N-Player Preemption Games *

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Abstract

This paper studies infinite horizon complete information preemption games with N players. We consider a continuous time model where firms have to choose a point in time at which they seize an opportunity to make an irreversible one-time investment. Upon investment, firms compete with other firms that have already invested. Flow profits are declining in the number of investors but the cost of investing declines over time. Our model captures environments such as new product introduction or entry into a growing market. We show that there exists a unique subgame perfect Nash equilibrium outcome. Payoffs are equalized. Firms' investments can be clustered, although coordination failures are ruled out and investment is rivalrous. Increasing the number of competitors in the investment game does not necessarily accelerate investment. In particular, the first investment in the two-player game is a lower bound for the first investment in any N-player game with linear Cournot competition.

KEYWORDS: Timing Games, Preemption, Technology Adoption, Dynamic Entry.

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1 Introduction

Many economic settings can be modeled as a game in which players' payoffs depend crucially on the timing of their actions. Examples include a firm's decision when to enter or exit a market, adopt a new technology, introduce a new product or discontinue an old product.

Consider the case of a dynamic game in which firms have to decide when to enter a market with growing demand: the timing of entry will determine the level of demand and the number of competitors a firm faces. If a firm enters the market prior to its rivals it will earn monopoly profits, but only until the first rival enters.

The literature distinguishes two classes of such *timing games*. In a *war of attrition*, delay exogenously decreases payoffs, but a player also has an incentive to wait because she prefers rivals to act before her. In a *preemption game* instead, delay exogenously increases payoffs, but a player also has an incentive to act early, because there is an early mover advantage. The war of attrition has been studied extensively, and the properties of a very general version of this game are well known (see for example Bulow and Klemperer (1999)). The preemption game on the other hand, has been studied primarily for the case of only two players following Fudenberg and Tirole (1985). We are often interested in how the degree of competition, such as the number of potential adopters of a new technology or the number of potential entrants into a market, affects the timing of actions and the distribution of payoffs. Thus, this paper studies the extension of preemption games to a finite number of players.

The analysis of two-player preemption games is greatly simplified by the fact that after the first player's action, the second player's decision problem becomes a standard single agent optimization problem. We develop an inductive technique that can be employed to solve the generalized game.

We study a dynamic, infinite-horizon investment game with a finite number of firms and observable actions. At the beginning of the game, a new investment opportunity becomes available. Investment can be interpreted as introduction of a new product or entry into a new market. There is a finite number of potential investors who have to decide when, if ever, to take this opportunity. Time is continuous, and at each moment in time, firms have to choose whether to invest or not. Investment is an irreversible stopping decision in our model, and each firm can invest at most once. The cost of investment is assumed to decrease over time.[1] A firm's flow payoff depends on whether it has invested or not, and how many rivals have already invested. In particular, we assume that the payoff of a firm increases when it invests, but investment is rivalrous: post-investment payoffs decline in the number of investors.

The incentives in this game can be summarized as follows: first, each firm would like to delay investment, as this reduces the cost of investing. Second, a firm's investment increases its flow payoffs, making early investment more attractive. Third, due to the rival nature of investment, a firm will want to invest before its rivals, thereby reducing their incentives to invest. This is the *preemption* incentive. Our model can be viewed as an extension of the duopoly preemption game studied by Fudenberg and Tirole (1985) to the case of a finite number of players. We are able to generalize the analysis to a finite number of players, because we assume that pre-investment payoffs are independent of the number of earlier investors. Thus the applicability of our model is limited to situations like new market entry and new product introduction, while Fudenberg and Tirole's (1985) model also captures process innovation.

We show that the subgame perfect Nash equilibrium outcome in this game is unique, up to a relabeling of players.[2] We employ an inductive technique to characterize this unique outcome. Fudenberg and Tirole (1985) find that for the case of two players, if investment does not affect the payoff of non-investors, then equilibrium payoffs are equalized and investment times are distinct.[3] We show that only the former result extends to a game with more than two players.

As in the two-player game, *payoffs are equalized*. In particular, equilibrium payoffs correspond to the equilibrium payoff of the last investor in an analogous game with unobservable actions, such as the one analyzed in Reinganum (1981*a*,*b*). An immediate consequence of this rent equalization result is that any regulatory intervention affecting flow payoffs before the last investment has occurred, will not affect the equilibrium payoff.

[1] We focus on this version of the model as it corresponds to that considered in Reinganum (1981a, 1981b) and Fudenberg and Tirole (1985), which will be the main benchmarks for our results. In Appendix A we present an alternative model where the real cost of investing is constant but payoffs are increasing over time. With assumptions modified in the appropriate places, all equilibrium properties derived in the main part of the paper are preserved.

[2] In the proof of the main result of this paper, we construct all the subgame-perfect Nash equilibria and show that they all yield the same outcome up to a permutation of players.

[3] If instead first investment by a player reduces the payoff of the player who has not invested yet, then Fudenberg and Tirole (1985) show that equilibria with late joint-investment are possible. A player who is preempted has an incentive to invest soon after the rival. This credible threat deters early investment and allows players to delay investment until it is optimal to act even if the rival acts simultaneously.

What distinguishes the game with more than two players from the duopoly game, is that equilibrium *investment times may be clustered*. In particular, we argue that there always exists a structure of flow profits which induces a cluster. This is surprising at first, as joint adoption or investment times are usually associated with the presence of coordination failures, informational spillovers, or network externalities, which are all absent in our model.

The intuition for our result can be described in the context of a three-player game. Once the first firm has invested, the ensuing subgame is a two-player preemption game, where the preemption motive accelerates the next investment. This implies that monopoly profits will be enjoyed by the first investor only for a short period of time, due to the acceleration of the next investment. Preemption thus becomes less attractive for the first investor, the stronger the preemption motive in the ensuing two-player subgame. If preemption in the two-player subgame leads to sufficiently early second investment, the first investor will have no desire to preempt the second investor. This results in a cluster of the first two firms investing simultaneously.

Another interesting feature of the preemption game is that increasing the number of players does not necessarily accelerate the first investment. In particular, we show that for the case of Cournot competition with linear demand and exponentially declining investment cost, the equilibrium first investment time in the 2-player game is a lower bound for the first investment time in any N-player game.

Related Literature

The properties of preemption games are very well understood for the special case of a game with only two players. Seminal papers by Reinganum (1981a) and Fudenberg and Tirole (1985) analyze its properties for the case of unobservable and observable actions, respectively. Numerous papers have studied variations of this preemption game with two players, changing the assumptions on the information structure or the payoff structure. For a survey, see Hoppe (2002). Hoppe and Lehman-Grube (2005) study a version of this game with a general payoff structure.

However, little is known about the properties of the general game with many players. Reinganum (1981b) derives the equilibrium of a game of technology adoption with N firms and unobservable actions.[4] Other special cases have been studied in the recent literature. Anderson and Engers (1994) study a modified game where there is a time window in which players decide to act and payoffs of a static game played only among those who have acted are collected after expiration of this time window. Park and Smith (2006) study general timing games, assuming that actions are unobservable. Goetz (1999) discusses the case of a continuum of firms where the preemption motive is absent. Levin and Peck (2003) develop a duopolistic model of market entry with private information about the cost of entry, in which the motivation for delay is not cost-saving, but the risk of coordination failure. They also present the extension to N firms, restricting the analysis to the Bertrand case where the game ends after one firm has entered. Our paper complements this literature by studying an environment with features that are present in many situations of interest: payoffs are affected immediately by agents' actions (such as technology adoption), actions by rivals are often observable (such as market entry), the preemption motive is important as long as the number of firms is finite, and markets can sustain more than one firm.

Strategic investment has also been studied in a real options framework. Greater uncertainty over the profitability of investment increases the option value of waiting and thus the tendency to delay investment. For recent examples see Weeds (2002) and references therein, as well as the survey by Hoppe (2002). In independent work that is closely related to ours, Bouis et al. (2006) study dynamic investment in oligopoly in a real options framework.[5] The advantage of the real options approach is that it allows for market level uncertainty in the payoff process, but comes at the cost of depending on a particular growth process for payoffs (a Brownian motion with drift).

The presence of clusters, or joint investment and adoption times, has sofar mainly been associated with coordination failures, as in Levin and Peck (2003), with the presence of positive network externalities, as in Mason and Weeds (2006), or with informational spillovers (e.g. Chamley and Gale (1994)), where rival investment signals a high profitability of investment. Brunnermeier and Morgan (2006) show that in equilibrium 'herding' occurs in a 'clock game' with preemption features, but attribute this herding effect to the fact that

[4] The assumption of unobservable actions is analogous to assuming that firms are able to pre-commit to an investment time. Thus the preemption motive is absent. While this paper is not concerned with pre-commitment games per se, the equilibrium investment times in the pre-commitment game turn out to be an important tool when constructing the equilibrium of our observable moves game.

[5] Our model of entry in a growing market presented in Appendix A coincides with their model without uncertainty, when we restrict ourselves to exponential growth. Bouis et al. (2006) do not prove provide an explicit proof of equilibrium existence, but derive comparative statics results conditional upon existence, which are similar to those obtained in this paper.

the private information of the first player to act is (partially) revealed by his decision to exit. In our model, clusters arise, although coordination failures are ruled out by assumption, and rival investment decreases the incentive to invest and has no informational content.[6]

The remainder of the paper is organized as follows. Section 2 introduces the model. We build on seminal work by Reinganum (1981*a*, 1981*b*) and Fudenberg and Tirole (1985) on a dynamic investment game, but consider the extension to a finite number of players. Following Hoppe and Lehmann-Grube (2005), we adopt the notion of time being continuous as being "discrete but with a grid that is infinitely fine" as introduced by Simon and Stinchcombe (1989).

In Section 3, we characterize the equilibrium outcome. In Section 3.1 we prove existence of symmetric and asymmetric subgame-perfect Nash equilibria, and that the outcome across all equilibria is unique, up to a relabeling of players. We present a simple algorithm to construct the outcome. With more than two players, the rent equalization results proved by Fudenberg and Tirole (1985) still holds, but investment times can be clustered. We discuss how equilibrium payoffs depend on the structure of flow payoffs, and compare equilibrium investment times and payoffs to those obtained in a game with pre-commitment. In Section 3.2 we discuss how equilibrium investment times depend on flow payoffs and characterize sufficient conditions for the presence of clusters. In Section 3.3 we analyze a slightly modified game in which we relax the initial assumption that game payoffs are sufficiently large so that all potential investors invest in finite time. Thus the number of eventual investors is determined endogenously. We show that in this case, equilibrium payoffs will still be equalized, but the more intense preemption motive drives their level to zero.

In Section 4, we consider a parametric version of our model in which the cost of investing declines exponentially, and investors sell a homogeneous good, competing in quantities and facing a linear demand curve. We show how instantaneous payoffs can be perturbed so that investment times are clustered. We compare the equilibrium investment times to those that would maximize industry profits and social welfare, respectively. Finally, we show that in this model, more competition in terms of increasing the number of players will not accelerate first investment. Section 5 concludes. All proofs are relegated to Appendix B.

[6]Quirmbach (1986, p. 36) argues that "declining incremental benefits are sufficient for the diffusion of adoptions" in a pre-commitment model.

2 Model

2.1 The Investment Game

We analyze an infinite horizon dynamic game in continuous time. At time 0, a new investment opportunity becomes available, and N players (firms) have to decide if, and when, to seize this opportunity. The investment opportunity can be interpreted as adoption of a new technology, or entry into a new market.

The set of firms is denoted by $\mathbf{N} = \{1, ..., N\}$ and a single firm is denoted by $i \in \mathbf{N}$.

The model corresponds to the one studied by Reinganum (1981*a*, 1981*b*) and Fudenberg and Tirole (1985) except for the following: Until a firm invests, it receives a constant flow of profits π_0 which we normalize to zero. This assumption that pre-investment payoffs are independent of the number of earlier investments will be essential for obtaining a unique outcome in each subgame, and rent equalization.[7] Upon investment, a firm earns flow profits of $\pi(m)$, where m is the number of firms who have already invested at a given point in time. Conditional upon the investment decision and the number of investors in the market, payoffs are symmetric.[8] Let $\boldsymbol{\pi} = [\pi(1), \pi(2), ..., \pi(N)]$ denote a flow profit structure.

Let $c(t)$ be the present value at time zero of the cost of investing at time t. Without loss of generality, we relabel firms so that j denotes the j-th investing firm. Let T^j denote the investment date of the j-th investor. We can write firm j's payoff as a function of investment times

$$V^j(T^1, T^2, ..., T^j, ..., T^N) = \sum_{m=j}^{N} \pi(m) \int_{T^m}^{T^{m+1}} e^{-rs} ds - c(T^j) \tag{1}$$

where r denotes the common discount rate, and $T^{N+1} \equiv +\infty$.

We introduce the following assumptions:

Assumption 1

(i) $\pi(m) > 0 \ \forall m$

(ii) $\pi(m)$ *is decreasing in* m

Assumption 1 states that investing always increases period payoffs for a firm, but the

[7] However, the normalization of pre-investment payoffs to zero, rather than any positive constant, will not affect the results of the paper.

[8] This assumption has been relaxed in the recent literature on two player games (e.g. Hoppe and Lehman-Grube (2005)).

benefits of investing decrease in the total number of investors: as more firms invest, competition among the investors becomes more intense.

Assumption 2

(i) $\left(c\left(t\right)e^{rt}\right)' < 0 \ \forall t$

(ii) $\left(c\left(t\right)e^{rt}\right)'' > 0 \ \forall t$

Assumption 2 determines the shape of the current value cost function $c\left(t\right)e^{rt}$. It states that the cost of investing declines over time. This may capture upstream process innovations or economies of learning and scale. Moreover, cost is assumed to decline at a decreasing rate.

Assumption 3

(i) $\frac{\pi(1)}{r} - c(0) < 0$

Assumption 3(i) guarantees that investing at time zero is too costly. No firm would invest immediately, even if it could thereby preempt all other firms and enjoy monopoly profits $\pi\left(1\right)$ forever.

Assumption 4

(i) $\exists \tau$ *such that* $c\left(\tau\right)e^{r\tau} < \frac{\pi(N)}{r}$

(ii) $\lim_{t\to\infty} c'(t)e^{rt} \in \left(-\pi\left(N\right), 0\right]$

Assumption 4(i) ensures that the value of investing becomes positive in finite time. The cost of investing eventually reaches a level sufficiently low, that it becomes profitable to invest, even for a firm facing maximum competition. Assumption 4(i) is necessary but not sufficient to guarantee that the last investment occurs in finite time. Assumption 4(ii) guarantees that the last investor does not have an incentive to delay investment indefinitely. We will relax this assumption in Section 3.3.

