Debt and Incomplete Financial Markets: A Case for Nominal GDP Targeting*

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Abstract

Financial markets are incomplete, thus for many households borrowing is possible only by accepting a financial contract that specifies a fixed repayment. However, the future income that will repay this debt is uncertain, so risk can be inefficiently distributed. This paper argues that a monetary policy of nominal GDP targeting can improve the functioning of incomplete financial markets when incomplete contracts are written in terms of money. By insulating households' nominal incomes from aggregate real shocks, this policy effectively completes financial markets by stabilizing the ratio of debt to income. The paper argues the objective of replicating complete financial markets should receive substantial weight even in an environment with other frictions that have been used to justify a policy of strict inflation targeting.

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1 Introduction

At the heart of any argument for a monetary policy strategy lies a view of what are the most important frictions or market failures that monetary policy should seek to mitigate. The canonical justification for inflation targeting as optimal monetary policy rests on the argument that pricing frictions in goods markets are of particular concern (see, for example, Woodford, 2003). With infrequent price adjustment owing to menu costs or other nominal rigidities, high or volatile inflation leads to relative price distortions that impair the efficient operation of markets, and which directly consumes time and resources in the process of setting prices. Inflation targeting is the appropriate policy response to such frictions because it is able to move the economy closer to, or even replicate, what the equilibrium would be if prices were flexible. In other words, inflation targeting is able to undo or partially circumvent the frictions created by nominal price stickiness.[1]

This paper argues that nominal price stickiness may not be the most serious friction that monetary policy has to contend with. While the use of money as a unit of account in setting infrequently adjusted goods prices is well documented, money's role as a unit of account in writing financial contracts is equally pervasive. Moreover, just as price stickiness means that nominal prices fail to be fully state contingent, financial contracts are typically not contingent on all possible future states of the world, for example, debt contracts that specify fixed nominal repayments. Financial contracts might not be fully contingent for a variety of reasons, but one explanation could be that transaction costs make it prohibitively expensive to write and enforce complicated and lengthy contracts. Many agents, such as households, would find it difficult to issue liabilities with state-contingent repayments resembling equity or derivatives, and must instead rely on simple debt contracts if they are to borrow. Thus, in a similar way to how menu costs can make prices sticky, transaction costs can make financial markets incomplete.

This paper studies the implications for optimal monetary policy of such financial-market incompleteness in the form of non-contingent nominal debt contracts.[2] The argument can be understood in terms of which monetary policy strategy is able to undo or mitigate the adverse consequences of financial-market incompleteness, just as inflation targeting can be understood as a means of circumventing the problem of nominal price stickiness. For both non-contingent nominal financial contracts and nominal price stickiness, it is money's role as a unit of account that is crucial, and in both cases, optimal monetary policy is essentially the choice of a particular nominal anchor that makes money best perform its unit-of-account function. But in spite of this formal similarity, the optimal nominal

[1]In addition to the theoretical case, the more practical merits of implementing inflation targeting are discussed in Bernanke, Laubach, Mishkin and Posen (1999).

[2]It is increasingly argued that monetary policy must take account of financial-market frictions such as collateral constraints or spreads between internal and external finance. These are different from the financial frictions emphasized in this paper. Starting from Bernanke, Gertler and Gilchrist (1999), there is now a substantial body of work that integrates credit frictions of the kind found in Bernanke and Gertler (1989) or Kiyotaki and Moore (1997) into monetary DSGE models. Recent work in this area includes Christiano, Motto and Rostagno (2010). These frictions can magnify the effects of both shocks and monetary policy actions and make these effects more persistent. But the existence of a quantitatively important credit channel does not in and of itself imply that optimal monetary policy is necessarily so different from inflation targeting unless new types of financial shocks are introduced (Faia and Monacelli, 2007, Carlstrom, Fuerst and Paustian, 2010, De Fiore and Tristani, 2012).

anchor turns out to be very different when the friction is financial-market incompleteness rather than sticky prices.

One problem of non-contingent debt contracts for risk-averse households is that when borrowing for long periods, there will be considerable uncertainty about the future income from which fixed debt repayments must be made. The issue is not only idiosyncratic uncertainty — households do not know the future course the economy will take, which will affect their labour income. Will there be a productivity slowdown, a deep and long-lasting recession, or even a 'lost decade' of poor economic performance to come? Or will unforeseen technological developments or terms-of-trade movements boost future incomes, and good economic management successfully steer the economy on a path of steady growth? Borrowers do not know what aggregate shocks are to come, but must fix their contractual repayments prior to this information being revealed.

The simplicity of non-contingent debt contracts can be seen as coming at the price of bundling together two fundamentally different transfers: a transfer of consumption from the future to the present for borrowers, but also a transfer of aggregate risk to borrowers. The future consumption of borrowers is paid for from the difference between their uncertain future incomes and their fixed debt repayments. The more debt they have, the more their future income is effectively leveraged, leading to greater consumption risk. The flip-side of borrowers' leverage is that savers are able to hold a risk-free asset, reducing their consumption risk.

To see the sense in which this bundling together of borrowing and a transfer of risk is inefficient, consider what would happen in complete financial markets. Individuals would buy or sell state-contingent bonds (Arrow-Debreu securities) that make payoffs conditional on particular states of the world (or equivalently, write loan contracts with different repayments across all states of the world). Risk-averse borrowers would want to sell relatively few bonds paying off in future states of the world where GDP and thus incomes are low, and sell relatively more in good states of the world. As a result, contingent bonds paying off in bad states would be relatively expensive and those paying off in good states relatively cheap. These price differences would entice savers to shift away from non-contingent bonds and take on more risk in their portfolios. Given that the economy has no risk-free technology for transferring goods over time, and as aggregate risk cannot be diversified away, the efficient outcome is for risk-averse households to share aggregate risk, and complete markets allow this to be unbundled from decisions about how much to borrow or save.

The efficient financial contract between risk-averse borrowers and savers in an economy subject to aggregate income risk (abstracting from idiosyncratic risk) turns out to have a close resemblance to an 'equity share' in GDP. In other words, borrowers' repayments should fall during recessions and rise during booms. This means the ratio of debt liabilities to GDP should be more stable than it would be in a world of incomplete financial markets where debt liabilities are fixed in value while GDP fluctuates.

With incomplete financial markets, monetary policy has a role to play in mitigating inefficiencies because private debt contracts are typically denominated in terms of money. Hence, the real degree of state-contingency in financial contracts is endogenous to monetary policy. If incomplete markets were the only source of inefficiency in the economy then the optimal monetary policy would aim to make nominally non-contingent debt contracts mimic through variation in the real value of the monetary unit of account the efficient financial contract that would be chosen with complete financial markets.

Given that the efficient financial contract between borrowers and savers resembles an equity share in GDP, it follows that a goal of monetary policy should be to stabilize the ratio of debt liabilities to GDP. With non-contingent nominal debt liabilities, this can be achieved by having a non-contingent level of nominal income, in other words, a monetary policy that targets nominal GDP. Nominal income thus replaces nominal goods prices as the optimal nominal anchor. The intuition is that while the central bank cannot eliminate uncertainty about future real GDP, it can in principle make the level of future nominal GDP (and hence the nominal income of an average household) perfectly predictable. Removing uncertainty about future nominal income thus alleviates the problem of nominal debt repayments being non-contingent.

A policy of nominal GDP targeting is generally in conflict with inflation targeting because any fluctuations in real GDP would lead to fluctuations in inflation of the same size and in the opposite direction. Recessions would feature higher inflation and booms would feature lower inflation, or even deflation. These inflation fluctuations can be helpful because they induce variation in the real value of the monetary unit of account, making it and the non-contingent debt contracts expressed in terms of it behave more like equity. This promotes efficient risk sharing. A policy of strict inflation targeting would fix the real value of the monetary unit of account, converting nominally non-contingent debt into real non-contingent debt, which would imply an uneven and generally inefficient distribution of risk.

The inflation fluctuations that occur with nominal GDP targeting would entail relative-price distortions if goods prices were sticky, so the benefit of efficient risk sharing is most likely not achieved without some cost.[3] It is ultimately a quantitative question whether the inefficiency caused by incomplete financial markets is more important than the inefficiency caused by relative-price distortions, and thus whether nominal GDP targeting is preferable to inflation targeting.

This paper presents a model that allows optimal monetary policy to be studied analytically in an incomplete-markets economy with heterogeneous households and aggregate risk, which can be straightforwardly calibrated for quantitative analysis. The model contains two types of households, relatively impatient households who will choose to become borrowers, and relatively patient households who will choose to become savers. Although households differ in their time preferences, they are all risk averse, and are all exposed to the same labour income risk. Real GDP is uncertain because of aggregate productivity shocks, but there are no idiosyncratic shocks. The economy is

[3]The model features both a trade-off between efficiency in goods markets and efficiency in financial markets, and a trade-off between the volatility of inflation and the volatility of financial-market variables. Such trade-offs are implicit in recent debates, though there is no widely accepted argument for why stabilizing prices in goods markets causes financial markets to malfunction. White (2009b) and Christiano, Ilut, Motto and Rostagno (2010) argue that stable inflation is no guarantee of financial stability, and may even create conditions for financial instability. Contrary to these arguments, the conventional view that monetary policy should not react to asset prices is advocated in Bernanke and Gertler (2001). Woodford (2011) makes the point that flexible inflation targeting can be adapted to accommodate financial stability concerns, and that it would be unwise to discard inflation targeting's role in providing a clear nominal anchor.

assumed to have no investment or storage technology, and is closed to international trade. There are no government bonds and no fiat money, and no taxes or fiscal transfers. In this world, patient households defer consumption by lending to impatient households, who can thus bring forwards consumption by borrowing. It is assumed the only financial contract available is a non-contingent nominal bond. The basic model contains no other frictions, initially assuming prices and wages are fully flexible.

The concept of a 'natural debt-to-GDP ratio' provides a useful benchmark for monetary policy analysis. This is defined as the ratio of (state-contingent) gross debt liabilities to GDP that would prevail were financial markets complete. This object is independent of monetary policy. The actual debt-to-GDP ratio in an economy with incomplete markets would coincide with the natural debt-to-GDP ratio if forecasts of future GDP were always correct ex post, but will in general fluctuate around it when the economy is hit by shocks. The natural debt-to-GDP ratio is thus analogous to concepts such as the natural rate of unemployment and the natural rate of interest.

If all movements in real GDP growth rates are unpredictable then the natural debt-to-GDP ratio turns out to be constant (or if utility functions are logarithmic, the ratio is constant irrespective of the statistical properties of GDP growth). Even when the natural debt-to-GDP ratio is not completely constant, plausible calibrations suggest it would have a low volatility relative to real GDP itself.

Since the equilibrium of an economy with complete financial markets would be Pareto efficient in the absence of other frictions, the natural debt-to-GDP ratio also has desirable welfare properties. A goal of monetary policy in an incomplete-markets economy is therefore to close the 'debt gap', defined as the difference between the actual and natural debt-to-GDP ratios. It is shown that doing so effectively 'completes the market' in the sense that the equilibrium with incomplete markets would then coincide with the hypothetical complete-markets equilibrium. Monetary policy can affect the actual debt-to-GDP ratio and thus the debt gap because that ratio is equal to nominal debt liabilities (which are non-contingent with incomplete markets) divided by nominal GDP, where the latter is under the control of monetary policy.

When the natural debt-to-GDP ratio is constant, or when the maturity of debt contracts is sufficiently long, closing the debt gap can be achieved by adopting a fixed target for the growth rate (or level) of nominal GDP. With this logic, the central bank uses nominal GDP as an intermediate target that achieves its ultimate goal of closing the debt gap. This turns out to be preferable to targeting the debt-to-GDP ratio directly because a monetary policy that targets only a real financial variable would leave the economy without a nominal anchor. Nominal GDP targeting uniquely pins down the nominal value of real incomes and thus provides the economy with a well-defined nominal anchor.

It is important to note that in an incomplete-markets economy hit by shocks, whatever action a central bank takes or fails to take will have distributional consequences. Ex post, there will be winners and losers from both the shocks themselves and the policy responses. Creditors lose out when inflation is unexpectedly high, while debtors suffer when inflation is unexpectedly low. It might then be thought surprising that inflation fluctuations would ever be desirable. However, the inflation fluctuations implied by a nominal GDP target are not arbitrary fluctuations — they are perfectly correlated with the real GDP fluctuations that are the ultimate source of uncertainty in the economy, and which themselves have distributional consequences when households are heterogeneous. For households to share risk, it must be possible to make transfers ex post that act as insurance from an ex-ante perspective. The result of the paper is that ex-ante efficient insurance requires inflation fluctuations that are negatively correlated with real GDP (a countercyclical price level) to generate the appropriate ex-post transfers between debtors and creditors.

It might be objected that there are infinitely many state-contingent consumption allocations which would equally well satisfy the criterion of ex-ante efficiency. However, only one of these — the hypothetical complete-markets equilibrium associated with the natural debt-to-GDP ratio — could ever be implemented through monetary policy. Thus for a policymaker solely interested in promoting efficiency, there is a unique optimal policy that does not require any explicit distributional preferences to be introduced.

The model also makes predictions about how different monetary policies will affect the volatility of financial-market variables such as the debt-to-GDP ratio. It is shown that policies implying an inefficient distribution of risk, for example, inflation targeting, are associated with near-random walk fluctuations in the debt-to-GDP ratio. On the other hand, with complete financial markets, the persistence of fluctuations in the debt-to-GDP ratio would be bounded by the persistence of shocks to real GDP growth. When a monetary policy is adopted that allows the economy to mimic complete financial markets, the actual debt-to-GDP ratio inherits these less persistent dynamics.

In a model with both nominal price rigidities and incomplete financial markets, these findings allow the tension between relative-price distortions and efficient risk sharing to be seen in more familiar terms as a trade-off between price stability and financial stability. Determining which of these objectives is the more important in practice can be done by studying a quantitative version of the model. Nominal price rigidity is introduced using the standard Calvo model of staggered price adjustment. With both incomplete financial markets and sticky prices, optimal monetary policy is a convex combination of the optimal monetary policies that are appropriate for each of the two frictions in isolation. After calibrating all the parameters of the model, the conclusion is that replicating complete financial markets should receive approximately 88% of the weight.

This paper is related to a number of areas of the literature on monetary policy and financial markets. First, there is the empirical work of Bach and Stephenson (1974), Cukierman, Lennan and Papadia (1985), and more recently, Doepke and Schneider (2006), who document the effects of inflation in redistributing wealth between debtors and creditors. The novelty here is in studying the implications for optimal monetary policy in an environment where inflation fluctuations with such distributional effects may actually be desirable precisely because of the incompleteness of financial markets.

The basic idea of this paper (though not the modelling or quantitative analysis) has many precedents in the literature. Selgin (1997) describes the ex-ante efficiency advantages of falling prices in good times and rising prices in bad times when financial contracts are non-contingent, though there is no formal modelling of the argument.[4] A survey reviewing the long history of this idea in monetary economics is given in Selgin (1995). In recent work, Koenig (2013) advances the risk-sharing argument for nominal GDP targeting in the context of a two-period model.[5] An earlier theoretical paper is Pescatori (2007), who studies optimal monetary policy in an economy with rich and poor households, in the sense of there being an exogenously specified distribution of assets among otherwise identical households. In that environment, both inflation and interest rate fluctuations have redistributional effects on rich and poor households, and the central bank optimally chooses the mix between them (there is a need to change interest rates because prices are sticky, with deviations from the natural rate of interest leading to undesirable fluctuations in output). A related paper is Lee (2010), who develops a model where heterogeneous households choose less than complete consumption insurance because of the presence of convex transaction costs in accessing financial markets. Inflation fluctuations expose households to idiosyncratic labour-income risk because households work in specific sectors of the economy, and sectoral relative prices are distorted by inflation when prices are sticky. This leads optimal monetary policy to put more weight on stabilizing inflation. Differently from those two papers, the argument here is that inflation fluctuations can actually play a positive role in completing otherwise incomplete financial markets.[6]

The idea that inflation fluctuations may have a positive role to play when financial markets are incomplete is now long-established in the literature on government debt (and has also been recently applied by Allen, Carletti and Gale (2011) in the context of the real value of the liquidity available to the banking system). Bohn (1988) developed the theory that non-contingent nominal government debt can be desirable because when combined with a suitable monetary policy, inflation can change the real value of the debt in response to fiscal shocks that would otherwise require fluctuations in distortionary tax rates.[7]

Quantitative analysis of optimal monetary policy of this kind was developed in Chari, Christiano and Kehoe (1991) and expanded further in Chari and Kehoe (1999). One finding was that inflation needs to be extremely volatile to complete financial markets. As a result, Schmitt-Grohé and Uribe (2004) and Siu (2004) argued that once some nominal price rigidity is considered so that

[4] Persson and Svensson (1989) is an early example of a model — in the context of an international portfolio allocation problem — where it is important how monetary policy affects the risk characteristics of nominal debt.

[5] Hoelle and Peiris (2013) study the efficiency properties of nominal GDP targeting in a large open economy with flexible prices, and explore the question of implementability through the central bank's balance sheet. Osorio-Rodríguez (2013) examines whether there remains a role for monetary policy in completing financial markets when nominal debt is denominated in terms of foreign currency.

[6] In other related work on incomplete markets and monetary policy, Akyol (2004) analyses optimal monetary policy in an incomplete-markets economy where households hold fiat money for self insurance against idiosyncratic shocks. Kryvtsov, Shukayev and Ueberfeldt (2011) study an overlapping generations model with fiat money where monetary policy can improve upon the suboptimal level of saving by varying the expected inflation rate and thus the returns to holding money.

[7] There is also a literature that emphasizes the impact of monetary policy on the financial positions of firms or entrepreneurs in an economy with incomplete financial markets. De Fiore, Teles and Tristani (2011) study a flexible-price economy where there is a costly state verification problem for entrepreneurs who issue short-term nominal bonds. Andrés, Arce and Thomas (2010) consider entrepreneurs facing a binding collateral constraint who issue short-term nominal bonds with an endogenously determined interest rate spread. Vlieghe (2010) also has entrepreneurs facing a collateral constraint, and even though they issue real bonds, monetary policy still has real effects on the wealth distribution because prices are sticky, so incomes are endogenous. In these papers, the wealth distribution matters because of its effects on the ability of entrepreneurs to finance their operations.

inflation fluctuations have a cost, the optimal policy becomes very close to strict inflation targeting. This paper shares the focus of that literature on using inflation fluctuations to complete markets, but comes to a different conclusion regarding the magnitude of the required inflation fluctuations and whether the costs of those fluctuations outweigh the benefits. First, the benefits of completing markets in this paper are linked to the degree of risk aversion and the degree of heterogeneity among households, which are in general unrelated to the benefits of avoiding fluctuations in distortionary tax rates, and which prove to be large in the calibrated model. Second, the earlier results assumed government debt with a very short maturity. With longer maturity debt (household debt in this paper), the costs of the inflation fluctuations needed to complete the market are much reduced.[8]

This paper is also related to the literature on household debt. Iacoviello (2005) examines the consequences of household borrowing constraints in a DSGE model, while Guerrieri and Lorenzoni (2011) and Eggertsson and Krugman (2012) study how a tightening of borrowing constraints for indebted households can push the economy into a liquidity trap. Differently from those papers, the focus here is on the implications of household debt for optimal monetary policy. Furthermore, the finding here that the presence of household debt substantially changes optimal monetary policy does not depend on there being borrowing constraints, or even the feedback effects from debt to aggregate output stressed in those papers. Cúrdia and Woodford (2009) also study optimal monetary policy in an economy with household borrowing and saving, but the focus there is on spreads between interest rates for borrowers and savers, while their model assumes an insurance facility that rules out the risk-sharing considerations studied here. Finally, the paper is related to the literature on nominal GDP targeting (Meade, 1978, Bean, 1983, Hall and Mankiw, 1994, and more recently, Sumner, 2012) but proposes a different argument in favour of that policy.

The plan of the paper is as follows. Section 2 sets out the basic model and derives the equilibrium conditions. Optimal monetary policy is studied in section 3 alongside the consequences of sub-optimal monetary policies. Section 4 introduces sticky prices and hence a trade-off between mitigating the incompleteness of financial markets and avoiding relative-price distortions. Section 5 presents an extension where incomplete financial markets are embedded into a full New Keynesian model in which monetary policy can also affect market incompleteness through 'financial repression', that is, changes in ex-ante real interest rates rather than ex-post real returns through inflation. Section 6 shows how the full model can be calibrated and presents a quantitative analysis of optimal monetary policy. Finally, section 7 draws some conclusions.

2 A model of a pure credit economy

There is an economy containing a measure-one population of households. Time is discrete and households are infinitely lived. There are equal numbers of two types of households, referred to as 'borrowers' and 'savers' and indexed by type $\tau \in \{\mathrm{b}, \mathrm{s}\}$.

[8]This point is made by Lustig, Sleet and Yeltekin (2008) in the context of government debt.

2.1 Preferences

A representative household of type τ has preferences given by the following utility function

$$\mathscr{U}_{\tau,t} = \sum_{\ell=0}^{\infty} \mathbb{E}_t \left[\left\{ \prod_{\jmath=0}^{\ell-1} \delta_{\tau,t+\jmath} \right\} \frac{C_{\tau,t+\ell}^{1-\alpha}}{1-\alpha} \right], \tag{2.1}$$

where $C_{\tau,t}$ is per-household consumption of a composite good by type-τ households at time t. The two types are distinguished by their subjective discount factors, with $\delta_{\tau,t}$ being the discount factor of type-τ households between time t and $t+1$. Both types have a constant coefficient of relative risk aversion given by α ($0 < \alpha < \infty$), and a constant elasticity of intertemporal substitution given by α^{-1}.[9]

Each household of type τ receives real income $Y_{\tau,t}$ at time t, to be specified below. The discount factor $\delta_{\tau,t}$ of type-τ households is assumed to be the following:

$$\delta_{\tau,t} = \delta_\tau \left(\frac{C_{\tau,t}}{Y_{\tau,t}} \right), \quad \text{where} \;\; \delta_\tau(c) = \Delta_\tau c^{-(1-\lambda)\alpha}, \tag{2.2}$$

and where the parameters Δ_{b}, Δ_{s}, and λ are such that $0 < \Delta_{\text{b}} < \Delta_{\text{s}} < \infty$ and $0 < \lambda < 1$. It is assumed individual households of type τ take $\delta_{\tau,t}$ as given, that is, they do not internalize the effect of their own consumption on the discount factor.

There are two differences compared to a representative-household model with a standard time-separable utility function. First, there is heterogeneity in discount factors because 'borrowers' are more impatient than 'savers' ($\Delta_{\text{b}} < \Delta_{\text{s}}$), all else equal. This is the key assumption that will give rise to borrowing and saving in equilibrium by the households that have been referred to as 'borrowers' and 'savers'. Second, discount factors display the marginal increasing impatience ($\lambda < 1$) property of Uzawa (1968), in that the discount factor is lower when consumption is higher (relative to income), all else equal. This assumption is invoked for technical reasons because it ensures the wealth distribution will be stationary around a well-defined non-stochastic steady state.[10] That households take discount factors as given is assumed for simplicity and is analogous to models of 'external' habits (see for example, Abel, 1990).[11]

2.2 Incomes

To begin with, real GDP Y_t is modelled as an exogenous endowment. The case of a production economy is developed later, but the main results do not depend on real GDP being endogenous, nor on any feedback from financial markets to real GDP. The real GDP growth rate $g_t = (Y_t - Y_{t-1})/Y_{t-1}$

[9] Risk aversion can be separated from intertemporal substitution with the utility function of Epstein and Zin (1989) and Weil (1989). The consequences of this extension were explored in an earlier working paper (Sheedy, 2013).

[10] None of the qualitative results depends on λ being significantly below one ($\lambda = 1$ is the standard case of fixed discount factors), and moreover, the quantitative importance of the results does not vanish when λ is arbitrarily close to one. The assumption is analogous to those employed in small open-economy models to ensure a stationary net foreign asset position (see Schmitt-Grohé and Uribe, 2003). An alternative is to work with an overlapping generations model where the utility function is entirely standard, which automatically has a stationary wealth distribution because households have finite lives. This avenue was explored in an earlier working paper (Sheedy, 2013).

[11] The assumption can be relaxed at the cost of more complicated algebra, but there is little impact on the results when λ is close to one. The changes to the results are described in appendix A.17.

is given by:

$$g_t = \bar{g} + \varsigma z_t, \quad \text{where} \;\; z_t \in [\underline{z}, \overline{z}], \quad \mathbb{E}z_t = 0, \quad \text{and} \;\; \mathbb{E}z_t^2 = 1, \qquad [2.3]$$

with z_t being a stationary stochastic process with bounded support. The real GDP growth rate has mean $\bar{g}$ and standard deviation ς. The parameters α, Δ_s, and $\bar{g}$ are assumed to satisfy the restriction $\Delta_s(1+\bar{g})^{1-\alpha} < 1$, which ensures that the utility of all households remains finite.

It is assumed for simplicity that the only difference between 'borrowers' and 'savers' is in their patience.[12] Both types are thus assumed to receive the same income:

$$Y_{\tau,t} = Y_t. \qquad [2.4]$$

2.3 Money

The economy is 'cash-less' in that money is not required for transactions, but money is used as a unit of account in writing financial contracts and in pricing goods. One unit of the composite good costs P_t units of money at time t, and $\pi_t = (P_t - P_{t-1})/P_{t-1}$ denotes the inflation rate of goods prices between $t-1$ and t. In the model, money is synonymous with interest-bearing reserves issued by the central bank. Reserves held between period t and $t+1$ pay a known nominal interest rate i_t.

2.4 Incomplete financial markets

Financial markets are incomplete. Households cannot sell state-contingent bonds, and in equilibrium, no such securities will be available to buy. The only liability that can be issued is a non-contingent nominal bond. Households can take positive or negative positions in this bond (save or borrow), and there is no limit on borrowing other than being able to repay in all states of the world given non-negativity constraints on consumption. With this restriction, no default will occur, and thus bonds are risk-free in nominal terms.[13] Furthermore, given the finite support of the income growth stochastic process [2.3] and the desire for borrowing as determined by the patience parameters Δ_τ, by ensuring that the standard deviation ς of real GDP growth is not too large, the natural borrowing limit will not be binding in equilibrium.[14]

The nominal bond has the following structure. One newly issued bond at time t makes a stream of coupon payments in subsequent time periods, paying 1 unit of money (a normalization) at time $t+1$, then γ units at $t+2$, γ^2 at $t+3$, and so on ($0 \leq \gamma < \infty$). The geometric structure of the coupon payments means that a bond issued at time $t-\ell$ is after its time-t coupon payment equivalent to a quantity γ^ℓ of new date-t bonds. If Q_t denotes the price in terms of money at time t of one new bond then the absence of arbitrage opportunities requires that bonds issued at date

[12] It is possible to derive similar results in the case where borrowing occurs because of differences in income over the life cycle, with no differences in the degree of impatience. This version of the model was studied in an earlier working paper (Sheedy, 2013).

[13] The model abstracts from the choice of default when repayment is feasible.

[14] It is common to assume that borrowers are subject to a tighter constraint than the natural borrowing limit, and furthermore, in many models, this tighter borrowing constraint is always binding (Iacoviello, 2005). To focus carefully on the implications of the lack of state-contingent forms of borrowing per se, this paper abstracts from additional restrictions on the ability of households to smooth consumption over time.

$t-\ell$ have price $\gamma^\ell Q_t$ at time t. It therefore suffices to track the overall quantity of bonds in terms of new-bond equivalents, rather than the quantities of each vintage separately.[15]

The flow budget identity at time t of households of type τ is:

$$C_{\tau,t} + \frac{Q_t B_{\tau,t}}{P_t} + \frac{M_{\tau,t}}{P_t} = Y_{\tau,t} + \frac{(1+\gamma Q_t)B_{\tau,t-1}}{P_t} + (1+i_{t-1})\frac{M_{\tau,t-1}}{P_t}, \qquad [2.5]$$

where $B_{\tau,t}$ denotes the outstanding quantity of bonds (in terms of new-bond equivalents) held (or issued, if negative) by type-τ households at the end of period t. The term $1+\gamma Q_t$ refers to the coupon payment plus the resale value of bonds acquired or issued in the past. Holdings of money (interest-bearing reserves) are denoted by $M_{\tau,t}$, which are subject to the non-negativity constraint $M_{\tau,t} \geq 0$ because reserves cannot be issued by households.

The Euler equation for maximizing the utility [2.1] of type-$\tau \in \{\mathrm{b}, \mathrm{s}\}$ households with respect to bond holdings $B_{\tau,t}$ subject to the flow budget identity [2.5] is:

$$C_{\tau,t}^{-\alpha} = \delta_{\tau,t}\mathbb{E}_t\left[\frac{(1+\gamma Q_{t+1})P_t}{Q_t P_{t+1}} C_{\tau,t+1}^{-\alpha}\right]. \qquad [2.6]$$

The Euler equation with respect to money holdings $M_{\tau,t}$ is:

$$C_{\tau,t}^{-\alpha} \geq \delta_{\tau,t}(1+i_t)\mathbb{E}_t\left[\frac{P_t}{P_{t+1}} C_{\tau,t+1}^{-\alpha}\right], \quad \text{with equality if } M_{\tau,t} > 0. \qquad [2.7]$$

If $B_{\tau,t+\ell}$ remains positive as $\ell \to \infty$ for some $\tau \in \{\mathrm{b}, \mathrm{s}\}$ then (since $M_{\tau,t} \geq 0$) the following transversality condition must hold in all those states of the world:

$$\lim_{\ell\to\infty} \left\{\prod_{\jmath=0}^{\ell-1} \delta_{\tau,t+\jmath}\right\} \frac{C_{\tau,t+\ell}^{-\alpha} Q_{t+\ell} B_{\tau,t+\ell}}{P_{t+\ell}} = 0. \qquad [2.8]$$

This condition rules out accumulating financial wealth that will not ultimately be spent. Finally, it is assumed that bond prices are 'bubble-free' in the sense of satisfying the transversality condition:

$$\lim_{\ell\to\infty} \gamma^\ell \mathbb{E}_t\left[\left\{\prod_{\jmath=0}^{\ell-1} \delta_{\tau,t+\jmath}\right\} \frac{C_{\tau,t+\ell}^{-\alpha} Q_{t+\ell}}{P_{t+\ell}}\right] = 0. \qquad [2.9]$$

2.5 Monetary policy

Specific monetary policies will be analysed subsequently, but first some restrictions are placed on the class of policies under consideration. Only conventional monetary policies are studied, namely those where the central bank aims to influence a nominal variable such as bond prices Q_t or goods prices P_t using as an instrument the short-term nominal interest rate i_t paid on reserves. The central bank does not engage in intermediation of private-sector credit and does not hold privately issued securities on its balance sheet (in equilibrium). Furthermore, it is assumed there is no fiscal authority in the economy, so the central bank does not hold government bonds. The analysis thus excludes the use of the central bank's balance sheet as a separate policy instrument. Letting M_t denote the supply of reserves and B_t central-bank net bond purchases, all monetary policies must

[15] Woodford (2001) uses this modelling device to study long-term government debt. See Garriga, Kydland and Šustek (2013) for a richer model of mortgage contracts.

feature the following balance sheet in equilibrium:[16]

$$M_t = 0, \quad \text{and} \;\; B_t = 0. \tag{2.10}$$

Monetary policy is also assumed to be conducted so that households holding bonds are indifferent between those bonds and interest-bearing reserves.[17] Thus, for τ with $B_{\tau,t} > 0$, the Euler equation for reserves [2.7] is:

$$C_{\tau,t}^{-\alpha} = \delta_{\tau,t}\mathbb{E}_t\left[\left(\frac{1+i_t}{1+\pi_{t+1}}\right)C_{\tau,t+1}^{-\alpha}\right], \tag{2.11}$$

where the definition of inflation π_t has been used. Monetary policy is also assumed to satisfy the restrictions that the interest rate i_t paid on reserves is a stationary stochastic process in equilibrium, and $\gamma < 1 + \bar{i}$, where $\bar{i}$ is the value of i_t in the absence of shocks.

2.6 Equilibrium

It is assumed that consumption is the only source of final demand (no investment, government spending, or international trade). Goods-market clearing thus requires:

$$C_t = Y_t, \quad \text{where} \;\; C_t = \frac{1}{2}C_{\mathrm{b},t} + \frac{1}{2}C_{\mathrm{s},t}, \tag{2.12}$$

with C_t denoting average per-household consumption across the measure $1/2$ of each type of household. With the central-bank balance sheet given by [2.10], bond- and money-market clearing require:

$$\frac{1}{2}B_{\mathrm{b},t} + \frac{1}{2}B_{\mathrm{s},t} = 0; \;\; \text{and} \tag{2.13}$$

$$\frac{1}{2}M_{\mathrm{b},t} + \frac{1}{2}M_{\mathrm{s},t} = 0, \quad \text{hence} \;\; M_{\tau,t} = 0, \tag{2.14}$$

where the second part of the second line follows because of the non-negativity constraints $M_{\tau,t} \geq 0$.

Now define the following variables D_t ('debt'), L_t ('loans'), and r_t ('real return'):

$$D_t = -\frac{1}{2}\frac{(1+\gamma Q_t)B_{\mathrm{b},t-1}}{P_t}, \quad L_t = -\frac{1}{2}\frac{Q_t B_{\mathrm{b},t}}{P_t}, \quad \text{and} \;\; 1 + r_t = \frac{(1+\gamma Q_t)P_{t-1}}{Q_{t-1}P_t}. \tag{2.15}$$

Assuming, as will be confirmed in equilibrium, that $B_{\mathrm{b},t} < 0$, the variable D_t represents the real value of gross debt liabilities per household (borrowers make up $1/2$ of the population) outstanding at the beginning of period t, and L_t represents the real value of ongoing loans (per household) at the end of period t. The variable r_t is defined as the ex-post real holding-period return on these loans between $t-1$ and t. The definitions in [2.15] imply the following accounting identity for debt dynamics:

$$D_t = (1 + r_t)L_{t-1}. \tag{2.16}$$

The flow budget identities [2.5] together with the assumptions on incomes in [2.4], the definitions

[16] Off the equilibrium path, the central bank may issue reserves and buy or sell securities if, for example, the bond price Q_t differed from its target, but the model abstracts from such implementation issues.

[17] This condition places no actual restrictions on what policy can achieve because the central bank can always choose to operate entirely in the market for long-term bonds if no households were willing to hold reserves. It is imposed simply because it is conventional to think of central banks as operating by setting short-term nominal interest rates.

of debt and loans in [2.15], the bond-market equilibrium condition [2.13] (hence $B_{\mathrm{s},t} = -B_{\mathrm{b},t}$), and the money-market equilibrium condition [2.14] imply:

$$C_{\mathrm{b},t} = Y_t - 2(D_t - L_t), \quad \text{and} \quad C_{\mathrm{s},t} = Y_t + 2(D_t - L_t). \qquad [2.17]$$

The Euler equations [2.6] can be stated in terms of expectations of the real return r_t using [2.15]:

$$C_{\tau,t}^{-\alpha} = \delta_{\tau,t}\mathbb{E}_t\left[(1+r_{t+1})C_{\tau,t+1}^{-\alpha}\right]. \qquad [2.18]$$

With the definition of loans L_t in [2.15] and the bond-market clearing condition [2.13], the transversality condition [2.8] can be stated as:

$$\lim_{\ell\to\infty}\left\{\prod_{\jmath=0}^{\ell-1}\delta_{\tau,t+\jmath}\right\}C_{\tau,t+\ell}^{-\alpha}L_{t+\ell} = 0, \qquad [2.19]$$

which must hold for both types τ and in all states of the world.

Rather than consider levels of consumption and debt directly, it is convenient to analyse these variables relative to real GDP Y_t. The debt-to-GDP ratio d_t, the loans-to-GDP ratio l_t, and the consumption-to-income ratios $c_{\tau,t}$ are defined as follows:

$$d_t = \frac{D_t}{Y_t}, \quad l_t = \frac{L_t}{Y_t}, \quad \text{and} \quad c_{\tau,t} = \frac{C_{\tau,t}}{Y_t}. \qquad [2.20]$$

Similarly, rather than consider the bond price Q_t directly, it is convenient to work with the yield-to-maturity $\jmath_t$. For any finite Q_t, the bond yield $\jmath_t$ is the solution of the following equation:

$$Q_t = \sum_{\ell=1}^{\infty}\frac{\gamma^{\ell-1}}{(1+\jmath_t)^{\ell}}, \quad \text{implying} \quad \jmath_t = \frac{1}{Q_t} - 1 + \gamma. \qquad [2.21]$$

With these definitions in hand, the full set of equilibrium conditions is collected below:

$$\rho_t = \mathbb{E}_t r_{t+1}; \qquad [2.22\text{a}]$$

$$d_t = \left(\frac{1+r_t}{1+g_t}\right) l_{t-1}; \qquad [2.22\text{b}]$$

$$c_{\mathrm{b},t} = 1 - 2(d_t - l_t), \quad \text{and} \quad c_{\mathrm{s},t} = 1 + 2(d_t - l_t); \qquad [2.22\text{c}]$$

$$1 = \delta_{\tau,t}\mathbb{E}_t\left[(1+r_{t+1})(1+g_{t+1})^{-\alpha}\left(\frac{c_{\tau,t+1}}{c_{\tau,t}}\right)^{-\alpha}\right]; \qquad [2.22\text{d}]$$

$$\delta_{\tau,t} = \Delta_\tau c_{\tau,t}^{-(1-\lambda)\alpha}; \qquad [2.22\text{e}]$$

$$\lim_{\ell\to\infty}\left\{\prod_{\jmath=0}^{\ell-1}\delta_{\tau,t+\jmath}(1+g_{t+1+\jmath})^{1-\alpha}\right\}c_{\tau,t+\ell}^{-\alpha}l_{t+\ell} = 0. \qquad [2.22\text{f}]$$

Equation [2.22a] is the definition of the real interest rate ρ_t (the ex-ante expected real return on bonds between t and $t+1$). Equation [2.22b] is the accounting identity for the debt-to-GDP ratio, which follows from [2.16], [2.20], and the definition of g_t. Equations [2.22c], [2.22d], [2.22e], and [2.22f] give the budget identities [2.17], the Euler equations [2.18], the discount factors [2.2], and the transversality condition [2.19] in terms of the variables defined in [2.20]. Finally, there is

$$\frac{1}{2}c_{\mathrm{b},t} + \frac{1}{2}c_{\mathrm{s},t} = 1, \qquad [2.23]$$

which is the goods-market clearing condition [2.12] (redundant by Walras' law, see [2.22c]).

Monetary policy is described by an as-yet-unspecified policy rule for the short-term nominal interest rate i_t and equation [2.11] requiring indifference between holding bonds or reserves:

$$1 = \delta_{\tau,t}\mathbb{E}_t\left[\left(\frac{1+i_t}{1+\pi_{t+1}}\right)(1+g_{t+1})^{-\alpha}\left(\frac{c_{\mathrm{s},t+1}}{c_{\mathrm{s},t}}\right)^{-\alpha}\right]. \qquad [2.24]$$

The remaining equilibrium conditions relate to the nominal bonds traded in the economy:

$$1 + r_t = \left(\frac{1+j_t}{1+\pi_t}\right)\left(\frac{1-\gamma+j_{t-1}}{1-\gamma+j_t}\right); \qquad [2.25a]$$

$$\lim_{\ell\to\infty}\gamma^{\ell}\mathbb{E}_t\left[\left\{\prod_{\jmath=1}^{\ell}\delta_{\tau,t+\jmath-1}\frac{(1+g_{t+\jmath})^{-\alpha}}{(1+\pi_{t+\jmath})}\left(\frac{c_{\tau,t+\jmath}}{c_{\tau,t+\jmath-1}}\right)^{-\alpha}\right\}(1-\gamma+j_{t+\ell})^{-1}\right] = 0. \qquad [2.25b]$$

Equation [2.25a] is derived from [2.15] using the price-yield relationship [2.21] and the definition of inflation π_t, and [2.25b] expresses the 'no-bubbles' transversality condition [2.9] in terms of the bond yield.[18] The asset-pricing equation that determines the bond yield j_t is implied by the consumption Euler equations [2.22d] together with [2.25a]:

$$(1-\gamma+j_t)^{-1} = \mathbb{E}_t\left[\delta_{\tau,t}(1+g_{t+1})^{-\alpha}\left(\frac{c_{\tau,t+1}}{c_{\tau,t}}\right)^{-\alpha}\left(\frac{1}{1+\pi_{t+1}}\right)\left(1+\gamma(1-\gamma+j_{t+1})^{-1}\right)\right].$$

The equilibrium j_t is obtained by iterating forwards and using the transversality condition [2.25b]:

$$j_t = \left(\mathbb{E}_t\left[\sum_{\ell=1}^{\infty}\gamma^{\ell-1}\left\{\prod_{\jmath=1}^{\ell}\delta_{\tau,t+\jmath-1}\frac{(1+g_{t+\jmath})^{-\alpha}}{(1+\pi_{t+\jmath})}\left(\frac{c_{\tau,t+\jmath}}{c_{\tau,t+\jmath-1}}\right)^{-\alpha}\right\}\right]\right)^{-1} + \gamma - 1. \qquad [2.26]$$

Before proceeding to the analysis of the economy with aggregate uncertainty, first consider the non-stochastic steady state (which corresponds to standard deviation $\varsigma = 0$ of real GDP growth [2.3] and the absence of any monetary policy shocks).

Proposition 1 *Suppose the economy faces no exogenous shocks ($g_t = \bar{g}$, $i_t = \bar{i}$).*

(i) There is a unique non-stochastic steady state of the system of equations [2.22a]–[2.22f]*:*

$$\bar{c}_{\mathrm{b}} = 1-\theta, \quad \bar{c}_{\mathrm{s}} = 1+\theta, \quad \textit{where } \theta = \frac{1-(\Delta_{\mathrm{b}}/\Delta_{\mathrm{s}})^{(1-\lambda)\alpha}}{1+(\Delta_{\mathrm{b}}/\Delta_{\mathrm{s}})^{(1-\lambda)\alpha}} \quad (0<\theta<1); \qquad [2.27a]$$

$$\bar{\delta}_{\mathrm{b}} = \bar{\delta}_{\mathrm{s}} = \delta, \quad \textit{where } \delta = \left(\left(\Delta_{\mathrm{b}}^{\frac{1}{(1-\lambda)\alpha}} + \Delta_{\mathrm{s}}^{\frac{1}{(1-\lambda)\alpha}}\right)/2\right)^{(1-\lambda)\alpha}; \quad \bar{\rho} = \bar{r} = \frac{1+\bar{g}}{\beta} - 1, \qquad [2.27b]$$

$$\bar{d} = \frac{\theta}{2(1-\beta)}, \quad \bar{l} = \frac{\beta\theta}{2(1-\beta)}, \quad \textit{where } \beta = \delta(1+\bar{g})^{1-\alpha} \quad (0<\beta<1). \qquad [2.27c]$$

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[18] This is generally independent of the transversality condition [2.22f] for financial wealth. However, the bond yield transversality condition [2.25b] can be derived from [2.22f] under the assumption that there is a minimum transaction size for bonds. Observe that [2.25b] is equivalent to

$$2c_{\tau,t}^{\alpha}P_tY_t\lim_{\ell\to\infty}\mathbb{E}_t\left[\left(\left\{\prod_{\jmath=1}^{\ell}\delta_{\tau,t+\jmath-1}(1+g_{t+\jmath})^{1-\alpha}\right\}c_{\tau,t+\ell}^{-\alpha}l_{t+\ell}\right)\left(\frac{\gamma^{\ell}}{B_{\mathrm{s},t+\ell}}\right)\right] = 0.$$

Thus, if $|B_{\tau,t}| > \underline{B}_t$ for some $\underline{B}_t$ satisfying $\gamma^t/\underline{B}_t < \infty$ as $t\to\infty$ when $B_{\tau,t} \neq 0$, then equation [2.22f] implies [2.25b].

(ii) *The steady state for real variables is independent of monetary policy. Given the steady-state nominal interest rate* $\bar{i}$, *equation* [2.24] *implies that inflation is* $\bar{\pi} = (1+\bar{i})/(1+\bar{\rho}) - 1$. *The steady-state bond yield consistent with* [2.25a] *and* [2.26] *is* $\bar{j} = \bar{i}$.

PROOF See appendix A.1. ■

In the steady state, households' consumption relative to income is determined as a function of θ, which depends on the relative patience Δ_b/Δ_s of the two household types and the utility-function parameters α and λ. In equilibrium, the discount factors of the two types are aligned at δ, which is effectively an average of the patience parameters Δ_b and Δ_s. The solutions for debt, loans, and interest rates depend on θ and a term β that represents the growth-adjusted market discount factor.

The model can be parameterized directly with β and θ rather than the two patience parameters Δ_b and Δ_s (leaving α, λ, and $\bar{g}$ to be chosen separately). The term β plays the usual role of the discount factor in a representative-household economy given its relationship with the real interest rate (with an adjustment for steady-state real GDP growth). The term θ quantifies the extent of heterogeneity between borrower and saver households, which is related to the amount of borrowing and saving that occurs in equilibrium, and hence to the debt-to-GDP ratio in [2.27c]. Given equation [2.27a], θ can be interpreted as the 'debt service ratio' because it is the net fraction of income transferred by borrowers to savers.[19] As will be seen, θ is a sufficient statistic for the extent of heterogeneity in the economy, with $\theta \to 0$ being the limiting case of a representative-household economy ($\Delta_b \to \Delta_s$).

Monetary policy has no effect on real variables in the steady state, but does determine nominal bond yields and inflation through the usual Fisher relationship and the indifference condition between bonds and reserves. Given Proposition 1 and equations [2.15] and [2.20], the steady-state fraction of total debt that is newly issued is $1-\mu$ with $\mu = \gamma/((1+\bar{\pi})(1+\bar{g}))$, so the coupon parameter γ is positively related to debt maturity and negatively related to the rate of refinancing.

3 Monetary policy in a pure credit economy

3.1 Complete financial markets and the natural debt-to-GDP ratio

As a benchmark for subsequent monetary policy analysis, first consider the hypothetical case where households have access to a complete set of state-contingent bonds (traded sequentially, period by period). All other assumptions of the model are unchanged. These bonds are denominated in real terms without loss of generality. Let $F^*_{\tau,t+1}$ denote the net portfolio of contingent bonds held between period t and $t+1$ by households of type τ (an asterisk is used to signify complete financial markets). The prices of these securities at time t relative to the conditional probabilities of the states and in real terms are denoted by K_{t+1}, so $\mathbb{E}_t[K_{t+1}F^*_{\tau,t+1}]$ is the date-t cost of the date $t+1$ portfolio $F^*_{\tau,t+1}$. The flow budget identities [2.5] are thus replaced by:

$$C_{\tau,t} + \mathbb{E}_t[K_{t+1}F^*_{\tau,t+1}] + \frac{M_{\tau,t}}{P_t} = Y_t + F^*_{\tau,t} + (1+i_{t-1})\frac{M_{\tau,t-1}}{P_t}. \qquad [3.1]$$

[19] Strictly speaking, when real GDP growth is different from zero, θ is the debt service ratio net of new borrowing.

Maximizing utility [2.1] with respect to $F^*_{\tau,t+1}$ subject to [3.1] implies the Euler equations and transversality conditions:

$$\delta_{\mathrm{b},t}\left(\frac{C^*_{\mathrm{b},t+1}}{C^*_{\mathrm{b},t}}\right)^{-\alpha} = K_{t+1} = \delta_{\mathrm{s},t}\left(\frac{C^*_{\mathrm{s},t+1}}{C^*_{\mathrm{s},t}}\right)^{-\alpha}, \quad \text{and} \quad \lim_{\ell\to\infty}\left\{\prod_{\jmath=0}^{\ell-1}\delta_{\tau,t+\jmath}\right\}\frac{K_{t+\ell+1}F^*_{\tau,t+\ell+1}}{{C^*}^{\alpha}_{\tau,t+\ell}} = 0, \quad [3.2]$$

where these equations hold in all states of the world. The Euler equation [2.7] for reserves is the same, and making the same assumptions on monetary policy, the equilibrium conditions [2.11] and [2.14] are unchanged. The market-clearing condition for nominal bonds [2.13] is replaced by a set of clearing conditions for all contingent bond markets:

$$\frac{1}{2}F^*_{\mathrm{b},t} + \frac{1}{2}F^*_{\mathrm{s},t} = 0. \quad [3.3]$$

The goods-market clearing condition [2.12] holds as in the incomplete-markets model.

To relate the economy with complete markets to its incomplete-markets equivalent, consider the following definitions of variables D^*_t, L^*_t, and r^*_t that will be seen to be the equivalents of debt D_t, loans L_t, and the ex-post real return r_t in the incomplete-markets economy:

$$D^*_t = -\frac{1}{2}F^*_{\mathrm{b},t}, \quad L^*_t = -\frac{1}{2}\mathbb{E}_t[K_{t+1}F^*_{\mathrm{b},t+1}], \quad \text{and} \quad 1 + r^*_t = \frac{F^*_{\mathrm{b},t}}{\mathbb{E}_{t-1}[K_t F^*_{\mathrm{b},t}]}. \quad [3.4]$$

Debt in an economy with complete financial markets refers to the total gross value of the contingent bonds repayable in the realized state of the world. Loans refers to the value of the whole portfolio of contingent bonds issued by borrowers, and the (gross) ex-post real return is the state-contingent value of the bonds repayable relative to the value of all the bonds previously issued.

With definitions [3.4], it is immediately apparent that the accounting identity [2.16] holds for D^*_t, L^*_t, and r^*_t. Similarly, using [2.14] and [3.3], the definitions from [3.4] substituted into [3.1] imply that [2.17] holds in terms of $C^*_{\mathrm{b},t}$, $C^*_{\mathrm{s},t}$, D^*_t, and L^*_t. The definitions in [3.4] also directly imply $1 = \mathbb{E}_t[(1 + r^*_{t+1})K_{t+1}]$ and hence using [3.2], the Euler equations [2.18] hold with $C^*_{\tau,t}$, and r^*_t. Similarly, the transversality condition [2.19] also holds with $C^*_{\tau,t}$ and L^*_t. Therefore, by following exactly the same steps as for the incomplete-markets economy with the definitions from [2.20], the set of equilibrium conditions [2.22a]–[2.22f] and equations [2.23] and [2.24] must hold, together with an equation for i_t describing monetary policy.

The difference between the equilibrium conditions with complete and incomplete financial markets is that r^*_t need not satisfy equation [2.25a].[20] This equation is replaced by a new equilibrium condition that follows from [3.2], which has no equivalent in the incomplete-markets economy:

$$\delta_{\mathrm{b},t-1}\left(\frac{c^*_{\mathrm{b},t}}{c^*_{\mathrm{b},t-1}}\right)^{-\alpha} = \delta_{\mathrm{s},t-1}\left(\frac{c^*_{\mathrm{s},t}}{c^*_{\mathrm{s},t-1}}\right)^{-\alpha}. \quad [3.5]$$

Proposition 2 *The complete-markets equilibrium is characterized by* [2.22a]–[2.22f] *and* [3.5].

(i) The equilibrium (which is independent of monetary policy) can be written in terms of the

[20] Equations [2.25a] and [2.25b] still hold in the sense that nominal bonds can be traded in a complete-markets economy, but there is no necessity for the real return r^*_t on the actual portfolio of contingent bonds to be tied to the ex-post real return r_t on nominal bonds, unlike in the incomplete-markets economy where other securities are not available.

stochastic process for real GDP growth g_t *and* δ *and* θ *defined in* [2.27a] *and* [2.27b]*:*

$$c^*_{\mathrm{b},t} = 1 - \theta, \quad c^*_{\mathrm{s},t} = 1 + \theta, \quad \textit{and} \;\; \delta^*_{\mathrm{b},t} = \delta^*_{\mathrm{s},t} = \delta; \tag{3.6a}$$

$$d^*_t = \frac{\theta}{2}\mathbb{E}_t\left[\sum_{\ell=0}^{\infty} \delta^{\ell} \prod_{\jmath=1}^{\ell} (1 + g_{t+\jmath})^{1-\alpha}\right], \quad \textit{and} \;\; l^*_t = d^*_t - \frac{\theta}{2}; \tag{3.6b}$$

$$r^*_t = \frac{(1+g_t)d^*_t}{d^*_{t-1} - \frac{\theta}{2}} - 1, \quad \textit{and} \;\; \rho^*_t = \frac{1}{\delta\mathbb{E}_t\left[(1+g_{t+1})^{-\alpha}\left\{\frac{(1+g_{t+1})d^*_{t+1}}{\mathbb{E}_t[(1+g_{t+1})d^*_{t+1}]}\right\}\right]} - 1. \tag{3.6c}$$

(ii) In the special cases where $\alpha = 1$ *or* $\{g_t\}$ *is i.i.d., equations* [3.6b] *and* [3.6c] *simplify to:*

$$d^*_t = \frac{\theta}{2(1-\beta^*)}, \quad l^*_t = \frac{\beta^*\theta}{2(1-\beta^*)}, \quad r^*_t = \frac{1+g_t}{\beta^*} - 1, \quad \textit{and} \;\; \rho^*_t = \frac{1 + \mathbb{E}_t g_{t+1}}{\beta^*} - 1, \tag{3.7}$$

where $\beta^* = \delta\mathbb{E}[(1+g_t)^{1-\alpha}]$.

(iii) With a representative household ($\Delta_\mathrm{b} = \Delta_\mathrm{s}$, *and hence* $\theta = 0$) *in the incomplete-markets economy, the equilibrium levels of debt, loans, and consumption always coincide with those of the complete-markets economy.*

Proof See appendix A.2. ■

The proposition gives an explicit solution for the equilibrium with complete financial markets. The equilibrium features full risk sharing between borrowers and savers, meaning that all households' consumption levels perfectly co-move in response to shocks. Complete financial markets therefore allocate consumption efficiently across states of the world, as well as over time. The equilibrium consumption-to-income ratios and discount factors are identical to their non-stochastic steady-state values given in Proposition 1. The equilibrium features a generally time-varying debt-to-GDP ratio d^*_t with associated values of l^*_t, r^*_t, and ρ^*_t, all of which are determined by utility-function parameters and the stochastic process for real GDP growth g_t. If the utility function is logarithmic or real GDP follows a random walk then d^*_t and l^*_t are constant over time.

The complete-markets debt-to-GDP ratio d^*_t is referred to as the 'natural debt-to-GDP ratio'. This terminology is motivated by analogy with such concepts as the natural rate of interest, the natural rate of unemployment, and the natural level of output. It shares many of the features of those concepts as will be seen in the analysis below. It represents an economic outcome that would be achieved in the absence of a particular friction (here, the friction is the inability to issue state-contingent bonds, in contrast to the frictions of imperfect information or nominal rigidities that affect the supply of output or labour in many models). It is independent of monetary policy. It is a desirable outcome in the sense that deviations of the actual debt-to-GDP ratio from its natural level are inefficient.[21] Finally, it is what the economy would eventually reach in the absence of shocks.

[21] In New Keynesian models of nominal rigidities, the 'natural' value of a variable is generally not optimal because of the presence of imperfect competition. Here, although financial markets are incomplete, they are modelled as perfectly competitive, so there is no equivalent of that distortion.

3.2 Replicating complete financial markets

The only difference between the equilibrium conditions with incomplete and complete markets is that the incomplete-markets economy includes [2.25a] instead of [3.5] in the complete-markets economy. Since there is one degree of freedom from monetary policy, the complete-markets equilibrium can be replicated in the incomplete-markets economy by choosing a monetary policy rule that makes equation [2.25a] equivalent to [3.5], in other words, a policy that ensures the actual and natural debt-to-GDP ratios coincide.[22] A characterization of such policies is given below in terms of nominal GDP $N_t = P_t Y_t$, with the nominal GDP growth rate denoted by $n_t = (N_t - N_{t-1})/N_{t-1}$.

Proposition 3 *There exist paths for nominal GDP N_t that imply the equilibrium of the hypothetical complete-markets economy is also an equilibrium of the incomplete-markets economy:*

(i) For logarithmic utility ($\alpha = 1$) or real GDP following a random walk (g_t is i.i.d.), any constant growth rate of nominal GDP ($n_t = n$).

(ii) For $\gamma > 0$, a constant growth rate of nominal GDP ($n_t = n$) given by $n = \gamma - 1$.

(iii) A state-contingent path for nominal GDP given by $N_t = (1+n)\left(\frac{d^*_{t-1} - \theta/2}{\beta d^*_t}\right) N_{t-1}$ for any n.

Proof See appendix A.3. ■

There are many possible ways of describing the monetary policy that replicates complete financial markets. However, it is natural to think of it as a nominal GDP target. First, if it is desirable to maintain a constant debt-to-GDP ratio, any constant level or growth path of nominal GDP is able to implement the complete-markets equilibrium. Intuitively, if the numerator of the debt-to-GDP ratio is fixed because nominal debt liabilities are not state contingent, the ratio can be stabilized by targeting the denominator. If the underlying target for the debt-to-GDP ratio is not constant then there exist time-varying paths for nominal GDP that replicate complete markets. These policies can operate with any average rate of nominal GDP growth. Even in the case where the natural debt-to-GDP ratio is time varying, if debt contracts have a sufficiently long term then it is possible to replicate complete financial markets with a particular constant growth rate of nominal GDP set low enough to avoid the need to refinance existing debt.[23] For example, with perpetuities ($\gamma = 1$), a zero nominal GDP growth rate would achieve this.

3.3 Equilibrium with incomplete financial markets

In cases where monetary policy follows one of the prescriptions from Proposition 3, the equilibrium of the incomplete-markets economy is the same as that of the equivalent complete-markets economy characterized in Proposition 2. This section characterizes the equilibrium of the economy when

[22] That monetary policy is able exactly to replicate complete financial markets is owing to the model having a 'representative borrower' and a 'representative saver'. With heterogeneity within these groups as well as between them, exact replication will generally not be possible.

[23] For low values of γ, the required declining path of nominal GDP may not be feasible if the nominal interest rate must be non-negative.

monetary policy fails to replicate complete markets, which can be used to compare the policies considered in Proposition 3 to other monetary policies such as inflation targeting. An exact analytical solution is not available in general, so this section resorts to finding the log-linear approximation to the equilibrium (the first-order perturbation around the non-stochastic steady state), which can be found analytically.

It is helpful to present the solution in terms of two variables v_t and e_t defined below:

$$v_t = \frac{1}{Y_t}\sum_{\ell=0}^{\infty}\mathbb{E}_t\left[\frac{Y_{t+\ell}}{\prod_{j=0}^{\ell-1}(1+\rho_{t+\ell})}\right], \quad \text{and} \quad e_t = \frac{1}{Y_t}\sum_{\ell=0}^{\infty}\left(\frac{\gamma}{1+\bar{n}}\right)^{\ell}\mathbb{E}_t\left[\frac{Y_{t+\ell}}{\prod_{j=0}^{\ell-1}(1+\rho_{t+\ell})}\right], \qquad [3.8]$$

where $\bar{n}$ is the steady-state growth rate of nominal GDP given by $1+\bar{n} = (1+\bar{\pi})(1+\bar{g})$. The variable v_t is the discounted value of future real GDP relative to current real GDP, effectively a 'price-earnings' ratio for the whole economy. The variable e_t is similar to v_t, but additionally discounts future real GDP to the extent that existing debt is refinanced ($\mu = \gamma/(1+\bar{n})$ is the steady-state fraction of debt that is not newly issued).

The log linearization of the equilibrium is presented below. The notational convention is that variables in a sans serif font denote log deviations of the equivalent variables in roman letters from their steady-state values as given in Proposition 1 (log deviations of interest rates, inflation rates, and growth rates are log deviations of the corresponding gross rates; for variables that have no steady state, the sans serif letter simply denotes the logarithm of that variable). It is also convenient to define 'gaps' relative to the complete-markets equilibrium, which are denoted by $\tilde{d}_t = d_t/d_t^*$, $\tilde{r}_t = (1+r_t)/(1+r_t^*)$, and so on.

