0	275.5

of capacity utilization. As usual, we define backlog for bidder i at date t as the sum of work remaining among projects l won by i up to t, where work remaining on project l at date t is defined as total project size (measured by the engineer's estimate) times the proportion of scheduled project days remaining at date t. Note that number of bids submitted by any given bidder is small relative to the number of auctions in the marketplace, with even large bidders competing in less than fifteen percent of total auctions on average.

Finally, Figure 1 plots the histogram (over all bidders i and lettings t) of the number of bids submitted by bidder i in letting t.[24] As evident from Figure 1, more than 55 percent of active bidders submit multiple bids in the same letting. Despite this, it is relatively uncommon for a typical bidder to compete in a large number of auctions; roughly 92 percent of bidders in our sample bid in 6 or fewer auctions and only 2.5 percent bid in more than 10.

[24] An observation for the purposes of Figure 1 is thus a bidder-letting pair.

Figure 1: Number of Simultaneous Bids Submitted, Bidder by Letting



4.2 Descriptive regressions

We next explore a series of simple regressions designed to explore the economic implications of simultaneous bidding in the MDOT marketplace. The unit of analysis in these regressions is a bidder-auction-round combination. The dependent variable is the log of bid submitted by bidder i in auction l in letting t, regressed on a vector of covariates intended to capture the effect of own- and cross-auction characteristics on i's bid in auction l.

Regression specification As usual, we control for a number of auction-level characteristics which we expect to be key direct determinants of i's bid in auction l: the size of auction l, captured by the MDOT engineer's estimate of project cost, the level of competition i faces in auction l, and the distance between project l and i's base of operations.[25] To control for the direct cost effects of capacity usage, we also include a standardized bidder-level backlog variable, derived from the backlog measure described above by subtracting the mean and

[25] We construct for each bidder-project pair the minimum straight-line distance (in miles) between any of i's plants and the centroid of the county in which project l is located. We take the shortest distance if bidder i owns multiple plants.

dividing by the standard deviation of backlogs for each bidder over time.

To explore cross-auction interaction in the MDOT marketplace, we seek a set of covariates relevant for combination payoffs but irrelevant for standalone valuations after conditioning on characteristics of auction l. Toward this end, we construct the following covariates.

To control for cross-auction competition which may shift combination win probabilities, we consider the total number of rivals across all auctions played by bidder i. The effect of cross-auction competition on i's bids in auction l is theoretically ambiguous, depending both on the sign of κ_i and on strategic responses by rival bidders. Heuristically, however, if objects are substitutes, we expect greater competition in auction k to increase marginal returns to winning auction l, and conversely if objects are complements.

To capture the presence of capacity constraints or diseconomies of scale, we consider two variables. First, as a direct measure of total project size, we consider the (log of) the sum of engineer's estimates across all auctions in which i is bidding. Second, as a measure of the degree of schedule overlap on projects for which i is bidding, we consider the total number of overlapping days for projects for which i submits bids, scaled by the sum of days scheduled for each of these projects. Insofar as marginal costs are increasing in capacity utilization, we expect the coefficients on these variables to be positive.

In principle, complementarities arising between similar projects may differ from those arising between different projects. To account for this possibility, we consider an index of concentration for the types of projects for which i is bidding, defined as a Herfindahl index over shares of each project type in i's participation set. A negative coefficient on this index is interpreted as a relative complementarity between similar projects.

Finally, to measure potential economies or diseconomies induced by distance between projects, we consider the (log of) total distance between the current project and each other project for which i bids, normalized by the total distance between each of these projects and the closest plant owned by bidder i. Insofar as relatively more distant projects potentially reduce economies of scale, we expect this variable to have a positive sign.

Regression results Table 4 reports OLS estimates for our baseline regression specifications: log bids on the own- and cross-auction characteristics defined above. We include a full set of bidder type, project type, and letting date indicators, with standard errors clustered by bidder and round to allow for correlation in elements of b_{it}. We also consider a specification with bidder identity rather than bidder type fixed effects.

Estimated effects of own-auction characteristics correspond closely both to our prior and to findings elsewhere in the literature. As expected, bids are increasing almost one for one in project size, with the coefficient on log engineer's estimate exceeding 0.97. Bidders facing more competition bid more aggressively, with one additional competitor associated with a 4 to 5 percent decrease in average bids. Finally, a one percent increase in i's distance to the project leads to about a 2 percent increase in i's bid on average.

More importantly, estimated cross-auction effects are also significant, with magnitudes stable across specifications and signs broadly consistent with our prior expectations. The positive coefficient on log sum of engineer's estimates suggests that competing for many large projects leads to a substantial decrease in aggressiveness by bidder i in auction l, with the negative coefficient on same-type projects suggesting that this effect is ameliorated slightly when the two projects are of the same type. Similarly, the positive sign on log distance among projects suggests that increasing distance to other projects reduces the synergies among them. Finally, the significant negative coefficient on total number of rivals in auctions participated by i suggests that facing more competition across auctions leads bidder i to bid more aggressively in auction l. Taken together, these results corroborate the hypothesis that simultaneous bidding induces strategic spillovers across auctions.

5 Structural estimation of complementarities

We now turn to this paper's primary interest: structural estimation of the function $\kappa_i(\cdot)$ describing preferences over combinations. In principle, the results in Section 3 support fully

Table 4: OLS Estimates of Cross-Auction Effects

$y = ln(bid)$	1	2
Log engineer's estimate	0.9708***	0.9763***
	(0.0018)	(0.0018)
Log number of rivals	-0.0496***	-0.0355***
	(0.0049)	(0.0047)
Log distance to project	0.0213***	0.0135***
	(0.0013)	(0.0012)
Log days to project start	0.004***	0.0036**
	(0.0016)	(0.0017)
Standardized backlog	0.0023**	0.0024***
	(0.0009)	(0.001)
Log number of big rivals faced	0.0016	0.0101*
	(0.0056)	(0.0054)
Log number of regular rivals faced	0.0238***	0.0229***
	(0.0041)	(0.004)
Multiple-bid indicator	-0.0897***	-0.175***
	(0.0273)	(0.0256)
Log sum engineer's estimate across played auctions	0.0058***	0.0114***
	(0.0019)	(0.0018)
Log sum number of rivals across played auctions	-0.0146***	-0.0123***
	(0.0028)	(0.0025)
Log distance across played projects	0.0037*	0.0037*
	(0.0022)	(0.002)
Fraction overlapping time across projects	0.0189***	0.0148***
	(0.0054)	(0.0055)
Same-type-auctions concentration index	-0.011*	-0.0284***
	(0.006)	(0.0058)
Big bidder	-	0.0093**
	-	(0.0043)
Regular Bidder	-	-0.0031
	-	(0.0024)
Year FE, Month FE, Auction type FE	YES	YES
Bidder type FE	NO	YES
Bidder ID FE	YES	NO
R-squared	98.02	97.78

Unit of analysis is bidder-auction-round, with standard errors clustered by bidder within each round. There are 41,524 observations. Variables *log of engineer's estimate*, *log of number of rivals in the auction* and *log of distance to the county centroid* measure size, strength of competition, and distance to project l respectively. Remaining variables capture cross-auction characteristics: number of rivals in other auctions, sum engineer's estimate, distance to auctions scaled by distance to project l in which i is competing and number of overlapping days among projects scaled by the total number of days to completion.

non-parametric estimation of κ_i. In practice, of course, the dimensionality of the problem renders this infeasible. We therefore implement our structural procedure in two steps. First, following Cantillon and Pesendorfer (2006) and Athey et al. (2011) among others, we estimate a parametric approximation to the equilibrium distribution G_{it} of bids submitted by each bidder i in letting t. Second, we map these estimates through the inverse bidding function (3) to obtain a set of moment conditions based on the exclusion restrictions discussed in Section 3, which we then use to estimate parameters in κ_i. Following Groeger (2014), we assume there is no binding reserve price.[26]

5.1 First step: estimation of G_{it}

As usual, the first step in our procedure is to estimate the conditional joint distribution G_{it} and pdf g_{it} of bids submitted by each bidder i in letting t. In view of the dimensionality of this problem, we follow Cantillon and Pesendorfer (2006) and Athey et al. (2011) in estimating a parametric approximation to this joint distribution, which we specify as follows. We model the $L_{it} \times 1$ bid vector b_{it} as drawn from a multivariate log-normal distribution characterized by mean vector μ_{it} and variance-covariance matrix Σ_{it}:

$$\ln(b_{it}) \sim g(\mu_{it}, \Sigma_{it}).$$

In theory, each bidder's equilibrium bid function depends not only on the bidder's own characteristics and the characteristics of the projects for which it bids, but but also on competitors' characteristics and the characteristics of all the auctions where they participate. In practice, it will be impossible to condition on all theoretically relevant variables, so we propose a parsimonious specification where we choose variables in each category guided by the reduced form analysis in Section 4. Thus we allow the parameters μ_{it} and Σ_{it} to depend on a vector of observables including i's characteristics Z_{it}, project characteristics

[26] When a bidder is the sole participant (which happens only 136 times out of 8824 auction analyzed), he will face MDOT that draws a completion cost from a fringe bidder's cost.

X_{lt}, combination characteristics W_{it}, and the number and types of rivals i faces within and across auctions. Specifically, for each auction $l = 1, ..., L_{it}$ played by i, we model the mean and variance of $\ln(b_{it,l})$ as $\mu_{it,l} = \alpha \cdot M^{\mu}_{it,l}$ and $\sigma^2_{it,l} = \exp(\beta \cdot M^{\sigma}_{it,l})$ respectively, where $M^{\mu}_{it,l}$ and $M^{\sigma}_{it,l}$ are vectors of covariates specified in Panels A and B of Table 5, and α and β are parameter vectors to be estimated. Meanwhile, we model the covariance $\rho_{it,kl}$ between distinct elements $\ln(b_{it,k})$ and $\ln(b_{it,l})$ of $\ln(b_{it})$ as

$$\rho_{it,kl} = \frac{\exp(\gamma \cdot M^{\rho}_{it,kl} - 1)}{\exp(\gamma \cdot M^{\rho}_{it,kl} + 1)},$$

where $M^{\rho}_{it,kl}$ is a vector of interactions between observable characteristics of projects k and l specified in Panel C of Table 5, and γ is a vector of parameters to be estimated.[27]

We estimate this first-step model by maximum likelihood, pooling data from bidders that participate in different numbers of auctions. The first-step log-likelihood function is therefore equal to:

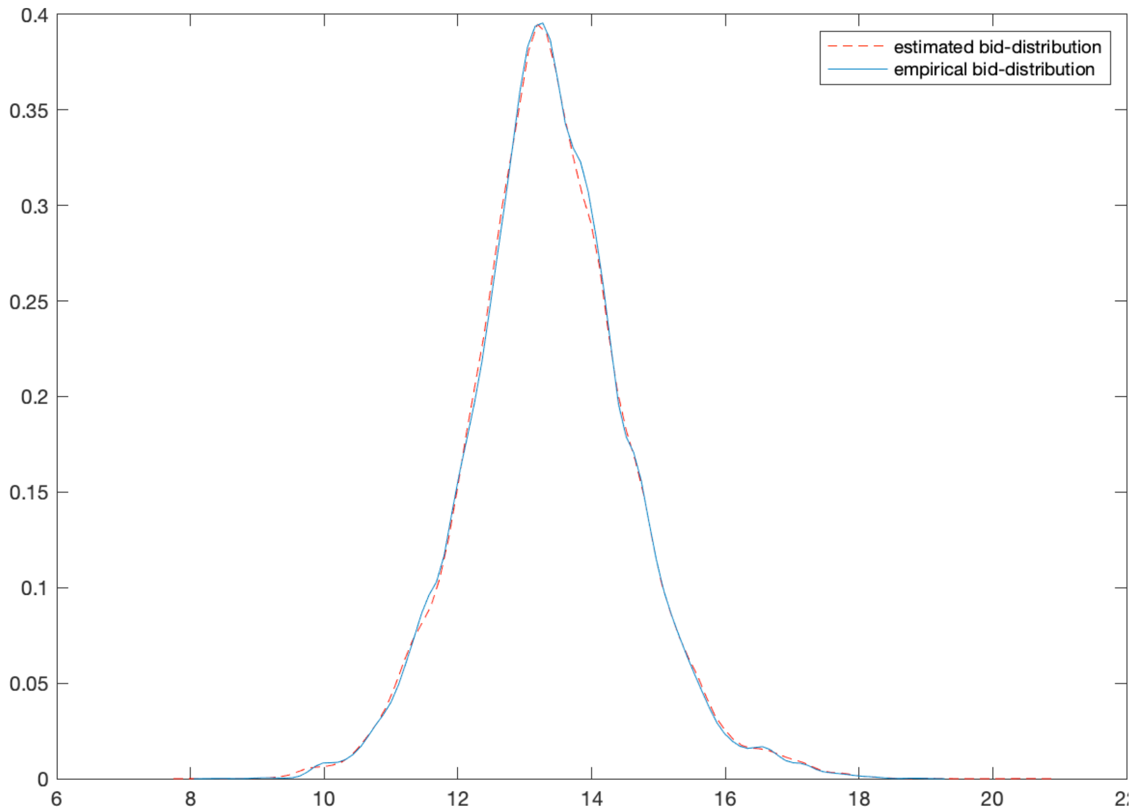
$$\sum_{t=1}^{T}\sum_{i=1}^{N_t} \log g(b_{it}|\mu_{it}, \Sigma_{it}).$$

Table 5 reports maximum likelihood estimates of the first-step parameters (α, β, γ) determining the distribution $g(\mu_{it}, \Sigma_{it})$. In Panel A, we report coefficient estimates $\hat{\alpha}$ on covariates $M^{\mu}_{it,l}$ appearing in the mean function μ_{it}; not surprisingly, these are very similar to coefficients in our descriptive regressions. Panel B reports coefficients $\hat{\beta}$ on covariates $M^{\sigma}_{it,l}$ in the variance function $\sigma^2_{it,l}$, which suggest that bidders competing in multiple auctions and for larger projects submit less dispersed bids.[28] Finally, in Panel C, we report coefficients $\hat{\gamma}$ on covariates $M^{\rho}_{it,kl}$ in the covariance function $\rho_{it,kl}$. These suggest at least two broad patterns in bidding behavior across auctions. First, bidders bid tend to bid more similarly for projects in the same county or of the same type. Second, when competing for projects

[27] Since at this stage we model the distribution of bids conditional on observables as continuous with respect to continuous characteristics, we implicitly assume continuity of the equilibrium selection mechanism, which represents a strengthening of Assumption 3. For further discussion on the continuity of equilibrium selection in games with multiple equilibria, see de Paula (2013) and Aguirregabiria and Mira (2008).

[28] While the parametrization of Σ_{ilt} does not imply its positive semi-definitiveness, the estimated variance-covariance matrix is positive semi-definite.

Figure 2: Predicted versus actual distribution of log bids



whose schedules overlap, bidders tend to bid for one more aggressively than the other. This is consistent with our prior that overlapping schedules exacerbate diseconomies of scale.

To evaluate goodness of fit of this first-step model, Figure 2 plots the observed distribution of log bids across all auctions and bidders, together with predicted distribution of log bids implied by the estimates in Table 5. As can be seen in Figure 2, the fit of our parametric approximation appears excellent, reinforcing confidence in the first-step estimates above.

5.2 Second step: estimation of complementarities

In view of our low-bid procurement application, we here reinterpret the general model in Section 2 as follows. Let V_{itl} be i's private standalone cost for completing project $l \in \mathcal{L}_{it}$, and $\kappa_i(Z_{it}, W_{it})$ be the vector of cost complementarities associated by bidder i with each combination $\omega \in \Omega_{it}$. We adopt the convention that $\kappa_i^{\omega}(Z_{it}, W_{it}) > 0$ means that winning

Table 5: First-step MLE estimates of parameters in G_i

Mean μ_{ilt}	$\hat{\alpha}$	MLE SEs	95% CI	
Auction l and bidder characteristics				
Constant	0.3766	0.0158	0.3456	0.4076
Log engineer's estimate	0.9769	0.0008	0.9753	0.9785
Log rivals in auction	-0.0352	0.0027	-0.0405	-0.0299
Log distance to project	0.0129	0.0009	0.0111	0.0147
Log days to the start	0.0039	0.0008	0.0023	0.0055
Standardize backlog	0.0024	0.001	0.0004	0.0044
Big bidder	0.0023	0.0044	-0.0063	0.0109
Regular bidder	-0.0023	0.0025	-0.0072	0.0026
Log number of big rivals faced	0.0104	0.003	0.0045	0.0163
Log number of regular rivals faced	0.0237	0.0021	0.0196	0.0278
Bidder Type FE	YES	-	-	-
Auction Type FE	YES	-	-	-
Other auctions characteristics				
Multiple bids dummy	-0.1728	0.0206	-0.2132	-0.1324
Same-type-auctions index	-0.0292	0.005	-0.039	-0.0194
Fraction overlapping time	0.0148	0.0035	0.0079	0.0217
Log sum engineer's (across l)	0.0113	0.0014	0.0086	0.014
Log sum rivals (across l)	-0.012	0.002	-0.0159	-0.0081
Log distance across played projects	0.0032	0.0017	-0.0001	0.0065
Year FE	YES	-	-	-
Month FE	YES	-	-	-

Variance σ^2_{ilt}	$\hat{\beta}$	MLE SEs	95% CI	
Constant	0.0652	0.0723	-0.0765	0.2069
Multiple bids dummy	-0.2211	0.0189	-0.2581	-0.1841
Log engineer's estimate	-0.2582	0.0053	-0.2686	-0.2478

Covariance ρ_{iklt}	$\hat{\gamma}$	MLE SEs	95% CI	
Constant	0.0052	0.0005	0.0042	0.0062
Same county projects	0.0033	0.0006	0.0021	0.0044
Same type projects	0.0019	0.0004	0.0011	0.0027
Fraction overlapping time	-0.001	0.0005	-0.002	0

combination ω increases bidder i's joint completion costs, while $\kappa_i^\omega(Z_{it}, W_{it}) < 0$ means that winning combination ω decreases bidder i's joint completion costs.

While, in theory our identification argument allows for $\kappa_i^\omega(Z_{it}, W_{it})$ to be fully nonparametric, in practice the high dimensionality of κ_i^ω renders nonparametric inference infeasible. We therefore adopt a parsimonious parametric structure in which the complementarity i associates with combination ω is modeled as a linear index of a $1 \times Q$ vector of bidder and combination-level observables M_{it}^ω which includes the total size, the distance among projects, the overlapping time between projects, the Herfindahl index of project types in combination ω, and a set of dummies for type of bidder i:

$$\kappa_i^\omega(Z_{it}, W_{it}) = M_{it}^\omega \theta_0, \tag{11}$$

where $\theta_0 \subset \Theta$ is a $Q \times 1$ vector of structural parameters to be estimated. With slight abuse of notation, let M_{it}^κ be the $2^{L_{it}} \times Q$ matrix whose rows collect covariate vectors M_{it}^ω describing each combination $\omega \in \Omega_{it}$.[29] By construction, under (11), we then have $\kappa_i(Z_{it}, W_{it}) = M_{it}^\kappa \theta_0$.

We consider estimation maintaining both Assumptions 5 and 6, exploiting variation in rival characteristics Z_{-it}, characteristics of other auctions X_{-lt}, and combination characteristics W_{it} to identify θ_0. Specifically, letting b_{it} denote the vector of bids submitted by bidder i in letting t, and rewriting this bidder's standalone cost realization $v_{itl} = E[V_{itl}|X_{lt}, Z_{it}] + \epsilon_{itl}$ without loss of generality, we have almost surely that

$$\begin{aligned} E[V_{itl}|X_{lt}, Z_{it}] + \epsilon_{itl} &= \Upsilon_{itl}(b_{itl}|M_t) - \Psi_{it,l}(b_{it}|M_t) \cdot \kappa_i(Z_{it}, W_{it}) \\ &= \Upsilon_{itl}(b_{itl}|M_t) - \Psi_{it,l}(b_{it}|M_t) M_{it}^\kappa \cdot \theta_0, \end{aligned} \tag{12}$$

where the second line substitutes the parametric form $\kappa_i(Z_{it}, W_{it}) = M_{it}^\kappa \theta_0$ assumed above.