In Appendix A we present an alternative model in which the current cost of investing is constant, but payoffs are exogenously increasing. This illustrates how our results can be used to study situations such as entry into a growing market.

2.2 Strategies in Continuous-Time Preemption Games

A fundamental problem with modelling timing games is that backwards induction cannot be applied in continuous time. We follow Hoppe and Lehmann-Grube (2005), who address this issue by adopting the framework introduced by Simon and Stinchcombe (1989) for modelling games with complete information in continuous time. Namely, we restrict play to pure strategies and interpret continuous time as "discrete time, but with a grid that is infinitely fine." The discussion below closely follows Hoppe and Lehmann-Grube (2005).

In this framework, the question of how to associate an outcome to a continuous-time strategy profile is addressed in the following way. A continuous-time strategy is interpreted as "a set of instructions about how to play the game on every conceivable discrete-time grid." For any continuous-time strategy profile, a sequence of outcomes is generated by restricting play to an arbitrary sequence of increasingly fine discrete-time grids, and the limit of this sequence of outcomes is defined as the continuous-time outcome of the profile.[9]

A second, well-known problem with modelling preemption games in continuous time is that typically games in this class do not have an equilibrium in pure strategies.[10] The problem is related to the possibility of coordination failures. In their seminal paper on preemption games, Fudenberg and Tirole (1985) address this issue modelling mixed strategy equilibria of games in continuous time as limits of mixed-strategy equilibria of games in discrete time. In their framework, coordination failures are ruled out in equilibrium.

Since we adopt the Simon and Stinchcombe (1989) framework, we need to explicitly rule out the possibility of coordination failures, and we do so using a randomization device as in Katz and Shapiro (1987), Dutta, Lach and Rustichini (1995), and Hoppe and Lehmann-Grube (2005):

Assumption 5
If n firms invest at the same instant t (with $n \in [2, N]$), then only one firm, each with probability $\frac{1}{n}$, succeeds.

Assumption 5 rules out the possibility of coordination failures and thus ensures existence of an equilibrium in pure strategies.

To understand how the randomization device operates, suppose that $N = 3$ and at a given time t all three firms would like to invest, provided that no more than two do so. Assumption 5 and the interpretation of continuous time as "discrete-time, but with a grid

[9] The continuous-time outcome is well-defined only if the sequence of discrete-time outcomes converges to a unique limit, which is independent from the specific sequence of grids. Simon and Stinchcombe (1989) identify conditions for the existence and the uniqueness of this limit. In our game, to satisfy these conditions we shall assume that strategies are piecewise continuous with respect to calendar time, and insensitive to slight differences in the times at which previous investments occurred. (Notice that the times at which previous investments took place are payoff-irrelevant).

[10] See the example in Simon and Stinchcombe (1989, p. 1178-1179).

that is infinitely fine" guarantee that the following happens at time t: at first, all three firms try to invest, and only one is successful, i.e. only one actually pays the cost $c(t)$ and starts receiving flow payoffs $\pi(1)$. Then, "consecutively but at the same moment in time" (Simon and Stinchcombe (1989), p. 1181) the two remaining firms try again to invest, and only one of them is successful. Finally, the third firm realizes that it is not optimal anymore to invest at t, and the game continues.[11]

3 Equilibrium Analysis

In this section, we first characterize the subgame-perfect Nash equilibria (SPNE) of the preemption game, and prove that the SPNE outcome is unique, up to a permutation of players. In equilibrium, all firms receive the same payoff. Clusters of simultaneous investments are possible. We then present comparative statics results in order to capture the intuition behind the presence, or absence, of clusters of simultaneous investments. Next, we provide a simple condition on the flow profit structure $\boldsymbol{\pi}$ that guarantees the presence of a cluster. Finally, we investigate the consequences of relaxing Assumption 4, and show that in a model where the number of players who eventually invest is determined endogenously, rents are equalized and driven to zero by preemption, and even the last investment can be clustered with one or more others.

3.1 Investment Times and Rent Equalization

In order to construct the SPNE of the preemption game, it is useful to first characterize the equilibrium investment times T_j^* ($j = 1, ..., N$) of a game analogous to ours but with unobservable moves.

The assumption that players' moves are unobservable (infinite information lags) is equivalent to the assumption that firms are able to pre-commit themselves to investment dates. Reinganum (1981*b*) proves that in every pure-strategy equilibrium of such a game, the equilibrium outcome is the same, up to a permutation of players. The j-th equilibrium investment time can be calculated by maximizing the payoff of a firm *conditional* on it

[11] For an interpretation of the randomization device, see Dutta, Lach and Rustichini (1995).

being the j-th investor:

$$T_j^* = \underset{t}{\text{argmax}} \quad \pi(j) \int_t^{T_{j+1}^*} e^{-rs} ds + \sum_{m=j+1}^{N} \pi(m) \int_{T_m^*}^{T_{m+1}^*} e^{-rs} ds - c(t)$$

where $T_{N+1}^* \equiv +\infty$. Assumption 2 guarantees that this objective function is strictly quasi-concave, and assumptions 1, 3 and 4 guarantee that it admits a maximum in a finite T_j^*, for every j. Moreover, Fudenberg and Tirole (1985) prove that equilibrium payoffs decline monotonically in the order of investment thus creating an incentive for late investors to preempt early investors. Formally:

Lemma 1 *Let T_j^*, $j = 1, ..., N$, denote the pre-commitment equilibrium investment times. They are implicitly defined by:*

$$-\pi(j)e^{-rT_j^*} - c'\left(T_j^*\right) = 0. \tag{2}$$

Equilibrium payoffs satisfy

$$V^i(T_i^*, T_{-i}^*) > V^j(T_j^*, T_{-j}^*)$$

for $i < j$.

Notice that our assumptions about the flow profit structure $\boldsymbol{\pi}$ and the cost function $c\,(t)$ guarantee that $T_i^* < T_j^*$ for $i < j$.

Consider now the game with observable moves. Given our assumptions, no player acts at time zero and all players invest in finite time. More precisely, denoting by t_j the SPNE investment time of the j-th investor for $j = 1, .., N$, the following Lemma holds:[12]

Lemma 2 *In any SPNE it holds that:*
(i) No firm invests at $t = 0$.
(ii) All firms invest in finite time, and $t_N = T_N^*$.

Assumptions 1(ii) and 3 guarantee that investment at time zero is too costly. This fact will be crucial in proving that all players receive the same payoff in equilibrium. Relaxing assumption 3, one could generate an equilibrium in which some players invest at time zero and receive a higher payoff than the remaining players, who would instead invest later and receive all the same, lower payoff.

[12] Existence of SPNE, and uniqueness of the equilibrium outcome up to a permutation of players, will be proved in Proposition 1.

The result that the last investment time in the SPNE is the same as in the pre-commitment equilibrium is not surprising. With observable moves, t_N solves the one-player optimization problem of a firm who is aware of being the last active player:

$$\max_t \pi(N) \int_t^\infty e^{-rt} dt - c(t).$$

As discussed above, Reinganum (1981*b*) shows that the solution to this problem is exactly the N-th equilibrium investment time in the game with observable moves. An immediate consequence of this result is that if we increase the number of potential investors to $N+1$, leaving $\pi(j)$ for $j = 1, .., N$, constant, and $\pi(N+1)$ satisfies Assumptions 1 and 4, the last investment will occur later in both the observable and unobservable move games.

Next, we present the main result of this section.

Proposition 1 *The preemption game with N players admits a unique subgame perfect equilibrium outcome, up to a permutation of players. All players receive the same payoff:*

$$\frac{\pi(N)}{r} e^{-rT_N^*} - c\left(T_N^*\right).$$

The equilibrium investment times $(t_1, t_2 ..., t_N)$ can be calculated recursively according to the following algorithm:
(i) $t_N = T_N^*$;
(ii) For $j = 1, ..., N-1$,
if $t_{j+1} > T_j^*$, then $t_j < t_{j+1}$, and t_j solves

$$\pi(j) \int_{t_j}^{t_{j+1}} e^{-rs} ds - c(t_j) + c(t_{j+1}) = 0,$$

if $t_{j+1} \leq T_j^*$, then $t_j = t_{j+1}$.

Proposition 1 states that there exists a unique SPNE outcome and provides the algorithm to construct it.

When pre-investment payoffs are independent of the number of earlier investors, Fudenberg and Tirole (1985) prove that for the case of two players, equilibrium payoffs are equalized and investment times are diffused. In Proposition 1, we establish that with a general number of players the rent equalization result still holds, but it is possible that at one or more points in the sequence of equilibrium investments two or more players act simultaneously.

To illustrate our results, we show how to construct the unique equilibrium outcome for the case of three players. Let (T_1^*, T_2^*, T_3^*) denote the pre-commitment investment times, which satisfy equation (2). From Lemma 2 we know that all firms will have invested by T_3^*. In any subgame starting at $\tau \geq T_3^*$, all firms will invest immediately. Thus the game can be solved backwards from T_3^* and we only need to consider subgames starting at $\tau < T_3^*$. First, we consider subgames with only one active firm left, i.e. subgames in which the first two firms have already invested. This one firm faces a single agent decision problem. It optimizes investing at T_3^*, which yields payoff $\frac{\pi(3)}{r} e^{-rT_3^*} - c\left(T_3^*\right)$.

Now consider subgames with two remaining active firms. Each of these subgames is analogous to the game in Fudenberg and Tirole (1985) for the case of π_0 constant. The two active firms know that, if one invests, the other will follow at T_3^*. By investing at some $t \in \left[\tau, T_3^*\right)$, a firm would receive

$$\pi(2) \int_t^{T_3^*} e^{-rs} ds + \frac{\pi(3)}{r} e^{-rT_3^*} - c(t).$$

Alternatively, it could wait until T_3^* and obtain $\frac{\pi(3)}{r} e^{-rT_3^*} - c\left(T_3^*\right)$. Let $D_2(t)$ denote the difference between these two functions:

$$D_2(t) = \pi(2) \int_t^{T_3^*} e^{-rs} ds - \left[c(t) - c\left(T_3^*\right)\right]. \tag{3}$$

This function describes the preemption incentive in the subgame played by the last two firms. The first term represents the advantage of being the second investor rather than the third, and the term in brackets is the associated cost. The function is plotted in Figure A. It is strictly quasiconcave, it has a unique global maximum in T_2^*, and is equal to zero at T_3^*. For small values of t, it is negative: the cost of investing is very high, so each firm would be better off waiting until T_3^* rather than investing immediately to preempt the rival. Later, the function $D_2(t)$ intersects zero from below at some point T_2 and reaches its maximum in $T_2^* \in \left(T_2, T_3^*\right)$. The convexity of the current value cost function $\left(c\left(t\right) e^{rt}\right)$ guarantees that $D_2(t)$ admits a maximum: In terms of current value, the marginal cost of waiting is constant, and given by the profit flow $\pi(2)$ that is forgone, while the marginal benefit, given by the reduction in the investment cost, is initially very high, but it decreases, and after T_2^*

becomes lower than the marginal cost of waiting.

The fact that the maximum of $D_2(t)$ is achieved exactly in T_2^*, the second equilibrium time of the game with precommittment might appear surprising at first. The reason is that the marginal cost and benefit to delay investment do not depend on either previous nor following investments. Hence the first-order condition of the problem

$$\max_t \ \pi(2)\int_t^{T_3^*} e^{-rs}ds + \pi(3)\int_{T_3^*}^{+\infty} e^{-rs}ds - c(t),$$

namely

$$-\pi(2)e^{-rT_2^*} - c'\left(T_2^*\right) = 0$$

that identifies the optimal time for the second investment in a game with pre-commitment, is also the first order condition of the problem

$$\max_t D_2(t) = \pi(2)\int_t^{T_3^*} e^{-rs}ds - \left[c(t) - c\left(T_3^*\right)\right].$$

In other words, even in the game without pre-commitment, if a player were guaranteed to be the second investor, he would choose to invest at T_2^*.

The fact that the function $D_2(t)$ is strictly positive in T_2^*, hence in the whole interval (T_2, T_3^*), is a direct consequence of Lemma 1: If firms invest at their pre-commitment equilibrium investment times, earlier investors receive a larger payoff than later investors. In particular, $V^2(T_1^*, T_2^*, T_3^*) > V^3(T_1^*, T_2^*, T_3^*)$. Since game payoffs do not depend on earlier investment times, this in turn implies that $V^2(T^1, T_2^*, T_3^*) > V^3(T^1, T_2^*, T_3^*)$. This creates the incentive for preemption and guarantees that the second SPNE investment time t_2 is exactly equal to T_2. None of the two active firms will invest before T_2, because for $t < T_2$ the payoff from being the second investor is smaller than the payoff from being the third investor. Also, the second investment cannot occur in the interval (T_2, T_3^*) because then the second investor would receive a higher payoff than the third, and last, investor. The latter would thus have an incentive to deviate and preempt the former. Therefore, the threat of preemption causes rent equalization and the strict quasi-concavity of expression (3), guaranteed by assumption 2, pins down the second investment time.

Now consider the problem the three firms are facing at the beginning of the game. First, it can be shown that the first investment cannot occur *after* t_2. In this case, one more firm would immediately invest after the first investment, since the payoff of the second investor is larger than the payoff of the third investor in the interval (t_2, T_3^*). But this cannot be an equilibrium because the first two investors would receive a higher payoff than the last investor, who would therefore have an incentive to preempt them.

At the beginning of the game, each firm therefore knows that it should either invest at some time t strictly before t_2, or wait exactly until t_2. In either case, the two remaining firms will invest at $t_2 = T_2$ and $t_3 = T_3^*$ respectively. They will both receive payoff $\frac{\pi(3)}{r} e^{-rT_3^*} - c\left(T_3^*\right)$. If the first investor invests at $t < t_2$ his payoff is

$$\pi(1) \int_t^{t_2} e^{-rs} ds + \pi(2) \int_{t_2}^{T_3^*} e^{-rs} ds + \pi(3) \int_{T_3^*}^{\infty} e^{-rs} ds - c(t). \tag{4}$$

If instead he invests at t_2, his payoff is

$$\pi(2) \int_{t_2}^{T_3^*} e^{-rs} ds + \pi(3) \int_{T_3^*}^{\infty} e^{-rs} ds - c(t_2). \tag{5}$$

The difference between (4) and (5) is

$$D_1(t) = \pi(1) \int_t^{t_2} e^{-rs} ds - [c(t) - c(t_2)] \tag{6}$$

where the first term measures the advantage of being first rather than second or third, and the second term represents the extra-cost associated. The properties of expression (6) are analogous to those of expression (3): It is strictly quasiconcave and has a maximum exactly at T_1^*, the first equilibrium investment time in the game with unobservable moves.