Proposition 4 *(i) A solution of the system of equations [2.22a]–[2.22f] and [3.8] must satisfy:*

$$\mathsf{\rho}_t = \alpha\mathbb{E}_t\mathsf{g}_{t+1}, \quad \textit{and} \quad \mathsf{v}_t = (1-\alpha)\sum_{\ell=1}^{\infty}\beta^{\ell}\mathbb{E}_t\mathsf{g}_{t+\ell}; \qquad [3.9a]$$

$$\mathsf{d}_t = \lambda\mathsf{d}_{t-1} + (\mathbb{E}_{t-1}\mathsf{v}_t - \lambda\mathsf{v}_{t-1}) + \upsilon_t, \quad \textit{for some } \upsilon_t \textit{ with } \mathbb{E}_{t-1}\upsilon_t = 0; \qquad [3.9b]$$

$$\mathsf{l}_t = \lambda\mathsf{d}_t + \left(\frac{1-\beta\lambda}{\beta}\right)\mathsf{v}_t, \quad \textit{and} \quad \mathsf{r}_t = \mathsf{g}_t + (\alpha-1)\mathbb{E}_{t-1}\mathsf{g}_t + (\mathsf{d}_t - \mathbb{E}_{t-1}\mathsf{d}_t). \qquad [3.9c]$$

(ii) Equation [2.24] implies that the policy rate i_t must satisfy:

$$\mathsf{i}_t = \mathsf{\rho}_t + \mathbb{E}_t\pi_{t+1}. \qquad [3.10]$$

(iii) If equation [3.5] holds (complete financial markets) then the equilibrium is [3.9a] and:

$$\mathsf{d}_t^* = \mathsf{v}_t, \quad \mathsf{l}_t^* = \beta^{-1}\mathsf{v}_t, \quad \textit{and} \quad \mathsf{r}_t^* = \mathsf{g}_t + (\mathsf{v}_t - \beta^{-1}\mathsf{v}_{t-1}), \qquad [3.11]$$

which is equivalent to equations [3.9b]–[3.9c] with $\upsilon_t = \upsilon_t^* = \mathsf{v}_t - \mathbb{E}_{t-1}\mathsf{v}_t$. Any solution of [2.22a]–[2.22f] can be expressed in the form of gaps relative to the solution [3.11] as follows:

$$\mathbb{E}_t\tilde{\mathsf{d}}_{t+1} = \lambda\tilde{\mathsf{d}}_t, \quad \tilde{\mathsf{l}}_t = \lambda\tilde{\mathsf{d}}_t, \quad \textit{and} \quad \tilde{\mathsf{r}}_t = \tilde{\mathsf{d}}_t - \mathbb{E}_{t-1}\tilde{\mathsf{d}}_t; \quad \textit{with} \qquad [3.12a]$$

$$\tilde{\mathsf{c}}_{b,t} = \mathsf{c}_{b,t} = -\left(\frac{\theta}{1-\theta}\right)\left(\frac{1-\beta\lambda}{1-\beta}\right)\tilde{\mathsf{d}}_t, \quad \textit{and} \quad \tilde{\mathsf{c}}_{s,t} = \mathsf{c}_{s,t} = \left(\frac{\theta}{1+\theta}\right)\left(\frac{1-\beta\lambda}{1-\beta}\right)\tilde{\mathsf{d}}_t. \qquad [3.12b]$$

(iv) If equations [2.25a] and [2.25b] hold (incomplete financial markets) then the bond yield j_t *must satisfy the transversality condition* $\lim_{\ell\to\infty}(\beta\mu)^\ell \mathbb{E}_t \mathsf{j}_{t+\ell} = 0$, *with* $\mu = \gamma/(1+\bar{n})$ *and* $0 \le \mu < \beta^{-1}$, *and the ex-post real return* r_t *is given by the following:*

$$\mathsf{r}_t = \frac{1}{1-\beta\mu}(\mathsf{j}_{t-1} - \beta\mu\mathsf{j}_t) - \pi_t, \quad \textit{where } \mathsf{j}_t = (1-\beta\mu)\sum_{\ell=0}^{\infty}(\beta\mu)^\ell \mathbb{E}_t[\rho_{t+\ell} + \pi_{t+1+\ell}]. \qquad [3.13]$$

The implied innovation to the debt-to-GDP gap $\tilde{\mathsf{d}}_t$ *can be expressed in terms of the surprise component of an average* $\mathcal{N}_t$ *of current and expected future nominal GDP together with the variables* v_t *and* e_t *from [3.8]:*

$$\tilde{\mathsf{d}}_t - \mathbb{E}_{t-1}\tilde{\mathsf{d}}_t = ((\mathsf{e}_t - \mathsf{v}_t) - \mathbb{E}_{t-1}[\mathsf{e}_t - \mathsf{v}_t]) - (\mathcal{N}_t - \mathbb{E}_{t-1}\mathcal{N}_t); \quad \textit{where} \qquad [3.14a]$$

$$\mathcal{N}_t = (1-\beta\mu)\sum_{\ell=0}^{\infty}(\beta\mu)^\ell \mathbb{E}_t \mathsf{N}_{t+\ell}, \quad \textit{and } \mathsf{e}_t = (1-\alpha)\sum_{\ell=1}^{\infty}(\beta\mu)^\ell \mathbb{E}_t \mathsf{g}_{t+\ell}. \qquad [3.14b]$$

Proof See appendix A.4. ■

The first two parts of the proposition study the implications of the equilibrium conditions [2.22a]–[2.22f] and [2.24] common to the incomplete- and complete-markets economies. These equations imply the ex-ante real interest rate is determined entirely by expectations of future real GDP growth, and is thus the same (to a first-order approximation) irrespective of the completeness of financial markets.[24] The policy rate i_t and inflation π_t can be determined by [3.10] given a description of monetary policy and the equilibrium real interest rate ρ_t from [3.9a].

The common block of equations leaves one degree of freedom in determining the debt-to-GDP ratio d_t, the loans-to-GDP ratio l_t, and the ex-post real return r_t. These variables depend on the 'price-earnings' ratio v_t (which is independent of the completeness of financial markets and depends only on expectations of future real GDP growth) and a serially uncorrelated sequence υ_t that must be determined by some equation outside of the block [2.22a]–[2.22f]. Once υ_t is known, the debt-to-GDP ratio d_t is determined, and with that, all other real variables such as l_t and r_t.

Since both the incomplete- and complete-markets economies satisfy [2.22a]–[2.22f], the difference between them must reduce to the sequence $\upsilon_t = \mathsf{d}_t - \mathbb{E}_{t-1}\mathsf{d}_t$, which is the unpredictable component of the debt-to-GDP ratio d_t. On the other hand, the predictable component $\mathbb{E}_{t-1}\mathsf{d}_t$ of the debt-to-GDP ratio is the same irrespective of the completeness of financial markets. Given the single degree of freedom allowed by the block of equilibrium conditions [2.22a]–[2.22f], all of the 'gaps' between the incomplete- and complete-markets economies are proportional to a single gap: that for the debt-to-GDP ratio. Furthermore, the expected future debt-to-GDP gap is equal to a multiple of the current gap, where the parameter λ controls the persistence of this gap over time. This means that with no shocks at any time, the economy would be at the natural debt-to-GDP ratio. Even when shocks occur, since $0 < \lambda < 1$, the economy is expected to converge in the long run to the natural debt-to-GDP ratio in the absence of further shocks.

[24] This is because the model's simplifying assumptions imply the marginal propensities to consume from financial wealth are (locally) identical for borrowers and savers. Note that the equation for ρ_t in [3.9a] is the same as the log-linearized Euler equation $\mathsf{Y}_t = \mathbb{E}_t\mathsf{Y}_{t+1} - \alpha^{-1}\rho_t$ in a representative-household economy.

With complete financial markets, the system [2.22a]–[2.22f] is closed by the addition of the risk-sharing condition [3.5]. This determines a particular value of the innovation to the debt-to-GDP ratio, denoted by $\upsilon_t^* = \mathsf{d}_t^* - \mathbb{E}_{t-1}\mathsf{d}_t^*$. It turns out that the debt-to-GDP ratio is then proportional to the economy's 'price-earnings' ratio v_t. On the other hand, with incomplete markets, the system of equations is closed with [2.25a] and [2.25b]. The implied ex-post real return r_t depends on inflation π_t and the bond yield j_t, with equations for both r_j and j_t given in [3.13], where $\mu = \gamma/(1+\bar{n})$ is the coupon parameter γ scaled by steady-state gross nominal GDP growth $1+\bar{n}$ (the rate of refinancing is $1-\mu$). Note that [3.13] reduces to the standard ex-post Fisher equation in the special case of short-term debt, $\mu = 0$.

The nominal bond yield j_t satisfies the expectations theory equation $\mathsf{j}_t = (1-\beta\mu)\sum_{\ell=0}^{\infty}(\beta\mu)^{\ell}\mathbb{E}_t\mathsf{i}_{t+\ell}$ in the sense of being equal to a weighted average of current and expected future policy rates, where the discount factor β and the maturity parameter μ determine the weights. Since j_t and π_t depend on monetary policy, the ex-post real return r_t is endogenous to policy. The implications of different monetary policies for the evolution of the debt-to-GDP gap $\tilde{\mathsf{d}}_t$ depend on the properties of $\mathcal{N}_t$, a weighted average of current and expected future levels of nominal GDP N_t. The other determinants of the debt-to-GDP gap $\tilde{\mathsf{d}}_t$ are the 'price-earnings' ratio variables v_t and e_t, which are independent of monetary policy.[25]

3.4 Consequences of optimal monetary policy for inflation

Optimal monetary policies that replicate complete financial markets have been characterized as nominal GDP targets in Proposition 3. By definition, stabilizing nominal GDP when there are fluctuations in real GDP entails fluctuations in inflation. That inflation fluctuates is not in itself the desirable feature of these policies, but rather that inflation displays a negative correlation with real GDP growth. In other words, there is an optimal degree of countercyclicality of the price level.

The following proposition shows that there is a time-invariant target for the relationship between the price level and real GDP that replicates complete financial markets (up to a first-order approximation). That is, even if optimal monetary policy cannot be described as a completely time-invariant target for nominal GDP (see part (iii) of Proposition 3), there is a time-invariant target for weighted nominal GDP $\mathsf{P}_t + \omega\mathsf{Y}_t$ which is optimal.

Proposition 5 *If real GDP growth g_t follows a stationary and invertible stochastic process then the complete-markets consumption allocation can be implemented (up to a first-order approximation) by a target for a fixed level or a constant growth rate of weighted nominal GDP $\mathsf{P}_t + \omega^*\mathsf{Y}_t$:*

$$\omega^* = 1 + (\alpha - 1)\left(1 - \frac{\Theta(\beta)}{\Theta(\beta\mu)}\right), \quad \textit{where } \mathsf{g}_t = \Theta(\mathbb{L})(\mathsf{Y}_t - \mathbb{E}_{t-1}\mathsf{Y}_t), \tag{3.15}$$

and $\Theta(\cdot)$ is a function of the lag operator $\mathbb{L}$.

[25] If there are no fluctuations in the difference between the 'price-earnings' ratios v_t and e_t then a policy of completely stabilizing nominal GDP around a deterministic path ensures the debt gap is always zero. The term $\mathsf{e}_t - \mathsf{v}_t$ is zero whenever $\alpha = 1$, $\mathbb{E}_t\mathsf{g}_{t+1} = 0$, or $\gamma = 1+\bar{n}$ ($\mu = 1$), which correspond to the cases analysed in parts (i) and (ii) of Proposition 3. When the 'price-earnings' term is not always zero, the nominal GDP target must be adjusted in response to shocks, as in part (iii) of Proposition 3.

Proof See appendix A.5. ■

The weighted nominal GDP target implies $\mathsf{P}_t = -\omega^* \mathsf{Y}_t$, so ω^* can be interpreted as how much the price level (or inflation) should respond to fluctuations in real GDP (or real GDP growth). The cases from Proposition 3 where a time-invariant target for (unweighted) nominal GDP is optimal correspond here to cases where $\omega^* = 1$.

As an example, consider the stochastic process $\Delta\mathsf{Y}_t = \phi\Delta\mathsf{Y}_{t-1} + \epsilon_t - (\phi + (1-\phi)\vartheta)\epsilon_{t-1}$, where the parameter ϑ can be interpreted as the percentage difference between the short-run and long-run effects of a shock to real GDP ($0 \leq \vartheta \leq 1$), and the parameter ϕ determines the speed of convergence to the long-run level of real GDP ($0 \leq \phi < 1$). Using [3.15], the optimal weight is

$$\omega^* = 1 + \frac{\beta\vartheta(1-\mu)(1-\phi)(\alpha-1)}{(1-\beta\phi)(1-\beta\mu(\phi+(1-\phi)\vartheta))}.$$

Under the assumptions that $\alpha > 1$ and $\mu < 1$, the optimal weight is increasing in ϑ, so to the extent that real GDP partially recovers following an initial shock, ω^* exceeds one.

Given Proposition 5, especially in the case where $\omega^* \neq 1$, it might be tempting to interpret the weighted nominal GDP target as simply a relabelling of so-called 'flexible inflation targeting'. But aside from the rationale being very different from that invoked to justify flexible inflation targeting, there is a fundamental difference between the two policies. Flexible inflation targeting can be formalized as a target criterion in both inflation and the output gap, whereas here, the target criterion includes the level of output, not the output gap. Monetary policy is effective in completing financial markets precisely because the level of output that appears in the target is not adjusted for any unexpected changes in potential output, even though these have consequences for inflation.

3.5 Consequences of sub-optimal monetary policy for financial markets

Following a monetary policy that replicates complete financial markets has consequences for fluctuations in inflation. Similarly, following a monetary policy that stabilizes inflation has observable implications for fluctuations in financial markets in addition to failing to replicate the risk sharing of complete financial markets. This is because the behaviour of variables such as the debt-to-GDP ratio is very different in a complete-markets economy than in an economy with incomplete markets and a sub-optimal monetary policy.

In the hypothetical complete-markets economy, the equilibrium debt-to-GDP ratio has some striking properties. Intuitively, the stock of outstanding debt relative to GDP ought to behave like a state variable because debt liabilities are predetermined. However, with access to complete financial markets, households' optimizing behaviour leads them to choose a portfolio of contingent securities such that the implied debt-to-GDP ratio is actually a purely forward-looking variable. In particular, the debt-to-GDP ratio moves in line with the 'price-earnings' ratio for a claim to a sequence of payments proportional to current and future GDP (see Proposition 4).[26] The intuition

[26] This result also reveals the type of security required in this simple economy to span complete financial markets in the absence of a suitable monetary policy to replicate those markets. For households to achieve this outcome directly, borrowers must issue GDP-linked perpetual bonds. This type of security differs from conventional debt contracts not

for this surprising result is that given the budget identities of households, full risk sharing requires the financial wealth of savers and the financial liabilities of borrowers to move in line with the value of the stream of current and future labour income. The debt-to-GDP ratio must therefore behave like an asset price rather than a state variable: it must reflect a forecast of the economy's future prospects rather than a record of past choices and shocks. It follows that in the complete-markets economy, past realizations of shocks would have no correlation with the current debt-to-GDP ratio except to the extent that these have predictive power for the economy's future fundamentals.

When monetary policy is used to replicate complete financial markets in an incomplete-markets economy, fluctuations of inflation are used to affect the real value of nominal debt so as to mimic the behaviour of the complete-markets debt-to-GDP ratio. But if monetary policy does not replicate complete financial markets then the debt-to-GDP behaves in line with what simple intuition suggests: it is intrinsically serially correlated, so past shocks have a persistent effect on the subsequent evolution of the level of debt. With incomplete financial markets, optimizing behaviour by households leads to consumption smoothing over time, but not generally across different states of the world. Thus, following a shock, households' financial wealth may diverge from the present discounted value of labour income. Moreover, when this divergence occurs, because households spread out the adjustment of consumption over time, the consequences of the shock for financial wealth and consumption are long lasting. While the model does not feature full consumption smoothing because of the endogenous discount factors in [2.2], when λ is close to one, the debt-to-GDP ratio displays near random-walk persistence.[27] These claims are confirmed in the proposition below.

Proposition 6 *(i) If real GDP growth* g_t *follows an MA*(q) *process then the natural debt-to-GDP ratio* d_t *follows an MA*$(\mathrm{q}-1)$ *process. If* g_t *follows an ARMA*(p,q) *process then the debt-to-GDP ratio is an ARMA*$(\mathrm{p},\max\{\mathrm{p}-1,\mathrm{q}-1\})$ *process with identical autoregressive roots to those of* g_t*.*

(ii) With incomplete financial markets and a monetary policy that does not replicate complete markets, the debt-to-GDP ratio possesses an autoregressive root λ *irrespective of the stochastic process for real GDP. This root is such that* $\lambda = 1-((1+\bar{\rho})/(1+\bar{g}))(MPC-(\bar{\rho}-\bar{g})/(1+\bar{\rho}))$*, where MPC is the marginal propensity to consume (of both borrowers and savers) from financial wealth.*

only in requiring contingent repayments but also in having a sufficiently long maturity to remove all refinancing risk.

[27]The stark difference compared to complete financial markets is analogous to some well-known results from the literature on optimal fiscal policy under different assumptions about the completeness of the financial markets a government has access to. That literature considers an environment where the government aims to find the least distortionary means of financing a stochastic sequence of government spending, given that available fiscal instruments entail distortions which are convex in tax rates. With incomplete financial markets (in the sense that the government can issue only non-contingent bonds), Barro (1979) finds the government should aim to smooth tax rates, which implies the stock of government debt follows a random walk. On the other hand, Lucas and Stokey (1983) assume the government can issue a full set of contingent bonds. In that case, the government now smooths taxes across states-of-the-world as well as time, and this means the value of outstanding government liabilities is now a purely forward-looking variable depending on expectations of future fiscal fundamentals. These findings for the behaviour of government debt mimic the findings for household debt here, with consumption smoothing (whether across time or also across states of the world) playing the role of tax smoothing in the analysis of optimal fiscal policy.

(iii) The difference between the average real interest rates under complete and incomplete financial markets is given by:

$$\frac{\mathbb{E}\rho_t - \mathbb{E}\rho_t^*}{1+\bar{\rho}} = \alpha \mathbb{E}\mathbb{C}_{t-1}[\mathsf{g}_t, \tilde{\mathsf{d}}_t] - \frac{\alpha\theta^2(1-\beta\lambda)^2}{2(1-\theta^2)(1-\beta)^2}(\alpha + (1+\lambda)^{-1})\mathbb{E}\mathbb{V}_{t-1}[\tilde{\mathsf{d}}_t] + \mathscr{O}^3, \qquad [3.16]$$

where $\mathbb{V}_{t-1}[\cdot]$ and $\mathbb{C}_{t-1}[\cdot,\cdot]$ denote variances and covariances conditional on time $t-1$ information, and $\mathscr{O}^3$ denotes terms third-order or higher in the standard deviation ς of real GDP growth.

Proof See appendix A.6. ■

The third part of the proposition shows that the completeness of financial markets also has implications for the average real interest rate. In equation [3.16] there are two terms that give the difference between the average real interest rate with complete financial markets (or a monetary policy that replicates those markets in an incomplete-markets economy) and with incomplete financial markets (and sub-optimal monetary policy). As will be confirmed (see Proposition 8 below), the first term is the negative of a risk premium that must be paid to savers for agreeing to the risk sharing found with complete markets (and is thus likely to be negative). The second term is due to the general-equilibrium effects of differences in precautionary saving between the two cases, which is unambiguously negative.[28] Sub-optimal monetary policy in an incomplete-markets economy therefore results in average real interest rates being inefficiently low.

3.6 Consequences of inflation indexation of bonds

Given that savers holding nominal bonds are exposed to the risk of inflation fluctuations, it might be thought desirable that bonds be indexed to inflation. Consider an otherwise identical economy with incomplete markets but where all bonds have coupons indexed to the price level P_t. Newly issued bonds at time t make a sequence of coupon payments from $t+1$ onwards of P_{t+1}/P_t, $\gamma^\dagger P_{t+2}/P_t$, ${\gamma^\dagger}^2 P_{t+3}/P_t$, …, where $\gamma^\dagger$ parameterizes the geometric sequence of real coupons (the superscript † signifies inflation indexation). Let $Q_t^\dagger$ denote the nominal price of newly issued indexed bonds at time t, with bonds issued at time $t-\ell$ being equivalent to ${\gamma^\dagger}^\ell P_t/P_{t-\ell}$ units of new bonds at time t. Let $B_{\tau,t}^\dagger$ denote the net bond position of type-τ households (expressed in terms of new-bond equivalents). If non-indexed bonds are not available, the budget identities in [2.5] are replaced by:

$$C_{\tau,t} + \frac{Q_t^\dagger B_{\tau,t}^\dagger}{P_t} + \frac{M_{\tau,t}}{P_t} = Y_{\tau,t} + \frac{(1+\gamma^\dagger Q_t^\dagger)B_{\tau,t-1}^\dagger}{P_{t-1}} + (1+i_{t-1})\frac{M_{\tau,t-1}}{P_t}, \qquad [3.17]$$

[28] Lower real interest rates in incomplete-markets economies is a widely obtained result (see, for example, Aiyagari, 1994), which follows from the incentive for precautionary saving to provide some self-insurance against shocks. This result is typically obtained in cases where risk is idiosyncratic to households and thus can be completely diversified away in the economy with complete markets, removing all incentives for precautionary saving. Here, the argument is somewhat different because the source of risk is aggregate shocks that cannot be diversified away even with complete markets. When risk is shared, incentives for precautionary saving decrease for some (borrowers), but increase for others (savers). It turns out that the effect on borrowers is the dominant one here, so the net effect goes in the same direction as in models with only idiosyncratic shocks.

and the bond-market clearing condition [2.13] by

$$\frac{1}{2}B^{\dagger}_{\mathrm{b},t} + \frac{1}{2}B^{\dagger}_{\mathrm{s},t} = 0. \tag{3.18}$$

Now consider definitions of debt $D^{\dagger}_t$, loans $L^{\dagger}_t$, and the real return $r^{\dagger}_t$ analogous to those in [2.15]:

$$D^{\dagger}_t = -\frac{1}{2}\frac{(1+\gamma^{\dagger}Q^{\dagger}_t)B^{\dagger}_{\mathrm{b},t-1}}{P_{t-1}}, \quad L^{\dagger}_t = -\frac{1}{2}\frac{Q^{\dagger}_t B^{\dagger}_{\mathrm{b},t}}{P_t}, \quad \text{and } 1+r^{\dagger}_t = \frac{1+\gamma^{\dagger}Q^{\dagger}_t}{Q^{\dagger}_{t-1}}. \tag{3.19}$$

The ex-post real return $r^{\dagger}_t$ can be written in terms of bond yields rather than bond prices by defining the notional real yield-to-maturity y_t on the indexed bond:

$$Q^{\dagger}_t = \sum_{\ell=1}^{\infty}\frac{{\gamma^{\dagger}}^{\ell-1}}{(1+y_t)^{\ell}} = \frac{1}{1-\gamma^{\dagger}+y_t}, \quad \text{and hence } 1+r^{\dagger}_t = (1+y_t)\left(\frac{1-\gamma^{\dagger}+y_{t-1}}{1-\gamma^{\dagger}+y_t}\right). \tag{3.20}$$

Substituting the expression for $r^{\dagger}_t$ into the Euler equations [2.22d] and solving forwards (imposing a transversality condition analogous to [2.25b]) leads to an expression for y_t:

$$y_t = \left(\mathbb{E}_t\left[\sum_{\ell=1}^{\infty}{\gamma^{\dagger}}^{\ell-1}\left\{\prod_{\jmath=1}^{\ell}\delta_{\tau,t+\jmath-1}(1+g_{t+\jmath})^{-\alpha}\left(\frac{c_{\tau,t+\jmath}}{c_{\tau,t+\jmath-1}}\right)^{-\alpha}\right\}\right]\right)^{-1} + \gamma^{\dagger} - 1. \tag{3.21}$$

Maximizing utility [2.1] subject to [3.17], using the definitions in [3.19], and imposing the market-clearing condition [3.18] and all other equilibrium conditions leads to a set of equations [2.22a]–[2.22f], and [2.24] as before, with [3.20] and the transversality condition used to derive [3.21] replacing equations [2.25a] and [2.25b], and the real yield y_t replacing the nominal yield $\jmath_t$ as a variable.

Proposition 7 *The equilibrium of the economy with indexed bonds has the following properties:*

(i) The equilibrium values of all real variables are independent of monetary policy.

(ii) The equilibrium is identical to that of an incomplete-markets economy with nominal bonds where the central bank follows a policy of strict inflation targeting ($\pi_t = \pi$) if $\gamma = (1+\pi)\gamma^{\dagger}$, that is, the nominal bonds have the same maturity as the indexed bonds (new debt is the same fraction of total debt for both types of bonds in the steady state).

PROOF See appendix A.7. ■

This result shows that moving to an economy with indexed bonds is generally worse if monetary policy aims to replicate complete financial markets. Now, whatever the central bank tries to do has no real effects, so there is no monetary policy intervention that can complete financial markets. All that happens is indexation locks in the generally sub-optimal outcome that would prevail with strict inflation targeting. This is in spite of the fact that savers are now protected from inflation fluctuations. The intuition for these findings is best understood by considering an economy in which both nominal and indexed bonds are available.

3.7 Portfolio choice with both nominal and indexed bonds

The analysis so far has assumed only a single type of bond is available. While the general case of many different assets is beyond the scope of this paper, it is helpful to explore the robustness of the results to the presence of both nominal and indexed bonds that can be bought or issued by households. Using the assumptions on the coupon structures of these bonds and the notation introduced earlier, the equations specific to this variant of the model are:

$$C_{\tau,t} + \frac{Q_t B_{\tau,t}}{P_t} + \frac{Q_t^\dagger B_{\tau,t}^\dagger}{P_t} + \frac{M_{\tau,t}}{P_t} = Y_t + \frac{(1+\gamma Q_t)B_{\tau,t-1}}{P_t} + \frac{(1+\gamma^\dagger Q_t^\dagger)B_{\tau,t-1}^\dagger}{P_{t-1}} + (1+i_{t-1})\frac{M_{\tau,t-1}}{P_t}; \tag{3.22a}$$

$$D_t^\ddagger = -\frac{1}{2}\frac{(1+\gamma Q_t)B_{\mathrm{b},t-1}}{P_t} - \frac{1}{2}\frac{(1+\gamma^\dagger Q_t^\dagger)B_{\mathrm{b},t-1}^\dagger}{P_{t-1}}, \quad L_t^\ddagger = -\frac{1}{2}\frac{Q_t B_{\mathrm{b},t} + Q_t^\dagger B_{\mathrm{b},t}^\dagger}{P_t}; \text{ and} \tag{3.22b}$$

$$r_t^\ddagger = (1-s_{t-1})r_t + s_{t-1}r_t^\dagger, \quad \text{where } s_t = \frac{Q_t^\dagger B_{\mathrm{s},t}^\dagger}{Q_t B_{\mathrm{s},t} + Q_t^\dagger B_{\mathrm{s},t}^\dagger}. \tag{3.22c}$$

Equation [3.22a] replaces the flow budget identities [2.5]. Equation [3.22b] gives the definitions of debt $D_t^\ddagger$ and loans $L_t^\ddagger$ in the two-bond economy (signified by the superscript ‡), which replace those in [2.15]. The ex-post real return $r_t^\ddagger$ on the combined portfolio in [3.22c] is a weighted average of the real returns r_t and $r_t^\dagger$ on the two bonds in [2.15] and [3.19], with s_t denoting the portfolio share in indexed bonds.

The set of equilibrium conditions is as follows. In place of equation [2.22a], there are now two real interest rates, $\rho_t = \mathbb{E}_t r_{t+1}$ for nominal bonds and $\rho_t^\dagger = \mathbb{E}_t r_{t+1}^\dagger$ for indexed bonds, with $\varpi_t = \rho_t - \rho_t^\dagger$ denoting the inflation risk premium. Using the bond-market clearing conditions [2.13] and [3.18] together with [3.22b] and [3.22c], equations [2.22b], [2.22c], and [2.22f] hold in terms of $d_t^\ddagger$, $l_t^\ddagger$, and $r_t^\ddagger$. The Euler equations [2.22d] must hold separately for the real returns r_t and $r_t^\dagger$ on both types of bonds. Equations [2.22e] and [2.24] are unchanged. The nominal bond yield j_t and the real return must satisfy [2.25a] and [2.25b], and the real yield y_t and real return $r_t^\dagger$ must satisfy [3.20] and the transversality condition used to derive [3.21]. The system of equations also includes [3.22c] and the portfolio share s_t as an additional variable.[29]

Proposition 8 *(i) With strict inflation targeting ($\pi_t = \pi$), when the two bonds have the same maturity ($\gamma = (1+\pi)\gamma^\dagger$), the equilibrium is identical (with any portfolio share s_t being consistent with equilibrium) to that where the same monetary policy is followed in an economy with only nominal bonds.*

(ii) Any monetary policy that replicates complete financial markets in the economy with only nominal bonds also replicates complete financial markets in the two-bond economy, and the resulting equilibrium is identical (the equilibrium portfolio share being $s_t = 0$).

(iii) With strict inflation targeting and bonds that have the same maturity ($\gamma = (1+\pi)\gamma^\dagger$), the inflation risk premium is $\varpi_t = 0$. With a monetary policy that replicates complete financial markets, the average inflation risk premium is (with no restriction on γ and $\gamma^\dagger$):

$$\mathbb{E}\varpi_t = (1+\bar{\rho})\alpha \sum_{\ell=0}^{\infty} \beta^\ell \left(1 - \alpha\left(1 - \mu^{\dagger\ell}\right)\right) \mathbb{E}\mathbb{C}_{t-1}[\mathrm{g}_t, \mathbb{E}_t \mathrm{g}_{t+\ell}] + \mathscr{O}^3, \tag{3.23}$$

[29] In equilibrium, the portfolio shares of savers and borrowers in indexed bonds must be equal.

where $\mu^\dagger = \gamma^\dagger/(1+\bar{g})$, $\mathbb{C}_{t-1}[\cdot,\cdot]$ *denotes a conditional covariance, and* $\mathscr{O}^3$ *denotes terms third-order or higher in the standard deviation* ς *of real GDP growth.*

(iv) Suppose nominal and indexed bonds have the same maturity ($\gamma = (1+\bar{\pi})\gamma^\dagger$).

If monetary policy is strict inflation targeting then the term $\mathbb{E}\mathbb{C}_{t-1}[\mathsf{g}_t, \tilde{\mathsf{d}}_t]$ *from the difference between the average real interest rates under incomplete and complete markets in* [3.16] *is negatively related to the average inflation risk premium* [3.23] *(associated with a monetary policy that replicates complete markets) according to* $\mathbb{E}\varpi_t = -(1+\bar{\rho})\alpha\mathbb{E}\mathbb{C}_{t-1}[\mathsf{g}_t, \tilde{\mathsf{d}}_t] + \mathscr{O}^3$.

Under the conditions of Proposition 5, the average inflation risk premium [3.23] *is also positively related to the optimal countercyclicality of the price level* ω^* *from* [3.15] *according to* $\mathbb{E}\varpi_t = (1+\bar{\rho})\alpha\Theta(\beta\mu)\omega^*\mathbb{E}\mathbb{V}_{t-1}[\mathsf{g}_t] + \mathscr{O}^3$.

Proof See appendix A.8. ∎

These results show that for both strict inflation targeting and a nominal GDP target that replicates complete financial markets, the equilibrium outcomes with both types of bonds are the same as those of an economy with only nominal bonds. For strict inflation targeting, assuming debt in the form of both bonds is refinanced at the same rate (so the bonds have the same maturity), this is simply because the absence of inflation fluctuations makes both types of bond equivalent, thus any portfolio share can be an equilibrium. Perhaps more surprisingly, the ability to hold indexed bonds does not change the equilibrium when monetary policy pursues a policy that replicates complete financial markets, which does entail fluctuations in inflation. The equilibrium portfolio share of indexed bonds is zero in this case, so savers do not attempt to protect themselves from inflation risk. The reason is the existence of an inflation risk premium, which means that savers earn higher average returns by holding nominal bonds, which compensate them for the risk they bear.

This perhaps shifts the question to why borrowers continue to issue nominal bonds when lower real interest rates are available on indexed bonds. However, for borrowers, it is the indexed bond that is riskier and the nominal bond that is safer because the former obliges the borrower to make the same real repayments irrespective of real income. When monetary policy replicates complete financial markets, the countercyclical price level makes nominal bonds behave like equity, so coupons have a lower real value when real incomes are low, providing insurance to borrowers, for which they are willing to pay a higher average real interest rate. The higher average real interest rate on nominal bonds can thus equally well be seen as an 'insurance premium' for borrowers as a 'risk premium' for savers, and this inflation risk premium is actually a desirable feature of monetary policy. The inflation fluctuations with nominal GDP targeting are not simply generating risk for savers; inflation is a hedge for borrowers against the underlying real risk in the economy.[30]

[30]It would also be possible to replicate complete financial markets in the two-bond economy with a lower variance of inflation as long as monetary policy deviates from strict inflation targeting, ensuring there is some correlation between inflation and real GDP growth. The reduction of the covariance of inflation and real GDP growth would require all households to hold a positive or negative position in both nominal and indexed bonds, with the size of the gross positions increasing as the covariance between inflation and real GDP growth shrinks. However, since the gross positions are larger, monetary policy errors (not generating exactly the covariance between inflation and real growth

3.8 Objectives, targets, and instruments

The argument for a nominal GDP target in this paper is that it replicates the debt-to-GDP ratio that would be found if the economy had complete financial markets. Thus, the debt-to-GDP ratio (and finally the implied consumption allocation) is the ultimate objective of policy. Nominal GDP is simply an intermediate target which helps achieve that goal. It might then be thought preferable to target the debt-to-GDP ratio directly if this is what monetary policy is actually seeking to influence. An obvious pitfall of a policy of this kind is that it fails to provide the economy with a nominal anchor because the debt-to-GDP ratio is a real variable.

Proposition 9 *Suppose monetary policy is conducted with a targeting rule for the debt-to-GDP ratio* $\mathsf{d}_t = \mathsf{d}_t^*$*. If* N_t^* *is a path of nominal GDP that is consistent with* $\mathsf{d}_t = \mathsf{d}_t^*$ *then so is*

$$\mathsf{N}_t' = \mathsf{z}_{t-1} - \beta\mu\mathsf{z}_t + \sum_{\ell=0}^{\infty}(\beta\mu)^{\ell}(\mathbb{E}_t\mathsf{N}_{t+\ell}^* - \mathbb{E}_{t-1}\mathsf{N}_{t+\ell}^*), \tag{3.24}$$

for any random variable z_t *that belongs to the date-t information set.*

Proof See appendix A.9. ■

Having adopted nominal GDP as an intermediate target, there is the further question of how this intermediate target is achieved using the short-term nominal interest rate as a policy instrument. The practicalities of implementation are beyond the scope of this paper, but it is worth noting how a nominal GDP target can in principle be implemented using an interest-rate feedback rule analogous to the Taylor rule.

Proposition 10 *Consider any (potentially state-contingent) path of nominal GDP* $\{\mathsf{N}_t^*\}$ *that is consistent with* $\mathsf{d}_t = \mathsf{d}_t^*$*, and suppose the interest rate* i_t *is set according to the following rule:*

$$\mathsf{i}_t = (\alpha - 1)\mathbb{E}_t\mathsf{g}_{t+1} + \mathbb{E}_t\mathsf{N}_{t+1}^* - \mathsf{N}_t^* + \zeta(\mathsf{N}_t - \mathsf{N}_t^*). \tag{3.25}$$

(i) If $\zeta > 0$ *then* $\mathsf{N}_t = \mathsf{N}_t^*$ *and* $\mathsf{d}_t = \mathsf{d}_t^*$ *is the only equilibrium of the economy in which nominal GDP growth remains bounded.*

(ii) If $\zeta = 0$ *then* $\mathsf{N}_t = \mathsf{N}_t^* + (\mathsf{N}_{t-1} - \mathsf{N}_{t-1}^*) + \epsilon_t$ *and* $\tilde{\mathsf{d}}_t = \lambda\tilde{\mathsf{d}}_{t-1} - \epsilon_t$ *is an equilibrium of the economy in which nominal GDP growth remains bounded for any martingale difference process* ϵ_t *(*$\mathbb{E}_{t-1}\epsilon_t = 0$*).*

Proof See appendix A.10. ■

The proposition shows that just as Taylor rules can be seen as a means of implementing inflation targeting, the nominal GDP target can be implemented as the unique equilibrium (with bounded inflation and nominal GDP growth) by having the policy rate i_t react to deviations of actual nominal GDP from its target level.[31] But simply setting a policy rate that is consistent with the target fails to rule out multiple (bounded) equilibria, which have real consequences.

that households were expecting) lead to larger deviations from complete financial markets than when households only need to buy or sell one type of bond. These issues are left for future research.

[31] The use of interest-rate feedback rules to determine inflation and the price level is studied by Woodford (2003). The determinacy properties of Taylor rules have been criticized by Cochrane (2011).

3.9 Welfare analysis

The earlier analysis took it for granted that replicating complete financial markets ought to be an objective of monetary policy. The implicit justification for this is the efficiency properties of the complete-markets equilibrium, which is Pareto efficient in the absence of any other distortions. Thus, in spite of policy having distributional effects, the optimal policy can be viewed as supporting (ex-ante) Pareto efficiency in much the same way that analyses of optimal monetary policy have pointed to other inefficiencies that policy might correct. However, this justification is incomplete in one important respect: while the complete-markets equilibrium is Pareto efficient, there are infinitely many other consumption allocations that would equally well satisfy the criterion of Pareto efficiency. What might support singling out one particular Pareto-efficient allocation as the target for policy when the choice among this set necessarily entails taking a stance on distributional questions?

Consider first a hypothetical social planner who has access to a full set of state-contingent transfers among households. With this complete set of instruments, the social planner would be able to implement any Pareto-efficient (first-best) allocation. The social planner's problem can be represented as maximizing a utilitarian social welfare function $\mathscr{W}_{t_0}$ from some time t_0 onwards, assigning Pareto weight $\mathit{\Omega}_{\tau|t_0}$ to each household of type τ:

$$\mathscr{W}_{t_0} = \frac{\mathit{\Omega}_{\mathrm{b}|t_0}}{2}\mathscr{U}_{\mathrm{b},t_0} + \frac{\mathit{\Omega}_{\mathrm{s}|t_0}}{2}\mathscr{U}_{\mathrm{s},t_0}, \tag{3.26}$$

where the utility function $\mathscr{U}_{\tau,t_0}$ is given in [2.1]. The full set of instruments allows the social planner to make a direct choice of state-contingent consumption allocations subject only to the economy's resource constraint, which follows from [2.12]:

$$\frac{1}{2}C_{\mathrm{b},t} + \frac{1}{2}C_{\mathrm{s},t} = Y_t. \tag{3.27}$$

A first-best allocation refers to a state-contingent consumption allocation $C_{\tau,t}$ for $t \geq t_0$ that maximizes the welfare function [3.26] subject to the resource constraint [3.27] for some Pareto weights $\mathit{\Omega}_{\tau|t_0}$, taking the discount factors $\delta_{\tau,t}$ as given (as households do), but with $\delta_{\tau,t}$ evaluated at the consumption allocation in accordance with [2.2].[32] Different choices of Pareto weights correspond to different first-best allocations. To focus on ex-ante efficiency, the set of admissible Pareto weights is restricted so that the weight assigned to borrowers relative to savers at time t_0 is not a random variable given information available immediately before time t_0.[33] Formally, $\mathit{\Omega}_{\mathrm{b}|t_0}/\mathit{\Omega}_{\mathrm{s}|t_0}$ must be $t_0 - 1$-measurable, that is, $\mathit{\Omega}_{\mathrm{b}|t_0}/\mathit{\Omega}_{\mathrm{s}|t_0} = \mathbb{E}_{t_0-1}[\mathit{\Omega}_{\mathrm{b}|t_0}/\mathit{\Omega}_{\mathrm{s}|t_0}]$.

In contrast to the hypothetical social planner, the central bank does not have access to fiscal instruments allowing direct transfers between households. This means the central bank is generally not able to implement any particular first-best allocation. A second-best allocation refers to a state-contingent consumption allocation that maximizes the welfare function [3.26] from t_0 onwards for

[32] Since the endogenous discount factors [2.2] are introduced only for technical reasons, and as the assumption that households take them as given is a simplification, it is helpful to have a benchmark where the social planner makes no attempt to correct for the fact that households themselves take the discount factors as given. The case where both households and the social planner internalize the discount factors is considered in appendix A.17.

[33] This is to avoid a vacuous notion of ex-post efficiency where arbitrary random redistributions can be labelled as efficient by introducing exogenous shocks to the Pareto weights assigned at time t_0.

some weights $\Omega_{\tau|t_0}$ (the ratio of the Pareto weights must be $t_0 - 1$-measurable, and the discount factors are taken as given, as in the social planner's problem) subject not only to the resource constraint [3.27], but also to a set of implementability constraints that reflect the constraints faced by households in a market economy and their optimizing decisions. First, there is the resource constraint [3.27], which represents goods-market equilibrium. Given this equation, only one of the budget identities [2.17] needs to be imposed. A single equation is obtained by subtracting one budget identity from the other, and the accounting identity [2.16] is imposed to eliminate the level of debt D_t, leaving the consumption difference $C_{\mathrm{s},t} - C_{\mathrm{b},t}$, loans L_t, and the real return r_t:

$$\frac{C_{\mathrm{s},t}}{2} - \frac{C_{\mathrm{b},t}}{2} = 2\left((1+r_t)L_{t-1} - L_t\right). \qquad [3.28]$$

The other implementability constraints are the pair of Euler equations in [2.18]. Given the limited set of instruments, a second-best allocation generally entails a trade-off between efficiency and a notion of 'fairness' associated with the particular Pareto weights that are assigned.

Since the maximization of the welfare function is considered from a specific starting date t_0, for comparison, it is helpful to define an equilibrium where complete financial markets are only open from a particular date onwards. Supposing that complete markets are available for trading securities paying off from date t_0 onwards, the set of equilibrium conditions comprises equations [2.22a]–[2.22f] at all times and equation [3.5] for all $t \geq t_0$. The levels of debt, loans, the real return, and the consumption ratios in this equilibrium are denoted by $d^*_{t|t_0}$, $l^*_{t|t_0}$, $r^*_{t|t_0}$, and $c^*_{\tau,t|t_0}$, where the subscript t_0 indicates dependence on the initial conditions before t_0 (for $t < t_0$, these variables are equal to the actual values of d_t, l_t, r_t, and $c_{\tau,t}$ realized prior to date t_0). The gaps between this equilibrium and the incomplete-markets equilibrium are denoted by $\tilde{d}_{t|t_0} = d_t/d^*_{t|t_0}$, $\tilde{r}_{t|t_0} = (1+r_t)/(1+r^*_{t|t_0})$, and so on.

The following proposition characterizes the connections between the sets of first- and second-best allocations and the complete-markets equilibrium.

Proposition 11 *Consider the problem of maximizing the welfare function $\mathscr{W}_{t_0}$ from [3.26].*

(i) An allocation is first best if and only if it satisfies [3.27] for all $t \geq t_0$, $C_{\mathrm{b},t_0}/C_{\mathrm{s},t_0}$ is $t_0 - 1$-measurable, and the following risk-sharing condition holds for all $t \geq t_0 + 1$:

$$\delta_{\mathrm{b},t-1}\left(\frac{C_{\mathrm{b},t}}{C_{\mathrm{b},t-1}}\right)^{-\alpha} = \delta_{\mathrm{s},t-1}\left(\frac{C_{\mathrm{s},t}}{C_{\mathrm{s},t-1}}\right)^{-\alpha}. \qquad [3.29]$$

(ii) The equilibrium with complete markets open from t_0 onwards is first best, but there are infinitely many other first-best allocations.

(iii) The equilibrium with complete markets open from t_0 onwards is implementable through monetary policy for any initial conditions at $t_0 - 1$. This is the only first-best allocation that also belongs to the set of second-best allocations implementable through monetary policy.

(iv) The optimal policy problem with Pareto weights that support the complete-markets equilibrium has a solution that is time consistent, unlike the second-best problem for a general choice of Pareto weights.

(v) A log-linear approximation of the equilibrium with complete markets open from t_0 onwards is:

$$\mathsf{d}^*_{t|t_0} = \mathsf{d}^*_t + \lambda^{t-t_0+1}(\mathsf{d}_{t_0-1} - \mathsf{d}^*_{t_0-1}), \quad \mathsf{l}^*_{t|t_0} = \mathsf{l}^*_t + \lambda^{t-t_0+2}(\mathsf{d}_{t_0-1} - \mathsf{d}^*_{t_0-1}), \quad \mathsf{r}^*_{t|t_0} = \mathsf{r}^*_t, \quad [3.30a]$$

with d^*_t, l^*_t, and r^*_t as given in [3.11]. The consumption ratios $c^*_{\tau,t|t_0}$ are non-stochastic, depending only on predetermined variables at date t_0, and can be written approximately as $\mathsf{c}^*_{\tau,t|t_0} = \lambda^{t-t_0+1}\mathsf{c}_{\tau,t_0-1}$. The gaps relative to this equilibrium are:

$$\mathbb{E}_t\tilde{\mathsf{d}}_{t+1|t_0} = \lambda\tilde{\mathsf{d}}_{t|t_0}, \quad \tilde{\mathsf{l}}_{t|t_0} = \lambda\tilde{\mathsf{d}}_{t|t_0}, \quad \text{and } \tilde{\mathsf{r}}_{t|t_0} = \tilde{\mathsf{r}}_t = \tilde{\mathsf{d}}_{t|t_0} - \mathbb{E}_{t-1}\tilde{\mathsf{d}}_{t|t_0}; \quad [3.30b]$$

$$\tilde{\mathsf{c}}_{\mathrm{b},t|t_0} = -\left(\frac{\theta}{1-\theta}\right)\left(\frac{1-\beta\lambda}{1-\beta}\right)\tilde{\mathsf{d}}_{t|t_0}, \quad \text{and } \tilde{\mathsf{c}}_{\mathrm{s},t|t_0} = \left(\frac{\theta}{1+\theta}\right)\left(\frac{1-\beta\lambda}{1-\beta}\right)\tilde{\mathsf{d}}_{t|t_0}.$$

(vi) The Pareto weights (normalized so that units of the welfare function are equivalent to percentages of initial real GDP) supporting the complete-markets equilibrium (from t_0) are $\Omega^*_{\tau|t_0} = c^{*\alpha}_{\tau,t_0|t_0}/Y^{1-\alpha}_{t_0}$. The welfare function (given the discount factors $\delta^*_{\tau,t|t_0}$ associated with $c^*_{\tau,t|t_0}$) can be written as

$$\mathscr{W}_{t_0} = -\frac{1}{2}\frac{\alpha\theta^2(1-\beta\lambda)^2}{(1-\theta^2)(1-\beta)^2}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\mathbb{E}_{t_0}\tilde{\mathsf{d}}^2_{t|t_0} + \mathscr{I}_{t_0} + \mathscr{O}^3, \quad [3.31]$$

where $\mathscr{I}_{t_0}$ denotes terms that are independent of monetary policy from t_0 onwards, and $\mathscr{O}^3$ denotes third- and higher-order terms in the standard deviation ς of real GDP growth.

Proof See appendix A.11. ■

The intersection between the sets of first-best allocations (Pareto efficient) and second-best allocations (welfare maximizing subject to the implementability constraints) is simply the complete-markets equilibrium. The intuition for this result is that the risk-sharing condition [3.29] is necessary for any allocation to be first best. It is also the only equation [3.5] that distinguishes the equilibrium conditions with complete markets from the equilibrium conditions with incomplete markets, with the latter being the implementability constraints on monetary policy in the second-best problem.

In what follows, it is assumed the policymaker has a lexicographic preference for efficiency over any explicit distributional concerns. Such a policymaker prefers any first-best allocation over one not in that set, and since there is only one implementable first-best allocation, the policymaker always adopts the Pareto weights associated with the complete-markets equilibrium (with markets open from the current date of welfare maximization). Proposition 11 also shows this is generally the only procedure for selecting Pareto weights that results in a time-consistent plan for policy.

Finally, Proposition 11 provides the log-linearization of the equilibrium with complete markets open from t_0 onwards, generalizing the results of Proposition 4. A second-order approximation of the welfare function is also derived, which quantifies the welfare losses (in terms of percentages of output) of deviating from the complete-markets equilibrium, for which the debt gap $\tilde{\mathsf{d}}_{t|t_0}$ is a sufficient statistic. It can be seen from [3.31] that these losses are increasing in risk aversion α and the degree of heterogeneity between borrower and saver households as measured by θ. Intuitively, greater risk aversion increases the importance of the risk sharing found in complete financial markets, while greater heterogeneity between households leads to larger financial positions of borrowers and savers, so a given percentage change in financial wealth has a larger impact on consumption.

3.10 Discussion

The importance of the arguments for nominal GDP targeting in this paper obviously depends on the plausibility of the incomplete-markets assumption in the context of household borrowing and saving. It seems reasonable to suppose that households will not find it easy to borrow by issuing Arrow-Debreu state-contingent bonds, but might there be other ways of reaching the same goal? Issuance of state-contingent bonds is equivalent to households agreeing loan contracts with financial intermediaries that specify a complete menu of state-contingent repayments. But such contracts would be much more time consuming to write, harder to understand, and more complicated to enforce than conventional non-contingent loan contracts, as well as making monitoring and assessment of default risk a more elaborate exercise.[34] Moreover, unlike firms, households cannot issue securities such as equity that feature state-contingent payments but do not require a complete description of the schedule of payments in advance.[35]

Another possibility is that even if households are restricted to non-contingent borrowing, they can hedge their exposure to future income risk by purchasing an asset with returns that are negatively correlated with GDP. But there are several pitfalls to this. First, it may not be clear which asset reliably has a negative correlation with GDP (even if 'GDP securities' of the type proposed by Shiller (1993) were available, borrowers would need a short position in these). Second, the required gross positions for hedging may be very large. Third, a household already intending to borrow will need to borrow even more to buy the asset for hedging purposes, and the amount of borrowing may be limited by an initial down-payment constraint and subsequent margin calls. In practice, a typical borrower does not have a significant portfolio of assets except for a house, and housing returns most likely lack the negative correlation with GDP required for hedging the relevant risks.

In spite of these difficulties, it might be argued the case for the incomplete markets assumption is overstated because the possibilities of renegotiation, default, and bankruptcy introduce some contingency into apparently non-contingent debt contracts. However, default and bankruptcy allow for only a crude form of contingency in extreme circumstances, and these options are not without their costs. Renegotiation is also not costless, and evidence from consumer mortgages in both the recent U.S. housing bust and the Great Depression suggests that the extent of renegotiation may be

[34] For examples of theoretical work on endogenizing the incompleteness of markets through limited enforcement of contracts or asymmetric information, see Kehoe and Levine (1993) and Cole and Kocherlakota (2001).

[35] Consider an individual owner of a business that generates a stream of risky profits. If the firm's only external finance is non-contingent debt then the individual bears all the risk (except in the case of default). If the individual wanted to share risk with other investors then one possibility would be to replace the non-contingent debt with state-contingent bonds where the payoffs on these bonds are positively related to the firm's profits. However, what is commonly observed is not issuance of state-contingent bonds but equity financing. Issuing equity also allows for risk sharing, but unlike state-contingent bonds does not need to spell out a schedule of payments in all states of the world. There is no right to any specific payment in any specific state at any specific time, only the right of being residual claimant. The lack of specific claims is balanced by control rights over the firm. However, there is no obvious way to be 'residual claimant' on or have 'control rights' over a household.

inefficiently low (White, 2009a, Piskorski, Seru and Vig, 2010, Ghent, 2011). Furthermore, even ex-post efficient renegotiation of a contract with no contingencies written in ex ante need not actually provide for efficient sharing of risk from an ex-ante perspective.

It is also possible to assess the completeness of markets indirectly through tests of the efficient risk-sharing condition, which is equivalent to perfect correlation between consumption growth rates of different households. These tests are the subject of a large literature (Cochrane, 1991, Nelson, 1994, Attanasio and Davis, 1996, Hayashi, Altonji and Kotlikoff, 1996), which has generally rejected the hypothesis of full risk sharing.

Finally, even if financial markets are incomplete, the assumption that contracts are written in terms of specifically nominal non-contingent payments is important for the analysis. The evidence presented in Doepke and Schneider (2006) indicates that household balance sheets contain significant quantities of nominal liabilities and assets (for assets, it is important to account for indirect exposure via households' ownership of firms and financial intermediaries). Furthermore, as pointed out by Shiller (1997), indexation of private debt contracts is extremely rare. This suggests the model's assumptions are not unrealistic.

The workings of nominal GDP targeting can also be seen from its implications for inflation and the real value of nominal liabilities. Indeed, nominal GDP targeting can be equivalently described as a policy of inducing a perfect negative correlation between the price level and real GDP, and ensuring these variables have the same volatility. When real GDP falls, inflation increases, which reduces the real value of fixed nominal liabilities in proportion to the fall in real income, and vice versa when real GDP rises. Thus the extent to which financial markets with non-contingent nominal assets are sufficiently complete to allow for efficient risk sharing is endogenous to the monetary policy regime: monetary policy can make the real value of fixed nominal repayments contingent on the realization of shocks. A strict policy of inflation targeting would be inefficient because it converts non-contingent nominal liabilities into non-contingent real liabilities. This points to an inherent tension between price stability and the efficient operation of financial markets.[36]

That optimal monetary policy in a non-representative-agent model should feature inflation fluctuations is perhaps surprising given the long tradition of regarding inflation-induced unpredictability in the real values of contractual payments as one of the most important of all inflation's costs. As discussed in Clarida, Galí and Gertler (1999), there is a widely held view that the difficulties this induces in long-term financial planning ought to be regarded as the most significant cost of inflation, above the relative price distortions, menu costs, and deviations from the Friedman rule that have been stressed in representative-agent models. The view that unanticipated inflation leads to inefficient or inequitable redistributions between debtors and creditors clearly presupposes a world of incomplete markets, otherwise inflation would not have these effects. How then to reconcile this argument with the result that the incompleteness of financial markets suggests nominal GDP targeting is desirable because it supports efficient risk sharing? (again, were markets complete, monetary

[36]In a more general setting where the incompleteness of financial markets is endogenized, inflation fluctuations induced by nominal GDP targeting may play a role in minimizing the costs of contract renegotiation or default when the economy is hit by an aggregate shock.

policy would be irrelevant to risk sharing because all opportunities would already be exploited)

While nominal GDP targeting does imply unpredictable inflation fluctuations, the resulting real transfers between debtors and creditors are not an *arbitrary* redistribution — they are perfectly correlated with the relevant fundamental shock: unpredictable movements in aggregate real incomes. Since future consumption uncertainty is affected by income risk as well as risk from fluctuations in the real value of nominal contracts, it is not necessarily the case that long-term financial planning is compromised by inflation fluctuations that have known correlations with the economy's fundamentals. An efficient distribution of risk requires just such fluctuations because the provision of insurance is impossible without the possibility of ex-post transfers that cannot be predicted ex ante. Unpredictable movements in inflation orthogonal to the economy's fundamentals (such as would occur in the presence of monetary-policy shocks) are inefficient from a risk-sharing perspective, but there is no contradiction with nominal GDP targeting because such movements would only occur if policy failed to stabilize nominal GDP.[37]

It might be objected that if debtors and creditors really wanted such contingent transfers then they would write them into the contracts they agree, and it would be wrong for the central bank to try to second-guess their intentions. But the absence of such contingencies from observed contracts may simply reflect market incompleteness rather than what would be rationally chosen in a frictionless world. Reconciling the non-contingent nature of financial contracts with complete markets is not impossible, but it would require both substantial differences in risk tolerance across households and a high correlation of risk tolerance with whether a household is a saver or a borrower. With assumptions on preferences that make borrowers risk neutral or savers extremely risk averse, it would not be efficient to share risk, even if no frictions prevented households writing contracts that implement it.

There are a number of problems with this alternative interpretation of the observed prevalence of non-contingent contracts. First, there is no compelling evidence to suggest that borrowers really are risk neutral or savers are extremely risk averse relative to borrowers. Second, while there is evidence suggesting considerable heterogeneity in individuals' risk tolerance (Barsky, Juster, Kimball and Shapiro, 1997, Cohen and Einav, 2007), most of this heterogeneity is not explained by observable characteristics such as age and net worth (even though many characteristics such as these have some correlation with risk tolerance). The dispersion in risk tolerance among individuals with similar observed characteristics also suggests there should be a wide range of types of financial contract with different degrees of contingency. Risk neutral borrowers would agree non-contingent contracts with risk-averse savers, but contingent contracts would be offered to risk-averse borrowers.

Another problem with the complete markets but different risk preferences interpretation relates to the behaviour of the price level over time. While nominal GDP has never been an explicit target of monetary policy, nominal GDP targeting's implication of a countercyclical price level has been largely true in the U.S. during the post-war period (Cooley and Ohanian, 1991), albeit with a correlation coefficient much smaller than one in absolute value, and a lower volatility relative to

[37] The loss function derived in Proposition 11 can be applied to study quantitatively the welfare costs of the arbitrary redistributions caused by inflation resulting from monetary-policy shocks.

real GDP. Whether by accident or design, U.S. monetary policy has had to a partial extent the features of nominal GDP targeting, resulting in the real values of fixed nominal payments positively co-moving with real GDP (but by less) on average. In a world of complete markets with extreme differences in risk tolerance between savers and borrowers, efficient contracts would undo the real contingency of payments brought about by the countercyclicality of the price level, for example, through indexation clauses. But as discussed in Shiller (1997), private nominal debt contracts have survived in this environment without any noticeable shift towards indexation. Furthermore, both the volatility of inflation and correlation of the price level with real GDP have changed significantly over time (the high volatility 1970s versus the 'Great Moderation', and the countercyclicality of the post-war price level versus its procyclicality during the inter-war period). The basic form of non-contingent nominal contracts has remained constant in spite of this change.[38]

Finally, while the policy recommendation of this paper goes against the long tradition of citing the avoidance of redistribution between debtors and creditors as an argument for price stability, it is worth noting that there is a similarly ancient tradition in monetary economics (which can be traced back at least to Bailey, 1837) of arguing that money prices should co-move inversely with productivity to promote 'fairness' between debtors and creditors. The idea is that if money prices fall when productivity rises, those savers who receive fixed nominal incomes are able to share in the gains, while the rise in prices at a time of falling productivity helps to ameliorate the burden of repayment for borrowers. This is equivalent to stabilizing the money value of incomes, in other words, nominal GDP targeting. The intellectual history of this idea (the 'productivity norm') is thoroughly surveyed in Selgin (1995). Like the older literature, this paper places distributional questions at the heart of monetary policy analysis, but studies policy through the lens of mitigating inefficiencies in incomplete financial markets, rather than with looser notions of fairness.

4 Policy trade-offs

The analysis thus far has been done using a model in which none of the usual welfare costs of inflation are present. This section adds sticky prices to the model to analyse the trade-off between monetary policies that mitigate incomplete financial markets and those that seek to minimize relative-price distortions.