We aim to estimate the parameters θ_0 governing complementarities. Toward this end, recall that the objects $\Upsilon_{itl}(b_{it,l}|M_t)$ and $\Psi_{it,l}(b_{it}|M_t) M_{it}^\kappa$ appearing on the right-hand side

[29] If ω contains only one object, then of course M_{it}^ω and $\kappa_i^\omega(Z_{it}, W_{it})$ are taken to be zero.

of (12) are "observable," in the sense that they depend deterministically on the observed bid vector b_{it}, the combination characteristics M_{it}^{κ}, and the equilibrium bid distribution estimated in Section 5.1. By contrast, both terms on the left-hand side of (12) are ex ante unknown. The expected standalone cost $E[V_{it,l}|X_{lt}, Z_{it}]$ is an unknown function of the characteristics (X_{lt}, Z_{it}) which we ultimately aim to eliminate through matched differences as described below. The residual ϵ_{itl} reflects the difference between bidder i's standalone cost realization v_{itl} and the average cost function $E[V_{itl}|X_{lt}, Z_{it}]$. Since i's equilibrium bid vector b_{it} depends on i's standalone cost vector v_{it}, the residual ϵ_{itl} will be correlated with the "observables" $\Upsilon_{itl}(b_{it,l}|M_t)$ and $\Psi_{it,l}(b_{it}|M_t)M_{it}^{\kappa}$. Under Assumptions 5 and 6, however, we have that $E[V_{itl}|M_t] = E[V_{itl}|X_{tl}, Z_{it}]$. These conditions in turn imply that the residuals ϵ_{itl} in (12) satisfy the key conditional moment restriction $E[\epsilon_{itl}|M_t] = E[\epsilon_{itl}|X_{lt}, Z_{it}] = 0$.

If the mean function $E[V_{itl}|X_{tl}, Z_{it}]$ were known, or were of known parametric form, then one could immediately translate these orthogonality restrictions on ϵ_{itl} into a GMM strategy for estimation of θ_0, using functions of the covariates M_{it}^{κ} and other elements of the market structure vector M_t as instruments for the endogenous "regressors" $[\Psi_{itl}(b_{it}|M_t)M_{it}^{\kappa}]$ multiplying θ_0 in (12). Observe that the covariates M_{it}^{κ} enter $[\Psi_{it,l}(b_{it}|M_t)M_{it}^{\kappa}]$ directly, while other market characteristics M_t enter $[\Psi_{it,l}(b_{it}|M_t)M_{it}^{\kappa}]$ through the combinatorial win probability gradient $\Psi_{it,l}(b_{it}|M_t)$. Thus any element of $(Z_{-i,t}, X_{-l,t}, W_t)$ which shifts $\Psi_{it,l}(b_{it}|M_t)$ will be a relevant instrument for the endogenous vector $[\Psi_{it,l}(b_{it}|M_t)M_{it}^{\kappa}]$ multiplying θ_0 in (12).

In practice, however, the mean function $E[V_{itl}|X_{tl}, Z_{it}]$ is unknown, and we aim to estimate without further parametric restrictions. To proceed, therefore, we must first eliminate the unknown function $E[V_{itl}|X_{tl}, Z_{it}]$ from the left-hand side of (12). Toward this end, paralleling our identification procedure, we employ a matched pairwise differencing strategy in the spirit of Honoré and Powell (2005) and Aradillas-Lopez et al. (2007). Specifically, for any distinct bidder-auction observations itl and $j\tau s$, define the differenced residual

$$\eta_{itl,j\tau s} = (\Upsilon_{il,t}(b_{il}|M_t) - \Upsilon_{js}(b_{js}|M_\tau)) - \left[\Psi_{it}(b_i|M_t)M_{it}^{\kappa} - \Psi_{j\tau}(b_{j\tau}|M_\tau)M_{j\tau}^{\kappa}\right] \cdot \theta_0. \qquad (13)$$

Then for any two observations itl, $j\tau s$ matched such that $x_{tl} = x_{\tau s}$ and $z_{it} = z_{j\tau}$, we have $\eta_{itl,j\tau s} = \epsilon_{itl} - \epsilon_{j\tau s}$, and therefore $E[\eta_{itl,j\tau s}|M_t, M_\tau] = 0$. At the same time, this allows unmatched variables $(Z_{-it}, X_{t,-l}, W_t)$ to differ from $(Z_{-j,\tau}, X_{\tau,-s}, W_\tau)$, with all of these variables relevant instruments for the endogenous difference term multiplying θ_0 in (13).

In implementing this pairwise differencing strategy, we match on both a set of discrete covariates denoted by y^d_{itl} and a set of continuous covariates denoted by y^c_{itl}. Discrete covariates y^d_{itl} incude year, month, regular, bidder type, project type, number of plants owned by each bidder, backlog[30] and a dummy indicating whether the project starts in the next 180 days. Meanwhile, continuous covariates y^c_{itl} are size and distance, all standardized to have mean zero and standard deviation one. For discrete covariates y^d, we employ exact matching, which effectively splits the whole dataset into a finite number of subgroups, among which we form all non-redundant matches. Let $\hat{D}_n$ be a subgroup defined by the discrete covariates and $\hat{\mathcal{D}} = \{\hat{D}_1, ..., \hat{D}_n, ..., \hat{D}_{|\hat{\mathcal{D}}|}\}$ the collection of these subgroups. Within each subgroup of discrete matches, we then use a Gaussian product kernel to assign weights to each potential match on the basis of differences in their continuous covariates y^c, scaling bandwidths for each covariate proportionally to Scott's rule of thumb based on the size of each subgroup.

Given the sample of weighted matched pairs thus formed, we proceed as follows. For each bidder in the estimation sample, we construct empirical analogs of the objects $\hat{\Upsilon}_{itl}$ and $\hat{\Psi}_{it}$ of the equilibrium objects $\Upsilon_i(b_{itl}|M_t)$ and $\Psi_i(b_{it}|M_t)$ from our first-step bid distribution estimates $(\hat{G}_i(\cdot|M_t))_{i=1}^{N_t}$, approximating gradients using finite differences.[31] Plugging in these first-step estimates $\hat{\Upsilon}_{itl}$ and $\hat{\Psi}_{it}$ into (13), we obtain an estimated residual $\hat{\eta}_{itl,j\tau s}$ for each pair of bidders in our matched sample. We form moments based on interactions between these weighted matched differenced residuals $\hat{\eta}_{itl,j\tau s}$ and a vector of instruments $I_{itl,j\tau s}$ (at least Q-dimensional) formed from M_t and M_τ.[32]

[30] For purposes of this matching procedure, we discretize backlog in three categories: low (up to 25th percentile), medium, and high (above 75th percentile).

[31] In practice, a small number of estimated $\hat{\Psi}_{it}$ and $\hat{\Upsilon}_{itl}$ are either very small or very large. To prevent bias from these outliers, we trim the top and bottom 2.5 percent of values in each of $\hat{\Psi}_{it}$ and $\hat{\Upsilon}_{itl}$.

[32] The instruments used are of three types: Z_i/W_i-type instruments such as the individual characteristics (big, regular, bidder type dummies), the sum of the characteristics for the maximum possible number of

Based on these moments, we estimate θ_0 using a two-step efficient GMM procedure. Formally, the resulting GMM objective function is

$$\hat{\theta}_0 = \arg\min_{\theta} \widehat{m}(\theta)' W^{-1} \widehat{m}(\theta), \tag{14}$$

where omitting the normalization factor for simplicity,

$$\widehat{m}(\theta) = \sum_{\hat{D}_n \in \hat{\mathcal{D}}} \sum_{\substack{x \in \hat{D}_n \\ k \neq x \in \hat{D}_n}} \frac{1}{h^{(1)}(y_x^d) \cdot h^{(2)}(y_x^d)} \times R\left(\frac{y_x^{c,(1)} - y_k^{c,(1)}}{h^{(1)}(y_x^d)}, \frac{y_x^{c,(2)} - y_k^{c,(2)}}{h^{(2)}(y_x^d)} \right) I'_{x,k} \hat{\eta}_{x,k} \tag{15}$$

where $x \equiv itl$ and $k \equiv j\tau s$ denote distinct bidder-letting-auction observations, $y_x^d = y_k^d$ for all $x \in D_n$ and $k \neq x \in D_n$, $h^{(1)}(y_x^d)$ and $h^{(2)}(y_x^d)$ are the bandwidths and R is a bivariate Gaussian product kernel defining continuous matching weights. Finally, the standard errors are adjusted to account for the first step estimation as in Newey and McFadden (1994).

Appendix H reports the results of two Monte Carlo simulation studies exploring the performance of this weighted matched-difference GMM procedure. These confirm that our matching procedure can recover informative estimates of complementarities even in moderately sized samples. We refer interested readers to Appendix H for further details.

5.3 The main result: structural estimates of θ_0

Table 6 reports the main structural estimates derived from the two-step matched-difference GMM procedure outlined above. Coefficient magnitudes are in millions of dollars, with negative signs reflecting lower costs and positive signs reflecting higher costs. Bearing these

auctions and the average of the combinatorial-auction characteristics for all possible combinations, the log sum of engineer estimates interacted with regular- and big-bidder dummies, log sum of distance and log overlapping time across the maximum number of auctions. Moreover, we use the total number of auctions in which the bidder participates, the Z_{-i}-type instruments such as the total number of rivals across auctions, the total number of big rivals across auctions, and the total number of regular rivals across auctions. Finally, we use X_{-l}-type instruments such as the log sum of engineer estimates and distance across all **other** auctions (but for the current auction).

Table 6: Estimated complementarity parameters θ_0

Combination characteristics (Elements of W)	$\hat{\theta}$	SE
Fraction overlapping time across projects	0.1002***	0.0385
Distance across played projects	-0.0001	0.00007
Sum engineer's estimate in millions	0.0272**	0.0115
Same-type-auctions index	-0.1616**	0.0776
Regular Bidder	-0.3830**	0.1666
Big Bidder	0.1507	0.1825
Bidder Type FE	YES	-

Coefficient magnitudes are in millions of dollars, positive coefficients imply higher completion costs associated with a combination win.

conventions in mind, these coefficients have the following economic interpretations.

The variable "Sum of engineer's estimates" reflects the total size of projects in a combination, with a positive coefficient suggesting that more total work renders a joint win less valuable, as we would expect in the presence of capacity constraints. The coefficient on "Fraction overlapping time" measures the effect of schedule overlap on costs, with magnitude suggesting that perfect schedule overlap increases the average completion costs by about $\$100,200$. Meanwhile, the coefficient on "Same-type auction index" suggests that more homogeneous combinations are less costly; a 0.1 change in the Herfindahl index of project types reduces costs by $\$16,160$. Regular bidders seem to have positive cost synergies. With the exception of the coefficient on distance between projects, which is negative although small and not statistically significant, these effects are all natural and consistent with our priors. While not reported in Table 6, we also include a vector of bidder type dummies in $\kappa(\cdot)$; signs on these vary, but suggest negative complementarities on aggregate as we quantify next.

To illustrate the economic significance of these parameter estimates, we next translate the parameter estimates $\hat{\theta}$ in Table 6 into estimates for the underlying complementarities $\kappa(\cdot)$ themselves. Specifically, we first construct, for each bidder i in the sample, the estimated complementarity associated with i winning the combination of all projects for which they bid. We then normalize this complementarity by the total size of projects in this combination,

Table 7: Empirical distribution of normalized complementarities across bidders

Decile rank	Decile of normalized complementarities
10th	-0.2380
20th	-0.1358
30th	-0.0794
40th	-0.0428
50th	-0.0165
60th	0.0038
70th	0.0211
80th	0.0512
90th	0.1079

"Normalized complementarity" denotes the estimated complementarity $\kappa^{\omega}(Z_i, W_i; \hat{\theta})$ among all projects bid by i divided by the sum of engineer's estimates for these projects, with deciles evaluated over the empirical distribution of (Z_i, W_i). Negative numbers mean lower costs.

and analyze the deciles of these normalized complementarities across bidders.

Table 7 summarizes the results of this procedure, reporting deciles of normalized complementarities for all bidders in our MDOT sample. As evident from Table 7, there is substantial heterogeneity in complementarities across bidders in the MDOT sample, with a joint win leading to cost *savings* of approximately 23.8 percent of combination size at the 10th quantile of complementarities across bidders, transitioning to cost *increases* of approximately 11 percent at the 90th quantile. Recalling the parameter estimates in Table 6, we view these pattern as consistent with an underlying U-shaped cost curve, with completion costs falling until firm resources are fully employed and rising thereafter.

We conclude this section with a note on interpretation of Tables 6 and 7 under endogenous entry. In Appendix B, we embed the bidding model considered here within a fully specified entry (interpreted as a process of value discovery upon costly entry) and bidding game, showing that our estimation strategy is robust to this extension. Hence the parameter estimates reported in Table 6 remain valid even under entry. In interpreting Table 7, however, it is important to note that the distribution of complementarities among projects *in which* *bidders enter* will differ from that which would arise if projects were randomly assigned. In particular, insofar as bidders tend to bid for complementary combinations, we would expect the distribution in Table 7 to be positively selected.

6 Counterfactuals

While the simultaneous first-price auction is clearly inefficient when bidders have combinatorial preferences, little is known about the magnitude of these inefficiencies in practice. Furthermore, little is known either theoretically or empirically about the revenue properties of the simultaneous first-price auction (FPA) mechanism relative to other feasible multi-object mechanisms such as the Vickrey-Clarke-Groves (VCG) auction.

As a first step toward answering these questions, we compare revenue and efficiency under MDOT's actual simultaneous low-price form with counterfactual outcomes which would have arisen under a combinatorial VCG auction. While the VCG mechanism is guaranteed to reduce social costs, its effect on payments to bidders is very much an open question. Especially in the context of multi-object auctions, the mechanism design literature has noted that VCG can yield poor outcomes for the auctioneer. See, for example, Ausubel and Milgrom (2006), who highlight several potential weaknesses of the VCG mechanism when bidder preferences involve synergies, including the possibility of zero revenue even with competitive bidding. Other relevant features of the auction environment, such as bidder asymmetry, may also lead FPA to generate higher revenue than VCG.[33] Theoretically, therefore, there are two open questions which this counterfactual aims to address. First, by how much does the VCG mechanism reduce social costs? Second, does VCG reduce or increase procurement costs relative to simultaneous FPA, and if so by how much?

Since both the number of combinations and the number of potential allocations increase exponentially in the number of auctions played, it is unfortunately infeasible to solve for VCG outcomes on the full MDOT sample. We therefore focus on the subsample of 5481 self-

[33] See, for example, Krishna (2009) for one example of this effect.

Table 8: Combinatorial VCG outcomes versus simultaneous FPA outcomes

Mechanism	Outcome	Estimate	Std Err
Combinatorial VCG	Completion cost per auction (in dollars)	$1,311,862$	$33,898$
	MDOT payments per auction (in dollars)	$1,617,044$	8436
Simultaneous FPA	Completion costs per auction (in dollars)	$1,396,127$	5686
	MDOT payments per auction (in dollars)	$1,599,995$	–

Results based on $R = 60$ draws of first-step estimated parameters, for the self-contained sample of auctions such that no bidder in any auction . For each draw, we

contained auctions such that no bidder is competing against a rival bidding in more than 12 auctions.[34] For this counterfactual sample, we consider $R = 60$ simulation replications. In each replication, we draw a new set of Step 1 distributional parameters from their asymptotic distribution and re-estimate complementarity parameters as in Step 2.[35] For each replication, we then estimate standalone costs for each bidder i and letting t in the counterfactual sample by mapping i's observed bid b_{it} through the inverse bid function (2), given the relevant complementarity estimates.[36] We simulate allocations, costs of project completion, and payments to bidders under both the baseline simultaneous FPA and the counterfactual combinatorial VCG, computing final completion costs inclusive of complementarities in both cases.[37] Finally, we take means and standard deviations of per-auction payments and costs across replications to obtain our final counterfactual results, reported in Table 8.

Two patterns emerge from this exercise. First, as expected, the simultaneous first-price mechanism is socially inefficient, generating expected social costs of roughly \$1.40 million

[34]To construct this self-contained sample, we first drop all bidders competing in more than 12 auctions. We then drop any bidder facing a rival (in any auction) who is dropped, and proceed recursively in this fashion until no further bidders are dropped. This recursive procedure alleviates the curse of dimensionality inherent in the VCG allocation problem while ensuring that counterfactual VCG outcomes are comparable to actual FPA outcomes, in the sense that every bidder in the VCG counterfactual is bidding in the same auctions against the same rivals as in the actual data. The resulting counterfactual sample consists of 5481 of our original 8224 auctions, representing approximately 24,000 of our original 41,000 bid-level observations.

[35]Since resampling bidders could lead to potentially large changes in the definition of our self-contained counterfactual subsample, we hold the sample of bidders fixed across counterfactual replications.

[36]In practice, a small fraction of estimated standalone costs are either negative or implausibly large. To prevent bias from these outliers, we windsorize standalone costs at thresholds derived from the 5th and 95th percentiles of relative standalone costs among single-auction bidders.

[37]In these simulations, we set MDOT's effective reserve price for each project equal to 200 percent of the MDOT engineer's cost estimate; other plausible values generate very similar results.

per auction, versus \$1.31 million per auction for the VCG mechanism. In other words, within our counterfactual sample, per-auction completion costs are roughly \$84,000 lower under the VCG mechanism than under the simultaneous FPA mechanism. In both level and percentage terms, this efficiency gain is nontrivial, suggesting that switching mechanisms could lower social project completion costs by approximately 6 percent.[38]

Second, although leading to substantially lower social costs, the VCG mechanism in fact slightly increases MDOT's payments to bidders: \$1.60 million per auction under the baseline simultaneous first-price auction, versus \$1.61 million per auction under VCG. In other words, even though the VCG mechanism is more efficient socially, our estimates suggest that it will increase MDOT's procurement costs by approximately 1 percent. From MDOT's perspective, the simultaneous first-price auction therefore appears to perform quite well relative to leading combinatorial alternatives such as VCG.[39]

As noted above, this pattern of higher procurement costs could arise either through revenue weaknesses of the VCG mechanism in the presence of synergies, or through other potential channels such as asymmetric bidders. To explore how these factors interact to shape VCG payment performance, we also simulated payments to bidders under the VCG mechanism using the same estimated costs as in the full counterfactual simulation, but setting complementarities to zero.[40] We find that, absent complementarities, VCG payments would increase by a further 1.1 percent relative to the baseline reported above. In other words, the overall payment increase observed under VCG reflects a roughly 2.1 percent baseline payment increase driven by asymmetric standalone costs alone, with about half of this baseline increase offset by lower payments induced by the presence of complementarities.[41]

[38] Across counterfactual replications, the standard deviation of per-auction savings is approximately \$34,000 in level terms and 2.1 percent in percentage terms.

[39] This analysis is of course only partial in that we effectively hold entry behavior fixed across mechanisms. By construction, since the VCG auction reduces social costs while increasing MDOT payments, it must also generate greater profit to bidders. In equilibrium, this should translate into greater entry, which could reduce procurement costs relative to our findings above. In contrast, since new entrants are by definition marginal, we expect efficiency gains net of entry to be similar to those reported above.

[40] Note that the complementarities we estimate above still affect estimated standalone costs. The goal is to determine how complementarities affect VCG revenue, holding standalone costs constant.