Whether the first investment occurs strictly earlier than the second, or exactly at the same time, depends on the following. The threat of preemption from the third investor guarantees that the second investment occurs earlier than T_2^*. In particular, the second investment time t_2 might fall either in (T_1^*, T_2^*) or even earlier than T_1^*. In the first case, illustrated in Figure A, the intuition from the subgame with two active players carries over: the payoff difference between the first and the second investor is negative for very small values of t, zero at points T_1 and at t_2, and positive in (T_1, t_2) by strict quasiconcavity. The threat of preemption then guarantees that the first investment will occur exactly at $t_1 = T_1$ and that all players earn the same equilibrium payoffs.

If instead the second investment occurs before T_1^*, as illustrated in Figure B, no firm will want to invest strictly before t_2. The intuition is the following: before T_1^*, the marginal benefit of delaying investment, namely the reduction in $c(t)$, outweighs the marginal cost of forgoing monopoly profits. This results in the function $D_1(t)$ being increasing at t_2. Thus each firm would rather wait until t_2 than invest immediately. We already argued that the first investment cannot happen after t_2. Hence, there is a cluster of two investments at t_2. Preemption in the final 2-player subgame accelerates the second investment to such an extent, that it is too costly for any player to invest even earlier.

Equilibrium Payoffs. As illustrated in Proposition 1, in equilibrium all firms receive the same payoff. Rent equalization in preemption games was first established by Fudenberg and Tirole (1985) for the case of two players. In their general setting, which allows for π_0 to depend on the number of previous investments, the game with two players can have multiple equilibrium outcomes, and this implies that rent equalization does not necessarily hold in an analogous game with three or more players. For example, different continuation payoffs might be specified, conditioning on the identity of the first investor. In our setting, in which π_0 does not depend on previous investments, the equilibrium outcome is unique in each subgame, and rent equalization holds for any finite number of players.

An interesting feature of this game is that a change in $\pi(m)$ for $m < N$ does not affect equilibrium payoffs. The last player optimally chooses to invest at T_N^*, and this determines his equilibrium payoff:

$$\frac{\pi(N)}{r}e^{-rT_N^*} - c(T_N^*).$$

In turn, the threat of preemption guarantees that in equilibrium every other player receives exactly the same payoff. Hence, a change in $\pi(m)$ for $m < N$ can at most imply a change in the equilibrium investment times, but not in the equilibrium payoffs.

If instead there is a change in $\pi(N)$, this directly affects not only the equilibrium investment time but also the equilibrium payoff of the last player, and thus, by rent equalization, of all the players in the game. In particular, it follows from the envelope theorem that an increase in $\pi(N)$ implies a higher equilibrium payoff for the last player, and hence for all players.

Formally, let $t_j(\boldsymbol{\pi})$ be the j-th equilibrium investment time for a given flow profit structure $\boldsymbol{\pi}$. Let

$$V^j(t_j(\boldsymbol{\pi}), t_{-j}(\boldsymbol{\pi}))$$

denote the corresponding equilibrium payoff of the j-th investor. Proposition 2 summarizes how equilibrium payoffs depend on the flow profit structure.

Proposition 2 *For every $j \in \{1, ..., N\}$, in the unique SPNE outcome of the N-player preemption game:*

(i) $\frac{\partial V^j(t_j(\boldsymbol{\pi}),t_{-j}(\boldsymbol{\pi}))}{\partial \pi(m)} = 0$ *for* $m < N$

(ii)$\frac{\partial V^j(t_j(\boldsymbol{\pi}),t_{-j}(\boldsymbol{\pi}))}{\partial \pi(N)} > 0.$

One interesting implication of this result is that any regulatory policy that affects post-investment profits $\pi(m)$ for $m < N$, for example a policy that reduces monopoly flow profits, does not affect players' equilibrium payoffs.

Now consider increasing the number of investors to $N+1$. Assume that $\pi(j)$ for $j = 1, ..., N$, are unchanged and that the amended flow profit structure $\boldsymbol{\pi} = [\pi(1), ..., \pi(N+1)]$

still satisfies Assumptions 1 to 4. Equilibrium payoffs will be lower for all firms. The reason is that in equilibrium, all players must receive the same payoff as the last investor. The last investor's flow profit is now smaller ($\pi(N+1)$ rather than $\pi(N)$), and the envelope theorem implies that she earns a smaller equilibrium payoff.

Pre-commitment versus SPNE. Proposition 1 enables us to compare equilibrium investment times and payoffs in the presence and in the absence of pre-commitment, respectively. The following Corollary states that allowing for preemption always accelerates investment and reduces payoffs relative to pre-commitment.

Corollary 1 *In every SPNE,*

(i) investment occurs earlier than in the pre-commitment equilibrium:

$t_j \leq T_j^*$ *for all* $j = 1, ..., N$ *;*

(ii) payoffs are lower than in the pre-commitment equilibrium:

$V^i(t_i, t_{-i}) \leq V^i(T_i^*, T_{-i}^*)$ *for all* $j = 1, .., N$*. The inequality is strict for all* $j = 1, .., N-1$*.*

The Corollary follows immediately from Proposition 1. For Part (i), note that the last investment occurs at the same time T_N^* in equilibrium, with or without pre-commitment. For all $j < N$, the preemption incentive is always positive at T_j^* when only $j-1$ firms have invested. Thus, preemption always accelerates investment times relative to the pre-commitment equilibrium. This result has an interesting implication. As we mentioned in the discussion of Proposition 1, even in the game without pre-commitment, if a player were guaranteed to be the j-th investor, he would choose to invest at T_j^*. Given the convexity of the investment cost function, for $t < T_j^*$, the marginal benefit of delaying the j-th investment, namely the reduction in the investment cost, outweighs the cost of delay (the profit flow that is forgone), while for $t > T_j^*$ the opposite is true. The result in Part (i) of the Corollary then implies that in any SPNE of the game without precommittment each player (except the last) would benefit from a marginal delay of his investment.

For Part (ii), observe that in any SPNE all investors obtain the same payoff, equal to the payoff of the last investor in the pre-commitment equilibrium. Since in the pre-commitment equilibrium the last investor receives the lowest payoff, every firm is worse off in the game with preemption.

3.2 Clusters and Diffusion

A key feature of the unique equilibrium outcome of the preemption game with N players is that it is possible that there are clusters of simultaneous investments at one or more points in the sequence of equilibrium investments, although coordination failures are ruled out by the randomization device we assumed in Assumption 5. In this section, we investigate under which conditions clusters arise.

First, we present a comparative statics result on the impact of a change in a given $\pi(m)$ on the equilibrium investment times. Then, we use this result to provide a simple condition on the flow profit structure $\boldsymbol{\pi}$ that is sufficient for the presence of a cluster.

In order to identify the mechanism that determines a cluster, we start by looking at the impact of a marginal change of one of the parameters $\pi(m)$ on the equilibrium investment times. A first observation is that a marginal change in $\pi(m)$ can only affect investment times t_1 to t_m. In other words, each equilibrium investment time t_j is affected only by the values of the parameters $\pi(m)$ for $m \geq j$. Consider, for example, the last two investment decisions. The last investment time t_N is the solution to a single player decision problem, which is affected only by $\pi(N)$. The $(N-1)$-th investment time t_{N-1} instead, is calculated by equating the extra cost of investing earlier than t_N to the extra benefit, which is the flow of profits $\pi(N-1)$ earned between t_{N-1} and t_N. Hence, the only parameters affecting this calculation are $\pi(N-1)$ and $\pi(N)$, which determines t_N. Similarly, the j-th investment time is determined by considerations that involve the threat of preemption by *later* investors, hence it can only be affected by $\pi(j)$ and by the parameters that affect *later* investment decisions, namely $\pi(m)$ for $m > j$.

Therefore the question arises of how exactly a change in $\pi(m)$ affects the equilibrium investment times of the m-th investor and of the previous investors.

Suppose the m-th and the $(m+1)$-th investment times are distinct, i.e. $t_m < t_{m+1}$, and let us consider the impact of a marginal change in $\pi(m)$ on t_m and t_{m-1}.

Proposition 3 *Suppose that for a given flow profit structure* $\boldsymbol{\pi}$, *the equilibrium investment times* t_m *and* t_{m+1} *satisfy* $t_m < t_{m+1}$. *It holds that*
(i) $\frac{\partial t_m}{\partial \pi(m)} < 0$.
Moreover,
(ii) if $t_{m-1} < t_m$, *then* $\frac{\partial t_{m-1}}{\partial \pi(m)} > 0$,
(iii) if $t_{m-1} = t_m$, *then* $\frac{\partial t_{m-1}}{\partial \pi(m)} < 0$.

If $\pi(m)$ increases, the m-th investment occurs earlier. The intuition relies on the principle of rent equalization. With a higher value of $\pi(m)$, if the m-th investment time did not change, the m-th investor would receive a higher payoff than later investors, whose investment times, and payoffs, are unchanged. This cannot happen in equilibrium, and the threat of preemption has to equate payoffs for all players. In the discussion of Corollary 1, we pointed out that since $t_m < T_m^*$, the m-th investor would benefit from a marginal *delay* of his investment. Therefore, the threat of preemption dissipates the extra payoff that the m-th investor would receive due to the increase of $\pi(m)$, by *accelerating* the m-th investment.

The impact on t_{m-1} instead, depends on whether the m-th and the $(m-1)$-th investment time are initially clustered or diffused. If they are clustered, since the increase in $\pi(m)$ accelerates t_m and does not affect T_{m-1}^*, the condition for a cluster, namely $t_m \leq T_{m-1}^*$, still holds. Therefore, the m-th and the $(m-1)$-th investments are still simultaneous, and as t_m decreases, t_{m-1} decreases as well.

If instead the m-th and the $(m-1)$-th investment are initially diffused, the $(m-1)$-th investment is delayed. The intuition is as follows. To guarantee rent equalization, the $(m-1)$-th investment must take place at the point in time where the incentive to preempt the rivals and be the $(m-1)$-th investor rather than the m-th, as measured by

$$D_{m-1}(t) = \pi(m-1) \int_t^{t_m} e^{-rs} ds - [c(t) - c(t_m)]$$

is exactly null. By part (i) of Proposition 3, an increase in $\pi(m)$ accelerates the m-th investment. This in turn decreases the preemption incentive $D_{m-1}(t)$ in every t: the $(m-1)$-

th investor now receives $\pi(m-1)$ for a shorter period, the m-th invests at a higher cost, and the first effect dominates the second because the point where they are evaluated, t_m, is larger than T^*_{m-1}, the point where the investment cost changes at a speed exactly equal to $\pi(m-1)$. Since the incentive to preempt the rivals and be the $(m-1)$-th investor is now lower in every t, and in particular is negative for the initial value of t_{m-1}, rent equalization requires that the $(m-1)$-th investment time changes in a way that exactly compensates for this decrease. From our discussion of Corollary 1, this change must be an *increase*: a marginal delay of the $(m-1)$-th investment would benefit the $(m-1)$-th investor while not affecting the m-th investor.

We now extend the previous comparative statics analysis, describing the impact of a change in $\pi(m)$ on all the first m equilibrium investment times.

Proposition 4 *If for a given flow profit structure* $\boldsymbol{\pi}$*, the first* m *investment times occur at* $n \leq m$ *distinct times indexed by* $[\bar{t}_1, \bar{t}_2, ., \bar{t}_{n-\ell}, .., \bar{t}_n]$*, where* $\bar{t}_n = t_m$*, and it holds that* $t_m < t_{m+1}$*, then*
(i) $\frac{\partial \bar{t}_{n-\ell}}{\partial \pi(m)} > 0$ *if* l *is odd*
(ii) $\frac{\partial \bar{t}_{n-\ell}}{\partial \pi(m)} < 0$ *if* l *is even*

Softening the competition among m investors, i.e. increasing $\pi(m)$, accelerates some of the previous investment times, and delays others. An important implication is that increasing the number of potential investors does not necessarily accelerate the first investment into the market. In Section 4, we will show that in the case of entry in a market with linear Cournot competition, increasing the number of entrants never accelerates first entry.

The previous comparative statics results will prove very helpful in understanding what determines the existence of a cluster of investments in our model. Suppose the $(m-1)$-th and the m-th investments are diffused for a given flow profit structure $\boldsymbol{\pi}$. Since an increase in $\pi(m)$ accelerates the m-th investment and delays the $(m-1)$-th investment, one would expect that for a sufficiently large $\pi(m)$, namely for a value of $\pi(m)$ sufficiently close to $\pi(m-1)$, the two investments would be simultaneous. In Proposition 5, we formalize this intuition and prove that there is a simple condition on the flow profit structure $\boldsymbol{\pi}$ that guarantees the presence of a cluster: in any equilibrium in which $t_{m-1} < t_m < t_{m+1}$, we can perturb either $\pi(m-1)$ or $\pi(m)$ in such a way that $[\pi(m), \pi(m-1)]$ becomes sufficiently small that in the new equilibrium the $(m-1)$-th and the m-th investment are clustered.

Proposition 5 *Suppose that for a given flow profit structure* $\boldsymbol{\pi}$ *the SPNE investment times satisfy* $t_{m-1} < t_m < t_{m+1}$. *Then:*
(i) there exists $\tilde{\pi}(m-1) \in (\pi(m), \pi(m-1))$ *such that the new SPNE involves* $t_{m-1} = t_m$.
(ii) there exists $\tilde{\pi}(m) \in (\pi(m), \pi(m-1))$ *such that the new SPNE involves* $t_{m-1} = t_m$.

We conclude this section by relaxing Assumption 4 and allowing for an endogenous number of investors.

3.3 Extension: Insufficient Profitability of Investment

Throughout our analysis, the number of players N participating in the preemption game was taken as an exogenously given parameter. In particular, Assumption 4 implied that the cost of investing eventually became sufficiently small, relative to the smallest possible flow payoff $\pi(N)$, that all N firms would invest in finite time. We now relax this assumption and explore how the number of investors could be determined endogenously as part of the equilibrium of an enriched game. We consider a variation of the basic game in which the maximum number of firms that can profitably invest is smaller than the number of potential investors. The market might be "too small" for all N firms to invest.

