Abstract: This paper applies a continuous-time model to study the equilibrium of an economy consisting one normal agent and one portfolio benchmarker who has to maintain his intermediate wealth above the maximum of a stochastic benchmark and a constant floor. After characterizing the optimization problems, I use the martingale approach to derive the equilibrium market dynamics in closed-form to compare with the normal economy. In the benchmarking economy before the constraint date, asset price, volatility and risk premium are higher than in the normal economy for very risky benchmarks. When the benchmark is relatively safe, asset price is higher but volatility, risk premium and the optimal fraction of wealth invested in the risky asset are decreased. In both cases, the effects are state-dependent. The derivation of equilibrium also reveals the benchmarker’s demand for index options.
This paper studies the effect of a portfolio benchmarking constraint on the equilibrium market dynamics. The economy has a normal agent with log utility over continuous consumption and a constrained agent (also with log utility) whose intermediate wealth must meet the maximum of a stochastic constraint and a constant floor. Using martingale approach, I solve this continuous-time consumption-based general equilibrium model explicitly to get the equilibrium assets prices, risk premium, volatility, optimal strategy and compare them with those in a normal economy. To my knowledge, this is the first paper to investigate the equilibrium effects of this benchmarking constraint.

The constraint studied here is a requirement that an agent’s wealth is no less than the maximum of a stochastic benchmark index and a constant floor. The problem of beating a floor has been studied by the portfolio insurance literature, e.g. the equilibrium analysis of portfolio insurance by Basak (1995) and Grossman and Zhou (1996). However, the portfolio insurance constraint only ensures the agent doesn’t lose more than a certain level without asking for a high return when the state is good. The inclusion of a stochastic benchmark index in the portfolio benchmarking constraint means the agent’s performance has to beat the index when market does well and beat the floor when market is bad. Therefore, imposing such a constraint on agent ensures good performance yet limits loss. Since there are only two assets in the economy, a risk-free asset and a risky stock, and markets are complete, the stochastic benchmark is a replication portfolio using the bond and stock. Then, the riskiness of the benchmark is measured by the positions in the risk-free asset. Since the benchmark index is achievable through passive management, it is considered as the minimum return that is required for active portfolio management.

After describing the economy, which apart from the existence of the constraint is the standard set-up following Lucas (1978), I first characterize the portfolio choice problems for the
normal agent and the constrained agent. The approach is the martingale representation approach as in Cox and Huang (1989) and Basak (1995). Then I get explicit solutions for the equilibrium market dynamics using the martingale approach for option pricing and Ito’s lemma. Similar procedure has been used by Basak (1995) but the addition of the stochastic benchmark complicates the situation to a significant degree.

I find that in the portfolio benchmarking economy before the constraint date, the risky asset price is higher than that in the normal economy because the constrained agent consumes less than if he’s not constrained. Since the risk-free asset is in zero net supply, the extra investment from the constrained agent goes into the risky asset market and drives up the risky asset’s price. After the constraint date, there’s no effect. The increase in stock price also reflects the agent’s preference for consumption and dividend after the constraint date. When the benchmark is very risky, which means the replication portfolio has a significant amount of borrowing, then the risk premium, volatility, and the optimal fraction of wealth invested in the risky asset by the constrained agent are increased by the presence of the constraint. However, for a given benchmark index, the constraint’s effect is more likely to be volatility-increasing in good states and volatility-decreasing in bad states. While if the benchmark is relatively safe, i.e. the replication portfolio contains positive position or limited negative position in the risk-free asset, then the risk premium, volatility, and the optimal fraction of wealth invested in the risky asset by the constrained agent are always decreased by the presence of the constraint. In both cases, the effects are state-dependent.

The rationale behind these findings is when there are more (less) demand for the risky asset from the constrained agent, the volatility has to increase (decrease) so as to clear the market. It is also because that the stock price in this economy is equal to the normal economy price plus the present value of two option payoffs that represents the effects
from the benchmark constraint and the floor constraint respectively. For the safe benchmark case, they are both volatility decreasing but for the risky benchmark case the effect from the benchmark constraint is volatility increasing and the floor constraint is still volatility-decreasing. Thus overall effect in the risky benchmark case is complicated and state-dependent. From the modeling aspect, the market price of risk is constant and the SPDs before the constraint date are not directly affected by the existence of the constraint due to the use of consumption as numeraire and the existence of intermediate dividend for consumption, which are not true in some other papers that have contrary conclusions for certain parts of the model, for instance Grossman and Zhou (1996). However, the conclusion of this paper is consistent with that for the portfolio insurance model of Basak (1995) which has the similar set-up and approach with this paper.

The closest literatures are the equilibrium analysis of portfolio insurance by Basak (1995) and Grossman and Zhou (1996). Basak (1995) builds a similar consumption-based general equilibrium model and compares the explicit expressions for equilibrium market dynamics in the portfolio insurance economy with those in the normal economy. The portfolio insurers’ strategies are similar to the synthetic put approach and the presence of the intermediate portfolio insurance constraints decreases the risk premium, volatility and optimal fraction of wealth invested in the risky asset. Although there are multiple normal and constrained agents, the selection of log utility ensures the SPDs are not affected by the constraints since they are derived by market clearing of intermediate consumption. In contrast, Grossman and Zhou (1996) adopts a different set-up in which the portfolio insurance constraint is on the final date and there’s no intermediate consumption so agents only care about consumption at the final date which is financed by a lump-sum of dividend. Therefore, the pricing kernels before the final date are directly affected by the constraint and that makes the overall effect of portfolio insurance to be increasing risk premium and volatility. However,
the use of bond price of numeraire results in different predictions with Basak (1995) and makes it impossible to be solved explicitly. As mentioned above, these two papers only consider the case of portfolio insurance which is benchmarking on a constant floor while Tepla (2001) studies the optimal strategy of an agent who performs against a stochastic benchmark but doesn’t derive the equilibrium.