[41] To gain further insight on factors affecting VCG versus FPA revenue performance, we also conducted

7 Conclusion

Motivated by an institutional framework common in procurement applications, we develop and estimate a structural model of bidding in simultaneous first-price auctions, to our knowledge the first such in the literature. We analyze identification of this model, showing that excluded variation in either characteristics of rival bidders or characteristics of other auctions supports nonparametric identification of cross-object complementarities. Finally, we apply this framework to data on Michigan Department of Transportation highway construction and maintenance auctions. Our estimates suggest that winning a two-auction combination generates cost effects ranging from roughly 23.8 percent cost savings (relative to combination size) at the 10th percentile to roughly 11 percent cost increases at the 90th percentile, with combination costs increasing in joint size of, scheduling overlap between, and heterogeneity in work types in the combination. Despite these substantial complementarities, we find that switching to an efficient Vickrey-Clarke-Groves mechanism would generate only modest social gains: roughly 6 percent savings in social costs of project completion, while slightly increasing MDOT's expected procurement costs. We view this as strong suggestive evidence that simultaneous first-price auctions can perform relatively well even in environments with economically important complementarities. This observation may partially rationalize the widespread popularity of simultaneous first-price auctions in practice.

Appendix A: Proof of Proposition 1

The proof of Proposition 1 rests on two key claims. First, the first-order system (2) must be well-defined for almost every b_i submitted by i, i.e. almost everywhere with respect to the measure induced by $G_i(\cdot|M)$. Second, at almost every b_i at which first order conditions hold, the matrix $\nabla\Gamma_i$ must be invertible. We establish each claim in turn.

several simple numerical simulations in a setting where two asymmetric bidders compete in two auctions, with one or both bidders having a positive complementarity. The results, reported in Appendix H.3, broadly confirm that either revenue ranking is possible depending on the interaction between asymmetry and complementarities, with asymmetry alone typically favoring FPA, and the effects of complementarities varying depending on whether these are assigned to the strong bidder, the weak bidder, or both.

First show that the first order system (2) is well-defined for almost every b_i submitted by i. Recall that we can write bidder i's objective as

$$\pi(v_i, b|K; M) = (\Omega v_i + K - \Omega b)^T P_i(b|M).$$

where v_i and K are given at the time of maximization. Note that the system (2) necessarily holds at any best respose where $\pi(v_i, \cdot|K; M)$ is differentiable and that Assumption 3 implies that each observed b_i is a best response. Hence the system (2) will be well defined for almost every b_i submitted by i if and only if $\pi(v_i, \cdot|K; M)$ is differentiable almost everywhere with respect to the measure on B_i induced by $G_i(\cdot|M)$. But under Assumption 4, $G_i(\cdot|M)$ is absolutely continuous. To establish the claim, it thus suffices to show differentiability of $\pi(v_i, \cdot|K; M)$ a.e. with respect to Lebesgue measure on $\mathcal{B}_i$.

Clearly $(\Omega v_i + K - \Omega b)$ is differentiable in b. Thus differentiability of $\pi(v_i, \cdot|K; M)$ at b is equivalent to differentiability of $P_i(\cdot|M)$ at b. Let B_{-i} be the $L_i \times 1$ random vector describing maximum rival bids in the set of auctions in which i participates. Again applying Assumption 4 to rule out ties, the probability i wins combination ω at bid b is

$$P^\omega(b|M) = \Pr(\{\cap_{\{m:\omega_m=1\}} 0 \le B_{-i,m} \le b_{i,m}\} \cap \{\cap_{\{m:\omega_m=0\}} b_{i,m} \le B_{-i,m} < \infty\}).$$

For each $\omega \in \Omega_i$, let b^ω be the $(\sum \omega) \times 1$ sub-vector of b describing i's bids for objects in ω, B^ω_{-i} be the $(\sum \omega) \times 1$ sub-vector of B_{-i} describing maximum rival bids for objects in ω, and $G^\omega_{-i}(b^\omega|M)$ be the equilibrium joint c.d.f. of B^ω_{-i}. Applying the formula for a rectangular probability and simplifying, we can then represent $P_i(\cdot|M)$ in the form

$$P^\omega_{-i}(b|M) = \sum_{\omega' \in \Omega} a^\omega_{\omega'} G^{\omega'}_{-i}(b^{\omega'}|M),$$

where each $a^\omega_{\omega'}$ is a known scalar (determined by ω, ω') taking values in $\{-1, 0, 1\}$. But by absolute continuity each c.d.f. $G^\omega_{-i}(\cdot|M)$ is differentiable a.e. (Lebesgue) in its support, and interpreted as a function from $\mathcal{B}_i$ to $\mathbb{R}^{L_i}$, each $b^{\omega'}$ is continuously differentiable in b. Thus interpreted as a function from $\mathcal{B}_i$ to $\mathbb{R}$, each $G^{\omega'}_{-i}(b^{\omega'}|M)$ is differentiable on a set of full Lebesgue measure in B_{-i}. The set of points in $\mathcal{B}_i$ at which all $G^{\omega'}_{-i}(b^{\omega'}|M)$ are differentiable is the intersection of points in $\mathcal{B}_i$ at which each $G^{\omega'}_{-i}(b^{\omega'}|M)$ is differentiable, i.e. the intersection of a finite collection of sets of full Lebesgue measure in $\mathcal{B}_i$. But from above differentiability of $G^{\omega'}_{-i}(b|Z, W)$ for all ω' implies differentiability of $P^\omega_{-i}(b|M)$. Hence $P^\omega_{-i}(\cdot|M)$ is differentiable on a set of full Lebesgue measure in $\mathcal{B}_i$. This in turn implies differentiability of $\pi(v_i, \cdot|K; M)$ a.e. with respect to the measure on $\mathcal{B}_i$ induced by $G_i(\cdot|M)$, as was to be shown.

We next establish that the first-order system (2) must yield a unique solution $\tilde{v}$ for almost every b_i submitted by i. Let $\tilde{B}_i$ be the set of points in $\mathcal{B}_i$ at which $\pi(\cdot, \cdot|K; W, Z)$ is differentiable in b; from above, $\tilde{B}_i$ is a subset of full Lebesgue measure in $\mathcal{B}_i$. Choosing any $b \in \tilde{B}_i$ and rearranging (2) yields

$$\nabla_b \Gamma_i(b|Z, W)\tilde{v} = \nabla_b \Gamma_i(b|Z, W)b + \Gamma_i(b|Z, W) - \nabla_b P_i(b|W, Z)^T K_i.$$

Hence uniqueness of $\tilde{v}$ is equivalent to invertibility of the $L_i \times L_i$ matrix $\nabla_b \Gamma_i(b|Z, W)$. Recall that $\Gamma_i(b|Z, W)$ is an $L_i \times 1$ vector whose lth element describes the probability that bid vector b wins auction l. Note that $b \in \tilde{B}_i$ rules out ties at b. Thus for $b \in \tilde{B}_i$ the mth element of $\Gamma_i(b|Z, W)$ is the marginal c.d.f. of the maximum rival bid $B_{-i,m}$ in auction m, from which it follows that $\nabla_b \Gamma_i(b|Z)$ is a diagonal matrix whose m, mth element is the marginal p.d.f. of $B_{-i,m}$. Denote this p.d.f. by $g_{-i,m}(b|Z,W)$; recall that by absolute continuity this p.d.f. is well defined. Then $\nabla_b \Gamma_i(b|Z,W)$ will be invertible at b if and only if $g_{-i,m}(b|Z,W) > 0$ for all $m = 1, ..., L_i$.

We aim to show that this latter property is an implication of equilibrium bidding under Assumption 4. Toward this end, recall that by hypothesis of equilibrium play, each submitted bid b_i is a best response to rival play at (Z,W) for some (v,K). Suppose that there exists an $\epsilon > 0$ such that $g_{-i,m}(\cdot|Z,W) = 0$ on $(b_{im} - \epsilon, b_i]$. Then player i could infinitesimally reduce b_{im} without affecting either Γ_i or P_i. Furthermore, if $\Gamma_{im}(b_i|M) > 0$, so that bidder i wins auction m with strictly positive probability at bid b_i, this deviation will *strictly* increase bidder i's profits. Hence we must have either $g_{-i,m}(\cdot|M) > 0$ or $\Gamma_{im}(B_i|M) = 0$ almost everywhere (Lebesgue) in the support of B_i. By absolute continuity of G_i, this in turn implies we must have either $g_{-i,m}(\cdot|M) > 0$ or $\Gamma_{im}(B_i|M) = 0$ for almost every b_i submitted by i. Furthermore, absolute continuity and common lower support jointly imply that we can have $\Gamma_{im}(B_i|M) = 0$ for at most a set of bids of G_i-measure zero. Hence we must have $g_{-i,m}(\cdot|Z,W) > 0$ for G_i-a.e. bid b_i submitted by i.

Since m was arbitrary, we must have $\nabla_b \Gamma_i(b_i|M)$ invertible for G_i-a.e. bid b_i submitted by i. Hence for almost every b_i submitted by i there will exist a unique $\tilde{v}$ satisfying (2) at b_i, given by

$$\tilde{v} = b_i + \nabla_b \Gamma_i(b_i|M)^{-1} \Gamma_i(b_i|M) + \nabla_b \Gamma_i(b_i|M)^{-1} \nabla_b P_i(b_i|M)^T K.$$

The RHS of this expression is identified up to K, establishing the claim.

Appendix B: Entry

In this Appendix, we formally embed the bidding model we describe above within a two-stage entry-plus-bidding model paralleling those considered by Levin and Smith (1994), Krasnokutskaya and Seim (2011), Athey et al. (2011), Moreno and Wooders (2011), Li and Zhang (2015), and Groeger (2014) among others. This discovery process proceeds as follows.

At the beginning of the game, each bidder i is endowed with a $L \times 1$ standalone valuation vector V_i^0 drawn by nature from a joint distribution F_i^0. However, realizations of V_i^0 are *ex ante unknown* to i and can be discovered by i only through costly entry.

Specifically, let $\mathcal{S}$ denote the power set of $\mathcal{L}$, i.e. the set of all sets of auctions in which bidder i could enter. Let $S \in \mathcal{S}$ denote a particular subset of auctions $S \in \mathcal{L}$. Suppose that, at the beginning of Stage 1, each bidder i observes a $2^L \times 1$ vector of private *combinatorial entry costs* C_i, with element C_i^S of C_i describing the total cost i must incur to enter the set of auctions $S \in \mathcal{S}$. This cost vector C_i satisfies the following properties:

Assumption 7 (Private Entry Costs)**.** *For each bidder i, C_i is drawn independently of V_i^0 from cost distribution $F_{C,i}$ with support on a compact, convex set $\mathcal{C}_i \subset R^{2^L}$, with C_i private information, $F_{C,i}$ common knowledge, and cost draws independent across bidders: $C_i \perp C_j$ for all i, j.*

Having observed C_i, bidder i chooses a set of auctions $\mathcal{L}_i \in \mathcal{S}$ in which to enter, pays the corresponding entry cost $C_i^{\mathcal{L}_i}$, and proceeds to Stage 2. Then at the beginning of Stage 2, Bidder i observes the realizations of her standalone valuation vector $V_i \equiv (V_{il}^0)$ for the auctions $l \in \mathcal{L}_i$ in which she has entered. Lastly, bidder i submits a single bid b_{il} for each object l in her entry set $\mathcal{L}_i$. The bidding subgame then proceeds exactly as described in the main text.

While the combinatorial nature of the entry problem renders notation somewhat involved, this model is in fact the natural combinatorial generalization of the canonical entry and bidding models cited above. Specifically, when valuations and entry costs are additively separable and the entry cost distributions $F_{C,i}$ is atomistic, we obtain the mixed-strategy entry models considered by Levin and Smith (1994), Krasnokutskaya and Seim (2011), and Athey et al. (2011). Alternatively, if auctions are separable and entry costs are continuously distributed, we obtain a pure-strategy value discovery model paralleling Moreno and Wooders (2011), Li and Zhang (2015), and Groeger (2014). The structure we consider here generalizes these canonical value-discovery models to our substantially richer setting involving both common-knowledge complementarities and (potentially) combinatorial entry costs.

We now turn to consider equilibrium entry behavior. Following Milgrom and Weber (1985), define a *distributional entry strategy* for player i as a measure σ_i^E over $\mathcal{C}_i \times \mathcal{S}$ whose marginal over $\mathcal{C}_i$ is $F_{C,i}$, with $\sigma^E = (\sigma_1^E, ..., \sigma_N^E)$ a profile of distributional entry strategies. Then any Bayes-Nash equilibrium must have the following form.

Suppressing generic object, bidder, and combination characteristics, let a market structure $M = (S_1, ..., S_N)$ now denote any vector of participation sets for each bidder. Each market structure M gives rise to a different simultaneous bidding subgame, the structure of which parallels that described in the main text. Let $\sigma^M = (\sigma_1^M, ..., \sigma_N^M)$ be any profile of Bayesian Nash equilibrium bidding strategies in the simultaneous bidding subgame arising under market structure M; as in the main text, we focus on pure strategies for simplicity, although this is inessential.

Recall that bidder i's standalone valuations V_i are unknown at the time of entry. Hence, taking subgame strategies σ^M as given, the ex ante (pre-entry) expected profit bidder i associates with market structure realization M is given by

$$\Pi_i(M) = \int_{V_i} \left[(V_i - \sigma_i^M(V_i))^T \Gamma_i(\sigma_i(V_i)|M) + \kappa_i^T P_i(\sigma_i^M(V_i)|M) \right] F_i(dV_i|M), \tag{16}$$

where, as in the main text, the vectors $\Gamma_i(b_i|M)$ and $P_i(b_i|M)$ denote the expected marginal and combinatorial win probabilities i associates with bid vector b_i, taking rival bidding strategies σ_{-i}^M as given.

Now consider the entry decision by bidder i. With slight abuse of notation, let $S_{-i} = (S_j)_{j \neq i}$ denote any realizations of participation sets for i's rivals, so that we may write market structure as $M = (S_i, S_{-i})$. Taking rival entry strategies σ_{-i}^E as given, we may write i's Stage 1 expected payoff from entering auction combination $S_i \in \mathcal{S}$ as

$$\Xi(S, \sigma_{-i}^E) = E[\Pi_i(S, S_{-i})|\sigma_{-i}^E],$$

where the expectation is over ex ante unknown rival entry decisions S_{-i}. Hence, in Stage 1, bidder i will optimally choose the participation set $\mathcal{L}_i$ maximizing her expected payoff net of entry costs:

$$\mathcal{L}_i = \arg\max_{S \in \mathcal{S}} \Xi(S, \sigma_{-i}^E) - C_i^S. \tag{17}$$

The Stage 1 action set for each bidder is the finite set $\mathcal{S}$, and private entry cost vectors C_i, C_j are independent across bidders i, j. Hence, taking the continuation payoff functions $\Pi_1(M), ..., \Pi_N(M)$ as given, by Proposition 1 of Milgrom and Weber (1985) there exists an equilibrium in distributional strategies for the Stage 1 entry game. So long as continuation payoffs $\Pi_1(M), ..., \Pi_N(M)$ are themselves generated from play of a Bayes-Nash equilibrium under every market structure M, this in turn will constitute an equilibrium of the overall entry and bidding game.

In general, equilibrium Stage 1 entry may be in either pure or mixed strategies. If, however, we add the restriction that $F_{C,i}$ is atomless on $\mathcal{C}_i$ for each i, then Proposition 4 of Milgrom and Weber (1985) implies existence of a equilibrium in which bidders play pure entry strategies.

How does this entry and bidding subgame shape our understanding of the bid-stage subgame we consider in the main text? Clearly, when entry is endogenous, each bidder i will tend to enter in combinations for which she expects either high standalone valuations, or positive complementarities, or both. Furthermore, as noted above, entry may be in either pure or mixed strateges, and there may exist many entry equilibria. So long as, however, only one set of equilibrium bidding strategies σ^M is played conditional on realization of each potential market structure M, we may proceed with identification maintaining Assumptions 1-5 as in the main text.

Furthermore, since $C_j \perp V_j$ for all bidders j, the distribution of i's post-entry private information is invariant to the realizations S_{-i} of entry decisions by i's competitors, variation in S_{-i} will be will be *effectively exogenous* and thus excludable with respect to F_i. In other words, so long as there exists at least some variation in rival entry decisions conditional (X, Z_i, W_i)—whether induced by variation in rival entry costs, mixed entry strategies as in Athey et al. (2011) and Levin and Smith (1994), or mixing across entry equilibria—then equilibrium entry will induce precisely the form of variation required for our identification argument. In this sense, the entry model described above provides a formal equilibrium justification for the key exclusion restriction (Assumption 5) on which we base our nonparametric identification argument.

Finally, note that the equilibrium entry conditions (17) in principle provide an additional set of identifying restrictions on complementarities. Specifically, we know that the combination which bidder i actually entered must yield the highest ex ante profit among all combinations which i could have entered. Furthermore, under the hypothesis $K_i = \kappa_i(Z_i, W_i)$, there exists a unique candidate $\hat{\Pi}_i(M, K_i)$ for the ex ante profit function $\Pi_i(M)$, which is identified up to K_i. Exploiting necessary conditions for optimality—for instance, that bidder i should not gain in expectation by adding or removing one auction from her participation set—one could translate (17) into a set of restrictions on complementarities K_i and entry costs C_i jointly. In principle, these would provide further identifying information on K_i. Unfortunately, these conditions also involve numerous high-dimensional integrals, evaluation of which would involve significant computational costs. In practice, we therefore focus on restrictions on K_i induced by the bidding model, without reference to additional restrictions induced by entry.

Appendix C: Unobserved auction heterogeneity

In this Appendix, we discuss how our identification results can be extended to allow for additively separable unobserved auction heterogeneity. Specifically, suppose that bidder i's standalone valuation V_{il} in auction l is

$$V_{il} = U_{il} + A_l,$$

where A_l is the unobserved heterogeneity in auction l, common knowledge to bidders, and U_{il} is private information for each bidder i. As is well known in the literature, if U_{il} are independent across i, independent of A_l, and each A_l has a log-concave density, then the vector of standalone valuations $(V_{1l}, \ldots, V_{Nl})$ is affiliated.

We further assume that the distribution of unobserved heterogeneity A_l in auction l depends only on the characteristics X_l of object l:

$$A_l \perp Z \mid X, W, \qquad A_l \perp W \mid X, \qquad U_{il} \perp A_l \mid X, Z, W, \tag{18}$$

$$A_l \perp X_{-l} \mid X_l, W, \qquad A_l \perp A_{\tilde{l}} \mid X, W \text{ for } l \neq \tilde{m}, \tag{19}$$

Denote $A = (A_1, \ldots, A_L)^\top$. Under regularity conditions analogous to those in our main text, bidding at observables X, Z, W and unobservable realization A must satisfy the F.O.C.

$$U_i + A = B_i + \nabla_b \Gamma_i(B_i|X,Z,W,A)^{-1} \times \Gamma_i(B_i|X,Z,W,A) - \nabla_b \Gamma_i(B_i|X,Z,W,A)^{-1} \times \nabla_b P_i(B_i|X,Z,W,A)^T K_i. \quad (20)$$

Also note that, if the strategy profile $\sigma^h(\cdot|X,Z,W)$ is a Bayes-Nash equilibrium at observable market characteristics (X,Z,W) and unobservables $A = 0$, then the strategy profile

$$\sigma(\cdot|X,Z,W,A) \equiv \sigma^h(\cdot|X,Z,W) + A$$

is a Bayes-Nash equilibrium at observables (X,Z,W) and unobserved heterogeneity realization A. To see why this is so, note that conditional on (X,Z,W) and A, the strategy profiles $\sigma(\cdot|X,Z,W,A)$ yields the same allocations and payoffs as does the strategy profile $\sigma^h(\cdot|X,Z,W)$. We assume throughout that, conditional on (X,Z,W), the same "fundamental" strategies $\sigma^h(\cdot|X,Z,W)$ are played for all realizations of A, so that variation in A shifts bids in each auction additively.

Under these assumptions, we can use techniques in Krasnokutskaya (2011) to establish that under suitable normalizations of mean bids within each auction l, both the distribution of B_{il} conditional on A_l, X, Z, W and the distribution of A_l conditional on X, W are identified from the distribution of B_{il} conditional on X, Z, W (recall that, conditional on X, W, A_l is independent of Z). In turn, by independence of A_l across auctions conditional on W, X, identification of each marginal $A_l|X,W$ implies identification of the distribution of the whole vector $A|X,W$.