Formally, we maintain the assumption that the number of players in the game is N, let Assumptions 1, 2, 3, and 5 still hold, and replace 4 by

Assumption 4'

(i) $\exists k \in \{2, N\}$ *such that* $\exists \tau$ *such that* $c(\tau) e^{r\tau} < \frac{\pi(k)}{r}$

(ii) $\lim_{t\to\infty} c'(t)e^{rt} \in (-\pi(N), 0]$

Let M denote the highest integer $k \in \{2, N\}$ such that $\exists \tau$ such that $c(\tau) e^{r\tau} < \frac{\pi(k)}{r}$: the investment cost decreases over time, but only to such an extent that at most M players can profitably invest. The following result holds:

Proposition 6

(a) *If* $M = N$, *then the unique SPNE outcome of the game is the one characterized in Proposition 1.*
(b) *If* $M < N$, *then the game admits a unique SPNE outcome in which*

- *only* M *firms invest,*
- *the equilibrium payoff is zero for all* N *players,*

- *the equilibrium investment times* $(t_1, t_2 ..., t_M)$ *can be calculated recursively according to the following algorithm:*

  (i) t_M *solves* $\frac{\pi(M)}{r} - c\,(t)\, e^{rt_M} = 0$

  (ii) For $j = 1, ..., M-1$,

  If $t_{j+1} > T_j^*$, *then* $t_j < t_{j+1}$ *and* t_j *solves*

$$\pi(j)\int_{t_j}^{t_{j+1}} e^{-rs}ds - c(t_j) + c(t_{j+1}) = 0$$

  If $t_{j+1} \leq T_j^*$, *then* $t_j = t_{j+1}$.

If the maximum number of players who can profitably invest is smaller than N, the equilibrium outcome has two novel features: first, all players receive a payoff of zero in equilibrium; second, while in the original model the last equilibrium investment time would never entail a cluster, it is now possible that the last two (or more) investments are clustered.

The intuition for the zero-profit result is very simple: some players will have to be inactive in equilibrium, and therefore make a payoff of zero. Through the usual mechanism, the threat of preemption guarantees rent equalization among all firms. In particular, this implies that the equilibrium payoff is zero even for the players who invest.

In the basic version of the model, after $N-1$ firms have invested the last active firm is not under threat of being preempted, hence it can wait till T_N^*, the optimal time to invest conditional on being the N-th investor, and make positive profits in equilibrium. If instead only $M < N$ firms can profitably invest, after $M-1$ firms have invested there are still $M-(N-1)$ players competing to be the last investor. Since $N-M$ of them will not be able to invest in equilibrium, and will receive a payoff of zero, rent equalization requires that the last investment time cannot be T_M^* (the optimal time to invest conditional on being the M-th investor) but has to be some $t_M < T_M^*$ that drives the profits of the last investor down to zero. This mechanism also allows for the possibility that the threat of preemption accelerates the last investment to such an extent that after $M-2$ investments no firm would find it profitable to invest before t_M and the last two investments would be clustered at t_M.

An interesting implication of Proposition 6 is that for the case of $M = 2$ and $N \geq 3$ it is possible that the two firms who successfully invest in equilibrium do so simultaneously. For the case of $M = N = 2$, Fudenberg and Tirole (1985) prove that, if pre-investment profits are independent of the number of previous investors, investment times will be diffused. In the extension considered in this section, we have shown that with $N \geq 3$ the possibility of clusters of simultaneous investments does not even require that more than two players invest in equilibrium: the preemption threat exerted by a third potential investor is sufficient to make clusters possible.

4 Example: Entry in a Market with Cournot Competition

We have sofar considered a general rivalrous flow profit structure and declining cost function. While we were able to characterize the equilibrium times and distribution of payoffs, the conclusions that can be reached regarding issues like efficiency of the equilibrium outcome, and the effects of increased competition on the time of first investment, are limited without specifying a cost function and the nature of post-investment competition. To illustrate that more precise results can be obtained with a specific flow profit and cost structure, we consider the case where investment is interpreted as entry in a market where investors compete in quantities facing linear demand, and the investment cost function is exponential.

The present value of the cost of entry at time t is

$$c(t) = c \cdot e^{-(\alpha+r)t} \tag{7}$$

where c is the cost of entering at time 0, r denotes the discount rate, and α is the exponential decay parameter. It is easy to verify that (7) satisfies Assumption 2. With this cost function, the vector of equilibrium investment times in a game with pre-commitment, as characterized in Lemma 1, can be calculated analytically as

$$T_m^* = -\frac{1}{\alpha} \cdot \log\left(\frac{\pi(m)}{(\alpha+r)\cdot c}\right)$$

for every $m = 1, .., N$.

After entering the market, firms compete *à la* Cournot and face a linear demand function $P(Q) = a - bQ$. Marginal cost is constant at k and identical for all firms. We assume that $a > k > 0$. The period payoff when m firms have invested then becomes $\pi(m) = \frac{(a-k)^2}{b(m+1)^2}$.

First, we would like to compare the equilibrium investment times to those that maximize industry profits. Consider the marginal effect of the m-th investment on industry flow profits

$$\pi(m) + (m-1)[\pi(m) - \pi(m-1)].$$

The second term, which corresponds to the business stealing effect, is clearly negative for $m \geq 2$, and null for $m = 1$ (the first entry has no business stealing impact because there are no incumbents in the market). The marginal private benefit of investing, on the other hand, is $\pi(m)$. Consequently, in a game with precommitment in which the j-th equilibrium investment time is calculated equating the private marginal benefit and the marginal cost of delaying entry, every investment but the first occurs too early from the industry's perspective. Part (i) in Corollary 1 states that without pre-commitment investment occurs even earlier: Therefore, from the industry's perspective, preemption aggravates the problem of early entry.

Next, we compare the equilibrium investment times to those that maximize social welfare. If the investment game is interpreted as a game of entry in a market, the acceleration of investment due to preemption does not just reduce industry profits, but also has a positive effect on consumer surplus. To evaluate the net effect, knowledge of demand and the nature of post-investment competition is required. In the Cournot example we are considering in this section, with precommittment the first investment occurs too late (the first entrant does not internalize the positive impact on consumer surplus) and every other investment occurs too early: firms do not internalize business stealing, nor the change in consumer surplus, and in this example the former is stronger than the latter. As we relax precommittment and allow for preemption, all investments are accelerated, therefore for $m \geq 2$ the m-th investment takes place too early from a social point of view, and the result on the first investment time is ambiguous. Under a simple condition, the first investment occurs too late even in the presence of precommittment.

Formally, denoting by T_m^W the welfare-maximizing m-th investment time, the following result holds:

Proposition 7 *Assume that flow profits $\pi(m)$ arise from Cournot competition among the investors, with a linear demand function and constant marginal cost. Then:*
(i) $t_m \leq T_m^* < T_m^W$ for $m \geq 2$.

(ii) $T_1^W < T_1^*$ and, if the cost of investment declines exponentially as in (7),

$\frac{r}{\alpha} > \widetilde{\left(\frac{r}{\alpha}\right)} \approx 1.275$ implies $T_1^W < t_1 \leq T_1^*$.

Next, we consider the impact of an increase in the number of potential investors N on the time of first entry. While it is not possible to derive a result that holds for any preemption game, for the case of exponential cost of entry into a market with Cournot competition and a linear demand function, the following holds:

Let $t_j(N)$ denote the j-th equilibrium investment time in a game with N potential investors.

Proposition 8 *If the cost of investment declines exponentially as in (7) and flow profits $\pi(m)$ arise from Cournot competition among the investors, with a linear demand function and constant marginal cost, then $t_1(N) > t_1(2)$.*

Proposition 8 states that increasing competition in linear Cournot by adding more potential entrants will not accelerate first entry. To capture the intuition, consider the case of going from two to three firms. Adding a third firm accelerates the second entry time, due to the preemption game played among the last two players. The question then becomes whether the second entry time is accelerated to such an extent that it takes place before T_1^*, and hence it is clustered with the first entry time, or not.

If this is not the case, then the first entry must take place at a time $t_1(N)$ that is later than $t_1(2)$. The intuition for this delay is that the acceleration of the second entry time decreases the preemption incentive to be the first, rather than the second entrant. This incentive is measured by the payoff difference

$$D_1(t) = \pi(1)\int_t^{t_2} e^{-rs}ds - [c(t) - c(t_2)]$$

which is increasing in t_2. As t_2 is accelerated by the presence of the third firm, the relative advantage from being first rather than second decreases, since monopoly profits are obtained for a shorter time, and the relative cost decreases as well, since now the second investor sustains a higher investment cost. Nonetheless, the first effect dominates the second, since they are evaluated in t_2, which is larger than T_1^*, the point where the two effects would be equated.

If instead $t_2(3)$ is accelerated to such an extent that it is clustered with $t_1(3)$, acceleration may in principle be strong enough that first and second entry occur earlier than the first entry in the two-player game. In our case of linear Cournot, even in the case of joint first and second entry, this cluster still occurs after the first entry time of the two-firm game.

Finally, we illustrate the mechanism behind Proposition 5. We present an example

	Case 1			Case 2		
m	$\pi(m)$	T_m^*	t_m	$\pi(m)$	T_m^*	t_m
1	4	32.1	28.2	4	32.1	31.9
2	$1\frac{7}{9}$	42.2	37.2	$2\frac{2}{5}$	38.4	31.9
3	1	49.4	49.4	1	49.4	49.4

Table 1: Period profits and adoption times

similar to the one that generated Figures A and B. Choosing parameters for the profit function $\pi(m)$ such that Assumptions 1 to 4 are satisfied,[13] we illustrate in Case 1 of Table 1 the SPNE investment times t relative to the pre-commitment equilibrium investment times T^*, as well as the flow profits $\pi(m)$ after m investments. In this case, the preemption incentive for the first investor at t_1 is still strong enough so that there is diffusion. Consider Case 2 in Table 1. We slightly increased duopoly profits $\pi(2)$ so that they now lie between monopoly profits and the original duopoly profits. Consider the effect of this change in the flow profit structure: The pre-commitment times T_1^* and T_3^* are unaffected, so that the last investment time remains unchanged. The second investment time t_2 is accelerated. In particular, the preemption incentive is so strong, such that t_2, the time that equalizes payoffs from investing second and third, is smaller than T_1^*, hence at time t_1 there is a cluster of two investments. Note also that the first equilibrium is now delayed, because the advantage of being first is diminished by the fact that the second firm would invest early.

5 Conclusion

The existing literature on timing games with complete information focuses on two-player preemption games. For the case where investment does not affect the payoff of non-investors, Fudenberg and Tirole (1985) prove that in a complete information preemption game investment times are diffused, and equilibrium payoffs equalized. We study the extension of that game to N players. We find that there exists a unique subgame-perfect Nash equilibrium outcome, up to a relabeling of players. Equilibrium payoffs are equalized across players, but investment times can be clustered. Clusters of adoption or entry times have often been ascribed to coordination failures, informational spillovers, or positive network externalities, but arise here in a game where all those elements are absent.

We also show that increasing the number of competitors in the investment game does

[13] We parameterize the demand and cost functions in the following way: $a = 5$, $b = 1$, $k = 1$, $c = 400$, $\alpha = .08$, $r = .05$.

not necessarily accelerate the first investment. In particular, we prove that for the case of Cournot competition with a linear demand, the first investment in the two-player game is a lower bound for the first investment in any N-player game.

While we are able to extend the analysis of the complete information game to N players, there are also limits to which environments we can study. Pre-investment payoffs are constant in our game. Thus, our model encompasses environments such as entry into a new market or product innovation, but it does not capture process innovation by technology adoption. The recent literature on two player games has also relaxed the assumption that post-investment profits are symmetric upon investment (for examples see Hoppe (2002)). The extension to the N-player case is left for future research.

Appendix A (Entry in a Growing Market)

In this Appendix, we present an alternative version of the model that captures competition among N firms that wait for the optimal moment to enter a market with growing demand. As in the version presented in the main text, firms face a trade-off that generates a preemption game among them. Firms have an incentive to delay, since demand in the market is increasing over time and the current cost of entry is fixed. At the same time, there is an early mover advantage, since entering before the rivals means facing fewer competitors in the market, at least for some time.

We present a set of assumptions that guarantee that all the results in the main text are preserved in this environment as well.

In this version of the model, "Investment" represents entry in a new market. As in the main model, a firm earns zero payoffs before investment. Upon investment, it earns flow profits of $A(t)\pi(m)$. Here, $A(t)$ stands for the current level of demand[14] in the market.

The current cost of investing K is constant over time. The present value at time 0 of the cost of entry is Ke^{-rt}.

Thus, the game payoff can now be written as:

$$V^j(T^1, T^2, .., T^j, ., T^N) = \sum_{m=j}^{N} \pi(m) \int_{T^m}^{T^{m+1}} A(s)e^{-rs}ds - Ke^{-rT_j}$$

where r denotes the common discount rate, and $T^{N+1} = \infty$.

We maintain assumptions 1, 5 and replace assumptions 2, 3, 4 regarding the cost function with analogous assumptions concerning the growth of demand:

Assumption A-2

(i) $A(t) > 0$, $\forall t$

(ii) $A'(t) > 0$, $\forall t$

Assumption A-2(i) states that demand is positive at all times and Assumption A-2 (ii) requires that demand is growing over time.

Assumption A-3

(i) $\pi(1) \int_0^{\infty} A(s)e^{-rs}ds - K < 0$

[14] $A(t)$ could also describe a productivity process so that the dynamics of profitability are driven by exogenously increasing efficiency of firms.

Assumption A-3(i) ensures that no firm enters at time zero, even if it could preempt all other firms indefinitely.

Assumption A-4

(i) $\exists \tau$ *such that* $K < \pi(N) \int_{\tau}^{\infty} A(s)e^{-r(s-\tau)}ds$

(ii) $\lim_{t\to\infty} A(t) \in \left(0, \frac{rK}{\pi(N)}\right)$

Assumption A-4(i) requires that demand becomes sufficiently large so that entry eventually becomes profitable. Assumption A-4(ii) ensures that no firm would wish to delay entry indefinitely.

An example satisfying Assumption A-2 to A-4 would be an exponentially growing market with $A(t) = A_0 e^{\rho t}$, where $\rho < r$.

It is easy to verify that the main results of the paper go through in this alternative game. The key is that the $D_j(t)$ curves will again be strictly quasi-concave, negative at time zero, and positive in some interval. Thus an algorithm corresponding to the one in Proposition 1 can be employed to compute the equilibrium entry times.