The rest of the paper is organized as follows. Section 1 presents the model and characterizes the optimization problems of agents. Section 2 solves for the equilibrium and provides the main results on the effect of the portfolio benchmarking constraint. Then Section 3 studies a special case of the main model and Section 4 derives the agent’s demand for options. Finally, Section 5 concludes.

1 The Model

1.1 The Economy

In a finite horizon $[0, T']$ pure-exchange economy that has a single consumption good, let $B$ denote a Brownian Motion on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Let $\{\mathcal{F}_t; t \in [0, T']\}$ be the augmentation by null sets of the filtration generated by $B$ which represents all uncertainties in the economy.

1.1.1 Securities

There are two securities. $S$ is a risky stock in constant net supply of 1 and pays dividend at rate $\delta_t$ in $[0, T']$. The dividend process follows a Geometric Brownian Motion.

$$d\delta_t = \delta_t(\mu dt + \sigma dB_t), \quad t \in [0, T']$$ (1)
\[ dS_t + \delta_t dt = S_t(\mu_t dt + \sigma_t dB_t + qA_t), t \in [0, T'] \] (2)

The stock price is a diffusion process with an \( F_T \)-measurable jump at time \( T \), which is the constraint date before the final date \( T' \).

\( S^0 \) is a risk-less asset (money market account) in zero net supply and also has a jump, which together with the jump in \( S_t \) account for the anticipated discontinuity in the equilibrium.

\[ dS^0_t = S^0_t(r_t dt + q^0 A_t), t \in [0, T'] \] (3)

Here, \( A_t \equiv 1_{t \geq T} \). The \( F_T \)-measurable random variable \( q \) is the jump size parameter. To rule out arbitrage, \( q = q^0 = \ln(S_T/S_{T-}) \) where \( S_{T-} \) denotes the left limit of \( S_T \).

The asset price is continuous semi-martingale because the demand for consumption will increase discontinuously immediately after the constraint date \( T \). Thus the demand for the risky asset will jump down, resulting in a downward jump in asset price. The jump is necessary for the equilibrium conditions to hold, which will be discussed in the section for equilibrium.

1.1.2 State Price Density

To apply the martingale approach, consider the state price density process (SPD),

\[ \pi_t = \frac{1}{S^0_t} \exp\left(-\int_o^t r_s ds - \int_o^t \theta_s dB_s - \frac{1}{2} \int_o^t \theta_s^2 ds - qA_t\right) \] (4)

where \( \theta_t = (\mu_t - r_t)/\sigma_t \) is the market price of risk. The SPD process also contains a jump. Apply Ito’s lemma to \( \pi_t \),

\[ d\pi_t = -\pi_t(r_t dt + \theta_t dB_t + qA_t), t \in [0, T'] \] (5)
and use the SPD process, the relationship between the stock price and future dividends is

\[ S_t = \frac{1}{\pi_t} E\left[ \int_t^{T'} \pi_s \delta_s ds \mid \mathcal{F}_t \right], t \in [0, T'] \] (6)

1.1.3 Agents

There are two agents in the economy, i.e. agent \( n \) and agent \( m \), each endowed with \( x_{n0} \) and \( x_{m0} \) at time zero. Let \( X_{it} \) denote the optimal invested wealth for agent \( i \) at time \( t \), \( i = n, m \).

\[ dX_{it} = (1 - \Phi_{it})X_{it}(\mu_t dt + qA_t) + \Phi_{it}X_{it}(\mu_t dt + \sigma_t dB_t + qA_t) - c_{it} dt, i = n, m, t \in [0, T'] \] (7)

where \( \Phi_{it} \) denotes the optimal fraction of wealth invested in the risky asset and \( c_{it} \) is the consumption process for agent \( i \). Both agents have a time-additive state-independent utility function for consumption. The function \( u(c_{it}) \) is the same for both agents and is continuous with continuous first derivatives, strictly increasing, strictly concave. Hereafter, all optimal quantities are denoted with a caret (\(^\hat{\cdot}\)).

\[ \hat{X}_{it} = \frac{1}{\pi_t} E\left[ \int_t^{T'} \pi_s \hat{c}_{is} ds \mid \mathcal{F}_t \right], t \in [0, T'] \] (8)

Agent \( n \) is the normal agent while agent \( m \) is the portfolio benchmarker who faces a portfolio benchmarking constraint at time \( T \) so that he has to maintain his time \( T \) wealth above the maximum of a stochastic benchmark \( aS_T + b \) and a constant floor \( K \). So the constraint is exogenously set to be \( X_{mT-} \geq \max[ aS_T + b, K ] \). Refine \( aS_0 + b = x_{m0} \) then the value of \( b \) characterize the riskiness of the benchmark and the choice of parameter values has important impact on the equilibrium properties.
1.2 The Optimization Problems

The normal agent solves the following problem:

\[
\max_{c_n} E\left[ \int_0^{T'} u(c_{ns})ds \right] \\
\text{subject to } E\left[ \int_0^{T'} \pi_s c_{ns}ds \right] \leq \pi_0 x_{n0}
\]

Assuming a solution exists, the solution is:

\[
\hat{c}_{nt} = I(\lambda_n \pi_t) \quad t \in [0, T']
\]  

(9)

where \( I(\cdot) \) is the inverse of \( u'(\cdot) \) and \( \lambda_n \) solves:

\[
E\left[ \int_0^{T'} \pi_s I(\lambda_n \pi_s)ds \right] = \pi_0 x_{n0}
\]  

(10)

The constrained agent \( m \) solves:

\[
\max_{c_m, X_{mT-}} E\left[ \int_0^{T'} u(c_{ms})ds \right] \\
\text{subject to } E\left[ \int_0^{T} \pi_s c_{ms}ds + \pi_{T-} X_{mT-} \right] \leq \pi_0 x_{n0} \\
E\left[ \int_T^{T'} \pi_s c_{ms}ds \mid \mathcal{F}_T \right] \leq \pi_{T-} x_{mT-} \text{ almost surely}, \\
X_{mT-} \geq \max(a_{ST} + b, K) \text{ almost surely}
\]