4.1 Households

The utility functions of households remain as given in [2.1], but now $C_{\tau,t}$ is interpreted explicitly as a consumption aggregator of a measure-one continuum of differentiated goods. Consumption of good $\imath \in [0,1]$ by households of type τ is denoted by $C_{\tau,t}(\imath)$, and the nominal price of this good is $P_t(\imath)$. Households allocate spending between goods to minimize the expenditure $P_t C_{\tau,t}$ required to obtain $C_{\tau,t}$ units of the consumption aggregator. The aggregator is the same for both types of

[38] It could be argued that part of the reluctance to adopt indexation is a desire to avoid eliminating the risk-sharing offered by nominal contracts when the price level is countercyclical.

households and features a constant elasticity of substitution ε between goods ($\varepsilon > 1$):

$$P_t C_{\tau,t} = \min_{\{C_{\tau,t}(\imath)\}} \int_{[0,1]} P_t(\imath) C_{\tau,t}(\imath) \mathrm{d}\imath \;\; \text{s.t.} \;\; C_{\tau,t} = \left(\int_{[0,1]} C_{\tau,t}(\imath)^{\frac{\varepsilon-1}{\varepsilon}} \mathrm{d}\imath \right)^{\frac{\varepsilon}{\varepsilon-1}}. \qquad [4.1]$$

The optimality conditions for this expenditure minimization problem are:

$$C_{\tau,t}(\imath) = \left(\frac{P_t(\imath)}{P_t} \right)^{-\varepsilon} C_{\tau,t}, \quad \text{where} \;\; P_t = \left(\int_{[0,1]} P_t(\imath)^{1-\varepsilon} \mathrm{d}\imath \right)^{\frac{1}{1-\varepsilon}}. \qquad [4.2]$$

The real income $Y_{\tau,t}$ of type-τ households now derives from two sources. First, each household supplies inelastically one homogeneous unit of labour, which earns real wage w_t.[39] Second, households have equal shareholdings in each of a measure-one continuum of firms, indexed by $\imath \in [0,1]$. Firm $\imath$ pays real dividend $J_t(\imath)$, and total real dividends are J_t. The assumptions on financial markets from section 2 are maintained, with the additional restriction that shares in firms are not tradable.[40] The maturity of bonds $\mu = \gamma/(1+\bar{n})$ is now taken to be a parameter rather than γ, and $\mu < 1$ is assumed so that some debt is refinanced each period (the rate of refinancing is $1-\mu$).[41] With H_t denoting hours of labour supplied, real income $Y_{\tau,t}$ is given by:

$$Y_{\tau,t} = w_t H_t + J_t, \quad \text{where} \;\; H_t = 1, \quad \text{and} \;\; J_t = \int_{[0,1]} J_t(\imath) \mathrm{d}\imath. \qquad [4.3]$$

4.2 Firms

Firm $\imath \in [0,1]$ is the monopoly producer of differentiated good $\imath$. Goods are produced using hours of a homogeneous labour input. Production of good $\imath$ is denoted by $Y_t(\imath)$, and firm $\imath$'s employment by $H_t(\imath)$. The firm pays out all real profits at time t as dividends $J_t(\imath)$:

$$J_t(\imath) = \frac{P_t(\imath)}{P_t} Y_t(\imath) - w_t H_t(\imath), \quad \text{where} \;\; Y_t(\imath) = A_t H_t(\imath)^{\frac{1}{1+\xi}}, \quad \text{and} \;\; Y_t(\imath) = \left(\frac{P_t(\imath)}{P_t} \right)^{-\varepsilon} C_t. \qquad [4.4]$$

The first equation following the definition of profits is the production function, with A_t denoting the common exogenous productivity level, and where the parameter ξ determines the extent of diminishing returns to labour ($\xi \geq 0$). The stochastic process for the growth rate of exogenous TFP A_t is assumed to have the same properties as that of the growth rate of the exogenous endowment in the model of section 2. The final equation in [4.4] is derived from the requirement that the quantity produced is the same as the quantity sold, given a measure one half of each type of household, equation [2.12], and the household demand functions in [4.2]. These two equations are constraints on the profits of firm $\imath$, and by substituting them into the definition of $J_t(\imath)$, profits can be written as a function of the price $P_t(\imath)$ of good $\imath$, the price level P_t, aggregate demand C_t, and a variable x_t denoting the level of real marginal cost for a hypothetical firm setting price equal to P_t:

$$J_t(\imath) = \left(\left(\frac{P_t(\imath)}{P_t} \right)^{1-\varepsilon} - \frac{x_t}{1+\xi} \left(\frac{P_t(\imath)}{P_t} \right)^{-\varepsilon(1+\xi)} \right) C_t, \quad \text{where} \;\; x_t = (1+\xi) \frac{w_t}{A_t} \left(\frac{C_t}{A_t} \right)^{\xi}. \qquad [4.5]$$

[39] The assumption of inelastic labour supply is relaxed in section 5.

[40] This is to avoid any interaction with the assumption that financial markets are incomplete.

[41] That is, given the fixed parameter μ and steady-state nominal GDP growth $\bar{n}$, a value of $\gamma = (1+\bar{n})\mu$ is used.

4.3 Sticky prices

Price adjustment is assumed to be staggered according to the Calvo (1983) pricing model. In each time period, there is a probability σ ($0 \leq \sigma < 1$) that firm $\imath$ must continue to use its previous nominal price $P_{t-1}(\imath)$ independent of other firms. If at time t a firm does receive an opportunity to change price, it sets a reset price denoted by $\hat{P}_t$ (all firms changing price at the same time choose the same reset price because the occurrence of opportunities for changing price is random and this is the only heterogeneity between firms). The reset price is set to maximize the current and expected future stream of profits. Future profits conditional on continuing to charge $\hat{P}_t$ are multiplied by the probability σ^ℓ that the reset price will actually remain in use ℓ periods ahead, and are discounted using the real interest rate ρ_t.[42] Using [4.5], $\hat{P}_t$ is determined by:

$$\max_{\hat{P}_t} \sum_{\ell=0}^{\infty} \mathbb{E}_t \left[\left\{ \frac{\sigma^\ell}{\prod_{\jmath=0}^{\ell-1}(1+\rho_{t+\jmath})} \right\} \left(\left(\frac{\hat{P}_t}{P_{t+\ell}} \right)^{1-\varepsilon} - \frac{x_{t+\ell}}{1+\xi} \left(\frac{\hat{P}_t}{P_{t+\ell}} \right)^{-\varepsilon(1+\xi)} \right) C_{t+\ell} \right]. \qquad [4.6]$$

The first-order condition for this maximization problem is:

$$\sum_{\ell=0}^{\infty} \mathbb{E}_t \left[\left\{ \frac{\sigma^\ell C_{t+\ell}}{\prod_{\jmath=0}^{\ell-1}(1+\rho_{t+\jmath})} \right\} \left(\frac{\hat{P}_t}{P_{t+\ell}} \right)^{-\varepsilon} \left(\frac{\hat{P}_t}{P_{t+\ell}} - \frac{\varepsilon}{\varepsilon-1} \left(\frac{\hat{P}_t}{P_{t+\ell}} \right)^{-\varepsilon\xi} x_{t+\ell} \right) \right] = 0. \qquad [4.7]$$

4.4 Equilibrium

In equilibrium, the total demand for each good $\imath$ must add up to production $Y_t(\imath)$, and the total demand for the homogeneous labour input must add up to its supply H_t:

$$\frac{1}{2}C_{\mathrm{b},t}(\imath) + \frac{1}{2}C_{\mathrm{s},t}(\imath) = Y_t(\imath), \quad \text{and} \quad \int_{[0,1]} H_t(\imath)\mathrm{d}\imath = H_t. \qquad [4.8]$$

Using the definition of total profits J_t from [4.3], by summing over profits $J_t(\imath)$ from [4.4] and using the definition of total consumption [2.12] and the market clearing conditions in [2.12] and [4.8], it follows that $J_t = Y_t - w_t H_t$. Together with [4.3], this means the distribution of total income given in [2.4] continues to hold.

Using the inelastic labour supply assumption from [4.3], the production functions and demand functions in [4.4], and the overall goods-market clearing condition [2.12] and labour-market clearing condition from [4.8], aggregate output Y_t is given by:

$$Y_t = \frac{A_t}{\Delta_t}, \quad \text{where } \Delta_t = \left(\int_{[0,1]} \left(\frac{P_t(\imath)}{P_t} \right)^{-\varepsilon(1+\xi)} \mathrm{d}\imath \right)^{\frac{1}{1+\xi}}. \qquad [4.9]$$

This is the economy's aggregate production function taking account of the fixed supply of labour and the distribution of relative prices. The term Δ_t represents the inefficiency caused by relative-price distortions ($\Delta_t \geq 1$, using equation [4.2]).

[42] The choice of discount factor has no consequences in this version of the model with inelastic labour supply. In the case of elastic labour supply, with a zero-inflation steady state, it turns out that only the steady-state value of the discount factor matters for a first-order accurate approximation to the equilibrium and to optimal policy. Given the results of Proposition 1, this means the choice of the ex-ante real interest rate to discount future profits as opposed to some combination of the stochastic discount factors of the households is of no consequence.

Given Calvo pricing, there is a stationary distribution of the ages of the reset prices used by firms. A fraction $(1-\sigma)\sigma^{\ell}$ uses a reset price set ℓ periods ago, and from equation [4.2], the price level P_t and the reset price $\hat{P}_t$ are related as follows:

$$P_t = \left(\sigma P_{t-1}^{1-\varepsilon} + (1-\sigma)\hat{P}_t^{1-\varepsilon}\right)^{\frac{1}{1-\varepsilon}}, \quad \text{hence } \hat{p}_t = \frac{\hat{P}_t}{P_t} = \left(\frac{1-\sigma(1+\pi_t)^{\varepsilon-1}}{1-\sigma}\right)^{\frac{1}{1-\varepsilon}}. \qquad [4.10]$$

The second equation gives the implied relationship between the relative reset price $\hat{p}_t$ and the inflation rate π_t. Similarly, the geometric age distribution of reset prices and the formula for $\varDelta_t$ from [4.9] imply:

$$(P_t^{-\varepsilon}\varDelta_t)^{1+\xi} = \sum_{\ell=0}^{\infty}(1-\sigma)\sigma^{\ell}\hat{P}_{t-\ell}^{-\varepsilon(1+\xi)}, \quad \text{or } (P_t^{-\varepsilon}\varDelta_t)^{1+\xi} = \sigma(P_{t-1}^{-\varepsilon}\varDelta_{t-1})^{1+\xi} + (1-\sigma)\hat{P}_t^{-\varepsilon(1+\xi)},$$

and by using [4.10], relative-price distortions $\varDelta_t$ and inflation π_t are related as follows:

$$\varDelta_t = \left(\sigma(1+\pi_t)^{\varepsilon(1+\xi)}\varDelta_{t-1}^{1+\xi} + (1-\sigma)\left(\frac{1-\sigma(1+\pi_t)^{\varepsilon-1}}{1-\sigma}\right)^{\frac{\varepsilon(1+\xi)}{\varepsilon-1}}\right)^{\frac{1}{1+\xi}}. \qquad [4.11]$$

Finally, using [4.7] and [4.10], inflation π_t and real marginal cost x_t must satisfy:

$$\sum_{\ell=0}^{\infty}\mathbb{E}_t\left[\left\{\frac{\sigma^{\ell}\prod_{\jmath=1}^{\ell}(1+g_{t+\jmath})}{\prod_{\jmath=0}^{\ell-1}(1+\rho_{t+\jmath})}\right\}\left(\frac{\left(\frac{1-\sigma(1+\pi_t)^{\varepsilon-1}}{1-\sigma}\right)^{\frac{1+\varepsilon\xi}{1-\varepsilon}}}{\prod_{\jmath=1}^{\ell}(1+\pi_{t+\jmath})^{1-\varepsilon}} - \frac{\frac{\varepsilon}{\varepsilon-1}x_{t+\ell}}{\prod_{\jmath=1}^{\ell}(1+\pi_{t+\jmath})^{-\varepsilon(1+\xi)}}\right)\right] = 0. \qquad [4.12]$$

4.5 Policy analysis

Consider first the special case of fully flexible prices ($\sigma = 0$) in an economy with incomplete financial markets and $\theta > 0$. It follows from [4.11] that $\hat{\varDelta}_t = 1$, and from [4.9] that $\hat{Y}_t = A_t$ ($\hat{Y}_t$ denotes the equilibrium value of Y_t when prices are fully flexible, and similarly for other variables). The flexible-price level of output $\hat{Y}_t$ is here equal to the first-best level of output and is independent of monetary policy. Thus, in this case, the only distortion in the economy is the incompleteness of financial markets, for which the optimal policy of nominal GDP targeting (or a countercyclical price level) has been characterized earlier (Proposition 3), conditional on the behaviour of the exogenous real GDP growth rate $\hat{g}_t$.

Second, consider the special case of complete financial markets (the modification of the assumptions in section 3.1) or a representative-household economy ($\theta = 0$). In this case, the presence of sticky prices ($\sigma > 0$) is the only distortion. It can be seen from [4.11] that a policy of strict inflation targeting with zero inflation implies $\varDelta_t = 1$ (assuming no initial relative-price distortions, $\varDelta_{t_0-1} = 1$), which given [4.9] achieves the first-best level of output $Y_t = \hat{Y}_t$. Therefore, strict inflation targeting is the optimal monetary policy.

In the general case there is a trade-off between addressing the problems of incomplete financial markets and sticky prices. The optimal trade-off is determined by maximizing the welfare function subject to the equilibrium conditions of the economy. The Pareto weights are set to those that support the equilibrium with complete markets open from t_0 onwards, assuming flexible prices.

This choice is motivated by the result from Proposition 11 that choosing other Pareto weights would lead to an additional trade-off between efficiency and fairness.

Proposition 12 *The Pareto weights in the welfare function [3.26] are set to* $\hat{\Omega}^*_{\tau|t_0} = c^{*\alpha}_{\tau,t_0|t_0}/\hat{Y}^{1-\alpha}_{t_0}$ *where* $c^*_{\tau,t|t_0}$ *is the consumption allocation in the complete-markets equilibrium from* t_0 *onwards and* $\hat{Y}_t$ *is output with flexible prices. The loss function*

$$\mathscr{L}_{t_0} = \frac{1}{2}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\mathbb{E}_{t_0}\left[\frac{\alpha\theta^2(1-\beta\lambda)^2}{(1-\theta^2)(1-\beta)^2}\tilde{\mathsf{d}}^2_{t|t_0} + \frac{\varepsilon(1+\varepsilon\xi)\sigma}{(1-\sigma)(1-\beta\sigma)}\pi^2_t\right], \qquad [4.13]$$

is a second-order accurate approximation of the welfare function [3.26] around the non-stochastic steady state with zero inflation ($\bar{\pi}=0$) *in that* $\mathscr{W}_{t_0} = -\mathscr{L}_{t_0} + \mathscr{I}_{t_0} + \mathscr{O}^3$, *where* $\mathscr{I}_{t_0}$ *denotes terms that are independent of monetary policy from* t_0 *onwards, and* $\mathscr{O}^3$ *terms that are third-order or higher in the standard deviation* ς *of real GDP growth. First-order accurate approximations of the constraints that monetary policy must satisfy are:*

$$\mathbb{E}_t\tilde{\mathsf{d}}_{t+1|t_0} = \lambda\tilde{\mathsf{d}}_{t|t_0}, \quad \frac{\mathsf{j}_{t-1}-\beta\mu\mathsf{j}_t}{1-\beta\mu} - \pi_t - \tilde{\mathsf{d}}_{t|t_0} + \lambda\tilde{\mathsf{d}}_{t-1|t_0} = \hat{\mathsf{r}}^*_t, \quad \textit{and} \quad \lim_{\ell\to\infty}(\beta\mu)^\ell\mathbb{E}_t\mathsf{j}_{t+\ell} = 0, \quad [4.14]$$

with $\mathsf{g}_t = \hat{\mathsf{g}}_t$ *(since* $\Delta_t = \mathscr{O}^2$*), up to terms of order* $\mathscr{O}^2$*, and where* $\mu = \gamma/(1+\bar{g})$.

Proof See appendix A.12. ■

The loss function includes the squared debt-to-GDP gap $\tilde{\mathsf{d}}_{t|t_0}$, which is a sufficient statistic for the welfare loss due to deviations from complete markets (the coefficient is the same as for the flexible-price economy). The second term is the squared inflation rate, which is a sufficient statistic for the welfare loss due to relative-price distortions. This is a well-known property of Calvo pricing, and the coefficient in the loss function is the same as in standard models (see Woodford, 2003).

The coefficient of inflation is increasing in the price elasticity of demand ε because a higher price sensitivity implies that a given amount of price dispersion now leads to greater dispersion of the quantities produced of different goods for which preferences and production technologies are identical. The coefficient is increasing in price stickiness σ because longer price spells imply that a given amount of inflation leads to greater relative-price distortions. The coefficient is also increasing in the parameter ξ (the output elasticity of marginal cost from [4.5]), which will be seen to determine the degree of real rigidity in the economy. Both nominal and real rigidity increase the welfare costs of inflation. Finally, note that a zero steady-state inflation rate is assumed. Given the presence of sticky prices, this is the optimal steady-state inflation rate.[43]

Since the steady state of the economy is not distorted (there are no linear terms in the loss function [4.13]), a first-order accurate approximation of optimal monetary policy can be obtained by minimizing the loss function subject to first-order accurate approximations of the economy's equilibrium conditions. These constraints are given in [4.14]. The first equation determines the

[43]With debt maturity $\mu = \gamma/(1+\bar{n})$ being a fixed parameter, the average rate of inflation $\bar{\pi}$ has no consequences for the incompleteness of financial markets. If the coupon parameter γ were fixed then the average rate of inflation would affect the constraints in [4.14] through $\bar{n}$ and hence μ, so it is not obvious what average inflation rate is optimal (the effect is absent in the special case of one-period debt contracts, $\gamma = 0$). This issue is left for future research.

predictable component of the debt-to-GDP gap $\tilde{\mathsf{d}}_{t|t_0}$, and the second equation links inflation π_t and nominal bond yields j_t to the unexpected component of the debt-to-GDP gap. The term $\hat{\mathsf{r}}^*_t$ denotes the ex-post real return with both complete markets and flexible prices (that is, when the real GDP growth rate is $\mathsf{g}_t = \hat{\mathsf{g}}_t$). Since $\hat{\mathsf{g}}_t$ is exogenous, $\hat{\mathsf{r}}^*_t$ can be determined as in Proposition 4, and this term plays the role of the exogenous shock in the system of equations [4.14]. The bond yield j_t is determined by equation [3.13], but this is not added as an additional constraint because it is implied by the second equation in [4.14] together with the third constraint, the transversality condition.

Finally, the policy rate i_t is linked to inflation via the Fisher equation $\mathsf{i}_t = \rho_t + \mathbb{E}_t\pi_{t+1}$ from [3.10], and to bond yields via the expectations theory equation $\mathsf{j}_t = (1-\beta\mu)\sum_{\ell=0}^{\infty}(\beta\mu)^\ell\mathbb{E}_t\mathsf{i}_{t+\ell}$ given [3.13]. Since the policy rate does not appear directly in the loss function [4.13] nor in the constraints [4.14], the optimal policy problem has the central bank directly choosing $\tilde{\mathsf{d}}_{t|t_0}$, π_t, and j_t subject to the constraints, with the required policy rate path then determined using the Fisher equation or the expectations equation $\mathsf{i}_t = (\mathsf{j}_t - \beta\mu\mathbb{E}_t\mathsf{j}_{t+1})/(1-\beta\mu)$ in recursive form.

Proposition 13 *Optimal monetary policy with commitment from date t_0 onwards minimizes the loss function [4.13] subject to the constraints in [4.14].*

(i) The optimal policy features fluctuations in the debt-to-GDP gap:

$$\tilde{\mathsf{d}}_{t|t_0} = \lambda\tilde{\mathsf{d}}_{t-1|t_0} - (1-\chi)\wp_t, \quad \textit{with} \;\; \wp_t = \sum_{\ell=0}^{\infty}\beta^\ell\left(1-\alpha(1-\mu^\ell)\right)\left(\mathbb{E}_t\hat{\mathsf{g}}_{t+\ell} - \mathbb{E}_{t-1}\hat{\mathsf{g}}_{t+\ell}\right), \tag{4.15a}$$

$$\textit{and} \;\; \chi = \left(1 + \frac{\varepsilon(1+\varepsilon\xi)\sigma(1-\theta^2)(1-\beta)^2(1-\beta\lambda^2)(1-\beta\mu^2)}{\alpha\theta^2(1-\sigma)(1-\beta\sigma)(1-\beta\lambda)^2}\right)^{-1}. \tag{4.15b}$$

(ii) The optimal policy features inflation fluctuations, with optimal inflation persistence determined by the maturity of debt contracts (the parameter μ):

$$\pi_t = \begin{cases} \mu\pi_{t-1} - (1-\beta\mu^2)\chi\wp_t & \textit{if } t \geq t_0 + 1 \\ -(1-\beta\mu^2)\chi\wp_t & \textit{if } t = t_0 \end{cases}. \tag{4.16}$$

(iii) If the growth rate of TFP $\mathsf{A}_t - \mathsf{A}_{t-1}$ is a stationary and invertible stochastic process then optimal monetary policy can be characterized as an inflation smoothing rule $\mathbb{E}_t\pi_{t+1} = \mu\pi_t$ together with either a long-run or a short-run target for weighted nominal GDP:

$$\textit{Long run}: \;\; \mathsf{P}_t + \vec{\omega}\mathsf{Y}_t \;\; \textit{stationary}, \quad \textit{with} \;\; \vec{\omega} = \chi\frac{(1-\beta\mu^2)}{(1-\mu)}\frac{\Theta(\beta\mu)}{\Theta(1)}\omega^*; \tag{4.17a}$$

$$\textit{Short run}: \;\; \mathsf{P}_t + \dot{\omega}\mathsf{Y}_t = \mathbb{E}_{t-1}[\mathsf{P}_t + \dot{\omega}\mathsf{Y}_t], \quad \textit{with} \;\; \dot{\omega} = \chi(1-\beta\mu^2)\frac{\Theta(\beta\mu)}{\Theta(0)}\omega^*, \tag{4.17b}$$

where $\Theta(\mathbb{L})$ is such that $\mathsf{g}_t = \Theta(\mathbb{L})(\mathsf{g}_t - \mathbb{E}_{t-1}\mathsf{g}_t)$ and ω^* is from [3.15] of Proposition 5.

Proof See appendix A.13. ■

Fluctuations in the growth rate $\hat{\mathsf{g}}_t$ of flexible-price output (due to shocks to TFP) lead to changes in the ex-post real return $\hat{\mathsf{r}}^*_t$ on the complete-markets portfolio. To replicate complete financial markets, the central bank needs to vary inflation and nominal bond yields so as to mimic this real return. Overall, what matters are unexpected changes in the discounted sum of current and future growth rates, adjusted for any mitigating (or aggravating) changes in real interest rates. This is the shock $\wp_t$ given in [4.15a]. The term $\alpha(1-\mu^{\ell})$ is the adjustment for changes in real interest rates caused by revised expectations of the economy's future growth prospects. The parameter α is the elasticity of the real interest rate with respect to expected real GDP growth (see [3.9a]). Since changes in (ex-ante) real interest rates only matter to the extent that debt is refinanced, for growth expectations ℓ periods ahead, the interest-rate effect is proportional to the fraction $1-\mu^{\ell}$ of existing debt that will be refinanced by then.

With sticky prices, replicating complete financial markets through variation in inflation is now costly, so the central bank tolerates some deviation from complete markets.[44] To the extent that χ in [4.15b] is less than one, a shock $\wp_t$ leads to fluctuations in the debt-to-GDP gap $\tilde{d}_t$. These fluctuations are persistent because of the first constraint in [4.14]: the serial correlation of the debt-to-GDP gap is λ. Once a non-zero debt gap arises at time t, there is no predictable future policy action that can undo its future consequences.

If χ were equal to 1, equation [4.15a] shows that the debt-to-GDP gap would be completely stabilized, and if χ were 0, equation [4.16] shows that inflation would be completely stabilized. Since $0 < \chi < 1$, optimal monetary policy can be interpreted as a convex combination of strict inflation targeting and a policy that replicates complete financial markets. As the responses of $\tilde{d}_{t|t_0}$ and π_t to the shock $\wp_t$ are linearly related to χ, the terms χ and $1-\chi$ can be interpreted respectively as the weights on completing financial markets and avoiding relative-price distortions. Comparing equations [4.13] and [4.15b], it can be seen that χ is positively related to the ratio of the coefficients of $\tilde{d}^2_{t|t_0}$ and π_t^2 in the loss function divided by $(1-\beta\lambda^2)$ and $(1-\beta\mu^2)$.

Greater risk aversion (α) or more heterogeneity and hence more borrowing (θ) increase the coefficient of $\tilde{d}^2_{t|t_0}$ and thus χ; a larger price elasticity of demand (ε), stickier prices (σ), or more real rigidities (ξ) increase the coefficient of π_t^2 and thus reduce χ. The optimal trade-off is also affected by the constraints in [4.14], which explains the presence of the terms $(1-\beta\lambda^2)$ and $(1-\beta\mu^2)$ in the formula for χ. A greater value of λ increases the persistence of the debt-to-GDP gap, which makes fluctuations in $\tilde{d}_{t|t_0}$ more costly than suggested by the loss function coefficient alone. The parameter μ affects the link between bond yields and the debt-to-GDP gap. It is seen that an increase in μ leads to a higher value of χ, the intuition for which is related to the optimal behaviour of inflation. Finally, note that while the optimal policy responses depend on the stochastic process for real GDP growth, the optimal weight χ does not.

Equation [4.16] shows that optimal monetary policy features inflation persistence with serial correlation given by the debt maturity parameter $\mu = \gamma/(1+\bar{g})$. The steady-state fractions of existing and newly issued debt are μ and $1-\mu$ respectively, so the result is that inflation should return to its average value at the same rate at which debt is refinanced. At the extremes, one-period

[44] If the nominal rigidity were sticky wages rather than sticky prices, the central bank would care about nominal wage inflation rather than price inflation (Erceg, Henderson and Levin, 2000). In that case, the tension with the goal of replicating complete financial markets is reduced because targeting nominal income growth is less likely to be in conflict with stabilizing nominal wage inflation than stabilizing price inflation.

debt ($\gamma = 0$ and $\mu = 0$) corresponds to serially uncorrelated inflation, while perpetuities ($\gamma = 1$, for which $\mu \approx 1$) correspond to near random-walk persistence of inflation.

To understand this, note that with one-period debt, the current bond yield j_t disappears from the constraints [4.14] (because [3.13] reduces to the standard ex-post Fisher equation in this case), thus the only way that policy can affect $\tilde{d}_{t|t_0}$ is through an unexpected change in current inflation. With longer maturity debt, the range of policy options increases. Changes in current bond yields j_t are also relevant in addition to current inflation, and it can be seen from [3.13] that the bond yield is affected by expectations of future inflation. The three constraints in [4.14] imply $\tilde{d}_{t|t_0} = \lambda\tilde{d}_{t-1|t_0} - \wp_t - \sum_{\ell=0}^{\infty}(\beta\mu)^{\ell}(\mathbb{E}_t\pi_{t+\ell} - \mathbb{E}_{t-1}\pi_{t+\ell})$, where μ^{ℓ} indicates the fraction of existing debt that will not have been refinanced after ℓ time periods. This shows it is now possible to use inflation that is spread out over time to influence the debt-to-GDP gap $\tilde{d}_{t|t_0}$, not only inflation surprises.

Furthermore, this 'inflation smoothing' is optimal because the welfare costs of inflation are convex (inflation appears in the loss function [4.13] as π_t^2), so the costs of a given cumulated amount of inflation are smaller when spread out over a number of quarters or years than when all the inflation occurs in just one quarter. This is analogous to the 'tax smoothing' argument of Barro (1979). Interestingly, the argument shows that high degrees of inflation persistence need not be interpreted as a failure of policy. Differently from the 'tax smoothing' argument, it is generally not optimal for inflation to display random walk or near-random walk persistence unless debt contracts are close to perpetuities. Equation [3.13] shows that as the maturity parameter μ is reduced and thus $\beta\mu$ falls significantly below one, expectations of inflation far in the future have a smaller effect on bond yields than inflation in the near future. The further in the future inflation is expected to occur, the less effective it is at influencing real returns and thus the debt-to-GDP ratio.

Even if optimal monetary policy places a substantial weight χ on the problem of incomplete markets compared to relative-price distortions, in what sense does monetary policy still resemble a nominal GDP target? Optimal inflation smoothing $\mathbb{E}_t\pi_{t+1} = \mu\pi_t$ (from [4.16]) pins down the predictable component of the inflation rate, but this in itself is not a complete description of policy because it leaves unspecified the unexpected component of inflation. Together with inflation smoothing, specifying how much inflation reacts to a shock on impact, or equivalently, how much cumulated inflation will follow a shock, completely characterizes the path of all nominal variables.

It turns out that optimal monetary policy retains the essential feature of nominal GDP targeting in generating a negative comovement between prices and output. However, because of the desire to smooth inflation, the central bank should not aim to stabilize nominal GDP (or a weighted measure of nominal GDP) exactly on a quarter-by-quarter basis. Instead, optimal policy can be formulated as a long-run target for weighted nominal GDP. When real GDP is non-stationary because TFP follows a non-stationary process, optimal monetary policy features cointegration between the price level and output. This cointegrating relationship can be interpreted a long-run target for weighted nominal GDP because there is some linear combination of prices and real output that is eventually returned to a constant or a deterministic trend following a shock to output.

The optimal monetary policy implications for inflation in [4.16] have one other noteworthy feature: time inconsistency in the case where $\mu > 0$. Starting from date t_0, optimal monetary policy commits to smooth out any inflation over future time periods, but chooses an inflation rate at date t_0 which is statistically independent of any past variables. In other words, it is optimal ex ante to commit to inflation smoothing, but optimal ex post to renege on those plans. The existence of this time-inconsistency problem depends on both sticky prices and debt maturities longer than one time period (time consistency with flexible prices is shown in Proposition 11, while time consistency with $\mu = 0$ is immediately apparent from [4.16]).

The intuition for time inconsistency comes from the timing of the benefits and costs of inflation fluctuations. As can be seen from the constraints [4.14], to close the gap with the complete-markets consumption allocation from time t onwards, policy (whether current actions or expectations of future actions) must affect the debt-to-GDP gap $\tilde{\mathsf{d}}_{t|t_0}$ at the time t. This reflects the forward-looking nature of household consumption and saving decisions and thus the forward-looking nature of financial markets. Accordingly, any future policy commitments announced in response to a shock affect bond yields, the debt-to-GDP ratio, and consumption immediately. When that commitment takes the form of inflation smoothing, some of the welfare costs of inflation are deferred until the future (with long-term debt, this is what allows overall welfare costs to be reduced). There is then an incentive to renege on the commitment once the benefits have been obtained and only the costs remain. In the absence of an ability to commit, the (Markovian) discretionary policy equilibrium is characterized below.

Proposition 14 *The Markovian discretionary policy equilibrium (reoptimization at all dates, with expectations of future policy actions restricted to depend on fundamental state variables only) is:*

$$\tilde{\mathsf{d}}_{t|t_0} = \lambda \tilde{\mathsf{d}}_{t-1|t_0} - (1-\chi')\wp_t, \quad \text{and} \quad \pi_t = -\chi'\wp_t, \quad \text{where} \tag{4.18a}$$

$$\chi' = \left(1 + \frac{\varepsilon(1+\varepsilon\xi)\sigma(1-\theta^2)(1-\beta)^2(1-\beta\lambda^2)}{\alpha\theta^2(1-\sigma)(1-\beta\sigma)(1-\beta\lambda)^2}\right)^{-1}. \tag{4.18b}$$

for arbitrary t_0 and all $t \geq t_0$, with $\wp_t$ as given in [4.15a]. The coefficient χ' is such that $\chi' < \chi$, where χ in [4.15b] corresponds to the case of commitment. When debt contracts have a maturity greater than one period ($\mu > 0$), the debt gap $\tilde{\mathsf{d}}_{t|t_0}$ has a higher standard deviation under discretion than commitment, but the same serial correlation in both cases. Inflation is serially uncorrelated under discretion, and hence less serially correlated than under commitment.

PROOF See appendix A.14. ■

This result might be described as the 'financial instability bias' of discretionary monetary policy. At a given point in time, the lowest-cost means of replicating the complete-markets consumption allocation is through affecting forward-looking financial variables such as bond yields, which depend on expectations of future policy (the longer the maturity of the bond, the more forward looking its yield is). However, commitments to change future inflation to affect those expectations are not credible. On the other hand, the welfare costs of inflation at that point in time depend on the actual rate of inflation, not expected future inflation.[45] Thus, the inability to commit makes it

[45]Note that even though price adjustment is staggered, the absence of an elastic labour supply decision means

difficult for monetary policy to affect current financial variables, but does not impede monetary policy controlling current inflation.

Discretion therefore leads monetary policy to focus too much on keeping inflation close to its optimal average rate and not enough on stabilizing variables such as the debt-to-GDP ratio that are best influenced through long-term bond yields. In equilibrium, the debt-to-GDP ratio is too volatile, while the central bank is always expected to return inflation to its average rate next period. Since how much the central bank cares about the debt-to-GDP ratio has not changed, inflation may actually end up being more volatile, though less serially correlated than under policy with commitment.

5 The ex-ante real interest rate and the output gap

By extending the model of section 4 to allow for a labour supply decision, monetary policy can affect two additional variables: the ex-ante real interest rate and the output gap. This turns out to add a number of interesting dimensions to the optimal monetary policy problem.

5.1 Households

The utility function [2.1] for households of type τ is replaced by the following, which now depends on hours $H_{\tau,t}$ of labour supplied in addition to consumption $C_{\tau,t}$:

$$\mathscr{U}_{\tau,t} = \sum_{\ell=0}^{\infty} \mathbb{E}_t \left[\left\{ \prod_{\jmath=1}^{\ell-1} \delta_{\tau,t+\jmath} \right\} \left(\frac{C_{\tau,t+\ell}^{1-\alpha}}{1-\alpha} - \frac{H_{\tau,t+\ell}^{1+\frac{1}{\eta_\tau}}}{1+\frac{1}{\eta_\tau}} \right) \right]. \qquad [5.1]$$

Households supply different types of labour, and η_τ is the potentially household-specific Frisch elasticity of labour supply ($0 < \eta_\tau < \infty$). The discount factors $\delta_{\tau,t}$ are as specified in [2.2] and are taken as given by individual households. The real wage received by type-τ households is $w_{\tau,t}$, all households own equal (non-tradable) shareholdings in each firm, and all households are assumed to face a common lump-sum tax T_t in real terms. Households of type τ thus have real income

$$Y_{\tau,t} = w_{\tau,t} H_{\tau,t} + J_t - T_t, \qquad [5.2]$$

where J_t is as defined in [4.3]. Since utility [5.1] is additively separable between consumption and hours, the Euler equations [2.6] and [2.7] are unchanged. Maximizing utility [5.1] with respect to hours $H_{\tau,t}$ subject to [2.5] and [5.2] implies the optimal labour supply condition:

$$C_{\tau,t}^{\alpha} H_{\tau,t}^{\frac{1}{\eta_\tau}} = w_{\tau,t}. \qquad [5.3]$$

there are no output gap fluctuations, and thus no New Keynesian Phillips curve as a constraint on monetary policy. The 'stabilization bias' analysed in Clarida, Galí and Gertler (1999) depends on the expectations of future inflation in the Phillips curve. It is shown in Proposition 16 below that the 'financial instability bias' remains even in a model with elastic labour supply where the standard New Keynesian Phillips curve is present as a constraint.

5.2 Firms

As before, there is a measure-one continuum of firms, each producing one of a continuum of differentiated goods. Two changes are introduced here: the labour input $H_t(\imath)$ of firm $\imath$ is now an aggregator of labour supplied by the two types of households, and firms now receive a proportional wage-bill subsidy at rate ε^{-1}. Firms choose labour inputs $H_{\tau,t}(\imath)$ to minimize the post-subsidy cost $w_t H_t(\imath)$ of obtaining a unit of the aggregate labour input $H_t(\imath)$:

$$w_t H_t(\imath) = \min_{H_{\tau,t}(\imath)} (1-\varepsilon^{-1})\left(w_{\mathrm{b},t}H_{\mathrm{b},t} + w_{\mathrm{s},t}H_{\mathrm{s},t}\right) \quad \text{s.t.} \quad H_t(\imath) = 2H_{\mathrm{b},t}(\imath)^{\frac{1}{2}}H_{\mathrm{s},t}(\imath)^{\frac{1}{2}}. \tag{5.4}$$

The labour aggregator has a Cobb-Douglas functional form, implying a unit elasticity of substitution between different labour types.[46] The cost-minimizing demand functions for individual labour inputs and the overall level of wage costs are:

$$H_{\tau,t}(\imath) = \frac{1}{2}\frac{w_t H_t(\imath)}{(1-\varepsilon^{-1})w_{\tau,t}}, \quad \text{where} \quad w_t = (1-\varepsilon^{-1})w_{\mathrm{b},t}^{\frac{1}{2}}w_{\mathrm{s},t}^{\frac{1}{2}}. \tag{5.5}$$

Apart from the reinterpretation of $H_t(\imath)$ and w_t, the production function, profit function, and the expression for real marginal cost x_t are the same as in [4.4] and [4.5].

5.3 Fiscal policy

The only role of fiscal policy here is to provide the wage-bill subsidy to firms by collecting equal amounts of a lump-sum tax from all households. It is assumed the fiscal budget is in balance, so taxes T_t are set at the level required to fund the current subsidy:[47]

$$T_t = \varepsilon^{-1}\int_{[0,1]} \left(w_{\mathrm{b},t}H_{\mathrm{b},t}(\imath) + w_{\mathrm{s},t}H_{\mathrm{s},t}(\imath)\right) \mathrm{d}\imath. \tag{5.6}$$

5.4 Equilibrium

There is now a separate market-clearing condition for each type of labour in addition to the equilibrium conditions introduced earlier in [4.8]:

$$\int_{[0,1]} H_{\tau,t}(\imath)\mathrm{d}\imath = \frac{1}{2}H_{\tau,t}, \quad \text{for all} \quad \tau \in \{\mathrm{b},\mathrm{s}\}. \tag{5.7}$$

Using the labour demand functions and wage index from [5.5] together with the market-clearing condition [5.7] and letting H_t be defined in accordance with [4.8] leads to:

$$H_{\tau,t} = \frac{w_t H_t}{(1-\varepsilon^{-1})w_{\tau,t}}, \quad \text{where} \quad H_t = H_{\mathrm{b},t}^{\frac{1}{2}}H_{\mathrm{s},t}^{\frac{1}{2}}, \quad \text{and} \quad Y_t = \frac{A_t H_t^{\frac{1}{1+\xi}}}{\varDelta_t}, \tag{5.8}$$

[46] The use of an aggregator of different labour types is the standard approach to studying the welfare costs of sticky wages (Erceg, Henderson and Levin, 2000). Here, a unit elasticity of substitution is the most analytically convenient assumption, though a priori it is not clear whether the labour of different household types is more substitutable or more complementary than this. In the model, being a borrower is simply a matter of being more impatient than savers, but empirically, the average borrower is likely to differ from the average saver in respects such as age that might mean their labour is not perfectly substitutable.

[47] The wage-bill subsidy is a standard assumption which ensures the economy's steady state is not distorted (Woodford, 2003). A balanced-budget rule is assumed to avoid any interactions between fiscal policy and financial markets.

where the final equation is derived using the same steps as [4.9].

With the same argument as before, the equilibrium conditions imply that per-household profits are $J_t = Y_t - w_t H_t$. Using equations [5.4] and [5.7], total taxes must be $T_t = \varepsilon^{-1} w_t H_t/(1-\varepsilon^{-1})$. Noting that [5.8] implies $w_{\tau,t} H_{\tau,t} = w_t H_t/(1-\varepsilon^{-1})$, it follows from [5.2] that the distribution of total income in [2.4] continues to hold.

The equilibrium of each of the labour markets can be derived conditional on aggregate variables and the consumption allocation by using equations [5.3] and [5.8]:

$$H_{\tau,t} = \left(\frac{w_t H_t}{(1-\varepsilon^{-1}) C_{\tau,t}^{\alpha}}\right)^{\frac{\eta_\tau}{1+\eta_\tau}}, \quad \text{and} \quad w_{\tau,t} = \left(\frac{w_t H_t C_{\tau,t}^{\alpha\eta_\tau}}{1-\varepsilon^{-1}}\right)^{\frac{1}{1+\eta_\tau}}, \tag{5.9}$$

and by substituting the expression for wages $w_{\tau,t}$ into equation [5.5], overall wage costs are:

$$w_t = (1-\varepsilon^{-1}) Y_t^{\alpha} H_t^{\frac{1}{\eta}} \left(c_{\mathrm{b},t}^{\frac{\eta_\mathrm{b}}{1+\eta_\mathrm{b}}} c_{\mathrm{s},t}^{\frac{\eta_\mathrm{s}}{1+\eta_\mathrm{s}}}\right)^{\frac{\alpha}{\frac{\eta_\mathrm{b}}{1+\eta_\mathrm{b}} + \frac{\eta_\mathrm{s}}{1+\eta_\mathrm{s}}}}, \quad \text{where } \eta = \frac{\frac{\eta_\mathrm{b}}{1+\eta_\mathrm{b}} + \frac{\eta_\mathrm{s}}{1+\eta_\mathrm{s}}}{\frac{1}{1+\eta_\mathrm{b}} + \frac{1}{1+\eta_\mathrm{s}}}. \tag{5.10}$$

The term η denotes an average of the Frisch elasticities of the two household types. By combining equations [4.5], [5.8], and [5.10], real marginal cost x_t (for a firm that has set a price equal to the price level P_t) is given by:

$$x_t = (1-\varepsilon^{-1})(1+\xi) \frac{\Delta_t^{\frac{1+\xi}{\eta}} Y_t^{\nu}}{A_t^{1+\xi+\frac{1+\xi}{\eta}}} \left(c_{\mathrm{b},t}^{\frac{\eta_\mathrm{b}}{1+\eta_\mathrm{b}}} c_{\mathrm{s},t}^{\frac{\eta_\mathrm{s}}{1+\eta_\mathrm{s}}}\right)^{\frac{\alpha}{\frac{\eta_\mathrm{b}}{1+\eta_\mathrm{b}} + \frac{\eta_\mathrm{s}}{1+\eta_\mathrm{s}}}}, \quad \text{with } \nu = \alpha + \xi + \frac{1+\xi}{\eta}, \tag{5.11}$$

where ν denotes the elasticity of real marginal cost with respect to aggregate output Y_t.

5.5 First-best benchmark

Now consider an economy with flexible prices and complete financial markets open from some date t_0 onwards. The equilibrium of this economy represents (one) first-best allocation. With complete financial markets, the risk-sharing condition [3.5] must hold from t_0 onwards (it is unaffected by the labour supply decision because utility [5.1] is additively separable). As shown in Proposition 11, the risk-sharing condition and the resource constraint determine the consumption-income ratios $c^*_{\tau,t|t_0}$ independently of the level of aggregate output (these ratios depend only on the initial wealth distribution at $t_0 - 1$). With flexible prices ($\sigma = 0$), it can be seen from [4.11] and [4.12] that $\hat{\Delta}_t = 1$ and $\hat{x}_t = 1 - \varepsilon^{-1}$. Hence, equation [5.11] can be used to obtain the level of output in this case, referred to as the natural level of output and denoted by $\hat{Y}^*_{t|t_0}$:

$$\hat{Y}^*_{t|t_0} = \left(\frac{1}{1+\xi}\left(c^{*\,\frac{\eta_\mathrm{b}}{1+\eta_\mathrm{b}}}_{\mathrm{b},t|t_0} c^{*\,\frac{\eta_\mathrm{s}}{1+\eta_\mathrm{s}}}_{\mathrm{s},t|t_0}\right)^{\frac{-\alpha}{\frac{\eta_\mathrm{b}}{1+\eta_\mathrm{b}} + \frac{\eta_\mathrm{s}}{1+\eta_\mathrm{s}}}} A_t^{1+\xi+\frac{1+\xi}{\eta}}\right)^{\frac{1}{\alpha+\xi+\frac{1+\xi}{\eta}}}. \tag{5.12}$$

This depends on the complete-markets consumption ratios $c^*_{\tau,t|t_0}$ because these reflect the initial $(t_0 - 1)$ wealth distribution, which has implications for labour supply decisions through wealth effects. Exogenous TFP is assumed to be such that the growth rate of the natural level of output satisfies the same restrictions as those on real GDP growth in the endowment economy of section 2.

Using the natural level of output [5.12], equation [5.11] for real marginal cost x_t can be reformulated in terms of the output gap $\tilde{Y}_{t|t_0} = Y_t/\hat{Y}^*_{t|t_0}$, relative-price distortions $\mathit{\Delta}_t$, and a wedge $\mathit{\Psi}_{t|t_0}$ that represents the effects on different households' labour supplies of deviations from the complete-markets wealth distribution (written in terms of the consumption gaps $\tilde{c}_{\tau,t|t_0} = c_{\tau,t}/c^*_{\tau,t|t_0}$ defined relative to consumption with complete financial markets open from t_0 onwards):

$$x_t = (1-\varepsilon^{-1})\mathit{\Psi}_{t|t_0}\mathit{\Delta}_t^{\frac{1+\xi}{\eta}}\tilde{Y}^{\nu}_{t|t_0}, \quad \text{where } \mathit{\Psi}_{t|t_0} = \left(\tilde{c}_{\mathrm{b},t|t_0}^{\frac{\eta_\mathrm{b}}{1+\eta_\mathrm{b}}}\tilde{c}_{\mathrm{s},t|t_0}^{\frac{\eta_\mathrm{s}}{1+\eta_\mathrm{s}}}\right)^{\frac{\alpha}{\frac{\eta_\mathrm{b}}{1+\eta_\mathrm{b}}+\frac{\eta_\mathrm{s}}{1+\eta_\mathrm{s}}}}. \qquad [5.13]$$

5.6 Policy analysis

It is instructive to consider again the special cases of flexible prices or complete financial markets. First suppose financial markets are incomplete but prices are fully flexible. With $\sigma = 0$, equations [4.11] and [4.12] imply $\mathit{\Delta}_t = 1$ and $x_t = 1 - \varepsilon^{-1}$, and it then follows from [5.13] that the output gap is given by $\tilde{Y}_{t|t_0} = \mathit{\Psi}_{t|t_0}^{-\frac{1}{\nu}}$. Using monetary policy to complete financial markets (from t_0 onwards, as in Proposition 11, assuming a real GDP growth rate of $\hat{g}^*_{t|t_0}$) results in $\tilde{c}_{\tau,t|t_0} = 1$, and from [5.13], $\mathit{\Psi}_{t|t_0} = 1$, which implies $\tilde{Y}_{t|t_0} = 1$. Thus, the complete-markets consumption allocation is achieved, the output gap is closed, and there are no relative price distortions, so the equilibrium is first best.[48] Therefore, all the policies considered in section 3 remain optimal when an elastic labour supply decision is added to an incomplete-markets model with flexible prices.

It is perhaps surprising at first that optimal monetary policy continues to replicate complete financial markets exactly as in the endowment economy. Since households can vary labour supply in response to shocks, it might be thought the need for the insurance provided by complete financial markets is reduced accordingly. However, if households had access to those insurance markets, they would prefer to use them to cushion their consumption following shocks, rather than to adjust their labour supply to achieve the same end. Intuitively, the utility function [5.1] has curvature in labour hours, so the desire for risk-sharing applies also to hours as well as consumption. Hence, if monetary policy can replicate insurance markets without cost (as it can in the economy with flexible prices), it remains optimal to do so.

The second special case is that of complete financial markets (the assumptions from section 3.1) or a representative household ($\theta = 0$), but where prices are sticky. Because financial markets are complete, there are no consumption gaps ($\tilde{c}_{\tau,t|t_0} = 1$), and hence from [5.13], $\mathit{\Psi}_{t|t_0} = 1$ holds automatically. With no initial relative-price distortions ($\mathit{\Delta}_{t_0-1} = 1$), strict inflation targeting (with a zero inflation target, $\pi_t = 0$) implies $\mathit{\Delta}_t = 1$ for all $t \geq t_0$ using equation [4.11]. Furthermore, with $\pi_t = 0$ the solution of [4.12] is $x_t = 1 - \varepsilon^{-1}$, and hence $\tilde{Y}_{t|t_0} = 1$ from [5.13]. Therefore, output is at its first-best level, there are no relative-price distortions, and the complete-markets consumption allocation is achieved, confirming that strict inflation targeting is optimal in this case.

In summary, with flexible prices, there is no trade-off between a policy that supports risk sharing and achieving the optimal level of aggregate output. With complete financial markets, there is no trade-off between avoiding relative-price distortions and achieving the optimal level of aggregate output. However, with both incomplete financial markets and sticky prices, all three objectives of

[48] As in the endowment economy, there are many first-best allocations, but only the complete-markets equilibrium can be implemented using monetary policy. This remains true in the production economy.

risk sharing, avoiding relative-price distortions, and closing the output gap are in conflict (in fact, any two of these three objectives are generally in conflict). The remainder of this section studies these trade-offs by deriving a loss function to approximate the welfare function, and approximations of the equilibrium conditions that represent the constraints on monetary policy.

Proposition 15 *The debt gap $\tilde{\mathsf{d}}_{t|t_0}$, inflation π_t, the output gap $\tilde{\mathsf{Y}}_{t|t_0}$, and the bond yield j_t must satisfy the following constraints (a Phillips curve and the equivalents of the constraints in [4.14] associated with incomplete financial markets) up to errors of order $\mathscr{O}^2$:*

$$\kappa(\pi_t - \beta\mathbb{E}_t\pi_{t+1}) = \nu\tilde{\mathsf{Y}}_{t|t_0} - \psi\tilde{\mathsf{d}}_{t|t_0}, \quad \textit{where} \;\; \kappa = \frac{\sigma(1+\varepsilon\xi)}{(1-\sigma)(1-\sigma\beta)}; \tag{5.14a}$$

$$\mathbb{E}_t\tilde{\mathsf{d}}_{t+1|t_0} = \lambda\tilde{\mathsf{d}}_{t|t_0}, \quad \frac{\mathsf{j}_{t-1} - \beta\mu\mathsf{j}_t}{1-\beta\mu} - \tilde{\mathsf{d}}_{t|t_0} + \lambda\tilde{\mathsf{d}}_{t-1|t_0} - \frac{(1-\alpha)(1-\beta)\psi}{(1-\beta\lambda)\nu}(\tilde{\mathsf{d}}_{t|t_0} - \lambda\tilde{\mathsf{d}}_{t-1|t_0}) - \alpha\tilde{\mathsf{Y}}_{t|t_0}$$
$$+\alpha\tilde{\mathsf{Y}}_{t-1|t_0} - \pi_t - \frac{(1-\alpha)(1-\beta)\kappa}{\nu}(\pi_t - \mathbb{E}_{t-1}\pi_t) = \hat{\mathsf{r}}^*_{t|t_0}, \quad \textit{and} \;\; \lim_{\ell\to\infty}(\beta\mu)^\ell\mathbb{E}_t\mathsf{j}_{t+\ell} = 0, \tag{5.14b}$$

where the coefficient ψ is defined below, and $\hat{\mathsf{r}}^*_{t|t_0}$ is the flexible-prices and complete-markets ex-post real return (that is, with real GDP growth equal to $\hat{\mathsf{g}}^*_{t|t_0}$). Using Pareto weights $\hat{\Omega}^*_{\tau|t_0} = c^{*\alpha}_{\tau,t_0|t_0}/\hat{Y}^{*\,1-\alpha}_{t_0}$, the welfare function [3.26] is equal to $\mathscr{W}_{t_0} = -\mathscr{L}_{t_0} + \mathscr{I}_{t_0} + \mathscr{O}^3$, where $\mathscr{I}_{t_0}$ denotes terms independent of policy from t_0 onwards, $\mathscr{O}^3$ is a term third-order or higher in the standard deviation ς of real GDP growth, and the loss function $\mathscr{L}_{t_0}$ is given by

$$\mathscr{L}_{t_0} = \frac{1}{2}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\mathbb{E}_{t_0}\left[\aleph_{\mathrm{d}}\tilde{\mathsf{d}}^2_{t|t_0} + \aleph_\pi\pi_t^2 + \aleph_{\mathrm{Y}}\tilde{\mathsf{Y}}^2_{t|t_0}\right], \quad \textit{where} \;\; \aleph_\pi = \varepsilon\kappa \;\; \textit{and} \;\; \aleph_{\mathrm{Y}} = \nu. \tag{5.15}$$

If all households have the same Frisch elasticity of labour supply η then $\aleph_{\mathrm{d}}$ and ψ are:

$$\aleph_{\mathrm{d}} = \frac{\alpha\theta^2(1-\beta\lambda)^2}{(1-\theta^2)(1-\beta)^2}\left(1 + \frac{\alpha\eta}{(1-\theta^2)(1+\xi)(1+\eta)}\right), \quad \textit{and} \;\; \psi = \frac{\alpha\theta^2(1-\beta\lambda)}{(1-\theta^2)(1-\beta)}, \tag{5.16a}$$

while if the Frisch elasticities are $\eta_{\mathrm{b}} = (1-\theta)\eta/(1+\theta\eta)$ and $\eta_{\mathrm{s}} = (1+\theta)\eta/(1-\theta\eta)$ (for $\eta < \theta^{-1}$):

$$\aleph_{\mathrm{d}} = \frac{\alpha\theta^2(1-\beta\lambda)^2}{(1-\theta^2)(1-\beta)^2}\left(1 + \frac{\alpha\eta}{(1+\xi)(1+\eta)}\right), \quad \textit{and} \;\; \psi = 0. \tag{5.16b}$$

Proof See appendix A.15. ■

Inflation and the output gap are connected by an equation [5.14a] similar to the New Keynesian Phillips curve. The coefficient κ captures the extent of both nominal and real rigidities, and ν (defined in [5.11]) is the elasticity of costs with respect to the output gap, both of which are as in the standard New Keynesian model (Woodford, 2003). With incomplete financial markets, the debt-to-GDP gap $\tilde{\mathsf{d}}_{t|t_0}$ also generally appears in the Phillips curve. The reason is that shocks affecting the wealth distribution have implications for labour supply through wealth effects. An increase in the debt-to-GDP ratio represents a negative wealth effect for borrowers, leading them to supply more labour, and a positive wealth effect for savers, leading them to supply less labour. These changes in labour supply then affect wages and thus costs for firms. The presence of this term in the Phillips curve means there is no longer an equivalence between stabilizing inflation and stabilizing the output gap.

The direction of the overall effect of debt on costs depends on whether the response of borrowers' or savers' labour supply is larger. If both borrowers and savers have the same Frisch elasticity of labour supply ($\eta_\tau = \eta$), the effect on borrowers is the dominant one and so an increase in the debt-to-GDP ratio reduces costs (the coefficient ψ of the debt gap in the Phillips curve [5.14a] is given in [5.16a], which is positive). However, if borrowers' and savers' Frisch elasticities differ then this might not be true. Proposition 15 provides an example where the wealth effects of changes in the debt-to-GDP ratio cancel out at the aggregate level, in which case the Phillips curve [5.14a] reduces to the standard New Keynesian Phillips curve ($\psi = 0$).

The assumption of elastic labour supply implies there can be first-order fluctuations in aggregate output, which creates a desire to stabilize the output gap $\tilde{\mathsf{Y}}_{t|t_0}$. The squared output gap now appears in the loss function [5.15] with the same coefficient as in the standard New Keynesian model (Woodford, 2003). Elastic labour supply also leads to an *increase* in the coefficient on the debt gap $\tilde{\mathsf{d}}_{t|t_0}$ (compared to equations [3.31] and [4.13]). This may appear counterintuitive because the ability to vary labour supply can be used to insulate consumption from shocks, which would reduce the welfare losses from financial markets providing less than perfect insurance. This effect is indeed present, but three forces work to offset it.

First, since shocks are aggregate shocks that affect many households simultaneously, labour supply cannot be increased without a reduction in wages in equilibrium, which reduces the efficacy of varying hours to provide consumption insurance (in the model, wage changes occur at the level of different labour types). Second, the curvature of the utility function in hours means households dislike variability in hours in the same way they dislike variability in consumption, and so the potential benefits of risk sharing in complete financial markets apply also to hours as well as consumption. Third, shocks affecting the wealth distribution change the relative supplies of labour of different households, and to the extent that labour inputs are imperfect substitutes, this represents a loss of productive efficiency compared to what could be achieved with complete financial markets (the argument is analogous to why fluctuations in wage inflation are undesirable in a model with sticky wages and imperfectly substitutable labour types, see Erceg, Henderson and Levin, 2000).

The other constraints on monetary policy in addition to the Phillips curve are given in [5.14b]. These are the elastic labour supply equivalents of those from [4.14]. The equations for the predictable component of the debt-to-GDP gap and the transversality condition for the bond yield are unchanged. However, with elastic labour supply, the surprise component of the debt-to-GDP gap is also affected by the output gap. The main reason for this is the connection between the output gap and the ex-ante real interest rate ρ_t. Using equation [3.9a], the real interest rate is given by $\rho_t = \alpha \mathbb{E}_t \tilde{\mathsf{Y}}_{t+1|t_0} - \alpha \tilde{\mathsf{Y}}_{t|t_0} + \hat{\rho}^*_{t|t_0}$, where $\hat{\rho}^*_{t|t_0} = \alpha \mathbb{E}_t[\hat{\mathsf{Y}}^*_{t+1|t_0} - \hat{\mathsf{Y}}^*_{t|t_0}]$ is the real interest rate at the natural level of output. This is the source of the coefficient α of $\tilde{\mathsf{Y}}_{t|t_0}$ in [5.14b]. Optimal monetary policy subject to these constraints is characterized below.

Proposition 16 *(i) The problem of minimizing the loss function* [5.15] *subject to the constraints in* [5.14a] *and* [5.14b] *(with commitment at some distant past date $t_0 \to -\infty$) has a solution with the following properties: the debt gap $\tilde{\mathsf{d}}_t$ is an AR(1) process with an innovation proportional to $\wp_t = \sum_{\ell=0}^{\infty} \beta^\ell (1 - \alpha(1 - \mu^\ell))(\mathbb{E}_t \hat{\mathsf{g}}^*_{t+\ell} - \mathbb{E}_{t-1} \hat{\mathsf{g}}^*_{t+\ell})$ and an autoregressive root λ; inflation*

π_t is an ARMA(3,2) process with an innovation proportional to $\wp_t$ and autoregressive roots λ, μ, and $\varkappa$ $(0 < \varkappa < 1)$ given by:

$$\varkappa = \frac{2}{1+\beta+\frac{\varepsilon\nu}{\kappa}+\sqrt{\left(1+\beta+\frac{\varepsilon\nu}{\kappa}\right)^2-4\beta}}. \qquad [5.17]$$

(ii) With strict inflation targeting, the debt gap $\tilde{\mathsf{d}}_t$ is an AR(1) process with an innovation proportional to $\wp_t$ and an autoregressive root λ, The solution for the debt gap is therefore a multiple of the debt gap under the optimal policy, and the optimal policy weight χ on incomplete financial markets is defined so that the debt gap under the optimal policy is equal to the debt gap under strict inflation targeting multiplied by $1-\chi$.

(iii) In the special case of short-term debt ($\mu = 0$) and where aggregate wealth effects on labour supply cancel out (with Frisch elasticities given in Proposition 15, so that $\psi = 0$ and the Phillips curve [5.14a] reduces to the standard New Keynesian Phillips curve), the solution of the optimal monetary policy problem is as follows. The debt gap is $\tilde{\mathsf{d}}_t = \lambda\tilde{\mathsf{d}}_{t-1} - (1-\chi)\wp_t$, where the optimal policy weight χ on incomplete financial markets is

$$\chi = \left(1 + \frac{\frac{(1-\theta^2)(1-\beta)^2(1-\beta\lambda^2)(1-\beta\varkappa)}{\alpha\theta^2(1-\beta\lambda)^2\left(1+\frac{\alpha\eta}{(1+\xi)(1+\eta)}\right)}}{\frac{\varkappa}{\nu}\left(\left(\frac{\nu}{\kappa}+(1-\beta)+\alpha\beta(1-\varkappa)\right)^2+\alpha^2\beta(1-\beta\varkappa)\right)}\right)^{-1}. \qquad [5.18a]$$

The solutions for inflation, the output gap, and the real interest rate gap are the following ARMA(1,1) processes:

$$\pi_t = \varkappa\pi_{t-1} - \frac{\frac{\nu}{\kappa}\left(\frac{\nu}{\kappa}+(1-\beta)+\alpha\beta(1-\varkappa)\right)\chi\left(\wp_t - \frac{\alpha(1-\beta\varkappa)}{\frac{\nu}{\kappa}+(1-\beta)+\alpha\beta(1-\varkappa)}\wp_{t-1}\right)}{\left(\frac{\nu}{\kappa}+(1-\beta)+\alpha\beta(1-\varkappa)\right)^2+\alpha^2\beta(1-\beta\varkappa)}; \qquad [5.18b]$$

$$\tilde{\mathsf{Y}}_t = \varkappa\tilde{\mathsf{Y}}_{t-1} - \frac{(1-\beta\varkappa)\chi\left(\left(\frac{\nu}{\kappa}+(1-\beta)+\alpha\beta(2-\varkappa)\right)\wp_t - \alpha\wp_{t-1}\right)}{\left(\frac{\nu}{\kappa}+(1-\beta)+\alpha\beta(1-\varkappa)\right)^2+\alpha^2\beta(1-\beta\varkappa)}; \qquad [5.18c]$$

$$\tilde{\rho}_t = \varkappa\tilde{\rho}_{t-1} + \frac{\alpha(1-\beta\varkappa)\chi\left(\left(\alpha+(1-\varkappa)\left(\frac{\nu}{\kappa}+(1-\beta)+\alpha\beta(2-\varkappa)\right)\right)\wp_t - \alpha\wp_{t-1}\right)}{\left(\frac{\nu}{\kappa}+(1-\beta)+\alpha\beta(1-\varkappa)\right)^2+\alpha^2\beta(1-\beta\varkappa)}. \qquad [5.18d]$$

(iv) For the parameters $\mu = 0$ and $\psi = 0$ considered in (iii), the (Markovian) discretionary policy equilibrium is $\tilde{\mathsf{d}}_t = \lambda\tilde{\mathsf{d}}_t - (1-\chi')\wp_t$, where $\chi' < \chi$ (generically). Inflation, the output gap, and the real interest rate gap are all serially uncorrelated processes proportional to $\wp_t$.