The next step is to identify the joint distribution of the bid vector B_i conditional on X, Z, W, A from the following information known to the econometrician: (i) the joint distribution of B_i conditional on Z, X, W; (ii) the distribution of $A|X,W$. Here we employ the fact that

$$B_i = B_i^h + A.$$

Thus, given identification of distributions of $B_i|X,Z,W$ and $A|X,Z,W$, the distribution of $B_i^h|X,Z,W$ is identified by a standard deconvolution argument. Note that the distribution of $B_i^h|X,Z,W$ does not depend on A and, thus, the distribution of $B_i|X,Z,W,A$ is simply a location shift (by A) of the distribution of now identified $B_i^h|X,Z,W$.

The rest of the identification strategy is analogous to our baseline case. Namely, we can apply either the identification strategy based on Assumption 5 (varying rival characteristics Z_{-i}), or the identification strategy based on Assumption 6 (varying characteristics of other auctions X_{-l}) to identify distributions of "undisturbed" standalone valuations U_{il} and complementarities K_i. Together with the conditions above on unobserved auction heterogeneity, the former approach would give us that $E[U_i + A\ |X,Z,W]$ does not depend on Z_{-i}, while the latter would give us that $E[U_{il} + A_l\ |X,Z,W]$ does not depend on X_{-l}. In either case, the system for determining $K_i(i, W_i)$ will be exactly the same as described in Section 3.

Appendix D: Complementarities depending on V

In this appendix, we explore prospects for generalizing our non-parametric identification results to the case where complementarities are additively separable functions of standalone valuations. In other words, conditional on Z, W, X the compementarities are stochastic but their randomness can be fully explained by the standalone valuations. As a special case, we consider a scenario when these functions are affine in standalone valuations. Such a case could arise if, for instance, winning two auctions together increases i's valuation for one or both objects by a fixed percentage.

Notation and definitions We say complementarities are *additively separable in standalone valuations* if for each ω that contains at least two non-zero components (that is, $\|\omega\|^2 \geq 2$), the complementarity for outcome ω is a function of the vector of standalone valuations $v_i = (v_{i1}, v_{i2}, \ldots, v_{iL_i})^T$ such that

$$K^\omega(v_i, Z_i, W_i) = \sum_{l:\,\omega_l=1} \phi_l(v_{il}, Z_i, W_i) + \bar{K}^\omega(Z_i, W_i) \tag{21}$$

for some functions ϕ_l, $l = 1, \ldots, L$. If each function ϕ_l is linear in v_{il}, then we obtain the special case of *complementarities affine in* v_i:

$$K^\omega(v_i, Z_i, W_i) = \sum_{l:\,\omega_l=1} \delta^l(Z_i, W_i) v_{il} + \bar{K}^\omega(Z_i, W_i), \qquad \text{if } \|\omega\|^2 \geq 2. \tag{22}$$

As usual, if ω contains at most one component equal to one (that is, $\|\omega\|^2 \leq 1$), then we set $K^\omega(v_i, Z_i, W_i) \equiv 0$. An interesting special case of (22) is when all δ^l are identical and $\bar{K}^\omega = 0$ for any ω. This case describes the situation of a constant relative complementarity – that is, when $K^\omega(v_i, Z_i, W_i)$ is a constant ratio of the additive valuation.

Now assume that complementarities are affine in v_i, and define an $L_i \times 1$ vector $\delta(Z_i, W_i)$ and an $L_i \times L_i$ matrix $D(\delta(Z_i, W_i))$ as follows:

$$\begin{aligned}
\delta(Z_i, W_i) &\equiv (\delta^1(Z_i, W_i), \delta^2(Z_i, W_i), \ldots, \delta^{L_i}(Z_i, W_i))^T \\
D(\delta(Z_i, W_i)) &\equiv \operatorname{diag}(\delta^1(Z_i, W_i), \delta^2(Z_i, W_i), \ldots, \delta^{L_i}(Z_i, W_i)).
\end{aligned}$$

To write this in a convenient vector-matrix notation, let A_i denote the $2^{L_i} \times 2^{L_i}$ matrix such that its submatrix $(a_{lj})_{l,j=L_i+2,\ldots,2^M}$ coincides with the identity matrix of size $2^{L_i} - L_i - 1$, with all the other elements of A_i being 0. We then have

$$K(v_i, Z_i, W_i) = A_i \Omega_i D(\delta(Z_i, W_i)) v_i + \bar{K}(Z_i, W_i),$$

where $\bar{K}(Z_i, W_i)$ denotes the $2^{L_i} \times 1$ vector of constant components in the complementarities (obviously, $\bar{K}(Z_i, W_i) \in \mathcal{K}_i$). Clearly, the rank of matrix $A_i \Omega_i$ is equal to L_i.

As can be seen, the functional form of complementarities does not depend on Z_{-i}. As we show below, under weak conditions there is enough variation in $Z_{-i} \mid Z_i, W, X$ to determine the linear (in v_{il}) part of complementarities as well as the constant part.

Non-parametric identification Using the first-order conditions and taking into account the form of $K(v_i, Z_i, W_i)$, obtain

$$v_i = b_i + [\nabla_b \Gamma_i(b_i|Z_{-i})]^{-1} \Gamma_i(b_i|Z_{-i}) - [\nabla_b \Gamma_i(b_i|Z_{-i})]^{-1} \nabla_b P_i(b_i|Z_{-i})^T \left[A_i \Omega_i D(\delta) v_i + \bar{K}\right],$$

where for notational simplicity conditioning on Z_i, W, X is omitted from the notation in the rest of this Appendix. Rewrite that system of equations by collecting all terms with v_i on the left-hand side:

$$\begin{aligned}
\Big(I_{L_i} + [\nabla_b \Gamma_i(b_i|Z_{-i})]^{-1} \nabla_b P_i(b_i|Z_{-i})^T A_i \Omega_i D(\delta)\Big) v_i = b_i + [\nabla_b \Gamma_i(b_i|Z_{-i})]^{-1} \Gamma_i(b_i|Z_{-i}) \\
- [\nabla_b \Gamma_i(b_i|Z_{-i})]^{-1} \nabla_b P_i(b_i|Z_{-i})^T \bar{K},
\end{aligned}$$

and introduce a notation for the matrix in front of v_i on the left-hand side:

$$\Pi(b_i, \delta, Z_{-i}) \equiv I_{L_i} + [\nabla_b \Gamma_i(b_i|Z_{-i})]^{-1} \nabla_b P_i(b_i|Z_{-i})^T A_i \Omega_i D(\delta).$$

Define $\Delta(Z_{-i})$ as the set of $\delta \in \Re^{L_i}$ such that

$$\Pi(b_i, \delta, Z_{-i}) \text{ is non-singular for almost all } b_i.$$

This set is non-empty as e.g. $0 \in \Delta(Z_{-i})$. If $\delta \in \Delta(Z_{-i})$, then we can multiply the system from the left by $\Pi(b_i, \delta, Z_{-i})^{-1}$ resulting in

$$\begin{aligned} v_i = {} & \Pi(b_i, \delta, Z_{-i})^{-1} b_i + \Pi(b_i, \delta, Z_{-i})^{-1} [\nabla_b \Gamma_i(b_i|Z_{-i})]^{-1} \Gamma_i(b_i|Z_{-i}) \\ & - \Pi(b_i, \delta, Z_{-i})^{-1} [\nabla_b \Gamma_i(b_i|Z_{-i})]^{-1} \nabla_b P_i(b_i|Z_{-i})^T \bar{K}. \end{aligned}$$

Assuming that $\delta \in \Delta(Z_{-i})$ and carrying on with fixed Z_i, W, X, let us denote

$$\begin{aligned} \mathcal{D}_1(\delta, Z_{-i}) &\equiv E_{B_i}\left[\Pi(B_i, \delta, Z_{-i})^{-1} B_i \big| Z_{-i}\right] + E_{B_i}\left[\Pi(B_i, \delta, Z_{-i})^{-1} [\nabla_b \Gamma_i(B_i|Z_{-i})]^{-1} \Gamma_i(B_i|Z_{-i}) \big| Z_{-i}\right], \\ \mathcal{D}_2(\delta, Z_{-i}) &\equiv E_{B_i}\left[\Pi(B_i, \delta, Z_{-i})^{-1} [\nabla_b \Gamma_i(B_i|Z_{-i})]^{-1} \nabla_b P_i(B_i|Z_{-i})^T \big| Z_{-i}\right]. \end{aligned}$$

Keeping Z_i, W, X fixed, let us draw another value Z'_{-i} from $\mathcal{Z}_{-i}|Z_i, W, X$. Due to the assumptions made on the distribution of the standalone valuations, $E[V_i|Z_i, Z_{-i}, W, X] = E[V_i|Z_i, Z'_{-i}, W, X]$. Therefore, for $\delta \in \Delta(Z_{-i}) \cap \Delta(Z'_{-i})$,

$$\mathcal{D}_1(\delta, Z'_{-i}) - \mathcal{D}_1(\delta, Z_{-i}) = \left(\mathcal{D}_2(\delta, Z'_{-i}) - \mathcal{D}_2(\delta, Z_{-i})\right) \bar{K}.$$

For fixed Z_i, W, X, this system has $2^{L_i} - 1$ unknowns (L_i in δ and $2^{L_i} - L_i - 1$ in $\bar{K}$) and L_i equations. This gives us the following result.

Proposition 3. *Suppose that for $(Z_i, W, X) \in \mathcal{Z}_i \times \mathcal{W} \times \mathcal{X}$, there exist $J+1 \geq (2^{L_i}-1)/L_i+1$ vectors $Z_{-i,0}, Z_{-i,1}, \ldots, Z_{-i,J}$ in the support $\mathcal{Z}_{-i}|Z_i, W, X$ such that there is a unique $\delta \in \bigcap_{j=0}^{J} \Delta(Z_{-i,j})$ and a unique $\kappa \in \mathcal{K}_i$ that solve the system of $J \cdot L_i$ equations*

$$\mathcal{D}_1(\delta, Z_{-i,j}) - \mathcal{D}_1(\delta, Z_{-i,0}) = (\mathcal{D}_2(\delta, Z_{-i,j}) - \mathcal{D}_2(\delta, Z_{-i,0}))\,\kappa, \quad j = 1, \ldots, J. \tag{23}$$

Then the values of $\delta(Z_i, W_i)$ and $\bar{K}(Z_i, W_i)$ are identified, and thus, the complementarity function is identified for these values of Z_i, W_i.

System (23) is non-linear in δ. However, for each fixed $\delta \in \bigcap_{j=0}^{J} \Delta(Z_{-i,j})$, this system is linear in κ. Proposition 3 implies that in the case of identification it is not possible to have a situation when for different δ_1 and δ_2, where $\delta_1, \delta_2 \in \bigcap_{j=0}^{J} \Delta(Z_{-i,j})$, system (23) has solutions $\kappa_1 \in \mathcal{K}_i$ and $\kappa_2 \in \mathcal{K}_i$, respectively. Thus, in this sense the question of identification of $\delta(Z_i, W_i)$ and $\bar{K}(Z_i, W_i)$ comes down to the question of the *existence* of a solution to a system of linear equations: (23) can have a solution κ for one δ only, and for that δ it has to be unique. Using the Kronecker-Capelli theorem, which gives the necessary and sufficient conditions for the existence of a solution to a system of linear equations, and also the necessary and sufficient conditions for the uniqueness of such a solution, we formulate the identification result in the Proposition 4 below.

Before we proceed to Proposition 4, let us rewrite (23) in a more convenient way. At the moment K_i has to satisfy certain restrictions (namely, the first $L_i + 1$ components of this vector are 0) and we first want to rewrite it through an unrestricted parameter to apply certain tools from algebra.

Let E_i denote the $2^{L_i} \times (2^{L_i} - L_i - 1)$ matrix such that its submatrix $(\tilde{e}_{ij})_{i=L_i+2,\ldots,2^{L_i}, j=1,\ldots,2^{L_i}-l_i-1}$ coincides with the identity matrix of size $2^{L_i} - L_i - 1$, and all its other elements (that is, all the elements in the first $L_i + 1$ rows) are equal to zero. For every $\kappa \in \mathcal{K}_i$ there is a unique $\breve{\kappa} \in \mathbb{R}^{2^{L_i}-L_i-1}$ such that

$$\kappa = E_i \breve{\kappa}.$$

Obviously, this $\breve{\kappa}$ is a parameter that does not have to satisfy any prior restrictions. It is formed by the last $2^{L_i} - L_i - 1$ values in κ. System (23) can equivalently be written as

$$\mathcal{D}_1(\delta, Z_{-i,j}) - \mathcal{D}_1(\delta, Z_{-i,0}) = (\mathcal{D}_2(\delta, Z_{-i,j})E_i - \mathcal{D}_2(\delta, Z_{-i,0})E_i)\ \breve{\kappa}, \qquad j = 1, \ldots, J, \tag{24}$$

with $\breve{\kappa} \in \mathbb{R}^{2^{L_i}-L_i-1}$. For a fixed δ, system (23) is linear in κ, has the $J \cdot L_i \times 2^{L_i}$ matrix of coefficients, and imposes restrictions on the solution κ by requiring that $\kappa \in \mathcal{K}_i$. Its equivalent representation (24) is linear in $\breve{\kappa}$ for a fixed δ, has the $J \cdot L_i \times (2^{L_i} - L_i - 1)$ matrix of coefficients, and does not impose any restrictions on the solution $\breve{\kappa} \in \mathbb{R}^{2^{L_i}-L_i-1}$. This allows us to apply the Kronecker-Capelli theorem to system (24) in a straightforward way.

Proposition 4. *Suppose that for $(Z_i, W_i, X) \in \mathcal{Z}_i \times \mathcal{W}_i \times \mathcal{X}$, there exist $J+1 \geq (2^{L_i}-1)/L_i + 1$ vectors $Z_{-i,0}, Z_{-i,1}, \ldots, Z_{-i,J}$ in the support $\mathcal{Z}_{-i}|Z_i, W, X$ such that there is a unique $\delta \in \bigcap_{j=0}^{J} \Delta(Z_{-i,j})$ that satisfies the following two conditions:*

1. *First,*
$$rank\left([\mathbf{M}_1(\delta) \,|\, \mathbf{M}_2(\delta)]\right) = rank\left(\mathbf{M}_2(\delta)\right), \tag{25}$$
where $\mathbf{M}_2(\delta)$ denotes the $J \cdot L_i \times (2^{L_i} - L_i - 1)$ matrix
$$\mathbf{M}_2(\delta) \equiv \begin{bmatrix} \mathcal{D}_2(\delta, Z_{-i,1})E_i - \mathcal{D}_2(\delta, Z_{-i,0})E_i \\ \vdots \\ \mathcal{D}_2(\delta, Z_{-i,J})E_i - \mathcal{D}_2(\delta, Z_{-i,0})E_i \end{bmatrix},$$
and $\mathbf{M}_1(\delta)$ denotes the $J \cdot L_i \times 1$ vector
$$\mathbf{M}_1(\delta) \equiv \begin{bmatrix} \mathcal{D}_1(\delta, Z_{-i,1}) - \mathcal{D}_1(\delta, Z_{-i,0}) \\ \vdots \\ \mathcal{D}_1(\delta, Z_{-i,J}) - \mathcal{D}_1(\delta, Z_{-i,0}) \end{bmatrix}.$$

2. *Moreover, this δ is such that $\mathbf{M}_2(\delta)$ has full column rank:*
$$rank\left(\mathbf{M}_2(\delta)\right) = 2^{L_i} - L_i - 1. \tag{26}$$

Then the values of $\delta(Z_i, W_i)$ and $\bar{K}(Z_i, W_i)$ are identified, and thus, the complementarity function is identified for these values of Z_i, W_i.

Condition (25) requires that in system (24), the rank of the matrix of coefficients $\mathbf{M}_2(\delta)$ is equal to the rank of the augmented matrix $[\mathbf{M}_1(\delta) \,|\, \mathbf{M}_2(\delta)]$ for one δ only. The Kronecker-Capelli theorem guarantees then that (24) has a solution $\breve{\kappa}$ for that δ only. Condition (26) then guarantees this $\breve{\kappa}$ is determined uniquely, and, thus, $\kappa = E_i \breve{\kappa}$ is determined uniquely.

Note that all the identification conditions in Proposition 4 are formulated in terms of δ. The closed form for $\delta(Z_i, W_i)$ cannot be found, but in practice one can find $\delta(Z_i, W_i)$ and $\bar{K}(Z_i, W_i)$ by solving, e.g., the following optimization problem:

$$\min_{\delta \in \bigcap_{j=0}^{J} \Delta(Z_{-i,j}),\, \breve{\kappa} \in \mathbb{R}^{2^{L_i}-L_i-1}} Q(\delta, \breve{\kappa}, Z_i, W, X),$$

where

$$Q(\delta, \breve{\kappa}, Z_i, W, X) \equiv (\mathbf{M}_1(\delta) - \mathbf{M}_2(\delta)\breve{\kappa})^T (\mathbf{M}_1(\delta) - \mathbf{M}_2(\delta)\breve{\kappa}).$$

Appendix E: Additional identifying restrictions induced by variation in characteristics of combinations

In Section 3, we show that excludable variation in charateristics of either other bidders or other objects can be used to obtain nonparametric identification of complementarities. But what about variation in combination characteristics W_i directly affecting complementarities? Intuitively, one might expect variation in W_i to be of considerable use in identifying complementarities. In this Appendix, we show that variation in W_i does indeed induce an additional set of partial differential equations which restrict the relationship between levels of $\kappa(Z, W_i)$ and changes in $\kappa(Z, W_i)$. Unfortunately, however, these restrictions alone will typically be insufficient to identify $\kappa(Z, W_i)$.

In this discussion, it is important to bear in mind one key caveat: in practice, W_i is typically a function of observables (X, Z). The extent to which one can vary W_i independently from (X, Z), or one element of W_i independently from others, may thus be mechanically limited by this dependence. Nevertheless, it may be natural to consider variation in W_i holding relevant features of (X, Z) constant—for instance, by changing the distance between two projects, holding i's distance to each project constant. What additional identifying restrictions are induced by such variation?

Toward this end, recall from Proposition 1 that we must have $V_i = \Upsilon_i(b_i|M) - \Psi_i(b_i|M) \cdot \kappa_i(Z, W_i)$ almost surely. This in turn implies the equilibrium identity

$$E[V_i|X, Z] = \bar{\Upsilon}_i(M) - \bar{\Psi}_i(M) \cdot \kappa_i(Z, W_i). \tag{27}$$

Suppose that element W_{ik} of W_i varies conditional on (X, Z). Assume for simplicity that the right-hand side of (27) is differentiable in W_{ik}; otherwise, we may proceed in finite differences. Differentiating both sides of (27) with respect to W_{ik}, noting that $E[V_i|X, Z]$ does not depend on W_{ik}, and rearranging, we obtain

$$\bar{\Psi}_i(M) \cdot \frac{\partial \kappa_i(Z, W_i)}{\partial W_{ik}} = \frac{\partial \bar{\Upsilon}_i(M)}{\partial W_{ik}} - \frac{\partial \bar{\Psi}_i(M)}{\partial W_{ik}} \cdot \kappa_i(Z, W_i). \tag{28}$$

For bidders competing in exactly two auctions, Equation (28) represents a system of $L_i = 2$ differential equations in the unknown scalar-valued function $\kappa_i(Z, W_i)$; in this case, variation in W_i alone may support nonparametric identification of $\kappa_i(Z, W_i)$. For bidders competing in more than two auctions, Equation (28) represents a system of L_i first-order differential equations in the unknown $(2^{L_i} - L_i - 1)$ dimensional vector-valued function $\kappa_i(Z, W_i)$. Even if the initial value $\kappa_i(Z, W_i)$ is given, it will generically be impossible to solve for the gradient $\frac{\partial \kappa_i(Z, W_i)}{\partial W_{ik}}$ uniquely from this system. Hence, absent additional *a priori* knowledge—for instance, that sufficiently many elements of $\frac{\partial \kappa_i(Z, W_i)}{\partial W_{ik}}$ are equal to zero—it will generally be impossible to uniquely identify even changes in $\kappa_i(Z, W_i)$ from variation in W_i alone.