**Lemma 1.**

Assuming a solution exists, the solution is:

\[
\hat{c}_{mt} = I(\lambda_{m1} \pi_t) \quad t \in [0, T) \tag{11}
\]

\[
\hat{c}_{mt} = I(\lambda_{m2} \pi_t) \quad t \in [T, T'] \tag{12}
\]

where \( \lambda_{m1} \) and \( \lambda_{m2} \) solve:

\[
E\left[ \int_0^{T} \pi_s I(\lambda_{m1} \pi_s)ds \right]
\]
\[ + \pi_T \max \{ \max(\alpha S_T + b, K), \frac{1}{\pi_T} E[\int_T^{T'} \pi_s I(\lambda_{m1}\pi_s)ds | \mathcal{F}_T] \} \]

\[ = \pi_0 x_{m0} \] \hspace{2cm} (13)

\[ E[\int_T^{T'} \pi_s I(\lambda_{m2}\pi_s)ds | \mathcal{F}_T] \]

\[ = \pi_T \max \{ \max(\alpha S_T + b, K), \frac{1}{\pi_T} E[\int_T^{T'} \pi_s I(\lambda_{m1}\pi_s)ds | \mathcal{F}_T] \} \] \hspace{2cm} (14)

N.B. (1) \( \lambda_n, \lambda_{m1} \) are constant, \( \lambda_{m2} \) is \( \mathcal{F}_T \)-measurable random variable. (2) If \( x_{n0} = x_{m0} \), then \( \lambda_{m1} \geq \lambda_n \). (3) \( \lambda_{m1} = \lambda_{m2} \) if the benchmarking constraint is not binding and \( \lambda_{m1} > \lambda_{m2} \) if it is binding.

For the optimal invested wealth at time \( T \),

\[ \dot{X}_{nT} = \frac{1}{\pi_{T'}} E[\int_T^{T'} \pi_s I(\lambda_n\pi_s)ds | \mathcal{F}_T] \] \hspace{2cm} (15)

\[ \dot{X}_{mT} = \max \{ \max(\alpha S_T + b, K), \frac{1}{\pi_{T'}} E[\int_T^{T'} \pi_s I(\lambda_{m1}\pi_s)ds | \mathcal{F}_T] \} \] \hspace{2cm} (16)

2 The Equilibrium

In the rest of this paper, I first solve for the equilibrium in this benchmarking economy and then compare it with the normal economy in which only normal agents exist. For simplicity, we only consider log-utility while similar approach can be applied to power and negative exponential utilities.

The equilibrium condition is given by market clearing:

\[ \delta_t = \dot{c}_{nt} + \dot{c}_{mt} \ \ t \in [0, T'] \] \hspace{2cm} (17)
Thus in this case, the derivation of SPD follows Basak (1995):

$$\delta_t = I(\lambda_n \pi_t) + I(\lambda_{m1} \pi_t) \quad t \in [0, T)$$ (18)

$$\delta_t = I(\lambda_n \pi_t) + I(\lambda_{m2} \pi_t) \quad t \in [T, T']$$ (19)

For $u(c) = \log(c)$, the solution for $\pi_t$ is:

$$\pi_t = \left( \frac{1}{\lambda_n} + \frac{1}{\lambda_{m1}} \right) \frac{1}{\delta_t} \quad t \in [0, T)$$ (20)

$$\pi_t = \left( \frac{1}{\lambda_n} + \frac{1}{\lambda_{m2}} \right) \frac{1}{\delta_t} \quad t \in [T, T']$$ (21)

Apply Ito’s lemma on $\pi_t$ which follows the dynamics in equation (5),

$$r_t = \mu - \sigma^2 = r$$ (22)

$$\theta_t = \sigma = \theta$$ (23)

$$q = \ln\left( \frac{1}{\lambda_n} + \frac{1}{\lambda_{m1}} \right) - \ln\left( \frac{1}{\lambda_n} + \frac{1}{\lambda_{m2}} \right) \leq 0$$ (24)

The short-rate and market price of risk are constant, while the SPD has a jump at time $T$.

If SPD is continuous, the normal agent’s demand for consumption will also be continuous.

But when agent $m$’s constraint is binding, since $\lambda_{m1} > \lambda_{m2}$ and $\hat{c}_{T-} = 1/\lambda_{m1} \pi_{T-}$, $\hat{c}_{mT} = 1/\lambda_{m2} \pi_T$, his demand for consumption will jump upwards immediately after time $T$, leading the aggregate demand for consumption jump upwards, which is impossible given a continuous dividend process. Hence, there must be a jump in SPD to smooth agent $m$’s demand for consumption. And the jump in SPD is upward because agent $m$ values consumption after the constraint date more than before the date.

Then, define the equity market as the total optimal wealth invested in the stock and denote it by $\hat{X}_{emt}$. Since the money market account is in zero net supply, the equity market is equal to the total optimal wealth invested by the two agents.

$$\hat{X}_{emt} = \hat{X}_{nt} + \hat{X}_{mt} \quad t \in [0, T']$$ (25)
Also, since the risky asset is in net supply of 1,

\[ S_t = \hat{X}_{emt} = \hat{X}_{nt} + \hat{X}_{mt} \quad t \in [0, T'] \quad (26) \]

### 2.1 Asset Prices

Now, we can derive explicit solutions for \( X_{emt}, S_t, \mu_t, \sigma_t, \phi_t \). To compare with the normal economy, denote the normal economy terms by \( \bar{X}_{emt}, \bar{S}_t, \bar{\mu}_t, \bar{\sigma}_t, \bar{\phi}_t \).

