Clustering in N-Player Preemption Games *

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Abstract

This paper studies N-player preemption games. A finite number of firms have to decide when to make an irreversible, observable investment. Upon investment, a firm receives flow profits declining in the number of firms that have already invested. The cost of investment declines over time. We characterize the subgame-perfect equilibrium outcome, which is unique up to a permutation of players. In games with more than two players, investment times are clustered when an investment is followed by an intense preemption race among the remaining active players. We characterize a necessary and sufficient condition for two or more investments to be clustered. The set or parameter values for which any two investments are clustered is shown to be non-empty. Our results show how clustering in the timing of firms' investment decisions can occur in the absence of coordination failures, informational spillovers or positive payoff externalities.

Keywords: Timing Games, Preemption, Technology Adoption, Dynamic Entry.

Jel Classification: C73, L13, O3.

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1 Introduction

Consider a game of timing in which players have to decide when to make an investment. The cost of investing declines over time. A firm earns a positive profit flow upon investment, but profit flows decline in the number of investors. This is a *preemption game*: Delay exogenously increases payoffs through lower investment cost, but each player also has an incentive to invest early, because there is an early mover advantage.

In a preemption game, investment by a player reduces the post-investment flow profit for later investors, and hence the incentive of the other players to invest. Therefore, our intuition lets us expect a period of delay until next investment occurs. This intuition is correct for the case of two players (Fudenberg and Tirole (1985)) but, as we show in this paper, may fail otherwise.

We study a general N-Player investment preemption game[1] and characterize conditions under which investment times are clustered. The presence of clusters in this game is surprising, as the timing game literature has attributed the presence of clusters mainly to coordination failures, as in Levin and Peck (2003), to the presence of positive network externalities, as in Mason and Weeds (2010), or to informational spillovers (e.g. Chamley and Gale (1994)), where rival investment signals a high profitability of investment. Brunnermeier and Morgan (2006) show that "herding" occurs in a preemptive "clock game", but attribute this herding effect to private information being (partially) revealed by the first player to act.

Also a large body of empirical literature has examined how rival adoption or market entry affects a the timing of a firm's own technology adoption or market entry. Several papers have found that adoption by a rival accelerates the adoption by remaining firms and interpreted this as evidence of positive payoff externalities or informational effects.[2] Our main result is that clusters, which are the most extreme form of acceleration, arise in a preemption game although coordination failures are ruled out by assumption, rival investment has no informational content, no positive externalities, and lowers the post-

[1] In two related papers, Park and Smith (2008,2010) analyze a timing game with a general payoff structure and more than two players. See Park and Smith (2008) for the case of unobservable actions, and Park and Smith (2010) for the case of observable actions and a continuum of players.

[2] These papers find that a firm's adoption hazard increases when a rival adopts. Karshenas and Stoneman (1993, p.521) interpret this acceleration of adoption as "epidemic effects" in the adoption of numerically controlled machine tools. Hannan and McDowell (1987, p.186) speak of "spillover effects" in the context of ATM adoption by banks, and Levin et al. (1992, p. 347) argue that there is an informational effect via customers in the case of optical scanners in grocery stores.

investment flow profit. We therefore provide an alternative interpretation of the empirical evidence.

The mechanism through which clusters arise in our model is based purely on preemption and backward induction. Suppose there are three firms: if being the second investor is profitable relative to being third, the preemption race to be the second investor is intense, and in equilibrium the second investment occurs early. In order to obtain monopoly profits for some time, a firm would have to invest even earlier. If monopoly profits are not much higher than duopoly profits, no firm wants to incur the extra-cost that is necessary to invest strictly before the second investor, and the first and second investments are clustered. In a game with more than three players, a similar mechanism can cause clusters at any point in the investment sequence.

We characterize the unique equilibrium outcome of the N-player game. We then provide a necessary and sufficient condition on the model parameters for two investments to be clustered. A similar condition determines whether a cluster involves more than two simultaneous investments. Finally, we show that the set of parameter values for which clusters occur is nonempty.

The mechanism that causes a cluster here is related to Bulow and Klemperer's (1994) model of a seller who has multiple identical objects and multiple buyers with independent private values. They show that if buyers' valuations are not too different, frenzies of simultaneous purchases can occur because a purchase by a buyer increases the remaining buyers' willingness to pay. In our model, investment by a player lowers the flow profit achievable by the next investor. Nonetheless, clusters are possible if this decrease is sufficiently small, and the ensuing preemption race to take the role of next investor is sufficiently intense.

In independent work, Bouis et al. (2009) study dynamic investment in oligopoly in a real options framework and find comparative statics result that are closely related to ours.[3] The real options approach allows aggregate uncertainty in the payoff process, but is restricted to a specific payoff growth process for payoffs (a Brownian motion with drift).

The remainder of the paper is organized as follows. The next section introduces the model. Section 3 characterizes the equilibrium outcome. Section 3.1 illustrates the benchmark case of a two-player game, in which investments are never clustered in equilibrium. Section 3.2 analyzes the three-player game, and shows under which conditions the first two

[3] Bouis et al. (2009) provide an explicit argument for existence of equilibrium, and for the existence of a cluster for the case of $N = 3$. For the general N-player game they provide comparative statics results under the assumption that equilibrium exists, and a numerical example.

investments are clustered. In Section 3.3 we characterize the equilibrium of the game with N players. The condition for the presence of a cluster is derived in Section 3.4. Section 3.5 discusses clusters involving more than two investments and the possibility of a cluster at the end of the investment sequence. Section 4 concludes.

2 Model

2.1 The investment game

We analyze an infinite horizon dynamic game in continuous time. At time 0, a new investment opportunity becomes available, and N identical players (firms) have to decide if, and when, to seize this opportunity. The investment opportunity can be interpreted as adoption of a new technology, or entry into a new market. Investment is observable and irreversible.

The set of firms is denoted by $\mathbf{N} = \{1, ..., N\}$ and a single firm is denoted by $i \in \mathbf{N}$.

The model corresponds to the one studied by Reinganum (1981a, 1981b) and Fudenberg and Tirole (1985) except for the following: Until a firm invests, it receives constant flow of profits π_0 which we normalize to zero. This assumption that pre-investment payoffs are independent of the number of earlier investments will be essential for obtaining a unique outcome in each subgame, which in turn guarantees rent equalization.[4] Upon investment, a firm earns flow profits of $\pi(m)$, where m is the number of firms that have already invested at a given point in time. Let $\boldsymbol{\pi} = (\pi(1), \pi(2), ..., \pi(N))$denote a flow profit structure.

Let $c(t)$ be the present value at time zero of the cost of investing at time t. If the outcome of the game is that the vector of investment times is T^j, for $j = 1, ...N$, and firm i is the j-th investor, firm i's payoff is the following:

$$V_i^j(T^1, T^2, ..., T^j, ..., T^N) = \sum_{m=j}^{N} \pi(m) \int_{T^m}^{T^{m+1}} e^{-rs} ds - c(T^j) \tag{1}$$

where r denotes the common discount rate, and $T^{N+1} \equiv +\infty$.

We introduce the following assumptions:

Assumption 1

Flow profits $\pi(m)$ are (i) strictly positive for any m and (ii) strictly decreasing in m.

[4] However, the normalization of pre-investment payoffs to zero, rather than any positive constant, will not affect the results of the paper.

Investing always increases period payoffs for a firm, but the benefits of investing decrease in the total number of investors: as more firms invest, competition among the investors becomes more intense.

Assumption 2
The current value cost function $\left(c\left(t\right)e^{rt}\right)$ *is (i) strictly decreasing and (ii) strictly convex.*

The cost of investing declines over time. This may capture upstream process innovations or economies of learning and scale. Moreover, cost declines at a decreasing rate.

Assumption 3
At time zero, (i) the investment cost exceeds discounted monopoly payoffs: $c(0) > \frac{\pi(1)}{r}$*, and (ii) there is an incentive to delay:* $\pi\left(1\right) < -c'\left(0\right)$*. Eventually, (iii) investment is profitable for all players:* $\exists\tau$ *such that* $c\left(\tau\right)e^{r\tau} < \frac{\pi(N)}{r}$*, and (iv) there is no incentive to delay:* $\lim_{t\to\infty} c'(t)e^{rt} \in \left(-\pi\left(N\right), 0\right]$.

Assumption 3(i) guarantees that investing at time zero is too costly. No firm would invest immediately, even if it could thereby preempt all other firms and enjoy monopoly profits $\pi\left(1\right)$ forever. Assumption 3(ii) guarantees that at time zero the marginal benefit from delaying investment, that is the reduction in the investment cost, is greater than the highest marginal cost from delay, that is the highest foregone profit flow. Assumption 3(iii) ensures that the value of investing becomes positive in finite time: The cost of investing eventually reaches a level sufficiently low, that it becomes profitable to invest, even for a firm facing maximum competition.[5] Assumption 3(iii) is necessary but not sufficient to guarantee that the last investment occurs in finite time. Assumption 3(iv) guarantees that the last investor does not have an incentive to delay investment indefinitely.

In what follows, we will denote by t_j the j-th equilibrium investment time.[6] The j-th and $(j+1)$-th investments are **clustered** if they occur at the same instant in time, i.e. $t_j = t_{j+1}$. Otherwise, they are **diffused**.

2.2 Strategies in continuous-time preemption games

We model strategies in a timing game with observable actions and continuous time following Hoppe and Lehmann-Grube (2005), who adopt the framework introduced by Simon and

[5] We will discuss the implications of relaxing this assumption in Section 3.5.
[6] In Proposition 1 we prove that the vector of equilibrium investment times $(t_1, ..., t_N)$ is unique.