Appendix B

Remark 1 *The function*

$$f(\tau_1) \equiv \int_{\tau_1}^{\tau_2} \pi(m) e^{-rs} ds + k - c(\tau_1)$$

is strictly quasi-concave in τ_1 *for any* $m \in \{1, 2..., N\}$, $\forall k \in R$, *and* $\forall \{\tau_1, \tau_2\} \in R_+^2$.

Proof. We prove the result by showing that in every critical point of the function the second derivative is strictly negative. The first derivative is

$$f'(\tau_1) = -\pi(m) e^{-r\tau_1} - c'(\tau_1)$$

and the second derivative is

$$f''(\tau_1) = r\pi(m) e^{-r\tau_1} - c''(\tau_1).$$

From Assumption $2(i)$ we know

$$e^{r\tau_1}\left[c'(\tau_1) + rc(\tau_1)\right] < 0$$

and from assumption $2(ii)$

$$e^{r\tau_1}\left[2c'(\tau_1) r + c(\tau_1) r^2 + c''(\tau_1)\right] > 0$$

which together imply

$$c'(\tau_1) r + c''(\tau_1) > 0.$$

Using $f'(\tau_1) = 0$ we can rewrite f'' evaluated at any critical point as

$$f''(\tau_1) = -c'(\tau_1) r - c''(\tau_1),$$

which is negative. ■

Proof of Lemma 1. Reinganum ($1981b$) proves that the equilibrium adoption times T_j^*

$(j = 1, ...N)$ of the game with unobservable actions are the solution to

$$\max_t g_j(t) \equiv \pi(j) \int_t^{T^*_{j+1}} e^{-rs} ds + \sum_{m=j+1}^{N} \pi(m) \int_{T^*_m}^{T^*_{m+1}} e^{-rs} ds - c(t) \tag{8}$$

for $j = 1, ...N$, where $T^*_{m+1} \equiv +\infty$. To guarantee that the proof still holds under our slightly different assumptions, we need to check that, given our assumptions, these quantities are well defined and that the associated payoff is greater or equal than the payoff from never investing. First, notice that for any $j = 1, ..., N$, the function $g_j(t)$ is strictly quasiconcave by Remark 1. By assumptions 3(i) and 1(ii), it is negative at $t = 0$. By assumption 4(i) and 2(i) it takes strictly positive values for every τ larger than some finite τ'. Finally, assumption 4(ii) guarantees that its derivative

$$g'_j(t) = -\pi(j)e^{-rt} - c'(t)$$

becomes negative for t sufficiently large. Hence, the function admits a unique maximum and its maximum value is strictly positive.

Next, we adapt the proof from Fudenberg-Tirole (1985) to prove the second claim. Let $i, j \in \{1, ..., N\}$ with $i < j$.

$$\begin{aligned} V^i(T^*_i, T^*_{-i}) &> V^i(T^*_1, ..., T^*_{i-1}, T^*_j, T^*_{i+1}, ..., T^*_j, ..., T^*_N) \\ &= V^j(T^*_1, ..., T^*_{i-1}, T^*_j, T^*_{i+1}, ..., T^*_j, ..., T^*_N) \\ &= V^j(T^*_1, ..., T^*_{i-1}, T^*_i, T^*_{i+1}, ..., T^*_j, ..., T^*_N) \\ &= V^j(T^*_j, T^*_{-j}) \end{aligned}$$

where the inequality holds because T^*_i is a strict best response, the first equality by symmetry of the payoff, and the second equality holds because the payoff of a player is independent of previous investment times. ■

Proof of Lemma 2. Assumptions 3(i) and 1(ii) guarantee that there is no investment at time zero: the cost of investing immediately is higher than the maximum amount of profits a firm can obtain in this game. For the second part of the claim, we first show that there exists a $\tau^* < \infty$ such that in equilibrium, in any decision node with one active firm and calendar time t, the firm plays:

$$\begin{aligned} \text{if } t &< \tau^*, \quad \text{WAIT} \\ \text{if } t &\geq \tau^*, \quad \text{INVEST} \end{aligned}$$

and that $\tau^* = T_N^*$ as defined in section 3.1.

Suppose $N-1$ firms have invested by time t. By Lemma 1, the optimal investment time for the last active firm is T_N^*. The function

$$g_N(t) = \pi(N) \int_t^{\infty} e^{-rs} ds - c(t)$$

is strictly quasi-concave, admits a unique maximum at T_N^* , it is strictly positive for every τ larger than some finite τ', and its derivative is negative for t sufficiently large. Therefore, we can conclude that if $N-1$ firms have already invested by time t and $t < T_N^*$ the last active firm will wait until T_N^* and then invest, while if $t \geq T_N^*$, then it will invest immediately. We conclude the proof of the second part of the claim by showing that all firms invest by T_N^*. Suppose the calendar time is $t \geq T_N^*$ and $N-2$ firms have invested. Any of the two remaining firms can choose whether to

(i) invest immediately: this would trigger immediate rival investment and yield payoff

$$\frac{\pi(N)}{r} e^{-rt} - c(t) > 0$$

(ii) never invest: the associated payoff would be zero.

(iii) wait until a finite $t+\varepsilon$. With this strategy, two cases are possible. If the rival waits until $t+\varepsilon$ as well, the payoff is

$$\frac{\pi(N)}{r} e^{-r(t+\varepsilon)} - c(t+\varepsilon) < \frac{\pi(N)}{r} e^{-rt} - c(t)$$

where the inequality holds by strict quasi-concavity and because $t \geq T_N^*$. If instead the rival invests at $\tau \in (t, t+\varepsilon)$, the firm would then have to invest immediately after the rival and get

$$\frac{\pi(N)}{r} e^{-r\tau} - c(\tau) < \frac{\pi(N)}{r} e^{-rt} - c(t)$$

where the inequality holds by strict quasi-concavity and because $\tau > t \geq T_N^*$. Therefore, the firm will invest immediately at t. Repeating the same argument, even if no firm has invested by time $t \geq T_N^*$, then all firms will invest immediately. ■

Proof of Proposition 1. We first prove that the game admits a unique symmetric equilibrium and characterize its outcome. Then, we analyze the asymmetric equilibria, and show that they induce the same outcome.

Symmetric equilibrium.

To construct the symmetric equilibria of the game, we introduce a mild symmetry assumption. Suppose that at any time t, if a firm is indifferent between being the m-th investor at t and the $(m+1)$-th investor, then it invests at t. The assumption guarantees that if at time t each of the $N-m+1$ active firms is indifferent between the payoff from investing immediately and being the m-th investor and the payoff from waiting and being the $(m+1)$-th investor, they all try to invest immediately, and the randomization device introduced in assumption 5 determines which of them is successful. Through a series of Lemmata we show that under this assumption the game admits a unique SPNE, which is symmetric, and characterize the associated outcome. The proof is articulated in the following steps:

• Denote by t_j the SPNE investment time of the j-th investor, for $j \in \{1, .., N\}$. In Definition 1, we introduce three functions, $L_j(t)$, $F_j(t)$, and their difference $D_j(t)$.

• In Lemma B-1 and Lemma B-2 we characterize their properties. Over a subset of their domain, $L_j(t)$ and $F_j(t)$ can be interpreted as the payoff of the j-th investor and the $(j+1)$-th investor, respectively, if the j-th investment takes place at t and the following investments take place at t_{j+1},..., t_N respectively. In the definition, the existence and uniqueness of the SPNE investment times is assumed. In the development of the proof, they will be proved. The existence and uniqueness of $t_N = T_N^*$ was proved in Lemma 2.

• In Lemma B-3, we prove that there exists a time $T_{N-1} < t_N$ in which $L_{N-1}(t) = F_{N-1}(t)$ and in Lemma B-4 we prove that this is the unique $(N-1)$-th equilibrium investment time, Therefore the equilibrium payoff of the last two investors is the same.

• Finally, in Lemma B-5, B-6 and B-7 we identify the algorithm for the construction of the equilibrium investment times t_j for $j \in \{1, ..., N-2\}$ and prove that rent equalization holds in equilibrium for all players. The argument is based on the induction principle. Lemma B-5 proves that there exists an algorithm to identify the unique t_{N-2}, given t_{N-1} and t_N, and that the equilibrium payoff of the last three investors is the same. Lemma B-6 shows that if an analogous algorithm can be used to identify a unique value of t_{N-k}, given

t_{N-k+1},...,t_N, and rent equalization holds for the last k players, then the same algorithm identifies a unique value for t_{N-k-1}, given t_{N-k},...,t_N, and rent equalization holds for the last $k+1$ players. Lemma B-7 concludes that, by the induction principle, the algorithm can be used to construct the SPNE investment times t_1,..., t_{N-2} and rent equalization holds for all players. This concludes the proof of the Proposition.

Definition 1 *For each $j \in \{1,..,N-1\}$, we define the following three functions over the interval $[0, T_N^*]$:*

$$L_j(t) \equiv \pi(j) \int_t^{t_{j+1}} e^{-rs} ds + \sum_{m=j+1}^{N} \pi(m) \int_{t_m}^{t_{m+1}} e^{-rs} ds - c(t)$$

where $t_{N+1} \equiv +\infty$,

$$F_j(t) \equiv \sum_{m=j+1}^{N} \pi(m) \int_{t_m}^{t_{m+1}} e^{-rs} ds - c(t_{j+1})$$

which is constant with respect to t, and

$$D_j(t) \equiv L_j(t) - F_j(t) = \pi(j) \int_t^{t_{j+1}} e^{-rs} ds - c(t) + c(t_{j+1}).$$

Notice that all these functions are clearly continuous.

Lemma B-1

(i) The functions $D_j(t)$ attain a unique global maximum in $T_j^* \in (0, T_N^*]$.

(ii) $T_1^* < T_2^* < ... < T_N^*$.

(iii) $D_j(T_j^*) \geq 0$ for $j \in \{1,..,N-1\}$.

Proof.

Part (i)**:** Notice that

$$D_j(t) = \pi(j) \int_t^{t_{j+1}} e^{-rs} ds - c(t) + c(t_{j+1}) \tag{9}$$

and

$$g_j(t) = \pi(j) \int_t^{T_{j+1}^*} e^{-rs} ds \sum_{m=j+1}^{N} +\pi(m) \int_{T_m^*}^{T_{m+1}^*} e^{-rs} ds - c(t) \tag{10}$$

differ by a finite constant. In the proof of Lemma 1 we showed that (10) attains a unique global maximum in $T_j^* \in (0, T_N^*]$, hence the same is true for (9).

Part (ii)**:** By the implicit function theorem

$$\frac{\partial T_j^*}{\partial \pi(j)} = -\frac{-e^{-rT_j^*}}{\partial(-\pi(j)\,e^{-rT_j^*} - c'\left(T_j^*\right))/\partial T_j^*} < 0,$$

where the inequality follows from the fact that the denominator is negative, since T_j^* is a maximum. Therefore, by Assumption 1(ii), it holds that $T_1^* < T_2^* < ... < T_N^*$.

Part (iii)**:** Since $D_j(t_{j+1}) = 0$ and T_j^* is the unique global maximizer, it holds that $D_j(T_j^*) \geq 0$. ■

Lemma B-2

(i) If $T_j^* \leq t_{j+1}$, *then* $\exists T_j \in (0 < T_j^*]$, *such that* $D_j(T_j) = 0$,

(ii) If $T_j^* > t_{j+1}$, *then* $D_j(t) < 0$ *and* $D_j'(t) > 0$ $\forall t < t_{j+1}$.

Proof. By Remark 1, $D_j(t)$ is strictly quasi-concave. Also, $D_j(0) < 0$, since

$$\begin{aligned} L_j(0) &= \pi(j)\int_0^{t_{j+1}} e^{-rs}ds + \sum_{m=j+1}^{N} \pi(m)\int_{t_m}^{t_{m+1}} e^{-rs}ds - c(0) < \frac{\pi(1)}{r} - c(0) < 0 \\ &\leq V^{j+1}(t_1, ...t_N) = \sum_{m=j+1}^{N} \pi(m)\int_{t_m}^{t_{m+1}} e^{-rs}ds - c(t_{j+1}) = F_j(0) \end{aligned}$$

Here the second inequality holds by assumption 3(i) and the third because no firm gets a negative payoff in equilibrium as it could always delay investment indefinitely ensuring a payoff of zero. Further, we have that $D_j(t_{j+1}) = 0$.

Therefore, two cases are possible: either $T_j^* \leq t_{j+1}$, in which case $\exists T_j \in (0 < T_j^*]$, such that $D_j(T_j) = 0$, and $D_j(t) > 0$ in the interval $t \in (T_j, t_{j+1})$, or $T_j^* > t_{j+1}$, in which case $D_j(t) < 0$ and $D_j'(t) > 0$ $\forall t < t_{j+1}$. ■

In the next Lemma, we show that for the case $j = N-1$, case (i) of Lemma B-2 applies.

Lemma B-3 $T_{N-1}^* < t_N = T_N^*$ *and* $\exists T_{N-1} < T_{N-1}^* < T_N^*$ *such that* $D_{N-1}(T_{N-1}) = 0$.

Proof. $T_{N-1}^* < T_N^*$ by Lemma B-1 and $t_N = T_N^*$ by Lemma 2. The rest of the claim follows from Lemma B-2. ■

In the next Lemma we show that in equilibrium the $(N-1)$-th firm investment time is T_{N-1}.

Lemma B-4 *In equilibrium, it holds that:*

in subgames starting at a decision node with calendar time $t \in [T_{N-1}, T_N^*)$, *each firm plays:*

(i) If there are $n > 1$ active firms, INVEST

(ii) If there is 1 active firm, WAIT $\forall \tau \in [t, T_N^*)$, INVEST $\forall \tau \geq T_N^*$

in subgames starting at a decision node with calendar time $t \in [0, T_{N-1})$, each firm plays:

(iii) If there are 2 active firms, WAIT $\forall \tau \in [t, T_{N-1})$ and INVEST $\forall \tau \geq T_{N-1}$

(iv) If there is 1 active firm, WAIT $\forall \tau \in [t, T_N^*)$, INVEST $\forall \tau \geq T_N^*$

Hence, $t_{N-1} = T_{N-1}$ and the equilibrium payoff of the last two investors is the same.

Proof. Parts (ii) and (iv) follow from the proof of Lemma 2.