PROOF See appendix A.16. ■

The proof of the proposition derives a closed-form solution for the stochastic processes of the debt gap $\tilde{\mathsf{d}}_t$, inflation π_t, and the output gap $\tilde{\mathsf{Y}}_t$ under the optimal monetary policy with commitment. The initial date t_0 of the commitment is set far in the past ($t_0 \to -\infty$) to abstract from time-consistency issues, so the t_0 subscript is dropped in what follows. The solution has the property that the optimal response of the debt gap to a shock $\wp_t$ is proportional to the response of the debt gap to $\wp_t$ under a policy of strict inflation targeting. This allows an equivalent of χ from Proposition 13 to be calculated that measures the weight optimal monetary policy places on stabilizing the debt gap relative to stabilizing inflation. The optimal response of the debt gap is $1-\chi$ multiplied by what it would be given strict inflation targeting, so $\chi = 0$ represents complete stabilization of inflation and $\chi = 1$ represents complete stabilization of the debt gap.

To understand the new aspects of the optimal monetary policy problem with elastic labour supply, note that policy can now influence three variables which have implications for the debt gap and thus the extent to which complete financial markets are replicated: inflation, real GDP, and the ex-ante real interest rate. Inflation affects the ex-post real return on nominal bonds and thus the value of existing debt (as before). Real GDP (and hence the output gap) affects the denominator of the debt-to-GDP ratio. The ex-ante real interest rate affects the ongoing costs of servicing debt relative to the stream of current and future labour income (formally, the ex-ante real interest rate influences the debt gap by changing the level of the debt-to-GDP ratio consistent with risk sharing).

By combining the four constraints in [5.14a] and [5.14b], the evolution of the debt gap depends on the shock $\wp_t$ and three terms related respectively to inflation, the output gap, and the ex-ante real interest rate.[49] The impact of shocks on the debt gap and potential policy responses through these three variables are scaled down to the extent that $\psi > 0$, so fluctuations in the debt gap are dampened when borrowers' labour supply response dominates that of savers. This is the effect discussed earlier where the ability to vary labour supply mitigates the impact of shocks when complete financial markets are not available. The effect of inflation on debt is related to the unexpected change in the term $\mathbb{E}_t[\pi_t + (\beta\mu)\pi_{t+1} + (\beta\mu)^2\pi_{t+2} + \cdots]$ as before, where μ^ℓ represents the stock of debt issued in the past that has not been refinanced ℓ periods after the shock.

An increase in the output gap $\tilde{Y}_t$ has the effect of directly boosting real GDP growth at time t and thus reducing the debt-to-GDP ratio, but the impact on the debt gap is more subtle. Since monetary policy has only a temporary influence on real GDP, extra growth now reduces overall growth in the future by exactly the same amount. Given the link between growth expectations and the debt-to-GDP ratio consistent with risk sharing (see equations [3.9a] and [3.11]), the effect of the output gap on the debt gap actually depends on $\tilde{Y}_t + \mathbb{E}_t[\beta(\tilde{Y}_{t+1} - \tilde{Y}_t) + \beta^2(\tilde{Y}_{t+2} - \tilde{Y}_{t+1}) + \cdots]$, not just $\tilde{Y}_t$. With the New Keynesian Phillips curve ([5.14a], setting $\psi = 0$ for now), it is seen that this term is equal to $(1-\beta)(\kappa/\nu)\pi_t$, the reciprocal of the long-run Phillips curve slope multiplied by current inflation. Since it is reasonable to set the discount factor β close to one (in which case the long-run Phillips curve is close to vertical), this term is negligible for all practical purposes (it is not exactly zero because future growth is discounted relative to current growth, so by bringing growth

[49] Using equations [5.14a] and [5.14b], the precise expression for the evolution of the debt-to-GDP gap $\tilde{d}_t$ is:

$$\tilde{d}_t = \lambda\tilde{d}_{t-1} - \left(1 + \left(\frac{1-\beta}{1-\beta\lambda} + \frac{\alpha\beta(1-\lambda)(1-\mu)}{(1-\beta\lambda)(1-\beta\lambda\mu)}\right)\frac{\psi}{\nu}\right)^{-1}\left(\wp_t + (\mathbb{E}_t - \mathbb{E}_{t-1})\sum_{\ell=0}^{\infty}(\beta\mu)^\ell\pi_{t+\ell}\right.$$
$$\left. + (1-\beta)\frac{\kappa}{\nu}(\mathbb{E}_t - \mathbb{E}_{t-1})\pi_t + \alpha\beta(1-\mu)\frac{\kappa}{\nu}(\mathbb{E}_t - \mathbb{E}_{t-1})\sum_{\ell=0}^{\infty}(\beta\mu)^\ell(\pi_{t+\ell} - \pi_{t+1+\ell})\right).$$

forwards, there is still a small positive effect). Monetary policy therefore cannot have a sustainable impact on the burden of debt simply through its temporary effect on the level of real GDP.

It does not follow however that expansionary or contractionary monetary policy has no effect on the debt burden beyond its implications for ex-post real returns through inflation. There remains the option of changing ex-ante real interest rates. Intuitively, expansionary monetary policy that reduces the ex-ante real interest rate is effectively a transfer from savers to borrowers, what might be labelled 'financial repression' (even though the transmission mechanism of monetary policy). While monetary policy cannot permanently affect real interest rates, there is no reason why cutting the real interest rates now means real interest rates in the future must be higher than they would otherwise have been (unlike real GDP growth, as discussed above).

Changing ex-ante real interest rates thus provides monetary policy with an alternative to influencing the debt gap through the effect of inflation on ex-post real returns. In contrast to the latter, which is effective only while debt contracts are *not* refinanced, the former is effective only when refinancing *does* take place. For debt refinanced ℓ periods after a shock at time t, the impact of monetary policy on the date-t debt burden is determined by the discounted sum of real interest gaps $\mathbb{E}_t[\beta^{\ell+1}\tilde{\rho}_{t+\ell} + \beta^{\ell+2}\tilde{\rho}_{t+\ell+1} + \beta^{\ell+3}\tilde{\rho}_{t+\ell+2} + \cdots]$ from $t+\ell$ onwards. Given the New Keynesian Phillips curve [5.14a] (with $\psi = 0$ for now) and the equation $\tilde{\rho}_t = \alpha\mathbb{E}_t[\tilde{Y}_{t+1} - \tilde{Y}_t]$ linking the real interest rate and output gaps, these terms reduce to $(\alpha\kappa/\nu)\beta^{\ell+1}\mathbb{E}_t[\pi_{t+\ell+1} - \pi_{t+\ell}]$, meaning that the *slope* of the inflation path over time is an indicator of financial repression through ex-ante real interest rates. With a (steady-state) fraction $(1-\mu)\mu^\ell$ of debt issued prior to date t being refinanced at $t+\ell$, the overall effect of changes in ex-ante real interest rates on the debt burden is given by the unexpected change in $(\alpha\kappa/\nu)(1-\mu)\beta\sum_{\ell=0}^{\infty}(\beta\mu)^\ell\mathbb{E}_t[\pi_{t+\ell+1} - \pi_{t+\ell}]$. Thus, the more the inflation trajectory is smoothed, the smaller is the effect of monetary policy on the ex-ante real interest rate.

As the maturity of debt increases, successful financial repression requires an inflation trajectory with a non-zero slope further in the future. Given the Phillips curve, the required inflation path entails output gap fluctuations over a longer horizon, increasing the losses from following such a policy. Financial repression is therefore not well suited to stabilizing the debt gap when debt contracts have a long maturity, in which case a policy of influencing ex-post real returns through inflation smoothly spread out over time is effective at a much lower cost in terms of the implied inflation and output gap fluctuations. But for short-maturity debt where only immediate inflation surprises can affect ex-post real returns, financial repression provides an additional tool for stabilizing the debt gap, with the losses from following this policy being small when the short-run Phillips curve is relatively flat.

For illustration, Proposition 16 gives an explicit solution of the optimal monetary policy problem with short-term debt ($\mu = 0$) and the standard New Keynesian Phillips curve ($\psi = 0$). Shocks that increase the debt gap now bring forth a monetary policy response that cuts real interest rates and increases the output gap.[50] Since expectations are important when using these tools, discretionary policy features too little stabilization of the debt gap relative to a policy with commitment.

[50]The autoregressive root $\varkappa$ from [5.17] in [5.18b]–[5.18d] is the same as the autoregressive root of the solution of the standard optimal monetary policy problem with cost-push shocks (see, Woodford, 2003).

6 Quantitative analysis of optimal monetary policy

This section presents a quantitative analysis of the nature of optimal monetary policy taking into account all the features of the full model from section 5.

6.1 Calibration

Let T denote the length in years of one discrete time period in the model. The numerical results presented here assume a quarterly frequency ($T = 1/4$), but the choice of frequency does not significantly affect the results. The parameters of the model are β, θ, α, λ, η, μ, ε, ξ, and σ. As far as possible, these parameters are set to match features of U.S. data.[51] The baseline calibration targets and the implied parameter values are given in Table 1 and justified below.

Table 1: *Baseline calibration*

Calibration targets			Implied parameter values		
Real GDP growth	g	1.7%			
Real interest rate	r	5%	Discount factor	β	0.992
Debt-to-GDP ratio	$\mathcal{D}$	130%	Debt service ratio	θ	8.6%
Coefficient of relative risk aversion				α	5
Marginal propensity to consume	m	6%	Discount factor elasticity	λ	0.993
Frisch elasticity of labour supply				η	2
Average duration of debt	T_{m}	5	Debt maturity parameter	μ	0.967
Price elasticity of demand				ε	3
Marginal cost elasticity w.r.t. output				ξ	0.5
Average duration of price stickiness	T_{p}	8/12	Calvo pricing parameter	σ	0.625

Notes: The parameters are derived from the calibration targets using equations [6.1]–[6.5]. The calibration targets are specified in annual time units; the parameter values assume a quarterly model ($T = 1/4$).
Sources: See discussion in section 6.1.

Consider first the parameters β and θ (the choice of these parameters is equivalent to specifying the patience parameters Δ_{b} and Δ_{s}, as seen in Proposition 1). These are calibrated to match evidence on the average price and quantity of household debt. The 'price' of debt is the average annual continuously compounded real interest rate r paid by households for loans. As seen in [2.27b], the steady-state growth-adjusted real interest rate is related to β. Let g denote the average annual real growth rate of the economy. Given the length of the discrete time period in the model, $1 + \bar{\rho} = e^{rT}$ and $1 + \bar{g} = e^{gT}$. Hence, using [2.27b], β can be set to:

$$\beta = e^{-(r-g)T}. \qquad [6.1]$$

From 1972 through to 2011 there was an average annual nominal interest rate of 8.8% on 30-year mortgages, 10% on 4-year auto loans, and 13.7% on two-year personal loans, while the average annual

[51] The source for all data referred to below is Federal Reserve Economic Data (http://research.stlouisfed.org/fred2).

change in the personal consumption expenditure (PCE) price index over the same time period was 3.8%. The average credit-card interest rate between 1995 and 2011 was 14%. For comparison, 30-year Treasury bonds had an average yield of 7.7% over the periods 1977–2001 and 2006–2011. The implied real interest rates are 4.2% on Treasury bonds, 5% on mortgages, 6.2% on auto loans, 9.9% on personal loans, and 12% on credit cards.[52] The baseline real interest rate is set to the 5% rate on mortgages as these constitute the bulk of household debt.[53] The sensitivity analysis considers values of r from 4% up to 7%. Over the period 1972–2011 used to calibrate the interest rate, the average annual growth rate of real GDP per capita was 1.7%. Together with the baseline real interest rate of 5%, this implies that $\beta \approx 0.992$ using [6.1]. Since many models used for monetary policy analysis are typically calibrated assuming zero real trend growth, for comparison the sensitivity analysis also considers values of g between 0% and 2%.

The relevant quantity variable for debt is the ratio of gross household debt to annual household income, denoted by $\mathcal{D}$.[54] This corresponds to what is defined as the loans-to-GDP ratio $\bar{l}$ in the model (the empirical debt ratio being based on the amount borrowed rather than the subsequent value of loans at maturity) after adjusting for the length of one time period (T years), hence $\mathcal{D} = \bar{l}T$. Using the expression for $\bar{l}$ in equation [2.27c] and given a value of β, the equation can be solved for the implied value of the debt service ratio θ:

$$\theta = \frac{2(1-\beta)\mathcal{D}}{\beta T}. \tag{6.2}$$

Note that in the model, all GDP is consumed, so for consistency between the data and the model's prediction for the debt-to-GDP ratio, either the numerator of the ratio should be total gross debt (not only household debt), or the denominator should be disposable personal income or private consumption. Since the model is designed to represent household borrowing, and because the implications of corporate and government debt may be different, the latter approach is taken.

In the U.S., like a number of other countries, the ratio of household debt to income has grown significantly in recent decades. To focus on the implications of levels of debt recently experienced, the model is calibrated to match average debt ratios during the five years from 2006 to 2010. The sensitivity analysis considers a wide range of possible debt ratios from 0% up to 200%. Over the period 2006–2010, the average ratio of gross household debt to disposable personal income was

[52] Average PCE inflation over the periods 1977–2001 and 2006–2011 was 3.5%, and 2% over the period 1995–2011.

[53] Mortgage debt was around 77% of all household debt on average during 2006–2010. The baseline real interest rate is close to the conventional calibration used in many real business cycle models (King and Rebelo, 1999). The mortgage rate implies a spread of 0.8% between the interest rates on loans to households and Treasury bonds of the same maturity. Cúrdia and Woodford (2009) consider a somewhat larger spread of 2% between interest rates for borrowers and savers.

[54] Given the heterogeneity between borrower and saver households, it would not make sense to net the financial assets of savers against the liabilities of borrowers. However, it might be thought appropriate to net assets and liabilities at the level of individual households, especially since a large fraction of household borrowing is to buy assets (houses). If the assets held by households had the same non-contingent return as their debt liabilities then this netting would be valid, but that is highly unlikely for assets such as housing, which experience significant fluctuations in value. In the model, non-contingent debt is repaid only from future income, not from the sale of assets, so the assumptions used in the calibration would be approximately correct if the value of the assets actually held by households were positively correlated with the value of GDP and had a similar volatility. If asset returns were more procyclical, the calibration would understate the problem of household leverage; if returns were less procyclical or countercyclical, the calibration would overstate leverage.

approximately 124%, while the ratio of debt to private consumption was approximately 135%. Taking the average of these numbers, the target chosen is a model-consistent debt-to-income ratio of 130%, which implies using [6.2] a debt service ratio of $\theta \approx 8.6\%$.[55]

In estimating the coefficient of relative risk aversion α, one possibility would be to choose values consistent with household portfolios of risky and safe assets. But since Mehra and Prescott (1985) it has been known that matching the equity risk premium may require a risk aversion coefficient above 30, while values in excess of 10 are considered by many to be highly implausible.[56] Subsequent analysis of the 'equity risk premium puzzle' has attempted to build models consistent with the large risk premium but with much more modest degrees of risk aversion.[57]

Alternative approaches to estimating risk aversion have made use of laboratory experiments, observed behaviour on game shows, and in a recent study, the choice of deductible for car insurance policies (Cohen and Einav, 2007).[58] The survey evidence presented by Barsky, Juster, Kimball and Shapiro (1997) potentially provides a way to measure risk aversion over stakes that are large as a fraction of lifetime income and wealth.[59] The results suggest considerable risk aversion, but most likely not in the high double-digit range for the majority of individuals. Overall, the weight of evidence from studies suggests a coefficient of relative risk aversion above one, but not significantly more than 10. A conservative baseline value of 5 is adopted, and the sensitivity analysis considers

[55]Empirically, a direct measure of the ratio of household debt payments to disposable personal income is available, though this is not directly comparable to the debt service ratio in the model. Between 2006 and 2010, the average debt service ratio was 12.7%. This measure includes both interest and amortization. For conventional T_{f}-year fixed-rate mortgages (where the borrower makes a sequence of equal repayments over the life of the loan) the share of amortization in total repayments (over the life of the loan, or over all cohorts of borrowers at a point in time) is approximately $(1 - e^{-rT_{\mathrm{f}}})/(rT_{\mathrm{f}})$, where r is the annual real interest rate. Taking $r \approx 5\%$ and $T_{\mathrm{f}} = 30$ years, this formula implies that interest payments are approximately 48% of total debt service costs, yielding an estimate of the interest-only debt-service ratio of around 6.1%. In the model, the debt service ratio θ is calculated only for borrowers, not for all households as in the data, and is net of new borrowing (which is positive when there is positive income growth). Using [2.27b], the model implies the interest-only debt service ratio over all households (including the 50% of savers with a zero debt service ratio) is given by $\bar{\rho}\bar{d}/(1+\bar{\rho})$, which is comparable to the interest-only adjustment of the empirical measure. The baseline calibration yields a debt service ratio of approximately 6.5%, close to the 6.1% in the data.

[56]Large values of α are also needed to generate a significant inflation risk premium. For illustration, suppose real GDP follows a random walk, and the standard deviation of annual real GDP growth is set to 2.1% as found in the data for the period 1972–2011. Under the flexible-price optimal policy of nominal GDP targeting (Proposition 3), the inflation risk premium would be approximately $0.044\alpha\%$ (at an annual rate) using Proposition 8. For $\alpha < 10$, the inflation risk premium can be no more than 0.44%, even though in this example inflation would be perfectly negatively correlated with real GDP growth and have the same standard deviation (of 2.1%, not too far below the actual standard deviation of PCE inflation of approximately 2.5% between 1972 and 2011).

[57]For example, Bansal and Yaron (2004) assume a relative risk aversion coefficient of 10, while Barro (2006) chooses a more conservative value of 4.

[58]Converting the estimates of absolute risk aversion into coefficients of relative risk aversion (using average annual after-tax income as a proxy for the relevant level of wealth) leads to a mean of 82 and a median of 0.4. The stakes are relatively small and many individuals are not far from being risk neutral, though a minority are extremely risk averse. As discussed in Cohen and Einav (2007), the estimated level of mean risk aversion is above that found in other studies, which are generally consistent with single-digit coefficients of relative risk aversion.

[59]Respondents to the U.S. Health and Retirement Study survey are asked a series of questions about whether they would be willing to leave a job bringing in a secure income for another job with a chance of either a 50% increase in income or a 50% fall. By asking a series of questions that vary the probabilities of these outcomes, the answers can in principle be used to elicit risk preferences. One finding is that approximately 65% of individuals' answers fall in a category for which the theoretically consistent coefficient of relative risk aversion is at least 3.8. The arithmetic mean coefficient is approximately 12, while the harmonic mean is 4.

values from zero up to 10.[60]

One approach to calibrating the discount factor elasticity parameter λ (from [2.2]) is to select a value on the basis of its implications for the marginal propensity to consume from financial wealth. Let m denote the increase in per-household (annual) consumption of savers from a marginal increase in their financial wealth.[61] Using the formula for λ from Proposition 6 and the expression for β in [2.27b], λ is given by:

$$\lambda = \frac{1 - mT}{\beta}. \quad [6.3]$$

Parker (1999) presents evidence to suggest that the marginal propensity to consume from wealth lies between 4% and 5% (for a survey of the literature on wealth and consumption, see Poterba, 2000). However, it is argued by Juster, Lupton, Smith and Stafford (2006) that the marginal propensity to consume varies between different forms of wealth. They find the marginal propensity to consume is lowest for housing wealth and larger for financial wealth. Given the focus on financial wealth in this paper, the baseline calibration assumes $m \approx 6\%$, which using [6.3] implies $\lambda \approx 0.993$.[62] The sensitivity analysis considers marginal propensities to consume from 4% to 8%.[63]

The range of available evidence on the Frisch elasticity of labour supply η is discussed by Hall (2009), who concludes that a value of approximately 2/3 is reasonable. However, both real business cycle and New Keynesian models have typically assumed Frisch elasticities significantly larger than this, often as high as 4 (see, King and Rebelo, 1999, Rotemberg and Woodford, 1997). The baseline calibration adopted here uses a Frisch elasticity of 2, and the sensitivity analysis considers a range of values for η from completely inelastic labour supply (as in the model of section 4) up to 4.[64]

The debt maturity parameter μ (which stands in for the parameter γ specifying the sequence

[60] The parameter α is also related to the elasticity of intertemporal substitution α^{-1}. Early estimates of intertemporal substitution suggested an elasticity somewhere between 1 and 2, such as those from the instrumental variables method applied by Hansen and Singleton (1982). Those estimates have been criticized for bias due to time aggregation by Hall (1988), who finds elasticities as low as 0.1 and often insignificantly different from zero. Using cohort data, Attanasio and Weber (1993) obtain values for the elasticity of intertemporal substitution in the range 0.7–0.8, while Beaudry and van Wincoop (1996) find an elasticity close to one using a panel of data from U.S. states. Contrary to these larger estimates, the survey evidence of Barsky, Juster, Kimball and Shapiro (1997) produces an estimate of 0.18. An earlier version of the model presented here (Sheedy, 2013) has separate parameters for risk aversion and intertemporal substitution, but quantitatively, the intertemporal substitution paramater is found to matter little for the results. For this reason, the calibration of α here focuses on its implications for risk aversion.

[61] A simplifying feature of the model is that borrowers have the same marginal propensity to consume from financial wealth as savers in the neighbourhood of the steady state.

[62] Together with the baseline calibration of β, λ, and α, the original patience parameters are $\Delta_b \approx 1.006$ and $\Delta_s \approx 1.012$, and the implied value of δ is 1.009. Thus, the exogenous difference between the annual rates of time preference of borrowers and savers is approximately 2.4%.

[63] A potential alternative approach to calibrating λ is to use its implications for the persistence of shocks to the wealth distribution. In the model, Proposition 4 shows that the impulse response of the debt-to-GDP gap is proportional to λ^{ℓ} after ℓ time periods have elapsed. The expected duration (in years) of the effects of shocks on the wealth distribution is thus $T_d = T \sum_{\ell=1}^{\infty} \ell(1-\lambda)\lambda^{\ell-1} = T/(1-\lambda)$, which can be used to obtain λ given an estimate of T_d. The baseline calibration is equivalent to $T_d \approx 36$ years. The sensitivity analysis for the marginal propensity to consume implies a range of λ values from 0.988 to 0.998, which is equivalent to considering values of T_d from approximately 21 to 139 years.

[64] The special case of different Frisch elasticities between borrowers and savers where aggregate wealth effects on labour supply cancel out is also considered (see Proposition 15). Using the baseline calibration, the required household-specific Frisch elasticities are $\eta_b \approx 1.6$ for borrowers and $\eta_s \approx 2.6$ for savers.

of coupon payments, given $\mu = \gamma/(1+\bar{n})$) is set to match the average maturity of household debt contracts. In the model, the average maturity of household debt is related to the duration of the bond that is traded in incomplete financial markets. Formally, duration T_{m} refers to the average of the maturities (in years) of each payment made by the bond weighted by its contribution to the present value of the bond. Given the geometric sequence of nominal coupon payments parameterized by γ, the bond duration (in steady state) is

$$T_{\mathrm{m}} = \sum_{\ell=1}^{\infty} \frac{\ell T}{\bar{Q}} \frac{\gamma^{\ell-1}}{(1+\bar{j})^{\ell}} = \frac{T}{1-\frac{\gamma}{1+\bar{j}}},$$

using the steady-state bond price (present value of the coupon payments) $\bar{Q} = (1-\gamma+\bar{j})^{-1}$ from [2.21].[65] Let j denote the average annualized nominal interest rate on household debt, with $1+\bar{j} = e^{jT}$. In the optimal policy analysis, the steady-state rate of inflation will be zero ($\bar{\pi} = 0$), hence nominal GDP growth is $\bar{n} = \bar{g}$, and so $\mu = \gamma/(1+\bar{g})$. It follows that γ and μ can be determined by:

$$\gamma = e^{jT}\left(1 - \frac{T}{T_{\mathrm{m}}}\right), \quad \text{and} \quad \mu = e^{-gT}\gamma. \tag{6.4}$$

Doepke and Schneider (2006) present evidence on the average duration of household nominal debt liabilities. Their analysis takes account of refinancing and prepayment of loans. For the most recent year in their data (2004), the duration lies between 5 and 6 years, while the duration has not been less than 4 years over the entire period covered by the study (1952–2004). This suggests a baseline duration of $T_{\mathrm{m}} \approx 5$ years, which using [6.4] implies $\mu \approx 0.967$.[66] The sensitivity analysis considers the effects of having durations as short as one quarter (one-period debt), and longer durations up to 10 years.[67]

There are two main strategies for calibrating the price elasticity of demand ε. The direct approach draws on studies estimating consumer responses to price differences within narrow consumption categories. A price elasticity of approximately three is typical of estimates at the retail level (see, for example, Nevo, 2001), while estimates of consumer substitution across broad consumption categories suggest much lower price elasticities, typically lower than one (Blundell, Pashardes and Weber, 1993). Indirect approaches estimate the price elasticity based on the implied markup

[65] Duration is equal to the percentage reduction in the real value of a nominal asset following a one percentage point (annualized) permanent rise in inflation. Since $\mu = \gamma/(1+\bar{n})$, $\bar{j} = \bar{i}$, $1+\bar{i} = (1+\bar{\rho})(1+\bar{g})$, $1+\bar{n} = (1+\bar{\pi})(1+\bar{g})$, and $\beta = (1+\bar{g})/(1+\bar{\rho})$, it follows that $T_{\mathrm{m}} = T/(1-\beta\mu)$. A permanent rise in inflation by one percentage point at an annualized rate is equivalent to increasing π_t by T in all time periods from some date onwards, and equation [3.13] shows that this reduces the ex-post real return on nominal bonds by $T/(1-\beta\mu)$, confirming the interpretation of T_{m}.

[66] A conventional T_{f}-year fixed-rate mortgage has a duration of $(e^{rT_{\mathrm{f}}} - 1 - rT_{\mathrm{f}})/(r(e^{rT_{\mathrm{f}}} - 1))$, which is approximately 11 years with $T_{\mathrm{f}} = 30$ and $r \approx 5\%$. The calibrated duration may seem short given the high share of mortgage debt in total household debt and the prevalence of 30-year fixed-rate mortgages, but refinancing shortens the duration of debt. In the model, the frequency of refinancing is determined by $1-\mu$. The baseline calibration implies that 12.6% of the total stock of debt is refinanced or newly issued each year.

[67] The baseline value of γ is 0.971. The calibration method implicitly assumes γ is a structural parameter that will remain invariant to changes in policy, including the change in the average rate of inflation. An alternative is to assume μ is the structural parameter, in which case μ is calibrated by dividing γ from [6.4] by $1+\bar{n}$, where $\bar{n}$ is the average of actual nominal GDP growth. This method leads to $\mu \approx 0.958$, which is well within the range of values of μ considered in the sensitivity analysis.

$1/(\varepsilon - 1)$, or as part of the estimation of a DSGE model. Rotemberg and Woodford (1997) estimate an elasticity of approximately 7.9 and point out this is consistent with the markups in the range of 10%–20%. Since it is the price elasticity of demand that directly matters for the welfare consequences of inflation rather than its implications for markups as such, the direct approach is preferred here and the baseline value of ε is set to 3. A range of values from the theoretical minimum elasticity of 1 up to 10 is considered in the sensitivity analysis.

The aggregate production function is given in equation [5.8]. If e denotes the elasticity of aggregate output with respect to hours then the elasticity ξ of real marginal cost with respect to output can be obtained from e using:

$$\xi = \frac{1-e}{e}.$$

A conventional value of $e \approx 2/3$ is adopted for the baseline calibration (this would be the labour share in a model with perfect competition), which implies $\xi \approx 0.5$. As discussed in Rotemberg and Woodford (1999), there may be reasons to expect an elasticity of marginal cost with respect to output higher than this (for example, if the elasticity of substitution between labour and other factors is less than one), so the sensitivity analysis examines the effects of higher values of ξ. An important implication of ξ is the strength of real rigidities (related to the term $1 + \varepsilon\xi$ appearing in the formula for κ in the Phillips curve [5.14a]), which are absent in the special case of a linear production function ($\xi = 0$).[68] The sensitivity analysis considers values of ξ between 0 and 1.

In the model, σ is the probability of not changing price in a given time period. The probability distribution of survival times for newly set prices is $(1-\sigma)\sigma^{\ell}$, and hence the expected duration of a price spell T_{p} (in years) is $T_{\mathrm{p}} = T\sum_{\ell=1}^{\infty}\ell(1-\sigma)\sigma^{\ell-1} = T/(1-\sigma)$. With data on T_{p}, the parameter σ can be inferred from:

$$\sigma = 1 - \frac{T}{T_{\mathrm{p}}}. \qquad [6.5]$$

There is now an extensive literature measuring the frequency of price adjustment across a representative sample of goods. Using the dataset underlying the U.S. CPI index, Nakamura and Steinsson (2008) find the median duration of a price spell is 7–9 months, excluding sales but including product substitutions. Klenow and Malin (2010) survey a wide range of studies reporting median durations in a range from 3–4 months to one year. The baseline duration is taken to be 8 months ($T_{\mathrm{p}} \approx 8/12$), implying $\sigma \approx 0.625$. The sensitivity analysis considers average durations from 3 to 15 months.[69]

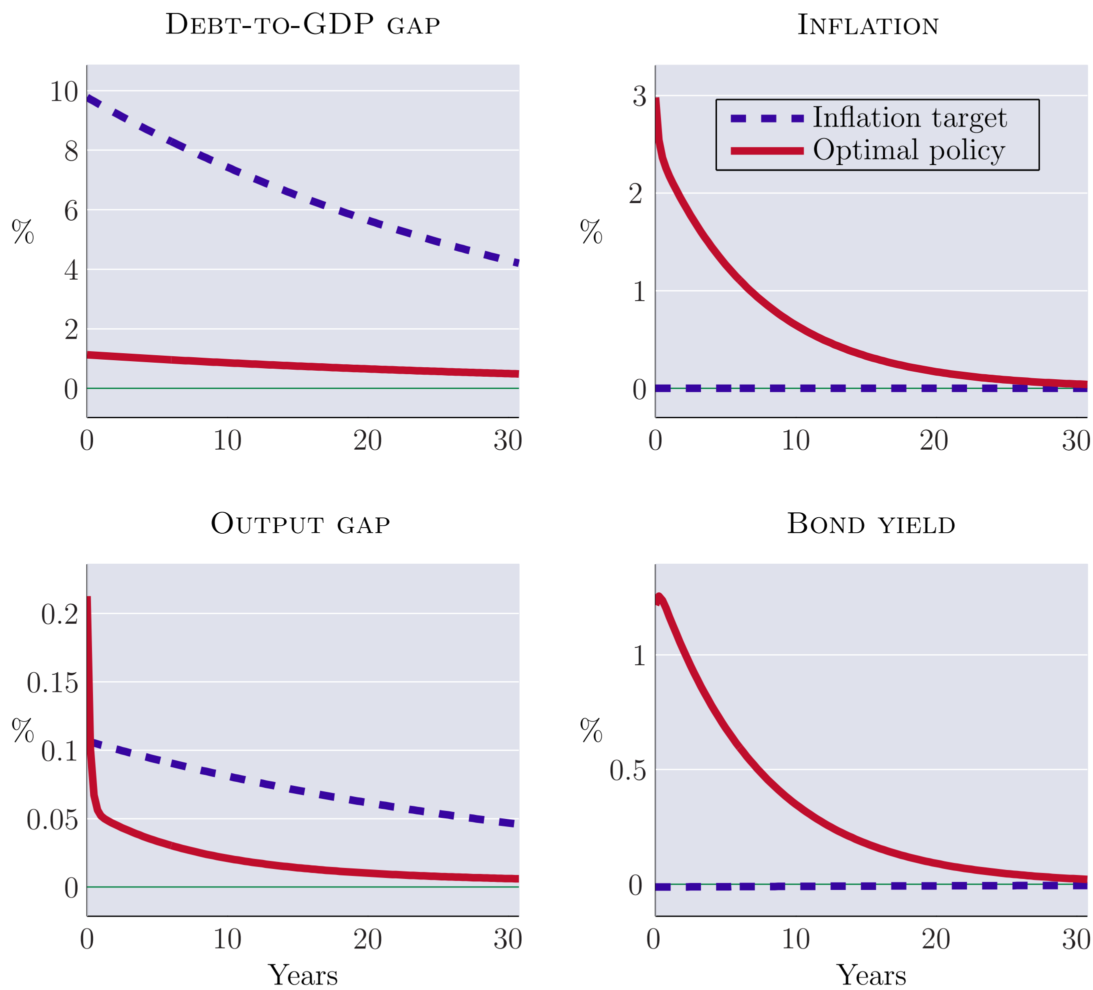
[68] It is conventional to assume a source of real rigidities in New Keynesian models, though Bils, Klenow and Malin (2012) present some critical evidence.

[69] An alternative approach to calibrating the parameters σ, ε, and ξ related to nominal and real rigidities would be to choose values consistent with estimates of the slope of the Phillips curve. The recent literature on estimating the New Keynesian Phillips curve studies the relationship $\pi_t = \beta\mathbb{E}_t\pi_{t+1} + (1/\kappa)\mathrm{x}_t$ between inflation π_t and real marginal cost x_t, where the latter is proxied by the labour share. The baseline calibration implies $1/\kappa \approx 0.091$. Galí and Gertler (1999) present a range of estimates of $1/\kappa$ lying between 0.02 and 0.04. Galí, Gertler and López-Salido (2001) estimate $1/\kappa$ to be in the range 0.03–0.04, while Sbordone (2002) obtains an estimate of 0.055. The Phillips curve implied by the baseline calibration is steeper than these estimates, but the sensitivity analysis for σ, ε, and ξ does allow for Phillips curve slopes in the range of econometric estimates. The maximum value of σ considered implies $1/\kappa \approx 0.021$, the maximum value of ε implies $1/\kappa \approx 0.038$, and the maximum value of ξ implies $1/\kappa \approx 0.057$.

6.2 Results

Consider an economy hit by an unexpected permanent fall in potential output. How should monetary policy react? In the basic New Keynesian model with sticky prices but either complete financial markets or a representative household, the optimal monetary policy response to a TFP shock is to keep inflation on target and allow actual output to fall in line with the loss of potential output. Using the baseline calibration from Table 1 and the solution to the optimal monetary policy problem given in Proposition 16, Figure 1 shows the impulse responses of the debt-to-GDP gap $\tilde{\mathsf{d}}_t$, inflation π_t, the output gap $\tilde{\mathsf{Y}}_t$, and the bond yield j_t under the optimal monetary policy and under a policy of strict inflation targeting for the 30 years following a 10% fall in potential output.

Figure 1: *Responses to a TFP shock, optimal monetary policy and strict inflation targeting*



Notes: The shock is an unexpected permanent TFP shock that reduces the natural level of output by 10% relative to its trend. The debt-to-GDP gap and the output gap are reported as percentage deviations, and inflation and bond yields are reported as annualized percentage rates. The parameters are set in accordance with the baseline calibration from Table 1.

With strict inflation targeting, the debt-to-GDP gap rises in line with the fall in output (10%) because the denominator of the debt-to-GDP ratio falls, while the numerator is unchanged. The effects of this shock on the wealth distribution and hence on consumption are long lasting (the half-life of the debt-to-GDP impulse response is around 25 years). The output gap is not completely stabilized because the disturbance to the wealth distribution leads to wealth effects on aggregate labour supply that insurance markets would eliminate, though this effect is found to be quantitatively small. Bond yields are almost completely stable because inflation is constant and the effects of the incompleteness of financial markets on ex-ante real interest rates are small.

The optimal monetary policy response is in complete contrast to strict inflation targeting. Optimal policy allows inflation to rise, which stabilizes nominal GDP over time in spite of the fall in real GDP. This helps to stabilize the debt-to-GDP ratio, moving the economy closer to the outcome with complete financial markets where borrowers would be insured against the shock and the value of debt liabilities would automatically move in line with income. The rise in the debt-to-GDP gap is very small (around 1%) compared to strict inflation targeting (10%). The rise in inflation is very persistent, lasting around two decades. The higher inflation called for is significant, but not dramatic: for the first two years, around 2–3% higher, for the next decade around 1–2% higher, and for the decade after that, around 0–1% higher. Inflation that is spread out over time is still effective in reducing the debt-to-GDP ratio because debt liabilities have a long average maturity. It is also significantly less costly in terms of relative-price distortions to have inflation spread out over a longer time than the typical durations of stickiness of individual prices.

The rise in inflation does lead to a disturbance to the output gap for the first one or two years, but this is short lived because the duration of the real effects of monetary policy through the traditional price-stickiness channel is brief compared to the relevant time scale of decades for the other variables. The effect is also quantitatively small because inflation is highly persistent, the rise in expected inflation closely matching the rise in actual inflation, so the Phillips curve implies little impact on the output gap. Over a longer horizon, the optimal policy actually performs better at stabilizing the output gap because it reduces the shock to the wealth distribution that distorts labour supply decisions in the case of strict inflation targeting. Finally, nominal bond yields show a persistent increase. It might seem surprising that yields do not fall as monetary policy is loosened, but the bonds in question are long-term bonds, and the effect on inflation expectations is dominant (there is a fall in real interest rates because the rise in bond yields is less than what would be implied by higher expected inflation alone).

As shown in Proposition 16, the impulse response function of the debt-to-GDP gap under the optimal policy is proportional to the impulse response under strict inflation targeting. Since the debt-to-GDP gap would be zero if incomplete financial markets were the only concern of the policymaker, this provides a measure of the weights attached by optimal policy to stabilizing the debt gap and to stabilizing inflation. The response of the debt gap under the optimal policy is approximately 11.6% of the response under strict inflation targeting, so the policy weight χ on debt gap stabilization is 88.4% and the policy weight $1-\chi$ on inflation stabilization is 11.6% (similarly, the inflation response under the optimal policy is around 88.4% of what would keep the debt gap exactly at zero).[70]

[70] There is a variant of the model where aggregate wealth effects on labour supply cancel out ($\psi = 0$, in which case the Phillips curve [5.14a] reduces to the standard New Keynesian Phillips curve). This entails setting different

The baseline calibration implies that addressing the problem of incomplete financial markets is quantitatively the main focus of optimal monetary policy rather than other objectives such as inflation stabilization. What explains this, and how sensitive is this conclusion to the particular calibration targets? Consider the exercise of varying each calibration target individually over the ranges discussed in section 6.1, holding all other targets constant. For each new target, the implied parameters are recalculated and the new policy weight χ is obtained using Proposition 16. Figure 2 plots the values of χ (the optimal policy weight on the debt-to-GDP gap) obtained for each target.

As can be seen in Figure 2, over the range of reasonable average real GDP growth rates there is almost no effect on the optimal policy weight. The range of real interest rates is somewhat larger (because there is less certainty about the appropriate real interest rate to assume for household borrowing) but the optimal policy weight on incomplete financial markets changes little. Both average real growth and average real interest rates affect the discount factor β, which enters the equations of the model in many places, but there is no intuitively obvious reason to expect it to have a large impact on the relative benefits and costs of achieving the various objectives of policy.

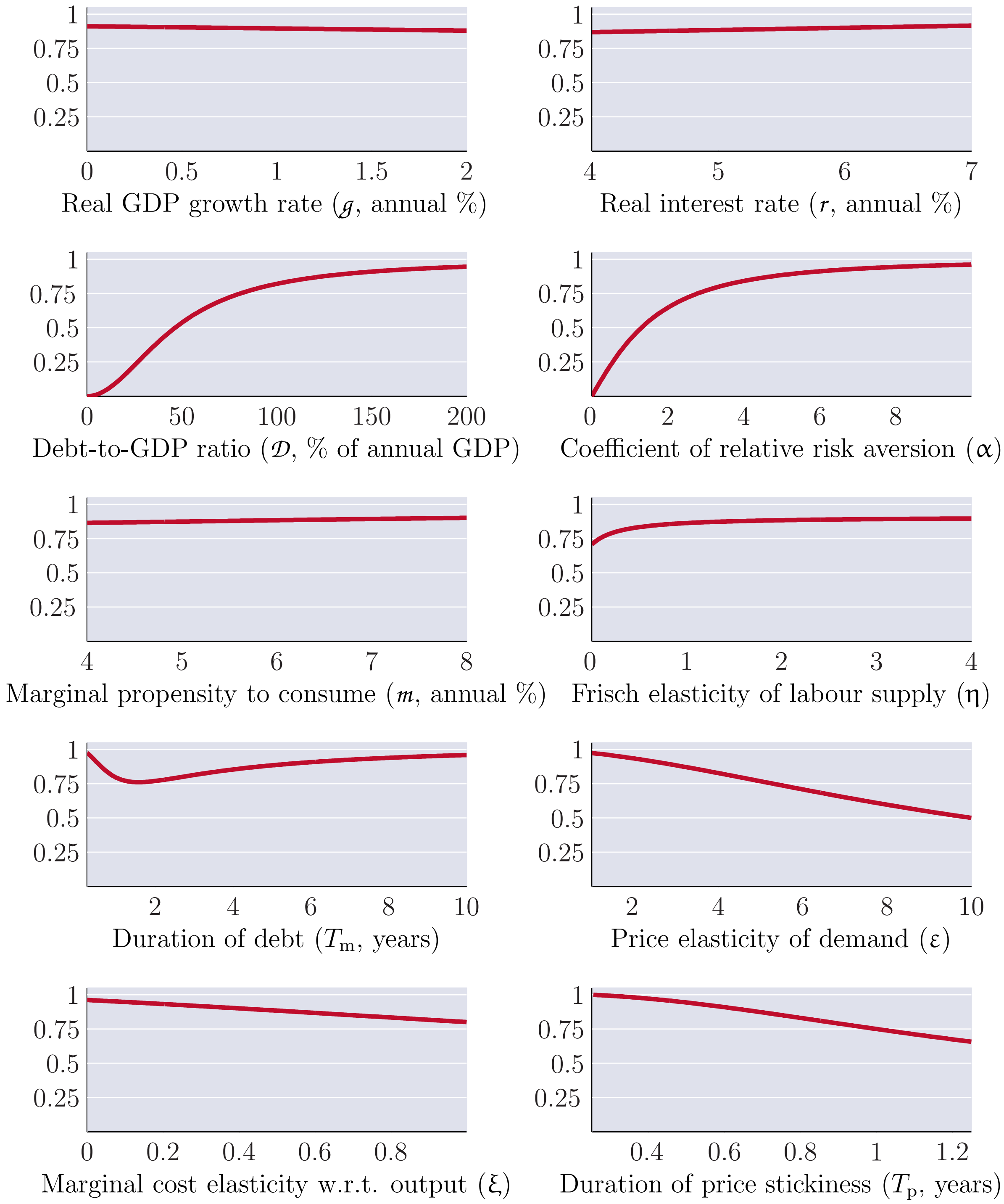
The results are most sensitive to the calibration targets for the average debt-to-GDP ratio and the coefficient of relative risk aversion. The average debt-to-GDP ratio proxies for the parameter θ, which is related to the difference in patience between borrowers and savers. It is not surprising that an economy with less debt in relation to income has less of a concern with the incompleteness of financial markets because it means the impact of shocks is felt more evenly by borrowers and savers. In the limiting case of a representative-household economy, the average debt-to-GDP ratio tends to zero, and the degree of completeness of financial markets becomes irrelevant. The debt gap receives more than half the weight in the optimal policy as long as the calibration target for the average debt-to-GDP ratio is not below 50%. It seems unlikely the U.S. would return to such low levels of household debt in the foreseeable future if the levels of borrowing experienced since the 1990s do indeed reflect the preferences and income profiles of borrowers and savers.

It is also not surprising that the results are sensitive to the coefficient of relative risk aversion. Since the only use for complete financial markets in the model is risk sharing, if households were risk neutral then there would be no loss from these markets being absent, as long as saving and borrowing incomplete financial markets remained possible. The baseline coefficient of relative risk aversion is higher than the typical value of 2 found in many macroeconomic models (though that number is usually relevant for intertemporal substitution in those models, not for attitudes to risk), but it is low compared to the values often assumed in finance models that seek to match risk premia (even the maximum value of 10 considered here would be insufficient to generate realistic risk premia without adding other features to the model). The optimal policy weight on the debt gap exceeds one half if the coefficient of relative risk aversion exceeds 1.3, so lower degrees of risk aversion do not necessarily overturn the conclusions of this paper.

The next most important calibration target is the price elasticity of demand. A higher price

Frisch elasticities for borrowers and savers with η being the average elasticity, as explained in Proposition 15. Using this version of the model and the baseline calibration, the optimal policy weight on the debt gap is 88.8%. Since the difference with the standard model is so small, this variant of the model is ignored in subsequent analysis.

Figure 2: *Sensitivity analysis for optimal policy weight χ on incomplete financial markets*



Notes: The response of the debt gap under the optimal policy is $1 - \chi$ multiplied by its response under strict inflation targeting. Each of the calibration targets in Table 1 is varied individually, holding all others at their baseline values. The baseline value of χ is 0.884.

elasticity increases the welfare costs of inflation. Welfare ultimately depends on quantities not prices, but the price elasticity determines how much quantities are distorted by dispersion of relative prices. To reduce the optimal policy weight on the debt gap below one half it is necessary to assume price elasticities in excess of 10. Such values would be outside the range typical in IO and microeconomic studies of demand, with 10 itself being at the high end of the range of values used in most macroeconomic models. The typical value of 6 often found in New Keynesian models only reduces χ to approximately 71%.

The results are largely insensitive to the marginal propensity to consume from financial wealth, which is used to determine the parameter λ in the specification of the endogenous discount factors. Since this feature of the model was introduced only for technical reasons, it is reassuring that it does not have a significant effect on the results within a wide range of reasonable parameter values. The Frisch elasticity of labour supply has a fairly small but not insignificant effect on the results, with the optimal policy weight on the debt gap increasing with the Frisch elasticity. A higher elasticity increases the welfare costs of shocks to wealth distribution by distorting the labour supply decisions of different households, as well as making it easier for monetary policy to influence the real value of debt by changing the ex-ante real interest rate in addition to inflation. An elastic labour supply does mean that inflation fluctuations lead to output gap fluctuations, which increases the importance of targeting inflation, but the first two effects turn out to be more important quantitatively.

The results are somewhat more sensitive to the average duration of a price spell and the elasticity of real marginal cost with respect to output. The first of these determines the importance of nominal price rigidities. Greater nominal rigidity leads to more dispersion of relative prices from a given amount of inflation, and thus reduces the optimal policy weight on the debt gap. A higher output elasticity of marginal cost implies that the production function has greater curvature, so a given dispersion of output levels across firms represents a more inefficient allocation of resources. However, the range of reasonable values for the duration of price stickiness does not reduce χ below 65%, and the range of marginal cost elasticities does not lead to any χ value below 80%.

Finally, there is the average duration of household debt, where the effects of this calibration target are more subtle. It might be expected that the longer the maturity of household debt, the higher is the optimal policy weight on the debt gap. This is because longer-term debt allows inflation to be spread out further over time, reducing the welfare costs of the inflation, yet still having an effect on the real value of debt. However, the sensitivity analysis shows the optimal policy weight is a non-monotonic function of debt maturity: either very short-term or long-term debt maturities lead to high values of χ, while debt of around 1.5 years maturity has the lowest value of χ (approximately 75%).

This puzzle is resolved by recalling there are two ways monetary policy can affect the real value of debt: inflation to change the ex-post real return on nominal debt, and changes in the ex-ante real interest rate ('financial repression'). As has been discussed, the first method is effective at a lower cost for long debt maturities. When labour supply is inelastic, the second method is not available, and the value of χ is then indeed a strictly increasing function of debt maturity (with the value of χ falling to 15% for the shortest-maturity debt).

When the ex-ante real interest rate method is available, it is most effective (taking account of the costs of using it in terms of inflation and output gap fluctuations) when debt maturities are short. This is because monetary policy can only affect ex-ante real interest rates for a few years at most (in line with reasonable calibrations of nominal and real rigidities). If debt is continually refinanced each quarter or comprises floating-rate instruments, monetary policy has significant power to affect its real value because nominal rigidities allow it to change the current real interest rate. However, if fixed-rate debt is rarely refinanced then, holding inflation constant (that is, ignoring the first method for affecting the real value of the debt), the real return is largely predetermined and thus insensitive to current monetary policy. Therefore, intermediate debt maturities correspond to the lowest optimal policy weights on stabilizing the debt gap because the maturity is too short for the inflation method to be effective at low cost, but too long for the ex-ante real interest rate method to work.

7 Conclusions

This paper has shown how a monetary policy of nominal GDP targeting facilitates efficient risk sharing in incomplete financial markets where contracts are denominated in terms of money. In an environment where risk derives from uncertainty about future real GDP, strict inflation targeting would lead to a very uneven distribution of risk, with leveraged borrowers' consumption highly exposed to any unexpected change in their incomes when monetary policy prevents any adjustment of the real value of their liabilities. Strict inflation targeting does provide savers with a risk-free real return, but fundamentally, the economy lacks any technology that delivers risk-free real returns, so the safety of savers' portfolios is simply the flip-side of borrowers' leverage and high levels of risk. Absent any changes in the physical investment technology available to the economy, aggregate risk cannot be annihilated, only redistributed.

That leaves the question of whether the distribution of risk is efficient. The combination of incomplete markets and strict inflation targeting implies a particularly inefficient distribution of risk when households are risk averse. If complete financial markets were available, borrowers would issue state-contingent debt where the contractual repayment is lower in a recession and higher in a boom. These securities would resemble equity shares in GDP, and they would have the effect of reducing the leverage of borrowers and hence distributing risk more evenly. In the absence of such financial markets, in particular because of the inability of households to sell such securities, a monetary policy of nominal GDP targeting can effectively complete financial markets even when only non-contingent nominal debt is available. Nominal GDP targeting operates by stabilizing the debt-to-GDP ratio. With financial contracts specifying liabilities fixed in terms of money, a policy that stabilizes the monetary value of real incomes ensures that borrowers are not forced to bear too much aggregate risk, converting nominal debt into real equity.

While the model is far too simple to apply to the recent financial crises and deep recessions experienced by a number of economies, one policy implication does resonate with the predicament of several economies faced with high levels of debt combined with stagnant or falling GDPs. Nominal GDP targeting is equivalent to a countercyclical price level, so the model suggests that higher inflation can be optimal in recessions. In other words, while each of the 'stagnation' and 'inflation' that make up the word 'stagflation' is bad in itself, if stagnation cannot immediately be remedied, some inflation might be a good idea to compensate for the inefficiency of incomplete financial markets. And even if policymakers were reluctant to abandon inflation targeting, the model does suggest that they have the strongest incentives to avoid deflation during recessions (a procyclical price level). Deflation would raise the real value of debt, which combined with falling real incomes would be the very opposite of the risk sharing stressed in this paper, and even worse than an unchanging inflation rate.

It is important to stress that the policy implications of the model in recessions are matched by equal and opposite prescriptions during an expansion. Thus, it is not just that optimal monetary policy tolerates higher inflation in a recession — it also requires lower inflation or even deflation during a period of high growth. Pursuing higher inflation in recessions without following a symmetric policy during an expansion is both inefficient and jeopardizes an environment of low inflation on average. Therefore the model also argues that more should be done by central banks to 'take away the punch bowl' during a boom even were inflation to be stable.

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A Appendices

A.1 Proof of Proposition 1

(i) Consider the system of equations [2.22a]–[2.22f]. Suppose there are no shocks to real GDP ($g_t = \bar{g}$, that is, $\varsigma = 0$ in [2.3], where $\bar{g}$ is an exogenous variable) and no uncertainty created by monetary policy ($i_t = \bar{i}$). A steady-state solution of the system of equations must be such that ex-ante and ex-post real returns are the same, that is, $\bar{\rho} = \bar{r}$, using [2.22a]. With $\bar{c}_\tau$ being time invariant, the Euler equations in [2.22d] imply that the following must hold:

$$\bar{\delta}_\mathrm{b}(1+\bar{\rho})(1+\bar{g})^{-\alpha} = \bar{\delta}_\mathrm{s}(1+\bar{\rho})(1+\bar{g})^{-\alpha} = 1, \tag{A.1.1}$$

and hence $\bar{\delta}_\mathrm{b} = \bar{\delta}_\mathrm{s}$. Using equation [2.22e], this requires

$$\Delta_\mathrm{b}\bar{c}_\mathrm{b}^{-(1-\lambda)\alpha} = \Delta_\mathrm{s}\bar{c}_\mathrm{s}^{-(1-\lambda)\alpha}. \tag{A.1.2}$$

The budget identities [2.22c] imply that equation [2.23] holds, and hence $\bar{c}_\mathrm{b}$ and $\bar{c}_\mathrm{s}$ must also satisfy:

$$\frac{1}{2}\bar{c}_\mathrm{b} + \frac{1}{2}\bar{c}_\mathrm{s} = 1. \tag{A.1.3}$$

Rearranging equation [A.1.2] yields

$$\frac{\bar{c}_\mathrm{b}}{\bar{c}_\mathrm{s}} = \left(\frac{\Delta_\mathrm{b}}{\Delta_\mathrm{s}}\right)^{\frac{1}{(1-\lambda)\alpha}},$$

which can be used to substitute for $\bar{c}_\mathrm{b}$ in terms of $\bar{c}_\mathrm{s}$ in equation [A.1.3] and thus solve for $\bar{c}_\mathrm{s}$:

$$\bar{c}_\mathrm{s} = \frac{2}{1+\left(\frac{\Delta_\mathrm{b}}{\Delta_\mathrm{s}}\right)^{\frac{1}{(1-\lambda)\alpha}}}. \tag{A.1.4}$$

This solution can be written as

$$\bar{c}_\mathrm{s} = 1+\theta, \quad \text{where } \theta = \frac{1-\left(\frac{\Delta_\mathrm{b}}{\Delta_\mathrm{s}}\right)^{\frac{1}{(1-\lambda)\alpha}}}{1+\left(\frac{\Delta_\mathrm{b}}{\Delta_\mathrm{s}}\right)^{\frac{1}{(1-\lambda)\alpha}}},$$

noting that the term θ satisfies $0 < \theta < 1$ because $0 < \Delta_\mathrm{b} < \Delta_\mathrm{s} < \infty$ and $0 < \lambda < 1$. Equation [A.1.3] then immediately implies $\bar{c}_\mathrm{b} = 1-\theta$, confirming [2.27a].

Let δ denote the common steady-state discount factor $\bar{\delta}_\tau$. Using equation [2.22e] and the solution [A.1.4] it follows that

$$\delta = \Delta_\mathrm{s}\left(\frac{2}{1+\left(\frac{\Delta_\mathrm{b}}{\Delta_\mathrm{s}}\right)^{\frac{1}{(1-\lambda)\alpha}}}\right)^{-(1-\lambda)\alpha} = \left(\frac{\Delta_\mathrm{b}^{\frac{1}{(1-\lambda)\alpha}} + \Delta_\mathrm{s}^{\frac{1}{(1-\lambda)\alpha}}}{2}\right)^{(1-\lambda)\alpha}, \tag{A.1.5}$$

where the latter expression is obtained by moving Δ_s inside the parentheses and simplifying. Together with [A.1.2], this shows that $\bar{\delta}_\mathrm{b} = \bar{\delta}_\mathrm{s} = \delta$. Observe that δ is a generalized average of Δ_b and Δ_s, and since $\Delta_\mathrm{b} < \Delta_\mathrm{s}$, it follows that $\Delta_\mathrm{b} < \delta < \Delta_\mathrm{s}$. Using the common steady-state discount factor, equation [A.1.1] can be used to obtain the steady-state real interest rate:

$$1+\bar{\rho} = \frac{(1+\bar{g})^\alpha}{\delta} = \frac{(1+\bar{g})}{\beta}, \quad \text{where } \beta = \delta(1+\bar{g})^{1-\alpha}. \tag{A.1.6}$$

Since $\bar{r} = \bar{\rho}$, this expression also gives the steady-state ex-post real return, and it can be seen that the Euler equations [2.22d] are satisfied at this rate of return, so equation [2.27b] is confirmed.

Using [A.1.6], the accounting identity [2.22b] provides a link between the debt-to-GDP $\bar{d}$ and loans-to-GDP $\bar{l}$ ratios in the steady state:

$$\bar{l} = \beta\bar{d}. \tag{A.1.7}$$

Using equation [2.27a] together with the budget identities in [2.22c] implies

$$\frac{\theta}{2} = \bar{d} - \bar{l},$$

and using [A.1.7] to substitute for $\bar{l}$, the expression for $\bar{d}$ in [2.27c] is obtained, and the expression for $\bar{l}$ follows immediately from [A.1.7]. Hence, in summary, the values of $\bar{c}_\mathrm{b}$, $\bar{c}_\mathrm{s}$, $\bar{\rho}$, $\bar{r}$, $\bar{d}$, and $\bar{l}$ satisfy [2.22a]–[2.22e]. Finally, note that the parameter restriction $\Delta_\mathrm{s}(1+\bar{g})^{1-\alpha} < 1$ implies $\delta(1+\bar{g})^{1-\alpha} < 1$ since δ in [A.1.5] is such that $\delta < \Delta_\mathrm{s}$. Thus, β defined in [A.1.6] satisfies $0 < \beta < 1$. Now observe that the transversality condition [2.22f] requires

$$\lim_{\ell\to\infty} \left(\delta(1+\bar{g})^{1-\alpha}\right)^{\ell} \bar{c}_\tau^{-\alpha}\bar{l} = 0, \quad \text{which is equivalent to} \quad \lim_{\ell\to\infty} \beta^{\ell} = 0,$$

with the definition of β in [A.1.6] and given that $\bar{c}_\tau \neq 0$, $\bar{l} \neq 0$. Since $0 < \beta < 1$, the second statement above holds, which shows that [2.22f] is satisfied.

The solution is given in terms of θ, β, and $\bar{g}$, which satisfy $0 < \theta < 1$ and $0 < \beta < 1$. Note that for any values of θ and β satisfying $0 < \theta < 1$ and $0 < \beta < 1$, there exist (given α, λ, and $\bar{g}$) values of the parameters Δ_b and Δ_s that satisfy $0 < \Delta_\mathrm{b} < \Delta_\mathrm{s} < \infty$ and generate an equilibrium consistent with θ and β. It follows from [2.22e] together with [2.27a]–[2.27c] that:

$$\Delta_\mathrm{b}(1-\theta)^{-(1-\lambda)\alpha} = \Delta_\mathrm{s}(1+\theta)^{-(1-\lambda)\alpha} = \frac{\beta}{(1+\bar{g})^{1-\alpha}},$$

which can be rearranged to obtain values of Δ_b and Δ_s:

$$\Delta_\mathrm{b} = \frac{\beta(1-\theta)^{(1-\lambda)\alpha}}{(1+\bar{g})^{1-\alpha}}, \quad \text{and} \quad \Delta_\mathrm{s} = \frac{\beta(1+\theta)^{(1-\lambda)\alpha}}{(1+\bar{g})^{1-\alpha}}. \qquad \text{[A.1.8]}$$

Since $0 < \theta < 1$ and $0 < \theta < 1$, these parameters satisfy $0 < \Delta_\mathrm{b} < \Delta_\mathrm{s} < \infty$.

(ii) The steady-state solution for $\bar{c}_\mathrm{b}$, $\bar{c}_\mathrm{s}$, $\bar{\delta}_\mathrm{b}$, $\bar{\delta}_\mathrm{s}$, $\bar{\rho}$, $\bar{r}$, $\bar{d}$, and $\bar{l}$ depends only on Δ_b, Δ_s, α, λ, and $\bar{g}$, and is thus independent of monetary policy. Given steady-state monetary policy $i_t = \bar{i}$, equation [2.24] requires that:

$$1 = \delta(1+\bar{g})^{-\alpha}\left(\frac{1+\bar{i}}{1+\bar{\pi}}\right),$$

and by using [A.1.6], the formula for $\bar{\pi}$ is obtained. Next, equation [2.25a] and $\bar{r} = \bar{\rho}$ imply:

$$1+\bar{\rho} = \frac{1+\bar{j}}{1+\bar{\pi}},$$

and given that $1+\bar{i} = (1+\bar{\rho})(1+\bar{\pi})$, this confirms $\bar{j} = \bar{i}$. With the parameter restriction $\gamma < 1+\bar{i}$, it follows that $1-\gamma+\bar{j} > 0$, so with equation [A.1.6], the transversality condition [2.25b] reduces to:

$$\lim_{\ell\to\infty} \frac{\gamma^{\ell}}{\left((1+\bar{\rho})(1+\bar{\pi})\right)^{\ell}} = \lim_{\ell\to\infty} \left(\frac{\gamma}{1+\bar{j}}\right)^{\ell} = 0.$$

The parameter restriction $\gamma < 1+\bar{i}$ together with $\bar{j} = \bar{i}$ implies $0 \leq \gamma/(1+\bar{j}) < 1$, so the limit above holds. This completes the proof.