Stinchcombe (1989). Namely, we restrict play to pure strategies and interpret continuous time as "discrete time, but with a grid that is infinitely fine."[7]

A well-known problem with modelling preemption games in continuous time is that typically games in this class do not have an equilibrium in pure strategies, due to the possibility of coordination failures. We explicitly rule out coordination failures introducing a randomization device as in Katz and Shapiro (1987), Dutta, Lach and Rustichini (1995), and Hoppe and Lehmann-Grube (2005):

Assumption 4

If n firms invest at the same instant t (with $n = 2, 3, ., N$), then only one firm, each with probability $\frac{1}{n}$, succeeds.

To understand how the randomization device operates, suppose that $N = 2$ and at a given time t each firm would like to invest, provided that the other does not. If at time t both firms try to invest, only one is successful, i.e. only one actually pays the cost $c(t)$ and starts receiving flow payoffs $\pi(1)$. Then, the remaining firm observes that the opponent has invested, and has the option to either invest "consecutively but at the same moment in time"[8] or to let the game continue. For an interpretation of this randomization device, we refer to Dutta, Lach and Rustichini (1995).[9]

2.3 The optimal "stand-alone" investment times

In this subsection we illustrate the basic trade-off of the investment problem, abstracting from strategic considerations. Consider the hypothetical problem of a firm who acts as a single decision maker and has to select the optimal time to make an investment which has cost $c\,(t)$ and guarantees flow payoff of $\pi\,(j)$ forever, for $j \in \{1, ..., N\}$, where $c\,(t)$ and $\pi\,(j)$ satisfy assumptions 1 to 3. This firm would choose t to maximize the following profit:

$$f_j\,(t) \equiv \frac{\pi\,(j)}{r} e^{-rt} - c\,(t)\,. \tag{2}$$

[7] See the discussion in section 2.1 of Hoppe and Lehmann-Grube (2005) for an illustration of the conditions that need to be satisfied in order to adopt the Simon and Stinchcombe (1989) framework.

[8] See Simon and Stinchcombe (1989), p. 1181.

[9] A possible alternative assumption is the one made by Dutta and Rustichini (1993). In that paper, if two firms stop simultaneously, both receive a convex combination of the payoff from being the only one to stop at that time, and the payoff from stopping later, at the optimal time for a "follower". This assumption would introduce the possibility of clusters of simultaneous investments through a mechanism that is unrelated to the one illustrated in our paper.

We denote the solution to this problem as T_j^*. Adopting the terminology in Katz and Shapiro (1987), we define it as the **stand-alone investment time for** $\pi(j)$. Observe that $f_j(t)$ is strictly quasi-concave and that T_j^* is well-defined[10] for every $j \in 1, ..., N$ as the solution to

$$f_j'(t) = 0 \Longleftrightarrow -\pi(j) e^{-rt} - c'(t) = 0. \tag{3}$$

At time T_j^*, the marginal benefit from delaying investment, that is the cost reduction $c'(t)$, is exactly equal to the marginal cost, that is the foregone discounted profit flow $\pi(j) e^{-rt}$. Before T_j^*, a player is willing to delay because the cost is decreasing at a speed that more than compensates the foregone profit flow. After T_j^*, a player would rather invest immediately than delay. It follows from the implicit function theorem that $T_j^* < T_{j'}^*$ for $j < j'$: For a larger foregone profit flow, that is for $j < j'$, the stand-alone time is earlier.

3 Equilibrium analysis

We now return to the strategic environment, and solve for the equilibria of the game. A feature of any equilibrium of the game that is built into our assumptions is the following:

Lemma 1 *In any SPNE, no firm invests at $t = 0$, all firms invest in finite time, and the last investment takes place at the stand-alone investment time T_N^*.*

Assumptions 1(ii) and 3(i) guarantee that investment at time zero is too costly.[11] Assumptions 2(i), 3(iii) and 3(iv) guarantee that all firms invest in finite time. The result that the last equilibrium investment time is exactly the stand-alone investment time T_N^* is not surprising: when only one active firm is left, it maximizes the profit (2) for $j = N$.

Next, we introduce our benchmark: the two-player investment game analyzed by Fudenberg and Tirole (1985).

3.1 The benchmark case: two firms, diffused investment

Suppose $N = 2$. The easiest way to capture the intuition for this game is to use a backward induction approach. By Lemma 1, both firms invest no later than the second stand-alone investment time T_2^*. Therefore, each firm anticipates that if it invests first at some time

[10] For a proof, see Claim 1 in the Appendix.

[11] This fact is crucial in proving that all players receive the same payoff in equilibrium. Relaxing assumption 3(i), one could generate an equilibrium in which some players invest at time zero and receive a higher payoff than the remaining players, who would instead invest later and receive all the same, lower payoff.

$t < T_2^*$, the opponent will follow at T_2^*. The payoff from this early investment will then be what Fudenberg and Tirole define as the *Leader Payoff*:

$$L(t) = \pi(1) \int_t^{T_2^*} e^{-rs} ds + \pi(2) \int_{T_2^*}^{\infty} e^{-rs} ds - c(t). \tag{4}$$

Alternatively, a firm could wait until T_2^* and receive the *Follower Payoff:*

$$F(t) = \pi(2) \int_{T_2^*}^{\infty} e^{-rs} ds - c(T_2^*). \tag{5}$$

The benefit from being the leader, rather than the follower, is that high profits $\pi(1)$ are earned for some period. The cost is that early investment is more expensive than late investment. The fact that the cost of investment, although initially prohibitive, is decreasing and convex, guarantees that the leader and follower payoff curves have the shape illustrated in Figure 1. Fudenberg and Tirole (1985) prove that the first time at which an investment occurs in equilibrium, t_1, is the earliest time when the two curves intersect. In equilibrium, firms invest at different points in time[12] (investments are *diffused*) and payoffs are the same for both firms (there is *rent equalization*).

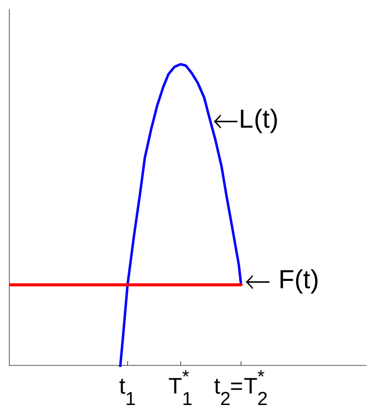

Figure 1: **Diffusion in the two-player game**. The LPC brings the first investment forward to t_1. Earlier preemption is not profitable: before t_1, $F(t)$ exceeds $L(t)$. The figure is drawn for cost function $c(t) = 2 \cdot 10^4 e^{-(\alpha+r)t}$ for $\alpha = 0.24$, $r = 0.1$, and flow profits $\boldsymbol{\pi} = (275, 214)$.

The mechanism at work is well-known: if unconstrained by strategic considerations, a single firm would like to invest at the stand-alone time T_1^*. Also in the presence of an opponent, each firm would like to invest first, at T_1^*. The opponent would then follow at T_2^*. The leader would receive a higher payoff than the follower. This cannot be an equilibrium

[12] The assumption that pre-investment payoffs are constant rules out the possibility of what Fudenberg and Tirole (1985) define as "late joint adoption equilibria".

because the firm who takes the role of the follower could profitably deviate and preempt the opponent by investing at $T_1^* - \varepsilon$. The presence of a second player introduces a **Leader Preemption Constraint (LPC)** on the time of first investment: Leader investment cannot take place at a time when earlier preemption is profitable. As a consequence, first investment must occur strictly earlier than T_1^*. In particular, it must occur weakly before the first intersection of the Leader and Follower payoff functions. Since the leader payoff function is increasing in that interval, first investment will occur at the latest time that satisfies the LPC, i.e. the first intersection of the two curves.

3.2 The three-firm game: When are the first two investments clustered?

Also in a three-player game, first investment must occur strictly earlier than T_1^*. The key difference from the two-player game is that t_1 is identified by the presence of *two* constraints. One is the LPC constraint discussed above. The second is what we call the **Follower Preemption Constraint (FPC)**. The latter reflects the fact that the first investment is followed by a preemption race among the remaining two players. This race among followers determines an upper bound on the time of first investment.

The aim of this section is twofold. First, we show that the first and second investments can be clustered or diffused. Which case occurs depends on which of the two constraints on t_1, the FPC or the LPC, is binding in equilibrium. Then, we derive the necessary and sufficient condition on the model primitives determining which constraint is binding. We proceed by solving the game by backward induction. Figure 2 illustrates.

The two-firm subgame

Suppose that the first investment has occurred, and consider the ensuing two-firm subgame. It is analogous to the two-firm game of Section 3.1. By Lemma 1, all firms must invest by T_3^*. Each firm anticipates that if it invests first at some time $t < T_3^*$, the opponent will follow at T_3^*. The *Leader Payoff* and *Follower Payoff* for this subgame are

$$L_2(t) = \pi(2)\int_t^{T_3^*} e^{-rs}ds + \pi(3)\int_{T_3^*}^{\infty} e^{-rs}ds - c(t) \text{ and} \tag{6}$$

$$F_2(t) = \pi(3)\int_{T_3^*}^{\infty} e^{-rs}ds - c(T_3^*) \tag{7}$$

respectively. The threat of preemption guarantees that the first investment in the subgame must take place at the earliest time when $L_2(t) = F_2(t)$. The second investment time in the game, t_2, coincides with this intersection. Last investment occurs at T_3^*.