In the normal economy:

\[ \bar{S}_t = \bar{X}_{emt} = \bar{X}_{nt} = 1 \bar{\pi}_t E \left[ \int_t^{T'} \frac{1}{\bar{\pi}_s} ds \mid \mathcal{F}_t \right] = (T' - t) \delta_t \quad t \in [0, T'] \quad (27) \]

In the benchmarking economy, for \( t \in [T, T'] \):

\[ S_t = X_{emt} = \hat{X}_{nt} + \hat{X}_{mt} = \frac{1}{\hat{\pi}_t} E \left[ \int_t^{T'} \frac{1}{\pi_n \hat{\pi}_s} ds \mid \mathcal{F}_t \right] + \frac{1}{\hat{\pi}_t} E \left[ \int_t^{T'} \frac{1}{\pi_m \hat{\pi}_s} ds \mid \mathcal{F}_t \right] = \frac{1}{(\frac{1}{\lambda_n} + \frac{1}{\lambda_m}) \frac{1}{\delta_t}} (\frac{1}{\lambda_n} + \frac{1}{\lambda_m})(T' - t) \]

\[ = (T' - t) \delta_t \quad (28) \]

So \( S_t = \bar{S}_t = (T' - t) \delta_t \), for \( t \in [T, T'] \). This is true because the benchmarking economy becomes the normal economy after \( T \). Then we study the economy before \( T \).
Lemma 2. For \( t \in [0, T) \):

\[
S_t = X_{emt} = \hat{X}_{nt} + \hat{X}_{mt} \\
= \frac{1}{\pi_t} E[\int_t^{T'} \pi_s \frac{1}{\lambda_n \pi_s} ds | \mathcal{F}_t] + \frac{1}{\pi_t} E[\int_t^{T} \pi_s \frac{1}{\lambda_{m1} \pi_s} ds + \pi_{T-} \hat{X}_{mT-} | \mathcal{F}_t]
\]

\[
= \frac{\lambda_{m1}}{\lambda_n + \lambda_{m1}} (T' - t) \delta_t + \frac{\lambda_n}{\lambda_n + \lambda_{m1}} (T - t) \delta_t \\
+ \frac{1}{\pi_t} E[\pi_{T-} \max\{\max(aS_T + b, K), \frac{1}{\pi_{T-}} E[\int_T^{T'} \pi_s I(\lambda_{m1} \pi_s) ds | \mathcal{F}_T] | \mathcal{F}_t]]
\]

\[
= \frac{\lambda_{m1}}{\lambda_n + \lambda_{m1}} (T' - t) \delta_t + \frac{\lambda_n}{\lambda_n + \lambda_{m1}} (T - t) \delta_t + \frac{\lambda_n}{\lambda_n + \lambda_{m1}} (T' - T) \delta_t \\
+ \frac{1}{\pi_t} E[\pi_{T-} \max\{\max(aS_T + b, K) - \frac{\lambda_n}{\lambda_n + \lambda_{m1}} (T' - T) \delta_{T-}, 0\} | \mathcal{F}_t]
\]

\[
= S_t + \frac{1}{\pi_t} E[\pi_{T-} \max\{\max(aS_T + b, K) - \frac{\lambda_n}{\lambda_n + \lambda_{m1}} (T' - T) \delta_{T-}, 0\} | \mathcal{F}_t]
\]

(29)

N.B. \( S_t \geq \bar{S}_t \), for \( t \in [0, T) \). This is true because before time \( T \) agent \( m \) consumes less than if he’s unconstrained. So he invests more than he would if unconstrained. Since the risk-free asset is in zero net supply, the extra investment from agent \( m \) goes into the stock market and drives the price up before time \( T \).

Now focus on the explicit solution of \( S_t \) for \( t \in [0, T) \). From the equation (28), \( S_T = (T' - T) \delta_T \), and by the continuity of \( \delta_t \), \( \delta_{T-} = \delta_T \). So, replace \( S_T \) by \( (T' - T) \delta_{T-} \), rewrite equation (29) as:

\[
S_t = (T' - t) \delta_t + \frac{1}{\pi_t} E[\pi_{T-} \max\{\max(a(T' - T) \delta_{T-} + b, K) - \frac{\lambda_n}{\lambda_n + \lambda_{m1}} (T' - T) \delta_{T-}, 0\}] | \mathcal{F}_t]
\]

\[
= (T' - t) \delta_t + \frac{1}{\pi_t} E[\pi_{T-} \max\{a \delta_{T-} + b, K - \beta \delta_{T-}, 0\}] | \mathcal{F}_t]
\]

(30)

where \( \alpha = (a - \frac{\lambda_n}{\lambda_n + \lambda_{m1}})(T' - T) \) and \( \beta = \frac{\lambda_n}{\lambda_n + \lambda_{m1}}(T' - T) \).

Since the process \( \delta_t \) is a Geometric Brownian Motion, we can apply the martingale approach to take the expectation in equation (30). The result depends on the values of the
parameters $\alpha, \beta, b, K$. Hence there can be several cases, among which we refine our attention to 2 most interesting ones in which anyone of the 3 terms in the max function can be the largest.

Namely, we study **case 1**: $b < -\alpha K/\beta < 0$, also known as the risky benchmark case. $b < 0$ means the benchmark is equivalent to investing more in the risky asset than the optimal unconstrained policy (which for this log utility case is to invest all wealth in risky asset) through borrowing. $b < 0$ also ensures $\alpha > 0$. In this case, when the market is very good, the benchmark constraint is binding; when the market is very poor, the floor constraint is binding; and when the market is in the middle region, the agent’s unconstrained strategy is the best and neither constraint will bind.

**case 2**: $b > -\alpha K/\beta > 0$ can be regarded as the safe benchmark case as the benchmark contains less investment in the risky asset than the unconstrained one by having positive investment in the risk-free asset. In this case, when the market is very good, the unconstrained strategy is the best; when the market is not so good nor too poor, the benchmark constraint will bind; and when the market is very poor, the floor constraint will be binding.