If, however, we additionally have excludable variation in either rival characteristics Z_{-i} or characteristics of other auctions X_{-l}, we can add the derivative restrictions (28) to the level equations

(10) considered in the main text. For example, by choosing a different rival structure Z'_{-i}, we could add an additional set of equations paralleling (28). With enough equations, under suitable regularity conditions, we can find a solution to this system that is unique up to a boundary condition on $\kappa_i(Z, W_i)$ when W_{ik} is fixed at some boundary value W_{ik0}. This boundary condition may depend on the values of Z_i and the values of other components in W_{ik}. It also has to be compatible with the boundary conditions when one considers similar systems using variation in W_{ir}, $r \neq k$. The knowledge of $\kappa_i(Z, W_i)$ at at some boundary value W_{ik0} may be achieved either through normalization, or through excludable variation in either Z_{-i} or X_{-l} as in Section 3.

Finally, in applications where $\kappa_i(Z, W_i)$ is modeled as parametric, so that levels and changes in $\kappa_i(Z, W_i)$ both depend on some underlying parameter θ, then equations of the form (28) will often be sufficient to identify θ even without further excluded variation in either Z_{-i} or X_{-l}. This is true, for example, in our application in Section 5, where a linear form for $\kappa_i(\cdot)$ would, in principle, permit identification even without excludable variation in Z_{-i}. In practice, however, we exploit both variation in W_i and variation in Z_{-i} in estimation, as both sources of variation add information on the underlying paramters of interest.

Appendix F: Identification relaxing conditions on equilibrium bidding in Assumption 4

Our point identification result for the vector-function of complementarities $\kappa_i(Z_i, W, X)$ and the conditional distribution of $V_i|Z_i, W, X$ relied on the first order conditions obtained from bidder's optimization of the payoff function. To derive those equations we employed the absolute continuity of the bid distribution functions G_i. That, in particular, eliminated the possibility of bidders playing atoms in the equilibrium. In this appendix, we analyze identification maintaining Assumptions 1-3 and 5, but dropping Assumption 4: that is, without assuming either absolute continuity or common support. We derive robust identified sets for $\kappa_i(Z_i, W_i)$ and the joint cumulative distribution function of $V_i|Z_i, X$ applicable to any data generating process satisfying these maintained assumptions, including those involving either atoms or non-common support.

Importantly, these robust identified sets collapse to point identification as in the main text if the data generating process additionally satisfies Assumption 4, even without maintaining Assumption 4 a priori. In other words, Assumption 4 may be viewed as a sufficient condition for point identification within the richer partial identification framework we develop below. In this sense, our fundamental identification insights are applicable with or without Assumption 4, although this assumption does sharpen and simplify the identification process.

We further show that the nature of partial identification depends critically on the specific violation of Assumption 4. In F.1 and F.2, we show that both κ_i and valuations are typically partially identified when the equilibrium bid distribution involves atoms, although identified sets may be quite informative.[42] Meanwhile, in F.4, we show that if bid distributions do not satisfy common support, but are differentiable for bids with interior probabilities of winning, one can

[42] For example, in Appendix H.1, we simulate a two-auction equilibria between a global bidder and several local bidders, restricting all bids to a discrete grid. Our main focus in Appendix H.1 is estimation based on first-order approximations to optimal bidding behavior, which we show perform very well even when underlying bids are actually discrete. For completeness, however, we also compute identified sets for κ_i explicitly accounting for discreteness as in F.2 below, assuming that the global bidder faces either one or three local rivals in each auction. The nonparametric identified superset is approximately $[0.04, 0.16]$ when $\kappa_i = 0.1$, $[-0.14, -0.03]$ when $\kappa_i = -0.1$, and $[-0.36, -0.22]$ when $\kappa_i = -0.3$.

typically identify κ_i and valuations corresponding to interior bids exactly, although valuations corresponding to always losing bids will still be only partially identified.[43] In particular, this latter analysis allows some bidder types to submit always-losing bids with positive probability—as could arise with asymmetric supports of standalone valuations, or with binding public reserve prices. Intuitively, in this case, one can base identification of κ_i on upper quantiles of standalone valuations, which will be identified and invariant to Z_{-i} even though lower quantiles will not.

F.1: Sharp identified set for κ_i

Let us fix $(Z_i, W_i, X) \in \mathcal{Z}_i \times \mathcal{W} \times \mathcal{X}$. For each realization (Z_{-i}, W_{-i}) in the support of $(Z_{-i}, W_{-i})|Z_i, W_i, X$, bidder maximizes the payoff function

$$\pi^M(v_i, b) = v_i^T \Gamma_i(b_i|M) - b_i^T \Gamma_i(b_i|M) + P_i(b_i|M)^T \kappa_i(Z_i, W_i)$$

with respect to $b_i \in \mathcal{B}_i$. Omitting fixed (X, Z_i, W_i) from the notation for ease of exposition, and letting κ_i denote the true complementarity vector $\kappa_i(Z_i, W_i)$ evaluated at the (fixed) observables (Z_i, W_i), every observed bid vector b_i must satisfy the best-response requirement

$$v_i^T \left(\Gamma_i(b_i|Z_{-i}) - \Gamma_i(b|Z_{-i})\right) + \left(P_i(b_i|Z_{-i}) - P_i(b|Z_{-i})\right)^T \kappa_i \geq b_i^T \Gamma_i(b_i|Z_{-i}) - b^T \Gamma_i(b|Z_{-i}) \quad \forall\, b \in \mathcal{B}_i. \quad (29)$$

For any conjectured complementarity vector κ_i, the system of linear inequalities (29) will define an identified, convex set of valuations rationalizing the observed bid vector b_i against rival characteristics Z_{-i}; we denote this set by $\mathcal{V}_i(b_i; Z_{-i}, \kappa_i)$. In principle, one may apply the results of Chesher and Rosen (2017) to characterize the sharp identified set for κ_i, using the set of solutions $\mathcal{V}_i(\cdot; Z_{-i}, \kappa_i)$ to (29) as the set-valued mapping implied by the bidding model, and rival characteristics Z_{-i} as instruments shifting this mapping but not the latent distribution F_i. In practice, however, this sharp identified set would be very difficult to operationalize numerically. We therefore proceed instead to derive analytically simpler identified supersets for κ_i, which nevertheless yield point identification when the data generating process satisfies Assumption 4.

F.2: Identified supersets based on single-bid deviations

First consider any bid vector b_i such that $\Gamma_{il}(b_{il}|Z_{-i}) < 1$; we return to the case of $\Gamma_{il}(b_{il}|Z_{-i}) = 1$ below. Let b be any bid vector such that $b = b_i + \epsilon e_l \in \mathcal{B}_i$ for some $\epsilon > 0$; i.e., any candidate deviation to a higher bid in auction l, holding bids in all other auctions constant. We then obtain from (29) that

$$v_{il} \left(\Gamma_{il}(b_{il}|Z_{-i}) - \Gamma_{il}(b_{il} + \epsilon|Z_{-i})\right) + \left(P_i(b_i|Z_{-i}) - P_i(b_i + \epsilon e_l|Z_{-i})\right)^T \kappa_i \geq b_{il}\Gamma_{il}(b_{il}|Z_{-i}) - (b_{il} + \epsilon)\Gamma_{il}(b_{il} + \epsilon|Z_{-i}),$$

[43] Intuitively, although the lower tail of standalone valuations will be unidentifiable in the presence of dominated bids, the upper tail of valuations will be identified up to κ_i above the (also identified) minimum threshold ensuring an interior bid. One can then form a continuum of identifying restrictions based on invariance of identified upper quantiles of V_{il} to rival characteristics Z_{-i}, which generically will substantially over-determine the finite-dimensional vector κ_i. The main technical challenge in formalizing this argument in our multi-auction context is that the threshold at which V_{il} is truncated generically depends on both Z_{-i} and on bids in other auctions. We provide a detailed resolution of this issue in F.4.

where we used the assumption that the ties are broken independently across auctions at b_i, and thus that a change in the lth component of b_i affects only the lth component of Γ_i. Since $\Gamma_{i,l}$ is (weakly) increasing in b_{il}, we have $\Gamma_{i,l}(b_{il}|Z_{-i}) - \Gamma_{i,l}(b_{il}+\epsilon|Z_{-i}) \leq 0$. Furthermore, since $\Gamma_{il}(b_{il}|Z_{-i}) < 1$, there will exist some $\epsilon > 0$ such that this inequality is strict. For any such $\epsilon > 0$, we can rearrange the best-response inequality above to obtain an informative upper bound on the standalone valuation v_{il} corresponding to bid b_i:

$$v_{il} \leq \frac{b_{il}\Gamma_{il}(b_{il}|Z_{-i}) - (b_{il}+\epsilon)\Gamma_{il}(b_{il}+\epsilon|Z_{-i})}{\Gamma_{il}(b_{il}|Z_{-i}) - \Gamma_{il}(b_{il}+\epsilon|Z_{-i})} - \left[\frac{P_i(b_i|Z_{-i}) - P_i(b_i+\epsilon e_l|Z_{-i})}{\Gamma_{il}(b_{il}|Z_{-i}) - \Gamma_{il}(b_{il}+\epsilon|Z_{-i})}\right]^T \cdot \kappa_i. \tag{30}$$

This upper bound is well defined for any bid vector b_i such that $\Gamma_{il}(b_{il}|Z_{-i}) < 1$, and identified up to the unknown complementarity vector κ_i.

Next consider any observed bid vector b_i such that $\Gamma_{il}(b_{il}|Z_{-i}) > 0$. For any such bid, there will exist some $\epsilon > 0$ such that $\Gamma_{il}(b_{il}|Z_{-i}) - \Gamma_{il}(b_{il}-\epsilon|Z_{-i}) > 0$. Proceeding analogously to the derivation above, but this time considering a deviation bid $b = b_i - \epsilon e_l \in \mathcal{B}_i$, we obtain an informative lower bound on the standalone valuation v_{il} corresponding to bid observed bid b_i:

$$v_{il} \geq \frac{b_{il}\Gamma_{il}(b_{il}|Z_{-i}) - (b_{il}-\epsilon)\Gamma_{il}(b_{il}-\epsilon|Z_{-i})}{\Gamma_{il}(b_{il}|Z_{-i}) - \Gamma_{il}(b_{il}-\epsilon|Z_{-i})} - \left[\frac{P_i(b_i|Z_{-i}) - P_i(b_i-\epsilon e_l|Z_{-i})}{\Gamma_{il}(b_{il}|Z_{-i}) - \Gamma_{il}(b_{il}-\epsilon|Z_{-i})}\right]^T \cdot \kappa_i. \tag{31}$$

As above, this lower bound is well defined so long as $\Gamma_{il}(b_{il}|Z_{-i}) > 0$, and identified up to the unknown vector κ_i describing complementarities.

The inequalities (30) and (31) will be the basis for our analysis. For any bid vector b_i such that $\Gamma_{il}(b_{il}|Z_{-i}) \in (0,1)$, these inequalities yield informative two-sided bounds on the set of standalone valuations v_{il} rationalizing b_i as an equilibrium bid. For bid vectors b_i such that $\Gamma_{il}(b_{il}|Z_{-i}) = 0$, only the upper bound (30) will be informative. Meanwhile, for bid vectors b_i such that $\Gamma_{il}(b_{il}|Z_{-i}) = 1$, only the lower bound (31) will be informative.

To handle bid vectors b_i for which one of (30) or (31) is uninformative, we assume there exist known (extended real) scalars $\underline{v}, \bar{v}$ such that $V_{il} \in [\underline{v}, \bar{v}]$. Unless noted otherwise, in the notation below, one may take $\underline{v} = -\infty$ and $\bar{v} = \infty$, which accommodates the case of completely uninformative bounds. For the expectations-based superset we derive below, however, we require both $\underline{v}$ and $\bar{v}$ to be finite (i.e., for the researcher to have some prior information on $\bar{V}_{il}$. We explicitly indicate finiteness of $\underline{v}$ and $\bar{v}$ when this is required.

Before proceeding to characterize the identified superset based on (30) and (31), we first introduce some notation. For any function f, let $\Delta^+_{\epsilon,l}[f(u)]$ and $\Delta^-_{\epsilon,l}[f(u)]$ denote differences in the values of $f(\cdot)$ associated with adding ϵ and $-\epsilon$ to the lth component of u respectively:

$$\Delta^+_{\epsilon,l}[f(u)] = f(u+\epsilon e_l) - f(u),$$
$$\Delta^-_{\epsilon,l}[f(u)] = f(u-\epsilon e_l) - f(u),$$

where e_l denotes the L_i-dimensional mth unit vector.

For each $b_i \in \mathcal{B}_i$, define $\Upsilon^+_{\epsilon,l}(b_i|Z_{-i})$ and $\Upsilon^-_{\epsilon,l}(b_i|Z_{-i})$ as follows:

$$\Upsilon^-_{\epsilon,l}(b_i|Z_{-i}) = \left\{\underline{v} \text{ if } \Delta^-_{\epsilon,l}[\Gamma_i(b_i|Z_{-i})] = 0,\ \frac{\Delta^-_{\epsilon,l}[b_i^T\Gamma_i(b_i|Z_{-i})]}{\Delta^-_{\epsilon,l}[\Gamma_{i,l}(b_i|Z_{-i})]} \text{ else}\right\};$$
$$\Upsilon^+_{\epsilon,l}(b_i|Z_{-i}) = \left\{\bar{v} \text{ if } \Delta^+_{\epsilon,l}[\Gamma_i(b_i|Z_{-i})] = 0,\ \frac{\Delta^+_{\epsilon,l}[b_i^T\Gamma_i(b_i|Z_{-i})]}{\Delta^+_{\epsilon,l}[\Gamma_{i,l}(b_i|Z_{-i})]} \text{ else}\right\}.$$

By construction, $\Upsilon^-_{\epsilon,l}(b_i|Z_{-i})$ and $\Upsilon^+_{\epsilon,l}(\cdot|Z_{-i})$ represent finite-difference analogs to the lth element of the vector $\Upsilon_\epsilon(b_i|Z_{-i})$ defined in the main text. Interpreted as functions of b_i, $\Upsilon^-_{\epsilon,l}(b_i|Z_{-i})$ and $\Upsilon^+_{\epsilon,l}(\cdot|Z_{-i})$ extend the first term in each of the bounds (30) and (31) to cases with potentially uninformative bounds. Furthermore, for each $\epsilon > 0$, both $\Upsilon^-_{\epsilon,l}(b_i|Z_{-i})$ and $\Upsilon^+_{\epsilon,l}(b_i|Z_{-i})$ are identified up to the researcher-specified bounds $\underline{v}, \bar{v}$.

Analogously, for each $b_i \in \mathcal{B}_i$, define the identified functions $\Psi^-_{\epsilon,l}(b_i|Z_{-i})$ and $\Psi^+_{\epsilon,l}(b_i|Z_{-i})$ by

$$\Psi^-_{\epsilon,l}(b_i|Z_{-i}) = \left\{ 0 \text{ if } \Delta^-_{\epsilon,l}[\Gamma_i(b_i|Z_{-i})] = 0, \ \frac{\Delta^-_{\epsilon,l}[P_i(b_i|Z_{-i})]}{\Delta^-_{\epsilon,l}[\Gamma_{i,l}(b_i|Z_{-i})]} \text{ else} \right\};$$

$$\Psi^+_{\epsilon,l}(b_i|Z_{-i}) = \left\{ 0 \text{ if } \Delta^+_{\epsilon,l}[\Gamma_i(b_i|Z_{-i})] = 0, \ \frac{\Delta^+_{\epsilon,l}[P_i(b_i|Z_{-i})]}{\Delta^+_{\epsilon,l}[\Gamma_{i,l}(b_i|Z_{-i})]} \text{ else} \right\}.$$

Note that $\Psi^-_{\epsilon,l}(b_i|Z_{-i})$ and $\Psi^-_{\epsilon,l}(b_i|Z_{-i})$ also represent finite-difference analogs to the lth column of the $2^{L_i} \times L_i$ matrix $\Psi_{\epsilon,l}(b_i|Z_{-i})$ defined in the main text. Interpreted as functions of b_i, $\Psi^-_{\epsilon,l}(b_i|Z_{-i})$ and $\Psi^+_{\epsilon,l}(b_i|Z_{-i})$ extend the terms multiplying κ_i in each bound (30) and (31) to accommodate potentially uninformative bounds.

Substituting the definitions above into the inequalities (30) and (31), we conclude that the following inequalities must hold for every $\epsilon > 0$ and almost every equilibrium bid b_i:

$$\Upsilon^-_{\epsilon,l}(b_i|Z_{-i}) - \Psi^-_{\epsilon,l}(b_i|Z_{-i})^T \kappa_i \leq v_i \leq \Upsilon^+_{\epsilon,l}(b_i|Z_{-i}) - \Psi^+_{\epsilon,l}(b_i|Z_{-i})^T \kappa_i.. \tag{32}$$

Identified supersets using bounds on marginal distribution functions By definition, we know that κ_i is a $2^{L_i} \times 1$ vector whose first $L_i \times 1$ components are zero. We further know that, evaluated at the true complementarity vector κ_i, the inequalities (32) must hold for all $\epsilon > 0$ and all b_i and Z_{-i} in the relevant equilibrium supports. In what follows, let $\mathcal{K}_i$ denote the set of candidates K for κ_i such that both of these conditions hold. Since, from above, failure of either condition implies $K \neq \kappa_i$, we can restrict attention to this candidate set $\mathcal{K}_i$ without loss of generality.

For each $K \in \mathcal{K}_i$, let $\tilde{F}^-_{il}(\cdot|K; Z_{-i})$ denote the c.d.f. of

$$\sup_{\epsilon > 0} \left(\Upsilon^-_{\epsilon,l}(b_i|Z_{-i}) - \Psi^-_{\epsilon,l}(b_i|Z_{-i})^T K \right),$$

and let $\tilde{F}^+_{il}(\cdot|K; Z_{-i})$ denote the c.d.f. of

$$\inf_{\epsilon > 0} \left(\Upsilon^+_{\epsilon,l}(b_i|Z_{-i}) - \Psi^+_{\epsilon,l}(b_i|Z_{-i})^T K \right).$$

By construction, $\tilde{F}^-_{il}(\cdot|K; Z_{-i})$ and $\tilde{F}^+_{il}(\cdot|K; Z_{-i})$ are identified for each candidate $K \in \mathcal{K}_i$. Furthermore, in view of (32), at the true complementarity vector $K = \kappa_i$, we must have

$$\tilde{F}^+_{il}(\cdot|\kappa_i; Z_{-i}) \leq F_{il}(\cdot) \leq \tilde{F}^-_{il}(\cdot|\kappa_i; Z_{-i})$$

for all possible rival characteristic vectors $Z_{-i} \in \mathcal{Z}_{-i}$.

Then inequalities (30) and (31) imply that a superset of the identified set of $\kappa(Z_i, W_i)$ for bidder i can be found as

$$\bigcap_{m=1}^{L_i} \tilde{\mathcal{K}}_{i,l}(Z_i, W_i),$$

where $\tilde{\mathcal{K}}_{i,l}(Z_i, W_i)$ is defined as

$$\tilde{\mathcal{K}}_{i,l}(Z_i, W_i) = \{K \in \mathcal{K}_i \mid \tilde{F}^+_{il}(\cdot|K; Z_{-i}) \leq \tilde{F}^-_{il}(\cdot|K; Z'_{-i}) \quad \forall\, Z_{-i}, Z'_{-i} \in \mathcal{Z}_{-i}|Z_i, W, X\}.$$

Let us denote this superset as $\mathcal{H}^{(1)}_{i,\kappa}(Z_i, W_i)$.

Supersets using bounds on expected values Finally, we translate the identified supersets based on invariant marginal distributions to identified supersets based on invariant expected valuations. Even though the resulting supersets will be larger than those discussed previously, they are easier to characterize numerically.