The Follower Preemption Constraint (FPC)

The conclusion above that second investment occurs at the earliest intersection of $L_2(t)$ and $F_2(t)$ clearly assumes that the first investment must occur weakly before this intersection. We show by contradiction that this must be the case in equilibrium. Suppose that the first investment took place strictly later, at some time $\tau \leq T_3^*$. A two-firm subgame would then start at τ. Because $L_2(\tau) > F_2(\tau)$, both firms would prefer to be leader rather than follower in this subgame. They would both try to invest at τ, one would succeed and the other would invest later at T_3^*. This cannot be an equilibrium because each of the last two investors receives a lottery between $L_2(\tau)$ and $F_2(\tau)$ while it could deviate and guarantee itself a payoff arbitrarily close to $L_2(\tau)$. Deviating by investing at $\tau - \varepsilon$, a firm would be the first investor in the game. It would trigger a two-firm subgame in which one more investment would occur at $\tau - \varepsilon$ and the last one at T_3^*. Therefore, the deviator would receive a payoff of $L_2(\tau - \varepsilon)$.

We have established that the time of first investment t_1 is constrained by the presence of a preemption race in the ensuing two-firm subgame: to guarantee that there is rent equalization in this race, t_1 must be no later than the first intersection of the leader and follower payoff curves of the two-firm subgame, namely $L_2(t)$ and $F_2(t)$. We call this the **Follower Preemption Constraint (FPC)**. As the second investment time t_2 coincides with this intersection, we say that the FPC is binding in equilibrium if the first investment occurs exactly at t_2, and not binding if it occurs strictly earlier than t_2.[13]

The Leader Preemption Constraint (LPC)

We now illustrate the LPC for the three-player game. It follows from the analysis above that the second and third investments will occur at t_2, with $L_2(t_2) = F_2(t_2)$, and $t_3 = T_3$ respectively. The FPC requires that the first investment occurs weakly earlier than the first intersection of $L_2(t)$ and $F_2(t)$. Therefore, the Leader payoff in the three-player game, i.e. the payoff of the first investor, is:

$$L_1(t) = \pi(1) \int_t^{t_2} e^{-rs} ds + \pi(2) \int_{t_2}^{T_3^*} e^{-rs} ds + \pi(3) \int_{T_3^*}^{\infty} e^{-rs} ds - c(t). \tag{8}$$

The Follower payoff in the three-firm game, i.e. the payoff from being either the second or

[13] Observe that in the two-player game in Section 3.1 this constraint is absent: After the first investment, there is a single follower left, whose investment time is not determined by a preemption race but by a single-agent decision problem.

the third investor, is

$$F_1(t) = \pi(2) \int_{t_2}^{T_3^*} e^{-rs} ds + \pi(3) \int_{T_3^*}^{\infty} e^{-rs} ds - c(t_2) = \pi(3) \int_{T_3^*}^{\infty} e^{-rs} ds - c(T_3^*). \quad (9)$$

As in the two-player game, there is a LPC: the first investment cannot occur at a time such that earlier preemption is profitable, because otherwise any of the followers would have a profitable deviation. We say that the LPC on t_1 is binding in equilibrium if given the subsequent investment times t_2 and t_3, $L_1(t) > F_1(t)$ for some $t < t_2$. Otherwise, preempting the leader is never profitable, and we say that the LPC is not binding.

The relationship between FPC and LPC

The FPC reflects the intensity of the follower preemption race that starts after the first investment: The more intense this race, the earlier is the first intersection of $L_2(t)$ and $F_2(t)$, i.e. the earlier is t_2. The LPC instead reflects the intensity of the race to be the first investor. To capture the relationship between the two, notice that the earlier is t_2, the shorter the period for which the first investor earns monopoly profits: Early t_2 makes the role of first investor less desirable. Therefore, the more intense the follower preemption race, the less intense race to be the leader of the three-firm game: the stronger the FPC, the weaker the LPC.

The key observation of our analysis is that for any given set of parameters, only one of the two constraints is binding, and which constraint is binding is equivalent to whether the first two investments are clustered or diffused. If the follower preemption race is sufficiently intense, only the FPC is binding, and investments are clustered. Otherwise, only the LPC is binding, and investments are diffused. We discuss these two cases below, and illustrate them in Figure 2.

First, observe that the leader payoff $L_1(t)$ is strictly quasiconcave, and maximized at T_1^*. By construction, it intersects $F_1(t)$ in t_2. Which constraint is binding depends on the relative position of t_2 with respect to T_1^*. The intuition for this is that T_1^*, being determined by $\pi(1)$, reflects the desirability of the role of first investor, hence the strength of the LPC, while t_2 reflects the strength of the FPC.

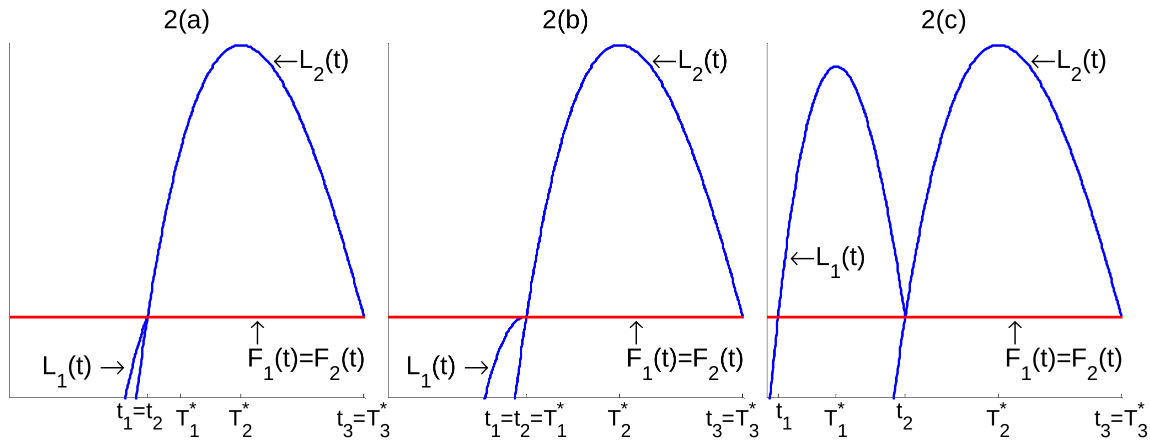

Figure 2: **Clustering vs. Diffusion**. In all three panels, the third investment occurs at T_3^* and the second at the first intersection of $L_2(t)$ and $F_2(t)$. Monopoly profits increase from panel 2(a) to 2(c) bringing forward the optimal stand-alone time T_1^*. In panels (2a) and (2b), $t_2 \leq T_1^*$, so in the race to be first, $F_1(t)$ exceeds $L_1(t)$ before t_2. Hence, only the FPC is binding and investments are clustered. Panel (2b) represents the knife-edge case with $t_2 = T_1^*$. In (2c), $t_2 > T_1^*$. Therefore, $L_1(t)$ exceeds $F_1(t)$ before t_2 and the LPC alone is binding. The cost function is as in Figure 1. Flow profits $\boldsymbol{\pi}$ are $(275, 214, 129)$ for (2a), $(315, 214, 129)$ for (2b), $(419, 214, 129)$ for (2c).

Case 1: $\mathbf{t_2 \leq T_1^*}$ (Figures 2a and 2b).

The LPC is not binding, because the follower payoff $F_1(t)$ exceeds the leader payoff $L_1(t)$ at any $t < t_2$. The first investment occurs exactly at t_2: the FPC is binding. The payoffs of all players are equalized. The first two investment times are *clustered*: $t_1 = t_2$.

Case 2: $\mathbf{t_2 > T_1^*}$ (Figure 2c).

The LPC is binding: The leader payoff $L_1(t)$ exceeds the follower payoff $F_1(t)$ to the left of t_2. The preemption race to be first investor brings t_1 forward to the earliest intersection of leader and follower payoffs. The payoffs of all players are equalized. The FPC instead, is not binding. The first two investments are *diffused:* $t_1 < T_1^* < t_2$.

How model primitives determine the presence of a cluster

As can be seen in Figure 2, whether case 1 or case 2 will occur is equivalent to whether $L_2(t) - F_2(t)$, the incentive to preempt in the 2-player subgame that follows the first investment, is positive or negative when evaluated at T_1^*.

In the knife-edge case of Figure (2b), the preemption incentive evaluated at T_1^* is zero: the first intersection of $L_2(t)$ and $F_2(t)$, that identifies t_2, coincides exactly with T_1^*. The follower preemption race is just intense enough to make the FPC binding and the LPC not binding. If instead the first intersection of $L_2(t)$ and $F_2(t)$ occurs to the left of T_1^*, as in Figure (2a), then the preemption incentive $L_2(t) - F_2(t)$ evaluated at T_1^* is strictly positive.

Conversely, if it occurs to the right of T_1^*, as in Figure (2c), then $L_2(t) - F_2(t)$ evaluated at T_1^* is strictly negative.