A.2 Proof of Proposition 2

(i) With complete financial markets, the risk-sharing condition [3.5] must hold at all times, where the discount factors $\delta_{\tau,t}$ are given in [2.22e]. The budget identities in [2.22c] imply equation [2.23] holds, so the consumption-income ratios $c^*_{\mathrm{b},t}$ and $c^*_{\mathrm{s},t}$ must satisfy the following pair of equations at all times and in all states of the world:

$$\Delta_\mathrm{b} {c^*}_{\mathrm{b},t-1}^{-(1-\lambda)\alpha}\left(\frac{c^*_{\mathrm{b},t}}{c^*_{\mathrm{b},t-1}}\right)^{-\alpha} = \Delta_\mathrm{s} {c^*}_{\mathrm{s},t-1}^{-(1-\lambda)\alpha}\left(\frac{c^*_{\mathrm{s},t}}{c^*_{\mathrm{s},t-1}}\right)^{-\alpha}, \quad \text{and} \quad \frac{1}{2}c^*_{\mathrm{b},t} + \frac{1}{2}c^*_{\mathrm{s},t} = 1. \qquad \text{[A.2.1]}$$

No exogenous variables appear in these equations, which are identical at all dates, so the solution will be time invariant. With $c^*_{\mathrm{b},t} = c^*_\mathrm{b}$ and $c^*_{\mathrm{s},t} = c^*_\mathrm{s}$, the pair of equations in [A.2.1] becomes identical to the pair of equations [A.1.2]–[A.1.3] characterizing the steady-state consumption ratios in Proposition 1. Since those equations have a unique solution, the complete-markets consumption ratios must be the same as those in the non-stochastic steady state. Hence, $c^*_{\mathrm{b},t} = \bar{c}_{\mathrm{b}} = 1 - \theta$ and $c^*_{\mathrm{s},t} = \bar{c}_{\mathrm{s}} = 1 + \theta$, where θ is as given in equation [2.27a]. With the same consumption ratios as the non-stochastic steady state, it follows from [2.22e] that the complete-markets discount factors $\delta^*_{\mathrm{b},t}$ and $\delta^*_{\mathrm{s},t}$ are equal to the steady-state discount factors, which are the same for both types of household and given by the expression for δ in [2.27b]. This confirms the results in [3.6a].

The accounting identity [2.22b] implies that the complete-markets debt and loans ratios d^*_t and l^*_t and ex-post real return r^*_t must satisfy:

$$1 + r^*_t = (1 + g_t)\frac{d^*_t}{l^*_{t-1}}. \qquad \text{[A.2.2]}$$

Substituting the consumption ratios from [3.6a] into the budget identities [2.22c] implies the following relationship between the debt and loans ratios d^*_t and l^*_t:

$$d^*_t - l^*_t = \frac{\theta}{2}. \qquad \text{[A.2.3]}$$

Since the consumption ratios and discount factors from [3.6a] are time invariant, the Euler equations [2.22d] reduce to:

$$\delta \mathbb{E}_t \left[(1 + r^*_{t+1})(1 + g_{t+1})^{-\alpha} \right] = 1. \qquad \text{[A.2.4]}$$

Now substitute equation [A.2.2] into [A.2.4] to obtain:

$$l^*_t = \delta \mathbb{E}_t \left[(1 + g_{t+1})^{1-\alpha} d^*_{t+1} \right], \qquad \text{[A.2.5]}$$

which together with equation [A.2.3] implies an expectational difference equation for the debt ratio d^*_t:

$$d^*_t = \frac{\theta}{2} + \delta \mathbb{E}_t \left[(1 + g_{t+1})^{1-\alpha} d^*_{t+1} \right]. \qquad \text{[A.2.6]}$$

Given the constant consumption ratios and discount factors from [3.6a], the transversality condition [2.22f] is equivalent to:

$$\bar{c}_\tau^{-\alpha} \lim_{\ell \to \infty} \left\{ \prod_{\jmath=0}^{\ell-1} \delta (1 + g_{t+1+\jmath})^{1-\alpha} \right\} l^*_{t+\ell} = 0,$$

and by cancelling $\bar{c}_\tau^{-\alpha} \neq 0$, substituting for $l^*_{t+\ell}$ using [A.2.5], and taking expectations conditional on period t information it becomes:

$$\lim_{\ell \to \infty} \delta^\ell \mathbb{E}_t \left[\left\{ \prod_{\jmath=1}^{\ell} (1 + g_{t+\jmath})^{1-\alpha} \right\} d^*_{t+\ell} \right] = 0. \qquad \text{[A.2.7]}$$

To solve for d^*_t, iterate equation [A.2.6] forwards to obtain:

$$d^*_t = \frac{\theta}{2} \mathbb{E}_t \left[\sum_{\ell=0}^{\infty} \delta^\ell \prod_{\jmath=1}^{\ell} (1 + g_{t+\jmath})^{1-\alpha} \right] + \lim_{\ell \to \infty} \delta^\ell \mathbb{E}_t \left[\left(\prod_{\jmath=1}^{\ell} (1 + g_{t+\jmath})^{1-\alpha} \right) d^*_{t+\ell} \right], \qquad \text{[A.2.8]}$$

and since the final term must be zero to satisfy [A.2.7], the expression for d^*_t in [3.6b] is obtained. Given that the parameters are such that $\delta(1 + \bar{g})^{1-\alpha} < 1$, the bounded support for the stochastic process [2.3] implies that the expression for d^*_t in [3.6b] is always finite for sufficiently small $\varsigma > 0$, and that the limit in [A.2.7] does indeed hold.

Given d^*_t, the solution for l^*_t in [3.6b] is confirmed immediately by rearranging equation [A.2.3]. The expression for r^*_t in [3.6c] is obtained by substituting l^*_t from [3.6b] into [A.2.2]. To solve for the real interest rate $\rho^*_t = \mathbb{E}_t r^*_{t+1}$ (given [2.22a]), note that the expectation of equation [A.2.2] at time $t+1$ conditional on date-t information is:

$$\mathbb{E}_t \left[(1 + g_{t+1}) d^*_{t+1} \right] = (1 + \rho^*_t) l^*_t. \qquad \text{[A.2.9]}$$

Substituting the solution for d_t^* from [3.6b] into [A.2.5] leads to an explicit expression for l_t^*:

$$l_t^* = \frac{\theta}{2}\mathbb{E}_t\left[\sum_{\ell=1}^{\infty}\delta^\ell \prod_{\jmath=1}^{\ell}(1+g_{t+\jmath})^{1-\alpha}\right],$$

from which one term can be factored out to write the formula as follows:

$$l_t^* = \mathbb{E}_t\left[\delta(1+g_{t+1})^{1-\alpha}\left(\frac{\theta}{2}\mathbb{E}_{t+1}\left[\sum_{\ell=0}^{\infty}\delta^\ell \prod_{\jmath=1}^{\ell}(1+g_{t+1+\ell})^{1-\alpha}\right]\right)\right].$$

Using the expression for d_t^* in [3.6b], the equation above becomes:

$$l_t^* = \delta\mathbb{E}_t\left[(1+g_{t+1})^{-\alpha}\left((1+g_{t+1})d_{t+1}^*\right)\right],$$

and by using this to substitute for l_t^* in [A.2.9]:

$$\mathbb{E}_t\left[(1+g_{t+1})d_{t+1}^*\right] = (1+\rho_t^*)\delta\mathbb{E}_t\left[(1+g_{t+1})^{-\alpha}\left((1+g_{t+1})d_{t+1}^*\right)\right].$$

Rearranging this equation yields the expression for ρ_t^* in [3.6c].

The solution for $c_{\mathrm{b},t}^*$, $c_{\mathrm{s},t}^*$, d_t^*, l_t^*, r_t^*, and ρ_t^* is seen to be independent of monetary policy. The equilibrium condition [2.24] involving i_t and π_t reduces to:

$$1 = \delta\mathbb{E}_t\left[(1+g_{t+1})^{-\alpha}\left(\frac{1+i_t}{1+\pi_{t+1}}\right)\right],$$

and this together with the equation for monetary policy determines inflation and nominal interest rates.

(ii) Now consider the special cases where either $\alpha = 1$ or $\{g_t\}$ is an i.i.d. stochastic process. First, in the case $\alpha = 1$, it is seen immediately that the expression for d_t^* in [3.6b] reduces to:

$$d_t^* = \frac{\theta}{2}\sum_{\ell=0}^{\infty}\delta^\ell = \frac{\theta}{2}\frac{1}{1-\delta}, \qquad \text{[A.2.10a]}$$

where the infinite sum is well defined because $\delta = \beta$ in the case $\alpha = 1$ (see [2.27c]) and $0 < \beta < 1$. Next, consider the case where $\{g_t\}$ is i.i.d., hence functions of real GDP growth rates at different dates are statistically independent, and conditional expectations of these are equal to the unconditional expectation. This means that the expression for d_t^* in [3.6b] becomes

$$d_t^* = \frac{\theta}{2}\sum_{\ell=0}^{\infty}\delta^\ell\prod_{\jmath=1}^{\infty}\mathbb{E}_t\left[(1+g_{t+\jmath})^{1-\alpha}\right] = \frac{\theta}{2}\sum_{\ell=0}^{\infty}\delta^\ell\left(\mathbb{E}\left[(1+g_t)^{1-\alpha}\right]\right)^\ell = \frac{\theta}{2}\frac{1}{1-\delta\mathbb{E}\left[(1+g_t)^{1-\alpha}\right]}, \qquad \text{[A.2.10b]}$$

where the infinite sum is well defined for sufficiently small $\varsigma > 0$ given the parameter restriction $\Delta_\mathrm{s}(1+\bar{g})^{1-\alpha} < 1$ and $\delta < \Delta_\mathrm{s}$. In either case, define the term β^* as follows:

$$\beta^* = \delta\mathbb{E}\left[(1+g_t)^{1-\alpha}\right], \qquad \text{[A.2.11]}$$

noting that $\beta^* = \delta$ in the case where $\alpha = 1$ and the right-hand side is time invariant in the case where g_t is independent over time. For each of the cases [A.2.10a] and [A.2.10b] it is seen that the expression for d_t^* is the one given in [3.7], and the expression for l_t^* follows immediately from [3.6b] using d_t^*. The solutions for d_t^* and l_t^* imply $l_{t-1}^* = \beta^* d_t^*$, and hence from [A.2.2] that $(1+r_t^*) = (1+g_t)/\beta^*$. Together with $\rho_t^* = \mathbb{E}_t r_{t+1}^*$, the expressions for r_t^* and ρ_t^* in [3.7] are confirmed.

(iii) As can be seen from [2.27a], the case of a representative-household economy ($\Delta_\mathrm{b} = \Delta_\mathrm{s}$) is equivalent to $\theta = 0$ (given that $\alpha > 0$ and $0 < \lambda < 1$). With complete financial markets, the equilibrium levels of debt, loans, and the consumption ratios given in [3.6b] are $d_t^* = 0$, $l_t^* = 0$, $c_{\mathrm{b},t}^* = 1$, and $c_{\mathrm{s},t}^* = 1$. These values satisfy equations [2.22b], [2.22c], and [2.22f]. Since $\Delta_\mathrm{b} = \Delta_\mathrm{s}$, the discount factors consistent with equation [2.22e] are $\delta_{\tau,t}^* = \delta = \Delta_\mathrm{b} = \Delta_\mathrm{s}$.

With incomplete financial markets in the case $\Delta_\mathrm{b} = \Delta_\mathrm{s}$, if $c_{\tau,t} = 1$ and $\delta_{\tau,t} = \delta$ for both types $\tau \in \{\mathrm{b},\mathrm{s}\}$ then the Euler equations in [2.22d] reduce to a single equation $1 = \delta\mathbb{E}_t\left[(1+r_{t+1})(1+g_{t+1})^{-\alpha}\right]$. Given a real return r_t that satisfies this equation, and $\rho_t = \mathbb{E}_t r_{t+1}$ as the implied real interest rate, all of the equilibrium conditions [2.22a]–[2.22f] hold with $d_t = 0$ and $l_t = 0$. This confirms that the equilibrium of the incomplete-markets economy is $c_{\tau,t} = 1$, $d_t = 0$, and $l_t = 0$ when $\Delta_b = \Delta_s$, completing the proof.

A.3 Proof of Proposition 3

The equilibrium conditions [2.22a]–[2.22f] and [2.24]–[2.25a] that hold in the incomplete-markets economy differ from those of the hypothetical complete-markets economy only by the absence of equation [3.5], which holds under complete markets, and by the presence of [2.25a] (and [2.25b]), which is absent from the complete-markets equilibrium conditions. The hypothetical complete-markets equilibrium determines a path $\{r_t^*\}$ for the ex-post real return, as shown in Proposition 2. In the incomplete-markets economy, since equation [2.25a] expresses r_t as a function of π_t and j_t, variables that do not appear in the other equilibrium conditions [2.22a]–[2.22f], it follows that if monetary policy can be used to influence π_t or j_t to ensure that $r_t = r_t^*$ then the consumption allocation in the incomplete-markets economy will coincide with the complete-markets equilibrium.

Here, it is assumed the monetary policy instrument can be used to determine a state-contingent path for the price level P_t, or equivalently, a path for nominal GDP $N_t = P_t Y_t$. The ex-post real return in equation [2.25a] can be expressed equivalently in terms of nominal GDP growth $n_t = (N_t - N_{t-1})/N_{t-1}$, real GDP growth g_t, and bond yields j_t:

$$1 + r_t = (1 + g_t)\left(\frac{1 + j_t}{1 + n_t}\right)\left(\frac{1 + j_{t-1} - \gamma}{1 + j_t - \gamma}\right). \qquad \text{[A.3.1]}$$

Similarly, the equilibrium bond yield j_t from [2.26] (given [2.25b]) can be expressed in terms of real variables and nominal GDP growth as follows:

$$j_t = \left(\mathbb{E}_t\left[\sum_{\ell=1}^{\infty}\gamma^{\ell-1}\left\{\prod_{\jmath=1}^{\ell}\delta_{\tau,t+\jmath-1}\frac{(1+g_{t+\jmath})^{1-\alpha}}{(1+n_{t+\jmath})}\left(\frac{c_{\tau,t+\jmath}}{c_{\tau,t+\jmath-1}}\right)^{-\alpha}\right\}\right]\right)^{-1} + \gamma - 1. \qquad \text{[A.3.2]}$$

If the complete-markets equilibrium is implemented then $\delta_{\tau,t} = \delta$ and $c_{\tau,t} = \bar{c}_\tau$ for all t according to Proposition 2. The expression in [A.3.2] for the equilibrium bond yield in this case then simplifies to:

$$j_t = \left(\mathbb{E}_t\left[\sum_{\ell=1}^{\infty}\gamma^{\ell-1}\delta^{\ell}\left\{\prod_{\jmath=1}^{\ell}\frac{(1+g_{t+\jmath})^{1-\alpha}}{(1+n_{t+\jmath})}\right\}\right]\right)^{-1} + \gamma - 1. \qquad \text{[A.3.3]}$$

(i) One of two special cases is considered: $\alpha = 1$ or $\{g_t\}$ is an i.i.d. stochastic process. With constant nominal GDP growth of $n_t = n$, equation [A.3.3] becomes:

$$j_t = \left(\frac{1}{1+n}\sum_{\ell=1}^{\infty}\left(\frac{\gamma}{1+n}\right)^{\ell-1}\mathbb{E}_t\left[\prod_{\jmath=1}^{\ell}\left\{\delta(1+g_{t+\jmath})^{1-\alpha}\right\}\right]\right)^{-1} + \gamma - 1. \qquad \text{[A.3.4]}$$

In the case $\alpha = 1$, $(1+g_t)^{1-\alpha}$ is a constant; in the case where $\{g_t\}$ is i.i.d., the conditional expectation of a product of functions of growth rates at different dates is simply equal to the product of the unconditional expectations. Thus:

$$\mathbb{E}_t\left[\prod_{\jmath=1}^{\ell}\left(\delta(1+g_{t+\jmath})^{1-\alpha}\right)\right] = \beta^{*\ell}, \quad \text{where } \beta^* = \delta\mathbb{E}\left[(1+g_t)^{1-\alpha}\right],$$

and using equation [A.3.4], the equilibrium bond yield is:

$$j_t = \left(\frac{\beta^*}{1+n}\sum_{\ell=1}^{\infty}\left(\frac{\gamma\beta^*}{1+n}\right)^{\ell-1}\right)^{-1} + \gamma - 1 = \left(\frac{\frac{\beta^*}{1+n}}{1 - \frac{\gamma\beta^*}{1+n}}\right)^{-1} + \gamma - 1 = \frac{1+n}{\beta^*} - 1, \qquad \text{[A.3.5]}$$

under the assumption that $\gamma\beta^* < 1+n$. For sufficiently small ς in [2.3], this follows from $\gamma\beta < 1+n$, which in turn follows from the parameter restriction $\gamma < 1+\bar{\imath}$ because $\beta = (1+\bar{\rho})/(1+\bar{g})$, $1+n = (1+\bar{g})(1+\bar{\pi})$, and $1+\bar{\imath} = (1+\bar{\rho})(1+\bar{\pi})$. Equation [A.3.5] implies $(1+j_t)/(1+n) = 1/\beta^*$, and substituting this into

[A.3.1] leads to the following ex-post real return r_t:

$$1 + r_t = \frac{1 + g_t}{\beta^*}.$$

According to Proposition 2, this is the ex-post real return in the complete-markets economy for either of the special cases, hence $r_t = r_t^*$, which confirms that any constant nominal GDP growth rate succeeds in replicating the complete-markets equilibrium.

(ii) Now consider the general case with no restriction on the parameter α or the stationary stochastic process g_t, but where the parameter γ is strictly positive. Suppose monetary policy sets a constant rate of nominal GDP growth equal to $n = \gamma - 1$, which is such that $n > -1$. Since $\gamma/(1 + n_t) = 1$, the expression for the bond yield in [A.3.3] becomes:

$$j_t = \left(\frac{1}{\gamma}\mathbb{E}_t\left[\sum_{\ell=1}^{\infty}\delta^\ell \prod_{j=1}^{\ell}(1 + g_{t+j})^{1-\alpha}\right]\right)^{-1} + \gamma - 1. \qquad \text{[A.3.6]}$$

Note that [3.6b] in Proposition 2 implies the complete-markets loans-to-GDP ratio l_t^* can be written as:

$$l_t^* = \frac{\theta}{2}\mathbb{E}_t\left[\sum_{\ell=1}^{\infty}\delta^\ell \prod_{j=1}^{\ell}(1 + g_{t+j})^{1-\alpha}\right],$$

and hence the expression for j_t from [A.3.6] is related to l_t^* as follows:

$$j_t = \gamma\left(1 + \frac{\theta}{2l_t^*}\right) - 1. \qquad \text{[A.3.7]}$$

Since equation [3.6b] implies that $d_t^* = (\theta/2) + l_t^*$, the bond yield j_t above can also be written as:

$$j_t = \frac{\gamma}{l_t^*}\left(\frac{\theta}{2} + l_t^*\right) - 1 = \frac{\gamma d_t^*}{l_t^*} - 1.$$

Together with equation [A.3.7], which implies $1 + j_t - \gamma = (\gamma\theta)/(2l_t^*)$, this expression for j_t can be substituted into [A.3.1] (with $1 + n_t = \gamma$) to obtain:

$$1 + r_t = (1 + g_t)\left(\frac{\frac{\gamma d_t^*}{l_t^*}}{\gamma}\right)\left(\frac{\frac{\gamma\theta}{2l_{t-1}^*}}{\frac{\gamma\theta}{2l_t^*}}\right) = (1 + g_t)\frac{d_t^*}{l_{t-1}^*} = (1 + g_t)\left(\frac{1 + r_t^*}{1 + g_t}\right) = 1 + r_t^*,$$

where the penultimate equality uses [2.22b]. This establishes $r_t = r_t^*$, and hence that the nominal GDP target succeeds in replicating the complete-markets equilibrium.

(iii) The state-contingent path for nominal GDP specified in this part is equivalent to the following nominal GDP growth rate:

$$n_t = (1 + n)\left(\frac{d_{t-1}^* - \frac{\theta}{2}}{\beta d_t^*}\right) - 1,$$

where d_t^* is the natural debt-to-GDP ratio characterized in Proposition 2. Using the steady-state results from Proposition 1, this specification of monetary policy can be seen to imply that nominal GDP growth fluctuates around a rate n in the non-stochastic steady state. By using equation [3.6b] to substitute for d_{t-1}^* in terms of l_{t-1}^*, and then by using the link between d_t^*, l_{t-1}^*, and r_t^* in [2.22b]:

$$1 + n_t = (1 + n)\left(\frac{l_{t-1}^*}{\beta d_t^*}\right) = \left(\frac{1 + n}{\beta}\right)\left(\frac{1 + g_t}{1 + r_t^*}\right). \qquad \text{[A.3.8]}$$

The bond yield is obtained by substituting this nominal GDP growth rate into equation [A.3.3]:

$$j_t = \left(\frac{\beta}{1 + n}\sum_{\ell=1}^{\infty}\left(\frac{\gamma\beta}{1 + n}\right)^{\ell-1}\mathbb{E}_t\left[\prod_{j=1}^{\ell}\left\{\delta(1 + g_{t+j})^{-\alpha}(1 + r_{t+j}^*)\right\}\right]\right)^{-1} + \gamma - 1. \qquad \text{[A.3.9]}$$

At the complete-markets equilibrium, $\delta_{\tau,t}^* = \delta$ and $c_{\tau,t}^* = \bar{c}_\tau$, in which case the Euler equations in [2.22d] reduce to:

$$\delta\mathbb{E}_t\left[(1+g_{t+1})^{-\alpha}(1+r_t^*)\right]=1,$$

and application of the law of iterated expectations shows that this implies for all $\ell \geq 1$:

$$\mathbb{E}_t\left[\prod_{\jmath=1}^{\ell}\left\{\delta(1+g_{t+\jmath})^{-\alpha}(1+r_{t+\jmath}^*)\right\}\right]=1.$$

Substituting this result into [A.3.9] shows that the bond yield j_t is given by:

$$j_t=\left(\frac{\beta}{1+n}\sum_{\ell=1}^{\infty}\left(\frac{\gamma\beta}{1+n}\right)^{\ell-1}\right)^{-1}+\gamma-1=\left(\frac{\frac{\beta}{1+n}}{1-\frac{\gamma\beta}{1+n}}\right)^{-1}+\gamma-1=\frac{1+n}{\beta}-1.$$

Convergence of the infinite sum requires $\gamma\beta<1+n$, which follows from the parameter restriction $\gamma<1+\bar{i}$ as in part (i) above. Since the equilibrium bond yield is constant over time with $1+j_t=(1+n)/\beta$, substitution of this and equation [A.3.8] into [A.3.1] implies that the ex-post real return is:

$$1+r_t=(1+g_t)\left(\frac{1+n}{\beta}\right)\left(\left(\frac{1+n}{\beta}\right)\left(\frac{1+g_t}{1+r_t^*}\right)\right)^{-1}=1+r_t^*.$$

With $r_t=r_t^*$, the specified nominal GDP path replicates the complete-markets equilibrium. This completes the proof.

A.4 Proof of Proposition 4

(i) Let $F_{\tau,t}$ denote beginning-of-period t per-household real financial wealth (in bonds) of type τ households and let $f_{\tau,t}$ denote this wealth relative to current real income:

$$F_{\tau,t}=\frac{(1+\gamma Q_t)B_{\tau,t-1}}{P_t},\quad \text{and} \quad f_{\tau,t}=\frac{F_{\tau,t}}{Y_{\tau,t}}. \qquad \text{[A.4.1]}$$

Given the definition of r_t in [2.15], the assumption on incomes in [2.4], and the equilibrium condition $M_{\tau,t}=0$ from [2.14], the flow budget identities [2.5] for each type of household can be stated as follows:

$$C_{\tau,t}+\frac{F_{\tau,t+1}}{1+r_{t+1}}=Y_t+F_{\tau,t},$$

which can be expressed in terms of the ratios $c_{\tau,t}=C_{\tau,t}/Y_t$ and $f_{\tau,t}=F_{\tau,t}/Y_t$ and real GDP growth g_t:

$$c_{\tau,t}+\left(\frac{1+g_{t+1}}{1+r_{t+1}}\right)f_{\tau,t+1}=1+f_{\tau,t}.$$

Writing this as an equation for $f_{\tau,t+1}$ and then multiplying both sides by $\delta_{\tau,t}(1+g_{t+1})^{1-\alpha}(c_{\tau,t+1}/c_{\tau,t})^{-\alpha}$ yields:

$$\delta_{\tau,t}(1+g_{t+1})^{1-\alpha}\left(\frac{c_{\tau,t+1}}{c_{\tau,t}}\right)^{-\alpha}f_{\tau,t+1}=\left(\delta_{\tau,t}(1+r_{t+1})(1+g_{t+1})^{-\alpha}\left(\frac{c_{\tau,t+1}}{c_{\tau,t}}\right)^{-\alpha}\right)(f_{\tau,t}+1-c_{\tau,t}).$$

Taking expectations of the above equation conditional on date-t information, using the Euler equation in [2.22d], and rearranging leads to:

$$f_{\tau,t}=(c_{\tau,t}-1)+\mathbb{E}_t\left[\delta_{\tau,t}(1+g_{t+1})^{1-\alpha}\left(\frac{c_{\tau,t+1}}{c_{\tau,t}}\right)^{-\alpha}f_{\tau,t+1}\right],$$

which can be iterated forwards to obtain:

$$f_{\tau,t} = \mathbb{E}_t\left[\sum_{\ell=0}^{\infty}\left\{\prod_{\jmath=1}^{\ell}\delta_{\tau,t+\jmath-1}(1+g_{t+\jmath})^{1-\alpha}\right\}\left(\frac{c_{\tau,t+\ell}}{c_{\tau,t}}\right)^{-\alpha}(c_{\tau,t+\ell}-1)\right] + \lim_{\ell\to\infty}\mathbb{E}_t\left[\left\{\prod_{\jmath=1}^{\ell}\delta_{\tau,t+\jmath-1}(1+g_{t+\jmath})^{1-\alpha}\right\}\left(\frac{c_{\tau,t+\ell}}{c_{\tau,t}}\right)^{-\alpha}f_{\tau,t+\ell}\right]. \quad \text{[A.4.2]}$$

The accounting identity [2.22b] and the law of iterated expectations can be used to deduce the following for any $\ell \geq 1$:

$$\mathbb{E}_t\left[\left\{\prod_{\jmath=1}^{\ell}\delta_{\tau,t+\jmath-1}(1+g_{t+\jmath})^{1-\alpha}\right\}c_{\tau,t+\ell}^{-\alpha}d_{t+\ell}\right] = \mathbb{E}_t\left[\left\{\prod_{\jmath=1}^{\ell-1}\delta_{\tau,t+\jmath-1}(1+g_{t+\jmath})^{1-\alpha}\right\}c_{\tau,t+\ell-1}^{-\alpha}l_{t+\ell-1}\mathbb{E}_{t+\ell-1}\left[\delta_{\tau,t+\ell-1}(1+g_{t+\ell})^{-\alpha}\left(\frac{c_{\tau,t+\ell}}{c_{\tau,t+\ell-1}}\right)^{-\alpha}\right]\right].$$

By using the Euler equations in [2.22d] and taking the limit of the above equation as $\ell \to \infty$, and then taking expectations of the transversality condition [2.22f] conditional on date-t information:

$$\lim_{\ell\to\infty}\mathbb{E}_t\left[\left\{\prod_{\jmath=1}^{\ell}\delta_{\tau,t+\jmath-1}(1+g_{t+\jmath})^{1-\alpha}\right\}c_{\tau,t+\ell}^{-\alpha}d_{t+\ell}\right] = 0. \quad \text{[A.4.3]}$$

Comparison of the definitions in [2.15] and [A.4.1] reveals $f_{\mathrm{b},t} = -2d_t$ and $f_{\mathrm{s},t} = 2d_t$, hence by using [A.4.3] it must be the case that:

$$\lim_{\ell\to\infty}\mathbb{E}_t\left[\left\{\prod_{\jmath=1}^{\ell}\delta_{\tau,t+\jmath-1}(1+g_{t+\jmath})^{1-\alpha}\right\}\left(\frac{c_{\tau,t+\ell}}{c_{\tau,t}}\right)^{-\alpha}f_{t+\ell}\right] = 0, \quad \text{[A.4.4]}$$

which implies the final term in equation [A.4.2] is zero, therefore:

$$f_{\tau,t} = \mathbb{E}_t\left[\sum_{\ell=0}^{\infty}\left\{\prod_{\jmath=1}^{\ell}\delta_{\tau,t+\jmath-1}(1+g_{t+\jmath})^{1-\alpha}\right\}\left(\frac{c_{\tau,t+\ell}}{c_{\tau,t}}\right)^{-\alpha}(c_{\tau,t+\ell}-1)\right]. \quad \text{[A.4.5]}$$

Now make the following definitions of variables $v_{\tau,t}$ and $m_{\tau,t}$ for each $\tau \in \{\mathrm{b},\mathrm{s}\}$:

$$v_{\tau,t} = \mathbb{E}_t\left[\sum_{\ell=0}^{\infty}\left\{\prod_{\jmath=1}^{\ell}\delta_{\tau,t+\jmath-1}(1+g_{t+\jmath})^{1-\alpha}\right\}\left(\frac{c_{\tau,t+\ell}}{c_{\tau,t}}\right)^{-\alpha}\right]; \text{ and} \quad \text{[A.4.6a]}$$

$$m_{\tau,t} = \left(\mathbb{E}_t\left[\sum_{\ell=0}^{\infty}\left\{\prod_{\jmath=1}^{\ell}\delta_{\tau,t+\jmath-1}(1+g_{t+\jmath})^{1-\alpha}\right\}\left(\frac{c_{\tau,t+\ell}}{c_{\tau,t}}\right)^{1-\alpha}\right]\right)^{-1}. \quad \text{[A.4.6b]}$$

With these definitions, equation [A.4.5] can be expressed concisely as

$$c_{\tau,t} = m_{\tau,t}(v_t + f_{\tau,t}), \quad \text{[A.4.7]}$$

which implicitly defines a consumption function for households of each type τ.

Now consider the log-linear approximation of the consumption functions [A.4.7]. First, take the log linearization of the Euler equations in [2.22d]:

$$\mathsf{\delta}_{\tau,t} + \mathbb{E}_t[\mathsf{r}_{t+1} - \alpha\mathsf{g}_{t+1} - \alpha(\mathsf{c}_{\tau,t+1} - \mathsf{c}_{\tau,t})] = 0, \quad \text{[A.4.8]}$$

where $\mathsf{\delta}_{\tau,t}$ is the log deviation of the discount factor $\delta_{\tau,t}$. The definition of the real interest rate in [2.22a] implies

$$\mathsf{\rho}_t = \mathbb{E}_t\mathsf{r}_{t+1}. \quad \text{[A.4.9]}$$

Together with this, equation [A.4.8] can be rearranged as follows:

$$\delta_{\tau,t} + (1-\alpha)\mathbb{E}_t \mathsf{g}_{t+1} - \alpha(\mathbb{E}_t \mathsf{c}_{\tau,t+1} - \mathsf{c}_{\tau,t}) = -(\rho_t - \mathbb{E}_t \mathsf{g}_{t+1}). \qquad \text{[A.4.10]}$$

Next, note that the formula for v_t in [3.8] can be written in terms of real GDP growth:

$$v_t = \sum_{\ell=0}^{\infty} \mathbb{E}_t \left[\prod_{\jmath=1}^{\ell} \left(\frac{1+g_{t+\jmath}}{1+\rho_{t+\jmath-1}} \right) \right].$$

Using Proposition 1, the steady-state value of this variable is $\bar{v} = \sum_{\ell=0}^{\infty} \beta^{\ell} = (1-\beta)^{-1}$, where β is as given in [2.27c], satisfying $0 < \beta < 1$. The expression for v_t above can then be log linearized as follows:

$$\mathsf{v}_t = (1-\beta) \sum_{\ell=0}^{\infty} \beta^{\ell+1} \sum_{\jmath=0}^{\ell} \mathbb{E}_t[\mathsf{g}_{t+\jmath+1} - \rho_{t+\jmath}]. \qquad \text{[A.4.11]}$$

Now, using again the non-stochastic steady state characterized in Proposition 1, the steady-state value of $v_{\tau,t}$ in [A.4.6a] is $\bar{v}_{\tau} = (1-\beta)^{-1}$ for all $\tau \in \{\mathrm{b}, \mathrm{s}\}$, and hence $v_{\tau,t}$ has the following log linearization:

$$\mathsf{v}_{\tau,t} = (1-\beta) \sum_{\ell=1}^{\infty} \beta^{\ell} \sum_{\jmath=1}^{\ell} \mathbb{E}_t \left[\delta_{\tau,t+\jmath-1} + (1-\alpha)\mathsf{g}_{t+\jmath} - \alpha(\mathsf{c}_{t+\jmath} - \mathsf{c}_{\tau,t+\jmath-1}) \right],$$

which can be expressed as below by substituting from equation [A.4.10]:

$$\mathsf{v}_{\tau,t} = -(1-\beta) \sum_{\ell=0}^{\infty} \beta^{\ell+1} \sum_{\jmath=0}^{\ell} \mathbb{E}_t[\rho_{t+\jmath} - \mathsf{g}_{t+\jmath+1}]. \qquad \text{[A.4.12]}$$

Note that the log-linear approximation is the same for all $\tau \in \{\mathrm{b}, \mathrm{s}\}$, and furthermore, is equal to the expression for v_t given in [A.4.11]. That formula can be simplified by changing the order of summation to obtain:

$$\mathsf{v}_t = \mathsf{v}_{\tau,t} = -\sum_{\ell=0}^{\infty} \beta^{\ell+1} \mathbb{E}_t[\rho_{t+\ell} - \mathsf{g}_{t+\ell+1}]. \qquad \text{[A.4.13]}$$

Now consider the variable $m_{\tau,t}$ defined in [A.4.6b]. Using Proposition 1, its value in the non-stochastic steady state is $\bar{m}_{\tau} = 1-\beta$ for all $\tau \in \{\mathrm{b}, \mathrm{s}\}$. Equation [A.4.6b] can then be log-linearized as follows:

$$\mathsf{m}_{\tau,t} = -(1-\beta) \sum_{\ell=1}^{\infty} \beta^{\ell} \sum_{\jmath=1}^{\ell} \mathbb{E}_t[\delta_{\tau,t+\jmath-1} + (1-\alpha)\mathsf{g}_{t+\jmath} + (1-\alpha)(\mathsf{c}_{\tau,t+\jmath} - \mathsf{c}_{\tau,t+\jmath-1})],$$

and by using [A.4.10], this can be written as:

$$\mathsf{m}_{\tau,t} = -(1-\beta) \sum_{\ell=1}^{\infty} \beta^{\ell} \sum_{\jmath=1}^{\ell} \mathbb{E}_t[\mathsf{c}_{\tau,t+\jmath} - \mathsf{c}_{\tau,t+\jmath-1}] + (1-\beta) \sum_{\ell=1}^{\infty} \beta^{\ell} \sum_{\jmath=1}^{\ell} \mathbb{E}_t[\rho_{t+\jmath-1} - \mathsf{g}_{t+\jmath}].$$

By simplifying the first term and using equation [A.4.11]:

$$\mathsf{m}_{\tau,t} = \mathsf{c}_{\tau,t} - (1-\beta) \sum_{\ell=0}^{\infty} \beta^{\ell} \mathbb{E}_t \mathsf{c}_{\tau,t+\ell} - \mathsf{v}_t. \qquad \text{[A.4.14]}$$

Next, note that the equation for the discount factors $\delta_{\tau,t}$ in [2.22e] has the following log-linear form:

$$\delta_{\tau,t} = -(1-\lambda)\alpha \mathsf{c}_{\tau,t}.$$

Substituting this into the log-linearized Euler equations in [A.4.8], then simplifying and dividing both sides by α leads to:

$$\mathbb{E}_t \mathsf{c}_{\tau,t+1} = \lambda \mathsf{c}_{\tau,t} + \alpha^{-1}\rho_t - \mathbb{E}_t \mathsf{g}_{t+1}.$$

This expectational difference equation can be iterated ℓ periods ahead to deduce:

$$\mathbb{E}_t \mathsf{c}_{\tau,t+\ell} = \lambda^{\ell} \mathsf{c}_{\tau,t} + \sum_{\jmath=0}^{\ell-1} \lambda^{\ell-1-\jmath} \mathbb{E}_t[\alpha^{-1}\rho_{t+\jmath} - \mathsf{g}_{t+\jmath+1}],$$

and by using this equation, an expression for the following infinite sum can be obtained:

$$(1-\beta)\sum_{\ell=0}^{\infty}\beta^{\ell}\mathbb{E}_t\mathsf{c}_{\tau,t+\ell} = (1-\beta)\sum_{\ell=0}^{\infty}\beta^{\ell}\left(\lambda^{\ell}\mathsf{c}_{\tau,t} + \sum_{\jmath=0}^{\ell-1}\lambda^{\ell-1-\jmath}\mathbb{E}_t[\alpha^{-1}\rho_{t+\jmath} - \mathsf{g}_{t+\jmath+1}]\right)$$

$$= \left(\frac{1-\beta}{1-\beta\lambda}\right)\left(\mathsf{c}_{\tau,t} + \sum_{\ell=0}^{\infty}\beta^{\ell+1}\mathbb{E}_t[\alpha^{-1}\rho_{t+\ell} - \mathsf{g}_{t+\ell+1}]\right),$$

where the second equality follows by changing the order of summation. Substituting this result into equation [A.4.14] and simplifying leads to:

$$\mathsf{m}_{\tau,t} = \frac{\beta(1-\lambda)}{(1-\beta\lambda)}\mathsf{c}_{\tau,t} - \mathsf{v}_t - \left(\frac{1-\beta}{1-\beta\lambda}\right)\sum_{\ell=0}^{\infty}\beta^{\ell+1}\mathbb{E}_t[\alpha^{-1}\rho_{t+\ell} - \mathsf{g}_{t+\ell+1}]. \qquad \text{[A.4.15]}$$

To complete the derivation of the consumption function, note from [2.15] and [A.4.1] that $f_{\mathrm{b},t} = -2d_t$ and $f_{\mathrm{s},t} = 2d_t$, and hence the equations in [A.4.7] can be written as follows:

$$c_{\mathrm{b},t} = m_{\mathrm{b},t}(v_{\mathrm{b},t} - 2d_t), \quad \text{and} \quad c_{\mathrm{s},t} = m_{\mathrm{s},t}(v_{\mathrm{s},t} + 2d_t).$$

Using $\bar{m}_\tau = 1-\beta$, $\bar{v}_\tau = (1-\beta)^{-1}$, the steady-state values of $c_{\tau,t}$ and d_t from Proposition 1, and equation [A.4.13], the log-linearizations of the equations above are:

$$\mathsf{c}_{\mathrm{b},t} = \mathsf{m}_{\mathrm{b},t} + \frac{1}{1-\theta}(\mathsf{v}_t - \theta\mathsf{d}_t), \quad \text{and} \quad \mathsf{c}_{\mathrm{s},t} = \mathsf{m}_{\mathrm{s},t} + \frac{1}{1+\theta}(\mathsf{v}_t + \theta\mathsf{d}_t). \qquad \text{[A.4.16]}$$

Substituting the expressions for $\mathsf{m}_{\tau,t}$ from [A.4.15] and rearranging and simplifying leads to the following consumption functions:

$$\mathsf{c}_{\mathrm{b},t} = -\left(\frac{1-\beta\lambda}{1-\beta}\right)\left(\frac{\theta}{1-\theta}\right)(\mathsf{d}_t - \mathsf{v}_t) - \beta\sum_{\ell=0}^{\infty}\beta^{\ell}\mathbb{E}_t[\alpha^{-1}\rho_{t+\ell} - \mathsf{g}_{t+\ell+1}]; \text{ and} \qquad \text{[A.4.17a]}$$

$$\mathsf{c}_{\mathrm{s},t} = \left(\frac{1-\beta\lambda}{1-\beta}\right)\left(\frac{\theta}{1+\theta}\right)(\mathsf{d}_t - \mathsf{v}_t) - \beta\sum_{\ell=0}^{\infty}\beta^{\ell}\mathbb{E}_t[\alpha^{-1}\rho_{t+\ell} - \mathsf{g}_{t+\ell+1}]. \qquad \text{[A.4.17b]}$$

Having derived the consumption functions, the equilibrium of the economy is found using the following steps. First, consider the budget identities in [2.22c]. Using the steady state characterized in Proposition 1, the log linearizations of those equations are:

$$\mathsf{c}_{\mathrm{b},t} = -\left(\frac{\theta}{1-\theta}\right)\left(\frac{1}{1-\beta}\right)(\mathsf{d}_t - \beta\mathsf{l}_t), \quad \text{and} \quad \mathsf{c}_{\mathrm{s},t} = \left(\frac{\theta}{1+\theta}\right)\left(\frac{1}{1-\beta}\right)(\mathsf{d}_t - \beta\mathsf{l}_t). \qquad \text{[A.4.18]}$$

Note that these equations imply

$$(1-\theta)\mathsf{c}_{\mathrm{b},t} + (1+\theta)\mathsf{c}_{\mathrm{s},t} = 0, \qquad \text{[A.4.19]}$$

which, given Proposition 1, is the log-linearized goods-market clearing condition [2.23]. By substituting the consumption functions from [A.4.17], equation [A.4.19] is satisfied if and only if the following holds for all t:

$$\sum_{\ell=0}^{\infty}\beta^{\ell}\mathbb{E}_t[\alpha^{-1}\rho_{t+\ell} - \mathsf{g}_{t+\ell+1}] = 0.$$

Since this equation must hold at any two consecutive dates, it follows that

$$0 = \left(\sum_{\ell=0}^{\infty}\beta^{\ell}\mathbb{E}_t[\alpha^{-1}\rho_{t+\ell} - \mathsf{g}_{t+\ell+1}]\right) - \beta\mathbb{E}_t\left[\sum_{\ell=0}^{\infty}\beta^{\ell}\mathbb{E}_{t+1}[\alpha^{-1}\rho_{t+1+\ell} - \mathsf{g}_{t+1+\ell+1}]\right] = \alpha^{-1}\rho_t - \mathbb{E}_t\mathsf{g}_{t+1},$$

where the law of iterated expectations is used, and hence the equilibrium real interest rate is $\rho_t = \alpha\mathbb{E}_t\mathsf{g}_{t+1}$, confirming the expression given in [3.9a]. Using equation [A.4.13], the equilibrium value of v_t is also that in [3.9a], confirming the results in that equation.

The equilibrium consumption ratios can be found in terms of v_t and the debt-to-GDP ratio d_t by substituting the equilibrium real interest rate from [3.9a] into the consumption functions from [A.4.17]:

$$\mathsf{c}_{\mathrm{b},t} = -\left(\frac{\theta}{1-\theta}\right)\left(\frac{1-\beta\lambda}{1-\beta}\right)(\mathsf{d}_t - \mathsf{v}_t), \quad \text{and} \quad \mathsf{c}_{\mathrm{s},t} = \left(\frac{\theta}{1+\theta}\right)\left(\frac{1-\beta\lambda}{1-\beta}\right)(\mathsf{d}_t - \mathsf{v}_t). \qquad [\text{A.4.20}]$$

By equating either $\mathsf{c}_{\mathrm{b},t}$ or $\mathsf{c}_{\mathrm{s},t}$ as they appear in [A.4.20] to the equivalent expression in [A.4.18], an equation for the loans-to-GDP ratio l_t is obtained (the same equation is obtained for both $\tau \in \{\mathrm{b}, \mathrm{s}\}$ because the expressions in [A.4.20] satisfy equation [A.4.19]):

$$(1-\beta\lambda)(\mathsf{d}_t - \mathsf{v}_t) = \mathsf{d}_t - \beta\mathsf{l}_t.$$

Rearranging this equation leads to the expression for l_t given in [3.9c].

The accounting identity [2.22b] has the following log linear form:

$$\mathsf{d}_t = (\mathsf{r}_t - \mathsf{g}_t) + \mathsf{l}_{t-1}. \qquad [\text{A.4.21}]$$

Taking expectations conditional on date-$t-1$ information implies that $\mathsf{l}_{t-1} = \mathbb{E}_{t-1}\mathsf{d}_t + \mathbb{E}_{t-1}\mathsf{g}_t - \mathbb{E}_{t-1}\mathsf{r}_t$, and by using [3.9a] and [A.4.9], it follows that $\mathsf{l}_{t-1} = (1-\alpha)\mathbb{E}_{t-1}\mathsf{g}_t + \mathbb{E}_{t-1}\mathsf{d}_t$. Substituting this result into [A.4.21] yields the expression for the ex-post real return r_t in [3.9c].

The expectation of equation [A.4.21] at date $t+1$ conditional on date-t information and equations [A.4.9] and [3.9a] imply that:

$$\mathbb{E}_t\mathsf{d}_{t+1} = (\alpha - 1)\mathbb{E}_t\mathsf{g}_{t+1} + \mathsf{l}_t. \qquad [\text{A.4.22}]$$

Observe that the solution for v_t in [3.9a] implies the following equation must hold:

$$\frac{\mathsf{v}_t - \beta\mathbb{E}_t\mathsf{v}_{t+1}}{\beta} = (1-\alpha)\mathbb{E}_t\mathsf{g}_{t+1}.$$

By substituting this and the expression for l_t from [3.9c] into [A.4.22]:

$$\mathbb{E}_t\mathsf{d}_{t+1} = \lambda\mathsf{d}_t + \left(\frac{1-\beta\lambda}{\beta}\right)\mathsf{v}_t - \left(\frac{\mathsf{v}_t - \beta\mathbb{E}_t\mathsf{v}_{t+1}}{\beta}\right),$$

and collecting terms and simplifying leads to the following expectational difference equation for the debt-to-GDP ratio:

$$\mathbb{E}_t\mathsf{d}_{t+1} = \lambda\mathsf{d}_t + \mathbb{E}_t\mathsf{v}_{t+1} - \lambda\mathsf{v}_t. \qquad [\text{A.4.23}]$$

Now take any martingale difference sequence $\{\upsilon_t\}$ that satisfies $\mathbb{E}_{t-1}\upsilon_t = 0$, and given the conditional expectation $\mathbb{E}_{t-1}\mathsf{d}_t$, let the actual debt-to-GDP ratio be such that $\mathsf{d}_t = \mathbb{E}_{t-1}\mathsf{d}_t + \upsilon_t$. Together with equation [A.4.23] at date $t-1$ this implies [3.9b] holds, and this construction is valid for any υ_t with $\mathbb{E}_{t-1}\upsilon_t = 0$. Thus, all the equilibrium conditions [2.22a]–[2.22f] are satisfied by the solution [3.9a]–[3.9c] for any martingale difference sequence $\{\upsilon_t\}$.

(ii) The equilibrium condition [2.24] can be log-linearized as follows:

$$(\mathsf{i}_t - \mathbb{E}_t\pi_{t+1}) + (\delta_{\mathrm{s},t} - \alpha\mathbb{E}_t\mathsf{g}_{t+1} - \alpha(\mathbb{E}_t\mathsf{c}_{\mathrm{s},t+1} - \mathsf{c}_{\mathrm{s},t})) = 0,$$

and by using the Euler equations [A.4.8] and the ex-ante real interest rate [A.4.9], this leads to the Fisher equation [3.10].

(iii) The non-stochastic steady state of the economy with complete financial markets is identical to that in Proposition 1 because the relevant set of equilibrium conditions [2.22a]–[2.22f] overlaps with the equilibrium conditions of the incomplete-markets economy. The expression in [3.6b] from Proposition 2 for the complete-markets debt-to-GDP ratio can then be log linearized as follows:

$$\mathsf{d}_t^* = (1-\alpha)(1-\beta)\sum_{\ell=1}^{\infty}\beta^\ell\sum_{\jmath=1}^{\ell}\mathbb{E}_t\mathsf{g}_{t+\jmath} = (1-\alpha)\sum_{\ell=1}^{\infty}\beta^\ell\mathbb{E}_t\mathsf{g}_{t+\ell},$$

where the second equality is obtained by changing the order of summation. This expression is identical to the formula for v_t in [3.9a], confirming the equation for d_t^* in [3.11]. The log-linearized loans-to-GDP ratio l_t^* given in [3.11] can be obtained immediately from [3.6b] with reference to the steady state characterized in Proposition 1. Similarly, the expression for r_t^* is obtained from [3.6c] using the expression for l_t^* in [3.6b] with reference again to the steady state and the formulas for d_t^* and l_t^* in [3.11]. The expression for the real interest rate in [3.6c] can be log linearized as follows

$$\rho_t^* = \alpha\mathbb{E}_t\mathsf{g}_{t+1} - \mathbb{E}_t\left[\mathsf{g}_{t+1} + \mathsf{d}_{t+1}^* - \mathbb{E}_t[\mathsf{g}_{t+1} + \mathsf{d}_{t+1}^*]\right] = \alpha\mathbb{E}_t\mathsf{g}_{t+1},$$

which shows that ρ_t^* is identical to the equilibrium value of ρ_t in [3.9a]. With identical ex-ante real interest rates, equation [A.4.13] implies that the equilibrium value of v_t is also the same as that in [3.9a]. This confirms the log linearization of the complete-markets equilibrium in [3.11].

The next claim is that these equilibrium values are identical to what would be obtained from equations [3.9b] and [3.9c] with $\upsilon_t = \upsilon_t^* = \mathsf{v}_t - \mathbb{E}_{t-1}\mathsf{v}_t$. Since $\mathbb{E}_{t-1}\upsilon_t^* = 0$, $\{\upsilon_t^*\}$ is a valid martingale difference sequence. With $\upsilon_t = \upsilon_t^*$, equation [3.9b] becomes:

$$\mathsf{d}_t = \lambda\mathsf{d}_{t-1} + (\mathsf{v}_t - \lambda\mathsf{v}_{t-1}),$$

and as this holds for all t, elimination of the common factor from the difference equation implies $\mathsf{d}_t = \mathsf{v}_t$. Since $\mathsf{v}_t = \mathsf{d}_t^*$, it follows that $\mathsf{d}_t = \mathsf{d}_t^*$. Substituting $\mathsf{d}_t = \mathsf{v}_t$ into the expression for l_t in [3.9c] implies $\mathsf{l}_t = \beta^{-1}\mathsf{v}_t$, and hence $\mathsf{l}_t = \mathsf{l}_t^*$. Similarly, substituting $\mathsf{d}_t = \mathsf{v}_t$ into the formula for r_t from [3.9c] and using the equilibrium value of v_t in [3.9a]:

$$\mathsf{r}_t = \mathsf{g}_t + \mathsf{v}_t + (\alpha-1)\mathbb{E}_{t-1}\mathsf{g}_t - (1-\alpha)\sum_{\ell=1}^{\infty}\beta^\ell\mathbb{E}_{t-1}\mathsf{g}_{t+\ell} = \mathsf{g}_t + \mathsf{v}_t - \beta^{-1}(1-\alpha)\sum_{\ell=1}^{\infty}\beta^\ell\mathbb{E}_{t-1}\mathsf{g}_{t-1+\ell}.$$

This shows that $\mathsf{r}_t = \mathsf{g}_t + \mathsf{v}_t - \beta^{-1}\mathsf{v}_{t-1}$, which by comparison with [3.11] confirms that $\mathsf{r}_t = \mathsf{r}_t^*$.

Now consider again the incomplete-markets economy and take any values of d_t, l_t, and r_t consistent with [3.9b]–[3.9c]. Equation [3.9b] implies $\mathbb{E}_t\mathsf{d}_{t+1} = \lambda\mathsf{d}_t + (\mathbb{E}_t\mathsf{v}_{t+1} - \lambda\mathsf{v}_t)$, and hence by using $\mathsf{d}_t^* = \mathsf{v}_t$:

$$\mathbb{E}_t[\mathsf{d}_{t+1} - \mathsf{d}_{t+1}^*] = \lambda(\mathsf{d}_t - \mathsf{d}_t^*).$$

This confirms the expectational difference equation for the debt gap $\tilde{\mathsf{d}}_t = \mathsf{d}_t - \mathsf{d}_t^*$ in [3.12a]. Again by using $\mathsf{d}_t^* = \mathsf{v}_t$ the equation for l_t in [3.9c] becomes:

$$\mathsf{l}_t = \beta^{-1}\mathsf{d}_t^* + \lambda(\mathsf{d}_t - \mathsf{d}_t^*),$$

which leads to the equation for $\tilde{\mathsf{l}}_t = \mathsf{l}_t - \mathsf{l}_t^*$ in [3.12a] by using $\mathsf{l}_t^* = \beta^{-1}\mathsf{d}_t^*$ from [3.11]. For the ex-post real return gap $\tilde{\mathsf{r}}_t = \mathsf{r}_t - \mathsf{r}_t^*$, note that by using [A.4.9], $\mathsf{r}_t = \rho_{t-1} + (\mathsf{r}_t - \mathbb{E}_{t-1}\mathsf{r}_t)$ and $\mathsf{r}_t^* = \rho_{t-1}^* + (\mathsf{r}_t^* - \mathbb{E}_{t-1}\mathsf{r}_t^*)$, and since $\rho_t = \rho_t^*$ it follows that $\mathbb{E}_{t-1}\tilde{\mathsf{r}}_t = 0$. Equations [3.9c] and [3.11] imply

$$\mathsf{r}_t - \mathbb{E}_{t-1}\mathsf{r}_t = (\mathsf{g}_t - \mathbb{E}_{t-1}\mathsf{g}_t) + (\mathsf{d}_t - \mathbb{E}_{t-1}\mathsf{d}_t), \quad \text{and} \quad \mathsf{r}_t^* - \mathbb{E}_{t-1}\mathsf{r}_t^* = (\mathsf{g}_t - \mathbb{E}_{t-1}\mathsf{g}_t) + (\mathsf{d}_t^* - \mathbb{E}_{t-1}\mathsf{d}_t^*),$$

and by putting these results together, the expression for $\tilde{\mathsf{r}}_t$ in [3.12a] is obtained. For the consumption gaps $\tilde{\mathsf{c}}_{\tau,t} = \mathsf{c}_{\tau,t} - \mathsf{c}_{\tau,t}^*$, first note that since $c_{\mathrm{b},t}^* = 1-\theta$ and $c_{\mathrm{s},t}^* = 1+\theta$ according to Proposition 2, the log linearization is $\mathsf{c}_{\tau,t}^* = 0$, and hence $\tilde{\mathsf{c}}_{\tau,t} = \mathsf{c}_{\tau,t}$. The expressions for the consumption gaps in equation [3.12b] then follow immediately from [A.4.20] using $\mathsf{v}_t = \mathsf{d}_t^*$.

(iv) Using $1+\bar{n} = (1+\bar{\pi})(1+\bar{g})$ (from the definition of nominal GDP growth), $1+\bar{\imath} = (1+\bar{\rho})(1+\bar{\pi})$ from Proposition 1, and the formula for β from [2.27c], the definition of μ in the proposition can be rewritten as:

$$\mu = \frac{\gamma}{1+\bar{n}} = \frac{\gamma}{(1+\bar{\pi})(1+\bar{g})} = \left(\frac{1+\bar{\rho}}{1+\bar{g}}\right)\left(\frac{\gamma}{1+\bar{\imath}}\right), \quad \text{and hence} \quad \beta\mu = \frac{\gamma}{1+\bar{\imath}}. \qquad \text{[A.4.24]}$$

Given the parameter restriction $\gamma < 1+\bar{\imath}$, it follows that $\beta\mu < 1$. Now consider the transversality condition [2.25b] for the bond yield. Noting that $\gamma/(1+\bar{\jmath}) = \beta\mu$ using [A.4.24] and $\bar{\jmath} = \bar{\imath}$ from Proposition 1, and $\gamma\delta(1+\bar{g})^{-\alpha}/(1+\bar{\pi}) = \delta(1+\bar{g})^{1-\alpha}\gamma/(1+\bar{n}) = \beta\mu$ using the definition of the nominal GDP growth rate and the expression for β from [2.27c], the transversality condition can be log linearized as follows:

$$\lim_{\ell\to\infty}(\beta\mu)^\ell\mathbb{E}_t\left[\sum_{\jmath=1}^{\ell}(\delta_{\tau,t+\jmath-1} - \alpha\mathsf{g}_{t+\jmath} - \pi_{t+\jmath} - \alpha(\mathsf{c}_{\tau,t+\jmath} - \mathsf{c}_{\tau,t+\jmath-1})) - \frac{1}{(1-\beta\mu)^2}\mathsf{j}_{t+\ell}\right] = 0.$$

Using the log-linearized Euler equation [A.4.8], equations [2.24] and [A.4.9], and the law of iterated expectations, the transversality condition is equivalent to:

$$\lim_{\ell\to\infty}(\beta\mu)^{\ell}\mathbb{E}_t \mathrm{j}_{t+\ell} = -(1-\beta\mu)^2 \lim_{\ell\to\infty}(\beta\mu)^{\ell}\sum_{\jmath=0}^{\ell-1}\mathbb{E}_t \mathrm{i}_{t+\jmath}.$$

Since $0 < \beta\mu < 1$, given the restriction to monetary policies where the nominal interest rate i_t is stationary, the right-hand side of the equation above is zero, so the log-linearized transversality condition is $\lim_{\ell\to\infty}(\beta\mu)^{\ell}\mathbb{E}_t \mathrm{j}_{t+\ell} = 0$ as claimed. The equation for the ex-post real return in [2.25a] can be log linearized as follows:

$$\mathrm{r}_t = (\mathrm{j}_t - \pi_t) + \frac{1}{1-\gamma/(1+\bar{\jmath})}(\mathrm{j}_{t-1} - \mathrm{j}_t), \qquad \text{[A.4.25]}$$

and the expression for r_t in [3.13] is obtained by collecting terms above and using the formula for $\beta\mu$ in [A.4.24] with $\bar{\jmath} = \bar{\imath}$ from Proposition 1. The equilibrium bond yield in [2.26] can also be log linearized as follows:

$$\mathrm{j}_t = -\left(1-\frac{\gamma}{1+\bar{\jmath}}\right)^2\sum_{\ell=1}^{\infty}\left(\frac{\gamma}{1+\bar{\jmath}}\right)^{\ell-1}\sum_{\jmath=1}^{\ell}\mathbb{E}_t\left[\delta_{\tau,t+\jmath-1} - \alpha \mathrm{g}_{t+\jmath} - \alpha(\mathrm{c}_{\tau,t+\jmath} - \mathrm{c}_{\tau,t+\jmath-1}) - \pi_{t+\jmath}\right].$$

By using the Euler equation [A.4.8], [A.4.9], $\bar{\jmath} = \bar{\imath}$, and the formula for $\beta\mu$ in [A.4.24], the above equation can be written as:

$$\mathrm{j}_t = (1-\beta\mu)^2\sum_{\ell=1}^{\infty}(\beta\mu)^{\ell-1}\sum_{\jmath=0}^{\ell-1}\mathbb{E}_t\left[\rho_{t+\jmath} + \pi_{t+\jmath+1}\right],$$

and changing the order of summation leads to the expression for j_t in [3.13] (the infinite sums converge because $0 < \beta\mu < 1$).