As mentioned above, each one of the three terms: the benchmark, the floor, and the unconstrained strategy can be the largest in the two cases we study. However, if $b \in [-\alpha K/\beta, 0)$, the agent can never beat the constraint. And if $b \in (0, -\alpha K/\beta]$, the model can be transformed to the portfolio insurance problem studied in Basak (1995). Therefore, we now restrict our attention to case 1 and case 2 only. In the end, the discussion section will summarize results together with the portfolio insurance case.
We have shown that $S_t \geq \bar{S}_t$ for $t < T$ for both cases, but the impact of the benchmarking constraint on equilibrium volatility, risk premium and optimal strategy is different for the two cases depending on the riskiness of the benchmark. Now, solve equation (30) for the two cases respectively, we have:

**Proposition 1.** In the benchmarking economy,

**case 1:** $b < -\alpha K/\beta < 0$

$$S_t = (T' - t)\delta_t + \alpha N(\hat{z}_1)\delta_t - \beta N(-z_1)\delta_t + be^{-r(T-t)}N(\bar{z}_2) + Ke^{-r(T-t)}N(-\check{z}_2)$$  \hspace{1cm} (31)

where

$$z_1 = z_2 + \sigma_\delta \sqrt{T - t}, \quad z_2 = \frac{\ln(\beta \delta_t / K) + (r - \frac{1}{2}\sigma_\delta^2)(T - t)}{\sigma_\delta \sqrt{T - t}}$$

$$\bar{z}_1 = \bar{z}_2 + \sigma_\delta \sqrt{T - t}, \quad \bar{z}_2 = \frac{\ln(\alpha \delta_t / -b) + (r - \frac{1}{2}\sigma_\delta^2)(T - t)}{\sigma_\delta \sqrt{T - t}}$$

**case 2:** $b > -\alpha K/\beta > 0$

$$S_t = (T' - t)\delta_t + \alpha [N(\bar{z}_1) - N(\hat{z}_1)]\delta_t - \beta N(-\bar{z}_1)\delta_t + be^{-r(T-t)}[N(\bar{z}_2) - N(\check{z}_2)] + Ke^{-r(T-t)}N(-\check{z}_2)$$ \hspace{1cm} (32)

where

$$\bar{z}_1 = \bar{z}_2 + \sigma_\delta \sqrt{T - t}, \quad \bar{z}_2 = \frac{\ln[(\alpha + \beta)\delta_t / (K - b)] + (r - \frac{1}{2}\sigma_\delta^2)(T - t)}{\sigma_\delta \sqrt{T - t}}$$

$$\check{z}_1 = \check{z}_2 + \sigma_\delta \sqrt{T - t}, \quad \check{z}_2 = \frac{\ln(\alpha \delta_t / -b) + (r - \frac{1}{2}\sigma_\delta^2)(T - t)}{\sigma_\delta \sqrt{T - t}}$$

In both cases, $N(\cdot)$ is the distribution function of normal random variable. And we also have:

$$\hat{X}_{mt} = S_t - \hat{X}_{nt} = S_t - \frac{\lambda_m}{\lambda_n + \lambda_{m1}}(T' - t)\delta_t$$ \hspace{1cm} (33)
2.2 Volatility and Risk Premium

Apply Ito’s lemma on $S_t$, we get explicit solutions for the volatility and risk premium in both the normal and benchmarking economies.

**Proposition 2.**

In the normal economy, for $t \in [0, T']$,

$$\bar{\sigma}_t = \sigma_\delta$$

$$\bar{\mu}_t - r = \sigma_\delta^2$$

In the benchmarking economy, for $t \in [T, T']$,

$$\sigma_t = \sigma_\delta$$

$$\mu_t - r = \sigma_\delta^2$$

While for $t \in [0, T)$,

case 1:

$$\sigma_t = \frac{1 - be^{-r(T-t)}N(\bar{z}_2) + Ke^{-r(T-t)}N(-\bar{z}_2)}{S_t}\sigma_\delta$$

$$\mu_t - r = \frac{1 - be^{-r(T-t)}N(\bar{z}_2) + Ke^{-r(T-t)}N(-\bar{z}_2)}{S_t}\sigma_\delta^2$$

case 2:

$$\sigma_t = \frac{1 - be^{-r(T-t)}[N(\bar{z}_2) - N(\bar{z}_2)] + Ke^{-r(T-t)}N(-\bar{z}_2)}{S_t}\sigma_\delta$$

$$\mu_t - r = \frac{1 - be^{-r(T-t)}[N(\bar{z}_2) - N(\bar{z}_2)] + Ke^{-r(T-t)}N(-\bar{z}_2)}{S_t}\sigma_\delta^2$$

Whether the volatility and risk premium in the benchmarking economy are greater or
less than those in the normal economy only depends on the sign of the fraction term inside the bracket. Therefore, for $t \in [0, T)$:

**Corollary 1.**

For case 1: the volatility and risk premium in the benchmarking economy are greater than those in the normal economy if $b < -K \frac{N(-z_2)}{N(z_2)}$ and vice versa.

For case 2: the volatility and risk premium in the benchmarking economy are always smaller than those in the normal economy.

The volatility and risk premium in the normal economy are constant. However, they’re conditional in the presence of the portfolio benchmarking constraint. In general, the volatility and risk premium will increase when the benchmark is very risky ($b$ small enough). But for a given risky benchmark, whether the effect is volatility increasing or decreasing is state-dependent; While for relatively safe benchmark, volatility and risk premium will always decrease. In both cases, the degree of the increase or decrease is also state-dependent. To better understand the results for volatility, I further investigate agent’s optimal strategy.

### 2.3 The Optimal Strategy

Denote the optimal fraction of wealth invested in the risky asset by $\hat{\Phi}_{nt}$ and $\hat{\Phi}_{mt}$ for agent $n$ and agent $m$. Apply Ito’s lemma on $\pi_t \hat{X}_{nt}$ and $\pi_t \hat{X}_{mt}$, we get explicit solutions for $\hat{\Phi}_{nt}$ and $\hat{\Phi}_{mt}$ in the benchmarking economy.