The previous observation allows us to identify the parameter range for which case 1 and case 2 occur, respectively. Evaluating the preemption incentive $L_2(t) - F_2(t)$ at T_1^*:

$$\begin{aligned} L_2(T_1^*) - F_2(T_1^*) &= \pi(2)\int_{T_1^*}^{T_3^*} e^{-rs}ds + \pi(3)\int_{T_3^*}^{\infty} e^{-rs}ds - c(T_1^*) - \left[\pi(3)\int_{T_3^*}^{\infty} e^{-rs}ds - c(T_3^*)\right] \\ &= \frac{\pi(2)}{r}\left[e^{-rT_1^*} - e^{-rT_3^*}\right] - \left[c(T_1^*) - c(T_3^*)\right] \end{aligned} \tag{10}$$

Recalling that T_1^* is a decreasing function of $\pi\left(1\right)$ and T_3^* is a decreasing function of $\pi\left(3\right)$, we can rewrite the preemption incentive evaluated at T_1^* as a function of the three profit parameters $\pi\left(1\right)$, $\pi\left(2\right)$, $\pi\left(3\right)$ only:

$$h_1\left(\pi\left(1\right), \pi\left(2\right), \pi\left(3\right)\right) \equiv \frac{\pi(2)}{r}\left[e^{-rT_1^*(\pi(1))} - e^{-rT_3^*(\pi(3))}\right] - \left[c(T_1^*(\pi\left(1\right))) - c(T_3^*(\pi\left(3\right)))\right] \tag{11}$$

For profit structures such that $h_1\left(\cdot\right)$ is nonnegative, the first two investments are clustered, for all other profit structures, investments are diffused.

The function $h_1\left(\cdot\right)$ is monotone in each of the profit parameters $\pi\left(1\right)$, $\pi\left(2\right)$ and $\pi\left(3\right)$.[14] The intuition is captured by looking at how each of them affects the relative strength of the two constraints. First, $h_1\left(\cdot\right)$ is decreasing in $\pi\left(1\right)$. Hence, starting from the knife-edge case, increasing $\pi\left(1\right)$ we fall into case 2 (diffusion). The intuition is that an increase in $\pi\left(1\right)$ makes the role of leader of the three-player preemption race more attractive. Hence, the LPC becomes binding.

Next, consider $\pi\left(2\right)$ and $\pi\left(3\right)$. The preemption incentive $h_1\left(\cdot\right)$ is increasing in $\pi\left(2\right)$ and decreasing in $\pi\left(3\right)$. Hence, starting from the knife-edge case, increasing $\pi\left(2\right)$ or decreasing $\pi(3)$ there continues to be a cluster. The intuition is that an increase in $\pi(2)$ or a decrease in $\pi(3)$ makes the role of second investor more attractive relative to the role of third investor. The preemption race among the followers becomes more intense, and this brings t_2 forward. The FPC constraint becomes stronger. At the same time, earlier t_2 makes the role of first investor less attractive so the LPC becomes weaker.

[14]For a proof, see Corollary 2.

3.3 The general case: N firms

In this section, we formalize and generalize our characterization of the equilibrium outcome of the game with three players to the general case of N players.

After the first $j-1$ investments have taken place, two constraints determine the next investment time t_j. First, the LPC: preempting the *leader* of the current subgame, i.e. the j-th investor, by investing earlier than t_j, must be unprofitable. Second, the FPC: t_j must be weakly earlier than the time of next investment, t_{j+1}, which is determined by the preemption race in the ensuing subgame that will be played by the *followers* after the j-th investment.[15]

Proposition 1 below establishes that the equilibrium outcome of the game is unique, that the rent equalization result is preserved even for a general number of players, and that the equilibrium investment times are constructed with the following backwards algorithm: The last investment time is equal to the last stand-alone investment time, as established in Lemma 1. For every $j < N$ instead, the equilibrium investment time t_j is determined comparing the $(j+1)$-th equilibrium investment time t_{j+1} with the j-th stand-alone investment time T_j^*. Two cases are possible: If the $(j+1)$-th investment takes place before T_j^*, the j-th investment will not be brought forward any further, and there will be a cluster of investments: $t_j = t_{j+1}$. Otherwise, the j-th investment will take place earlier than t_{j+1}, at the earliest time that guarantees rent equalization. In the first case, analogous to Case 1 in Section 3.2, the FPC alone is binding. In the second case, analogous to Case 2 in Section 3.2, the LPC alone is binding.

Proposition 1 *The game admits a unique SPNE outcome, up to a permutation of players. All players receive the same payoff:* $\frac{\pi(N)}{r}e^{-rT_N^*} - c\left(T_N^*\right)$.
The equilibrium investment times $(t_1, t_2..., t_N)$ *can be calculated recursively as follows:*
The last investment time is equal to the last stand-alone investment time: $t_N = T_N^*$;
For $j = 1, ..., N-1$, *two cases are possible:*
Case 1. *The* $(j+1)$*-th investment time* t_{j+1} *is weakly earlier than the* j*-th stand-alone investment time* T_j^*. *Then, the* j*-th and the* $(j+1)$*-th investments are clustered:* $t_j = t_{j+1}$.
Case 2. *The* $(j+1)$*-th investment time* t_{j+1} *is strictly later than the* j*-th stand-alone investment time* T_j^*. *Then, the* j*-th and the* $(j+1)$*-th investments are diffused, and* t_j *is*

[15] The above discussion holds for $j \leq N-2$. In equilibrium, the last two investment times are the equilibrium outcome of a subgame that resembles the two-firm game illustrated in Section 3.1.

the smallest solution to: $\pi(j)\int_{t_j}^{t_{j+1}} e^{-rs}ds - c(t_j) + c(t_{j+1}) = 0.$

Proposition 1 has two implications that go beyond the features of the three-player example. First, for $N > 3$ clusters can include more than two simultaneous investments: suppose that the preemption race for the role of $(j+1)$-th investor is sufficiently intense that not only $t_{j+1} < T_j^*$, but $t_{j+1} < T_{j-1}^* < T_j^*$: in this case, the $(j-1)$-th, j-th and $(j+1)$-th investments will be clustered. In Section 3.5 we illustrate how model primitives affect the size of a cluster.

Second, clusters can occur not only at the beginning, but at any point of the investment sequence, except for the last. In Section 3.5 we discuss under which alternative assumption the last investment can be part of a cluster.

3.4 The condition for a cluster

In this section, we characterize the set of parameter vectors for which the j-th and $(j+1)$-th investments are clustered. Moreover, we show that the set of parameter values for which a cluster occurs is nonempty.

Proposition 1 establishes that the j-th and the $(j+1)$-th investments are clustered if and only if t_{j+1} is weakly earlier than T_j^*. As in our discussion of Figure 2, we start from the observation that the relative position of t_{j+1} with respect to T_j^* is related to the preemption incentive in the follower preemption race that follows the j-th investment, evaluated at T_j^*. We will derive an expression for this preemption incentive, and use it to formulate a necessary and sufficient condition on the model primitives for a cluster $t_j = t_{j+1}$.[16]

Consider the follower preemption race that follows the j-th investment: As in the three-player game, the intensity of this race determines an upper bound t_{j+1} on t_j. There is one difference from the special case of a three-player game: The upper bound t_2 on t_1 is always strictly earlier than the third investment time T_3^*. In the N-player game instead, the upper bound t_{j+1} may coincide with one or more subsequent investments. To account for this, we introduce the following definition.

Definition 1 *For a given flow profit structure* $\boldsymbol{\pi}$*, consider the subgame starting in equilibrium after the* j*-th investment. Let* $k \geq 1$ *be the number of investments occurring at the first investment time in this subgame, so that* $t_{j+1} = ... = t_{j+k}$.

[16] The analysis closely follows the discussion presented in section 3.2 for $N = 3$.

The leader and follower payoffs in the race to be one of the k leaders of the follower preemption race that follows the j-th investment are given by:

$$L_{j+k}(t) = \pi(j+k)\int_t^{t_{j+k+1}} e^{-rs}ds + \sum_{m=j+k+1}^{N} \pi(m)\int_{t_m}^{t_{m+1}} e^{-rs}ds - c(t) \quad (12)$$

$$F_{j+k}(t) = \sum_{m=j+k+1}^{N} \pi(m)\int_{t_m}^{t_{m+1}} e^{-rs}ds - c(t_{j+k+1}) \quad (13)$$

respectively,[17] where $t_{N+1} \equiv +\infty$. The incentive to preempt at time t is given by their difference: $L_{j+k}(t) - F_{j+k}(t)$. The point $t_{j+1} = t_{j+k}$ is the earliest time when this incentive is null. The relative position of t_{j+1} with respect to T_j^*, hence the presence or absence of a cluster of t_j and t_{j+1}, depends on the sign of this preemption incentive evaluated at T_j^*:[18]

Corollary 1 *In equilibrium, the j-th investment is clustered with the $(j+1)$-th if and only if the preemption incentive in the follower preemption race evaluated at T_j^* is nonnegative: $L_{j+k}(T_j^*) - F_{j+k}(T_j^*) \geq 0$.*

The condition in Corollary 1 allows us to characterize a necessary and sufficient condition on the profit parameters for a cluster $t_j = t_{j+1}$. The preemption incentive $L_{j+k}(T_j^*) - F_{j+k}(T_j^*)$ can be rewritten as follows:

$$L_{j+k}(T_j^*) - F_{j+k}(T_j^*) = \frac{\pi(j+k)}{r}\left[e^{-rT_j^*} - e^{-rt_{j+k+1}}\right] - \left[c(T_j^*) - c(t_{j+k+1})\right]. \quad (14)$$

Compare (14), the preemption incentive at T_j^*, to its formulation for a three-player game, for $j = 1$, by recalling equation (10) from Section 3.2:

$$L_2(T_1^*) - F_2(T_1^*) = \frac{\pi(2)}{r}\left[e^{-rT_1^*} - e^{-rT_3^*}\right] - [c(T_1^*) - c(T_3^*)]$$

The key difference is that in the three player-game t_3 coincides with the last stand-alone investment time T_3^*, hence only depends on $\pi(3)$. Instead, in the general N-player game, t_{j+k+1} can be determined by the outcome of a preemption race. Hence, t_{j+k+1} is a function of a subvector of the profit structure $\boldsymbol{\pi}$ that not only includes $\pi(j+k+1)$, but also profit parameters associated with later investments. However, the recursive nature of the equilibrium characterized in Proposition 1 implies that it is *only* this subvector $\left(\pi(m)_{m=j+k+1}^{N}\right)$

[17] Note that for $N = 3$, $j = 1$ it always holds that $k = 1$, and equations 12 and 13 correspond to equations 6 and 7.