Toward this end, suppose that the researcher's a priori bounds $\underline{v}, \bar{v}$ on V_{il} are finite; note that this is the only step in which we need finiteness of $\underline{v}$ and $\bar{v}$. Define $L_i \times 1$ vectors $\bar{\Upsilon}^-_\epsilon(Z_{-i})$, $\bar{\Upsilon}^+_\epsilon(Z_{-i})$ and $L_i \times 2^{L_i}$ matrices $\bar{\Psi}^-_\epsilon(Z_{-i})$, $\bar{\Psi}^+_\epsilon(Z_{-i})$ as follows:

$$\begin{aligned}
\bar{\Upsilon}^-_\epsilon(Z_{-i}) &\equiv \left[E[\Upsilon^-_{\epsilon,l}(B_i|Z_{-i})|Z_{-i}]\right]_{l=1}^{L_i} \\
\bar{\Upsilon}^+_\epsilon(Z_{-i}) &\equiv \left[E[\Upsilon^+_{\epsilon,l}(B_i|Z_{-i})|Z_{-i}]\right]_{l=1}^{L_i} \\
\bar{\Psi}^-_\epsilon(Z_{-i}) &\equiv \left[E[\Psi^-_{\epsilon,l}(B_i|Z_{-i})|Z_{-i}]^T\right]_{l=1}^{L_i} \\
\bar{\Psi}^+_\epsilon(Z_{-i}) &\equiv \left[E[\Psi^+_{\epsilon,l}(B_i|Z_{-i})|Z_{-i}]^T\right]_{l=1}^{L_i}.
\end{aligned}$$

Then, applying the expectation over the distribution of bids conditional on Z_i, W, X to inequalities (30) and (31) and pooling restrictions across Z_{-i}, Z'_{-i} and $l = 1, \ldots, L_i$, we establish that a superset of the identified set for $\kappa(Z_i, W_i)$ can be found in the following way:

$$\mathcal{H}^{(2)}_{i,\kappa}(Z_i, W_i) = \bigcap_{\epsilon > 0} \hat{\mathcal{K}}^\epsilon_i(Z_i, W_i),$$

where $\hat{\mathcal{K}}^\epsilon_i(Z_i, W_i)$ is defined as

$$\begin{aligned}
\hat{\mathcal{K}}^\epsilon_i(Z_i, W_i) \equiv \Big\{K \in \mathcal{K}_i \;\Big|\; &\left(\bar{\Upsilon}^-_\epsilon(Z_{-i}) - \bar{\Upsilon}^+_\epsilon(Z'_{-i})\right) \\
&- \left(\bar{\Psi}^-_\epsilon(Z_{-i}) - \bar{\Psi}^+_\epsilon(Z'_{-i})\right) K \leq 0 \text{ for all } Z_{-i}, Z'_{-i} \in \mathcal{Z}_{-i}|Z_i, W, X\Big\}. \quad (33)
\end{aligned}$$

Observe that the identified superset $\mathcal{H}^{(2)}_{i,\kappa}(Z_i, W_i)$ can be represented as the intersection of a set of half-spaces in $\mathcal{K}_i$, where half-spaces are bounded by hyperplanes involving slope vectors $(\bar{\Psi}^-_{\epsilon,l}(Z_{-i}) - \bar{\Psi}^+_{\epsilon,l}(Z'_{-i}))$ and intercepts $(\bar{\Upsilon}^-_{\epsilon,l}(Z_{-i}) - \bar{\Upsilon}^+_{\epsilon,l}(Z'_{-i}))$, and the intersection is taken over collections of $(Z_{-i}, Z'_{-i}, \epsilon, l)$. It follows immediately that $\mathcal{H}^{(2)}_{i,\kappa}(Z_i, W_i)$ is convex. Furthermore, $\mathcal{H}^{(2)}_{i,\kappa}(Z_i, W_i)$ will be bounded so long as rival characteristics induce sufficient variation in $(\bar{\Upsilon}^-_{\epsilon,l}(Z_{-i}) - \bar{\Upsilon}^+_{\epsilon,l}(Z'_{-i}))$, in the sense that there exists a collection $\{(Z_{-i,k}, Z'_{-i,k})\}_{k=1}^K$ of competition structures such that the cone spanned by the rows of the matrix

$$\begin{bmatrix} \bar{\Upsilon}^+_{\epsilon,l}(Z_{-i,1}) - \bar{\Upsilon}^-_{\epsilon,l}(Z'_{-i,1}) \\ \vdots \\ \bar{\Upsilon}^+_{\epsilon,l}(Z_{-i,K}) - \bar{\Upsilon}^-_{\epsilon,l}(Z'_{-i,K}) \end{bmatrix}$$

contains the space of candidate complementarity vectors $\mathcal{K}_i$. This latter condition is essentially the generalization of the rank conditions for identification in the main text to the system of half-spaces defined by (33). Notice, however, that in the absence of absolute continuity this condition may require somewhat more variation in Z_{-i}, as variation in Z_{-i} will additionally need to overcome mechanical differences between $\bar{\Upsilon}^{+}_{\epsilon,l}(Z_{-i,k})$ and $\bar{\Upsilon}^{-}_{\epsilon,l}(Z_{-i,k})$ induced by discreteness.

Supersets for the distribution of standalone values Finally, we construct supersets of the identified sets for the distributions of standalone valuations. Let $\mathcal{F}_c(\mathbb{R}^p)$ denote the set of all continuous cumulative distribution functions on $\mathbb{R}^p$, and let $\mathcal{H}_{i,\kappa}(Z_i, W_i)$ be either of the identified supersets for κ_i defined above.

A superset of the identified set for the c.d.f. of the standalone valuation V_{il} conditional on Z_i, X can be found as the set of univariate functions $F_{il}(\cdot) \in \mathcal{F}_c(\mathbb{R})$ such that for any $\eta \in \mathbb{R}$,

$$F_{il}(\eta) \in \bigcap_{W \in \mathcal{W}|Z_i,X} \bigcap_{\kappa_0 \in \mathcal{H}_{i,\kappa}(Z_i,W_i)} \bigcap_{Z_{-i},Z'_{-i} \in \mathcal{Z}_{-i}|Z_i,W,X} [\tilde{F}^{+}_{il}(\eta|\kappa_0; Z_{-i}), \tilde{F}^{-}_{il}(\eta|\kappa_0; Z'_{-i})]\}. \tag{34}$$

Here we applied the exclusion restriction that the distribution of standalone valuations conditional on Z_i, W_i, X does not depend on W_i. Let us denote this superset as $\mathcal{H}_{i,F_l}(Z_i, X)$.

Our final step is to construct a superset $\mathcal{H}_{i,F}(Z_i, X)$ for the identified set for the joint distribution of the vector of standalone valuations. $\mathcal{H}_{i,F}(Z_i, X)$ can be found as the set of L_i-variate functions $F_i(\cdot) \in \mathcal{F}_c(\mathbb{R}^{L_i})$ such that each lth marginal distribution function generated by $F_i(\cdot)$ belongs to $\mathcal{H}_{i,F_l}(Z_i, X)$, $l = 1, \ldots, L_i$. Moreover, for any $\eta = (\eta_1, \ldots, \eta_{L_i})$,

$$F_i(\eta) \leq \min_{l=1,\ldots,L_i} \inf_{W \in \mathcal{W}|Z_i,X} \inf_{\kappa_0 \in \mathcal{H}_{i,\kappa}(Z_i,W_i)} \inf_{Z_{-i} \in \mathcal{Z}_{-i}|Z_i,W,X} \tilde{F}^{+}_{il}(\eta_l|\kappa_0; Z_{-i}), \tag{35}$$

$$F_i(\eta) \geq \max\left\{\sum_{m=1}^{L_i} \sup_{W \in \mathcal{W}|Z_i,X} \sup_{\kappa_0 \in \mathcal{H}_{i,\kappa}(Z_i,W_i)} \sup_{Z_{-i} \in \mathcal{Z}_{-i}|Z_i,W,X} \tilde{F}^{-}_{il}(\eta_l|\kappa_0; Z_{-i}) - L_i + 1, 0\right\}, \tag{36}$$

where we employed the well known result on sharp Frechet-Hoeffding bounds for joint distributions.

F.3: Robust supersets yield point identification when the data generating process satisfies Assumption 4

We next show that analysis based on the supersets above yields point identification if the underlying data generating process additionally satisfies the conditions in Assumption 4. In other words, from an identification perspective, the partial identification analysis described here in fact loses no information relative to the simpler first-order approach described in the main text; if the data generating process satisfies the relevant smoothness and support conditions, then the identified supersets derived above will collapse to point identification, even when these conditions are not maintained a priori. In this sense, one may essentially view Assumption 4 as a sufficient condition for point identification within the robust partial identification analysis described above.

Toward this end, suppose that in addition to Assumptions 1-3 and 5 maintained throughout this Appendix, the data generating process satisfies the absolute continuity and common support conditions of Assumption 4. Since G_i is absolutely continuous for each bidder, then bidder i's objective function will be differentiable for almost every observed b_i. Furthermore, if bids satisfy common support, then combined with absolute continuity it follows that we can have $\Gamma_i(b_i|Z_{-i}) = 0$ or $\Gamma_i(b_i|Z_{-i}) = 1$ on at most a set of bids b_i having measure zero with respect to G_i. Hence for almost every bid b_i submitted by bidder i, we will have for all l

$$\lim_{\epsilon \to 0} \frac{\Delta^-_{\epsilon,l} b_i \Gamma_i(b_i|Z_{-i})}{\Delta^-_{\epsilon,l} \Gamma_{i,l}(b_i|Z_{-i})} = \lim_{\epsilon \to 0} \frac{\Delta^-_{\epsilon,l} b_i \Gamma_i(b_i|Z_{-i})/\epsilon}{\Delta^-_{\epsilon,l} \Gamma_{i,l}(b_i|Z_{-i})/\epsilon} = \frac{\partial(b_i \Gamma_i(b_i|Z_{-i}))/\partial b_{il}}{d\Gamma_{i,l}(b_i|Z_{-i}))/db_{il}},$$

and therefore $\Upsilon^-_{\epsilon,l}(b_i|Z_{-i}) \to \Upsilon_{\epsilon,l}(b_i|Z_{-i})$. Analogously, $\Upsilon^+_{\epsilon,l}(b_i|Z_{-i}) \to \Upsilon_{\epsilon,l}(b_i|Z_{-i})$, $\Psi^-_{\epsilon,l}(b_i|Z_{-i}) \to \Psi_{\epsilon,l}(b_i|Z_{-i})$, and $\Psi^+_{\epsilon,l}(b_i|Z_{-i}) \to \Psi_{\epsilon,l}(b_i|Z_{-i})$ almost surely. Applying the expectations operator, it follows that $\bar{\Upsilon}^-_\epsilon(\cdot) \to \bar{\Upsilon}(\cdot)$, $\bar{\Upsilon}^+_\epsilon(\cdot) \to \bar{\Upsilon}(\cdot)$, $\bar{\Psi}^-_\epsilon \to \bar{\Psi}(\cdot)$, and $\bar{\Psi}^+_\epsilon \to \bar{\Psi}(\cdot)$.

Hence, returning to the inequality (33) defining the robust expectations-based identified superset for the unknown complementarity vector κ_i, it follows that

$$\bar{\Upsilon}(Z_{-i}) - \bar{\Psi}(Z_{-i})\kappa_i \leq \bar{\Upsilon}(Z'_{-i}) - \bar{\Psi}(Z'_{-i})\kappa_i \quad \forall\, Z_{-i}, Z'_{-i} \in \mathcal{Z}_{-i}|Z_i, W, X.$$

Noting that Z_{-i}, Z'_{-i} are interchangeable, we thus have for any $Z_{-i}, Z'_{-i} \in \mathcal{Z}_{-i}|Z_i, W, X$:

$$\begin{aligned} \bar{\Upsilon}(Z_{-i}) - \bar{\Psi}(Z_{-i})\kappa_i &\leq \bar{\Upsilon}(Z'_{-i}) - \bar{\Psi}(Z'_{-i})\kappa_i \\ \bar{\Upsilon}(Z'_{-i}) - \bar{\Psi}(Z'_{-i})\kappa_i &\leq \bar{\Upsilon}(Z_{-i}) - \bar{\Psi}(Z_{-i})\kappa_i, \end{aligned}$$

or equivalently

$$\bar{\Upsilon}(Z_{-i}) - \bar{\Psi}(Z_{-i})\kappa_i = \bar{\Upsilon}(Z'_{-i}) - \bar{\Psi}(Z'_{-i})\kappa_i \quad \forall\, Z_{-i}, Z'_{-i} \in \mathcal{Z}_{-i}|Z_i, W, X.$$

But this is exactly the identification restriction invoked in Proposition 3 in the main text. In other words, even the coarsest identified superset we derive maintaining neither absolute continuity nor smoothness a priori will collapse to point identification as in the main text under Assumption 4. In this sense, Assumption 4 may be viewed as a sufficient (though not necessary) condition for point identification within a robust identification analysis maintaining only Assumptions 1-3 and 5.

F.4: Point identification of κ_i without common support

Finally, we consider identification dropping the common support condition of Assumption 4, but maintaining the hypothesis that marginal bid distributions G_{il} are absolutely continuous for every bid vector b_i with a positive probability of winning in auction l. More precisely, for each bidder i and each combination $\omega \in \Omega_i$, we assume that marginal distribution $G^\omega_i(b_{il}|Z_{il})$ of i's bids in combination ω admits a density for almost every bid vector such that $\Gamma_{il}(b_{il}|Z_{-i}) > 0$ in each auction l such that $\omega_l > 0$. In other words, we assume that i's bid distribution is smooth almost everywhere i's bids are relevant, although potentially involving mass points at bids which win with probability zero.[44] Such a case could arise, for example, if asymmetric bidders have different supports of standalone valuations, in which case some types of weak bidders may have no strictly profitable bid. Similarly, although we do not model reserve prices explicitly, a binding public reserve price in auction l could lead at least some types of some bidders to submit null bids in auction l. Interpreting these as bids below the reserve price, this would be isomorphic in our notation to a situation where bidder i submits a positive mass of bids with $\Gamma_{il}(b_{il}|Z_{-i}) = 0$.

Toward this end, maintain the smoothness conditions above, and consider any bid vector b_i

[44] The methods discussed here could also apply to bidders who bid such that $\Gamma_{il}(b_{il}|Z_{-i}) = 1$ with positive probability, but (except in cases of extreme asymmetry) such bids are typically difficult to rationalize as an equilibrium phenomenon. For ease of exposition, we therefore focus on cases where $\Gamma_{il}(b_{il}|Z_{-i}) = 0$, which are likely to arise more frequently in practice.

such that $\Gamma_{il}(b_{il}|Z_{-i}) > 0$. Bearing in mind that rival bid distributions are differentiable almost surely over all relevant marginals involving auction l, we then have $\Upsilon^-_{l,\epsilon}(b_{il}|Z_{-i}) \to \Upsilon_l(b_{il}|Z_{-i})$, $\Upsilon^+_{l,\epsilon}(b_{il}|Z_{-i}) \to \Upsilon_l(b_{il}|Z_{-i})$, $\Psi^-_{l,\epsilon}(b_{il}|Z_{-i}) \to \Psi_l(b_{il}|Z_{-i})$, and $\Psi^+_{l,\epsilon}(b_{il}|Z_{-i}) \to \Psi_l(b_{il}|Z_{-i})$ as $\epsilon \to 0$. Consequently, as in the last section, the two-sided bounds (32) collapse to the equality

$$v_{il} = \Upsilon_l(b_{il}|Z_{-i}) - \Psi_l(b_{il}|Z_{-i})^T \kappa_i,$$

where both $\Upsilon_l(b_{il}|Z_{-i})$ and $\Psi_l(b_{il}|Z_{-i})$ are identified from limits of identified objects as above. In other words, for each bid vector b_i such that $\Gamma_{il}(b_{il}|Z_{-i}) > 0$, there exists only one standalone valuation v_{il} consistent with the bid b_i given the complementarity vector κ_i.

Next, let b_i, b'_i be any two bid vectors in the support of G_i which differ only in their lth element; i.e., such that $b_{il} = b'_{ik}$ for all $k \neq l$. Without loss of generality, take $b'_{il} > b_{il}$, and suppose that $\Gamma_{il}(b_{il}|Z_{-i}) < \Gamma_{il}(b'_{il}|Z_{-i})$. Let K be any element of the set $\mathcal{K}_i$ of candidates for κ_i, and consider any candidates v_{il}, v'_{il} consistent with the bounds (32) evaluated at $\kappa_i = K$.[45] Observe that, taking $\epsilon = b'_{il} - b_{il}$, the bounds (32) satisfy the following marginal monotonicity property:

$$\begin{aligned} v'_{il} &\geq \Upsilon^-_{\epsilon,l}(b'_i|Z_{-i}) - \Psi^-_{\epsilon,l}(b'_i|Z_{-i})^T K \\ &= \Upsilon^+_{\epsilon,l}(b_i|Z_{-i}) - \Psi^+_{\epsilon,l}(b_i|Z_{-i})^T K \\ &\geq v_{il}, \end{aligned}$$

where the middle equality follows since, by definition, $\Upsilon^+_{\epsilon,l}(b_i|Z_{-i}) = \Upsilon^-_{\epsilon,l}(b'_i|Z_{-i})$ and $\Psi^+_{\epsilon,l}(b_i|Z_{-i}) = \Psi^-_{\epsilon,l}(b'_i|Z_{-i})$. In other words, fixing i's bids in other auctions, we require a higher standalone valuation in auction l to rationalize a higher (nontrivial) bid in auction l.

Now let $\underline{b}_{il} = \inf\{b_l : \Gamma_{il}(b_l|Z_{-i}) > 0\}$ be the infimum of bids at which i has a positive probability of winning auction l, and consider any bid vector b_i such that $b_{il} \leq \underline{b}_{il}$; note that $\underline{b}_{il}$ is implicitly determined by Z_{-i}, although we suppress this in notation. For any such bid, the lower bound (31) will be uninformative, but there will exist some $\epsilon > 0$ such that the upper bound (30) will be informative. Furthermore, we know this upper bound is bounded from above by the standalone valuation rationalizing the bid vector $(\underline{b}_{il}, b_{i,-l})$ in which i's actual bid b_{il} is replaced by the infimum undominated bid $\underline{b}_{il}$ in auction l. In other words, defining

$$\underline{v}_{il}(b_{i,-l}; Z_{-i}, \kappa_i) = \inf_{\epsilon > 0}\{\Upsilon^+_{l,\epsilon}(\underline{b}_{il}, b_{i,-l}|Z_{-i}) - \Psi^+_{l,\epsilon}(\underline{b}_{il}, b_{i,-l}|Z_{-i})^T \kappa_i\},$$

$b_{il} \leq \underline{b}_{il}$ implies $v_{il} \leq \underline{v}_{il}(b_{i,-l}; Z_{-i}, \kappa_i)$ almost surely, where $\underline{v}_{il}(b_{i,-l}; Z_{-i}, \kappa_i)$ is identified up to κ_i.

Finally, suppose that the support of i's undominated valuations is sufficiently rich in the following sense. For each auction l, define the maximum and minimum *marginal* complementarities i associates with object l as follows:

$$\begin{aligned} \Delta\kappa^+_{il} &= \max_{\omega} \kappa^{\omega + e_l} - \kappa^{\omega} : \omega_l = 0 \\ \Delta^-\kappa_{il} &= \min_{\omega} \kappa^{\omega + e_l} - \kappa^{\omega} : \omega_l = 1. \end{aligned}$$

Note that, by definition, $\Delta\kappa^+_{il}$ represents the maximum change in complementarity κ^{ω}_i induced by winning object l versus not, while $\Delta\kappa^-_{il}$ reflects the minimum such change. We then maintain the following assumption on the support of the conditional distribution of V_{il} given $V_{-i,-l}$:

[45] Recall that we restrict attention to the set $\mathcal{K}_i$ such that such candidate valuation vectors exist almost surely.