Part (i). Notice that in the interval $t \in [T_{N-1}, T_N^*]$ it holds that $D_{N-1}(t) \geq 0$ and that in the open interval $t \in (T_{N-1}, T_N^*)$ it holds that $D_{N-1}(t) > 0$. Consider a decision node with calendar time $t \in [T_{N-1}, T_N^*]$, with $n > 1$ active firms. First we show that the actions prescribed in part (i) are compatible with equilibrium, then we rule out other actions.

If they follow the actions prescribed in part (i), all firms try to invest immediately, until only one is left, which will then wait to invest until time T_N^* (see part (ii) and Lemma 2). The associated payoff is :

$$\frac{1}{n}\left[(n-1)L_{N-1}(t) + F_{N-1}(t)\right]$$

Suppose one firm deviates and plays WAIT at t. This deviation cannot be profitable, since it increases the probability of receiving payoff $F_{N-1}(t)$ and reduces the probability of receiving payoff $L_{N-1}(t) > F_{N-1}(t)$, therefore the actions prescribed in part (i) are compatible with equilibrium.

Next we will show that strategies profiles that prescribe actions different from those prescribed in part (i) for nodes with calendar time $t \in [T_{N-1}, T_N^*)$ and more than one active firm cannot be an equilibrium. We develop the argument by induction. First we show it holds for $n = 2$, active firms, then we show that if it holds for some $n \geq 2$, then it holds for $n + 1$, and then we conclude it holds for any $n \geq 2$ by the induction principle.

- $n = 2$.

We need to consider two classes of strategy profiles: one with first investment at t by one firm only, and one in which both firms play WAIT at time t, hence the first investment is made at some $\tau \in (t, T_N^*]$, by one or two firms.

- First, consider any strategy profile with first investment at t by one firm only.

For $t \in (T_{N-1}, T_N^*)$, this cannot be an equilibrium, because then the remaining firm gets at most $F_{N-1}(t)$, while she could try to invest at t as well and get $\frac{1}{2}\left[L_{N-1}(t) + F_{N-1}(t)\right] >$

$F_{N-1}(t)$. For $t = T_{N-1}$, investment by only one of the active firms is ruled out by our symmetry assumption, which requires that if the two active firms are indifferent between the payoff from successfully investing immediately and being the $(N-1)$-th investor and the payoff from waiting and being the last investor, they both try to invest immediately.

- Next, consider any strategy profile with first investment at $\tau \in (t, T_N^*]$, by $\nu = 1$ or 2 firms.

This cannot be an equilibrium for $\nu = 2$, because both firms get

$$\frac{1}{2}\left[L_{N-1}(\tau) + F_{N-1}(\tau)\right]$$

while each of them could deviate by investing at $\tau - \varepsilon$ and get $L_{N-1}(\tau - \varepsilon)$. By continuity, $\exists \varepsilon > 0$ small enough that this is profitable.

Also, this cannot be an equilibrium with $\nu = 1$ because the "late investor" gets $F_{N-1}(\tau)$ while she could deviate by investing at $\tau - \varepsilon$ and get

$$L_{N-1}(\tau - \varepsilon) > F_{N-1}(\tau).$$

- $n \Rightarrow n+1$

Suppose $n+1$ firms are active. Again, we need to consider two classes of candidate equilibria, with first investment at t by $\nu < n+1$ firms, and with first investment at at $\tau > t$, by $\nu \leq n+1$ firms, respectively.

- First, consider any strategy profile with first investment at t by $\nu < n+1$ firms.

Observe that after the first successful investment, only n active firms are left. In the development of the induction argument, we are now assuming that with n active firms, in this class of subgames, all firms try to invest immediately until only one is left. Consider then one of the firms who do not (try to) invest immediately. It gets expected payoff:

$$\frac{1}{n}\left[(n-1)L_{N-1}(t) + F_{N-1}(t)\right]$$

while she could profitably deviate by trying to invest at the beginning of the subgame. This deviation would increase the probability of receiving $L_{N-1}(t)$ and decrease the probability of receiving $F_{N-1}(t) < L_{N-1}(t)$. Therefore, this is not an equilibrium.

- Next, consider any strategy profile with first investment at $\tau > t$, by $\nu = 2, ..., n+1$ firms.

If at time τ one firm successfully invests, only n active firms remain, and again, because we assume that the argument holds for n, we know that they all try to invest immediately until only one is left.

This implies that any of the firms who try to invest at τ gets

$$\frac{1}{\nu}L_{N-1}(\tau)+\frac{\nu-1}{\nu}\left[\frac{1}{n}\left[(n-1)L_{N-1}(\tau)+F_{N-1}(\tau)\right]\right]$$

while she could deviate by investing at $\tau-\varepsilon$ and get $L_{N-1}(\tau-\varepsilon)$. By continuity, $\exists\varepsilon>0$ small enough that this deviation would be profitable. Therefore, this cannot be an equilibrium either.

- Finally; consider any strategy profile with first investment at $\tau>t$, by $\nu=1$ firm only. This cannot be an equilibrium either, because all the n "late investors" try to invest immediately after the "early investor" and receive expected payoff get

$$\frac{1}{n}\left[(n-1)L_{N-1}(\tau)+F_{N-1}(\tau)\right]$$

while any of them could deviate by investing at $\tau-\varepsilon$ and get

$$L_{N-1}(\tau-\varepsilon)>\frac{1}{n}\left[(n-1)L_{N-1}(\tau)+F_{N-1}(\tau)\right].$$

This completes the induction argument and we can conclude that strategies profiles that prescribe actions different from those prescribed in part (i) for nodes with calendar time $t\in[T_{N-1},T_N^*)$ cannot be an equilibrium.

Part (iii). First, notice that it holds by part (i) that if there are 2 active firms and the calendar time is greater or equal than T_{N-1}, both active forms will play INVEST. Next, consider decision nodes with calendar time $t\in[0,T_{N-1})$ and two active firms. The payoff in the candidate equilibrium is

$$\frac{1}{2}\left[L_{N-1}(T_{N-1})+F_{N-1}(T_{N-1})\right]=L_{N-1}(T_{N-1})=F_{N-1}(T_{N-1})$$

where the equality comes from the definition of T_{N-1}. The deviation payoff from investing at some τ before T_{N-1} is

$$L_{N-1}(\tau)<L_{N-1}(T_{N-1})$$

where the inequality comes from the fact that $L_{N-1}(\cdot)$ is increasing in $[0,T_{N-1}]$.

The deviation consisting in playing WAIT at $t=T_{N-1}$ is not profitable by the proof of part

(i) of this Lemma. Therefore, there is no profitable deviation.

Next, we show that there is no other strategy profile compatible with equilibrium for decision nodes with calendar time $t \in [0, T_{N-1})$ and 2 active firms. Suppose there is first investment at $\tau < T_{N-1}$. Let $\nu \leq 2$ be the number of firms who play INVEST at τ. The equilibrium payoff for any of these early investors is

$$\frac{1}{\nu}\left[L_{N-1}(\tau) + (\nu - 1)F_{N-1}(\tau)\right]$$

Each of them could profitably deviate by playing WAIT at τ, since $F_{N-1}(\tau) > L_{N-1}(\tau)$. Therefore, there are no strategy profiles that prescribe action different from (iii) and are compatible with equilibrium. This concludes the proof of part (iii).

The conclusion that $t_N = T_{N-1}$ follows directly from parts (i), (ii), (iii) and (iv). By construction of T_{N-1}, this implies rent equalization for the last two firms. ■

We now identify the algorithm for the construction of the equilibrium investment times t_j for $j \in \{1, ..., N-1\}$. The argument is based on the induction principle. Lemma B-5 contains a statement for $j = N-2$. Lemma B-6 shows that if the same statement holds for $j = N-k$, then it holds for $j = N-k-1$. Lemma B-7 concludes that, by the induction principle, the statement holds for a general j.

Lemma B-5 *Given $t_N = T^*_N$ and $t_{N-1} = T_{N-1}$, t_{N-2} can be constructed as follows.*
Part (a) *Suppose $T^*_{N-2} < T_{N-1}$. In equilibrium it holds that*

in subgames starting at a decision node with calendar time $t \in [T_{N-2}, T_{N-1})$ each firm plays:

(i) If there are $n > 2$ active firms, INVEST

(ii) If there are $n = 2$ active firms, WAIT $\forall\tau \in [t, T_{N-1})$ and INVEST at T_{N-1};

(iii) If there is 1 active firm, WAIT $\forall\tau \in [t, T^*_N)$, INVEST at $\tau = T^*_N$

in subgames starting at a decision node with calendar time $t \in [0, T_{N-2})$, each firm plays:

(iv) If there are 3 active firms, WAIT $\forall\tau \in [0, T_{N-2})$ and INVEST at T_{N-2}

(v) If there are 2 active firms, WAIT $\forall\tau \in [t, T_{N-1})$ and INVEST at T_{N-1},

(vi) If there is 1 active firm, WAIT $\forall\tau \in [t, T^*_N)$, INVEST at $\tau = T^*_N$.

Therefore, $t_{N-2} = T_{N-2}$ and the payoff of the last 3 investors is equalized.

Part (b) *Suppose $T^*_{N-2} \geq T_{N-1}$. In equilibrium it holds that*
In subgames starting at a decision node with calendar time $t \in [0, T_{N-1}]$, each firm plays

(i) If there are $n = 3$ *active firms, WAIT* $\forall \tau \in [t, T_{N-1})$ *and INVEST at* T_{N-1}

(ii) If there are $n = 2$ *active firms, WAIT* $\forall \tau \in [t, T_{N-1})$ *and INVEST at* T_{N-1}

(iii) If there is 1 *active firm, WAIT* $\forall \tau \in [t, T_N^*)$*, INVEST at* $\tau = T_N^*$[15]

Therefore, $t_{N-2} = T_{N-1}$ *and the payoff of the last 3 investors is equalized.*

Proof.

Part (a): From Lemma B-4, $t_{N-1} = T$. Hence, from Lemma B-2 (i), it follows that $\exists T_{N-2} \in (0 < T_{N-2}^*]$, such that $D_{N-2}(T_{N-2}) = 0$. Parts (iii) and (vi) follow from the proof of Lemma 2. Parts (ii) and (v) are immediate from Lemma B-4. The proof of part (i) follows from arguments similar to the proof of part (i) of Lemma B-4. The proof of part (iv) follows from arguments similar to the proof of part (iii) of Lemma B-4. The conclusion that $t_{N-2} = T_{N-2}$ follows directly from parts (i) to (vi). By construction of T_{N-2}, this implies rent equalization for the last three firms.

Part (b): From Lemma B-4, $t_{N-1} = T$. Hence, from Lemma B-2 (ii), it follows that $D_{N-2}(t) < 0$ and $D'_{N-2}(t) > 0$ $\forall t < T_{N-1}$. As in the proof of part (a), (ii) follows from Lemma B-4, and (iii) follows from Lemma 2.

We now prove part (i). First we show that the actions prescribed in part (i) are compatible with equilibrium, then we rule out different actions.

If a firm invests at $\tau < T_{N-1}$, its payoff is $L_{N-2}(\tau)$. The equilibrium payoff instead is $L_{N-2}(T_{N-1})$ (In this candidate equilibrium, there is rent equalization among the last 3 firms: $t_{N-2} = t_{N-1}$ guarantees that the first two receive the same payoff and, by construction of T_{N-1}, the last two also receive the same profits). The deviation is not profitable because $D_{N-2}(\tau) < 0 = D_{N-2}(T_{N-1})$ implies $L_{N-2}(\tau) < L_{N-2}(T_{N-1})$, so this is compatible with equilibrium.

Next we will show that strategies profiles that prescribe actions different from those prescribed in part (i) cannot be an equilibrium. In other words, we rule out strategy profiles that prescribe that if at $\tau < T_{N-1}$ there are three active firms, $\nu \leq 3$ of them play INVEST. If $\nu = 1$, the early investor gets payoff $L_{N-2}(\tau)$ while he could profitably deviate by waiting until T_{N-1} and getting $F_{N-2}(T_{N-1}) = L_{N-2}(T_{N-1}) > L_{N-2}(\tau)$.

If instead $\nu \in \{2, 3\}$, at least one firm gets $L_{N-2}(\tau)$ with positive probability while she

[15] Notice that it has to be the case that the first N-3 firms invest by T_{N-1} by Lemma B-4 (i).

could profitably deviate by playing WAIT at τ, thus getting

$$F_{N-2}(\tau) > L_{N-2}(\tau)$$

where the inequality holds because $D_{N-2}(\tau) < 0$.

The conclusion that $t_{N-2} = t_{N-1} = T_{N-1}$ follows directly from parts (i), (ii) and (iii). By construction of T_{N-1}, this implies rent equalization for the last three firms. ■

Lemma B-6 *If the following statement holds for $j = N - k$, with $k \geq 2$, then it holds for $j = N - k - 1$.*

Given the last $N - j$ equilibrium investment times $(t_{j+1}, ..., t_N)$, the j-th equilibrium investment time t_j can be constructed as follows:

Part (a) *Suppose $T_j^* < t_{j+1}$. In equilibrium it holds that*

in subgames starting at a decision node with calendar time $t \in [T_j, t_{j+1})$, each firm plays:

(i) If there are $n > N - j$ active firms, INVEST $\forall \tau \geq t$

(ii) If there are $n \leq N - j$ active firms, WAIT $\forall \tau \in [t, t_{N-n+1})$ and INVEST at t_{N-n+1}.

in subgames starting at a decision node with calendar time $t \in [0, T_j)$, each firm plays

(iii) If there are $n = N - j + 1$ active firms, WAIT $\forall \tau \in [0, T_j)$ and INVEST at T_j

(iv) If there are $n < N - j + 1$ active firms, WAIT $\forall \tau \in [0, t_{N-n+1})$ and INVEST at t_{N-n+1}.

Therefore, $t_j = T_j$ and the payoff of the last $N - j + 1$ investors is equalized.

Part (b) *Suppose $T_j^* \geq t_{j+1}$. In equilibrium it holds that*

in subgames starting at a decision node with calendar time $t \in [0, t_{j+1}]$ each firm plays:

(i) If there are $n = N - j + 1$ firms left, WAIT $\forall \tau \in [t, t_{j+1})$ and INVEST at t_{j+1}

(ii) If there are $n = N - j$ remaining firms left, WAIT $\forall \tau \in [t, t_{j+1})$ and INVEST at t_{j+1}

(iii) If there are $n < N - j$ remaining firms left, WAIT $\forall \tau \in [t, t_{N-n+1})$ and INVEST at t_{N-n+1}

Therefore $t_j = t_{j+1}$ and the payoff of the last $N - j + 1$ investors is equalized.