[18] This result formalizes the observation in section 3.2, for $N = 3$ and $j = 1$.

that pins down t_{j+k+1}. The two equations are otherwise very similar: the preemption incentive (14) is a function of T_j^*, hence of the profit flow $\pi(j)$. Moreover, it depends directly on the profit flow $\pi(j+k)$ earned by each of the k leaders of the preemption subgame that starts after the j-th investment.

The next Proposition states these insights formally: The preemption incentive in (14) can be rewritten as a function of only the profit flows associated with having j or more competitors. Hence, the condition for a cluster from Corollary 1 can be expressed as a necessary and sufficient condition on the sign of a function of model primitives.

Proposition 2 *The preemption incentive in the follower preemption race evaluated at T_j^* is a function of the $N-j+1$ smallest profit parameters only:*

$$h_j\left(\pi(m)_{m=j}^N\right) \equiv L_{j+k}(T_j^*) - F_{j+k}(T_j^*).$$

The j-th and the $(j+1)$-th investments are clustered if and only if $h_j\left(\pi(m)_{m=j}^N\right) \geq 0$.

To gain a more transparent insight on how the presence or absence of a cluster is related to the profit flows, we examine the comparative statics of the function $h_j(\cdot)$ with respect to its arguments.

Corollary 2 *The function $h_j\left(\pi(m)_{m=j}^N\right)$ is weakly monotone in each of its arguments. In particular, it is strictly decreasing in $\pi(j)$, and strictly increasing in $\pi(j+k)$.*

The intuition for a cluster can be summarized as follows. The more intense the follower preemption race in the subgame that starts after the j-th investment, the earlier the upper bound from the FPC, t_{j+1}, becomes. In turn, the earlier this upper bound is, the weaker the LPC becomes: if high profits $\pi(j)$ are achieved for a shorter interval of time, the role of leader, i.e. of j-th investor, is less attractive.

Therefore, any change of the parameters $\pi(m)_{m=j}^N$ that makes the follower preemption race more intense, makes it more likely that the FPC alone is binding, and a cluster occurs. I.e., it increases $h_j\left(\pi(m)_{m=j}^N\right)$, making it more likely that it is positive. Similarly, any change that makes the role of leader, i.e. of j-th investor, more attractive, makes it more likely that the LPC alone is binding, and that investments are diffused. In other words, it decreases $h_j\left(\pi(m)_{m=j}^N\right)$.

Consider a change in the flow-profit $\pi(j+k)$ obtained by the first k firms to invest in the follower preemption race that starts after the j-th investment. The larger this profit flow, the more intense the race to obtain it. Therefore, we expect that the higher $\pi(j+k)$, the more likely a cluster is. This is confirmed by Corollary 2 which states that $h_j(\cdot)$ is strictly increasing in $\pi(j+k)$.

Similarly, consider a decrease in $\pi(j)$. It does not affect the follower preemption race that determines the FPC, but it makes the role of leader, i.e. of j-th investor, less attractive. Hence, we expect that the smaller $\pi(j)$, the more likely it is that the LPC is not binding, and the FPC is. I.e., that there is a cluster. This intuition is confirmed by Corollary 2: $h_j(\cdot)$ is strictly decreasing in $\pi(j)$.

We characterized the set of parameter vectors for which a cluster $t_j = t_{j+1}$ occurs in Proposition 2. We conclude this section by showing that this set is nonempty. Proposition 3 below states that starting from any parameter vector $\boldsymbol{\pi}$ for which $t_j < t_{j+1}$, one can modify $\pi(j)$ or $\pi(j+1)$ individually in such a way that j-th and the $(j+1)$-th investments are clustered.

Proposition 3 *Suppose that for a given flow profit structure* $\boldsymbol{\pi}$ *the SPNE investment times* t_j *and* t_{j+1} *are diffused. Then:*
(i) there exists a sufficiently large $\tilde{\pi}(j+1) \in (\pi(j+1), \pi(j))$ *such that in the new SPNE outcome there is a cluster:* $t_j = t_{j+1}$.
(ii) if, in addition, t_{j+1} *and* t_{j+2} *are diffused, there also exists a sufficiently small* $\tilde{\pi}(j) \in (\pi(j+1), \pi(j))$ *such that in the new SPNE outcome there is a cluster:* $t_j = t_{j+1}$.

3.5 Discussion

Size of clusters

Section 3.4 focused on the question of whether two specific investments, the j-th and $(j+1)$-th, are clustered. The previous analysis can be extended to find a necessary and sufficient condition on the model primitives for a cluster of two or more investments.

Suppose $t_{j+1} < t_{j+2}$. We ask the question whether t_{j+1} coincides with t_j and, more generally, with one or more previous investments. Proposition 1 implies that there is cluster in t_{j+1} if and only if $t_{j+1} \leq T^*_{j-v}$ for some $v \geq 0$. There is a cluster of the j-th and $(j+1)$-th investments if and only if the upper bound t_{j+1} from the FPC is weakly earlier than the stand-alone investment time T^*_j. If the follower preemption race advances t_{j+1} even ahead of some previous stand-alone investment time T^*_{j-v}, then the cluster will include the $(j-v)$-th investment as well: $t_{j-v} = ... = t_j = t_{j+1}$.

It follows from the analysis in Section 3.4 that a condition equivalent to $t_{j+1} \leq T^*_{j-v}$ is that the preemption incentive in the follower preemption race, evaluated at T^*_{j-v}, is positive:

$$L_{j+1}(T^*_{j-v}) - F_{j+1}(T^*_{j-v}) = \frac{\pi(j+1)}{r}\left[e^{-rT^*_{j-v}} - e^{-rt_{j+2}}\right] - \left[c(T^*_{j-v}) - c(t_{j+2})\right]. \quad (15)$$

For $v = 0$, this expression corresponds to expression (14). In general, the size of the cluster is given by $\bar{v} + 2$, where $\bar{v}$ is the largest $v \geq 0$ for which (15) is non-negative.

The mechanism relating profit flows to the presence of “large clusters” is similar to above. The preemption incentive in (15) is a function of $\pi(j - v)$ and of all the profit flows smaller or equal than $\pi(j + 1)$.

The larger $\pi(j + 1)$, the earlier t_{j+1}, and the more likely that it advances ahead of earlier stand-alone investment times. Conversely, a decrease in a profit flow $\pi(j - v)$ delays the corresponding stand-alone investment time T^*_{j-v} so that it becomes more likely that $t_{j+1} \leq T^*_{j-v}$ and t_{j-v} is part of the cluster.

Clusters including the last investment

An implication of Proposition 1 is that the last two investments are diffused. Suppose we relax Assumption 3(iii) so that only $M < N$ can profitably invest. After the $(M - 1)$-th investment, there is a race among the remaining active players to secure the last profitable investment possibility. By rent equalization, the last investor (as well as all the predecessors) must earn the same equilibrium payoff as the $(N - M)$ players who do not invest, which is zero. The last investment is thus brought forward to a time t_M earlier than the stand-alone time T^*_M. If $t_M < T^*_{M-1}$, then there is cluster at $t_{M-1} = t_M$. In the baseline model instead, the last investment time t_N is T^*_N, therefore it is always strictly later than T^*_{N-1}. Thus allowing for endogenous entry, can generate a cluster at the end of the investment sequence.

4 Conclusions

We extended the preemption game introduced by Fudenberg and Tirole (1985) to the case of N players, assuming players' payoff before investing are constant (and normalized to zero). We find that in this environment the rent equalization result of the two-player game is preserved, but clusters of simultaneous investments are possible, even in the absence of coordination failures, positive externalities, and informational spillovers, and given the fact that investment by a player reduces the flow profit achievable by the remaining active players. If the preemption race among the last $N - j$ firms is so intense that the $(j+1)$-th investment takes place when the investment cost still decreases at a rate higher than flow-profits $\pi(j)$, the j-th investment will not be brought forward any further and there will be a cluster of the j-th and $(j+1)$-th investments.

We characterize a necessary and sufficient condition on the model primitives for investments to be clustered. We also prove that the set of parameter values for which any two investments are clustered is nonempty. Our results imply that the observation of clustering of market entry or adoption times need not reflect informational spillovers or positive externalities.

Appendix

Claim 1 *The function $f_j(t)$ is strictly quasi-concave in t for any $j \in \{1, 2..., N\}$ and admits a unique global maximum in T_j^*, defined as the solution to*

$$f_j'(t) = 0 \Longleftrightarrow -\pi(j) e^{-rt} - c'(t) = 0.$$

Moreover, $f_j\left(T_j^*\right) > 0$ and $T_j^* < T_{j'}^*$ for $j < j'$.