Assumption 8. *There exists $\alpha > 0$ such that for all realizations $v_{i,-l}$ in the support of $V_{i,-l}$,*

$$\Pr(V_{il} + \Delta\kappa_{il}^{-} \geq \max_{j \neq i}\{\underline{v}_{jl} + \Delta\kappa_{jl}^{+}\} | V_{i,-l} = v_{i,-l}) \geq \alpha,$$

where $\underline{v}_{jl}$ is the infimum of the support of bidder j's valuation in auction l. In other words, bidder i's minimum marginal valuation (over combinations) exceeds the minimum of each rival j's maximal marginal valuation (over combinations) with probability bounded away from zero.

Observe that bidder i would always be willing to win auction l at any price below $V_{il} + \Delta\kappa_{il}^{-}$, while bidder j would never be willing to win auction l at any price above $V_{jl} + \Delta\kappa_{jl}^{+}$. Hence, under Assumption 8, at least fraction α of i's bids in auction l will be undominated, regardless of how i bids in other auctions. For these bids, we may identify candidate standalone valuations v_{il} exactly for any $K \in \mathcal{K}_i$. Furthermore, by marginal monotonicity, we know that these candidates will always contain (at least) the α highest quantiles of the marginal distribution of V_{il}, although the lower tail of V_{il} may be truncated by a threshold $\underline{v}_{il}(b_{i,-l}; Z_{-i}, K)$ which varies with $b_{i,-l}$.

Bearing these facts in mind, let $\tilde{V}_{il}$ be the random variable defined by

$$\tilde{V}_{il} = \begin{cases} \underline{v}_{il}(B_{i,-l}; Z_{-i}, K), & B_{il} < \underline{b}_{il} \\ \Upsilon_l(B_{il}|Z_{-i}) - \Psi_l(B_{il}|Z_{-i})^T K, & B_{il} \geq \underline{b}_{il} \end{cases},$$

and let $\tilde{F}_{il}(\cdot|Z_{-i}; K)$ denote the marginal c.d.f. of $\tilde{V}_{il}$. In view of the analysis above, at $K = \kappa_i$, we must have (at least)

$$\tilde{F}_{il}(v_l|Z_{-i}; K) = F_{il}(v) \quad \text{for all } v_l \geq \underline{\nu}_{il}(Z_{-i}, K),$$

where $\tilde{F}(\cdot|Z_{-i}; K)$ is identified up to the candidate complementarity vector $K \in \mathcal{K}_i$, and

$$\underline{\nu}_{il}(Z_{-i}, K) = \sup_{b_{i,-l}} \underline{v}_{il}(b_{i,-l}; Z_{-i}, K)$$

denotes the identified supremum of $\underline{v}_{il}(b_{i,-l}; Z_{-i}, K)$ across bids submitted by i in other auctions.

Furthermore, recall that the true distribution $F_{il}(\cdot)$ is invariant to Z_{-i}. Hence, for any two competition structures Z_{-i}, Z'_{-i}, we must have

$$\tilde{F}_{il}(v_l|Z_{-i}; \kappa_i) = \tilde{F}_{il}(v_l|Z'_{-i}; \kappa_i) \quad \text{for all } v_l \geq \max\{\underline{\nu}_{il}(Z_{-i}, \kappa_i), \underline{\nu}_{il}(Z'_{-i}, \kappa_i)\}, \tag{37}$$

where all objects in (37) are identified up to κ_i.

Since, under Assumption 8, both candidate CDFs must assign probability of at least $1 - \alpha$ to valuations above their respective truncation points, the system (37) will define a continuum of equations in the unknown vector κ_i for each distinct pair of competition structures Z_i and Z'_{-i}. Intuitively, these require that at $K = \kappa_i$, (at least) the α highest quantiles of the conjectured distribution $\tilde{F}_{il}(\cdot|Z_{-i}; K)$ be invariant to Z_{-i}. Since each such quantile is a nonlinear function of K, it is difficult to provide formal sufficient conditions for a unique solution. Nevertheless, it is clear that this system will generically be overdetermined. Hence so long as the support of standalone valuations is sufficiently rich and the distribution of undominated bids is differentiable a.e., one will typically obtain point identification of κ_i even without common support, although one will be able to point-identify $F_i(\cdot)$ only over the region of valuations leading to undominated bids.

Appendix G: Nonparametric tests for complementarities

While our main analysis focuses on identification, this Appendix briefly discuss approaches to testing for the presence of complementarities based on the bid data together with bidder-specific and individual-specific characteristics. Taking the complementarities function as deterministic, the condition of the absence of complementarities can be written as

$$\kappa_i(Z, W_i) = 0 \quad \text{for each } i.$$

We can rely on testing the following testable implications on the distribution of bids:

$$H_0 : E\left[B_i|X, Z, W\right] = E\left[B_i|X, Z\right] \quad a.e.,$$

which can be equivalently written as

$$E\left[B_i - E\left[B_i|X, Z\right] \mid X, Z, W\right] = 0 \quad a.e.$$

The alternative hypothesis H_1 is the negation of the null.

There are a variety of tests in econometrics and statistics for testing the null hypothesis above. We could, e.g., use the test of Delgado and Manteiga (2003), which requires residuals obtained from nonparametric estimation under the null. We could also employ tests in Lavergne (2001) or Neumeyer and Dette (2003) that develop general tests for the equality of two nonparametric regression curves (the former work uses smoothing techniques whereas the latter is non-smoothing test). A test proposed in Racine (1997) looks as whether the partial derivatives of the regression function with respect to the variables being tested are zero.

Many of these tests require i.i.d. observations. In light of this, we can focus only on the bidders that participate in two (or some other fixed number of) auctions and construct test statistics by picking one such bidder per letting date.

Appendix H: Monte Carlo simulation study

Finally, we report results of two Monte Carlo experiments evaluating the estimators developed in Section 5. In practice, the main constraint on design of these simulations is solving for equilibrium bidding strategies; numerically speaking, this is a very challenging problem, since standard ordinary-differential-equation methods for solving equilibrium bids no longer apply in simultaneous auctions with nonseparable preferences. In view of this constraint, we explore two distinct Monte Carlo designs, both considering one global bidder against many local bidders. In the first, we approximate equilibrium in a two-auction market numerically over a finite grid. In the second, we solve best responses exactly for a global bidder competing in three auctions against local rivals who bid according to log-normal distributions. In each case, we specify complementarities as linear in combination characteristics W_{it}: $\kappa_{it} = W_{it}\theta_0$. Our objective is to estimate θ_0.

Toward this end, we employ the matching GMM procedure outlined in Section 5. Specifically, differencing the key first-order condition (12) across distinct bidder-letting-auction pairs itl and $j\tau\ell$, we obtain matched residual differences of the form

$$\epsilon_{it,l} - \epsilon_{j\tau,\ell} = (\Upsilon_{it,l} - \Upsilon_{j\tau,\ell}) - (\Psi^T_{it,l}W_{it} - \Psi^T_{j\tau,\ell}W_{j\tau}) \cdot \theta_0. \tag{38}$$

Our maintained exclusion restrictions imply that, conditional on matching observations such that $Z_{it} = Z_{j\tau}$ and $X_{tl} = X_{\tau\ell}$, the differenced residuals $\epsilon_{itl} - \epsilon_{j\tau\ell}$ are independent of the instrument vector $I_{itl,j\tau\ell} = [Z_{-i,t}, Z_{-j,\tau}, X_{t,-l}, X_{\tau,-\ell}, W_{it}, W_{j\tau}]$. Specializing to mean independence, this implies

$$E[I_{itl,j\tau\ell} \cdot (\epsilon_{itl} - \epsilon_{j\tau\ell})|Z_{it} = Z_{j\tau}, X_{tl} = X_{\tau\ell}] = 0. \tag{39}$$

Furthermore, since rival characteristic $Z_{-i,t}$ shift $\Psi_{it,l}^T$, while combination characteristics W_{it} shift the product $\Psi_{it,l}^T W_{it}$ both directly and indirectly through Ψ_{it}, the excluded variables $I_{itl,j\tau\ell}$ are relevant instruments for the endogenous "regressors" $(\Psi_{it}W_{it} - \Psi_{j\tau}W_{j\tau})$ multiplying θ_0.

In practice, we implement estimation based on (39) in two steps. In the first step, we substitute estimates $\hat{\Upsilon}_{it,l}, \hat{\Psi}_{it}$ for the directly identified objects $\Upsilon_{it,l}, \Psi_{it}$. In the second step, we form all nonredundant (or randomly selected) matched differences of the form (38) across relevant distinct observations $j\tau \neq it$. We kernel weight these matched differences based on distance between (Z_{it}, X_{tl}) and $(Z_{j\tau}, X_{\tau\ell})$, using a Gaussian kernel for continuous covariates. We then estimate Then estimate θ_0 using weighted optimal GMM on the resulting synthetic sample of matched differences, based on orthogonality conditions of the form $E[(\epsilon_{itl} - \epsilon_{j\tau\ell})W_{it}, Z_{-i,t}, W_{j\tau}, Z_{j\tau})] = 0$.

Appendix H.1: Equilibrium bidding on a finite grid

Our first Monte Carlo design considers one global bidder competing in two auctions against different sets of local rivals who bid in one auction only. Specifically, the global bidder faces n_1 local rivals in auction 1, and n_2 local rivals in auction 2, where each of n_1, n_2 vary on $\{1, 3\}$. Standalone valuations for each bidder are drawn i.i.d. across bidders and auctions from a log-normal distribution with mean parameter zero and scale parameter 0.5, with each valuation shifted up by one so that the lower limit of support is $\underline{v} = 1$. The complementarity for the global bidder is given by $\kappa_{12} = \theta_1 + W\theta_2$, where W is a scalar observable taking values in $\{-0.3, -0.1., 0.1\}$, the true parameters are $\theta_1 = 0$ and $\theta_2 = 1$, and negative complementarities imply lower joint valuations. The true complementarity κ_{12} thus takes three possible values: $\kappa_{12} \in \{-0.3, -0.1, 0.1\}$. For this exercise, we abstract from covariates X shifting the distribution of valuations; we introduce these in our second Monte Carlo exercise below. There are thus 12 possible configurations $\{n_1, n_2, \kappa_{12}\}$ of the data generating process.

Even in this simplified setting, solving for equilibrium on a continuous bid space would be a formidable computational challenge, requiring solution of a system of partial differential equations which themselves depend on integrals over non-rectangular, a priori unknown, subsets within the space of valuations. In view of this challenge, we instead solve for equilibrium restricting bidding to a finite grid. Specifically, in each auction, bidders may bid on a 100-element grid evenly spaced on the interval between $\underline{v} - \min\{\kappa_{12}, 0\}$ (the lowest possible marginal valuation) and 2.8 (a non-binding upper limit). If there is a unique high bidder, this bidder wins; otherwise, ties are broken i.i.d. across bidders and auctions.[46]

For each configuration $\{n_1, n_2, \kappa_{12}\}$, we approximate equilibrium by iterating over simulated best responses. For each bidder, we first draw 120,000 standalone valuations from the ex ante distribution. For each simulated draw for each bidder, we calculate best-responses against the simulated distribution of rival bids at the last iteration. We then update a randomly selected fraction of bids for each bidder (in practice, 5 percent) to their best response values and proceed.

[46] In practice, to ensure strictly positive winning probabilities, we further assume that each bidder perceives a small probability (0.001) of exogenously facing no rivals in each auction. This is purely for purposes of numerical stability, and should not affect the estimator otherwise.

At each iteration, we compute each bidder's proportional profit lost, defined as the ratio of simulated average profit at current bids to simulated average profit at best-response bids, assuming rivals bid as in the last iteration. We stop when each bidder's proportional profit lost falls below 10^{-6}, implying that no bidder can increase average profit by more than one ten-thousandth of a percent by unilaterally changing their bidding strategy. While more efficient algorithms could doubtless be developed, this simple iterative scheme leads to steady convergence for all DGPs we consider, typically reaching an approximate equilibrium in between 2000 and 3000 iterations.

Having solved for approximate equilibria, we implement $R = 1000$ Monte Carlo replications as follows. For each replication $r = 1, ..., R$ and each of the 12 possible configurations of $\{n_1, n_2, \kappa_{12}\}$, we draw a subsample of T bids (with replacement) for each bidder from our original sample of 120,000 simulated equilibrium bids. For each bidder, we estimate perceived winning probabilities on the discrete bid grid using the empirical distribution of resampled rival bids. We smooth these estimated probabilities using third-order polynomial interpolation across the discrete grid. We then form estimates $(\hat{\Upsilon}_i, \hat{\Psi}_i)$ for the terms (Υ_i, Ψ_i) appearing in (38) on levels and gradients of this polynomial interpolation. For a small number of observations in the tails of the distribution, imprecise first-step estimates of winning probabilities lead to either very large or very small estimated $(\hat{\Upsilon}_i, \hat{\Psi}_i)$; we account for this by dropping observations having values of $\hat{\Upsilon}_i$ in the top and bottom 0.01 percent of the estimated distribution of $\hat{\Upsilon}_i$ for the global bidder. We then estimate θ_0 using matched-difference GMM as described above.

Since, in this simple exercise, all bidders draw from the same distribution of standalone valuations in all auctions, any two observations may in principle be matched with each other. In practice, we treat auction identifier (auction 1 or auction 2) as a dummy auction type, and perform matching based on this "covariate". To assess how the number of matches affects the performance of the estimator, we match each observation with $m \in \{1, 10, 100\}$ counterfactual observations selected at random from bidders participating in the same auction. We first estimate allowing matches only among observations for the global bidder, then also allowing matches between the global bidder and local bidders competing in the same-type auction.

The results of this exercise are reported in Table 9. On net, these are quite encouraging, and confirm that our theoretical estimators perform well. Estimates of θ_1 are essentially unbiased across all sample sizes, although precision improves considerably as sample size increases. Estimates of θ_2 are biased upward by approximately 10 percent in the smallest sample, with this bias dissipating to less than 1 percent in the largest sample. Not surprisingly, increasing the number of matches per bidder reduces mean squared error of both parameter estimates, especially in the largest sample considered. Encouragingly, however, this reduction is relatively small—moving from $m = 1$ match per observation to $m = 100$ matches per observation reduces root mean squared error by roughly 20 percent, with most of this gain appearing by $m = 10$—suggesting that our matching procedure can perform well even if relatively few matches per observation are available.

Interestingly, as we quadruple sample size from $T = 250$ to $T = 1000$ samples per configuration of (n_1, n_2, W), both standard deviation and root MSE fall by factors of roughly 3. This reduction is larger than the factor of 2 expected from the $\sqrt{T}$-convergence predicted for standard GMM, consistent with the fact that larger samples reduce error in both first-step distribution estimates and second-step GMM estimates. To assess which of these channels was more important, we started from the largest $T = 1000$ sample, but estimating two sets of first-step parameters on samples of size $T = 500$ for each configuration, finding bias closer to the $T = 500$ case than the $T = 1000$ case. This suggests that, at least in small to moderate samples, gains from improving first-step estimates are relatively larger than gains from more second-step GMM observations.

We emphasize that this estimation approach represents a continuous first-order approximation

Table 9: Matched Difference GMM estimates of complementarity parameters, using a continuous first-order approximation to data simulated on a discrete equilibrium. True complementarities for the global bidder are given by $\kappa_{12} = \theta_1 + \theta_2 W$, where $\theta_1 = 0$, $\theta_2 = 1$, and $W \in \{-0.3, -0.1, 0.1\}$. The global bidder faces $n_l \in \{1, 3\}$ local rivals in each auction $l = 1, 2$. In each Monte Carlo replication, estimates are based on T observations per configuration of $\{n_1, n_2, \kappa_{12}\}$, drawn randomly from a sample of 120,000 simulated equilibrium bids, using m randomly drawn matches per original bid.

Sample size: $T = 250$ lettings per configuration of (n_1, n_2, W)

		Mean Bias	Median Bias	Std Dev	Root MSE
Matched differences GMM, $m = 1$	θ_1	0.0032	0.0108	0.2209	0.2209
	θ_2	0.1054	0.0979	0.3305	0.3469
Matched differences GMM, $m = 10$	θ_1	0.0030	0.0147	0.1956	0.1956
	θ_2	0.1050	0.0975	0.2947	0.3128
Matched differences GMM, $m = 100$	θ_1	0.0045	0.0164	0.1955	0.1955
	θ_2	0.1040	0.1008	0.2932	0.3111

Sample size: $T = 500$ lettings per configuration of (n_1, n_2, W)

		Mean Bias	Median Bias	Std Dev	Root MSE
Matched differences GMM, $m = 1$	θ_1	0.0097	0.0165	0.1596	0.1599
	θ_2	0.0486	0.0480	0.2409	0.2457
Matched differences GMM, $m = 10$	θ_1	0.0107	0.0148	0.1622	0.1626
	θ_2	0.0432	0.0340	0.2441	0.2479
Matched differences GMM, $m = 100$	θ_1	0.0091	0.0172	0.1697	0.1700
	θ_2	0.0437	0.0390	0.2460	0.2498

Sample size: $T = 1000$ lettings per configuration of (n_1, n_2, W)

		Mean Bias	Median Bias	Std Dev	Root MSE
Matched differences GMM, $m = 1$	θ_1	-0.0056	-0.0019	0.0905	0.0906
	θ_2	0.0105	0.0124	0.1247	0.1251
Matched differences GMM, $m = 10$	θ_1	-0.0047	-0.0011	0.0764	0.0765
	θ_2	0.0083	0.0098	0.1121	0.1124
Matched differences GMM, $m = 100$	θ_1	-0.0046	-0.0024	0.0773	0.0774
	θ_2	0.0075	0.0115	0.1093	0.1095

to a discrete equilibrium; bidders are truly choosing bids to maximize profit on a finite grid, but we model them as choosing bids to satisfy a first-order condition on a continuous bid space. Furthermore, the granularity of the bid space in this simulation (less than 100 relevant bid points) is many orders of magnitude coarser than that in our application (in which the bid interval is one cent, with bids in hundreds of thousands or millions). One byproduct of this exercise is therefore to confirm that the canonical first-order approach we pursue in the main text can continue to perform well, even in the presence of much more discreteness than is typical in practice.

Finally, for completeness, we also compute identified sets for κ_i explicitly accounting for discreteness as described in Appendix F.2 above. Encouragingly, we find that these bounds are quite tight, even when the global bidder faces only two possible competition levels ($n_l = 1$ or $n_l = 3$) in each auction. Specifically, we consider nonparametric identified supersets based on the quantile invariance criterion described in Appendix F.2. We find supersets of approximately $[0.04, 0.16]$ when $\kappa_i = 0.1$, $[-0.14, -0.03]$ when $\kappa_i = -0.1$, and $[-0.36, -0.22]$ when $\kappa_i = -0.3$. Given the difficulties surrounding set inference, we do not consider estimation based on these sets in detail. Nevertheless, this exercise confirms that the robust identification sets we describe above in fact convey considerable information on κ_i, even in the face of significant atoms in bids.

Appendix H.2: Best response bidding against parametric rival distributions

While our first Monte Carlo exercise confirms that the matching estimator in Section 5 can yield informative estimates of complementarities, the computational challenges involved in solving for equilibrium render it difficult to extend the simulation exercise beyond a simple two-auction market. As a second Monte Carlo exercise, we therefore explore how our estimator performs in a richer environment with three simultaneous low-price sealed bid auctions, in which standalone costs for the global bidder are correlated across projects and both standalone costs and complementarities depend on project size. We consider one global bidder competing against many local rivals, assuming that the global bidder bids optimally against local rivals who bid according to log-normal distributions. As in Section 5, we estimate parameters of this distribution in a first step, then estimate complementarities for the global bidder using the kernel-weighted mean- and matched-difference estimators described above.

