General Equilibrium Analysis of Portfolio Benchmarking

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Outline

Introduction

The Model

Equilibrium

Discussion of Results

Conclusion

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Introduction

- This paper studies the equilibrium effect of portfolio benchmarking on a heterogeneous-agent economy.
- I use a continuous time model extended from Basak (1995) to derive the equilibrium market dynamics in closed-form.
- The asset price, volatility and risk premium are higher than those in the normal economy for very risky benchmarks.
- However, when the benchmark is relatively safe, asset price is higher but volatility and risk premium are decreased by the presence of the constraint.
Basak (1995)

- Continuous time consumption-based general equilibrium model
- Intermediate dividend and consumption
- Intermediate constraint date
- Closed-form solutions
- Presence of the portfolio insurance constraint increases stock price, decreases the risk premium, volatility and optimal fraction of wealth invested in the stock by the insurer.
Grossman and Zhou (1996)

- Consumption and dividend only at final date
- Constraint at final date
- Bond as numeraire
- Numerical solutions
- Time-varying market price of risk
- Presence of the portfolio insurance constraint decreases stock price, increases risk premium and volatility
Portfolio Benchmarking

- Wealth at time $T$ no less than $\max[aS_T + b, K]$
- When market is good, agent has to do better than a benchmark index
- When market is bad, agent must not lose too much
- Ensures good performance yet limits loss
- Riskiness of the benchmark index is measured by $b$
- For risky benchmarks, constraint can be binding even when market is good
Model Set-up: Markets

Finite horizon \([0, T']\), one consumption good as numeraire. Stock \(S\) in constant net supply of 1 and pays exogenous dividend at rate \(\delta_t\) in \([0, T']\). Dividend process is GBM.

\[
d\delta_t = \delta_t(\mu \delta dt + \sigma \delta dB_t), \quad t \in [0, T']
\]

\[
dS_t + \delta_t dt = S_t(\mu dt + \sigma dB_t + qA_t), \quad t \in [0, T']
\]

There is an \(\mathcal{F}_T\)-measurable jump at the constraint date \(T\) before \(T'\). Here, \(A_t \equiv 1_{t \geq T}\). \(q\) is the jump size parameter.

A risk-less bond \(S^0\) in zero net supply (money market account).

\[
dS^0_t = S^0_t(r_t dt + q^0 A_t), \quad t \in [0, T']
\]

\(q = q^0 = \ln(S_T/S_{T-})\) to rule out arbitrage, where \(S_{T-}\) denotes the left limit of \(S_T\).
SPD and Prices

The state price density process is:

\[ \pi_t = \frac{1}{S_0} \exp \left( - \int_0^t r_s ds - \int_0^t \theta_s dB_s - \frac{1}{2} \int_0^t \theta_s^2 ds - qA_t \right) \]  \hspace{1cm} (4)

where \( \theta_t = \frac{\mu_t - r_t}{\sigma_t} \) is the market price of risk. The SPD process also contains a jump. Apply Ito’s lemma to \( \pi_t \),

\[ d\pi_t = -\pi_t (r_t dt + \theta_t dB_t + qA_t), \quad t \in [0, T'] \]  \hspace{1cm} (5)

and use the SPD process, we have

\[ S_t = \frac{1}{\pi_t} E \left[ \int_t^{T'} \pi_s \delta_s ds \mid \mathcal{F}_t \right], \quad t \in [0, T'] \]  \hspace{1cm} (6)
Model Set-up: Agents

Two agents: agent $n$ and agent $m$, endowed with $x_{n0}$ and $x_{m0}$ at time zero. The utility function $u(c_{it})$ is the same for both agents.

- Agent $n$ is the normal agent
- Agent $m$ is the portfolio benchmarker, whose time $T$ wealth must remain above $\max [aS_T + b, K]$
Optimization of the Unconstrained

The unconstrained agent \( n \) solves:

\[
\max_{c_n} E\left[ \int_0^{T'} u(c_{ns}) ds \right]
\]

subject to \( E\left[ \int_0^{T'} \pi_s c_{ns} ds \right] \leq \pi_0 x_{n0} \)

Assuming a solution exists:

\[
\hat{c}_{nt} = I(\lambda_n \pi_t) \quad t \in [0, T']
\]  \hspace{1cm} (7)

where \( I(\cdot) \) is the inverse of \( u'(\cdot) \) and \( \lambda_n \) solves:

\[
E\left[ \int_0^{T'} \pi_s I(\lambda_n \pi_s) ds \right] = \pi_0 x_{n0}
\]  \hspace{1cm} (8)
Optimization of the Constrained