**Proposition 3.**

In both case 1 and 2, for $t \in [0, T')$,

$$\hat{\Phi}_{nt} = (\mu_t - r)/\sigma^2_t$$  \hfill (42)
While for $\hat{\Phi}_{mt}$, when $t \in [T, T']$,

$$\hat{\Phi}_{mt} = \hat{\Phi}_{nt} = \frac{(\mu_t - r)}{\sigma_t^2}$$

(43)

When $t \in [0, T)$,

**case 1:**

$$\hat{\Phi}_{mt} = \left(1 - \frac{be^{-r(T-t)}N(\bar{z}_2) + Ke^{-r(T-t)}N(-\bar{z}_2)}{\hat{X}_{mt}}\right)\frac{\mu_t - r}{\sigma_t^2}$$

(44)

**case 2:**

$$\hat{\Phi}_{mt} = \left\{1 - \frac{be^{-r(T-t)}[N(\bar{z}_2) - N(\bar{z}_2)] + Ke^{-r(T-t)}N(-\bar{z}_2)}{\hat{X}_{mt}}\right\}\frac{\mu_t - r}{\sigma_t^2}$$

(45)

Similar to the analysis on volatility and risk premium, we have:

**Corollary 2.**

For **case 1:** the constrained agent invests more fraction of wealth in the risky asset than the normal agent if $b < -K\frac{N(-z_2)}{N(z_2)}$ and vice versa.

For **case 2:** the constrained agent always invests less fraction of wealth in the risky asset than the normal agent.

### 2.4 Discussion of the Equilibrium

We have seen that for case 1, when the benchmark is equivalent to borrowing a significant amount of money and investing in the risky asset, the existence of this very risky benchmark increases risk premium, volatility conditionally. While for case 2 and the other part of case 1, when the benchmark is relatively safe, those terms are always smaller in the benchmarking economy than in the normal economy. Moreover, when $b \in (0, -\alpha K/\beta]$, the model transforms to Basak (1995)’s portfolio insurance model in which the risk premium,
volatility and the optimal fraction of wealth invested in the risky asset are smaller than in the normal economy. Therefore, we can summarize the results of this model as:

**Proposition 4.**

In an economy with a very risky benchmark such that $b < \min\left[-N(-z_2)K/N(\tilde{z}_2), -\alpha K/\beta\right]$, the stock price, risk premium and volatility are increased by the presence of the portfolio benchmarking constraint, the constrained agent invests more fraction of wealth in the stock than the unconstrained agent and the equilibrium effect is state-dependent.

And in an economy with a safe benchmark such that $b > 0$, the stock price is increased but the risk premium and volatility are decreased by the presence of the portfolio benchmarking constraint, the constrained agent invests less fraction of wealth in the stock than the unconstrained agent.

Such findings may be contrary to conventional wisdom, e.g. Grossman and Zhou (1996), which says the inclusion of the portfolio insurance constrained agent should increase volatility and risk premium. However, the results here are understandable in the following ways:

Firstly, seen from proposition 3 and corollary 2, the effect of volatility can be explained by the optimal fraction of wealth invested in the stock by both agent. In the normal economy, both agents optimally invest all their wealth into the stock (as implied by the log-utility). In the safe benchmark case, the constrained agent has to invest less in the stock so as to hold some risk-free asset. To clear the market, it means the unconstrained agent has to take more than he would in the normal economy. Since we have,

$$\hat{\Phi}_{nt} = (\mu_t - r)/\sigma_t^2 = \theta/\sigma_t$$

(46)
so the only way to make the unconstrained agent be willing to hold more stock is to decrease \( \sigma_t \). While if the benchmark in the constraint is very risky, the constrained agent may have to hold more stock. So the volatility has to increase to induce the unconstrained agent to sell. This suggests that in this economy which has a constant market price of risk \((\theta)\), the volatility of the stock has to decrease (increase) so as to make it more (less) attractive to the unconstrained agent.

Furthermore, in the risky benchmark case, the constrained agent is more likely to demand for more stock if the market is good and the benchmark constraint is likely to bind. So the overall effect of the portfolio benchmarking constraint is more likely to be volatility-increasing in good states. On the other hand, it is more likely to be volatility-decreasing in bad states in which the floor constraint is likely to bind so the constrained agent has to demand for less stock. In the safe benchmark case, the constrained agent always has to hold less stock than in the normal economy whether the benchmark constraint or the floor constraint is more likely to bind. So the effect on volatility will always be a decrease. In both case, since the constrained agent has to adjust his demand for the stock conditionally, the degree to which the volatility is increased or decreased is therefore state-dependent.

In another attempt to explain the effect on volatility, let’s look at the expression for \( S_t \) before \( T \). We can re-write \( S_t \) as:

\[
S_t = \bar{S}_t + \frac{1}{\pi_t} E[\pi T - \max\{\max[(a - \frac{\lambda_n}{\lambda_n + \lambda_m})S_T + b, 0], \\max[K - \frac{\lambda_n}{\lambda_n + \lambda_m}S_T, 0]\}] |\mathcal{F}_t]
\]

(47)

The stock price in the benchmarking economy is the normal economy price plus an expectation term which is the present value of the maximum of two option payoffs. The later one is always a put option payoff while the former one is a call option payoff when the
benchmark is very risky, and it is a put option payoff when the benchmark is safe. For the safe benchmark case, when the market goes down, the put option value goes up, which helps stabilize stock price. So the volatility is decreased in the presence of the constraint. However, in the risky benchmark case, when the market goes down, the call option value goes down as well and the put option value goes up. If the benchmark is very risky, then the effect from the call option is more significant so the value of the expectation term also goes down. Therefore, the stock price falls even further and the constraint increases volatility.

Thus we can view the overall effect of the portfolio benchmarking constraint as the mixture of the effect from the benchmark constraint and the effect from the floor constraint. The individual effect of the floor constraint is volatility-decreasing, so does that for a safe benchmark constraint. But the effect of a risky benchmark constraint tends to be volatility-increasing. To better illustrate this point, I consider a special case in the next part in which \( K = 0 \), so we can single out the effect of the benchmark constraint for discussion.