Proof. Assume that the statement holds for $j = N - k$. This implies that either $t_{N-k} = T_{N-k}$ or $t_{N-k} = t_{N-k+1}$, and that in both cases payoffs of the last $k + 1$ investors are equalized.

Now we need to prove that the statement holds for $j = N - k - 1$.

Part (a): Parts (ii) and (iv) follow from the assumption that the statement holds for $j = N - k$.

To prove part (i), first notice that in the interval $t \in [T_{N-k-1}, t_{N-k})$ it holds that $D_{N-k-1}(t) \geq 0$ and that in the open interval $t \in (T_{N-k-1}, t_{N-k})$ it holds that $D_{N-k-1}(t) > 0$. The rest of the proof follows arguments similar to the proof of part (i) of Lemma B-4.

To prove part (iii), notice that $D_{N-k-1}(t) < 0$ and $D'_{N-k-1}(t) > 0$ $\forall t < T_{N-k-1}$. The rest of the proof of part (iii) follows from arguments similar to the proof of part (iii) of Lemma B-4.

The conclusion that $t_{N-k-1} = T_{N-k-1}$ follows directly from parts (i) to (iv). By construction of T_{N-k-1}, this implies rent equalization for the last $k+2$ firms.

Part (b): Parts (ii) and (iii) follow from the assumption that the statement holds for $j = N - k$.

To prove part (i), first notice that, by Lemma B-2, in the interval $t \in [0, t_{N-k}]$ it holds that $D_{N-k-1}(t) < 0$ and $D'_{N-k-1}(t) > 0$. The rest of the proof follows from arguments similar to the proof of part (b) (i) of Lemma B-5.

The conclusion that $t_{N-k-1} = T_{N-k}$ follows directly from parts (i), (ii), (iii). By construction of T_{N-k}, this implies rent equalization for the last $k+2$ firms. ■

Lemma B-7 *The statement in Lemma B-6 holds for any $j \leq N - 2$.*

Proof. The result follows from Lemmata B-5 and B-6 by the induction principle. ■

Asymmetric equilibria.

We now relax our symmetry assumption. In other words, we do not require that all active firms play INVEST when they are indifferent between successfully investing today as the m-th investor, and investing later as the $(m+1)$-th investor. This implies that players' strategies can differ in the action they prescribe at times T_j, for $j = 1, ..., N$. It follows that the characterization of all the asymmetric equilibria is analogous to the the previous analysis, with only the following changes:

- In Lemma B-4, part (i), in subgames starting at a decision node with calendar time T_{N-1}, with $n > 1$ active firms, it is admissible that in equilibrium any number of firms $\nu \leq n - 1$ play WAIT, and the remaining firms play INVEST.

- Similarly, in Lemma B-5, part (a), in subgames starting at a decision node with calendar time T_{N-2}, with $n > 2$ active firms, it is admissible that in equilibrium any number of firms $\nu \leq n-1$ play WAIT, and the remaining firms play INVEST.

- In Lemma B-6, the statement that holds for $j = N-k-1$, provided it holds for $j = N-k$, can be modified allowing for the fact that in subgames starting at a decision node with calendar time T_j, with $n > N-j$ active firms, in equilibrium any number of firms $\nu \leq n-1$ play WAIT, and the remaining firms play INVEST.

Therefore, the game admits asymmetric equilibria with the same equilibrium investment times and the same equilibrium payoffs as the symmetric equilibrium, in which the only difference is that at times t_j, for $j = 1, ..., N-1$, the number of firms playing INVEST is any number between 1 and $N-j+1$, while in the symmetric equilibrium all the $N-j+1$ active firms play INVEST. ■

Proof of Proposition 2. By Proposition 1, equilibrium payoffs are equal to

$$\frac{\pi(N)}{r} e^{-rT_N^*} - c\left(T_N^*\right)$$

for all players. Part (i) follows immediately, part (ii) follows from the envelope theorem. ■

Proof of Proposition 3. Consider part (i). Given $t_m < t_{m+1}$, it has to be the case, from Proposition 1, that t_m solves

$$\frac{\pi\left(m\right)}{r}\left[e^{-rt_m} - e^{-rt_{m+1}}\right] - \left[c\left(t_m\right) - c\left(t_{m+1}\right)\right] = 0$$

Differentiating implicitly yields:

$$\frac{\partial t_m}{\partial \pi_m} = -\frac{\left[e^{-rt_m} - e^{-rt_{m+1}}\right]/r}{-\pi\left(m\right)e^{-rt_m} - c'\left(t_m\right)}$$

Notice that the numerator is positive, because $t_m < t_{m+1}$. The denominator is positive as well, because $t_m < T_m^*$. Therefore, t_m decreases in $\pi\left(m\right)$. Next, consider part (ii). Given t_m and the absence of a cluster, it follows from Proposition 1 that t_{m-1} solves

$$\frac{\pi\left(m-1\right)}{r}\left[e^{-rt_{m-1}} - e^{-rt_m}\right] - \left[c\left(t_{m-1}\right) - c\left(t_m\right)\right] = 0$$

Using the chain rule and differentiating implicitly the expression above:

$$\frac{\partial t_{m-1}}{\partial \pi_m} = \frac{\partial t_{m-1}}{\partial t_m} \cdot \frac{\partial t_m}{\partial \pi_m} = -\frac{\pi\left(m-1\right)e^{-rt_m} + c'\left(t_m\right)}{-\pi\left(m-1\right)e^{-rt_{m-1}} - c'\left(t_{m-1}\right)} \cdot \frac{\partial t_m}{\partial \pi_m}$$

Consider the first term. The denominator is positive because $t_{m-1} < T^*_{m-1}$. By assumption $t_m > T^*_{m-1}$, so the numerator is positive as well. The second term has been shown to be negative in part (i). Therefore, the whole expression is positive. This concludes the proof of part (ii). Finally, consider part (iii). Since $t_{m-1} = t_m$, it has to be the case that $t_m < T^*_{m-1}$. If $\pi(m)$ increases, t_m decreases (by part (i)), while T^*_{m-1} doesn't change. Therefore, it is still true that $t_m < T^*_{m-1}$.and that $t_{m-1} = t_m$. Consequently, since t_m decreases, then t_{m-1} decreases as well. ■

Proof of Proposition 5. First, notice that investment times $[t_{m+1,}t_{m+2}, ..., t_N]$ are constant in $\pi(m-1)$ and $\pi(m)$. Consider part (i). We first show that T^*_{m-1} decreases in $\pi(m-1)$. By definition, T^*_{m-1} solves

$$-\pi(m-1)\,e^{-rT^*_{m-1}} - c'\left(T^*_{m-1}\right) = 0$$

Using the implicit function theorem:

$$\frac{\partial T^*_{m-1}}{\partial \pi(m-1)} = -\frac{-e^{-rT^*_{m-1}}}{\partial(-\pi(m-1)\,e^{-rT^*_{m-1}} - c'\left(T^*_{m-1}\right))/\partial T^*_{m-1}} < 0$$

where the inequality follows from the fact that the denominator is negative since T^*_{m-1} is a maximum. Next, notice that

$$\lim_{\pi(m-1)\to\pi(m)} T^*_{m-1} = T^*_m > t_m$$

By continuity, this implies that for small enough $\pi(m-1)$ it will be the case that $T^*_{m-1} > t_m$ and there is clustering at t_m. This completes the proof of part (i).

Now consider part (ii). By Proposition 3, t_m decreases in $\pi(m)$. Next, notice that

$$\lim_{\pi(m)\to\pi(m-1)} T^*_m = T^*_{m-1}.$$

Since $t_m < T^*_m$, this implies that for large enough $\pi(m-1)$ it will be $t_m < T^*_{m-1}$. ■

Proof of Proposition 4. First, assume that for a given period payoff vector $[\pi(1), ..., \pi(m), .., \pi(N)]$ the SPNE investment times satisfy $t_{m-j} < t_{m-j+1} ... < t_m$. We prove the argument by induction. First, notice that the statement is true for $j = 0, 1$. For $j = 0$, $\frac{\partial t_m}{\partial \pi_m} < 0$ follows from Proposition3(i). For $j = 1$, $\frac{\partial t_{m-1}}{\partial \pi_m} > 0$ follows from Proposition 3(ii). We next show that if the statement holds for $j-1$ it also holds for j. Given the absence of clusters, it follows from Proposition 1 that t_{m-j} solves

$$\frac{\pi(m-j)}{r}\left[e^{-rt_{m-j}}-e^{-rt_{m-j+1}}\right]-\left[c\left(t_{m-j}\right)-c\left(t_{m-j+1}\right)\right]=0.$$

Applying the chain rule and differentiating implicitly yields:

$$\frac{\partial t_{m-j}}{\partial \pi_m}=\frac{\partial t_{m-j}}{\partial t_{m-j+1}}\cdot\frac{\partial t_{m-j+1}}{\partial \pi_m}=-\frac{\pi(m-j)\,e^{-rt_{m-j+1}}+c'\left(t_{m-j+1}\right)}{-\pi(m-j)\,e^{-rt_{m-j}}-c'\left(t_{m-j}\right)}\cdot\frac{\partial t_{m-j+1}}{\partial \pi_m}.$$

Consider the first term. The denominator is positive because $t_{m-j}<T^*_{m-j}$. By assumption, $t_{m-j+1}>T^*_{m-j}$, so the numerator is positive as well. Hence, the first term is negative. If j is odd, and the statement is true for $j-1$, then t_{m-j+1} is decreasing in $\pi(m)$, so the second term is negative as well and $\frac{\partial t_{m-j}}{\partial \pi_m}$ is positive. Similarly, if j is even the second term is positive, and $\frac{\partial t_{m-j}}{\partial \pi_m}$ is negative. The statement follows from the induction principle. For the general case in which for a given period payoff vector $[\pi(1),...,\pi(m),..,\pi(N)]$ the first m investment times occur at $n\leq m$ distinct times indexed by $[\bar{t}_1,\bar{t}_2,.,\bar{t}_{n-\ell},..,\bar{t}_n]$, where $\bar{t}_n=t_m$, and it holds that $t_m<t_{m+1}$, then the proof is the same as above, taking into account the distinction between an adoption t_{m-j} and a distinct adoption time $\bar{t}_{n-\ell}$. ■

Proof of Proposition 6. Part (a) follows from Proposition 1. We prove part (b) through a series of Lemmata. The proof is articulated through the following steps:

- Suppose the symmetry assumption introduced in the construction of the symmetric equilibrium of the basic game, in proof of Proposition 1, holds. Lemmas B-8 to B-12 establish that in equilibrium exactly M firms invest, the last investment occurs at T_M, and there is rent equalization at zero-profits for the firms who never enter and the last entrant.

- Given the previous results, it follows immediately that an algorithm analogous to the algorithm constructed in Proposition 1 permits to construct the unique symmetric equilibrium.

- Next, suppose that the symmetry assumption is relaxed. Then the game admits asymmetric equilibria as well. As for the case of Proposition 1, these asymmetric equilibria have the same equilibrium investment times and payoffs as the symmetric equilibrium. The only difference between symmetric and asymmetric equilibria in this case is that at times t_j, for $j=1,...,M$, the number of firms playing INVEST is any number between 1 and $M-j+1$, while in the symmetric equilibrium all the $M-j+1$ active firms play INVEST.

Lemma B-8 *In equilibrium, in any subgame starting in a node with calendar time t, with*

$n \leq N - M$ active firms, each firm plays WAIT for every $\tau \geq t$.

Proof. Investment in this subgame yields at most

$$\frac{\pi\left(N-n+1\right)}{r}e^{-r\tau}-c\left(\tau\right)\leq\frac{\pi\left(M+1\right)}{r}e^{-r\tau}-c\left(t\right)<0\quad\forall\tau.$$

Hence no firm will invest. ■

Lemma B-9 *In equilibrium there can be at most M active firms.*

Proof. Follows immediately from the previous Lemma. ■

Lemma B-10 *The function*

$$L_M\left(t\right)=\pi(M)\int_t^{\infty}e^{-rs}ds-c(t)=\frac{\pi(M)}{r}e^{-rt}-c(t)$$

is strictly quasi-concave, admits a unique maximum at T_M^* , is equal to zero in some $T_M < T_M^*$ (which exists by the assumptions given, in particular by continuity, 3(i) and 4(i)) and hence its maximum value is strictly positive. It is nonnegative for $t > T_M$.

Proof. The first statement follows from the proof of Lemma 2. The fact that $L_M\left(t\right)$ nonnegative for $t > T_M$ follows from the assumptions made about the function $c\left(t\right)e^{rt}$, since the sign of $L_M\left(t\right)$ is the opposite of the sign of $\left[c\left(t\right)e^{rt}-\frac{\pi(M)}{r}\right]$ and $\frac{\pi(M)}{r}$ is a constant w.r.t. t. ■

Lemma B-11 *In equilibrium it holds that in all subgames starting at a decision node with calendar time $t \geq T_M$, each firm plays:*
(i) if there are $n < N - (M-1)$ active firms, WAIT for every $\tau \geq t$
(ii)If there are $n \geq N - (M-1)$ active firms, INVEST.

Proof. Part (i) follows from Lemma B-8. For part (ii), notice that in a subgame with $[N-(M-1)]$ active firms, if a firm successfully invests, a subgame with $N-M$ active firms starts, in which, in equilibrium, no active firms ever invests. Hence, the successful investor gets $L_M\left(t\right)$ and every other firm gets $F_M\left(t\right)=0$

In a subgame that starts at $t \geq T_M$, it holds that $L_M\left(t\right) \geq F_M\left(t\right)$. The latter inequality is strict for all $t > T_M$. By the usual arguments, in equilibrium all active firms play INVEST and

$$\frac{1}{n}L_M\left(t\right)+(n-1)F_M\left(t\right)$$

is the expected payoff. ■

Consider now subgames starting at a decision node with calendar time $t \in [0, T_M)$ with $[N-(M-1)]$ active firms.

Lemma B-12 *In equilibrium, it holds that:*

In all subgames starting at a decision node with calendar time $t \in [0, T_M)$, each firm plays: If there are $n = N-(M-1)$, active firms, WAIT for every $\tau \in [t, T_M)$, and INVEST for every $\tau \geq T_M$.