The problem for the constrained agent $m$ is:

$$\max_{c_m, X_mT-} E\left[\int_0^{T'} u(c_{ms}) ds\right]$$

subject to $E\left[\int_0^T \pi_s c_{ms} ds + \pi_T - X_mT-\right] \leq \pi_0 x_{m0}$

$$E\left[\int_T^{T'} \pi_s c_{ms} ds | \mathcal{F}_T\right] \leq \pi_T - X_mT-$$

$$X_mT- \geq \max(aS_T + b, K)$$
Optimization of the Constrained: Lemma 1

For agent $m$:

**Lemma 1.**

Assuming a solution exists ($x_{m0}$ large enough):

\[
\hat{c}_{mt} = I(\lambda_m \pi_t) \quad t \in [0, T)
\]

\[
\hat{c}_{mt} = I(\lambda_m \pi_t) \quad t \in [T, T']
\]  

(continues next page)
Lemma 1 continued

\(\lambda_{m1}\) and \(\lambda_{m2}\) solve:

\[
E\left[\int_{0}^{T} \pi_s I(\lambda_{m1}\pi_s)ds + \pi_{T-}\max\{\max(aS_T + b, K), \right.
\]

\[
\frac{1}{\pi_{T-}}E\left[\int_{T}^{T'} \pi_s I(\lambda_{m1}\pi_s)ds | \mathcal{F}_T\right]\} = \pi_0 x_{m0} \tag{11}
\]

\[
E\left[\int_{T}^{T'} \pi_s I(\lambda_{m2}\pi_s)ds | \mathcal{F}_T\right]\]

\[
= \pi_{T-}\max\{\max(aS_T + b, K), \frac{1}{\pi_{T-}}E\left[\int_{T}^{T'} \pi_s I(\lambda_{m1}\pi_s)ds | \mathcal{F}_T\right]\} \tag{12}
\]
Discussion of the Solution and Consumption Behavior

From the solution we have:

- $\lambda_n, \lambda_{m1}$ are constant, $\lambda_{m2}$ is $\mathcal{F}_T$-measurable random variable
- If $x_{n0} = x_{m0}$, then $\lambda_{m1} \geq \lambda_n$
- $\lambda_{m1} = \lambda_{m2}$ if the constraint is not binding and $\lambda_{m1} > \lambda_{m2}$ if it is binding

Consumption behaviors implied by the multipliers:

- agent $m$ consumes less than if he’s unconstrained
- agent $m$ consumes more after time $T$ if constraint is binding
Equilibrium Conditions and SPDs

By market clearing of consumption:

\[ \delta_t = \hat{c}_{nt} + \hat{c}_{mt} \quad t \in [0, T'] \]  

(13)

For \( u(c) = \log(c) \), SPDs are:

\[ \pi_t = \left( \frac{1}{\lambda_n} + \frac{1}{\lambda_{m1}} \right) \frac{1}{\delta_t} \quad t \in [0, T) \]  

(14)

\[ \pi_t = \left( \frac{1}{\lambda_n} + \frac{1}{\lambda_{m2}} \right) \frac{1}{\delta_t} \quad t \in [T, T'] \]  

(15)
Characteristics of the Equilibrium

Apply Ito’s lemma on $\pi_t$:

\[
\begin{align*}
    r_t &= \mu_\delta - \sigma_\delta^2 = r \\
    \theta_t &= \sigma_\delta = \theta \\
    q &= ln\left(\frac{1}{\lambda_n} + \frac{1}{\lambda_{m1}}\right) - ln\left(\frac{1}{\lambda_n} + \frac{1}{\lambda_{m2}}\right) \leq 0
\end{align*}
\]

- The short-rate and market price of risk are constant
- An upward jump in the SPD is required due to the discontinuity in consumption
- It also reflects investor’s different valuation for consumptions before and after time $T$
Stock Price in General

Denote the total optimal wealth invested in the stock by \( \hat{X}_{emt} \), the optimal wealth of agent \( n \) and \( m \) by \( \hat{X}_{nt} \) and \( \hat{X}_{mt} \).

\[
S_t = \hat{X}_{emt} = \hat{X}_{nt} + \hat{X}_{mt} \quad t \in [0, T']
\] (19)

In the normal economy:

\[
\bar{S}_t = (T' - t)\delta_t \quad t \in [0, T']
\] (20)

In the benchmarking economy after the constraint date:

\[
S_t = (T' - t)\delta_t
\] (21)

So \( S_t = \bar{S}_t \) for \( t \in [T, T'] \).
Solving for Stock Price

Then we study the benchmarking economy before $T$.