The results for the safe benchmark case is consistent with Basak (1995) which has the similar set-up in using consumption as numeraire and having intermediate consumption, dividend and constraint date. These features result in a constant market price of risk as compared to the price of consumption. But in Grossman and Zhou (1996), the market price of risk is time-varying as they used bond price as numeraire and there’s no intermediate consumption so the pre-constraint pricing kernels are conditional expectations of the final one and therefore directly affected by the constraint. Therefore, the volatility is affected by both the change in market price of risk and the change in agent’s risk aversion, and they show that the overall effect is a higher volatility.
3 Pure Benchmark Constraint

Now consider a special case in which $K = 0$. This means the constrained agent effectively only has to compete with the stochastic benchmark $a S_T + b$. Without elaborating the derivation, simply replace the above results on equilibrium market dynamics with $K = 0$, we have:

**Proposition 5.** If $K = 0$, for log-utility,

In an economy with a risky benchmark such that $b < 0$, the stock price, the risk premium and volatility are increased by the presence of the benchmarking constraint.

In an economy with a safe benchmark such that $b > 0$, the stock price is increased but the risk premium and volatility are decreased by the presence of the portfolio benchmarking constraint.

This suggests the individual equilibrium effect of a pure benchmark constraint is volatility-increasing if the benchmark is risky and volatility-decreasing if it’s safe. In the main model where there’s also a floor constraint, the individual effects from the benchmark constraint and the floor constraint are in the same direction if the benchmark is safe; and they’re in opposite directions in the benchmark is risky. Therefore, when market is good, a risky benchmark constraint is more likely to bind. Its volatility-increasing effect overshadows the volatility-decreasing effect of the floor constraint so the overall effect is volatility-increasing. But when market is bad, the floor constraint’s effect is stronger as it’s more like to bind, so the overall effect is volatility-decreasing. For the safe benchmark case, the two individual effects are both volatility-decreasing, so does the overall effect.
4 The Demand and Pricing of Index Options

Now come back to the main model, the inclusion of a constrained agent introduces heterogeneity in agents’ risk aversion, which according to Leland (1980) should make the constrained agent demand for options. Indeed, I identify agent’s demand for index options in closed form.

Now consider two European call options on the risky asset with maturity at time $T$. Option 1 has exercise price of $\frac{K-b}{a}$ and Option 2 has exercise price of $\frac{k}{\beta}$. Denote the time $t$ value of these two options by $\text{Call}_1^t$ and $\text{Call}_2^t$ respectively. Following similar approach by which we derived $S_t$ earlier, we have:

**Corollary 4.** For $t \in [0, T)$, case 1:

$$S_t = a \frac{N(\tilde{z}_1)}{N(\tilde{z}_1)} \cdot \text{Call}_1^t + \beta \left[ 1 - \frac{N(\tilde{z}_1)}{N(z_1)} \right] \cdot \text{Call}_2^t + (T' - t)\delta_t - \beta \delta_t$$

$$+ b e^{-r(T-t)} [N(\tilde{z}_2) - \frac{N(\tilde{z}_2)N(\tilde{z}_1)}{N(\tilde{z}_1)}] + Ke^{-r(T-t)} \left[ 1 + \frac{N(\tilde{z}_2)N(\tilde{z}_1)}{N(z_1)} - \frac{N(\tilde{z}_2)N(\tilde{z}_1)}{N(z_1)} \right]$$

$$\hat{X}_{mt} = a \frac{N(\tilde{z}_1)}{N(\tilde{z}_1)} \cdot \text{Call}_1^t + \beta \left[ 1 - \frac{N(\tilde{z}_1)}{N(z_1)} \right] \cdot \text{Call}_2^t + \frac{\lambda_n}{\lambda_n + \lambda_m} (T - t)\delta_t$$

$$+ b e^{-r(T-t)} [N(\tilde{z}_2) - \frac{N(\tilde{z}_2)N(\tilde{z}_1)}{N(\tilde{z}_1)}] + Ke^{-r(T-t)} \left[ 1 + \frac{N(\tilde{z}_2)N(\tilde{z}_1)}{N(z_1)} - \frac{N(\tilde{z}_2)N(\tilde{z}_1)}{N(z_1)} \right]$$

(49)
case 2:

\[
S_t = a\left[1 - \frac{N(\tilde{z}_1)}{N(\tilde{z}_2)}\right] \cdot \text{Call}_1^1 + \beta \frac{N(\tilde{z}_1)}{N(z_1)} \cdot \text{Call}_1^2 + (T' - t)\delta_t - \beta \delta_t
+ be^{-r(T-t)} \left[\frac{N(\tilde{z}_2)N(\tilde{z}_1)}{N(\tilde{z}_1)} - N(\tilde{z}_2)\right] + Ke^{-r(T-t)} \left[1 - \frac{N(\tilde{z}_2)N(\tilde{z}_1)}{N(\tilde{z}_1)} + \frac{N(z_2)N(\tilde{z}_1)}{N(z_1)}\right]
\]

(50)

\[
\hat{X}_{mt} = a\left[1 - \frac{N(\tilde{z}_1)}{N(\tilde{z}_2)}\right] \cdot \text{Call}_1^1 + \beta \frac{N(\tilde{z}_1)}{N(z_1)} \cdot \text{Call}_1^2 + \frac{\lambda_n}{\lambda_n + \lambda_m} (T - t)\delta_t
+ be^{-r(T-t)} \left[\frac{N(\tilde{z}_2)N(\tilde{z}_1)}{N(\tilde{z}_1)} - N(\tilde{z}_2)\right] + Ke^{-r(T-t)} \left[1 - \frac{N(\tilde{z}_2)N(\tilde{z}_1)}{N(\tilde{z}_1)} + \frac{N(z_2)N(\tilde{z}_1)}{N(z_1)}\right]
\]

(51)

Therefore, for both cases, the constrained agent’s optimal invested wealth has long positions in two call options on the risky asset. Option 1 will be exercised when the benchmark beats the floor and Option 2 will be exercised when the optimal unconstrained strategy beats the floor. So the combined position gives agent \( m \) full protection that his time \( T \) wealth is no less than the maximum of the benchmark and the floor. The emergence of agent \( m \)'s demand for long positions in call options is consistent with Proposition 2 in Leland (1980) and also provides an example of the end users’ motives to buy index options mentioned in Garleanu, Pedersen and Poteshman (2007).