The data generating process for this exercise involves a single global bidder, competing in three auctions $l = 1, 2, 3$ against n_l symmetric local rivals, where the number of rivals n_l varies independently across auctions on $\{2, 4, 6\}$. Each project has a size x_l which is common knowledge to all bidders, drawn independently from a log-normal distribution with mean parameter $\mu_x = 0$ and scale parameter $\sigma_x = 0.5$. The global bidder's standalone cost for competing project l is given by $c_l = \exp(x_l + e_l)$, where e_l is a standalone cost shock whose marginal distribution is truncated normal with mean parameter $\mu_e = 2$ and standard deviation parameter $\sigma_e = 0.5$, truncated on the interval $[0, 4]$. To allow for correlation in the global bidder's cost shocks, we further incorporate dependence between $[e_1, e_2, e_3]$ via a Gaussian copula with correlation parameter $\rho = 0.5$. Except for the fact that the marginal distributions are truncated at 4 standard deviations, $[e_1, e_2, e_3]$ are thus nearly jointly normal with correlation $\rho = 0.5$. The complementarity the global bidder associates with a combination ω varies with the sum of the sizes of projects won: $\kappa^\omega = \mathbb{I}\left[\sum \omega \geq 2\right] \cdot [\theta_1 + \theta_2(\omega^T x)]$, where $\theta_1 = -0.5$ reflects the economies of scope arising from a small joint win, and $\theta_2 = 0.2$ reflects the rate at which these economies of scope decrease as combination size $\omega^T x$ increases.

Each local rival in auction $l = 1, 2, 3$ bids according to a log-normal distribution which depends on the size x_l of project l and the number of bidders $n_l + 1$ in auction l, with the mean and scale

Table 10: Matched Difference GMM estimates of complementarity parameters for a global bidder competing against local rivals who bid according to log-normal distributions. True parameters are $\theta_1 = -0.5$, $\theta_2 = 0.2$. Results based on $T = \{250, 500, 1000\}$ three-auction lettings.

		Median Bias	Mean Bias	Std Dev	Root MSE
Matched differences GMM, $T = 250$	θ_1	0.0211	0.0312	0.2749	0.2755
	θ_2	-0.0148	-0.0180	0.2839	0.2841
Matched differences GMM, $T = 500$	θ_1	0.0096	0.0135	0.1924	0.1925
	θ_2	-0.0076	-0.0144	0.1933	0.1934
Matched differences GMM, $T = 1000$	θ_1	0.0089	0.0111	0.1405	0.1407
	θ_2	0.0074	0.0028	0.1352	0.1353

parameters of this distribution calibrated to match the mean and standard deviation of log bids that would arise among $n_l + 1$ local bidders drawing standalone costs c_l distributed as above.[47] Meanwhile, the global bidder chooses bids $[b_1, b_2, b_3]$ optimally against anticipated play by local rivals, given its private cost information $e = [e_1, e_2, e_3]$, the commonly known vector of project sizes $x = [x_1, x_2, x_3]$, and the set of local rivals $z = [n_1, n_2, n_3]$ faced in each auction.

We simulate T observations for the global bidder under the data generating process above, drawing numbers of local rivals, costs for the global bidder, and bids for local rivals according the the process above, then solving for the global bidder's optimal bids $[b_1, b_2, b_3]$ against the expected distribution of play by local rivals. Using this sample of S simulated lettings, we estimate the parameters of the bid distribution among local rivals by maximum likelihood. We translate these first-step parameters into estimates $(\hat{\Upsilon}_i,\ \hat{\Psi}_i)$ for the terms $(\Upsilon_i,\ \Psi_i)$ appearing in the first-order condition of the global bidder. We then estimate (θ_1, θ_2) using optimal GMM on kernel matched differences as above, weighting potential matches by differences in standalone project size based on a Gaussian kernel. As instruments in the second-step GMM, we employ the number of rivals in the current auction, the size of the current auction, the sum of rivals faced in other auctions, the sum of sizes of other projects, the product of size and rivals in the current auction, and the product of size and rivals in other auctions.

Table 10 reports the results of this exercise for samples of $T = 250$, $T = 500$, and $T = 1000$ three-auction lettings respectively. Encouragingly, our estimator performs well even in moderately sized samples. Even for the smallest sample, $T = 250$, estimates exhibit little bias, although standard deviations are somewhat imprecise. As sample size increases, both bias and standard deviation fall, with θ significantly different from zero for both $T = 500$ and $T = 1000$ (although for these parameters θ_2 becomes individually significant only for sample sizes larger than $T = 1000$). Bearing in mind these sample sizes are quite small relative to our empirical application (in which we observe approximately $T = 6000$ multi-auction bidders), this exercise offers encouraging evidence regarding the finite-sample properties of our matching GMM estimator.

[47] That is, we calibrate parameters of the log-normal distribution to approximate the distribution of equilibrium bids that would arise if cross-auction complementarities were zero.

Table 11: Simulated comparison of VCG versus FPA revenue

(a) Same bidder strong in both auctions, $K = 0$ for both bidders

Degree of Asymmetry	FPA Revenue	VCG Revenue
$V_s \sim U[0,1]$, $V_w \sim U[0,1]$	0.6647	0.6668
$V_s \sim U[0,1.5]$, $V_w \sim U[0,1]$	0.8100	0.7782
$V_s \sim U[0,2]$, $V_w \sim U[0,1]$	0.9153	0.8331

(b) Same bidder strong in both auctions, $K = 0.3$ for weak bidder

Degree of Asymmetry	FPA Revenue	VCG Revenue
$V_s \sim U[0,1]$, $V_w \sim U[0,1]$	0.8141	0.8472
$V_s \sim U[0,1.5]$, $V_w \sim U[0,1]$	0.9412	1.0235
$V_s \sim U[0,2]$, $V_w \sim U[0,1]$	1.0357	1.1019

(c) Same bidder strong in both auctions, $K = 0.3$ for strong bidder

Degree of Asymmetry	FPA Revenue	VCG Revenue
$V_s \sim U[0,1]$, $V_w \sim U[0,1]$	0.8138	0.8472
$V_s \sim U[0,1.5]$, $V_w \sim U[0,1]$	0.9687	0.9075
$V_s \sim U[0,2]$, $V_w \sim U[0,1]$	1.0802	0.9333

(d) Each bidder strong in one auction, $K = 0.3$ for both bidders

Degree of Asymmetry	FPA Revenue	VCG Revenue
$V_s \sim U[0,1]$, $V_w \sim U[0,1]$	0.9708	1.0507
$V_s \sim U[0,1.5]$, $V_w \sim U[0,1]$	1.1179	1.1934
$V_s \sim U[0,2]$, $V_w \sim U[0,1]$	1.2347	1.2799

Appendix H.3: Numerical simulations of VCG versus FPA revenue

To explore how asymmetry and complementarities interact to shape VCG revenue performance, we also implemented a series of simple numerical simulations involving two potentially asymmetric bidders competing in two auctions, where one or both bidders assign a positive complementarity to winning both objects together. Specifically, we considered two bidders competing for two auctions, drawing valuations from uniform distributions with potentially asymmetric upper supports (representing asymmetric bidder strengths). In each auction, at least one bidder was "weak", drawing valuations from $U[0,1]$, while the other was "strong," drawing valuations from either $U[0,1]$ (in which case valuations are symmetric), $U[0,1.5]$, or $U[0,2]$. We considered two cases: one bidder strong in both auctions, and each bidder strong in one auction. We also endowed one or both bidders with a complementarity $K = 0.3$ for winning both objects together, varying whether this was assigned to neither bidder, the bidder weak in both auctions, the bidder strong in both auctions, or symmetrically to horizontally differentiated bidders. We then simulated VCG versus FPA revenue in each case, using the iterative algorithm described in Appendix H.1 to solve for an (approximate) FPA equilibrium on a grid of 40 possible bid values.

Table 11 summarizes the results of this simulation exercise. These confirm that either FPA or

VCG can yield higher revenue, depending on the interaction between asymmetry and complementarities. Intuitively, in this example, asymmetry alone favors FPA relative to VCG, by inducing aggressive bidding from the weak bidder. A positive complementarity for the strong bidder enhances this effect by increasing the strong bidder's incentives to bid aggressively to win both objects, leading to even higher revenue under FPA than VCG. By contrast, a complementarity for a relatively weak bidder has a relatively small effect on FPA bids (since the weak bidder seldom wins both objects), while substantially increasing the strong bidder's expected payments under VCG.

References

Aguirregabiria, V. and P. Mira (2008): "Structural Estimation of Games with Multiple Equilibria in the Data," Working Paper.

Aradillas-Lopez, A., B. Honoré, and J. Powell (2007): "Pairwise Difference Estimation with Nonparametric Control Variables," *International Economic Review*, 48, 1119–1158.

Athey, S. (2004): "Single Crossing Properties and Existence of Pure Strategy Equilibria in Games of Asymmetric Information," *Econometrica*, 69, 861–889.

Athey, S., J. Levin, and E. Seira (2011): "Comparing Open and Sealed Bid Auctions: Evidence from Timber Auctions," *The Quarterly Journal of Economics*, 126, 207 –257.

Ausubel, L. and P. Cramton (2004): "Auctioning Many Divisible Goods," *Journal of the European Economic Association*, 480–493.

Ausubel, L., P. Cramton, P. McAfee, and J. McMillan (1997): "Synergies in Wireless Telephony: Evidence from the Broadband PCS Auctions," *Journal of Economics and Management Strategy*, 6, 497–527.

Ausubel, L. and P. Milgrom (2006): "The Lovely but Lonely Vickrey Auction," in *Combinatorial Auctions*, ed. by Y. S. P. Cramton and R. Steinberg, Cambridge, MA: MIT Press, chap. 1.

Ausubel, L. M. and P. Milgrom (2002): "Ascending Auctions with Package Bidding," *Frontiers in Theoretical Economics*, 1.

Bajari, P. and J. Fox (2013): "Measuring the Efficiency of an FCC Spectrum Auction," *American Economic Journal: Microeconomics*, 5, 100–146.

Bajari, P. and L. Ye (2003): "Deciding between Competition and Collusion," *Review of Economics and Statistics*, 85, 971?989.

Balat, J. (2015): "Highway Procurement and the Stimulus Package: Identification and Estimation of Dynamic Auctions with Unobserved Heterogeneity," Working paper, University of Texas at Austin.

Cantillon, E. and M. Pesendorfer (2006): "Sequential Auctions with Synergy and Affiliation Across Auctions," Working paper, London School of Economics.

Chapman, J., D. McAdams, and H. Paarsch (2007): "Bounding Revenue Comparisons across Multi-Unit Auction Formats under e-Best Response," *American Economic Review*, 97, 455–458.

Chesher, A. and A. Rosen (2017): *Econometrica*, 85, 959–989.

Cramton, P. (1997): "The FCC Spectrum Auctions: An Early Assessment," *Journal of Economics and Management Strategy*, 6, 647–675.

——— (2006): "Simultaneous Ascending Auctions," in *Combinatorial Auctions*, ed. by Y. S. P. Cramton and R. Steinberg, Cambridge, MA: MIT Press, chap. 4, 99–114.

Cramton, P., Y. Shoham, and R. Steinberg (2006): *Combinatorial Auctions*, Cambridge, MA: MIT Press.

de Paula, A. (2013): "Econometric Analysis of Games with Multiple Equilibria," *Annual Review of Economics*, 5, 107–131.

De Silva, D. (2005): "Synergies in Recurring Procurement Auctions: An Empirical Investigation," *Economic Inquiry*, 43, 55–66.

De Silva, D., T. Jeitschko, and G. Kosmopolou (2005): "Stochastic Synergies in Sequential Auctions," *International Journal of Industrial Organization*, 23, 183–201.

De Vreis, S. and R. Vorha (2003): "Combinatorial Auctions: A Survey," *INFORMS Journal of Computing*, 13, 284–309.

Decarolis, F. (2017): "Comparing Public Procurement Auctions," *International Economic Review*, forthcoming.

Delgado, M. and W. Manteiga (2003): "Significance Testing in Nonparametric Regression Based on the Bootstrap," *The Annals of Statistics*, 29, 1469–1507.

Feldman, M., H. Fu, N. Gravin, and B. Lucier (2012): "Simultaneous Auctions are (Almost) Efficient," Working Paper, *http://arxiv.org/abs/1209.4703*.

Fevrier, P., R. Preget, and M. Visser (2004): "Econometrics of Share Auctions," Center for Research in Economics and Statistics (CREST) Working Paper.

Flambard, V. and I. Perrigne (2006): "Asymmetry in Procurement Auctions: Evidence from Snow Removal Contracts," *The Economic Journal*, 116, 1014–1036.

Gentry, M., T. Komarova, P. Schiraldi, and W. Shin (2019): "On Monotone Strategy Equilibria in Simultaneous Auctions for Complementary Goods," *Journal of Mathematical Economics*, Forthcoming.

Gentry, M. and T. Li (2014): "Identification in Auctions with Selective Entry," *Econometrica*, 82, 315–344.

Ghosh, G. (2012): "Simultaneous First-Price Auctions with Budget-Constrained Bidders," Working paper.

Groeger, J. R. (2014): "A Study of Participation in Dynamic Auctions," *International Economic Review*, 55, 1129–1154.

Guerre, E., I. Perrigne, and Q. Vuong (2000): "Optimal Nonparametric Estimation of First-Price Auctions," *Econometrica*, 68, 525–574.

——— (2009): "Nonparametric Identification of Risk Aversion in First-Price Auctions Under Exclusion Restrictions," *Econometrica*, 77, 1193–1227.

Hendricks, K., J. Pinkse, and R. Porter (2003): "Empirical Implications of Equilibrium Bidding in First Price, Symmetric, Common Value Auctions," *Review of Economic Studies*, 70, 115–145.

Honoré, B. and J. Powell (2005): "Pairwise Difference Estimation of Nonlinear Models," *Identification and Inference in Econometric Models*, 520–553.

Hortacsu, A. (2011): "Recent Progress in the Empirical Analysis of Multi-Unit Auctions," *International Journal of Industrial Organization*, 29, 345–349.

Hortacsu, A. and D. McAdams (2010): "Mechanism Choice and Strategic Bidding in Divisible Good Auctions: An Empirical Analysis of the Turkish Treasury Auction Market," *Journal of Political Economy*, 118.

Hortacsu, A. and S. Puller (2008): "Understanding Strategic Bidding in Multi-Unit Auctions: A Case Study of the Texas Electricity Spot Market," *RAND Journal of Economics*, 39, 86–114.

Jackson, M. O., L. K. Simon, J. M. Swinkels, and W. R. Zame (2002): "Communication and Equilibrium in Discontinuous Games of Incomplete Information," *Econometrica*, 70, 1711–1740.

Jofre-Bonet, M. and M. Pesendorfer (2003): "Estimation of a Dynamic Auction Game," *Econometrica*, 71, 1443–1489.

Kastl, J. (2011): "Discrete Bids and Empirical Inference in Divisible Good Auctions," *Review of Economic Studies*, 974–1014.

Kawai, K. (2011): "Auction Design and the Incentives to Invest: Evidence from Procurement Auctions," Working paper, UCLA.

Kim, S., M. Olivares, and G. Weintraub (2014): "Measuring the Performance of Large-Scale Combinatorial Auctions: A Structural Estimation Approach," *Management Science*, 60, 1180–1201.

Klemperer, P. (2008): "A New Auction for Substitutes: Central Bank Liquidity Auctions, the U.S. TARP, and Variable Product-Mix Auctions." Working paper, Oxford University.

——— (2010): "The Product-Mix Auction: A new auction design for differentiated goods," *Journal of the European Economic Association*, 8, 526–536.

Kong, Y. (2018): "Sequential Auctions with Synergy and Affiliation Across Auctions," Working paper, Rice University.

Krasnokutskaya, E. (2011): "Identification and Estimation in Highway Procurement Auctions under Unobserved Auction Heterogeneity," *Review of Economic Studies*, 78, 293–327.

Krasnokutskaya, E. and K. Seim (2011): "Bid Preference Programs and Participation in Highway Procurement Auctions," *American Economic Review*, 101, 2653–2686.

Krishna, V. (2009): *Auction Theory*, Academic Press.

Krishna, V. and R. W. Rosenthal (1996): "Simultaneous Auctions with Synergies," *Games and Economic Behavior*, 17, 1–31.

Lavergne, P. (2001): "An Equality Test across Nonparametric Regressions," *Journal of Econometrics*, 103, 307–344.

Lebrun, B. (1999): "First Price Auctions in the Asymmetric N Bidder Case," *International Economic Review*, 40, 125–142.

Levin, D. and J. L. Smith (1994): "Equilibrium in Auctions with Entry," *The American Economic Review*, 84, 585–599.

Lewis, G. and P. Bajari (2011): "Procurement Contracting With Time Incentives: Theory and Evidence," *The Quarterly Journal of Economics*, 126, 1173–1211.

Li, T. and B. Zhang (2010): "Testing for Affiliation in First-Price Auctions Using Entry Behavior," *International Economic Review*, 51, 837–850.

——— (2015): "Affliation and Entry in First-Price Auctions with Heterogeneous Bidders: An Analysis of Merger Effects," *American Economic Journal: Microeconomics*, 7.

Li, T. and X. Zheng (2009): "Entry and Competition Effects in First-Price Auctions: Theory and Evidence from Procurement Auctions," *Review of Economic Studies*, 76, 1397–1429.

Lu, J. and I. Perrigne (2008): "Estimating Risk Aversion from Ascending and Sealed-Bid Auctions: The Case of Timber Auction Data," *Journal of Applied Econometrics*, 23, 871–896.

Lunander, A. and S. Lundberg (2013): "Bids and Costs in Combinatorial and Noncombinatorial Procurement Auctions – Evidence from Procurement of Public Cleaning Contracts," *Contemporary Economic Policy*, 31, 733–745.

McAdams, D. (2006): "Monotone Equilibrium in Multi-Unit Auctions," *Review of Economic Studies*, 73, 1039–1056.

Milgrom, P. (2000a): "Putting Auction Theory to Work: Ascending Auctions with Package Bidding," Working Paper, Stanford University.

——— (2000b): "Putting Auction Theory to Work: The Simultaneous Ascending Auction," *Journal of Political Economy*, 108, 245–272.

Milgrom, P. and R. Weber (1985): "Distributional Strategies for Games with Incomplete Information," *Mathematics of Operations Research*, 10, 619–632.

Moreno, D. and J. Wooders (2011): "Auctions with heterogeneous entry costs," *The RAND Journal of Economics*, 42, 313–336.

Moreton, P. and P. Spiller (1998): "Whats in the Air: Interlicense Synergies and Their Impact on FCC Broadband PCS Spectrum Auctions," *Journal of Law and Economics*, 41, 677–716.

Neumeyer, N. and H. Dette (2003): "Nonparametric Comparison of Regression Curves: An Empirical Process Approach," *The Annals of Statistics*, 31, 880–920.

Newey, W. and D. McFadden (1994): "Consistent Significance Testing for Nonparametric Regression," *Handbook of Econometrics*, 4, 2111–2245.

Racine, J. (1997): "Large sample estimation and hypothesis testing," *Journal of Business and Economic Statistics*, 15, 369–378.

Reguant, M. (2014): "Complementary Bidding Mechanisms and Startup Costs in Electricity Markets," *Review of Economic Studies*, 81, 1708–1742.

Reny, P. J. (2011): "On the Existence of Monotone Pure?Strategy Equilibria in Bayesian Games," *Econometrica*, 79, 499–553.

Roberts, J. and A. Sweeting (2013): "When Should Sellers Use Auctions?" *American Economic Review*, forthcoming.

Rosenthal, R. and R. Wang (1996): "Simultaneous Auctions with Synergies and Common Values," *Games and Economic Behavior*, 17, 32–55.

Somaini, P. (2015): "Competition and Interdependent Costs in Highway Procurement," Working Paper, Stanford University.

Syrgkanis, V. (2012): "Bayesian Games and the Smoothness Framework," Working paper, *http://arxiv.org/abs/1203.5155.*

Szentes, B. and R. Rosenthal (2003): "Three-object Two-bidder Simultaneous Auctions: Chopsticks and Tetrahedra," *Games and Economic Behavior*, 44, 114–133.

Wolak, F. (2007): "Quantifying the Supply-Side Benefits from Forward Contracting in Wholesale Electricity Markets," *Journal of Applied Econometrics*, 22, 1179–1209.