**Lemma 2.** For $t \in [0, T)$:

$$S_t = \tilde{S}_t + \frac{1}{\pi_t} E[\pi T_- \max\{\max(aS_T + b, K)$$

$$- \frac{\lambda_n}{\lambda_n + \lambda_{m1}} (T' - T)\delta_{T-}, 0\} | \mathcal{F}_t]$$

(22)

Therefore for $t \in [0, T)$, $S_t \geq \tilde{S}_t$. Re-write the above formula as:

$$S_t = (T' - t)\delta_t + \frac{1}{\pi_t} E[\pi T_- \max\{\alpha\delta_{T-} + b, K - \beta\delta_{T-}, 0\} | \mathcal{F}_t]$$

(23)

where $\alpha = (a - \frac{\lambda_n}{\lambda_n + \lambda_{m1}})(T' - T)$ and $\beta = \frac{\lambda_n}{\lambda_n + \lambda_{m1}}(T' - T)$. $\delta_t$ is a GBM, so apply the martingale approach to take the expectation.
Stock Price in Closed-form

The result depends on the values of \( \alpha, \beta, b, K \). There can be several cases. We study the 2 most interesting ones in which anyone of the 3 terms in the max function can be the largest.

- **case 1:** \( b < -\alpha K / \beta < 0 \), the risky benchmark case.
- **case 2:** \( b > -\alpha K / \beta > 0 \), the safe benchmark case

(There’ll be further classification later within the two cases)

For the two cases, I derived closed-form solution for \( S_t \), which is a function of \( \delta_t \) and \( N(\cdot) \). The solution is similar to the Black-Scholes formula for option pricing.
Risk Premium and Volatility: Solutions

Apply Ito’s lemma on $S_t$,

**Proposition 2.**

In the normal economy, for $t \in [0, T']$,

\[
\bar{\sigma}_t = \sigma \delta \\
\bar{\mu}_t - r = \sigma^2 \delta
\]

(24) \hspace{1cm} (25)

In the benchmarking economy, for $t \in [T, T']$,

\[
\sigma_t = \bar{\sigma}_t = \sigma \delta \\
\mu_t - r = \bar{\mu}_t - r = \sigma^2 \delta
\]

(26) \hspace{1cm} (27)
Proposition 2 Continued

For $t \in [0, T)$,

**case 1:**

$$\sigma_t = \left[1 - \frac{be^{-r(T-t)} N(\tilde{z}_2) + Ke^{-r(T-t)} N(-z_2)}{S_t}\right] \sigma_\delta$$

$$\mu_t - r = \left[1 - \frac{be^{-r(T-t)} N(\tilde{z}_2) + Ke^{-r(T-t)} N(-z_2)}{S_t}\right] \sigma_\delta^2$$

(28)

**case 2:**

$$\sigma_t = \left[1 - \frac{be^{-r(T-t)} [N(\tilde{z}_2) - N(\bar{z}_2)] + Ke^{-r(T-t)} N(-\bar{z}_2)}{S_t}\right] \sigma_\delta$$

$$\mu_t - r = \left[1 - \frac{be^{-r(T-t)} [N(\tilde{z}_2) - N(\bar{z}_2)] + Ke^{-r(T-t)} N(-\bar{z}_2)}{S_t}\right] \sigma_\delta^2$$

(29)
Risk Premium and Volatility: Results

Summarizing the above results, for \( t \in [0, T) \) we have:

**Corollary 1.**

- For **case 1:** the volatility and risk premium in the benchmarking economy are greater than those in the normal economy if \( b < -N(-z_2)K/N(\tilde{z}_2) \) and vice versa.

- For **case 2:** the volatility and risk premium in the benchmarking economy are always smaller than those in the normal economy.