5 Concluding Remarks

This paper studies the equilibrium effect of a portfolio benchmarking constraint. The economy has a normal agent with log utility over continuous consumption and a constrained agent whose intermediate wealth must meet the maximum of a stochastic constraint and a constant floor. Using martingale approach, I solve this consumption-based general equilibrium model explicitly to get the equilibrium assets prices, risk premium, volatility, optimal
strategy and compare them with those in a normal economy. The problem of the equilibrium effects of portfolio insurance studied by Basak (1995) and Grossman and Zhou (1996) is a special case of the problem studied here.

In the portfolio benchmarking economy before the constraint date, the risky asset price is higher than that in the normal economy because the constrained agent consumes less than if he’s not constrained. Since the risk-free asset is in zero net supply, the extra investment flows into the risky asset market and drives up the risky asset’s price. When the benchmark is chosen to be a very risky one, i.e. can be replicated by borrowing a significant amount of money to invest in the risky asset, then the risk premium, volatility are increased by the presence of the constraint. For a given risky benchmark index, the overall effect of the constraint is more likely to be volatility-increasing in good states and volatility-decreasing in bad states. While if the benchmark is relatively safe, i.e. the replication portfolio of which contains limited short position or any positive position in the risk-free asset, the risk premium, volatility are decreased. In both cases, the degree of increase or decrease are state-dependent.

The rationale behind these findings is when there are more (less) demand for the risky asset from the constrained agent, the volatility has to increase (decrease) so as to clear the market. This is true because in this model the market price of risk is constant and the SPDs before the constraint date are not directly affected by the existence of the constraint due to the existence of intermediate dividend for consumption. The overall effect of the portfolio benchmarking constraint is the mixture of the individual effect from the benchmark constraint, which is volatility-increasing if its risky and volatility-decreasing if it’s safe, and the individual effect from the floor constraint which is always volatility-decreasing. So the direction of the overall effect is state-dependent in the risky benchmark case and is always
volatility decreasing in the safe benchmark case, which has consistent conclusion with the portfolio insurance model of Basak (1995) that has the similar set-up and approach with this paper.

The paper also makes two extensions of the main model, one is to single out the individual effect of the benchmark constraint by considering a special case where the floor is zero. Another extension is a variation of the derivation of the equilibrium risky asset price which reveals that the constrained agent’s optimal investment before the constraint date contains long positions in two call options on the risky asset with maturity on the constraint date.
6 Appendix (very brief proof of some results)

Proof of Lemma 1: see Basak (1995)’s proof of lemma 3, replace \( K \) by \( \max aS_T + b, K \).

Proof of Proposition 1:

case 1 In this case, we have \( 0 < \frac{K}{\beta} < \frac{K-b}{\alpha+\beta} < \frac{-b}{\alpha} \). When \( \delta_{T-} \in (0, \frac{K}{\beta}] \), \( K - \beta \delta_{T-} \) is the largest in the Max function; When \( \delta_{T-} \in (\frac{K}{\beta}, \frac{-b}{\alpha}] \), 0 is the largest; And when \( \delta_{T-} > \frac{-b}{\alpha} \), \( \alpha \delta_{T-} + b \) is the largest. Then, apply the martingale approach for option pricing, we have the explicit solution for \( S_t \).

case 2 In this case, \( 0 < \frac{K-b}{\alpha+\beta} < \frac{K}{\beta} < \frac{-b}{\alpha} \). So, when \( \delta_{T-} \in (0, \frac{K-b}{\alpha+\beta}] \), \( K - \beta \delta_{T-} \) is the largest in the Max function; When \( \delta_{T-} \in (\frac{K-b}{\alpha+\beta}, \frac{-b}{\alpha}] \), \( \alpha \delta_{T-} + b \) is the largest; And when \( \delta_{T-} > \frac{-b}{\alpha} \), 0 is the largest.

Proof of Proposition 2: apply Ito’s lemma on the explicit solution for \( \hat{S}_t \) and \( S_t \), get the expression for the drift and diffusion terms. Also, we have

\[
dS_t + \delta_t dt = S_t(\mu_t dt + \sigma_t dB_t, t \in [0, T])
\]

by definition, so we can solve for \( \mu_t - r \) and \( \sigma_t \). When applying Ito’s lemma on \( S_t \), we can take advantage of the fact that we can rewrite \( S_t \) in terms of combination of Black-Scholes European call option prices (treat \( \delta_t \) as the underlying), so the \( \partial S/\partial \delta \) term is not as complicated as it seems. After getting the explicit solution for \( \sigma_t \), we can then use the relationship that \( (\mu_t - r)/\sigma_t = \theta_t = \sigma_\delta \) to derive \( \mu_t - r \).

Proof of Proposition 3: Applying Ito’s lemma on the product \( \pi_t \hat{X}_t \) gives

\[
d(\pi_t \hat{X}_t) + \pi_t \hat{c}_t dt = \pi_t \hat{X}_t[\hat{\Phi}_t \sigma_t - \theta_t] d\hat{B}_t
\]

again apply Ito’s lemma on the explicit solutions of \( \hat{X}_{nt} \) and \( \hat{X}_{mt} \), equalizing the diffusion term solves \( \hat{\Phi}_{nt} \) and \( \hat{\Phi}_{mt} \).
7 References