Now consider the variable e_t defined in equation [3.8]. Using Proposition 1 and the expressions for β and μ from [2.27c] and [A.4.24], the steady-state value of this variable is $\bar{e} = (1-\beta\mu)^{-1}$, and the log linearization is:

$$\mathrm{e}_t = (1-\beta\mu)\sum_{\ell=0}^{\infty}(\beta\mu)^{\ell+1}\sum_{\jmath=0}^{\ell}\mathbb{E}_t[\mathrm{g}_{t+\jmath+1} - \rho_{t+\jmath}] = -\sum_{\ell=0}^{\infty}(\beta\mu)^{\ell+1}\mathbb{E}_t[\rho_{t+\ell} - \mathrm{g}_{t+\ell+1}], \qquad \text{[A.4.26]}$$

where the second equality is obtained by changing the order of summation (with convergence because $0 < \beta\mu < 1$). By using the equilibrium real interest rate in [3.9a], the equilibrium value of e_t in [3.14b] is deduced. Next, observe that the expression for j_t in [3.13] implies:

$$\frac{\beta\mu}{1-\beta\mu}\mathrm{j}_t = \sum_{\ell=0}^{\infty}(\beta\mu)^{\ell+1}\mathbb{E}_t[\rho_{t+\ell} - \mathrm{g}_{t+\ell+1}] + \sum_{\ell=0}^{\infty}(\beta\mu)^{\ell+1}\mathbb{E}_t[\pi_{t+\ell+1} + \mathrm{g}_{t+\ell+1}]. \qquad \text{[A.4.27]}$$

Using the definitions of inflation $\pi_t = \mathrm{P}_t - \mathrm{P}_{t-1}$ and real GDP growth $\mathrm{g}_t = \mathrm{Y}_t - \mathrm{Y}_{t-1}$, the following infinite sums can be obtained by collecting terms in the levels of prices and real GDP:

$$\sum_{\ell=0}^{\infty}(\beta\mu)^{\ell+1}\mathbb{E}_t\pi_{t+\ell+1} = (1-\beta\mu)\sum_{\ell=0}^{\infty}(\beta\mu)^{\ell}\mathbb{E}_t\mathrm{P}_{t+\ell} - \mathrm{P}_t; \quad \text{and}$$

$$\sum_{\ell=0}^{\infty}(\beta\mu)^{\ell+1}\mathbb{E}_t\mathrm{g}_{t+\ell+1} = (1-\beta\mu)\sum_{\ell=0}^{\infty}(\beta\mu)^{\ell}\mathbb{E}_t\mathrm{Y}_{t+\ell} - \mathrm{Y}_t.$$

Since nominal GDP is $\mathrm{N}_t = \mathrm{P}_t + \mathrm{Y}_t$, using the equations above together with the definition of $\mathcal{N}_t$ in [3.14b], and then substituting into equation [A.4.27] leads to:

$$\frac{\beta\mu}{1-\beta\mu}\mathrm{j}_t = \mathcal{N}_t - \mathrm{N}_t - \mathrm{e}_t. \qquad \text{[A.4.28]}$$

Equation [3.11] implies that the innovation to the complete-markets ex-post real return is

$$\mathrm{r}^*_t - \mathbb{E}_{t-1}\mathrm{r}^*_t = (\mathrm{g}_t - \mathbb{E}_{t-1}\mathrm{g}_t) + (\mathrm{v}_t - \mathbb{E}_{t-1}\mathrm{v}_t). \qquad \text{[A.4.29]}$$

The innovation to the ex-post real return with incomplete markets is obtained by using equation [3.13]:

$$\mathsf{r}_t - \mathbb{E}_{t-1}\mathsf{r}_t = -(\pi_t - \mathbb{E}_{t-1}\pi_t) - \frac{\beta\mu}{1-\beta\mu}(\mathsf{j}_t - \mathbb{E}_{t-1}\mathsf{j}_t).$$

Using the equation above, and [A.4.28] and [A.4.29] together with the definition of the ex-post return gap $\tilde{\mathsf{r}}_t = \mathsf{r}_t - \mathsf{r}_t^*$, it follows that:

$$\tilde{\mathsf{r}}_t - \mathbb{E}_{t-1}\tilde{\mathsf{r}}_t = -(\pi_t - \mathbb{E}_{t-1}\pi_t) - (\mathsf{g}_t - \mathbb{E}_{t-1}\mathsf{g}_t) - (\mathsf{v}_t - \mathbb{E}_{t-1}\mathsf{v}_t) + (\mathsf{e}_t - \mathbb{E}_{t-1}\mathsf{e}_t) \\ - (\mathcal{N}_t - \mathbb{E}_{t-1}\mathcal{N}_t) + (\mathsf{N}_t - \mathbb{E}_{t-1}\mathsf{N}_t). \quad \text{[A.4.30]}$$

Note that the following holds by definition

$$(\pi_t - \mathbb{E}_{t-1}\pi_t) + (\mathsf{g}_t - \mathbb{E}_{t-1}\mathsf{g}_t) = (\mathsf{N}_t - \mathsf{N}_{t-1}) - \mathbb{E}_{t-1}[\mathsf{N}_t - \mathsf{N}_{t-1}] = \mathsf{N}_t - \mathbb{E}_{t-1}\mathsf{N}_t,$$

and by substituting this into [A.4.30] and using [3.12a], the expression for $\tilde{\mathsf{d}}_t - \mathbb{E}_{t-1}\tilde{\mathsf{d}}_t$ in [3.14a] is obtained. This completes the proof.

A.5 Proof of Proposition 5

Let $\epsilon_t = \mathsf{g}_t - \mathbb{E}_{t-1}\mathsf{g}_t = \mathsf{Y}_t - \mathbb{E}_{t-1}\mathsf{Y}_t$ denote the innovation to real GDP growth and the level of real GDP (with $\mathsf{g}_t = \mathsf{Y}_t - \mathsf{Y}_{t-1}$). If real GDP growth g_t is stationary and invertible then it can be expressed as $\mathsf{g}_t = \sum_{\ell=0}^{\infty}\vartheta_\ell\epsilon_{t-\ell}$ for a sequence $\{\vartheta_\ell\}$ with $\vartheta_0 = 1$ and where the innovations $\epsilon_{t-\ell}$ belong to the time-t information set for all $\ell \geq 0$. It follows that $\mathbb{E}_t\mathsf{g}_{t+\ell} = \sum_{\jmath=0}^{\infty}\vartheta_{\ell+\jmath}\epsilon_{t-\jmath}$, and by substituting this into the expression for v_t in [3.9a]:

$$\mathsf{v}_t = (1-\alpha)\sum_{\ell=1}^{\infty}\beta^\ell\left(\sum_{\jmath=0}^{\infty}\vartheta_{\ell+\jmath}\epsilon_{t-\jmath}\right), \quad \text{and hence } \mathsf{v}_t - \mathbb{E}_{t-1}\mathsf{v}_t = (1-\alpha)\left(\sum_{\ell=1}^{\infty}\beta^\ell\vartheta_\ell\right)\epsilon_t.$$

Let $\Theta(z) = \sum_{\ell=0}^{\infty}\vartheta_\ell z^\ell$ denote the z-transform of the sequence $\{\vartheta_\ell\}$, using which the stochastic process for real GDP growth can be written as $\mathsf{g}_t = \Theta(\mathbb{L})\epsilon_t$, where $\mathbb{L}$ is the lag operator. The equation above can then be expressed in terms of the function $\Theta(z)$ as follows:

$$\mathsf{v}_t - \mathbb{E}_{t-1}\mathsf{v}_t = (1-\alpha)(\Theta(\beta) - \Theta(0))\epsilon_t. \quad \text{[A.5.1]}$$

For the variable e_t from [3.14b], following exactly the same method used to derive [A.5.1], the innovation to e_t is given by:

$$\mathsf{e}_t - \mathbb{E}_{t-1}\mathsf{e}_t = (1-\alpha)(\Theta(\beta\mu) - \Theta(0))\epsilon_t,$$

and together with equation [A.5.1] this implies:

$$(\mathsf{e}_t - \mathsf{v}_t) - \mathbb{E}_{t-1}[\mathsf{e}_t - \mathsf{v}_t] = (1-\alpha)(\Theta(\beta\mu) - \Theta(\beta))\epsilon_t. \quad \text{[A.5.2]}$$

Using the identity $\mathsf{g}_t = \mathsf{Y}_t - \mathsf{Y}_{t-1}$, note the following result which is obtained by changing the order of summation:

$$(1-\beta\mu)\sum_{\ell=0}^{\infty}(\beta\mu)^\ell\mathbb{E}_t\mathsf{Y}_{t+\ell} = (1-\beta\mu)\sum_{\ell=0}^{\infty}(\beta\mu)^\ell\mathbb{E}_t\left[\mathsf{Y}_{t-1} + \sum_{\jmath=0}^{\ell}\mathsf{g}_{t+\jmath}\right] = \mathsf{Y}_{t-1} + \sum_{\ell=0}^{\infty}(\beta\mu)^\ell\mathbb{E}_t\mathsf{g}_{t+\ell}. \quad \text{[A.5.3]}$$

Following again the same method used to derive equation [A.5.1], the innovation to a sum of expected growth rates can be expressed as

$$\sum_{\ell=0}^{\infty}(\beta\mu)^\ell\mathbb{E}_t\mathsf{g}_{t+\ell} - \mathbb{E}_{t-1}\left[\sum_{\ell=0}^{\infty}(\beta\mu)^\ell\mathbb{E}_t\mathsf{g}_{t+\ell}\right] = \Theta(\beta\mu)\epsilon_t,$$

and combining this equation with [A.5.3] yields:

$$(1-\beta\mu)\sum_{\ell=0}^{\infty}(\beta\mu)^\ell\left(\mathbb{E}_t\mathsf{Y}_{t+\ell} - \mathbb{E}_{t-1}\mathsf{Y}_t\right) = \Theta(\beta\mu)\epsilon_t. \quad \text{[A.5.4]}$$

Now suppose that monetary policy implements the weighted nominal GDP target $\mathsf{N}_{\omega,t} = 0$ at all times for some ω, where $\mathsf{N}_{\omega,t} = \mathsf{P}_t + \omega\mathsf{Y}_t$. Under this policy, unweighted nominal GDP $\mathsf{N}_t = \mathsf{P}_t + \mathsf{Y}_t$ is given by

$\mathsf{N}_t = (1-\omega)\mathsf{Y}_t$, and thus $\mathcal{N}_t$ defined in [3.14b] satisfies

$$\mathcal{N}_t = (1-\omega)(1-\beta\mu)\sum_{\ell=0}^{\infty}(\beta\mu)^{\ell}\mathbb{E}_t\mathsf{Y}_{t+\ell}, \quad \text{and hence } \mathcal{N}_t - \mathbb{E}_{t-1}\mathcal{N}_t = (1-\omega)\Theta(\beta\mu)\epsilon_t, \tag{A.5.5}$$

where the second equation follows from [A.5.4]. To achieve $\tilde{\mathsf{d}}_t = 0$ (meaning that $\mathsf{d}_t = \mathsf{d}_t^*$), equation [3.12a] shows that this is equivalent to achieving $\tilde{\mathsf{d}}_t - \mathbb{E}_{t-1}\tilde{\mathsf{d}}_t = 0$ at all times. Using [3.14a] and equations [A.5.2] and [A.5.5], in the presence of uncertainty about real GDP this is achieved by the target $\mathsf{N}_{\omega^*,t} = 0$ if and only if ω^* satisfies:

$$(1-\omega^*)\Theta(\beta\mu) = (1-\alpha)(\Theta(\beta\mu) - \Theta(\beta)).$$

Solving the equation for ω^* yields the expression in [3.15]. The invertibility of the stochastic process ensures $\Theta(\beta\mu) \neq 0$ given that $0 < \beta\mu < 1$ because $\Theta(z)$ cannot have any roots with modulus less than one. This completes the proof.

A.6 Proof of Proposition 6

(i) Suppose real GDP growth g_t is a stationary stochastic process with $\mathsf{g}_t = \sum_{\ell=0}^{\infty}\vartheta_\ell\epsilon_{t-\ell}$ for some sequence of independent random variables $\{\epsilon_t\}$, where $\epsilon_{t-\ell}$ belongs to the time-t information set for all $\ell \geq 0$. Since $\mathbb{E}_t\mathsf{g}_{t+\ell} = \sum_{\jmath=0}^{\infty}\vartheta_{\ell+\jmath}\epsilon_{t-\jmath}$, equations [3.9a] and [3.11] can be used to deduce the following expression for the natural debt-to-GDP ratio:

$$\mathsf{d}_t^* = (1-\alpha)\sum_{\ell=1}^{\infty}\beta^{\ell}\left(\sum_{\jmath=0}^{\infty}\vartheta_{\ell+\jmath}\epsilon_{t-\jmath}\right) = (1-\alpha)\sum_{\ell=0}^{\infty}\left(\sum_{\jmath=1}^{\infty}\beta^{\jmath}\vartheta_{\ell+\jmath}\right)\epsilon_{t-\ell}, \tag{A.6.1}$$

where the second equality follows by changing the order of summation. If real GDP growth g_t is an MA(q) process then $\vartheta_\jmath = 0$ for all $\jmath > \mathrm{q}$. It can be seen from [A.6.1] that the expression for d_t^* then has zero coefficients of $\epsilon_{t-\ell}$ for all $\ell \geq \mathrm{q}$, implying d_t^* is an MA($\mathrm{q}-1$) process.

Now suppose real GDP growth g_t follows a stationary ARMA(p, q) process, that is, $\mathsf{g}_t - \sum_{\ell=1}^{\mathrm{p}}\phi_\ell\mathsf{g}_{t-\ell} = \sum_{\ell=0}^{\mathrm{q}}\vartheta_\ell\epsilon_{t-\ell}$. Stationarity means the process can also be written in MA(∞) form as $\mathsf{g}_t = \sum_{\ell=0}^{\infty}\vartheta'_\ell\epsilon_{t-\ell}$ for some sequence $\{\vartheta'_\ell\}$. Observe that

$$\mathsf{d}_t^* - \sum_{\ell=1}^{\mathrm{p}}\phi_\ell\mathsf{d}_{t-\ell}^* = \mathbb{E}_{t-\mathrm{p}}\left[\mathsf{d}_t^* - \sum_{\ell=1}^{\mathrm{p}}\phi_\ell\mathsf{d}_{t-\ell}^*\right] + \sum_{\jmath=0}^{\mathrm{p}-1}\left(\mathbb{E}_{t-\jmath}\left[\mathsf{d}_t^* - \sum_{\ell=1}^{\jmath}\phi_\ell\mathsf{d}_{t-\ell}^*\right] - \mathbb{E}_{t-\jmath-1}\left[\mathsf{d}_t^* - \sum_{\ell=1}^{\jmath}\phi_\ell\mathsf{d}_{t-\ell}^*\right]\right), \tag{A.6.2}$$

where the second term is a moving-average process of order $\mathrm{p}-1$ because the terms in the summation are multiples of $\epsilon_t, \ldots, \epsilon_{t-\mathrm{p}+1}$ (as can be seen from equation [A.6.1]). Using equations [3.9a] and [3.11] and the law of iterated expectations, the first term on the right-hand side is given by:

$$\begin{aligned}\mathbb{E}_{t-\mathrm{p}}\left[\mathsf{d}_t^* - \sum_{\ell=1}^{\mathrm{p}}\phi_\ell\mathsf{d}_{t-\ell}^*\right] &= \mathbb{E}_{t-\mathrm{p}}\left[(1-\alpha)\sum_{\jmath=1}^{\infty}\beta^{\jmath}\mathbb{E}_t\mathsf{g}_{t+\jmath} - \sum_{\ell=1}^{\mathrm{p}}\phi_\ell(1-\alpha)\sum_{\jmath=1}^{\infty}\beta^{\jmath}\mathbb{E}_t\mathsf{g}_{t-\ell+\jmath}\right] \\ &= (1-\alpha)\sum_{\jmath=1}^{\infty}\beta^{\jmath}\mathbb{E}_{t-\mathrm{p}}\left[\mathsf{g}_{t+\jmath} - \sum_{\ell=1}^{\mathrm{p}}\phi_{\mathrm{p}}\mathsf{g}_{t+\jmath-\ell}\right] = (1-\alpha)\sum_{\jmath=1}^{\infty}\beta^{\jmath}\mathbb{E}_{t-\mathrm{p}}\left[\sum_{\ell=0}^{\mathrm{q}}\vartheta_\ell\epsilon_{t+\jmath-\ell}\right].\end{aligned} \tag{A.6.3}$$

Since $\mathbb{E}_{t-\mathrm{p}}\epsilon_{t+\jmath-\ell} = \epsilon_{t+\jmath-\ell}$ when $\jmath \leq \ell - \mathrm{p}$ and equals zero otherwise, the final expression contains multiples of $\epsilon_{t-\mathrm{p}}, \ldots, \epsilon_{t-\mathrm{q}+1}$ if $\mathrm{p} < \mathrm{q}$, an MA($\mathrm{q}-1$) process, and no terms if $\mathrm{p} \geq \mathrm{q}$. Equations [A.6.2] and [A.6.3] together show that $\mathsf{d}_t^* - \sum_{\ell=1}^{\mathrm{p}}\phi_\ell\mathsf{d}_{t-\ell}^*$ is a moving-average process of order $\max\{\mathrm{p}-1, \mathrm{q}-1\}$. Thus, the natural debt-to-GDP ratio d_t^* is an ARMA($\mathrm{p}, \max\{\mathrm{p}-1, \mathrm{q}-1\}$) process. Since the autoregressive coefficients ϕ_ℓ are the same as those of the real GDP process, the autoregressive roots of d_t^* are the same as those of g_t.

(ii) With incomplete financial markets, the debt-to-GDP ratio d_t must satisfy equation [3.9b] for some martingale difference sequence $\{\upsilon_t\}$ (with $\mathbb{E}_{t-1}\upsilon_t = 0$), that is:

$$\mathsf{d}_t - \lambda \mathsf{d}_{t-1} = \upsilon_t + \mathbb{E}_{t-1}\mathsf{d}_t^* - \lambda \mathsf{d}_{t-1}^*,$$

which uses the result $\mathsf{v}_t = \mathsf{d}_t^*$ from [3.11]. This equation can be written as:

$$\mathsf{d}_t - \mathsf{d}_t^* = \lambda(\mathsf{d}_{t-1} - \mathsf{d}_{t-1}^*) + (\upsilon_t - (\mathsf{d}_t^* - \mathbb{E}_{t-1}\mathsf{d}_t^*)),$$

and since $\mathbb{E}_{t-1}[\upsilon_t - (\mathsf{d}_t^* - \mathbb{E}_{t-1}\mathsf{d}_t^*)] = 0$ using the martingale difference property of υ_t, the sequence $\upsilon_t - (\mathsf{d}_t^* - \mathbb{E}_{t-1}\mathsf{d}_t^*)$ is serially uncorrelated. This shows that $\mathsf{d}_t - \mathsf{d}_t^*$ is an autoregressive process with root λ whenever $\upsilon_t \neq \mathsf{d}_t^* - \mathbb{E}_{t-1}\mathsf{d}_t^*$, that is, whenever the incomplete-markets equilibrium does not coincide with the complete-markets equilibrium. Since part (i) shows that the autoregressive roots of d_t^* are the same as those of the exogenous stochastic process g_t, it follows that d_t must have an autoregressive root of λ.

For the interpretation of λ, note that [A.4.17] in the proof of Proposition 4 implies:

$$\frac{\partial \log c_{\mathrm{b},t}}{\partial \log d_t} = -\left(\frac{1-\beta\lambda}{1-\beta}\right)\left(\frac{\theta}{1-\theta}\right), \quad \text{and} \quad \frac{\partial \log c_{\mathrm{s},t}}{\partial \log d_t} = \left(\frac{1-\beta\lambda}{1-\beta}\right)\left(\frac{\theta}{1+\theta}\right),$$

where the partial derivatives are evaluated at the steady state characterized in Proposition 1, holding real GDP and interest rates constant. If $F_{\tau,t}$ denotes per-household financial wealth of a type-τ household (as in [A.4.1]) then $F_{\mathrm{b},t} = -2d_tY_t$ and $F_{\mathrm{s},t} = 2d_tY_t$ (by comparison with [2.15] and [2.20]). Using the steady-state values $\bar{c}_{\mathrm{b}} = 1-\theta$, $\bar{c}_{\mathrm{s}} = 1+\theta$, and $\bar{d} = \theta/2(1-\beta)$ from Proposition 1, it follows that the marginal propensity to consume (MPC) from financial wealth is:

$$\mathrm{MPC} = \frac{\partial C_{\mathrm{b},t}}{\partial F_{\mathrm{b},t}} = \frac{\partial C_{\mathrm{s},t}}{\partial F_{\mathrm{s},t}} = 1 - \beta\lambda,$$

where the partial derivatives hold income and interest rates constant and are evaluated at the steady state. This can be rearranged to write λ as a function of β and MPC:

$$1 - \lambda = \frac{\mathrm{MPC} - (1-\beta)}{\beta},$$

and the expression for λ in the proposition can be obtained using the formula for β from [2.27b].

(iii) Substituting the discount factors from [2.22e] into the Euler equations [2.22d] leads to

$$1 = \Delta_\tau \mathbb{E}_t\left[(1+r_{t+1})(1+g_{t+1})^{-\alpha}\left(\frac{c_{\tau,t+1}}{c_{\tau,t}^{\lambda}}\right)^{-\alpha}\right],$$

and second-order accurate approximations of these equations around the non-stochastic steady state given in Proposition 1 are:

$$\mathbb{E}_t\left[\mathsf{r}_{t+1} - \alpha\mathsf{g}_{t+1} - \alpha(\mathsf{c}_{\tau,t+1} - \lambda\mathsf{c}_{\tau,t})\right] + \frac{1}{2}\mathbb{E}_t\left[(\mathsf{r}_{t+1} - \alpha\mathsf{g}_{t+1} - \alpha(\mathsf{c}_{\tau,t+1} - \lambda\mathsf{c}_{\tau,t}))^2\right] = \mathscr{O}^3,$$

where $\mathscr{O}^3$ denotes terms third-order or higher in the standard deviation ς of the real GDP stochastic process [2.3]. Expanding the brackets and simplifying yields the following equations:

$$\mathbb{E}_t\left[\mathsf{r}_{t+1} + \frac{1}{2}\mathsf{r}_{t+1}^2\right] = \mathbb{E}_t\Bigg[\alpha\mathsf{g}_{t+1} - \frac{\alpha^2}{2}\mathsf{g}_{t+1}^2 + \alpha\mathsf{g}_{t+1}\mathsf{r}_{t+1} - \frac{\alpha^2}{2}(\mathsf{c}_{\tau,t+1} - \lambda\mathsf{c}_{\tau,t})^2$$
$$+ \alpha(\mathsf{c}_{\tau,t+1} - \lambda\mathsf{c}_{\tau,t}) + \alpha(\mathsf{r}_{t+1} - \alpha\mathsf{g}_{t+1})(\mathsf{c}_{\tau,t+1} - \lambda\mathsf{c}_{\tau,t})\Bigg] + \mathscr{O}^3. \quad \text{[A.6.4]}$$

Using the expressions for steady-state consumption from Proposition 1, the second-order approximation of the goods-market clearing condition [2.23] is

$$\frac{(1-\theta)}{2}\mathsf{c}_{\mathrm{b},t} + \frac{(1+\theta)}{2}\mathsf{c}_{\mathrm{s},t} = -\frac{1}{2}\left(\frac{(1-\theta)}{2}\mathsf{c}_{\mathrm{b},t}^2 + \frac{(1+\theta)}{2}\mathsf{c}_{\mathrm{s},t}^2\right) + \mathscr{O}^3. \quad \text{[A.6.5]}$$

Multiplying equation [A.6.4] by $(1-\theta)/2$ for $\tau = \mathrm{b}$ and $(1+\theta)/2$ for $\tau = \mathrm{s}$, summing the equations, and using [A.6.5] implies:

$$\mathbb{E}_t\left[\mathsf{r}_{t+1}+\frac{1}{2}\mathsf{r}_{t+1}^2\right]=\mathbb{E}_t\Bigg[\alpha\mathsf{g}_{t+1}-\frac{\alpha^2}{2}\mathsf{g}_{t+1}^2+\alpha\mathsf{g}_{t+1}\mathsf{r}_{t+1}-\frac{\alpha^2}{2}\frac{(1-\theta)}{2}(\mathsf{c}_{\mathrm{b},t+1}-\lambda\mathsf{c}_{\mathrm{b},t})^2$$
$$-\frac{\alpha^2}{2}\frac{(1+\theta)}{2}(\mathsf{c}_{\mathrm{s},t+1}-\lambda\mathsf{c}_{\mathrm{s},t})^2-\frac{\alpha}{2}\left(\frac{(1-\theta)}{2}(\mathsf{c}_{\mathrm{b},t+1}^2-\lambda\mathsf{c}_{\mathrm{b},t}^2)+\frac{(1+\theta)}{2}(\mathsf{c}_{\mathrm{s},t+1}^2-\lambda\mathsf{c}_{\mathrm{s},t}^2)\right)\Bigg]+\mathscr{O}^3. \quad \text{[A.6.6]}$$

The equations in [3.12b] imply that $\mathsf{c}_{\mathrm{b},t}$ and $\mathsf{c}_{\mathrm{s},t}$ are proportional to $\tilde{\mathsf{d}}_t$ up to an error of order $\mathscr{O}^2$. By noting the following algebra

$$(1-\theta)\left(\frac{\theta}{1-\theta}\frac{(1-\beta\lambda)}{(1-\beta)}\right)^2+(1+\theta)\left(\frac{\theta}{1+\theta}\frac{(1-\beta\lambda)}{(1-\beta)}\right)^2=\frac{\theta^2(1-\beta\lambda)^2}{(1-\theta^2)(1-\beta)^2},$$

equation [3.12b] implies that [A.6.6] can be written as:

$$\mathbb{E}_t\left[\mathsf{r}_{t+1}+\frac{1}{2}\mathsf{r}_{t+1}^2\right]=\mathbb{E}_t\Bigg[\alpha\mathsf{g}_{t+1}-\frac{\alpha^2}{2}\mathsf{g}_{t+1}^2+\alpha\mathsf{g}_{t+1}\mathsf{r}_{t+1}-\frac{\alpha^2}{2}\frac{\theta^2}{(1-\theta^2)}\frac{(1-\beta\lambda)^2}{(1-\beta)^2}(\tilde{\mathsf{d}}_{t+1}-\lambda\tilde{\mathsf{d}}_t)^2$$
$$-\frac{\alpha}{2}\frac{\theta^2}{(1-\theta^2)}\frac{(1-\beta\lambda)^2}{(1-\beta)^2}(\tilde{\mathsf{d}}_{t+1}^2-\lambda\tilde{\mathsf{d}}_t^2)\Bigg]+\mathscr{O}^3. \quad \text{[A.6.7]}$$

The first equation in [3.12a] (which holds up to an error of order $\mathscr{O}^2$) can be used to deduce the following:

$$\mathbb{E}_t[(\tilde{\mathsf{d}}_{t+1}-\lambda\tilde{\mathsf{d}}_t)^2]=\mathbb{E}_t[(\tilde{\mathsf{d}}_{t+1}-\mathbb{E}_t\tilde{\mathsf{d}}_{t+1})^2]+\mathscr{O}^3, \quad \text{and}$$
$$\mathbb{E}_t[\tilde{\mathsf{d}}_{t+1}^2-\lambda\tilde{\mathsf{d}}_t^2]=\mathbb{E}_t[((\tilde{\mathsf{d}}_{t+1}-\mathbb{E}_t\tilde{\mathsf{d}}_{t+1})+\lambda\tilde{\mathsf{d}}_t)^2]-\lambda\tilde{\mathsf{d}}_t^2+\mathscr{O}^3=\mathbb{E}_t[(\tilde{\mathsf{d}}_{t+1}-\mathbb{E}_t\tilde{\mathsf{d}}_{t+1})^2]-\lambda(1-\lambda)\tilde{\mathsf{d}}_t^2+\mathscr{O}^3,$$

where the second equality uses the law of iterated expectations, and by substituting these into [A.6.7]:

$$\mathbb{E}_t\left[\mathsf{r}_{t+1}+\frac{1}{2}\mathsf{r}_{t+1}^2\right]=\mathbb{E}_t\Bigg[\alpha\mathsf{g}_{t+1}-\frac{\alpha^2}{2}\mathsf{g}_{t+1}^2+\alpha\mathsf{g}_{t+1}\mathsf{r}_{t+1}-\frac{\alpha(\alpha+1)}{2}\frac{\theta^2}{(1-\theta^2)}\frac{(1-\beta\lambda)^2}{(1-\beta)^2}(\tilde{\mathsf{d}}_{t+1}-\mathbb{E}_t\tilde{\mathsf{d}}_{t+1})^2$$
$$+\frac{\alpha}{2}\frac{\theta^2}{1-\theta^2}\frac{(1-\beta\lambda)^2}{(1-\beta)^2}\lambda(1-\lambda)\tilde{\mathsf{d}}_t^2\Bigg]+\mathscr{O}^3. \quad \text{[A.6.8]}$$

The Euler equations [2.22d] must also hold with the ex-post real return r_t^* for the complete-markets equilibrium with consumption ratios $c_{\tau,t}^*=\bar{c}_\tau$ and discount factors $\delta_{\tau,t}=\delta$ from Proposition 2, hence:

$$1=\delta\mathbb{E}_t\left[(1+r_{t+1}^*)(1+g_{t+1})^{-\alpha}\right],$$

which has the following second-order approximation around the non-stochastic steady state:

$$\mathbb{E}_t\left[\mathsf{r}_{t+1}^*+\frac{1}{2}\mathsf{r}_{t+1}^{*2}\right]=\mathbb{E}_t\left[\alpha\mathsf{g}_{t+1}-\frac{\alpha^2}{2}\mathsf{g}_{t+1}^2+\alpha\mathsf{g}_{t+1}\mathsf{r}_{t+1}^*\right]+\mathscr{O}^3. \quad \text{[A.6.9]}$$

The definitions of the real interest rates ρ_t and ρ_t^* from [2.22a] and the definitions of the log deviations r_t and r_t^* imply

$$\rho_t-\rho_t^*=(1+\bar{\rho})\mathbb{E}_t\left[\left(\mathsf{r}_{t+1}+\frac{1}{2}\mathsf{r}_{t+1}^2\right)-\left(\mathsf{r}_{t+1}^*+\frac{1}{2}\mathsf{r}_{t+1}^{*2}\right)\right]+\mathscr{O}^3,$$

and by substituting from equations [A.6.8] and [A.6.9], it follows that:

$$\frac{\rho_t-\rho_t^*}{1+\bar{\rho}}=\mathbb{E}_t\left[\alpha\mathsf{g}_{t+1}\tilde{\mathsf{r}}_{t+1}-\frac{\alpha\theta^2(1-\beta\lambda)^2}{2(1-\theta^2)(1-\beta)^2}((\alpha+1)(\tilde{\mathsf{d}}_{t+1}-\mathbb{E}_t\tilde{\mathsf{d}}_{t+1})^2-\lambda(1-\lambda)\tilde{\mathsf{d}}_t^2)\right]+\mathscr{O}^3, \quad \text{[A.6.10]}$$

where the definition $\tilde{\mathsf{r}}_t=\mathsf{r}_t-\mathsf{r}_t^*$ has been used. The third result in [3.12a] (which holds up to an error of order $\mathscr{O}^2$) can be used to deduce:

$$\mathbb{E}_t[\mathsf{g}_{t+1}\tilde{\mathsf{r}}_{t+1}]=\mathbb{E}_t[\mathsf{g}_{t+1}(\tilde{\mathsf{d}}_{t+1}-\mathbb{E}_t\tilde{\mathsf{d}}_{t+1}+\mathscr{O}^2)]=\mathbb{E}_t[(\mathsf{g}_{t+1}-\mathbb{E}_t\mathsf{g}_{t+1})(\tilde{\mathsf{d}}_{t+1}-\mathbb{E}_t\tilde{\mathsf{d}}_{t+1})]+\mathscr{O}^3, \quad \text{[A.6.11]}$$

where the second equality uses the law of iterated expectations. By iterating equation [3.12a] backwards, the debt gap $\tilde{\mathsf{d}}_t$ is given by:

$$\tilde{\mathsf{d}}_t=\sum_{\ell=0}^{\infty}\lambda^\ell(\tilde{\mathsf{d}}_{t-\ell}-\mathbb{E}_{t-\ell-1}\tilde{\mathsf{d}}_{t-\ell})+\mathscr{O}^2, \quad \text{and hence } \mathbb{E}[\tilde{\mathsf{d}}_t^2]=\frac{\mathbb{E}[(\tilde{\mathsf{d}}_t-\mathbb{E}_{t-1}\tilde{\mathsf{d}}_t)^2]}{1-\lambda^2}+\mathscr{O}^3, \quad \text{[A.6.12]}$$

which follows because the terms of the summation are serially uncorrelated, and their unconditional expectations do not depend on time as g_t is stationary. Equation [3.16] can be deduced by taking the unconditional expectation of equation [A.6.10] and substituting from [A.6.11] and [A.6.12], noting the definitions of the conditional moments $\mathbb{V}_{t-1}[\tilde{\mathsf{d}}_t] = \mathbb{E}_{t-1}[(\tilde{\mathsf{d}}_t - \mathbb{E}_{t-1}\tilde{\mathsf{d}}_t)^2]$ and $\mathbb{C}_{t-1}[\mathsf{g}_t, \tilde{\mathsf{d}}_t] = \mathbb{E}_{t-1}[(\mathsf{g}_t - \mathbb{E}_{t-1}\mathsf{g}_t)(\tilde{\mathsf{d}}_t - \mathbb{E}_{t-1}\tilde{\mathsf{d}}_t)]$. This completes the proof.

A.7 Proof of Proposition 7

(i) Observe that the system of equations [2.22a]–[2.22f] and [3.20] (and the implied equation [3.21]) includes no nominal variables, and hence no variables that can be directly influenced by monetary policy. Since the equations characterizing the real variables form a closed system, the resulting real equilibrium is independent of monetary policy.

(ii) Consider the equilibrium of an economy with nominal bonds in which monetary policy is strict inflation targeting, that is, $\pi_t = \pi$, and where the coupon parameter is such that $\gamma = (1+\pi)\gamma^\dagger$. The equilibrium nominal bond yield j_t in such an economy is given by equation [2.26]. Now consider the economy with only inflation-indexed bonds (coupon parameter $\gamma^\dagger$). Using equation [3.21] and $\gamma^\dagger = \gamma/(1+\pi)$, the yield on the inflation-indexed bonds is:

$$y_t = \left((1+\pi)\mathbb{E}_t\left[\sum_{\ell=1}^{\infty}\gamma^{\ell-1}\left\{\prod_{\jmath=1}^{\ell}\delta_{\tau,t+\jmath-1}\frac{(1+g_{t+\jmath})^{-\alpha}}{(1+\pi)}\left(\frac{c_{\tau,t+\jmath}}{c_{\tau,t+\jmath-1}}\right)^{-\alpha}\right\}\right]\right)^{-1} + \frac{\gamma}{1+\pi} - 1.$$

Conjecturing that the equilibrium of the economy with indexed bonds features the same consumption allocation as the economy with nominal bonds, a comparison of the equation above to [2.26] with $\pi_t = \pi$ implies:

$$y_t = \left((1+\pi)(1+j_t-\gamma)^{-1}\right)^{-1} + \frac{\gamma}{1+\pi} - 1 = \frac{1+j_t}{1+\pi} - 1, \tag{A.7.1}$$

where j_t is the equilibrium bond yield in the nominal-bond economy. Now note that the ex-post real return [2.25a] in the nominal-bond economy under strict inflation targeting $\pi_t = \pi$ can be written as:

$$1 + r_t = \left(\frac{1+j_{t-1}}{1+\pi_t}\right)\left(\frac{1-\frac{\gamma}{1+j_{t-1}}}{1-\frac{\gamma}{1+j_t}}\right), \tag{A.7.2}$$

and similarly, the ex-post real return [3.20] of the inflation-indexed bond can also be reformulated as:

$$1 + r_t^\dagger = (1+y_{t-1})\left(\frac{1-\frac{\gamma^\dagger}{1+y_{t-1}}}{1-\frac{\gamma^\dagger}{1+y_t}}\right).$$

With the coupon parameters γ and $\gamma^\dagger$ being related by $\gamma^\dagger = \gamma/(1+\pi)$, the equation above becomes:

$$1 + r_t^\dagger = \left(\frac{(1+\pi)(1+y_t)}{1+\pi}\right)\left(\frac{1-\frac{\gamma}{(1+\pi)(1+y_{t-1})}}{1-\frac{1}{(1+\pi)(1+y_t)}}\right). \tag{A.7.3}$$

Under the supposition that the two economies have the same consumption allocation, the bond yields are related according to [A.7.1], and hence $1 + j_t = (1+\pi)(1+y_t)$ for all t. A comparison of equations [A.7.2] and [A.7.3] then shows that $r_t = r_t^\dagger$. The equilibrium of the incomplete-markets economy with nominal bonds under a policy of strict inflation targeting is characterized by equations [2.22a]–[2.22f] together with [A.7.2]. The equilibrium of the incomplete-markets economy with inflation-indexed bonds with coupon parameter $\gamma^\dagger = \gamma/(1+\pi)$ is characterized by equations [2.22a]–[2.22f] together with [A.7.3]. Note that y_t and j_t do not appear directly in [2.22a]–[2.22f], and [A.7.2] and [A.7.3] contain no other variables apart from $r_t^\dagger$ and r_t. Thus, since $r_t = r_t^\dagger$, the resulting equilibrium is the same for both economies. This completes the proof.

A.8 Proof of Proposition 8

(i) Suppose monetary policy is strict inflation targeting, namely $\pi_t = \pi$ at all times, and that nominal and inflation-indexed bonds have the same maturity, that is, $\gamma = (1+\pi)\gamma^\dagger$. Supposing that the equilibrium consumption allocation of the economy with both bonds is the same as that of the economy with only nominal bonds under the same monetary policy, it follows that since both economies have the same consumption levels and the same inflation rate, they must have the same nominal bond yield j_t.

With $\gamma^\dagger = \gamma/(1+\pi)$, the argument in the proof of Proposition 7 that led to equation [A.7.1] applies, and the inflation-indexed bond yield y_t is related to the nominal bond yield j_t in the economy with both bonds according to $1 + j_t = (1+\pi)(1+y_t)$. Again following the argument of Proposition 7 and using equations [A.7.2] and [A.7.3], this means that $r_t = r_t^\dagger$. From equation [3.22c] it then follows that $r_t^\ddagger = r_t$ for any choice of s_t. With $r_t^\ddagger = r_t$, the system of equations [2.22a]–[2.22f] and [3.20] is identical that characterizing the equilibrium of the economy with only nominal bonds if monetary policy is strict inflation targeting. Hence, that equilibrium is also the equilibrium of the economy with both types of bonds. Any portfolio share s_t is consistent with the equilibrium conditions.

(ii) Consider a monetary policy that replicates the complete-markets equilibrium in the economy with only nominal bonds (coupon parameter γ). The existence of such monetary policies was demonstrated in Proposition 3. Such a monetary policy must imply a path for inflation π_t for which, together with the equilibrium nominal bond yield given by [2.26], the ex-post real return in [2.25a] is $r_t = r_t^*$. The complete-markets ex-post real return r_t^* is characterized in terms of exogenous variables by Proposition 2. Now suppose a monetary policy that generates the same path for inflation is implemented in the economy with both nominal and inflation-indexed bonds (the latter with coupon parameter $\gamma^\dagger$).

It is conjectured this monetary policy leads to the same consumption allocation as in the economy with only nominal bonds. If so, since the path for inflation is the same, the equilibrium nominal bond yield j_t from [2.26] would be the same, and thus the ex-post real return r_t from [2.25a] would then satisfy $r_t = r_t^*$, namely what the monetary policy achieves in the economy with only nominal bonds. Now suppose there is an equilibrium with $s_t = 0$, in which case, from equation [3.22c], the ex-post real return on the overall portfolio of bonds is $r_t^\ddagger = r_t = r_t^*$. It then follows immediately that the complete-markets consumption allocation, discount factors, debt and loan ratios, and real interest rate satisfy equations [2.22a]–[2.22f].

Since [3.22c] is satisfied with $s_t = 0$ and $r_t^\ddagger = r_t$, the remaining equilibrium conditions are related to the ex-post real return $r_t^\dagger$ on inflation-indexed bonds and the associated real interest rate $\rho_t^\dagger = \mathbb{E}_t r_{t+1}^\dagger$. This definition of $\rho_t^\dagger$ is in accordance with [2.22a]. There are then the two Euler equations in [2.22d] that must be satisfied by $r_{t+1}^\dagger$. As Proposition 2 establishes, the complete-markets allocation features consumption ratios $c_{\tau,t}^* = \bar{c}_\tau$ that are constant over time and discount factors $\delta_{\tau,t}^* = \delta$ that are equal across both types of households. This means the two Euler equations in [2.22d] collapse to a single equilibrium condition for $r_{t+1}^\dagger$:

$$1 = \delta \mathbb{E}_t \left[(1 + r_{t+1}^\dagger)(1 + g_{t+1})^{-\alpha} \right], \tag{A.8.1}$$

and by using [3.20], the equilibrium yield [3.21] on inflation-indexed bonds simplifies to:

$$y_t = \left(\mathbb{E}_t \left[\sum_{\ell=1}^{\infty} {\gamma^\dagger}^{\ell-1} \delta^\ell \prod_{\jmath=1}^{\ell} (1 + g_{t+\jmath})^{-\alpha} \right] \right)^{-1} + \gamma^\dagger - 1. \tag{A.8.2}$$

With this bond yield y_t, the ex-post real return $r_t^\dagger$ on inflation-indexed bonds is determined by [3.20], which then satisfies [A.8.1], and determines the real interest rate $\rho_t^\dagger = \mathbb{E}_t r_{t+1}^\dagger$. This confirms that all equilibrium conditions are satisfied at the conjectured consumption allocation, which is supported by $r_t^\ddagger = r_t^*$. Thus, the equilibrium of the economy with both types of bonds coincides with the equilibrium of the economy with only nominal bonds (for any coupon parameters γ and $\gamma^\dagger$). In equilibrium, the portfolio share of inflation-indexed bonds is zero ($s_t = 0$).

(iii) First consider the economy with strict inflation targeting ($\pi_t = \pi$) and $\gamma = (1+\pi)\gamma^\dagger$. As shown in part (i), $r_t = r_t^\dagger$, and hence $\rho_t = \rho_t^\dagger$. The inflation risk premium is thus $\varpi_t = 0$.

Now consider an economy where monetary policy replicates the complete-markets equilibrium. This means that the equilibrium is given by Proposition 2, and hence the consumption-income ratios and discount factors are $c^*_{\tau,t} = \bar{c}_\tau$ and $\delta^*_{\tau,t} = \delta$. The Euler equations [2.22d] associated with holdings of nominal bonds then reduce to:

$$1 = \delta\mathbb{E}_t\left[(1 + r_{t+1})(1 + g_{t+1})^{-\alpha}\right].$$

In the non-stochastic steady state characterized in Proposition 1, equation [3.21] implies $\bar{y} = \bar{\rho}$, equation [3.20] implies $\bar{r}^\dagger = \bar{y}$, and $\rho^\dagger_t = \mathbb{E}_t r^\dagger_{t+1}$ implies $\bar{\rho}^\dagger = \bar{r}^\dagger$. Second-order accurate approximations around this non-stochastic steady state of the equation above and the Euler equation [A.8.1] for inflation-indexed bonds are:

$$\mathbb{E}_t\left[\mathsf{r}_{t+1} + \frac{1}{2}\mathsf{r}^2_{t+1}\right] = \alpha\mathbb{E}_t\left[\mathsf{g}_{t+1} - \frac{\alpha}{2}\mathsf{g}^2_{t+1} + \mathsf{g}_{t+1}\mathsf{r}_{t+1}\right] + \mathscr{O}^3, \quad \text{and} \tag{A.8.3a}$$

$$\mathbb{E}_t\left[\mathsf{r}^\dagger_{t+1} + \frac{1}{2}\mathsf{r}^{\dagger 2}_{t+1}\right] = \alpha\mathbb{E}_t\left[\mathsf{g}_{t+1} - \frac{\alpha}{2}\mathsf{g}^2_{t+1} + \mathsf{g}_{t+1}\mathsf{r}^\dagger_{t+1}\right] + \mathscr{O}^3. \tag{A.8.3b}$$

The definitions of the inflation risk premium ϖ_t and the real interest rates ρ_t and $\rho^\dagger_t$ imply $\varpi_t = \mathbb{E}_t r_{t+1} - \mathbb{E}_t r^\dagger_{t+1}$, and hence a second-order accurate approximation of ϖ_t in terms of the log deviations r_{t+1} and $\mathsf{r}^\dagger_{t+1}$ is:

$$\varpi_t = (1 + \bar{\rho})\left(\mathbb{E}_t\left[\mathsf{r}_{t+1} + \frac{1}{2}\mathsf{r}^2_{t+1}\right] - \mathbb{E}_t\left[\mathsf{r}^\dagger_{t+1} + \frac{1}{2}\mathsf{r}^{\dagger 2}_{t+1}\right]\right) + \mathscr{O}^3.$$

Substituting from equation [A.8.3] implies the second-order approximation reduces to:

$$\varpi_t = (1 + \bar{\rho})\alpha\mathbb{E}_t[\mathsf{g}_{t+1}(\mathsf{r}_{t+1} - \mathsf{r}^\dagger_{t+1})] + \mathscr{O}^3. \tag{A.8.4}$$

With monetary policy conducted to replicate complete financial markets, it must be the case that $r_t = r^*_t$ and hence $\mathsf{r}_t = \mathsf{r}^*_t$. Using the equations for v_t and r_t in [3.9a] and [3.11] (which hold up to an error of order $\mathscr{O}^2$), it follows that:

$$\mathsf{r}_t = \mathsf{g}_t + (1 - \alpha)\sum_{\ell=1}^{\infty}\beta^\ell\mathbb{E}_t\mathsf{g}_{t+\ell} - (1 - \alpha)\sum_{\ell=1}^{\infty}\beta^{\ell-1}\mathbb{E}_{t-1}\mathsf{g}_{t-1+\ell} + \mathscr{O}^2,$$

which can be rearranged to obtain:

$$\mathsf{r}_t = \alpha\mathsf{g}_t + (1 - \alpha)\sum_{\ell=0}^{\infty}\beta^\ell(\mathbb{E}_t\mathsf{g}_{t+\ell} - \mathbb{E}_{t-1}\mathsf{g}_{t+\ell}) + \mathscr{O}^2. \tag{A.8.5}$$

Since the complete-markets consumption allocation is achieved, the equilibrium real bond yield y_t is given by equation [A.8.2]. Using $\bar{y} = \bar{\rho}$ in the non-stochastic steady state and noting the definition $\mu^\dagger = \gamma^\dagger/(1+\bar{g})$, equation [A.8.2] can be log linearized as follows:

$$\frac{1}{(1 - \beta\mu^\dagger)^2}\mathsf{y}_t = \alpha\sum_{\ell=1}^{\infty}(\beta\mu^\dagger)^{\ell-1}\sum_{\jmath=1}^{\ell}\mathbb{E}_t\mathsf{g}_{t+\jmath} + \mathscr{O}^2,$$

using the expression for β given in [2.27c]. Changing the order of summation and simplifying leads to:

$$\mathsf{y}_t = \alpha(1 - \beta\mu^\dagger)\sum_{\ell=0}^{\infty}(\beta\mu^\dagger)^\ell\mathbb{E}_t\mathsf{g}_{t+1+\ell} + \mathscr{O}^2. \tag{A.8.6}$$

Since $\bar{r}^\dagger = \bar{\rho}$ in the non-stochastic steady state, equation [3.20] for the ex-post real return $r^\dagger_t$ on the indexed bond has the following log-linear approximation:

$$\mathsf{r}^\dagger_t = \frac{1}{1 - \beta\mu^\dagger}\mathsf{y}_{t-1} - \frac{\beta\mu^\dagger}{1 - \beta\mu^\dagger}\mathsf{y}_t + \mathscr{O}^2.$$

Substituting the equilibrium bond yield y_t from [A.8.6] into the equation above leads to:

$$\mathsf{r}^\dagger_t = \alpha\sum_{\ell=0}^{\infty}(\beta\mu^\dagger)^\ell\mathbb{E}_{t-1}\mathsf{g}_{t+\ell} - \alpha\sum_{\ell=1}^{\infty}(\beta\mu^\dagger)^\ell\mathbb{E}_t\mathsf{g}_{t+\ell} + \mathscr{O}^2 = \alpha\mathsf{g}_t - \alpha\sum_{\ell=0}^{\infty}(\beta\mu^\dagger)^\ell(\mathbb{E}_t\mathsf{g}_{t+\ell} - \mathbb{E}_{t-1}\mathsf{g}_{t+\ell}) + \mathscr{O}^2,$$

where the second equality is obtained by grouping terms in real GDP growth at the same date. Using this equation together with [A.8.4] and [A.8.5], the average inflation risk premium $\mathbb{E}\varpi_t$ is given by:

$$\mathbb{E}\varpi_t = (1+\bar{\rho})\alpha\mathbb{E}\left[\mathbb{E}_{t-1}\left[g_t\left\{\sum_{\ell=0}^{\infty}\beta^{\ell}\left(1-\alpha\left(1-\mu^{\dagger\ell}\right)\right)(\mathbb{E}_t g_{t+\ell}-\mathbb{E}_{t-1}g_{t+\ell})\right\}\right]\right]+\mathscr{O}^3. \quad \text{[A.8.7]}$$

Using the law of iterated expectations and the definition of the conditional moments $\mathbb{C}_{t-1}[g_t, \mathbb{E}_t g_{t+\ell}] = \mathbb{E}_{t-1}[(g_t - \mathbb{E}_{t-1}g_t)(\mathbb{E}_t g_{t+\ell} - \mathbb{E}_{t-1}g_{t+\ell})]$, this confirms the expression given in equation [3.23].

(iv) Consider the case where nominal bonds and inflation-indexed bonds have the same maturity, that is, $\gamma = (1+\bar{\pi})\gamma^{\dagger}$. Given the definitions $\mu = \gamma/(1+\bar{n})$, $\mu^{\dagger} = \gamma^{\dagger}/(1+\bar{g})$, and $1+\bar{n} = (1+\bar{\pi})(1+\bar{g})$, this means that $\mu = \mu^{\dagger}$.

Now suppose that monetary policy is strict inflation targeting ($\pi_t = \pi$ at all times). Consider either the economy with only nominal bonds or the economy with both types of bonds (given the result in part (i), the equilibrium is the same in both cases). Using the expressions for v_t and r_t^* in equations [3.9a] and [3.11] (which hold up to an error of order $\mathscr{O}^2$), it follows that $r_t^* - \mathbb{E}_{t-1}r_t^* = (g_t - \mathbb{E}_{t-1}g_t) + (v_t - \mathbb{E}_{t-1}v_t) + \mathscr{O}^2$ and hence:

$$r_t^* - \mathbb{E}_{t-1}r_t^* = \sum_{\ell=0}^{\infty}\beta^{\ell}(\mathbb{E}_t g_{t+\ell} - \mathbb{E}_{t-1}g_{t+\ell}) - \alpha\sum_{\ell=1}^{\infty}\beta^{\ell}(\mathbb{E}_t g_{t+\ell} - \mathbb{E}_{t-1}g_{t+\ell}) + \mathscr{O}^2. \quad \text{[A.8.8]}$$

With strict inflation targeting ($\pi_t = 0$), equations [3.9a] and [3.13] imply that the equilibrium nominal bond yield is given by:

$$j_t = \alpha(1-\beta\mu)\sum_{\ell=0}^{\infty}(\beta\mu)^{\ell}\mathbb{E}_t g_{t+\ell+1} + \mathscr{O}^2.$$

Using this equation, $\pi_t = 0$, and the expression for the ex-post real return on nominal bonds in [3.13] leads to:

$$r_t - \mathbb{E}_{t-1}r_t = -\frac{\beta\mu}{1-\beta\mu}(j_t - \mathbb{E}_{t-1}j_t) + \mathscr{O}^2 = -\alpha\sum_{\ell=1}^{\infty}(\beta\mu)^{\ell}(\mathbb{E}_t g_{t+\ell} - \mathbb{E}_{t-1}g_{t+\ell}) + \mathscr{O}^2. \quad \text{[A.8.9]}$$

Equations [A.8.8] and [A.8.9] together with the definition of the ex-post real return gap $\tilde{r}_t = r_t - r_t^*$ imply:

$$\tilde{r}_t - \mathbb{E}_{t-1}\tilde{r}_t = -\sum_{\ell=0}^{\infty}\beta^{\ell}\left(1-\alpha\left(1-\mu^{\ell}\right)\right)(\mathbb{E}_t g_{t+\ell} - \mathbb{E}_{t-1}g_{t+\ell}) + \mathscr{O}^2. \quad \text{[A.8.10]}$$

Since $\tilde{r}_t = \tilde{d}_t - \mathbb{E}_{t-1}\tilde{d}_t + \mathscr{O}^2$ from equation [3.12a], it follows that $\mathbb{C}_{t-1}[g_t, \tilde{d}_t] = \mathbb{E}_{t-1}[g_t(\tilde{r}_t - \mathbb{E}_{t-1}\tilde{r}_t)] + \mathscr{O}^3$. Hence, given $\mu = \mu^{\dagger}$ and equation [A.8.10]:

$$\mathbb{E}\mathbb{C}_{t-1}[g_t, \tilde{d}_t] = -\mathbb{E}\left[\mathbb{E}_{t-1}\left[g_t\left\{\sum_{\ell=0}^{\infty}\beta^{\ell}\left(1-\alpha\left(1-\mu^{\dagger\ell}\right)\right)(\mathbb{E}_t g_{t+\ell}-\mathbb{E}_{t-1}g_{t+\ell})\right\}\right]\right]+\mathscr{O}^3,$$

and comparison with equation [A.8.7] shows that $\mathbb{E}\varpi_t = -(1+\bar{\rho})\alpha\mathbb{E}\mathbb{C}_{t-1}[g_t, \tilde{d}_t] + \mathscr{O}^3$.

Now consider an economy where monetary policy replicates complete financial markets. Under the conditions assumed in Proposition 5, there is an optimal degree of countercyclicality of the price level, namely the term ω^* given in equation [3.15]. This is expressed in terms of the function $\Theta(z) = \sum_{\ell=0}^{\infty}\vartheta_{\ell}z^{\ell}$ such that the stochastic process for real GDP growth is $g_t = \sum_{\ell=0}^{\ell}\vartheta_{\ell}\epsilon_{t-\ell}$, and where the innovation $\epsilon_t = g_t - \mathbb{E}_{t-1}g_t$ is such that $\epsilon_{t-\ell}$ belongs to the date-t information set for all $\ell \geq 0$, and $\{\vartheta_{\ell}\}$ is a sequence with $\vartheta_0 = 1$.

Since $\mathbb{E}_t g_{t+\ell} - \mathbb{E}_{t-1}g_{t+\ell} = \vartheta_{\ell}\epsilon_t$ for all $\ell \geq 0$, using the definition of ϵ_t it follows that:

$$\mathbb{E}\mathbb{C}_{t-1}[g_t, \mathbb{E}_t g_{t+\ell}] = \mathbb{E}\mathbb{E}_{t-1}\left[(g_t - \mathbb{E}_{t-1}g_t)(\mathbb{E}_t g_{t+\ell} - \mathbb{E}_{t-1}g_{t+\ell})\right] = \vartheta_{\ell}\mathbb{E}\epsilon_t^2 = \vartheta_{\ell}\mathbb{E}\mathbb{V}_{t-1}[g_t].$$

Substituting this into the expression for the average inflation risk premium in [3.23] and using $\mu^{\dagger} = \mu$ leads to:

$$\mathbb{E}\varpi_t = (1+\bar{\rho})\alpha\left(\sum_{\ell=0}^{\infty}\beta^{\ell}\left(1-\alpha(1-\mu^{\ell})\right)\vartheta_{\ell}\right)\mathbb{E}\mathbb{V}_{t-1}[g_t] + \mathscr{O}^3.$$

Using the definition of the z-transform $\Theta(z)$ of the sequence $\{\vartheta_\ell\}$, the equation above can be written as:

$$\mathbb{E}\varpi_t = (1+\bar{\rho})\alpha\left((1-\alpha)\Theta(\beta) + \alpha\Theta(\beta\mu)\right)\mathbb{E}\mathbb{V}_{t-1}[\mathsf{g}_t] + \mathscr{O}^3,$$

and comparison with the expression for ω^* in [3.15] yields the equation for $\mathbb{E}\varpi_t$ given in the proposition (invertibility of the stochastic process for g_t ensuring $\Theta(\beta\mu) \neq 0$ since $0 < \beta\mu < 1$). This completes the proof.

A.9 Proof of Proposition 9

Suppose N^*_t is a path for nominal GDP that achieves the target $\mathsf{d}_t = \mathsf{d}^*_t$. Since this path is consistent with $\tilde{\mathsf{d}}_t = 0$, it follows from equations [3.14a] and [3.14b] that

$$(1-\beta\mu)\sum_{\ell=0}^{\infty}(\beta\mu)^\ell(\mathbb{E}_t\mathsf{N}^*_{t+\ell} - \mathbb{E}_{t-1}\mathsf{N}^*_{t+\ell}) = (\mathsf{e}_t - \mathsf{v}_t) - \mathbb{E}_{t-1}[\mathsf{e}_t - \mathsf{v}_t]. \qquad \text{[A.9.1]}$$

Now observe from [3.24] that $\mathbb{E}_t\mathsf{N}'_{t+\ell} = \mathbb{E}_t[\mathsf{z}_{t-1+\ell} - \beta\mu\mathsf{z}_{t+\ell}]$ for all $\ell \geq 1$, and note that:

$$\sum_{\ell=0}^{\infty}(\beta\mu)^\ell\mathbb{E}_t[\mathsf{z}_{t-1+\ell} - \beta\mu\mathsf{z}_{t+\ell}] = \mathsf{z}_{t-1}.$$

These observations imply

$$\sum_{\ell=0}^{\infty}(\beta\mu)^\ell\mathbb{E}_t\mathsf{N}'_{t+\ell} = \mathsf{z}_{t-1} + \sum_{\ell=0}^{\infty}(\beta\mu)^\ell(\mathbb{E}_t\mathsf{N}^*_{t+\ell} - \mathbb{E}_{t-1}\mathsf{N}^*_{t+\ell}),$$

and hence using [A.9.1] that $\mathcal{N}'_t = (1-\beta\mu)\sum_{\ell=0}^{\infty}(\beta\mu)^\ell\mathbb{E}_t\mathsf{N}'_{t+\ell}$ satisfies $\mathcal{N}'_t - \mathbb{E}_{t-1}\mathcal{N}'_t = (\mathsf{e}_t - \mathsf{v}_t) - \mathbb{E}_{t-1}[\mathsf{e}_t - \mathsf{v}_t]$. Therefore, given [3.14a], the path of nominal GDP N'_t is also consistent with $\tilde{\mathsf{d}}_t - \mathbb{E}_{t-1}\tilde{\mathsf{d}}_t = 0$, and hence by using [3.12a] with $\mathsf{d}_t = \mathsf{d}^*_t$ for all t. This completes the proof.

A.10 Proof of Proposition 10

Suppose the nominal interest rate i_t is set according to the rule [3.25]. In equilibrium, the nominal interest rate must satisfy the Fisher equation [3.10], and the real interest rate must be as given in [3.9a]. Hence, the following equation must hold:

$$(\alpha - 1)\mathbb{E}_t\mathsf{g}_{t+1} + \mathbb{E}_t\mathsf{N}^*_{t+1} - \mathsf{N}^*_t + \zeta(\mathsf{N}_t - \mathsf{N}^*_t) = \alpha\mathbb{E}_t\mathsf{g}_{t+1} + \mathbb{E}_t\pi_{t+1}. \qquad \text{[A.10.1]}$$

Using the definition of nominal GDP $\mathsf{N}_t = \mathsf{P}_t + \mathsf{Y}_t$:

$$\mathbb{E}_t\pi_{t+1} = \mathbb{E}_t[\mathsf{N}_{t+1} - \mathsf{N}_t] - \mathbb{E}_t\mathsf{g}_{t+1},$$

and by substituting this into equation [A.10.1]:

$$\mathbb{E}_t\mathsf{N}^*_{t+1} - \mathsf{N}^*_t + \zeta(\mathsf{N}_t - \mathsf{N}^*_t) = \mathbb{E}_t[\mathsf{N}_{t+1} - \mathsf{N}_t].$$

Using the nominal GDP path N^*_t, define the gap $\tilde{\mathsf{N}}_t = \mathsf{N}_t - \mathsf{N}^*_t$ between the actual and the desired path, and note that the equation above can be written as:

$$\mathbb{E}_t\tilde{\mathsf{N}}_{t+1} = (1+\zeta)\tilde{\mathsf{N}}_t.$$

Now define a martingale difference sequence $\epsilon_t = \tilde{\mathsf{N}}_t - \mathbb{E}_{t-1}\tilde{\mathsf{N}}_t$, noting that the expectational difference equation above can be written equivalently as:

$$\tilde{\mathsf{N}}_t = (1+\zeta)\tilde{\mathsf{N}}_{t-1} + \epsilon_t, \qquad \text{[A.10.2]}$$

and iterating this equation ℓ periods forward leads to:

$$\tilde{\mathsf{N}}_{t+\ell} = (1+\zeta)^\ell\tilde{\mathsf{N}}_t + \sum_{\jmath=1}^{\ell}(1+\zeta)^{\ell-\jmath}\epsilon_{t+\jmath}. \qquad \text{[A.10.3]}$$

The variable $\mathcal{N}_t$ is as defined in [3.14b], and let $\mathcal{N}_t^*$ denote the equivalent variable defined in terms of N_t^*, and $\tilde{\mathcal{N}}_t = \mathcal{N}_t - \mathcal{N}_t^*$. With these definitions, equation [3.14b] implies that:

$$\tilde{\mathcal{N}}_t = (1-\beta\mu)\sum_{\ell=0}^{\infty}(\beta\mu)^{\ell}\mathbb{E}_t\tilde{\mathsf{N}}_t. \tag{A.10.4}$$

Given equation [3.14a], the characteristic of the path N_t^* that ensures $\mathsf{d}_t = \mathsf{d}_t^*$ is $\mathcal{N}_t - \mathbb{E}_{t-1}\mathcal{N}_t = (\mathsf{e}_t - \mathsf{v}_t) - \mathbb{E}_{t-1}[\mathsf{e}_t - \mathsf{v}_t]$. Therefore, $\tilde{\mathsf{d}}_t$ and $\tilde{\mathcal{N}}_t$ are related as follows

$$\tilde{\mathsf{d}}_t - \mathbb{E}_{t-1}\tilde{\mathsf{d}}_t = -(\tilde{\mathcal{N}}_t - \mathbb{E}_{t-1}\tilde{\mathcal{N}}_t), \tag{A.10.5}$$

which again uses equation [3.14a].