See the next part for intuition.
Optimal Strategy: Solutions

Denote the optimal fraction of wealth invested in the stock by $\hat{\Phi}_{nt}$ and $\hat{\Phi}_{mt}$. Apply Ito’s lemma on $\pi_t \hat{X}_{nt}$ and $\pi_t \hat{X}_{mt}$.

**Proposition 3.**

In both case 1 and 2, for $t \in [0, T']$,

$$\hat{\Phi}_{nt} = (\mu_t - r)/\sigma_t^2$$

While for $\hat{\Phi}_{mt}$, when $t \in [T, T']$,

$$\hat{\Phi}_{mt} = (\mu_t - r)/\sigma_t^2$$
Proposition 3 Continued

For $t \in [0, T)$,

**case 1:**

$$\hat{\Phi}_{mt} = [1 - \frac{be^{-r(T-t)}N(\tilde{z}_2) + Ke^{-r(T-t)}N(-z_2)}{\hat{X}_{mt}}] \frac{\mu_t - r}{\sigma_t^2} \quad (32)$$

**case 2:**

$$\hat{\Phi}_{mt} = \left\{1 - \frac{be^{-r(T-t)}[N(\bar{z}_2) - N(\tilde{z}_2)] + Ke^{-r(T-t)}N(-\bar{z}_2)}{\hat{X}_{mt}} \right\} \frac{\mu_t - r}{\sigma_t^2} \quad (33)$$
Optimal Strategy: Results

Summarizing the above results and using results for $\mu_t - r$ and $\sigma_t$, for $t \in [0, T)$ we have:

**Corollary 2.**

- For **case 1**: the constrained agent invests more fraction of wealth in the risky asset than the normal agent if $b < -N(-z_2)K/N(\tilde{z}_2)$ and vice versa.
- For **case 2**: the constrained agent always invests less fraction of wealth in the risky asset than the normal agent.
Discussion of Equilibrium: Main Results

Proposition 4.

▶ In an economy with a very risky benchmark such that \( b < \min[-N(-z_2)K/N(\tilde{z}_2), -\alpha K/\beta] \), the stock price, risk premium and volatility are increased by the presence of the portfolio benchmarking constraint, the constrained agent invests more fraction of wealth in the stock than the unconstrained agent and the equilibrium effect is state-dependent.

▶ In an economy with a safe benchmark such that \( b > 0 \), the stock price is increased but the risk premium and volatility are decreased by the presence of the portfolio benchmarking constraint, the constrained agent invests less fraction of wealth in the stock than the unconstrained agent.
Explaining by the Optimal Strategy

The effects on volatility can be explained by the unconstrained agent $n$’s optimal strategy.

$$\hat{\Phi}_{nt} = \frac{(\mu_t - r)}{\sigma_t^2} = \frac{\sigma_\delta}{\sigma_t}$$  \hspace{1cm} (34)

- For very risky benchmarks
- For relatively safe benchmarks
Explaining by Stock Price

\[
S_t = S_t + \frac{1}{\pi_t} E[\pi_T - \max\{\max[ (a - \frac{\lambda_n}{\lambda_n + \lambda_{m1}} ) S_T + b, 0 ], \max[K - \frac{\lambda_n}{\lambda_n + \lambda_{m1}} S_T, 0]\} | \mathcal{F}_t]
\]

(35)

- The effect is a combination of effect from the benchmark index constraint and effect from the floor constraint.

- For a given risky benchmark index, the overall effect is more likely to be volatility-increasing in good states and volatility-decreasing in bad states.
A Special Case: Pure Benchmarking ($K = 0$)

If $K = 0$, the constrained agent effectively only has to compete with the stochastic benchmark $aS_T + b$. Replace the above results with $K = 0$, we have:

**Proposition 5.** If $K = 0$,

- In an economy with a risky benchmark such that $b < 0$, the stock price, the risk premium and volatility are increased by the presence of the benchmarking constraint.
- In an economy with a safe benchmark such that $b > 0$, the stock price is increased but the risk premium and volatility are decreased by the presence of the portfolio benchmarking constraint.
Conclusion

- In the portfolio benchmarking economy before the constraint date, the risky asset price is higher than that in the normal economy.
- When the benchmark is chosen to be a very risky one, the risk premium and volatility are increased in the presence of the constraint and the effect is state-dependent.
- A constraint with a given risky benchmark is more likely to be volatility-increasing in good states and volatility-decreasing in bad states.
- If the benchmark is relatively safe, the risk premium and volatility are always decreased by the constraint.