Proof of Claim 1. First, we prove that the function is strictly quasiconcave, by showing that in every critical point of the function the second derivative is strictly negative. The first derivative $f_j'(t)$ is equal to $\left[-\pi(j) e^{-rt} - c'(t)\right]$ and the second derivative $f_j''(t)$ is equal to$\left[r\pi(j) e^{-rt} - c''(t)\right]$. Using $f_j'(t) = 0$ we can rewrite f_j'' evaluated at any critical point as

$$f_j''(t) = -c'(t) r - c''(t). \tag{16}$$

By Assumption 2:

$$\begin{aligned} e^{rt}\left[c'(t) + rc(t)\right] &< 0 \\ e^{rt}\left[2c'(t) r + c(t) r^2 + c''(t)\right] &> 0 \end{aligned}$$

which together imply that expression (16) is negative.

Next, we prove that the first order condition $f_j'(t) = 0$ characterizes the unique global maximum of the function. Assumption 3(ii) guarantees that $f_j'(0) > 0$ and assumption 3(iv) guarantees that $f_j'(t) < 0$ for t sufficiently large.

Next, assumptions 3(iii) and 2(i) imply that $f_N(t)$ is strictly positive for any $t \geq T_N^*$. Thus, Assumption 1(ii) implies that $f_j\left(T_j^*\right) > 0$ for all j.

Finally, note that by the implicit function theorem

$$\frac{\partial T_j^*}{\partial \pi(j)} = -\frac{-e^{-rT_j^*}}{f_j''\left(T_j^*\right)} < 0,$$

where the inequality holds because T_j^* is a maximum, hence the denominator is negative. Therefore, Assumption 1(ii) implies that $T_1^* < T_2^* < ... < T_N^*$.■

Proof of Lemma 1. Assumptions 3(i) and 1(ii) guarantee that there is no investment at time zero: the cost of investing immediately is higher than the maximum amount of profits a firm can obtain in this game. This proves part (i).

For part (ii), we first show that in equilibrium, at any decision node with one active firm and calendar time t, the firm plays WAIT if $t < T_N^*$ and INVEST otherwise. Then, we show that in any decision node with $t \geq T_N^*$ any number of active firms play INVEST.

If at time t there is only one active firm, its payoff from investing is $f_N(t)$ as defined in equation (2). From Claim 1, $f_N(t)$ has a strict global maximum in T_N^* and its maximum value is strictly positive. Therefore, a single active firm will optimally play WAIT if $t < T_N^*$ and INVEST otherwise.

Next, consider decision nodes with $t \geq T_N^*$ and two active firms. In equilibrium, at least one of them must play INVEST. If this is the case, the game enters a subgame with one active firm, who invests immediately, and both firms receive payoff $f_N(t)$. If both play INVEST, and one of them deviates and plays WAIT, the payoff is unchanged. If only one of them plays INVEST and the other plays WAIT, the latter has no incentive to deviate because its payoff would be unchanged, and the former has no incentive to deviate because it would then get either zero or $f_N(\tau)$ for some $\tau > t$ and $f_N(\cdot)$ is strictly decreasing in the interval considered. By a similar argument, in equilibrium it cannot be the case that both of them play WAIT because each of them would be better off by deviating.

Repeating the same argument for $\ell = 3, ..., N$, it follows that in any SPNE, at any decision node with $t \geq T_N^*$ and any number ℓ of active firms, at least one of them plays INVEST, and the claim follows immediately. ■

Proof of Proposition 1. Through a series of Lemmata we show that the game admits a unique SPNE outcome, and characterize it. The proof is articulated in the following steps:

• Denote by t_j the SPNE investment time of the j-th investor, for $j \in \{1, .., N\}$. In Definition 2, we introduce three functions, $L_j(t)$, $F_j(t)$, and their difference $D_j(t)$.

• In Lemma 2 and Lemma 3 we characterize their properties. Over a well-defined subset of their domain, $L_j(t)$ and $F_j(t)$ can be interpreted as the payoff of the j-th investor and the $(j+1)$-th investor, respectively, if the j-th investment takes place at t and the following investments take place at t_{j+1},..., t_N respectively. In the definition, the existence and uniqueness of the SPNE investment times is assumed. In the development of the proof, they will be proved. The existence and uniqueness of $t_N = T_N^*$ was proved in Lemma 1.

• In Lemma 4 we establish that in any subgame with one active firm, it plays WAIT before T_N^* and INVEST from T_N^* on.

• In Lemma 5, we prove that there exists a time $T_{N-1} < T_N^*$ in which $L_{N-1}(t) = F_{N-1}(t)$ and in Lemma 6 we prove that this is the unique $(N-1)$-th equilibrium investment time. Therefore, the equilibrium payoff of the last two investors is the same.

• Finally, in Lemmata 7,8 and 9 we identify the algorithm for the construction of the equilibrium investment times t_j for $j \in \{1, ..., N-2\}$ and prove that rent equalization holds in equilibrium for all players. The argument is based on the induction principle. Lemma 7 proves that there exists an algorithm to identify the unique t_{N-2}, given t_{N-1} and t_N, and that the equilibrium payoff of the last three investors is the same. Lemma 8 shows that if an analogous algorithm can be used to identify a unique value of t_{N-l}, given t_{N-l+1},...,t_N, and rent equalization holds for the last l players, then the same algorithm identifies a unique value for t_{N-l-1}, given t_{N-l},...,t_N, and rent equalization holds for the last $l+1$ players. Lemma 9 applies the induction principle to prove that the algorithm can be used to construct the SPNE investment times t_1,..., t_{N-2} and rent equalization holds for all players. This concludes the proof of the Proposition.

Definition 2 *For each $j \in \{1, .., N-1\}$, we define the following three functions over the interval $[0, T_N^*]$:*

$$\begin{aligned} L_j(t) &\equiv \pi(j) \int_t^{t_{j+1}} e^{-rs} ds + \sum_{m=j+1}^{N} \pi(m) \int_{t_m}^{t_{m+1}} e^{-rs} ds - c(t) \\ F_j(t) &\equiv \sum_{m=j+1}^{N} \pi(m) \int_{t_m}^{t_{m+1}} e^{-rs} ds - c(t_{j+1}) \\ D_j(t) &\equiv L_j(t) - F_j(t) = \pi(j) \int_t^{t_{j+1}} e^{-rs} ds - c(t) + c(t_{j+1}) \end{aligned}$$

where $t_{N+1} \equiv +\infty$.

Notice that $F_j(t)$ is constant with respect to t.

Lemma 2

(i) The function $D_j(t)$ attains a unique global maximum in T_j^*, for $j \in \{1, .., N-1\}$

(ii) $D_j(T_j^*) \geq 0$ for $j \in \{1, .., N-1\}$.

Proof.

Part (i)**.** Notice that

$$D_j(t) = \pi(j) \int_t^{t_{j+1}} e^{-rs} ds - c(t) + c(t_{j+1}) \tag{17}$$

and $f_j(t)$ as defined in equation (2) differ by a finite constant. By Claim 1, $f_j(t)$ attains a unique global maximum in T_j^*, hence the same is true for $D_j(t)$.

Part (ii)**.** Since $D_j(t_{j+1}) = 0$ and T_j^* is the unique global maximizer, it holds that $D_j(T_j^*) \geq 0$. ■

Lemma 3

(i) If $T_j^* \leq t_{j+1}$*, then* $\exists T_j \in (0, T_j^*]$*, such that* $D_j(T_j) = 0$*,*

(ii) If $T_j^* > t_{j+1}$*, then* $D_j(t) < 0$ *and* $D_j'(t) > 0$ $\forall t < t_{j+1}$*.*

Proof. Since $D_j(t)$ and $f_j(t)$ differ by a finite constant, Claim 1 implies that $D_j(t)$ is strictly quasi-concave. Also, $D_j(0) < 0$, since

$$\begin{aligned} L_j(0) &= \pi(j) \int_0^{t_{j+1}} e^{-rs} ds + \sum_{m=j+1}^{N} \pi(m) \int_{t_m}^{t_{m+1}} e^{-rs} ds - c(0) < \frac{\pi(1)}{r} - c(0) < 0 \\ &\leq V^{j+1}(t_1, ...t_N) = \sum_{m=j+1}^{N} \pi(m) \int_{t_m}^{t_{m+1}} e^{-rs} ds - c(t_{j+1}) = F_j(0) \end{aligned}$$

Here the second inequality holds by assumption 3(i) and the third because no firm gets a negative payoff in equilibrium as it could always delay investment indefinitely ensuring a payoff of zero. Moreover, $D_j(t_{j+1}) = 0$. Therefore, two cases are possible:

(i): $T_j^* \leq t_{j+1}$, in which case $\exists T_j \in (0, T_j^*]$, such that $D_j(T_j) = 0$, and $D_j(t) > 0$ in the interval $t \in (T_j, t_{j+1})$,

(ii) $T_j^* > t_{j+1}$, in which case $D_j(t) < 0$ and $D_j'(t) > 0$ $\forall t < t_{j+1}$. ■

In the next Lemma, we analyze decision nodes with one active firm.

Lemma 4 *In equilibrium, if at time* t *there is one active firm, it plays WAIT if* $t < T_N^*$ *and INVEST if* $t \geq T_N^*$*.*

Proof. The result follows immediately from the proof of Lemma 1. ■

In the next Lemma, we show that for the case $j = N-1$, case (i) of Lemma 3 applies.