Proof. The proof follows the arguments presented in Lemma B-4. ■

Proof of Proposition 7. Part (i): In the case of Cournot competition with a linear demand function and constant marginal cost, profits are

$$\pi(m) = \frac{(a-k)^2}{b(m+1)^2}, \tag{11}$$

consumer surplus is

$$CS(m) = \frac{m^2\,(a-k)^2}{2b(m+1)^2},$$

and total surplus is:

$$TS(m) = \frac{(a-k)^2}{(m+1)^2}\frac{m(m+2)}{2b}.$$

The change in (instantaneous) welfare after the m-th entry is:

$$\Delta TS(m) = (TS(m) - TS(m-1)) = \frac{(a-k)^2}{2b}\frac{2m+1}{m^2(m+1)^2}.$$

Therefore, total welfare is

$$W(T) = \sum_{m=1}^{N}\int_{T^m}^{T^{m+1}} TS(m)e^{-rt}dt - \sum_{m=1}^{N} c(T^m).$$

The socially optimal m-th investment time is given by the following first order condition:

$$TS(m-1)e^{-rT^m} - TS(m)e^{-rT^m} - c'(T^m) = 0$$

or

$$\Delta TS(m)e^{-rT^m} = \left(TS(m) - TS(m-1)\right)e^{-rT^m} = -c'(T^m) \tag{12}$$

We can compare this condition to the first order condition that characterizes the pre-commitment investment times:

$$\pi(m)e^{-rT^m} = -c'(T^m) \tag{13}$$

If the left hand side of 12 is less than the left hand side of 13,

$$\Delta TS(m) < \pi(m),$$

the m-th precommitment entry time is earlier than the corresponding welfare maximizing time. Substituting the expressions for $TS(m)$ and $\pi(m)$, this is the case if

$$\frac{(a-k)^2}{2b}\frac{2m+1}{m^2(m+1)^2} < \frac{(a-k)^2}{b(m+1)^2}$$

or

$$2m < 2m^2 - 1,$$

which holds for all $m \geq 2$. Hence, by corollary 1, the equilibrium entry times in the preemption game are earlier than the welfare maximizing times for all $m \geq 2$.

Part(ii): It follows immediately from the analysis above that for $m = 1$:

$$\Delta TS(1) > \pi(1)$$

Thus the first pre-commitment time is later than the socially optimal one. We want to characterize the conditions under which not even preemption eliminates this delay. From Proposition 8, we know that $t_1(N) > t_1(2)$ so if we find a condition that guarantees that the first investment happens too late, from a social point of view, in a game with two players, that also guarantees that it happens too late in a game with more than two players. We know that $t_1(2)$ solves

$$\pi_1 \int_{T_1}^{T_2^*} e^{-rs} ds - c(T_1) + c\left(T_2^*\right) = 0.$$

The expression

$$\pi_1 \int_{T_1^W}^{T_2^*} e^{-rs} ds - c(T_1^W) + c\left(T_2^*\right) \tag{14}$$

describes the preemption incentive to be the first, rather than the second entrant, evaluated atT_1^W, the welfare maximizing time of first entry. If expression 14 is negative, the first equilibrium investment time is too late from a social point of view.

With linear Cournot and exponential cost, it holds that

$$T_1^W = -\frac{1}{\alpha} \cdot \log\left(\frac{3\,(a-k)^2}{8b \cdot (\alpha + r)c}\right)$$

and expression (14) becomes

$$\begin{aligned} &\frac{(a-k)^2}{4rb}\left[\left(\frac{3\,(a-k)^2}{8b \cdot (\alpha+r)c}\right)^{\frac{r}{\alpha}} - \left(\frac{(a-k)^2}{9b \cdot (\alpha+r) \cdot c}\right)^{\frac{r}{\alpha}}\right] - \\ &-c\left[\left(\frac{3\,(a-k)^2}{8b \cdot (\alpha+r)c}\right)^{\frac{r+\alpha}{\alpha}} - \left(\frac{(a-k)^2}{9b \cdot (\alpha+r) \cdot c}\right)^{\frac{r+\alpha}{\alpha}}\right], \end{aligned} \tag{15}$$

which can be simplified to

$$\left(\frac{(a-k)^2}{b(\alpha+r)}\right)^{1+\frac{r}{\alpha}} \left(\frac{1}{c}\right)^{\frac{r}{\alpha}} \left\{\frac{1}{4}\left(\frac{\alpha+r}{r}\right)\left[\left(\frac{3}{8}\right)^{\frac{r}{\alpha}} - \left(\frac{1}{9}\right)^{\frac{r}{\alpha}}\right] - \left[\left(\frac{3}{8}\right)^{1+\frac{r}{\alpha}} - \left(\frac{1}{9}\right)^{1+\frac{r}{\alpha}}\right]\right\}.$$

Since

$$\left(\frac{(a-k)^2}{b(\alpha+r)}\right)^{1+\frac{r}{\alpha}} \left(\frac{1}{c}\right)^{\frac{r}{\alpha}} > 0,$$

we can conclude that expression 14 has the same sign as the expression in curly brackets, which is negative for $\frac{r}{\alpha} > \widetilde{\left(\frac{r}{\alpha}\right)} \approx 1.275$. ■

Proof of Proposition 8. First, we show that for any cost function and payoff structure $\boldsymbol{\pi}$ that satisfy Assumptions 1 to 4, $t_1(N) < t_1(2)$ implies $t_1(N) = t_2(N)$. Suppose $t_1(N) < t_2(N)$. It follows from the proof of Proposition 3 (ii) that $\frac{\partial t_1(N)}{\partial t_2(N)} < 0$. Then, since $t_2(N) \leq T_2^* = t_2(2)$, it must be the case that $t_1(N) \geq t_1(2)$.

Next we show that even if $t_1(N) = t_2(N)$, there is a sufficient condition for $t_1(N) \geq t_1(2)$ that is satisfied in the case of exponential cost function and profits arising from Cournot competition among the investors facing a linear demand function. Let $M \in \{2, ..., N-1\}$ be the size of the initial cluster:

$$t_1(N) = t_2(N) = ... = t_M(N) < T_M^* < t_{M+1}(N)$$

Then, $t_1(N) \geq t_1(2)$ iff

$$\pi(1)\int_t^{T_2^*} e^{-rs}ds - c(t) + c(T_2^*) \geq \pi(M)\int_t^{t_{M+1}(N)} e^{-rs}ds - c(t) + c(t_{M+1}(N))$$

for every $t < T_1^*$. The above condition can be rewritten as:

$$[\pi(1) - \pi(M)] \int_t^{T_1^*} e^{-rs} ds + [\pi(1) - \pi(M)] \int_{T_1^*}^{T_2^*} e^{-rs} ds \geq$$
$$\pi(M) \int_{T_2^*}^{t_{M+1}(N)} e^{-rs} ds + [c\,(t_{M+1}\,(N)) - c\,(T_2^*)] \tag{16}$$

Since

$$[\pi(1) - \pi(M)] \int_t^{T_1^*} e^{-rs} ds > 0 > [c\,(t_{M+1}\,(N)) - c\,(T_2^*)]\,,$$

condition (16) will hold if

$$[\pi(1) - \pi(M)] \int_{T_1^*}^{T_2^*} e^{-rs} ds > \pi(M) \int_{T_2^*}^{t_{M+1}(N)} e^{-rs} ds. \tag{17}$$

By Corollary 1, $t_{M+1}\,(N) < T_{M+1}^*$. Hence, condition (17) will hold if

$$[\pi(1) - \pi(M)] \int_{T_1^*}^{T_2^*} e^{-rs} ds > \pi(M) \int_{T_2^*}^{T_{M+1}^*} e^{-rs} ds. \tag{18}$$

Since e^{-rs} is decreasing in s, a sufficient condition for (18) is that

$$[\pi(1) - \pi(M)]\,[T_2^* - T_1^*] > [\pi(M)]\,\left[T_{M+1}^* - T_2^*\right] \tag{19}$$

which can be rewritten as

$$\pi(1)\,[T_2^* - T_1^*] > \pi(M)\,\left[T_{M+1}^* - T_1^*\right]. \tag{20}$$

For an exponential cost function

$$c(t) = c \cdot \exp\{-(\alpha + r)t\}$$

condition (20) becomes

$$\frac{\pi(1)}{\pi(M)} > \frac{\ln\left(\frac{\pi(1)}{\pi(M+1)}\right)}{\ln\left(\frac{\pi(1)}{\pi(2)}\right)}. \tag{21}$$

If flow profits arise from Cournot competition among the investors, with a linear demand function $P(Q) = a - bQ$, and constant, identical marginal cost $k > 0$, flow profits are given by (11) and condition (21) becomes

$$\frac{(M+1)^2}{2^2} > \frac{\ln\left(\frac{M+2}{2}\right)}{\ln\left(\frac{3}{2}\right)},$$

which holds for any $M \geq 2$. ■

References

[1] Anderson, S.P. and M. Engers (1994), Strategic Investment and Timing of Entry, *International Economic Review*, 35, 833-853.

[2] Bouis, R., Huisman, K.J.M., and P.M. Kort (2006), Investment in Oligopoly under Uncertainty: The Accordion Effect, Discussion Paper 69, Tilburg University, Center for Economic Research.

[3] Brunnermeier, M.K. and J. Morgan (2006), Clock Games: Theory and Experiments, mimeo, Princeton.

[4] Bulow, J., and P. Klemperer (1999), The Generalized War of Attrition, *American Economic Review*, 89(1), 175-189.

[5] Chamley, C., and D. Gale (1994), Information Revelation and Strategic Delay in a Model of Investment, *Econometrica*, 5, 1065-1085.

[6] Dutta, P.J, Lach, S., and A. Rustichini (1995), Better Late than Early: Vertical Differentiation in the Adoption of a New Technology, *Journal of Economics and Management Strategy*, 4, 563-589.

[7] Fudenberg, D., and J. Tirole (1985), Preemption and Rent Equalization in the Adoption of a New Technology, *Review of Economic Studies*, 52, 383-401.

[8] Götz, G. (1999), Monopolistic Competition and the Diffusion of New Technology, *Rand Journal of Economics*, 30, 679-693.

[9] Hoppe, H.C. (2002), The Timing of New Technology Adoption: Theoretical Models and Empirical Evidence, *Manchester School, Special Issue Industrial Organization*, 56-76.T

[10] Hoppe, H.C. and U. Lehmann-Grube (2005), Innovation Timing Games: A General Framework with Applications, *Journal of Economic Theory*, Vol. 121, 30-50.

[11] Katz, M. and C. Shapiro (1987), R & D Rivalry with Licensing or Imitation, *American Economic Review*, 77, 402-420.

[12] Levin, D. and J. Peck (2003), To Grab for the Market or to Bide One's Time, *Rand Journal of Economics*, 34, 536-556.

[13] Mason, R. and H. Weeds (2006), Can Greater Uncertainty Hasten Investment?, *mimeo*, University of Southampton and University of Essex

[14] Park, A. and L. Smith (2006), Caller Number Five: Timing Games that Morph from One Form to Another, *mimeo*, University of Michigan and University of Toronto.

[15] Quirmbach, H.C. (1986), The Diffusion of New Technology and the Market for an Innovation, *Rand Journal of Economics*, 17, 33-47.

[16] Reinganum, J.F. (1981*a*), On the Diffusion of New Technology: a Game-Theoretic Approach, *Review of Economic Studies*, 153, 395-406.

[17] Reinganum, J.F. (1981*b*), Market Structure and the Diffusion of New Technology, *Bell Journal of Economics*, 12, 618-624.

[18] Simon, L., and M. Stinchcombe (1989), Extensive Form Games in Continuous Time: Pure Strategies, *Econometrica*, 57, 1171-1214.

[19] Weeds, H. (2002), Strategic Delay in a Real Options Model of R&D Competition, *Review of Economic Studies*, 69, 729-747.

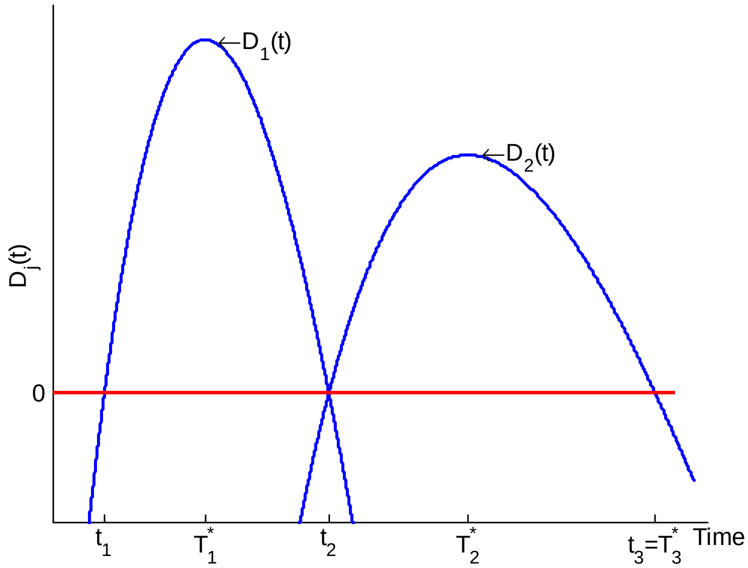

Figure A: Case 1 (no cluster)

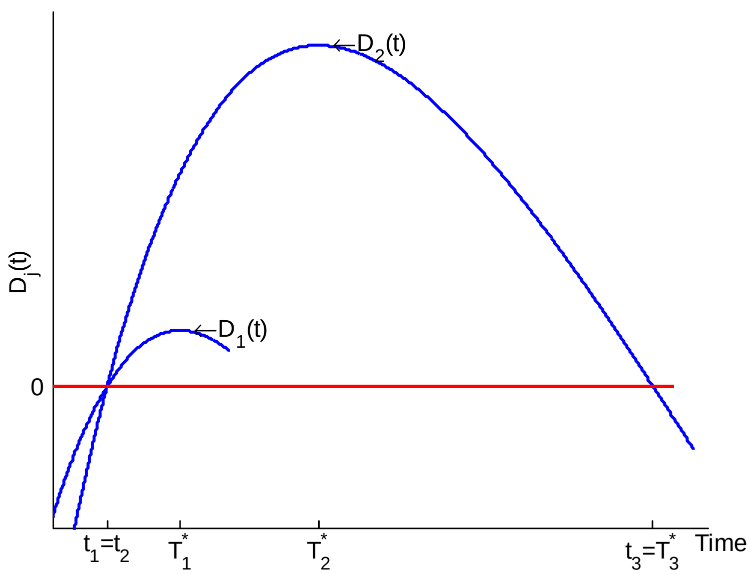

Figure B: Case 2 (cluster)