(i) Assuming $\zeta > 0$, it is clear from [A.10.3] (given $\mathbb{E}_t\epsilon_{t+\ell} = 0$ for all $\ell \geq 1$) that $|\tilde{\mathsf{N}}_{t+\ell}| \to \infty$ with probability one as $\ell \to \infty$ unless $\tilde{\mathsf{N}}_t = 0$ for all t. Since [A.10.2] implies that nominal GDP growth $\mathsf{n}_t = \mathsf{N}_t - \mathsf{N}_{t-1}$ is $\mathsf{n}_t = \mathsf{n}_t^* + \zeta\tilde{\mathsf{N}}_{t-1} + \epsilon_t$, any $\tilde{\mathsf{N}}_t \neq 0$ would imply an unbounded path of nominal GDP growth (and inflation). Ruling out such cases as possible equilibria leaves only $\tilde{\mathsf{N}}_t = 0$ as an equilibrium. It follows from [A.10.4] that this equilibrium must feature $\tilde{\mathcal{N}}_t = 0$, and hence from [A.10.5] that $\tilde{\mathsf{d}}_t = 0$. Thus, there is a unique (bounded) equilibrium with $\mathsf{N}_t = \mathsf{N}_t^*$ and $\mathsf{d}_t = \mathsf{d}_t^*$.

(ii) Assuming $\zeta = 0$, the equilibrium condition [A.10.2] reduces to $\tilde{\mathsf{N}}_t = \tilde{\mathsf{N}}_{t-1} + \epsilon_t$, which implies that nominal GDP growth $\mathsf{n}_t = \mathsf{N}_t - \mathsf{N}_{t-1}$ is given by $\mathsf{n}_t = \mathsf{n}_t^* + \epsilon_t$. Since ϵ_t is bounded, the implied rate of nominal GDP growth is also bounded, so any martingale difference sequence ϵ_t satisfying $\mathbb{E}_{t-1}\epsilon_t = 0$ is consistent with equilibrium. Since $\tilde{\mathsf{N}}_t$ follows a random walk, its expected future value is $\mathbb{E}_t\tilde{\mathsf{N}}_{t+\ell} = \tilde{\mathsf{N}}_t$ for any $\ell \geq 0$. Therefore, by substituting this into equation [A.10.4], it follows that $\tilde{\mathcal{N}}_t = \tilde{\mathsf{N}}_t$, and hence $\tilde{\mathcal{N}}_t - \mathbb{E}_{t-1}\tilde{\mathcal{N}}_t = \epsilon_t$. By using equation [A.10.5] it is seen that $\tilde{\mathsf{d}}_t - \mathbb{E}_{t-1}\tilde{\mathsf{d}}_t = -\epsilon_t$, and together with [3.12a], the difference equation for $\tilde{\mathsf{d}}_t$ is obtained. This completes the proof.

A.11 Proof of Proposition 11

Taking the utility functions of each household type from equation [2.1], the welfare function from [3.26] can be written explicitly as:

$$\mathscr{W}_{t_0} = \mathbb{E}_{t_0}\left[\frac{\Omega_{\mathrm{b}|t_0}}{2}\sum_{t=t_0}^{\infty}\left\{\prod_{\ell=1}^{t-t_0}\delta_{\mathrm{b},t-\ell}\right\}\frac{C_{\mathrm{b},t}^{1-\alpha}}{1-\alpha} + \frac{\Omega_{\mathrm{s}|t_0}}{2}\sum_{t=t_0}^{\infty}\left\{\prod_{\ell=1}^{t-t_0}\delta_{\mathrm{s},t-\ell}\right\}\frac{C_{\mathrm{s},t}^{1-\alpha}}{1-\alpha}\right]. \tag{A.11.1}$$

(i) Start from particular Pareto weights $\Omega_{\tau|t_0}$, where the ratio $\Omega_{\mathrm{b}|t_0}/\Omega_{\mathrm{s}|t_0}$ is measurable with respect to $t_0 - 1$ information. The corresponding first-best allocation is the one that maximizes [A.11.1] subject to the resource constraint [3.27], taking the discount factors $\delta_{\tau,t}$ as given (as individual households do), with the given values of the discount factors equal to [2.2] evaluated at the first-best consumption allocation. This social planner's problem assumes that a full set of state-contingent transfers are available. The Lagrangian for the constrained maximization problem is:

$$\mathcal{L}_{t_0} = \mathscr{W}_{t_0} + \mathbb{E}_{t_0}\left[\sum_{t=t_0}^{\infty}\Phi_{t|t_0}\left\{Y_t - \frac{C_{\mathrm{b},t}}{2} - \frac{C_{\mathrm{s},t}}{2}\right\}\right], \tag{A.11.2}$$

where $\mathscr{W}_{t_0}$ is the welfare function from [A.11.1] and $\Phi_{t|t_0}$ is the sequence of state-contingent Lagrangian multipliers on the resource constraints [3.27] starting from $t = t_0$ onwards. The first-order conditions with respect to $C_{\tau,t}$ for $t \geq t_0$ are:

$$\Omega_{\tau|t_0}\left\{\prod_{\ell=1}^{t-t_0}\delta^{\star}_{\tau,t-\ell}\right\}C^{\star\,-\alpha}_{\tau,t} = \Phi^{\star}_{t|t_0}, \tag{A.11.3}$$

where $\delta^{\star}_{\tau,t}$ denotes the discount factor [2.2] evaluated at $C^{\star}_{\tau,t}$. For households of type τ, taking the ratio of the first-order condition at time $t \geq t_0 + 1$ and the first-order condition at time $t-1$ yields:

$$\delta^{\star}_{\tau,t-1}\left(\frac{C^{\star}_{\tau,t}}{C^{\star}_{\tau,t-1}}\right)^{-\alpha} = \frac{\Phi^{\star}_{t|t_0}}{\Phi^{\star}_{t-1|t_0}}.$$

Since this equation must hold for all $\tau \in \{\mathrm{b}, \mathrm{s}\}$, the following risk-sharing condition is obtained for all $t \geq t_0 + 1$:

$$\delta^\star_{\mathrm{b},t-1}\left(\frac{C^\star_{\mathrm{b},t}}{C^\star_{\mathrm{b},t-1}}\right)^{-\alpha} = \delta^\star_{\mathrm{s},t-1}\left(\frac{C^\star_{\mathrm{s},t}}{C^\star_{\mathrm{s},t-1}}\right)^{-\alpha}, \qquad \text{[A.11.4]}$$

which confirms equation [3.29]. Now take the ratio of the first-order conditions [A.11.3] at time $t = t_0$ for borrowers and savers:

$$\frac{\Omega_{\mathrm{b}|t_0} C^{\star\,-\alpha}_{\mathrm{b},t_0}}{\Omega_{\mathrm{s}|t_0} C^{\star\,-\alpha}_{\mathrm{s},t_0}} = 1, \quad \text{and hence} \quad \frac{C^\star_{\mathrm{b},t_0}}{C^\star_{\mathrm{s},t_0}} = \left(\frac{\Omega_{\mathrm{b}|t_0}}{\Omega_{\mathrm{s}|t_0}}\right)^{\frac{1}{\alpha}}. \qquad \text{[A.11.5]}$$

Since $\Omega_{\mathrm{b}|t_0}/\Omega_{\mathrm{s}|t_0}$ is measurable with respect to $t_0 - 1$ information, it follows that $C^\star_{\mathrm{b},t_0}/C^\star_{\mathrm{s},t_0}$ is also measurable at $t_0 - 1$. As this result and [A.11.4] are obtained for any admissible Pareto weights $\Omega_{\tau|t_0}$, it follows that [3.27] for all $t \geq t_0$, [3.29] for all $t \geq t_0 + 1$, and $C^\star_{\mathrm{b},t_0}/C^\star_{\mathrm{s},t_0}$ being $t_0 - 1$ measurable are necessary conditions for $C^\star_{\tau,t}$ to be a first-best consumption allocation.

Now consider the converse. Take a state-contingent consumption allocation $C^\star_{\tau,t}$ satisfying [3.27] for all $t \geq t_0$, [3.29] for all $t \geq t_0 + 1$, and where the ratio $C^\star_{\mathrm{b},t_0}/C^\star_{\mathrm{s},t_0}$ is measurable at time $t_0 - 1$. It is now shown that this is a first-best allocation in the sense of solving the planner's problem for some Pareto weights $\Omega^\star_{\tau|t_0}$, where $\Omega^\star_{\mathrm{b}|t_0}/\Omega^\star_{\mathrm{s}|t_0}$ is $t_0 - 1$ measurable. Set the weights $\Omega^\star_{\tau|t_0}$ as follows:

$$\Omega^\star_{\tau|t_0} = C^{\star\,\alpha}_{\tau,t_0}, \qquad \text{[A.11.6]}$$

where this choice of weights implies that:

$$\frac{\Omega^\star_{\mathrm{b}|t_0}}{\Omega^\star_{\mathrm{s}|t_0}} = \left(\frac{C^\star_{\mathrm{b},t_0}}{C^\star_{\mathrm{s},t_0}}\right)^{-\alpha},$$

and hence the measurability restriction on the Pareto weights follows from the measurability condition satisfied by the consumption allocation at t_0. Now the sequence of supporting Lagrangian multipliers $\Phi^\star_{t|t_0}$ is constructed recursively. Start from a normalization $\Phi^\star_{t_0|t_0} = 1$. For $t \geq t_0 + 1$, the next iteration of the sequence is determined by:

$$\Phi^\star_{t|t_0} = \delta^\star_{\mathrm{b},t-1}\left(\frac{C^\star_{\mathrm{b},t}}{C^\star_{\mathrm{b},t-1}}\right)^{-\alpha}\Phi^\star_{t-1|t_0} = \delta^\star_{\mathrm{s},t-1}\left(\frac{C^\star_{\mathrm{s},t}}{C^\star_{\mathrm{s},t-1}}\right)^{-\alpha}\Phi^\star_{t-1|t_0},$$

where $\delta^\star_{\tau,t}$ is set according to [2.2] evaluated at the consumption allocation $C^\star_{\tau,t}$. The two expressions above return the same value because the allocation satisfies equation [A.11.4] for all $t \geq t_0 + 1$. Iterating this equation forwards yields two valid expressions for $\Phi^\star_{t|t_0}$:

$$\Phi^\star_{t|t_0} = C^{\star\,\alpha}_{\mathrm{b},t_0}\left\{\prod_{\ell=1}^{t-t_0}\delta^\star_{\mathrm{b},t-\ell}\right\}C^{\star\,-\alpha}_{\mathrm{b},t} = C^{\star\,\alpha}_{\mathrm{s},t_0}\left\{\prod_{\ell=1}^{t-t_0}\delta^\star_{\mathrm{s},t-\ell}\right\}C^{\star\,-\alpha}_{\mathrm{s},t},$$

and by using equation [A.11.6] it follows immediately that the first-order condition [A.11.3] is satisfied for all τ and all $t \geq t_0$. Since the allocation also satisfies the resource constraint [3.27] for all $t \geq t_0$, it follows that it is the solution to the planner's problem for some weights [A.11.6] that satisfy the measurability restriction. This establishes that the resource constraint [2.23] for all $t \geq t_0$, the risk-sharing condition [3.29] for all $t \geq t_0 + 1$, and the measurability restriction on $C^\star_{\mathrm{b},t_0}/C^\star_{\mathrm{s},t_0}$ are necessary and sufficient conditions that characterize the set of first-best consumption allocations.

(ii) The consumption allocation corresponding to the equilibrium with complete financial markets open for securities paying off from time t_0 onwards is such that the consumption ratios $c^*_{\tau,t|t_0}$ satisfy the goods-market clearing condition [2.23] for all $t \geq t_0$ and the risk-sharing condition [3.5] for all $t \geq t_0$. Given the definition $c_{\tau,t} = C_{\tau,t}/Y_t$, this is equivalent to $C^*_{\tau,t|t_0}$ satisfying [3.27] for all $t \geq t_0$ and [3.29] for all $t \geq t_0 + 1$. Note that the risk-sharing condition [3.5] at time $t = t_0$ implies:

$$\frac{C^*_{\mathrm{b},t_0|t_0}}{C^*_{\mathrm{s},t_0|t_0}} = \frac{c^*_{\mathrm{b},t_0|t_0}}{c^*_{\mathrm{s},t_0|t_0}} = \left(\frac{\delta_{\mathrm{b},t_0-1}}{\delta_{\mathrm{s},t_0-1}}\right)^{\frac{1}{\alpha}}\left(\frac{c_{\mathrm{b},t_0-1}}{c_{\mathrm{s},t_0-1}}\right),$$

and hence the measurability restriction on $C^*_{\mathrm{b},t_0|t_0}/C^*_{\mathrm{s},t_0|t_0}$ is satisfied. This allocation thus belongs to the set of first-best allocations. However, the measurability restriction does not imply that [3.5] holds for $t = t_0$, so the converse is not true. There are infinitely many first-best consumption allocations. Any allocation starting with a non-stochastic ratio $C^\star_{\mathrm{b},t_0}/C^\star_{\mathrm{s},t_0}$ will be among the first-best allocations subject to the overall consumption levels satisfying the resource constraint [3.27] and the subsequent evolution of the consumption paths for $t \geq t_0 + 1$ being consistent with the risk-sharing condition [3.29].

(iii) Given the complete-markets equilibrium $d^*_{t|t_0}$ and $l^*_{t|t_0}$ from t_0 onwards, consider the path for the nominal GDP growth rate $n_t = (N_t - N_{t-1})/N_{t-1}$ specified below that is to be followed from time t_0:

$$n_t = (1+n)\left(\frac{l^*_{t-1|t_0}}{\beta d^*_{t|t_0}}\right) - 1, \tag{A.11.7}$$

where n is any steady-state rate of nominal GDP growth consistent with the parameter restriction $\gamma < 1 + \bar{\imath}$. The consumption ratios and implied discount factors in the complete-market equilibrium from t_0 are denoted by $c^*_{\tau,t|t_0}$ and $\delta^*_{\tau,t|t_0}$. Using the definition of nominal GDP growth n_t, inflation is given by $1 + \pi_t = (1 + n_t)/(1 + g_t)$. Hence, if the complete-markets consumption allocation is implemented, the equilibrium bond yield j_t from [2.26] can be written as:

$$j_t = \left(\mathbb{E}_t\left[\sum_{\ell=1}^{\infty}\gamma^{\ell-1}\left\{\prod_{\jmath=1}^{\ell}\delta^*_{\tau,t+\jmath-1|t_0}\frac{(1+g_{t+\jmath})^{1-\alpha}}{(1+n_{t+\jmath})}\left(\frac{c^*_{\tau,t+\jmath|t_0}}{c^*_{\tau,t+\jmath-1|t_0}}\right)^{-\alpha}\right\}\right]\right)^{-1} + \gamma - 1. \tag{A.11.8}$$

Using equation [2.22b], the path for the nominal GDP growth rate in [A.11.7] implies

$$1 + n_t = \left(\frac{1+n}{\beta}\right)\left(\frac{1+g_t}{1+r^*_{t|t_0}}\right), \tag{A.11.9}$$

where $r^*_{t|t_0}$ is the complete-markets (from t_0 onwards) ex-post real return, and substituting this equation into [A.11.8] leads to:

$$j_t = \left(\frac{\beta}{1+n}\sum_{\ell=1}^{\infty}\left(\frac{\gamma\beta}{1+n}\right)^{\ell-1}\mathbb{E}_t\left[\prod_{\jmath=1}^{\ell}\delta^*_{\tau,t+\jmath-1|t_0}(1+r^*_{t|t_0})(1+g_{t+\jmath})^{-\alpha}\left(\frac{c^*_{\tau,t+\jmath|t_0}}{c^*_{\tau,t+\jmath-1|t_0}}\right)^{-\alpha}\right]\right)^{-1} + \gamma - 1. \tag{A.11.10}$$

The complete-markets equilibrium must satisfy the Euler equations [2.22d], thus the following equation must hold for all $\ell \geq 1$:

$$\mathbb{E}_t\left[\prod_{\jmath=1}^{\ell}\delta^*_{\tau,t+\jmath-1|t_0}(1+r^*_{t|t_0})(1+g_{t+\jmath})^{-\alpha}\left(\frac{c^*_{\tau,t+\jmath|t_0}}{c^*_{\tau,t+\jmath-1|t_0}}\right)^{-\alpha}\right] = 1,$$

which implies $1 + j_t = (1+n)/\beta$ when substituted into [A.11.10] (as explained in the proof of Proposition 3, the infinite sum converges because the parameter restriction $\gamma < 1 + \bar{\imath}$ implies $\gamma\beta < 1 + n$). Using the definition of nominal GDP growth $1 + n_t = (1 + \pi_t)(1 + g_t)$, equation [2.25a], the description of policy in [A.11.9], and the constant bond yield j_t, the ex-post real return with incomplete financial markets must be such that $r_t = r^*_{t|t_0}$. This demonstrates that the equilibrium with complete markets open from t_0 can be replicating using monetary policy in the incomplete-markets economy starting from any initial conditions at $t_0 - 1$.

To show that the equilibrium with complete markets open from t_0 onwards is the only first-best allocation implementable using monetary policy starting from date t_0, consider the welfare maximization problem of the policymaker at the central bank who does not have access to the full set of instruments implicit in the social planner's problem. In an economy with nominal bonds, the central bank can use its policy instrument to determine the state-contingent path of the price level, and is thus able to affect the ex-post real return r_t on bonds. The central bank then maximizes the welfare function [A.11.1] for some Pareto weights $\Omega_{\tau|t_0}$ (with $\Omega_{\mathrm{b}|t_0}/\Omega_{\mathrm{s}|t_0}$ being $t_0 - 1$ measurable) starting from t_0 onwards. The constraints comprise the equilibrium conditions of the incomplete-markets economy. These can be reduced to the resource constraint [3.27] of the social planner's problem, a single equation [3.28] that represents the budget identities of the households (given the resource constraint, one budget identity is redundant), and the Euler equations [2.18] for each household type. Starting at time t_0, these equations are constraints on the central bank that must hold for all $t \geq t_0$. The endogenous variables are the consumption allocation $C_{\tau,t}$, the end-of-period wealth distribution as captured by the real value of loans L_t, and the ex-post real return on bonds r_t.

The Lagrangian for the policymaker's constrained maximization problem is

$$\mathcal{L}_{t_0} = \mathscr{W}_{t_0} + \sum_{t=t_0}^{\infty} \mathbb{E}_{t_0}\left[\varPhi_{t|t_0}\left\{Y_t - \frac{C_{\mathrm{b},t}}{2} - \frac{C_{\mathrm{s},t}}{2}\right\} + \Im_{t|t_0}\left\{2((1+r_t)L_{t-1} - L_t) + \frac{C_{\mathrm{b},t}}{2} - \frac{C_{\mathrm{s},t}}{2}\right\}\right]$$
$$+ \sum_{t=t_0}^{\infty} \mathbb{E}_{t_0}\left[\frac{\Re_{\mathrm{b},t|t_0}}{2}\left\{C_{\mathrm{b},t}^{-\alpha} - \delta_{\mathrm{b},t}(1+r_{t+1})C_{\mathrm{b},t+1}^{-\alpha}\right\} + \frac{\Re_{\mathrm{s},t|t_0}}{2}\left\{C_{\mathrm{s},t}^{-\alpha} - \delta_{\mathrm{s},t}(1+r_{t+1})C_{\mathrm{s},t+1}^{-\alpha}\right\}\right], \quad \text{[A.11.11]}$$

where $\varPhi_{t|t_0}$ is the Lagrangian multiplier on the resource constraint [3.27], $\Im_{t|t_0}$ is the multiplier on the budget identity [3.28], and $\Re_{\tau,t|t_0}$ are the multipliers on the Euler equations. The first-order conditions with respect to $C_{\mathrm{b},t}$, $C_{\mathrm{s},t}$, L_t, and r_t are:

$$\varOmega_{\mathrm{b}|t_0}\left\{\prod_{\ell=1}^{t-t_0}\delta^{\star}_{\mathrm{b},t-\ell}\right\}C^{\star}{}_{\mathrm{b},t}^{-\alpha} = \varPhi^{\star}_{t|t_0} - \Im^{\star}_{t|t_0} + \alpha\left(\Re^{\star}_{\mathrm{b},t|t_0} - \delta^{\star}_{\mathrm{b},t-1}(1+r^{\star}_t)\Re^{\star}_{\mathrm{b},t-1|t_0}\right)C^{\star}{}_{\mathrm{b},t}^{-\alpha-1}; \quad \text{[A.11.12a]}$$

$$\varOmega_{\mathrm{s}|t_0}\left\{\prod_{\ell=1}^{t-t_0}\delta^{\star}_{\mathrm{s},t-\ell}\right\}C^{\star}{}_{\mathrm{s},t}^{-\alpha} = \varPhi^{\star}_{t|t_0} + \Im^{\star}_{t|t_0} + \alpha\left(\Re^{\star}_{\mathrm{s},t|t_0} - \delta^{\star}_{\mathrm{s},t-1}(1+r^{\star}_t)\Re^{\star}_{\mathrm{s},t-1|t_0}\right)C^{\star}{}_{\mathrm{s},t}^{-\alpha-1}; \quad \text{[A.11.12b]}$$

$$\Im^{\star}_{t|t_0} = \mathbb{E}_t\left[(1+r^{\star}_{t+1})\Im^{\star}_{t+1|t_0}\right]; \text{ and} \quad \text{[A.11.12c]}$$

$$2L^{\star}_{t-1}\Im^{\star}_{t|t_0} = \alpha\left(\frac{\delta^{\star}_{\mathrm{b},t-1}\Re^{\star}_{\mathrm{b},t-1|t_0}C^{\star}{}_{\mathrm{b},t}^{-\alpha-1}}{2} + \frac{\delta^{\star}_{\mathrm{s},t-1}\Re^{\star}_{\mathrm{s},t-1|t_0}C^{\star}{}_{\mathrm{s},t}^{-\alpha-1}}{2}\right), \quad \text{[A.11.12d]}$$

which must hold for all $t \geq t_0$, with the notational convention that $\Re_{\tau,t|t_0} = 0$ if $t < t_0$, indicating that equilibrium conditions prior to t_0 are not taken as constraints. If a consumption allocation is a solution of this constrained maximization problem for some admissible Pareto weights (ones satisfying the measurability restriction) then it is referred to as a second-best allocation.

Now take any second-best allocation $C^{\star}_{\tau,t}$ (associated with Pareto weights $\varOmega^{\star}_{\tau|t_0}$) that also belongs to the set of first-best allocations. Since it is second-best, it must satisfy the equilibrium conditions of the incomplete-markets economy, with associated real loans $L^{\star}_t$ and ex-post real return $r^{\star}_t$. The allocation must satisfy both the social planner's first-order conditions [A.11.3] and the policymaker's first-order conditions in [A.11.12]. A comparison of these sets of equations reveals that the Lagrangian multipliers on the extra implementability constraints must be zero at all times, that is, $\Im^{\star}_{t|t_0} = 0$ and $\Re^{\star}_{\tau,t|t_0} = 0$, meaning that given the Pareto weights $\varOmega^{\star}_{\tau|t_0}$, the chosen allocation would respect the implementability constraints even if these were not imposed explicitly.

Since the allocation satisfies [A.11.3], equation [A.11.5] must hold at time t_0, and this equation can be multiplied by $(1+r^{\star}_{t_0})C^{\star}{}_{\mathrm{s},t_0}$ to obtain:

$$\frac{\varOmega^{\star}_{\mathrm{b}|t_0}}{\varOmega^{\star}_{\mathrm{s}|t_0}}(1+r^{\star}_{t_0})C^{\star}{}_{\mathrm{b},t_0}^{-\alpha} = (1+r^{\star}_{t_0})C^{\star}{}_{\mathrm{s},t_0}^{-\alpha}.$$

The restriction on the Pareto weights requires $\varOmega^{\star}_{\mathrm{b}|t_0}/\varOmega^{\star}_{\mathrm{s}|t_0}$ to be measurable with respect to period $t_0 - 1$ information, so by taking expectations of both sides of the above equation conditional on date-$t_0 - 1$ information, it follows that:

$$\frac{\varOmega^{\star}_{\mathrm{b}|t_0}}{\varOmega^{\star}_{\mathrm{s}|t_0}} = \frac{\mathbb{E}_{t_0-1}\left[(1+r^{\star}_{t_0})C^{\star}{}_{\mathrm{s},t_0}^{-\alpha}\right]}{\mathbb{E}_{t_0-1}\left[(1+r^{\star}_{t_0})C^{\star}{}_{\mathrm{b},t_0}^{-\alpha}\right]}. \quad \text{[A.11.13]}$$

In equilibrium, the Euler equations [2.18] must hold at time $t_0 - 1$ (though these are not a constraint in the policymaker's problem at time t_0), hence by using the Euler equations of both household types:

$$\delta_{\mathrm{b},t_0-1}\mathbb{E}_{t_0-1}\left[(1+r^\star_{t_0})\left(\frac{C^\star_{\mathrm{b},t_0}}{C_{\mathrm{b},t_0-1}}\right)^{-\alpha}\right]=\delta_{\mathrm{s},t_0-1}\mathbb{E}_{t_0-1}\left[(1+r^\star_{t_0})\left(\frac{C^\star_{\mathrm{s},t_0}}{C_{\mathrm{s},t_0-1}}\right)^{-\alpha}\right],$$

and extracting terms dated t_0-1 from the conditional expectations leads to:

$$\frac{\mathbb{E}_{t_0-1}\left[(1+r^\star_{t_0}){C^\star}^{-\alpha}_{\mathrm{s},t_0}\right]}{\mathbb{E}_{t_0-1}\left[(1+r^\star_{t_0}){C^\star}^{-\alpha}_{\mathrm{b},t_0}\right]}=\frac{\delta_{\mathrm{b},t_0-1}}{\delta_{\mathrm{s},t_0-1}}\left(\frac{C_{\mathrm{b},t_0-1}}{C_{\mathrm{s},t_0-1}}\right)^{\alpha}.$$

By combining this with equation [A.11.13], the Pareto weights for this consumption allocation must be such that

$$\frac{\Omega^\star_{\mathrm{b}|t_0}}{\Omega^\star_{\mathrm{s}|t_0}}=\frac{\delta_{\mathrm{b},t_0-1}}{\delta_{\mathrm{s},t_0-1}}\left(\frac{C_{\mathrm{b},t_0-1}}{C_{\mathrm{s},t_0-1}}\right)^{\alpha}, \qquad \text{[A.11.14]}$$

and substituting this expression for the ratio of weights back into [A.11.5] implies:

$$\delta_{\mathrm{b},t_0-1}\left(\frac{C^\star_{\mathrm{b},t_0}}{C_{\mathrm{b},t_0-1}}\right)^{-\alpha}=\delta_{\mathrm{s},t_0-1}\left(\frac{C^\star_{\mathrm{s},t_0}}{C_{\mathrm{s},t_0-1}}\right)^{-\alpha}.$$

This is the risk-sharing condition [A.11.4] at time $t=t_0$. In summary, since the allocation is second best, all the market clearing and budget identities must hold. As the allocation is also first best, the risk-sharing condition must hold for all $t\geq t_0+1$. Finally, as shown above, since the allocation is first best, implementable, and has Pareto weights satisfying the measurability restriction, the risk-sharing condition must also hold at time t_0. Thus, all the equilibrium conditions hold for the economy where complete financial markets are open from time t_0 onwards. The consumption allocation must then coincide with the equilibrium of that economy. This demonstrates that this equilibrium is the only first-best allocation implementable by the policymaker at the central bank.

(iv) Now suppose that whenever a new state-contingent plan for monetary policy from some time t_0 onwards is made, the policymaker solves the constrained maximization problem with Pareto weights that support the one implementable first-best allocation. This entails using Pareto weights with the ratio $\Omega^*_{\mathrm{b}|t_0}/\Omega^*_{\mathrm{s}|t_0}$ as given in equation [A.11.14]. As shown in part (iii), the consumption allocation implied by this plan coincides with the equilibrium of the economy with complete markets open for securities paying off from t_0 onwards.

Suppose that monetary policy successfully replicates this consumption allocation between dates t_0 and $t_0'-1$ for some $t_0'>t_0$. Taking the initial conditions now prevailing at date $t_0'-1$ as predetermined, the equilibrium with complete markets open from t_0' features the same consumption allocation for all $t\geq t_0'$ as does the complete-markets equilibrium from t_0 onwards. Thus, if in the future there is a choice of a new plan selected on the same principles then the ongoing consumption allocation would coincide with that specified by the earlier plan, assuming the earlier plan had been followed up to that point (the relative Pareto weight $\Omega^*_{\mathrm{b}|t_0'}/\Omega^*_{\mathrm{s}|t_0'}$ remains unchanged at any reoptimization date t_0'). This implies the policymaker's plan is time consistent when the Pareto weights are chosen to support the one implementable first-best allocation.

Now suppose that whenever a new state-contingent plan for monetary policy is chosen, the ratio of the Pareto weights is set differently from that described above, so it does not support the complete-markets equilibrium consumption allocation. Using the results of part (iii), the consumption allocation resulting from the plan does not belong to the set of first-best allocations. It is the solution of the constrained maximization problem represented by the Lagrangian [A.11.11] for some Lagrangian multipliers with $\Re_{\tau,t|t_0}\neq 0$ (if $\Re_{\tau,t|t_0}=0$ for all τ and t then [A.11.12d] implies $\Im_{t|t_0}=0$ for all t, in which case the allocation would be a solution of the social planner's problem and thus first best). However, since any reoptimization at time $t_0'>t_0$ sets the Lagrangian multipliers on constraints prior to t_0' to zero, non-zero terms in $\Re_{\tau,t_0'-1|t_0}$ in the original problem that appear in the first-order conditions [A.11.12a], [A.11.12b], and [A.11.12d] would be replaced by zero terms $\Re_{\tau,t_0'-1|t_0'}$ after the reoptimization. This implies that second-best allocations are generally time inconsistent even if the relative Pareto weights of households do not change.

(v) With complete financial markets open for securities paying off from t_0 onwards, equations [2.22e], [2.23], and [3.5] hold for all $t \geq t_0$. These equations form a closed system in the consumption ratios $c_{\tau,t}$ and the discount factors $\delta_{\tau,t}$. Taking the past consumption ratios c_{τ,t_0-1} as state variables, these equations determine a path for $c^*_{\tau,t|t_0}$ and $\delta^*_{\tau,t|t_0}$. Given that no exogenous shocks appear in this set of equations, the resulting paths are non-stochastic, predetermined by the initial conditions at $t_0 - 1$.

The log-linear approximation of the equilibrium can be derived by first noting that all the log linearized equilibrium conditions from part (i) of Proposition 4 must hold, as must those from part (iii) of that proposition characterizing the gaps relative to the equilibrium with complete markets open at all dates. In addition, complete markets being open from $t \geq t_0$ onwards implies the risk-sharing condition [3.5] holds for all $t \geq t_0$. That equation can be log linearized as follows

$$\delta_{\mathrm{b},t-1} - \alpha(\mathsf{c}_{\mathrm{b},t} - \mathsf{c}_{\mathrm{b},t-1}) = \delta_{\mathrm{s},t-1} - \alpha(\mathsf{c}_{\mathrm{s},t} - \mathsf{c}_{\mathrm{s},t-1}),$$

and by subtracting the equation derived by taking expectations of both sides conditional on period-$t-1$ information, the following must hold for all $t \geq t_0$:

$$\mathsf{c}^*_{\mathrm{b},t|t_0} - \mathbb{E}_{t-1}\mathsf{c}^*_{\mathrm{b},t|t_0} = \mathsf{c}^*_{\mathrm{s},t|t_0} - \mathbb{E}_{t-1}\mathsf{c}^*_{\mathrm{s},t|t_0}.$$

Using [3.12b] with $\mathsf{c}_{\tau,t} = \tilde{\mathsf{c}}^*_{\tau,t|t_0}$, the above equation implies:

$$\left\{\left(\frac{\theta}{1+\theta}\right)\left(\frac{1-\beta\lambda}{1-\beta}\right) + \left(\frac{\theta}{1-\theta}\right)\left(\frac{1-\beta\lambda}{1-\beta}\right)\right\}\left((\mathsf{d}^*_{t|t_0} - \mathsf{d}^*_t) - \mathbb{E}_{t-1}[\mathsf{d}^*_{t|t_0} - \mathsf{d}^*_t]\right) = 0,$$

and since the coefficient is strictly positive, the condition below must hold for all $t \geq t_0$:

$$\mathsf{d}^*_{t|t_0} - \mathbb{E}_{t-1}\mathsf{d}^*_{t|t_0} = \mathsf{d}_t - \mathbb{E}_{t-1}\mathsf{d}_t.$$

With $\mathsf{d}_t = \mathsf{d}^*_{t|t_0}$, combining the equation above with [3.9b] and using the result $\mathsf{v}_t = \mathsf{d}^*_t$ from [3.11]:

$$\mathsf{d}^*_{t|t_0} = \lambda\mathsf{d}^*_{t-1|t_0} + (\mathbb{E}_{t-1}\mathsf{d}^*_t - \lambda\mathsf{d}^*_{t-1}) + (\mathsf{d}^*_t - \mathbb{E}_{t-1}\mathsf{d}^*_t),$$

which simplifies to:

$$\mathsf{d}^*_{t|t_0} = \mathsf{d}^*_t + \lambda(\mathsf{d}^*_{t-1|t_0} - \mathsf{d}^*_{t-1}).$$

The notational convention $\mathsf{d}^*_{t_0-1|t_0} = \mathsf{d}_{t_0-1}$ is adopted for variables that are predetermined at date t_0. Iterating the equation above backwards yields the solution for $\mathsf{d}^*_{t|t_0}$ in [3.30a], which is valid for all $t \geq t_0$. This is given in terms of d^*_t, a function of exogenous variables from Proposition 4, and $\mathsf{d}_{t_0-1} - \mathsf{d}^*_{t_0-1}$, which is predetermined from t_0 onwards. Now use $\mathsf{l}_t = \mathsf{l}^*_{t|t_0}$ in [3.9c] together with $\mathsf{v}_t = \mathsf{d}^*_t$ and $\mathsf{l}^*_t = \beta^{-1}\mathsf{d}^*_t$ from [3.11] to deduce:

$$\mathsf{l}^*_{t|t_0} = \mathsf{l}^*_t + \lambda(\mathsf{d}^*_{t|t_0} - \mathsf{d}^*_t),$$

and substituting the solution for $\mathsf{d}^*_{t|t_0}$ from [3.30a] yields the solution for $\mathsf{l}^*_{t|t_0}$ in [3.30a]. Since the solution is such that $\mathsf{d}^*_{t|t_0} - \mathbb{E}_{t-1}\mathsf{d}^*_{t|t_0} = \mathsf{d}^*_t - \mathbb{E}_{t-1}\mathsf{d}^*_t$ for all $t \geq t_0$, it follows from [3.9c] that $\mathsf{r}^*_{t|t_0} = \mathsf{r}^*_t$, confirming the result in [3.30a]. Substituting the solution for $\mathsf{d}^*_{t|t_0}$ from [3.30a] into [3.12b] shows that the consumption ratios $\mathsf{c}^*_{\tau,t|t_0}$ are given by:

$$\mathsf{c}^*_{\mathrm{b},t|t_0} = -\left(\frac{\theta}{1-\theta}\right)\left(\frac{1-\beta\lambda}{1-\beta}\right)\lambda^{t-t_0+1}(\mathsf{d}_{t_0-1} - \mathsf{d}^*_{t_0-1}); \quad \text{and}$$

$$\mathsf{c}^*_{\mathrm{s},t|t_0} = \left(\frac{\theta}{1+\theta}\right)\left(\frac{1-\beta\lambda}{1-\beta}\right)\lambda^{t-t_0+1}(\mathsf{d}_{t_0-1} - \mathsf{d}^*_{t-t_0+1}).$$

Since $\mathsf{c}_{\mathrm{b},t_0-1}$ and $\mathsf{c}_{\mathrm{s},t_0-1}$ also satisfy [3.12b], it follows from the equations above that $\mathsf{c}^*_{\tau,t|t_0} = \lambda^{t-t_0+1}\mathsf{c}_{\tau,t_0-1}$, confirming the claim in the proposition.

Finally, because equations [3.9b] and [3.9c] are satisfied by d_t and $\mathsf{d}^*_{t|t_0}$, l_t and $\mathsf{l}^*_{t|t_0}$, and r_t and $\mathsf{r}^*_{t|t_0}$, the equation for the gaps $\tilde{\mathsf{d}}_{t|t_0}$, $\tilde{\mathsf{l}}_{t|t_0}$, and $\tilde{\mathsf{r}}_{t|t_0}$ in [3.30b] follow immediately. The equations for the consumption gaps in [3.30b] also follow from [3.12b].

(vi) The welfare function [3.26] starting from time t_0 is evaluated using the Pareto weights that support the only implementable first-best consumption allocation from that date. From part (iii), this is the equilibrium with complete financial markets open from t_0 onwards. Let $c^*_{\tau,t|t_0}$ denote the consumption ratios in the complete-markets equilibrium from t_0 onwards, and $\delta^*_{\tau,|t_0}$ the associated discount factors that are taken as given by the policymaker, but are consistent with the consumption allocation $c^*_{\tau,t|t_0}$. The required Pareto weights $\mathit{\Omega}^*_{\tau|t_0}$ satisfy [A.11.3] with an associated sequence of Lagrangian multipliers $\mathit{\Phi}^*_{t|t_0}$ on the resource constraint. This equation can be written in terms of the consumption ratios as follows:

$$\mathit{\Omega}^*_{\tau|t_0}\left\{\prod_{\ell=1}^{t-t_0}\delta^*_{\tau,t-\ell|t_0}\right\} = {c^*}^{\alpha}_{\tau,t|t_0}\mathit{\Phi}^*_{t|t_0}Y_t^{\alpha}. \qquad \text{[A.11.15]}$$

Since all these equations are homogeneous in the Pareto weights and the Lagrangian multipliers, the values that support the complete-markets equilibrium are determined only up to scale. This degree of freedom is used to determine the units of the welfare function. By the envelope theorem, the Lagrangian multiplier $\mathit{\Phi}_{t_0|t_0}$ represents the increment to the period-t_0 welfare function from a marginal relaxation of the period-t_0 resource constraint. By setting $\mathit{\Phi}_{t_0|t_0} = 1/Y_{t_0}$, this means that one unit of the welfare measure is equivalent to a 1% increase in the initial level of output. Therefore, the following normalization is adopted:

$$\mathit{\Phi}^*_{t_0|t_0} = \frac{1}{Y_{t_0}}, \quad \text{and hence} \quad \mathit{\Omega}^*_{\tau|t_0} = \frac{{c^*}^{\alpha}_{\tau,t_0|t_0}}{Y_{t_0}^{1-\alpha}}, \qquad \text{[A.11.16]}$$

where the second equation follows from [A.11.15] at $t = t_0$, and which confirms the Pareto weights specified in the proposition.

In what follows, the variable $\varphi_{t|t_0}$ is defined as a transformation of the sequence of Lagrangian multipliers $\mathit{\Phi}_{t|t_0}$:

$$\varphi^*_{t|t_0} = \frac{Y_t\mathit{\Phi}^*_{t|t_0}}{\beta^{t-t_0}}, \quad \text{which satisfies} \quad \mathit{\Omega}^*_{\tau|t_0}\left\{\prod_{\ell=1}^{t-t_0}\delta^*_{\tau,t-\ell|t_0}\right\} = \beta^{t-t_0}\frac{\varphi^*_{t|t_0}{c^*}^{\alpha}_{\tau,t|t_0}}{Y_t^{1-\alpha}}, \qquad \text{[A.11.17]}$$

where the latter equation follows from [A.11.15]. Using that equation, with Pareto weights [A.11.16] and given discount factors $\delta^*_{\tau,t|t_0}$, the welfare function [A.11.1] can be expressed as follows:

$$\mathscr{W}_{t_0} = \sum_{t=t_0}^{\infty}\beta^{t-t_0}\mathbb{E}_{t_0}\left[\varphi^*_{t|t_0}\left\{\frac{1}{2}\frac{{c^*}^{\alpha}_{\mathrm{b},t|t_0}c^{1-\alpha}_{\mathrm{b},t}}{1-\alpha} + \frac{1}{2}\frac{{c^*}^{\alpha}_{\mathrm{s},t|t_0}c^{1-\alpha}_{\mathrm{s},t}}{1-\alpha}\right\}\right].$$

The welfare function is now written in terms of a variable $\varUpsilon_{t|t_0}$ that depends on the actual consumption ratios $c_{\tau,t}$ and their values $c^*_{\tau,t|t_0}$ if financial markets were complete from date t_0 onwards:

$$\mathscr{W}_{t_0} = \sum_{t=t_0}^{\infty}\beta^{t-t_0}\mathbb{E}_{t_0}\left[\varphi^*_{t|t_0}\frac{\varUpsilon^{-(1-\alpha)}_{t|t_0}}{1-\alpha}\right], \quad \text{where} \quad \varUpsilon_{t|t_0} = \left(\frac{1}{2}{c^*}^{\alpha}_{\mathrm{b},t|t_0}c^{1-\alpha}_{\mathrm{b},t} + \frac{1}{2}{c^*}^{\alpha}_{\mathrm{s},t|t_0}c^{1-\alpha}_{\mathrm{s},t}\right)^{-\frac{1}{1-\alpha}}. \qquad \text{[A.11.18]}$$

Now consider a second-order accurate approximation of the welfare function in [A.11.18] around the non-stochastic steady state characterized in Proposition 1. First, combining equation [A.11.17] for the transformed Lagrangian multipliers $\varphi^*_{t|t_0}$ and the normalization in [A.11.16] implies:

$$\varphi^*_{t|t_0} = \left(\frac{c^*_{\tau,t_0|t_0}}{c^*_{\tau,t|t_0}}\right)^{\alpha}\frac{\prod_{\ell=1}^{t-t_0}\delta^*_{\tau,t-\ell|t_0}(1+g_{t-\ell})^{1-\alpha}}{\beta^{t-t_0}}. \qquad \text{[A.11.19]}$$

Given that β is defined in [2.27c] such that $\beta = \delta(1+\bar{g})^{1-\alpha}$, and given the steady state from Proposition 1, it is clear that the transformed Lagrangian multipliers $\varphi^*_{t|t_0}$ have a well defined value $\bar{\varphi} = 1$ in the non-stochastic steady state. With the result from part (v), the consumption ratios $c^*_{\tau,t|t_0}$ with complete financial markets open from t_0 onwards depend only on variables that are predetermined at time t_0 (and so do the discount factors $\delta^*_{\tau,t|t_0}$ given that these are evaluated at the complete-markets equilibrium). Therefore, the transformed multipliers $\varphi^*_{t|t_0}$ are independent of policy decisions from t_0 onwards.

Using the definition of $\varUpsilon_{t|t_0}$ in [A.11.18] and the non-stochastic steady state characterized in Proposition 1 (which is the same in the case of complete markets) with $\bar{c}_{\mathrm{b}} = \bar{c}^*_{\mathrm{b}} = 1-\theta$ and $\bar{c}_{\mathrm{s}} = \bar{c}^*_{\mathrm{s}} = 1+\theta$, it can be seen that $\varUpsilon_{t|t_0}$ has a well defined non-stochastic steady-state value $\bar{\varUpsilon} = 1$. Taking each component of $\Upsilon_{t|t_0}$ from [A.11.18] in turn, note the following second-order accurate approximation:

$$c^{*\alpha}_{\tau,t|t_0}c^{1-\alpha}_{\tau,t} = \bar{c}_\tau\left(1+(1-\alpha)\mathsf{c}_{\tau,t}+\alpha\mathsf{c}^*_{\tau,t|t_0}+\frac{1}{2}(1-\alpha)^2\mathsf{c}^2_{\tau,t}+\alpha(1-\alpha)\mathsf{c}^*_{\tau,t|t_0}\mathsf{c}_{\tau,t}+\frac{1}{2}\alpha^2\mathsf{c}^{*2}_{\tau,t|t_0}\right)+\mathscr{O}^3,$$

which can be written as follows by rearranging terms:

$$c^{*\alpha}_{\tau,t|t_0}c^{1-\alpha}_{\tau,t} = \bar{c}_\tau\left(1+(1-\alpha)\left(\mathsf{c}_{\tau,t}+\frac{1}{2}\mathsf{c}^2_{\tau,t}\right)-\frac{\alpha(1-\alpha)}{2}\left(\mathsf{c}_{\tau,t}-\mathsf{c}^*_{\tau,t|t_0}\right)^2+\alpha\mathsf{c}^*_{\tau,t|t_0}\right.$$
$$\left.+\frac{\alpha^2}{2}\mathsf{c}^{*2}_{\tau,t|t_0}+\frac{\alpha(1-\alpha)}{2}\mathsf{c}^{*2}_{\tau,t|t_0}\right)+\mathscr{O}^3.$$

This expression can be simplified by stating it in terms of the consumption gaps $\tilde{\mathsf{c}}_{\tau,t|t_0}=\mathsf{c}_{\tau,t}-\mathsf{c}^*_{\tau,t|t_0}$:

$$c^{*\alpha}_{\tau,t}c^{1-\alpha}_{\tau,t} = \bar{c}_\tau\left(1-\frac{\alpha(1-\alpha)}{2}\tilde{\mathsf{c}}^2_{\tau,t|t_0}+(1-\alpha)\left(\mathsf{c}_{\tau,t}+\frac{1}{2}\mathsf{c}^2_{\tau,t}\right)+\alpha\left(\mathsf{c}^*_{\tau,t|t_0}+\frac{1}{2}\mathsf{c}^{*2}_{\tau,t|t_0}\right)\right)+\mathscr{O}^3.$$

Therefore, the second-order accurate approximation of equation [A.11.18] for $\Upsilon_{t|t_0}$ is:

$$\Upsilon_{t|t_0}=\frac{\alpha}{2}\left(\frac{(1-\theta)}{2}\tilde{\mathsf{c}}^2_{\mathrm{b},t|t_0}+\frac{(1+\theta)}{2}\tilde{\mathsf{c}}^2_{\mathrm{s},t|t_0}\right)-\left((1-\theta)\left(\mathsf{c}_{\mathrm{b},t}+\frac{1}{2}\mathsf{c}^2_{\mathrm{b},t}\right)+(1+\theta)\left(\mathsf{c}_{\mathrm{s},t}+\frac{1}{2}\mathsf{c}^2_{\mathrm{s},t}\right)\right)$$
$$-\frac{\alpha}{1-\alpha}\left((1-\theta)\left(\mathsf{c}^*_{\mathrm{b},t|t_0}+\frac{1}{2}\mathsf{c}^{*2}_{\mathrm{b},t|t_0}\right)+(1+\theta)\left(\mathsf{c}^*_{\mathrm{s},t}+\frac{1}{2}\mathsf{c}^{*2}_{\mathrm{s},t|t_0}\right)\right)+\frac{1-\alpha}{2}\Upsilon^2_{t|t_0}+\mathscr{O}^3. \quad \text{[A.11.20]}$$

The consumption ratios of the actual economy and the hypothetical complete-markets economy must both satisfy [2.23], where a second-order accurate approximation of that equation is:

$$(1-\theta)\left(\mathsf{c}_{\mathrm{b},t}+\frac{1}{2}\mathsf{c}^2_{\mathrm{b},t}\right)+(1+\theta)\left(\mathsf{c}_{\mathrm{s},t}+\frac{1}{2}\mathsf{c}^2_{\mathrm{s},t}\right)=\mathscr{O}^3,$$

and by substituting this into [A.11.20], all first-order terms are eliminated from the right-hand side. This implies that $\Upsilon_{t|t_0}=\mathscr{O}^2$, and hence the term in $\Upsilon^2_{t|t_0}$ can be included among the $\mathscr{O}^3$ (third-order or higher) terms. Therefore, the second-order accurate approximation is:

$$\Upsilon_{t|t_0}=\frac{\alpha}{2}\left(\frac{(1-\theta)}{2}\tilde{\mathsf{c}}^2_{\mathrm{b},t|t_0}+\frac{(1+\theta)}{2}\tilde{\mathsf{c}}^2_{\mathrm{s},t|t_0}\right)+\mathscr{O}^3.$$

Substituting the equations for the consumption gaps from [3.30b] (which hold up to an error of order $\mathscr{O}^2$), the equation above can be written solely in terms of the debt gap $\tilde{\mathsf{d}}_{t|t_0}$:

$$\Upsilon_{t|t_0}=\frac{\alpha}{2}\frac{1}{2}\left(\frac{1-\beta\lambda}{1-\beta}\right)^2\left(\frac{(1-\theta)\theta^2}{(1-\theta)^2}+\frac{(1+\theta)\theta^2}{(1+\theta)^2}\right)\tilde{\mathsf{d}}^2_t+\mathscr{O}^3,$$

and by simplifying the coefficient:

$$\Upsilon_{t|t_0}=\frac{\aleph_{\mathrm{d}}}{2}\tilde{\mathsf{d}}^2_t+\mathscr{O}^3,\quad\text{where }\aleph_{\mathrm{d}}=\alpha\left(\frac{\theta^2}{1-\theta^2}\right)\left(\frac{1-\beta\lambda}{1-\beta}\right)^2. \quad \text{[A.11.21]}$$

A second-order accurate approximation of equation [A.11.18] for the welfare function in terms of the log deviations $\varphi^*_{t|t_0}$ and $\Upsilon_{t|t_0}$ is

$$\mathscr{W}_{t_0}=\sum_{t=t_0}^{\infty}\beta^{t-t_0}\mathbb{E}_{t_0}\left[\frac{1}{1-\alpha}\left(1+\varphi^*_{t|t_0}+\frac{1}{2}\varphi^{*2}_{t|t_0}\right)-\Upsilon_{t|t_0}-\varphi^*_{t|t_0}\Upsilon_{t|t_0}+\frac{(1-\alpha)}{2}\Upsilon^2_{t|t_0}\right]+\mathscr{O}^3. \quad \text{[A.11.22]}$$

Given the parameter restrictions, Proposition 1 shows that $0<\beta<1$. Equation [A.11.21] shows that $\Upsilon_{t|t_0}$ depends only on the stationary variable $\tilde{\mathsf{d}}_{t|t_0}$. Using equation [A.11.19], $\varphi^*_{t|t_0}$ is given by:

$$\varphi^*_{t|t_0}=\alpha(\mathsf{c}^*_{\tau,t_0|t_0}-\mathsf{c}^*_{\tau,t|t_0})+\sum_{\ell=1}^{t-t_0}\left(\delta^*_{\tau,t-\ell|t_0}+(1-\alpha)\mathsf{g}_{t-\ell}\right),$$

which is a sum of stationary variables. The variance may increase with t, but since all terms dated t in [A.11.22] are multiplied by β^{t-t_0}, and as $0<\beta<1$, the value of the welfare function is well defined. With $\Upsilon_{t|t_0}=\mathscr{O}^2$ and $\varphi^*_{t|t_0}$ being independent of policy from t_0 onwards (denoted by $\mathscr{I}_{t_0}$), equation [A.11.22] reduces to:

$$\mathscr{W}_{t_0} = -\sum_{t=t_0}^{\infty} \beta^{t-t_0}\mathbb{E}_{t_0}\Upsilon_{t|t_0} + \mathscr{I}_{t_0} + \mathscr{O}^3.$$

Hence, the welfare function can be written as $\mathscr{W}_{t_0} = -\mathscr{L}_{t_0} + \mathscr{I}_{t_0} + \mathscr{O}^3$, where the loss function $\mathscr{L}_{t_0}$ is as defined in [3.31]. The coefficient of the debt gap is taken from [A.11.21]. This completes the proof.

A.12 Proof of Proposition 12

The welfare function [3.26] is evaluated at the Pareto weights associated with the first-best level of output $\hat{Y}_t$ and complete financial markets open for securities paying off from t_0 onwards. Given the first-best level of output, part (iii) of Proposition 11 shows that the complete-markets equilibrium is the only first-best consumption allocation that can be implemented using monetary policy. Following the proof of part (vi) of Proposition 11, these Pareto weights $\hat{\Omega}^*_{\tau|t_0}$ must satisfy the equivalent of equation [A.11.15]:

$$\hat{\Omega}^*_{\tau|t_0}\left\{\prod_{\ell=1}^{t-t_0}\delta^*_{\tau,t-\ell|t_0}\right\} = c^{*\alpha}_{\tau,t|t_0}\hat{\Phi}^*_{t|t_0}\hat{Y}^{\alpha}_t, \quad \text{with normalization } \hat{\Phi}^*_{t_0|t_0} = \frac{1}{\hat{Y}_{t_0}}, \tag{A.12.1}$$

where $\hat{\Phi}^*_{t|t_0}$ is the sequence of Lagrangian multipliers on the resource constraint in the social planner's problem. Since the Pareto weights and Lagrangian multipliers are determined only up to scale, the normalization in [A.12.1] is imposed, which means that one unit of the welfare function is equivalent to a 1% increase in the initial first-best level of output (see Proposition 11).

Equation [A.12.1] makes use of the result from part (v) of Proposition 11 that the consumption ratios $c^*_{\tau,t|t_0}$ in the equilibrium with complete markets open from t_0 onwards depend only on variables that are predetermined at date t_0, not on the stochastic process for real GDP. This means that these ratios are the same irrespective of whether actual real GDP Y_t is equal to the first-best level of output $\hat{Y}_t$, formally $\hat{c}^*_{\tau,t|t_0} = c^*_{\tau,t|t_0}$. The discount factors $\delta^*_{\tau,t|t_0}$ in [A.12.1] are taken as given by the policymaker, but are evaluated at the consumption allocation $c^*_{\tau,t|t_0}$. These too are independent of the actual value of real GDP because of the same property of $c^*_{\tau,t|t_0}$, hence $\hat{\delta}^*_{\tau,t|t_0} = \delta^*_{\tau,t|t_0}$. Using equation [A.12.1] at $t = t_0$, the required Pareto weights are seen to be $\hat{\Omega}^*_{\tau|t_0} = c^{*\alpha}_{\tau,t_0|t_0}/\hat{Y}^{1-\alpha}_{t_0}$, confirming the statement in the proposition.

Following the proof of part (vi) of Proposition 11, define a transformation $\hat{\varphi}^*_{t|t_0}$ of the Lagrangian multipliers $\hat{\Phi}^*_{t|t_0}$, the equivalent of equation [A.11.17]:

$$\hat{\varphi}^*_{t|t_0} = \frac{\hat{Y}_t\hat{\Phi}^*_{t|t_0}}{\beta^{t-t_0}}, \quad \text{and hence } \hat{\Omega}^*_{\tau|t_0}\left\{\prod_{\ell=1}^{t-t_0}\delta^*_{\tau,t-\ell|t_0}\right\} = \beta^{t-t_0}\frac{\hat{\varphi}^*_{t|t_0}c^{*\alpha}_{\tau,t|t_0}}{\hat{Y}^{1-\alpha}_t}, \tag{A.12.2}$$

where the latter equation uses [A.12.1]. That equation implies $\hat{\varphi}^*_{t|t_0}$ is given by:

$$\hat{\varphi}^*_{t|t_0} = \left(\frac{c^*_{\tau,t_0|t_0}}{c^*_{\tau,t|t_0}}\right)^{\alpha}\frac{\prod_{\ell=1}^{t-t_0}\delta^*_{\tau,t-\ell|t_0}(1+\hat{g}_{t-\ell})^{1-\alpha}}{\beta^{t-t_0}}, \tag{A.12.3}$$

where $\hat{g}_t$ is the growth rate of the first-best level of output $\hat{Y}_t$ (equal to output with flexible prices). Since none of the complete-markets consumption ratios $c^*_{\tau,t|t_0}$, discount factors $\delta^*_{\tau,t|t_0}$, nor $\hat{g}_t$ depend on policy from t_0 onwards, neither does $\hat{\varphi}^*_{t|t_0}$.

Substituting [A.12.2] into the welfare function [3.26] and using the utility function in [2.1] leads to:

$$\mathscr{W}_{t_0} = \sum_{t=t_0}^{\infty}\beta^{t-t_0}\mathbb{E}_{t_0}\left[\frac{\hat{\varphi}^*_{t|t_0}}{\hat{Y}^{1-\alpha}_t}\left\{\frac{1}{2}\frac{c^{*\alpha}_{\mathrm{b},t|t_0}C^{1-\alpha}_{\mathrm{b},t}}{1-\alpha} + \frac{1}{2}\frac{c^{*\alpha}_{\mathrm{s},t|t_0}C^{1-\alpha}_{\mathrm{s},t}}{1-\alpha}\right\}\right].$$

Since flexible-price output is $\hat{Y}_t = A_t$, it follows from [4.9] that $\Delta_t = \hat{Y}_t/Y_t$. Using this and the definition of $\Upsilon_{t|t_0}$ in equation [A.11.18] of the proof of Proposition 11, the welfare function can be written as follows:

$$\mathscr{W}_{t_0} = \sum_{t=t_0}^{\infty} \beta^{t-t_0} \mathbb{E}_{t_0} \left[\frac{\hat{\varphi}^*_{t|t_0} \Delta_t^{-(1-\alpha)} \Upsilon_{t|t_0}^{-(1-\alpha)}}{1-\alpha} \right]. \qquad \text{[A.12.4]}$$

The link between Δ_t and inflation π_t is given in [4.11]. With a zero-inflation non-stochastic steady state ($\bar{\pi} = 0$), the steady-state value of Δ_t is $\bar{\Delta} = 1$, which implies $\hat{g}_t$ and g_t have the same steady-state value. This means the non-stochastic steady state of Proposition 1 is the same in this version of the model, and the log linearizations of part (v) of Proposition 11 continue to hold. The term $\Upsilon_{t|t_0}$ is identical to that appearing in the proof of Proposition 11, and as there is no change to the non-stochastic steady state here, it has steady-state value $\bar{\Upsilon} = 1$ and its second-order approximation is as given in [A.11.21]. With the same steady state as before, equation [A.12.3] confirms that $\bar{\varphi} = 1$, as in Proposition 11. The second-order approximation of the welfare function in [A.12.4] in terms of log deviations of variables from their non-stochastic steady-state values is therefore:

$$\mathscr{W}_{t_0} = \sum_{t=t_0}^{\infty} \beta^{t-t_0} \mathbb{E}_{t_0} \Bigg[\frac{1}{1-\alpha} \left(1 + \hat{\varphi}^*_{t|t_0} + \frac{1}{2} \hat{\varphi}^{*2}_{t|t_0} \right) - (\Delta_t + \Upsilon_{t|t_0}) - \hat{\varphi}^*_{t|t_0} (\Delta_t + \Upsilon_{t|t_0}) + \frac{(1-\alpha)}{2} (\Delta_t + \Upsilon_{t|t_0})^2 \Bigg] + \mathscr{O}^3. \qquad \text{[A.12.5]}$$

A second-order accurate approximation of equation [4.11] around the zero-inflation steady state leads to the following recursion for Δ_t:

$$\Delta_t = \sigma \Delta_{t-1} + \frac{\varepsilon(1+\varepsilon\xi)}{2} \frac{\sigma}{(1-\sigma)} \pi_t^2 + \mathscr{O}^3. \qquad \text{[A.12.6]}$$

This implies that $\Delta_t = \sigma\Delta_{t-1} + \mathscr{O}^2$, and hence $\Delta_t = \mathscr{O}^2$ under the assumption that initial relative price distortions are second order ($\Delta_{t_0-1} = \mathscr{O}^2$). Without introducing first-order relative-price distortions exogenously, equation [A.12.6] implies these would never emerge endogenously. It is known from [A.11.21] in Proposition 11 that $\Upsilon_{t|t_0} = \mathscr{O}^2$, and equation [A.12.3] confirms $\hat{\varphi}^*_{t|t_0}$ is independent of policy choices from t_0 onwards. Thus, the second-order approximation in [A.12.5] reduces to:

$$\mathscr{W}_{t_0} = -\sum_{t=t_0}^{\infty} \beta^{t-t_0} \mathbb{E}_{t_0} [\Upsilon_{t|t_0} + \Delta_t] + \mathscr{I}_{t_0} + \mathscr{O}^3, \qquad \text{[A.12.7]}$$

where $\mathscr{I}_{t_0}$ denotes terms independent of policy choices from t_0 onwards.