Lemma 5 $T_{N-1}^* < t_N = T_N^*$ *and* $\exists T_{N-1} < T_{N-1}^* < T_N^*$ *such that* $D_{N-1}(T_{N-1}) = 0$*.*

Proof. $T^*_{N-1} < T^*_N$ by Claim 1 and $t_N = T^*_N$ by Lemma 1. The rest of the statement follows from Lemma 3. ■

In the next Lemma we show that the $(N-1)$-th equilibrium investment time is T_{N-1}.

Lemma 6 *In equilibrium, it holds that:*
(i) At any time $t \in [T_{N-1}, T^*_N)$, if there are $n > 1$ active firms, at least one investment takes place at t.
(ii) At any time $t \in [0, T_{N-1})$, if there are $n = 2$ active firms, each of them plays WAIT.
(iii) $t_{N-1} = T_{N-1}$ and the equilibrium payoff of the last two investors is the same.

Proof.
Part (i). In the interval $t \in [T_{N-1}, T^*_N)$, $D_{N-1}(t) \geq 0$. It is an equilibrium for a subgame starting in this interval for all active firms to play INVEST: all n active firms try to invest immediately at t, until only one is left, which will then wait to invest until time T^*_N (see Lemma 4). The associated payoff is a lottery between $L_{N-1}(t)$ and $F_{N-1}(t)$. Suppose one firm deviates and plays WAIT at t. This deviation is not profitable, since it increases the probability of receiving payoff $F_{N-1}(t)$ and reduces the probability of receiving payoff $L_{N-1}(t)$. It is also an equilibrium for all firms to play WAIT at t and INVEST at any $\tau > t$, as the continuous time outcome of this profile is the same as the outcome of the previous profile.

In the interval $t \in (T_{N-1}, T^*_N)$, no other strategy profile constitutes an equilibrium. First, suppose that at time t only some of the active firms play INVEST. Then, any of the remaining firms receives payoff $F_{N-1}(t)$ with probability one and can profitably deviate by playing INVEST thus receiving a lottery between $F_{N-1}(t)$ and $L_{N-1}(t)$. Next, suppose that all firms play WAIT in an interval starting at time t, and first investment takes place at some time τ strictly larger than t. By Lemma 1 τ must be smaller or equal than T^*_N. If all active firms play INVEST at τ, then each of them receives a lottery between $F_{N-1}(\tau)$ and $L_{N-1}(\tau)$ and could profitably deviate by playing INVEST at $\tau - \varepsilon$ and receiving $L_{N-1}(\tau - \varepsilon)$, since by continuity $\exists \varepsilon > 0$ small enough that this is profitable. If instead some active firms play WAIT at τ, which by Lemma 1 can happen only if $\tau < T^*_N$, then each of them receives $F_{N-1}(\tau)$. It could deviate by playing INVEST at $\tau - \varepsilon$ and receive $L_{N-1}(\tau - \varepsilon)$. By continuity $\exists \varepsilon > 0$ small enough that this is profitable.

For subgames starting at $t = T_{N-1}$, it is also an equilibrium that if $n > 1$ firms are active, any positive number $\nu < n$ of firms play INVEST at t. The outcome is that all except one invest at T_{N-1}, the last one invests at T_N^*, and all receive payoff $L_{N-1}(T_{N-1}) = F_{N-1}(T_{N-1})$, hence no deviation is profitable.

Part (ii). Notice that in the interval $t \in [0, T_{N-1})$ it holds that $L_{N-1}(t) < F_{N-1}(t)$ and that $L_{N-1}(t)$ is increasing. Given part (i) of this Lemma, and Lemma 4, if firms follow the actions prescribed in part (ii) the expected payoff for each of them is

$$\frac{1}{2}\left[L_{N-1}(T_{N-1}) + F_{N-1}(T_{N-1})\right] = L_{N-1}(T_{N-1}) = F_{N-1}(T_{N-1})$$

where the equality comes from the definition of T_{N-1}. The deviation payoff from investing at some τ before T_{N-1} is $L_{N-1}(\tau) < L_{N-1}(T_{N-1})$, hence this is an equilibrium.

Next, we show that there is no other action profile compatible with equilibrium. Suppose that $\nu \leq 2$ firms play INVEST at τ. The equilibrium payoff for any of these early investors is

$$\frac{1}{\nu}\left[L_{N-1}(\tau) + (\nu - 1)F_{N-1}(\tau)\right]$$

Each of them could profitably deviate by playing WAIT at τ, since $F_{N-1}(\tau) > L_{N-1}(\tau)$. This concludes the proof of part (ii).

Part (iii). The conclusion that $t_{N-1} = T_{N-1}$ follows directly from parts (i) and (ii) and from Lemma 4. By construction of T_{N-1}, this implies rent equalization for the last two investors. ■

We now identify the algorithm for the construction of the equilibrium investment times t_j for $j \in \{1, ..., N-1\}$. The argument is based on the induction principle. Lemma 7 contains a statement for $j = N-2$. Lemma 8 shows that if the same statement holds for $j = N - l$, then it holds for $j = N - l - 1$. Lemma 9 concludes that, by the induction principle, the statement holds for a general j.

Lemma 7 *Given $t_N = T_N^*$ and $t_{N-1} = T_{N-1}$, t_{N-2} can be constructed as follows.*

Part (a) *Suppose $T_{N-2}^* < T_{N-1}$. In equilibrium it holds that:*

(i) At any time $t \in [T_{N-2}, T_{N-1})$, if there are $n > 2$ active firms, at least one investment takes place at t.

(ii)At any time $t \in [0, T_{N-2})$, if there are 3 active firms, each of them plays WAIT

(iii) $t_{N-2} = T_{N-2}$ and the payoff of the last 3 investors is equalized.

Part (b) *Suppose $T_{N-2}^* \geq T_{N-1}$. In equilibrium it holds that:*

(i) at any time $t \in [0, T_{N-1})$, if there are $n = 3$ active firms, each of them plays WAIT

(ii) $t_{N-2} = T_{N-1}$ and the payoff of the last 3 investors is equalized.

Proof.

Part (a): By Lemma 3 (i), $\exists T_{N-2} \in (0 < T^*_{N-2}]$, such that $D_{N-2}(T_{N-2}) = 0$. The proofs of parts (i) and (ii) follow from arguments similar to the proofs of parts (i) and (ii) of Lemma 6, respectively.

The conclusion that $t_{N-2} = T_{N-2}$ follows directly from parts (i) and (ii) and from Lemmata 6 and 4. By construction of T_{N-2}, this implies rent equalization for the last three firms.

Part (b): By Lemma 3 (ii), $L_{N-2}(t) < F_{N-2}(t)$ and $L'_{N-2}(t) > 0$ $\forall t < T_{N-1}$. The proof of part (i) follows from arguments analogous to the proof of part (ii) of Lemma 6. The conclusion that $t_{N-2} = t_{N-1} = T_{N-1}$ follows directly from part (i) and from Lemmata 4 and 6. By construction of T_{N-1}, this implies rent equalization for the last three firms. ■

Lemma 8 *If the following statement holds for $j = N - l$, with $l \geq 2$, then it holds for $j = N - l - 1$.*

Given the last $N - j$ equilibrium investment times $(t_{j+1}, ..., t_N)$, and given rent equalization for the last $(N - j)$ investors, the j-th equilibrium investment time t_j can be constructed as follows:

Part (a) *Suppose $T^*_j < t_{j+1}$. In equilibrium it holds that:*

(i) At any time $t \in [T_j, t_{j+1})$, if there are $n > N - j$ active firms, at least one investment takes place at t.

(ii) At any time $t \in [0, T_j)$, if there are $n = N - j + 1$ active firms, each of them plays WAIT

(iii) $t_j = T_j$ and the payoff of the last $N - j + 1$ investors is equalized.

Part (b) *Suppose $T^*_j \geq t_{j+1}$. In equilibrium it holds that:*

(i)At any time $t \in [0, t_{j+1}]$, if there are $n = N - j + 1$ active firms, each of them plays WAIT:

(ii) $t_j = t_{j+1}$ and the payoff of the last $N - j + 1$ investors is equalized.

Proof. Assume that the statement holds for $j = N - l$. This implies that either $t_{N-l} = T_{N-l}$ or $t_{N-l} = t_{N-l+1}$, and that in both cases payoffs of the last $l + 1$ investors are equalized.

Now we need to prove that the statement holds for $j = N - l - 1$.

Part (a): By Lemma 3 (i), $\exists T_{N-l-1} \in (0 < T^*_{N-l-1}]$, such that $D_{N-l-1}(T_{N-l-1}) = 0$. The proofs of parts (i) and (ii) follow from arguments similar to the proofs of parts (i) and (ii) of Lemma 6, respectively. For part (iii), the conclusion that $t_{N-l-1} = T_{N-l-1}$ follows directly from parts (i) and (ii) and from the assumptions. By construction of T_{N-l-1}, this implies rent equalization for the last $l + 2$ firms.

Part (b): By Lemma 3 (ii), $D_{N-l-1}(t) < 0$ and $D'_{N-l-1}(t) > 0$ $\forall t < t_{N-l}$. The proof of part (i) follows from arguments analogous to the proof of part (ii) of Lemma 6. The conclusion that $t_{N-l-1} = T_{N-l}$ follows directly from part (i) and from the assumptions. By construction of T_{N-l}, this implies rent equalization for the last $l + 2$ firms. ■

Lemma 9 *The statement in Lemma 8 holds for any* $j \leq N - 2$.

Proof. The result follows from Lemmata 7 and 8 by the induction principle. ■

Proof of Corollary 1. From Proposition 1, $t_j = t_{j+1}$ if and only if $T^*_j \geq t_{j+1}$. We prove that $T^*_j \geq t_{j+1}$ if and only if $L_{j+k}(T^*_j) - F_{j+k}(T^*_j) \geq 0$. First, notice that by Lemmata 1 and 2 $L_{j+k}(t) - F_{j+k}(t) = D_{j+k}(t)$ is strictly quasiconcave and attains a unique global maximum in T^*_{j+k}, which is larger than T^*_j by Claim 1. Moreover, by construction, $L_{j+k}(t) - F_{j+k}(t)$ is null in $t_{j+k} = t_{j+1}$. It follows that either $t_{j+1} \leq T^*_j < T^*_{j+k}$, in which case $L_{j+k}(T^*_j) - F_{j+k}(T^*_j) \geq 0$, or $T^*_j < t_{j+1} < T^*_{j+k}$, in which case $L_{j+k}(T^*_j) - F_{j+k}(T^*_j) < 0$.

Proof of Proposition 2. Given equation (14), the first statement follows immediately from the definition of T^*_j and from the observation that the recursive algorithm identified in Proposition 1 is such that t_{j+k+1} is not a function of the profit flows $(\pi(m))^{j+k}_{m=1}$. The second statement follows from Corollary 1.

Proof of Corollary 2. First, consider monotonicity with respect to $\pi(j)$. Differentiating yields:

$$\frac{\partial h_j\left(\pi(m)^N_{m=j}\right)}{\partial \pi(j)} = \frac{\partial h_j\left(\pi(m)^N_{m=j}\right)}{\partial T^*_j}\frac{\partial T^*_j}{\partial \pi(j)} = \left[-\pi(j+k)e^{-rT^*_j} - c'(T^*_j)\right]\frac{\partial T^*_j}{\partial \pi(j)} < 0.$$

To see this, observe that $[-\pi(j+k)e^{-rT^*_j} - c'(T^*_j)] > 0$ because $T^*_j < T^*_{j+k}$, and that $\frac{\partial T^*_j}{\partial \pi(j)} < 0$ by the proof of Claim 1.

Second, consider monotonicity with respect to $\pi(\ell)$ for $\ell = j + 1, ., j + k - 1$. The algorithm in Proposition 1 implies that t_{j+k+1} is a function at most of all the profit flows

$(\pi(m))_{m=j+k+1}^{N}$. It follows that the derivative of $h_j(\cdot)$ with respect to $\pi(\ell)$ is null.

Third, consider monotonicity with respect to $\pi(j+k)$. Differentiating yields:

$$\frac{\partial h_j\left(\pi(m)_{m=j}^N\right)}{\partial \pi(j+k)} = \frac{e^{-rT_j^*} - e^{-rt_{j+k+1}}}{r}.$$

Notice that $T_{j+k}^* < t_{j+k+1}$ by definition of k, because if $T_{j+k}^* \geq t_{j+k+1}$, the $(j+k)$-th and the $(j+k+1)$-th investment would be clustered. Because $T_j^* < T_{j+k}^*$ by Claim 1, it follows that $T_j^* < t_{j+k+1}$. Hence, $e^{-rT_j^*} - e^{-rt_{j+k+1}} > 0$ and this derivative is positive.

Finally, consider monotonicity with respect to $\pi(\ell)$ for $\ell = j+k+1, ., N$. Differentiating yields:

$$\begin{aligned}
\frac{\partial h_j\left(\pi(m)_{m=j}^N\right)}{\partial \pi(\ell)} &= \frac{\partial h_j\left(\pi(m)_{m=j}^N\right)}{\partial t_{j+k+1}} \frac{\partial t_{j+k+1}}{\partial \pi(\ell)} \\
&= \left[\pi(j+k)e^{-rt_{j+k+1}} + c'(t_{j+k+1})\right] \frac{\partial t_{j+k+1}}{\partial \pi(\ell)}.
\end{aligned}$$

Since $T_{j+k}^* < t_{j+k+1}$, the term in square brackets is positive by definition of T_{j+k}^*. Hence, the sign of the derivative coincides with the sign of the second term.

Suppose that in equilibrium all the investments from the $(j+k+1)$-th to the last are diffused. Then, the second term is nonzero for every $\ell = j+k+1, ., N$. We now prove that for $\ell = j+k+1, ..., N$, $\frac{\partial t_\ell}{\partial \pi(\ell)} < 0$. By construction, t_ℓ solves:

$$\frac{\pi(\ell)}{r}\left[e^{-rt_\ell} - e^{-rt_{\ell+1}}\right] - \left[c(t_\ell) - c(t_{\ell+1})\right] = 0.$$

Differentiating implicitly yields: $\frac{\partial t_\ell}{\partial \pi(\ell)} = -\frac{\left[e^{-rt_\ell} - e^{-rt_{\ell+1}}\right]/r}{-\pi(\ell)e^{-rt_\ell} - c'(t_\ell)}$. The numerator is positive, because $t_\ell < t_{\ell+1}$. The denominator is positive as well, because in equilibrium $t_\ell < T_\ell^*$. Therefore, t_ℓ is decreasing in $\pi(\ell)$. In particular, this proves that $\frac{\partial t_{j+k+1}}{\partial \pi(j+k+1)} < 0$. Next, consider $\ell > j+k+1$. By definition t_{j+k+1} solves:

$$\frac{\pi(j+k+1)}{r}\left[e^{-rt_{j+k+1}} - e^{-rt_{j+k+2}}\right] - \left[c(t_{j+k+1}) - c(t_{j+k+2})\right] = 0.$$

Applying the chain rule and differentiating implicitly yields:

$$\frac{\partial t_{j+k+1}}{\partial \pi(\ell)} = \left(\prod_{\tilde{\ell}=j+k+1}^{\ell-1} \frac{\partial t_{\tilde{\ell}}}{\partial t_{\tilde{\ell}+1}}\right) \cdot \frac{\partial t_\ell}{\partial \pi(\ell)}.$$

We proved above that the second term is negative. For the first term, differentiating implicitly yields: $\frac{\partial t_{\widetilde{\ell}}}{\partial t_{\widetilde{\ell}+1}} = -\frac{\pi(\widetilde{\ell})e^{-rt_{\widetilde{\ell}+1}}+c'(t_{\widetilde{\ell}+1})}{-\pi(\widetilde{\ell})e^{-rt_{\widetilde{\ell}}}-c'(t_{\widetilde{\ell}})}$. The denominator is positive because $t_{\widetilde{\ell}} < T^*_{\widetilde{\ell}}$. The numerator is also positive because $t_{\widetilde{\ell}+1} > T^*_{\widetilde{\ell}}$. Therefore, $\prod_{\widetilde{\ell}=j+k+1}^{\ell-1} \frac{\partial t_{\widetilde{\ell}}}{\partial t_{\widetilde{\ell}+1}}$ is strictly positive for $\ell - k - j - 1$ even and strictly negative for $\ell - k - j - 1$ odd. This implies that $\frac{\partial t_{j+k+1}}{\partial \pi(\ell)}$, and hence $\frac{\partial h_j(\pi(m)_{m=j}^N)}{\partial \pi(\ell)}$ is strictly negative for $\ell - k - j - 1$ even and strictly positive for $\ell - k - j - 1$ odd.

If instead for a given flow profit structure $\boldsymbol{\pi}$ some of the investments from the $(j+k+1)$-th to the last are clustered, then t_{j+k+1} is a function of a subset of the profit flows $(\pi(m))_{m=j+k+1}^N$. Arguments analogous to those presented above show that also in this case t_{j+k+1} is monotone in each profit flow in this subset, and so is $h_j\left(\pi(m)_{m=j}^N\right)$.

Proof of Proposition 3. First, notice that investment times $[t_{j+k+1}, ..., t_N]$ are constant in $\pi(j)$ and $\pi(j+k)$.

Part (i). For the profit structure $\boldsymbol{\pi}$, let $k \geq 1$ be the number of investments occurring jointly in equilibrium at the beginning of the subgame played among the last $N-j$ players, so that $t_{j+1} = ... = t_{j+k}$. Suppose $k = 1$. The proof of Corollary 2 shows that t_{j+1} decreases in $\pi(j+1)$. Moreover, $\lim_{\pi(j+1)\to\pi(j)} T^*_{j+1} = T^*_j$. Since $t_{j+1} < T^*_{j+1}$, there exists a $\widetilde{\pi}(j+1)$ large enough that $t_{j+1} < T^*_j$. Now suppose that $k > 1$. Since t_j and t_{j+1} are diffused, $T^*_j < t_{j+1}$. Since T^*_{j+1} is strictly decreasing in $\pi(j+1)$ and $\lim_{\pi(j+1)\to\pi(j)} T^*_{j+1} = T^*_j < t_{j+2}$, there exists a $\widehat{\pi}(j+1)$ sufficiently close to $\pi(j)$ such that $t_{j+1} < t_{j+2}$. Then, the proof for the case $k = 1$ implies that there exists a $\widetilde{\pi}(j+1)$ large enough such that $t_{j+1} < T^*_j$ and t_j and t_{j+1} are clustered.

Part (ii). Observe that T^*_j is decreasing in $\pi(j)$ and that $\lim_{\pi(j)\to\pi(j+1)} T^*_j = T^*_{j+1}$. Since $t_{j+1} < t_{j+2}$, it must hold that $T^*_{j+1} > t_{j+1}$. Continuity implies that there exists a $\widetilde{\pi}(j)$ small enough that $T^*_j > t_{j+1}$ and t_j and t_{j+1} are clustered. ■

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