The recursion for Δ_t in [A.12.6] can be iterated backwards to obtain:

$$\Delta_t = \frac{\varepsilon}{2} \frac{\sigma}{(1-\sigma)} \sum_{\ell=0}^{t-t_0} \sigma^\ell \pi_{t-\ell}^2 + \mathscr{I}_{t_0} + \mathscr{O}^3,$$

where the term $\sigma^{t-t_0+1}\Delta_{t_0-1}$ is independent of policy from t_0 onwards (this depends only on predetermined variables). Hence, the expected discounted sum of current and future relative-price distortions is:

$$\sum_{t=t_0}^{\infty} \beta^{t-t_0} \mathbb{E}_{t_0} \Delta_t = \frac{\varepsilon(1+\varepsilon\xi)}{2} \frac{\sigma}{(1-\sigma)} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \sum_{\ell=0}^{t-t_0} \sigma^\ell \mathbb{E}_{t_0} \pi_{t-\ell}^2 + \mathscr{I}_{t_0} + \mathscr{O}^3,$$

and by changing the order of summation this can be written as:

$$\sum_{t=t_0}^{\infty} \beta^{t-t_0} \mathbb{E}_{t_0} \Delta_t = \frac{\varepsilon\kappa}{2} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \mathbb{E}_{t_0} \pi_t^2 + \mathscr{I}_{t_0} + \mathscr{O}^3, \quad \text{where } \kappa = \frac{\sigma(1+\varepsilon\xi)}{(1-\sigma)(1-\beta\sigma)}. \qquad \text{[A.12.8]}$$

Substituting this result and equation [A.11.21] into [A.12.7] confirms the expression for the loss function $\mathscr{L}_{t_0}$, where $\mathscr{W}_{t_0} = -\mathscr{L}_{t_0} + \mathscr{I}_{t_0} + \mathscr{O}^3$.

Since the log-linearizations of Proposition 4 and Proposition 11 continue to hold, the first constraint in [4.14] follows immediately from [3.30b], and the transversality condition on j_t follows from part (iii) of Proposition 4. Relative-price distortions $\Delta_t = \hat{Y}_t/Y_t$ are such that $\Delta_t = \mathscr{O}^2$, hence $\mathsf{g}_t = \hat{\mathsf{g}}_t + \mathscr{O}^2$. Since [3.9a] and [3.11] imply that r^*_t depends only on the stochastic process for real GDP, it must be the case that

$\mathsf{r}^*_t = \hat{\mathsf{r}}^*_t$. Equation [3.30a] also implies $\hat{\mathsf{r}}^*_t = \hat{\mathsf{r}}^*_{t|t_0}$. Together with the equation for $\tilde{\mathsf{r}}_{t|t_0} = \mathsf{r}_t - \mathsf{r}^*_{t|t_0}$ in [3.30b], the second constraint in [4.14] is deduced. This completes the proof.

A.13 Proof of Proposition 13

(i) Optimal monetary policy with commitment starting from date t_0 minimizes the loss function [4.13] subject to the constraints in [4.14], which must hold for all $t \geq t_0$. The endogenous variables are the debt gap $\tilde{\mathsf{d}}_{t|t_0}$, inflation π_t, and the bond yield j_t. The exogenous variable is the natural real GDP growth rate $\hat{\mathsf{g}}_t$ (with flexible prices), which determines (using [3.11]) the complete-markets ex-post real return $\hat{\mathsf{r}}^*_t$ associated with this sequence of real GDP growth rates.

The Lagrangian for the constrained minimization problem is

$$\mathcal{L}_{t_0} = \frac{1}{2}\sum_{t=t_0}^{\infty} \beta^{t-t_0}\mathbb{E}_{t_0}\left[\aleph_{\mathsf{d}}\tilde{\mathsf{d}}^2_{t|t_0} + \aleph_\pi \pi_t^2\right] + \sum_{t=t_0}^{\infty}\beta^{t-t_0}\mathbb{E}_{t_0}\left[\daleth_{t|t_0}\left\{\lambda\tilde{\mathsf{d}}_{t|t_0} - \tilde{\mathsf{d}}_{t+1|t_0}\right\}\right] + \sum_{t=t_0}^{\infty}\beta^{t-t_0}\mathbb{E}_{t_0}\Bigg[\beth_{t|t_0}\Bigg\{ \frac{1}{1-\beta\mu}\mathsf{j}_{t-1} - \frac{\beta\mu}{1-\beta\mu}\mathsf{j}_t - \pi_t - \tilde{\mathsf{d}}_{t|t_0} + \lambda\tilde{\mathsf{d}}_{t-1|t_0} - \hat{\mathsf{r}}^*_t\Bigg\}\Bigg] + \Gamma_{t_0}\lim_{t\to\infty}(\beta\mu)^{t-t_0}\mathbb{E}_{t_0}\mathsf{j}_t, \quad \text{[A.13.1]}$$

with the coefficients of the debt gap and inflation in the loss function [4.13] denoted by $\aleph_{\mathsf{d}}$ and $\aleph_\pi$, and where the Lagrangian multipliers $\daleth_{t|t_0}$ and $\beth_{t|t_0}$ are for convenience expressed in current-value terms by scaling by β^{t-t_0}. The first-order conditions with respect to $\tilde{\mathsf{d}}_{t|t_0}$, π_t, and j_t are:

$$\aleph_{\mathsf{d}}\tilde{\mathsf{d}}_{t|t_0} + \lambda\daleth_{t|t_0} - \beta^{-1}\daleth_{t-1|t_0} - \beth_{t|t_0} + \beta\lambda\mathbb{E}_t\beth_{t+1|t_0} = 0; \quad \text{[A.13.2a]}$$

$$\aleph_\pi\pi_t - \beth_{t|t_0} = 0; \quad \text{[A.13.2b]}$$

$$\beta\mathbb{E}_t\beth_{t+1|t_0} - \beta\mu\beth_{t|t_0} = 0, \quad \text{[A.13.2c]}$$

where the notational convention is adopted that $\daleth_{t|t_0} = 0$ for all $t < t_0$ (the policymaker does not consider constraints prior to t_0). Furthermore, the transversality conditions of the constrained minimization problem [A.13.1] that must be satisfied by the multipliers $\daleth_{t|t_0}$ and Γ_{t_0} are:

$$\lim_{t\to\infty}\beta^{t-t_0}\daleth_{t|t_0} = 0, \quad \text{and} \quad \Gamma_{t_0} = \frac{\beta\mu}{1-\beta\mu}\lim_{t\to\infty}\frac{\mathbb{E}_{t_0}\beth_{t|t_0}}{\mu^{t-t_0}}. \quad \text{[A.13.3]}$$

Equation [A.13.2c] implies $\mathbb{E}_t\beth_{t+1|t_0} = \mu\beth_{t|t_0}$ for all $t \geq t_0$, and equation [A.13.2b] implies $\beth_{t|t_0} = \aleph_\pi\pi_t$ for all $t \geq t_0$. With $\aleph_\pi > 0$ because $\sigma > 0$ (from [4.13]), combining these equations implies for all $t \geq t_0$:

$$\mathbb{E}_t\pi_{t+1} = \mu\pi_t. \quad \text{[A.13.4]}$$

This establishes that optimal monetary policy must feature inflation persistence with autoregressive coefficient μ.

Iterating forwards the first constraint in [4.14] and the optimality condition [A.13.4], the expected future paths of $\tilde{\mathsf{d}}_{t|t_0}$ and π_t for any $t \geq t_0$ must be as follows for all $\ell \geq 0$:

$$\mathbb{E}_t\tilde{\mathsf{d}}_{t+\ell|t_0} = \lambda^\ell\tilde{\mathsf{d}}_{t|t_0}, \quad \text{and} \quad \mathbb{E}_t\pi_{t+\ell} = \mu^\ell\pi_t. \quad \text{[A.13.5]}$$

Using [A.13.2b], [A.13.2c], and [A.13.3], the Lagrangian multipliers $\beth_{t|t_0}$ and Γ_{t_0} can be expressed in terms of inflation:

$$\beth_{t|t_0} = \aleph_\pi\pi_t, \quad \text{and} \quad \Gamma_{t_0} = \frac{\beta\mu}{1-\beta\mu}\aleph_\pi\pi_{t_0}.$$

Now consider the first-order condition [A.13.2a]. Multiplying both sides by $\beta > 0$ and substituting terms in inflation π_t for those in $\beth_{t|t_0}$ using [A.13.2b] yields the following equation that holds for all $t \geq t_0$:

$$\daleth_{t-1|t_0} = \beta\lambda\daleth_{t|t_0} + \beta\aleph_{\mathsf{d}}\tilde{\mathsf{d}}_{t|t_0} - \beta\aleph_\pi\pi_t + \beta^2\lambda\aleph_\pi\mathbb{E}_t\pi_{t+1}.$$

Using [A.13.5], the expectations of future inflation can be replaced by terms in current inflation:

$$\daleth_{t-1|t_0} = \beta\lambda\daleth_{t|t_0} + \beta\aleph_{\mathsf{d}}\tilde{\mathsf{d}}_{t|t_0} - \beta(1-\beta\mu\lambda)\aleph_\pi\pi_t.$$

Taking expectations of this equation at time $t+1$ conditional on date-t information and making use of

[A.13.5] implies that the following must hold for all $t \geq t_0$:

$$\daleth_{t|t_0} = \beta\lambda\mathbb{E}_t\daleth_{t+1|t_0} + \beta\lambda\aleph_{\mathrm{d}}\tilde{\mathrm{d}}_{t|t_0} - \beta\mu(1-\beta\mu\lambda)\aleph_\pi\pi_t.$$

This equation can be iterated forwards to deduce:

$$\daleth_{t|t_0} = \sum_{\ell=0}^{\infty}(\beta\lambda)^\ell\mathbb{E}_t\left[\beta\lambda\aleph_{\mathrm{d}}\tilde{\mathrm{d}}_{t+\ell|t_0} - \beta\mu(1-\beta\mu\lambda)\aleph_\pi\pi_{t+\ell}\right] + \lim_{\ell\to\infty}(\beta\lambda)^\ell\mathbb{E}_t\daleth_{t+\ell|t_0},$$

and by taking expectations of the transversality condition [A.13.3] and noting that $0 < \lambda < 1$, the final term must be zero. Thus, $\daleth_{t|t_0}$ is given by

$$\daleth_{t|t_0} = \beta\lambda\aleph_{\mathrm{d}}\sum_{\ell=0}^{\infty}(\beta\lambda)^\ell\mathbb{E}_t\tilde{\mathrm{d}}_{t+\ell|t_0} - \beta\mu(1-\beta\mu\lambda)\aleph_\pi\sum_{\ell=0}^{\infty}(\beta\lambda)^\ell\mathbb{E}_t\pi_{t+\ell}.$$

By using the results in [A.13.5], this yields the following expression which holds for all $t \geq t_0$:

$$\daleth_{t|t_0} = \beta\lambda\aleph_{\mathrm{d}}\left(\sum_{\ell=0}^{\infty}(\beta\lambda^2)^\ell\right)\tilde{\mathrm{d}}_{t|t_0} - \beta\mu(1-\beta\mu\lambda)\aleph_\pi\left(\sum_{\ell=0}^{\infty}(\beta\mu\lambda)^\ell\right)\pi_t$$

$$= \aleph_{\mathrm{d}}\frac{\beta\lambda}{1-\beta\lambda^2}\tilde{\mathrm{d}}_{t|t_0} - \aleph_\pi\beta\mu\pi_t. \quad \text{[A.13.6]}$$

With the expression for $\daleth_{t|t_0}$ in [A.13.6] and noting the initial condition $\daleth_{t_0-1|t_0} = 0$, the first-order condition [A.13.2a] at date $t = t_0$ is:

$$0 = \beta\lambda\left(\aleph_{\mathrm{d}}\frac{\beta\lambda}{1-\beta\lambda^2}\tilde{\mathrm{d}}_{t_0|t_0} - \aleph_\pi\beta\mu\pi_{t_0}\right) + \beta\aleph_{\mathrm{d}}\tilde{\mathrm{d}}_{t_0|t_0} - \beta(1-\beta\mu\lambda)\aleph_\pi\pi_{t_0},$$

which simplifies to:

$$\frac{\aleph_{\mathrm{d}}}{(1-\beta\lambda^2)}\tilde{\mathrm{d}}_{t_0|t_0} = \aleph_\pi\pi_{t_0}, \quad \text{or} \quad \frac{\aleph_{\mathrm{d}}}{(1-\beta\lambda^2)}(\tilde{\mathrm{d}}_{t_0|t_0} - \lambda\tilde{\mathrm{d}}_{t_0-1|t_0}) = \aleph_\pi\pi_{t_0}, \quad \text{[A.13.7]}$$

where the latter holds because of the initial condition $\tilde{\mathrm{d}}_{t_0-1|t_0} = 0$ (gaps are zero for any t before t_0). Now consider any $t \geq t_0 + 1$. The first-order condition [A.13.2a] together with equation [A.13.6] implies:

$$\aleph_{\mathrm{d}}\frac{\beta\lambda}{1-\beta\lambda^2}\tilde{\mathrm{d}}_{t-1|t_0} - \aleph_\pi\beta\mu\pi_{t-1} = \beta\aleph_{\mathrm{d}}\tilde{\mathrm{d}}_{t|t_0} - \beta(1-\beta\mu\lambda)\aleph_\pi\pi_t + \beta\lambda\left(\aleph_{\mathrm{d}}\frac{\beta\lambda}{1-\beta\lambda^2}\tilde{\mathrm{d}}_{t|t_0} - \aleph_\pi\beta\mu\pi_t\right),$$

which simplifies to the following, which holds for all $t \geq t_0 + 1$:

$$\frac{\aleph_{\mathrm{d}}}{(1-\beta\lambda^2)}(\tilde{\mathrm{d}}_{t|t_0} - \lambda\tilde{\mathrm{d}}_{t-1|t_0}) = \aleph_\pi(\pi_t - \mu\pi_{t-1}). \quad \text{[A.13.8]}$$

Taking the differences between equations [A.13.7] and [A.13.8] and their expectations conditional on information available one period earlier, the following optimality condition is obtained that must hold for all $t \geq t_0$:

$$\frac{\aleph_{\mathrm{d}}}{(1-\beta\lambda^2)}(\tilde{\mathrm{d}}_{t|t_0} - \mathbb{E}_{t-1}\tilde{\mathrm{d}}_{t|t_0}) = \aleph_\pi(\pi_t - \mathbb{E}_{t-1}\pi_t). \quad \text{[A.13.9]}$$

For any $t \geq t_0$, by taking expectations of the second constraint in [4.14] at time $t+1$ conditional on date-t information and using the first constraint in [4.14] to cancel terms from the equation, the bond yield j_t must satisfy:

$$\mathrm{j}_t = \beta\mu\mathbb{E}_t\mathrm{j}_{t+1} + (1-\beta\mu)(\mathbb{E}_t\pi_{t+1} + \mathbb{E}_t\hat{\mathrm{r}}^*_{t+1}).$$

Iterating this equation forwards implies

$$\mathrm{j}_t = (1-\beta\mu)\sum_{\ell=0}^{\infty}(\beta\mu)^\ell\mathbb{E}_t[\pi_{t+1+\ell} + \hat{\mathrm{r}}^*_{t+\ell}] + \lim_{\ell\to\infty}(\beta\mu)^\ell\mathbb{E}_t\mathrm{j}_{t+\ell},$$

and since the final term is zero given the transversality condition for j_t in [4.14], the bond yield j_t for any

$t \geq t_0$ is:

$$\mathsf{j}_t = (1-\beta\mu)\sum_{\ell=0}^{\infty}(\beta\mu)^{\ell}\mathbb{E}_t[\pi_{t+1+\ell} + \hat{\mathsf{r}}^*_{t+1+\ell}]. \tag{A.13.10}$$

Taking the difference between the second constraint in [4.14] and its expectation conditional on information available at date $t-1$, the innovation to the debt gap for any $t \geq t_0$ must satisfy

$$\tilde{\mathsf{d}}_{t|t_0} - \mathbb{E}_{t-1}\tilde{\mathsf{d}}_{t|t_0} = -(\pi_t - \mathbb{E}_{t-1}\pi_t) - \frac{1}{1-\beta\mu}(\mathsf{j}_t - \mathbb{E}_{t-1}\mathsf{j}_t) - (\hat{\mathsf{r}}^*_t - \mathbb{E}_{t-1}\hat{\mathsf{r}}^*_t). \tag{A.13.11}$$

Using equation [A.13.10], the innovation to the bond yield j_t is:

$$\mathsf{j}_t - \mathbb{E}_{t-1}\mathsf{j}_t = (1-\beta\mu)\sum_{\ell=0}^{\infty}(\beta\mu)^{\ell}\left(\mathbb{E}_t[\pi_{t+1+\ell} + \hat{\mathsf{r}}^*_{t+1+\ell}] - \mathbb{E}_{t-1}[\pi_{t+1+\ell} + \hat{\mathsf{r}}^*_{t+1+\ell}]\right),$$

and by substituting this into [A.13.11], the following must hold for all $t \geq t_0$:

$$\tilde{\mathsf{d}}_{t|t_0} - \mathbb{E}_{t-1}\tilde{\mathsf{d}}_{t|t_0} = -\sum_{\ell=0}^{\infty}(\beta\mu)^{\ell}\left(\mathbb{E}_t[\pi_{t+\ell} + \hat{\mathsf{r}}^*_{t+\ell}] - \mathbb{E}_{t-1}[\pi_{t+\ell} + \hat{\mathsf{r}}^*_{t+\ell}]\right). \tag{A.13.12}$$

In what follows, define a variable $\wp_t$:

$$\wp_t = \sum_{\ell=0}^{\infty}(\beta\mu)^{\ell}\left(\mathbb{E}_t\hat{\mathsf{r}}^*_{t+\ell} - \mathbb{E}_{t-1}\hat{\mathsf{r}}^*_{t+\ell}\right), \tag{A.13.13}$$

where this variable is an exogenous martingale difference sequence ($\mathbb{E}_{t-1}\wp_t = 0$) by construction. Using the formula for expectations of future inflation in [A.13.5], the discounted sum of future inflation is given by

$$\sum_{\ell=0}^{\infty}(\beta\mu)^{\ell}\mathbb{E}_t\pi_{t+\ell} = \left(\sum_{\ell=0}^{\infty}\left(\beta\mu^2\right)^{\ell}\right)\pi_t = \frac{1}{1-\beta\mu^2}\pi_t,$$

which is valid for all $t \geq t_0$. This equation in conjunction with [A.13.12] and the definition of $\wp_t$ in [A.13.13] implies that the innovation to the debt gap satisfies the following for all $t \geq t_0$:

$$\tilde{\mathsf{d}}_{t|t_0} - \mathbb{E}_{t-1}\tilde{\mathsf{d}}_{t|t_0} = -\frac{1}{1-\beta\mu^2}(\pi_t - \mathbb{E}_{t-1}\pi_t) - \wp_t. \tag{A.13.14}$$

Combining this equation with the optimality condition [A.13.9] implies the following equation for the unexpected component of inflation

$$(1-\beta\lambda^2)\frac{\aleph_\pi}{\aleph_\mathsf{d}}(\pi_t - \mathbb{E}_{t-1}\pi_t) = -\frac{1}{1-\beta\mu^2}(\pi_t - \mathbb{E}_{t-1}\pi_t) - \wp_t,$$

which has the solution below for all $t \geq t_0$:

$$\pi_t - \mathbb{E}_{t-1}\pi_t = -(1-\beta\mu^2)\chi\wp_t, \quad \text{where } \chi = \left(1 + (1-\beta\mu^2)(1-\beta\lambda^2)\frac{\aleph_\pi}{\aleph_\mathsf{d}}\right)^{-1}. \tag{A.13.15a}$$

Substituting this into [A.13.14] yields the solution for the unexpected component of the debt gap:

$$\tilde{\mathsf{d}}_{t|t_0} - \mathbb{E}_{t-1}\tilde{\mathsf{d}}_{t|t_0} = -(1-\chi)\wp_t. \tag{A.13.15b}$$

Combining this with the first constraint from [4.14] confirms the solution for the optimal evolution of the debt gap in [4.15a]. By using equations [3.9a] and [3.11], the complete-markets real return $\hat{\mathsf{r}}^*_t$ with flexible prices (that is, when $\mathsf{g}_t = \hat{\mathsf{g}}_t$) is given by

$$\hat{\mathsf{r}}^*_t = \alpha\hat{\mathsf{g}}_t + (1-\alpha)\sum_{\ell=0}^{\infty}\beta^{\ell}\left(\mathbb{E}_t\hat{\mathsf{g}}_{t+\ell} - \mathbb{E}_{t-1}\hat{\mathsf{g}}_{t+\ell}\right),$$

and by substituting this into [A.13.13], $\wp_t$ can be expressed as:

$$\wp_t = (1-\alpha)\sum_{\ell=0}^{\infty}\beta^{\ell}\left(\mathbb{E}_t\hat{\mathsf{g}}_{t+\ell} - \mathbb{E}_{t-1}\hat{\mathsf{g}}_{t+\ell}\right) + \alpha\sum_{\ell=0}^{\infty}(\beta\mu)^{\ell}\left(\mathbb{E}_t\hat{\mathsf{g}}_{t+\ell} - \mathbb{E}_{t-1}\hat{\mathsf{g}}_{t+\ell}\right). \tag{A.13.16}$$

This confirms the expression for $\wp_t$ in [4.15a]. The formula for χ in [4.15b] is verified by substituting for the loss function coefficients $\aleph_{\mathrm{d}}$ and $\aleph_\pi$ in [A.13.15a] using [4.13].

(ii) Substituting the solution [4.15a] for the debt gap into equations [A.13.7] and [A.13.8] and using the formula for χ from [A.13.15a] leads to the solution for inflation given in [4.16].

(iii) With $\hat{\mathsf{Y}}_t = \mathsf{A}_t$, it follows that $\hat{\mathsf{g}}_t = \mathsf{A}_t - \mathsf{A}_{t-1}$, and hence $\hat{\mathsf{g}}_t$ is a stationary and invertible stochastic process under the assumptions in the proposition. The stochastic process for $\hat{\mathsf{g}}_t$ can be expressed in the form:

$$\hat{\mathsf{g}}_t = \Theta(\mathbb{L})\epsilon_t, \quad \text{where } \Theta(z) = \sum_{\ell=0}^{\infty} \vartheta_\ell z^\ell \text{ and } \epsilon_t = \hat{\mathsf{g}}_t - \mathbb{E}_{t-1}\hat{\mathsf{g}}_t, \tag{A.13.17}$$

for some sequence $\{\vartheta_\ell\}$ with $\vartheta_0 = 1$. The innovation ϵ_t is such that $\epsilon_{t-\ell}$ belongs to the date-t information set for all $\ell \geq 0$. The expectation of $\hat{\mathsf{g}}_{t+\ell}$ conditional on information available at time t is:

$$\mathbb{E}_t \hat{\mathsf{g}}_{t+\ell} = \sum_{\jmath=0}^{\infty} \vartheta_{\ell+\jmath}\epsilon_{t-\jmath},$$

and hence using [A.13.16], [A.13.17], and the definition of the z-transform $\Theta(z)$, the term $\wp_t$ is given by:

$$\wp_t = (\alpha\Theta(\beta\mu) + (1-\alpha)\Theta(\beta))\,\epsilon_t. \tag{A.13.18}$$

Now consider the level of weighted nominal GDP $\mathsf{N}_{\omega,t} = \mathsf{P}_t + \omega\mathsf{Y}_t$. Noting that $\mathsf{N}_{\omega,t} - \mathbb{E}_{t-1}\mathsf{N}_{\omega,t} = (\pi_t - \mathbb{E}_{t-1}\pi_t) + \omega(\mathsf{g}_t - \mathbb{E}_{t-1}\mathsf{g}_t)$, by using the solution [A.13.15a] and [A.13.17], the condition $\mathsf{N}_{\omega,t} = \mathbb{E}_{t-1}\mathsf{N}_{\omega,t}$ is equivalent to

$$\omega\Theta(0)\epsilon_t - (1-\beta\mu^2)\chi\wp_t = 0.$$

Substituting the expression for $\wp_t$ from [A.13.18] it can be seen there is a weight on real output (denoted by $\dot{\omega}$) such that weighted nominal GDP is insulated from shocks on impact (though not subsequently). This weight satisfies the following equation:

$$\Theta(0)\dot{\omega} = \chi(1-\beta\mu^2)\,(\alpha\Theta(\beta\mu) + (1-\alpha)\Theta(\beta))\,. \tag{A.13.19}$$

The weight ω^* from equation [3.15] of Proposition 5 satisfies the following equation

$$\Theta(\beta\mu)\omega^* = \alpha\Theta(\beta\mu) + (1-\alpha)\Theta(\beta), \tag{A.13.20}$$

and hence the solution of [A.13.19] is as given in [4.17b] ($\Theta(0) \neq 0$ due to the invertibility of the real GDP growth stochastic process). As this argument shows, [4.17b] is equivalent to the surprise component of inflation in [A.13.15a], and together with $\mathbb{E}_t\pi_{t+1} = \mu\pi_t$ for all $t > t_0$, the implied solution for inflation is identical to that given in [4.16].

Consider now a weight such that weighted nominal GDP is insulated from shocks in the long run, or in other words, where it is a stationary variable (or equivalently, the price level P_t and the level of real GDP Y_t are cointegrated). First, note that the solution for inflation in [4.16] can be written explicitly as follows:

$$\pi_t = -(1-\beta\mu^2)\chi\sum_{\ell=0}^{t-t_0}\mu^\ell\wp_{t-\ell},$$

or equivalently in terms of an initial condition represented by the variable z_{t_0}:

$$\pi_t = -(1-\beta\mu^2)\chi\sum_{\ell=0}^{\infty}\mu^\ell\wp_{t-\ell} - \mu^{t-t_0}\mathsf{z}_{t_0-1}, \quad \text{where } \mathsf{z}_{t_0-1} = -(1-\beta\mu^2)\chi\sum_{\ell=1}^{\infty}\mu^\ell\wp_{t_0-\ell}. \tag{A.13.21}$$

Using equation [A.13.18], inflation is thus

$$\pi_t = -\chi(1-\beta\mu^2)\,(\alpha\Theta(\beta\mu) + (1-\alpha)\Theta(\beta))\,(\mathbb{I} - \mu\mathbb{L})^{-1}\epsilon_t - \mu^{t-t_0}\mathsf{z}_{t_0-1},$$

where $\mathbb{L}$ is the lag operator and $\mathbb{I}$ is the identity operator. Together with [A.13.17], the change in weighted nominal GDP is $\Delta\mathsf{N}_{\omega,t} = \pi_t + \omega\mathsf{g}_t$ is:

$$\Delta\mathsf{N}_{\omega,t} = \left(\omega\Theta(\mathbb{L}) - \chi(1-\beta\mu^2)\,(\alpha\Theta(\beta\mu) + (1-\alpha)\Theta(\beta))\,(\mathbb{I} - \mu\mathbb{L})^{-1}\right)\epsilon_t - \mu^{t-t_0}\mathsf{z}_{t_0-1}.$$

Since $0 < \mu < 1$ and given the definition in [A.13.21], the term $\mu^{t-t_0}\mathsf{z}_{t_0-1}$ tends to zero as $t \to \infty$ for fixed t_0. Weighted nominal GDP is therefore stationary if the function of the lag operator in the equation above has a root at unity. This occurs when $\omega = \vec{\omega}$, which satisfies:

$$\Theta(1)\vec{\omega} = \frac{\chi(1-\beta\mu^2)\left(\alpha\Theta(\beta\mu) + (1-\alpha)\Theta(\beta)\right)}{1-\mu},$$

and by using equation [A.13.20] that is satisfied by ω^* from [3.15], the solution is as given in [4.17a]. The term $\Theta(1)$ in the denominator is non-zero because of the invertibility of the real GDP growth stochastic process. This completes the proof.

A.14 Proof of Proposition 14

In each time period t, the policymaker chooses inflation π_t, unconstrained by any past announcements, and at all future dates the policymaker will have the same freedom. The objective at each date t is to minimize the continuation value of the loss function [4.13]:

$$\mathscr{L}_t = \frac{1}{2}\sum_{\ell=0}^{\infty}\beta^{\ell}\mathbb{E}_t\left[\aleph_{\mathsf{d}}\tilde{\mathsf{d}}^2_{t+\ell|t} + \aleph_\pi\pi^2_{t+\ell}\right], \qquad \text{[A.14.1]}$$

where $\aleph_{\mathsf{d}}$ and $\aleph_\pi$ denote the coefficients of the debt gap and inflation in [4.13]. Policy decisions at time t must be consistent with the constraints [4.14] from t onwards, taking variables determined before time t as given. The constraints are:

$$\mathbb{E}_t\tilde{\mathsf{d}}_{t+1|t} = \lambda\tilde{\mathsf{d}}_{t|t}, \quad \frac{1}{1-\beta\mu}\mathsf{j}_{t-1} - \frac{\beta\mu}{1-\beta\mu}\mathsf{j}_t - \pi_t - \tilde{\mathsf{d}}_{t|t} = \hat{\mathsf{r}}^*_t, \quad \text{and} \quad \lim_{\ell\to\infty}(\beta\mu)^{\ell}\mathbb{E}_t\mathsf{j}_{t+\ell} = 0, \qquad \text{[A.14.2]}$$

where the definition $\tilde{\mathsf{d}}_{t-1|t} = 0$ has been used. The debt gap $\tilde{\mathsf{d}}_{t+\ell|t} = \mathsf{d}_{t+\ell} - \mathsf{d}^*_{t+\ell|t}$ is defined relative to the debt-to-GDP ratio $\mathsf{d}^*_{t+\ell|t}$ with complete markets open from t onwards. Using equation [3.30a] of Proposition 11 at $t+\ell$ with $t_0 = t$, for each $\ell \geq 0$, $\mathsf{d}^*_{t+\ell|t}$ is given by:

$$\mathsf{d}^*_{t+\ell|t} = \mathsf{d}^*_t + \lambda^{\ell+1}(\mathsf{d}_{t-1} - \mathsf{d}^*_{t-1}),$$

with d^*_t from [3.11]. Repeated use of this equation together with the definition of the debt gap allows the following to be deduced for any $\ell \geq 0$:

$$\begin{aligned}\tilde{\mathsf{d}}_{t+\ell|t} &= \mathsf{d}_{t+\ell} - \mathsf{d}^*_{t+\ell|t} = (\mathsf{d}_{t+\ell} - \mathsf{d}^*_{t+\ell|t+\ell}) + (\mathsf{d}^*_{t+\ell|t+\ell} - \mathsf{d}^*_{t+\ell|t}) = \tilde{\mathsf{d}}_{t+\ell|t+\ell} \\ &+ \left(\mathsf{d}^*_{t+\ell} + \lambda(\mathsf{d}_{t+\ell-1} - \mathsf{d}^*_{t+\ell-1})\right) - \left(\mathsf{d}^*_{t+\ell} + \lambda^{\ell+1}(\mathsf{d}_{t-1} - \mathsf{d}^*_{t-1})\right) = \tilde{\mathsf{d}}_{t+\ell|t+\ell} \\ &\qquad + \lambda\left(\mathsf{d}_{t+\ell-1} - \left(\mathsf{d}^*_{t+\ell-1} + \lambda^{\ell}(\mathsf{d}_{t-1} - \mathsf{d}^*_{t-1})\right)\right) = \tilde{\mathsf{d}}_{t+\ell|t+\ell} + \lambda\tilde{\mathsf{d}}_{t+\ell-1|t}. \end{aligned} \qquad \text{[A.14.3]}$$

Taking expectations of this equation in the case of $\ell = 1$ conditional on date-t information implies $\mathbb{E}_t\tilde{\mathsf{d}}_{t+1|t+1} = \mathbb{E}_t\tilde{\mathsf{d}}_{t+1|t} - \lambda\tilde{\mathsf{d}}_{t|t} = 0$, using the first constraint in [A.14.2]. The law of iterated expectations then implies for all $\ell \geq 1$:

$$\mathbb{E}_t\tilde{\mathsf{d}}_{t+\ell|t+\ell} = 0. \qquad \text{[A.14.4]}$$

Equation [A.14.3] can be iterated backwards until $\tilde{\mathsf{d}}_{t-1|t} = 0$ is reached to obtain the following:

$$\tilde{\mathsf{d}}_{t+\ell|t} = \sum_{\jmath=0}^{\ell}\lambda^{\ell-\jmath}\tilde{\mathsf{d}}_{t+\jmath|t+\jmath}. \qquad \text{[A.14.5]}$$

This equation can be used to convert the sum of terms in $\tilde{\mathsf{d}}_{t+\ell|t}$ in the continuation loss function [A.14.6] into a sum of terms in $\tilde{\mathsf{d}}_{t+\ell|t+\ell}$:

$$\sum_{\ell=0}^{\infty}\beta^{\ell}\mathbb{E}_t\tilde{\mathsf{d}}^2_{t+\ell|t} = \sum_{\ell=0}^{\infty}\beta^{\ell}\mathbb{E}_t\left[\left(\sum_{\jmath=0}^{\ell}\lambda^{\ell-\jmath}\tilde{\mathsf{d}}_{t+\jmath|t+\jmath}\right)^2\right]$$
$$= \sum_{\ell=0}^{\infty}\beta^{\ell}\mathbb{E}_t\left[\sum_{\jmath=0}^{\ell}(\lambda^2)^{\ell-\jmath}\tilde{\mathsf{d}}^2_{t+\jmath|t+\jmath} + 2\sum_{\jmath=1}^{\ell}\sum_{\imath=0}^{\jmath-1}\lambda^{\ell-\jmath}\lambda^{\ell-\imath}\tilde{\mathsf{d}}_{t+\imath|t+\imath}\tilde{\mathsf{d}}_{t+\jmath|t+\jmath}\right] = \left(\sum_{\jmath=0}^{\infty}(\beta\lambda^2)^{\jmath}\right)\sum_{\ell=0}^{\infty}\beta^{\ell}\mathbb{E}_t\tilde{\mathsf{d}}^2_{t+\ell|t+\ell},$$

where the second line uses the law of iterated expectations together with $\mathbb{E}_{t+\imath}\tilde{\mathsf{d}}_{t+\jmath|t+\jmath} = 0$ for $0 \leq \imath < \jmath$ (from [A.14.4]), and then the order of summation is changed for the remaining terms. Therefore, the continuation loss function [A.14.6] at time t can be expressed as:

$$\mathscr{L}_t = \frac{1}{2}\sum_{\ell=0}^{\infty}\beta^{\ell}\mathbb{E}_t\left[\frac{\aleph_{\mathrm{d}}}{1-\beta\lambda^2}\tilde{\mathsf{d}}^2_{t+\ell|t+\ell} + \aleph_{\pi}\pi^2_{t+\ell}\right]. \quad \text{[A.14.6]}$$

Taking expectations of the second constraint in [A.14.2] at $t+1$ conditional on date-t information, and using the result from [A.14.4], all bond yields from t onwards must satisfy

$$\mathsf{j}_t = \beta\mu\mathbb{E}_t\mathsf{j}_{t+1} + (1-\beta\mu)\mathbb{E}_t[\pi_{t+1} + \hat{\mathsf{r}}^*_{t+1}].$$

By iterating this equation forwards and using the transversality condition from [A.14.2], the policymaker faces the following constraint on the determination of bond yields:

$$\mathsf{j}_t = (1-\beta\mu)\sum_{\ell=1}^{\infty}(\beta\mu)^{\ell-1}\mathbb{E}_t[\pi_{t+\ell} + \hat{\mathsf{r}}^*_{t+\ell}]. \quad \text{[A.14.7]}$$

The implied value of $\tilde{\mathsf{d}}_{t|t}$ can be deduced by substituting this constraint on the bond yield into the second equation from [A.14.2] and using the definition of $\wp_t$ from [A.13.13]:

$$\tilde{\mathsf{d}}_{t|t} = -(\pi_t - \mathbb{E}_{t-1}\pi_t) - \sum_{\ell=1}^{\infty}(\beta\mu)^{\ell}(\mathbb{E}_t\pi_{t+\ell} - \mathbb{E}_{t-1}\pi_{t+\ell}) - \wp_t. \quad \text{[A.14.8]}$$

Given a sequence of inflation rates, if the debt gap $\tilde{\mathsf{d}}_{t|t}$ is determined by the equation above at all dates then all the constraints in [A.14.2] are satisfied. Together with the continuation loss function in [A.14.6], equation [A.14.8] allows the discretionary policy problem to be reduced to a choice of two endogenous variables $\tilde{\mathsf{d}}_{t|t}$ and π_t subject to a single constraint [A.14.8] at each date with $\wp_t$ as an exogenous variable. The definition in [A.13.13] implies $\wp_t$ is a martingale difference sequence ($\mathbb{E}_{t-1}\wp_t = 0$) that depends on revisions to forecasts of the exogenous variable $\hat{\mathsf{r}}^*_t$.

As can be seen from equation [A.14.8], $\tilde{\mathsf{d}}_t$ is not a state variable. It depends only on the surprise component of inflation and any forecast revisions to future inflation, together with the exogenous variable $\wp_t$. Since there is no policy action at time t that can affect future inflation surprises or future forecast revisions of inflation, current policy has no effect on the sequences of future debt gaps $\tilde{\mathsf{d}}_{t+\ell|t+\ell}$ and inflation rates $\pi_{t+\ell}$ that satisfy the constraints from $t+\ell$ onwards for any $\ell \geq 1$. With the continuation loss function $\mathscr{L}_{t+\ell}$ (from [A.14.6]) independent of policy actions at time t for $\ell \geq 1$, it follows that in a Markovian discretionary policy equilibrium, current policy actions have no effect on date-t expectations of $\tilde{\mathsf{d}}_{t+\ell|t+\ell}$ or $\pi_{t+\ell}$ for any $\ell \geq 1$. These expectations can be taken as given when the date-t policy decision is made.

The loss function $\mathscr{L}_t$ is thus separated into a component $\mathfrak{L}_t$ that depends on current policy choices and a component depending on future expectations that are taken as given:

$$\mathscr{L}_t = \mathfrak{L}_t + \frac{1}{2}\sum_{\ell=1}^{\infty}\beta^{\ell}\mathbb{E}_t\left[\frac{\aleph_{\mathrm{d}}}{1-\beta\lambda^2}\tilde{\mathsf{d}}^2_{t+\ell|t+\ell} + \aleph_{\pi}\pi^2_{t+\ell}\right], \quad \text{where } \mathfrak{L}_t = \frac{1}{2}\left(\frac{\aleph_{\mathrm{d}}}{1-\beta\lambda^2}\tilde{\mathsf{d}}^2_{t|t} + \aleph_{\pi}\pi^2_t\right). \quad \text{[A.14.9]}$$

Discretionary policy at date t therefore minimizes the period loss function $\mathfrak{L}_t$ subject to the constraint [A.14.8] at date t, taking all expectations of future variables as given. Using [A.14.8] to substitute for $\tilde{\mathsf{d}}_{t|t}$

in $\mathfrak{L}_t$ from [A.14.9]:

$$\mathfrak{L}_t = \frac{1}{2}\left(\frac{\aleph_{\mathrm{d}}}{1-\beta\lambda^2}\left(-(\pi_t - \mathbb{E}_{t-1}\pi_t) - \sum_{\ell=1}^{\infty}(\beta\mu)^{\ell}(\mathbb{E}_t\pi_{t+\ell} - \mathbb{E}_{t-1}\pi_{t+\ell}) - \wp_t\right)^2 + \aleph_\pi\pi_t^2\right).$$

Taking expectations of inflation as given, the first-order condition with respect to current inflation π_t is:

$$\frac{\aleph_{\mathrm{d}}}{1-\beta\lambda^2}\left(-(\pi_t - \mathbb{E}_{t-1}\pi_t) - \sum_{\ell=1}^{\infty}(\beta\mu)^{\ell}(\mathbb{E}_t\pi_{t+\ell} - \mathbb{E}_{t-1}\pi_{t+\ell}) - \wp_t\right) = \aleph_\pi\pi_t,$$

and this implies the following using the constraint [A.14.8] again:

$$\tilde{\mathsf{d}}_{t|t} = (1-\beta\lambda^2)\frac{\aleph_\pi}{\aleph_{\mathrm{d}}}\pi_t. \qquad \text{[A.14.10]}$$

The Markovian discretionary policy equilibrium is characterized by the constraint [A.14.8] and the first-order condition [A.14.10]. Using the law of iterated expectations and $\mathbb{E}_{t-1}\wp_t = 0$, equation [A.14.8] implies $\mathbb{E}_{t-1}\tilde{\mathsf{d}}_{t|t} = 0$. Together with [A.14.10], it follows that $\mathbb{E}_{t-1}\pi_t = 0$ for all t, and hence inflation π_t must be a martingale difference sequence in equilibrium (and therefore serially uncorrelated). The law of iterated expectations then implies $\mathbb{E}_t\pi_{t+\ell} = 0$ for all $\ell \geq 1$, and so equation [A.14.7] reduces to:

$$\tilde{\mathsf{d}}_{t|t} = -\pi_t - \wp_t.$$

Combining this equation with the first-order condition [A.14.10], the Markovian discretionary policy equilibrium is:

$$\tilde{\mathsf{d}}_{t|t} = -(1-\chi')\wp_t, \quad \text{and} \quad \pi_t = -\chi'\wp_t, \quad \text{where } \chi' = \left(1 + (1-\beta\lambda^2)\frac{\aleph_\pi}{\aleph_{\mathrm{d}}}\right)^{-1}. \qquad \text{[A.14.11]}$$

This confirms the solution for inflation in [4.18a]. The solution for $\tilde{\mathsf{d}}_{t|t}$ above can be converted into a solution for $\tilde{\mathsf{d}}_{t|t_0}$ for any arbitrarily fixed date t_0 (even though reoptimization occurs at all dates with discretion). Using equation [A.14.3] and [A.14.11], the stochastic process for $\tilde{\mathsf{d}}_{t|t_0}$ in [4.18a] is obtained. Substituting the loss function coefficients $\aleph_{\mathrm{d}}$ and $\aleph_\pi$ from [4.13] into the formula for χ' in [A.14.11] confirms the expression for χ' in [4.18b].

By comparing χ and χ' in [4.15b] and [4.18b] it can be seen that $\chi' < \chi$ whenever $\mu > 0$. The solutions in [4.15a] and [4.18a] show that the debt gap $\tilde{\mathsf{d}}_{t|t_0}$ has serial correlation coefficient λ under both commitment and discretion. Since $\chi' < \chi$, it follows that the standard deviation of $\tilde{\mathsf{d}}_{t|t_0}$ is higher under discretion than under commitment. The solutions in [4.16] and [4.18a] show that optimal monetary policy with commitment implies inflation has serial correlation coefficient μ, while the Markovian discretionary policy equilibrium [A.14.11] features serially uncorrelated inflation. Thus, inflation is less persistent with discretion whenever $\mu > 0$. This completes the proof.

A.15 Proof of Proposition 15

With a zero-inflation non-stochastic steady state ($\bar{\pi} = 0$), it can be seen from equation [4.12] that the implied steady-state value of x_t is $\bar{x} = 1 - \varepsilon^{-1}$. Making use of this, equation [4.12] can be log linearized around the steady state as follows:

$$\sum_{\ell=0}^{\infty}(\sigma\beta)^{\ell}\mathbb{E}_t\left[\left((1+\varepsilon\xi)\left(\frac{\sigma}{1-\sigma}\right)\pi_t - (1-\varepsilon)\sum_{\jmath=1}^{\ell}\pi_{t+\jmath}\right) - \left(\mathsf{x}_{t+\ell} + \varepsilon(1+\xi)\sum_{\jmath=1}^{\ell}\pi_{t+\jmath}\right)\right] = 0,$$

which simplifies to the following equation:

$$\sum_{\ell=0}^{\infty}(\sigma\beta)^{\ell}\mathbb{E}_t\left[(1+\varepsilon\xi)\left(\frac{\sigma}{1-\sigma}\pi_t - \sum_{\jmath=1}^{\ell}\pi_{t+\jmath}\right) - \mathsf{x}_{t+\ell}\right] = 0.$$

By changing the order of summation, the equation is equivalent to:

$$(1+\varepsilon\xi)\left(\frac{\sigma}{(1-\sigma)(1-\sigma\beta)}\pi_t - \frac{\sigma\beta}{1-\sigma\beta}\sum_{\ell=0}^{\infty}(\sigma\beta)^{\ell}\mathbb{E}_t\pi_{t+1+\ell}\right) - \sum_{\ell=0}^{\infty}(\sigma\beta)^{\ell}\mathbb{E}_t x_{t+\ell} = 0,$$

and since this must hold for all t, subtracting from the above $\sigma\beta$ multiplied by the expectation of the equation at time $t+1$ conditional on date-t information leads to:

$$\frac{\sigma(1+\varepsilon\xi)}{(1-\sigma)(1-\sigma\beta)}\left((\pi_t - \sigma\beta\mathbb{E}_t\pi_{t+1}) - \beta(1-\sigma)\mathbb{E}_t\pi_{t+1}\right) - x_t = 0.$$

Therefore, the following Phillips curve is obtained:

$$\kappa(\pi_t - \beta\mathbb{E}_t\pi_{t+1}) = x_t, \quad \text{where} \ \ \kappa = \frac{\sigma(1+\varepsilon\xi)}{(1-\sigma)(1-\sigma\beta)}. \qquad [A.15.1]$$

Now consider the expression for real marginal cost in [5.13]. This has the following log-linear form

$$x_t = \Psi_{t|t_0} + \left(\frac{1+\xi}{\eta}\right)\Delta_t + \nu\tilde{Y}_{t|t_0}, \qquad [A.15.2]$$

as does the wedge $\Psi_{t|t_0}$ also defined in [5.13]:

$$\Psi_{t|t_0} = \frac{\alpha}{\frac{\eta_b}{1+\eta_b} + \frac{\eta_s}{1+\eta_s}}\left(\frac{\eta_b}{1+\eta_b}\tilde{c}_{b,t|t_0} + \frac{\eta_s}{1+\eta_s}\tilde{c}_{s,t|t_0}\right). \qquad [A.15.3]$$

Suppose the Frisch elasticities are the same for both types of household ($\eta_\tau = \eta$), in which case equation [A.15.3] reduces to:

$$\Psi_{t|t_0} = \alpha\left(\frac{1}{2}\tilde{c}_{b,t|t_0} + \frac{1}{2}\tilde{c}_{s,t|t_0}\right).$$

The log linearizations derived in Proposition 11 remain valid here, and substituting the equations for the consumption gaps from [3.30b] allows the above to be stated in terms of the debt gap $\tilde{d}_{t|t_0}$:

$$\Psi_{t|t_0} = \alpha\left(\frac{1}{2}\left(\frac{\theta}{1+\theta}\right) - \frac{1}{2}\left(\frac{\theta}{1-\theta}\right)\right)\left(\frac{1-\beta\lambda}{1-\beta}\right)\tilde{d}_{t|t_0} = -\alpha\left(\frac{\theta^2}{1-\theta^2}\right)\left(\frac{1-\beta\lambda}{1-\beta}\right)\tilde{d}_{t|t_0}. \qquad [A.15.4]$$

Now consider the case where the Frisch elasticities are heterogeneous, in particular:

$$\eta_b = \frac{(1-\theta)\eta}{1+\theta\eta}, \quad \text{and} \ \ \eta_s = \frac{(1+\theta)\eta}{1-\theta\eta}, \qquad [A.15.5]$$

which are well defined if $\eta < \theta^{-1}$. These formulas imply

$$\frac{\eta_b}{1+\eta_b} = \frac{\eta(1-\theta)}{1+\eta}, \quad \text{and} \quad \frac{\eta_s}{1+\eta_s} = \frac{\eta(1+\theta)}{1+\eta}, \qquad [A.15.6]$$

and by substituting the above into equation [A.15.3]:

$$\Psi_{t|t_0} = \frac{\alpha}{\frac{2\eta}{1+\eta}}\left(\frac{\eta(1-\theta)}{1+\eta}\tilde{c}_{b,t|t_0} + \frac{\eta(1+\theta)}{1+\eta}\tilde{c}_{s,t|t_0}\right) = \alpha\left(\frac{(1-\theta)}{2}\tilde{c}_{b,t|t_0} + \frac{(1+\theta)}{2}\tilde{c}_{s,t|t_0}\right) = 0. \qquad [A.15.7]$$

In summary, the log linearization of $\Psi_{t|t_0}$ is:

$$\Psi_{t|t_0} = -\psi\tilde{d}_{t|t_0}, \quad \text{where} \ \ \psi = \begin{cases} \alpha\left(\frac{\theta^2}{1-\theta^2}\right)\left(\frac{1-\beta\lambda}{1-\beta}\right) & \text{if } \eta_\tau = \eta \\ 0 & \text{if } \eta_\tau \text{ as given in [A.15.5]} \end{cases}. \qquad [A.15.8]$$

Hence, using this equation and the result $\Delta_t = \mathscr{O}^2$ from Proposition 12, equation [A.15.2] becomes:

$$x_t = \nu\tilde{Y}_{t|t_0} - \psi\tilde{d}_{t|t_0}. \qquad [A.15.9]$$

The Phillips curve [5.14a] is then obtained by combining equations [A.15.1] and [A.15.9], with the expressions in [5.16a] and [5.16b] for ψ confirmed using [A.15.8].

Using the results from Proposition 4, the real return $r^*_t = r^*_{t|t_0}$ in the hypothetical case of complete markets given the actual real GDP growth rate g_t is:

$$\mathsf{r}_t^* = \alpha\mathsf{g}_t + (1-\alpha)\sum_{\ell=0}^{\infty}\beta^{\ell}\left(\mathbb{E}_t\mathsf{g}_{t+\ell} - \mathbb{E}_{t-1}\mathsf{g}_{t+\ell}\right). \qquad [A.15.10]$$

Using the definition of the growth rate $\mathsf{g}_t = \mathsf{Y}_t - \mathsf{Y}_{t-1}$ and the output gap $\tilde{\mathsf{Y}}_{t|t_0} = \mathsf{Y}_t - \mathsf{Y}^*_{t|t_0}$ it follows that $\mathsf{g}_t = \tilde{\mathsf{Y}}_t - \tilde{\mathsf{Y}}_{t-1} + \hat{\mathsf{g}}^*_{t|t_0}$, and hence:

$$\begin{aligned}\sum_{\ell=0}^{\infty}\beta^{\ell}\mathbb{E}_t\mathsf{g}_{t+\ell} &= \sum_{\ell=0}^{\infty}\beta^{\ell}\mathbb{E}_t\left[\tilde{\mathsf{Y}}_{t+\ell|t_0} - \tilde{\mathsf{Y}}_{t+\ell-1|t_0}\right] + \sum_{\ell=0}^{\infty}\beta^{\ell}\mathbb{E}_t\hat{\mathsf{g}}^*_{t+\ell|t_0} \\ &= (1-\beta)\left(\sum_{\ell=0}^{\infty}\beta^{\ell}\mathbb{E}_t\tilde{\mathsf{Y}}_{t+\ell|t_0}\right) - \tilde{\mathsf{Y}}_{t-1|t_0} + \sum_{\ell=0}^{\infty}\beta^{\ell}\mathbb{E}_t\hat{\mathsf{g}}^*_{t+\ell|t_0}, \qquad [A.15.11]\end{aligned}$$

where the second line is obtained by collecting terms in $\tilde{\mathsf{Y}}_{t|t_0}$. Iterating forwards the Phillips curve equation [5.14a] leads to:

$$\frac{\kappa}{\nu}\left(\pi_t - \lim_{\ell\to\infty}\beta^{\ell}\mathbb{E}_t\pi_{t+\ell}\right) = \sum_{\ell=0}^{\infty}\beta^{\ell}\mathbb{E}_t\tilde{\mathsf{Y}}_{t+\ell|t_0} - \frac{\psi}{\nu}\sum_{\ell=0}^{\infty}\beta^{\ell}\mathbb{E}_t\tilde{\mathsf{d}}_{t+\ell|t_0}.$$

Given that $\mathsf{i}_t = \rho_t + \mathbb{E}_t\pi_{t+1}$ and $\rho_t = \alpha\mathbb{E}_t\mathsf{g}_{t+1}$ from [3.9a] and [3.10], since real GDP growth g_t is stationary, the restriction that i_t is stationary can only be satisfied if inflation π_t is stationary. This requires $\lim_{\ell\to\infty}\beta^{\ell}\mathbb{E}_t\pi_{t+\ell} = 0$. The stationarity restriction is seen below to be without loss of generality in the optimal monetary policy problem. Using the limit together with $\mathbb{E}_t\tilde{\mathsf{d}}_{t+\ell|t_0} = \lambda^{\ell}\tilde{\mathsf{d}}_{t|t_0}$ from [3.30b], the equation above becomes:

$$(1-\beta)\sum_{\ell=0}^{\infty}\beta^{\ell}\mathbb{E}_t\tilde{\mathsf{Y}}_{t+\ell|t_0} = \frac{(1-\beta)\kappa}{\nu}\pi_t + \frac{\psi}{\nu}(1-\beta)\left(\sum_{\ell=0}^{\infty}(\beta\lambda)^{\ell}\right)\tilde{\mathsf{d}}_{t|t_0} = \frac{(1-\beta)\kappa}{\nu}\pi_t + \frac{\psi(1-\beta)}{\nu(1-\beta\lambda)}\tilde{\mathsf{d}}_{t|t_0},$$

and substituting this result into [A.15.11] leads to:

$$\sum_{\ell=0}^{\infty}\beta^{\ell}\mathbb{E}_t\mathsf{g}_{t+\ell} = \frac{(1-\beta)\kappa}{\nu}\pi_t + \frac{\psi(1-\beta)}{\nu(1-\beta\lambda)}\tilde{\mathsf{d}}_{t|t_0} - \tilde{\mathsf{Y}}_{t-1|t_0} + \sum_{\ell=0}^{\infty}\beta^{\ell}\mathbb{E}_t\hat{\mathsf{g}}^*_{t+\ell|t_0}. \qquad [A.15.12]$$

Now let $\hat{\mathsf{r}}^*_{t|t_0}$ denote the value of r^*_t from [A.15.10] in the hypothetical case where g_t is equal to $\hat{\mathsf{g}}^*_{t|t_0}$ for all $t \geq t_0$:

$$\hat{\mathsf{r}}^*_{t|t_0} = \alpha\hat{\mathsf{g}}^*_{t|t_0} + (1-\alpha)\sum_{\ell=0}^{\infty}\beta^{\ell}\left(\mathbb{E}_t\hat{\mathsf{g}}^*_{t+\ell|t_0} - \mathbb{E}_{t-1}\hat{\mathsf{g}}^*_{t+\ell|t_0}\right).$$

Combining this with equations [A.15.10] and [A.15.12] it follows that r^*_t is:

$$\mathsf{r}^*_t = \alpha(\tilde{\mathsf{Y}}_{t|t_0} - \tilde{\mathsf{Y}}_{t-1|t_0}) + \frac{(1-\alpha)(1-\beta)\kappa}{\nu}(\pi_t - \mathbb{E}_{t-1}\pi_t) + \frac{(1-\alpha)(1-\beta)\psi}{\nu(1-\beta\lambda)}(\tilde{\mathsf{d}}_t - \mathbb{E}_{t-1}\tilde{\mathsf{d}}_t) + \hat{\mathsf{r}}^*_{t|t_0},$$

and by substituting this into [3.13], the second equation of [5.14b] is obtained. The first and third equations follow respectively from the results in Proposition 11 and Proposition 4.

The welfare function [3.26] is evaluated at Pareto weights $\hat{\Omega}^*_{\tau|t_0} = {c^*}^{\alpha}_{\tau,t|t_0}/{\hat{Y}^*}^{1-\alpha}_{t|t_0}$ associated with the consumption allocation $c^*_{\tau,t|t_0}$ with complete financial markets from t_0 onwards (which depends only on variables that are predetermined at date t_0, as shown in Proposition 11), and the level of real GDP $\hat{Y}^*_{t|t_0}$ with flexible prices and complete markets from [5.12]. With $\delta^*_{\tau,t|t_0}$ denoting the given discount factors associated with the complete-markets consumption allocation, these variables must satisfy:

$$\hat{\Omega}^*_{\tau|t_0}\left\{\prod_{\ell=1}^{t-t_0}\delta^*_{\tau,t-\ell|t_0}\right\} = {c^*}^{\alpha}_{\tau,t|t_0}\hat{\Phi}^*_{t|t_0}{\hat{Y}^*}^{\alpha}_{t|t_0},$$

where $\hat{\Phi}^*_{t|t_0}$ denotes the sequence of Lagrangian multipliers on the resource constraint in the social planner's problem. Following the same steps as in Proposition 12, with the normalization $\hat{\Phi}^*_{t_0|t_0} = 1/\hat{Y}^*_{t_0|t_0}$ and the definition of the transformed multipliers $\hat{\varphi}^*_{t|t_0} = \hat{Y}^*_{t|t_0}\hat{\Phi}^*_{t|t_0}/\beta^{t-t_0}$, by substituting the utility function [5.1] into [3.26], the welfare function is:

$$\mathscr{W}_{t_0} = \sum_{t=t_0}^{\infty}\beta^{t-t_0}\mathbb{E}_{t_0}\left[\frac{\hat{\varphi}^*_{t|t_0}}{\hat{Y^*}^{1-\alpha}_{t|t_0}}\left\{\frac{{c^*}^{\alpha}_{\mathrm{b},t|t_0}}{2}\left(\frac{C^{1-\alpha}_{\mathrm{b},t}}{1-\alpha}-\frac{H^{1+\frac{1}{\eta_\mathrm{b}}}_{\mathrm{b},t}}{1+\frac{1}{\eta_\mathrm{b}}}\right)+\frac{{c^*}^{\alpha}_{\mathrm{s},t|t_0}}{2}\left(\frac{C^{1-\alpha}_{\mathrm{s},t}}{1-\alpha}-\frac{H^{1+\frac{1}{\eta_\mathrm{s}}}_{\mathrm{s},t}}{1+\frac{1}{\eta_\mathrm{s}}}\right)\right\}\right]. \quad [\text{A.15.13}]$$

The transformed Lagrangian multipliers $\hat{\varphi}^*_{t|t_0}$ have a well-defined steady-state value and are independent of policy from t_0 onwards following the argument in Proposition 11 and Proposition 12.

Using the definition of the variable $\Upsilon_{t|t_0}$ in equation [A.11.18], the terms in consumption from [A.15.13] can be written as

$$\frac{{c^*}^{\alpha}_{\mathrm{b},t|t_0}}{2}\frac{C^{1-\alpha}_{\mathrm{b},t}}{1-\alpha}+\frac{{c^*}^{\alpha}_{\mathrm{s},t|t_0}}{2}\frac{C^{1-\alpha}_{\mathrm{s},t}}{1-\alpha} = \frac{Y^{1-\alpha}_{t|t_0}}{1-\alpha}\left(\frac{{c^*}^{\alpha}_{\mathrm{b},t|t_0}c^{1-\alpha}_{\mathrm{b},t}+{c^*}^{\alpha}_{\mathrm{s},t|t_0}c^{1-\alpha}_{\mathrm{s},t}}{2}\right) = \frac{\Upsilon^{-(1-\alpha)}_{t|t_0}Y^{1-\alpha}_{t|t_0}}{1-\alpha}. \quad [\text{A.15.14}]$$

The terms in hours from [A.15.13] can be analysed using the labour supply and demand equations in [5.3] and [5.8] as follows:

$$\frac{w_t H_t}{1-\varepsilon^{-1}} = w_{\tau,t}H_{\tau,t} = C^{\alpha}_{\tau,t}H^{1+\frac{1}{\eta_\tau}}_{\tau,t}, \quad \text{and hence} \quad H^{1+\frac{1}{\eta_\tau}}_{\tau,t} = c^{-\alpha}_{\tau,t}\frac{w_t H_t Y^{-\alpha}_t}{1-\varepsilon^{-1}}. \quad [\text{A.15.15}]$$

The terms in aggregate variables on the right-hand side of the second equation can be rewritten using the aggregate production function from [5.8] and the expression for wages in [5.10]:

$$\frac{w_t H_t Y^{-\alpha}_t}{1-\varepsilon^{-1}} = \left(\frac{\Delta_t Y_t}{A_t}\right)^{(1+\xi)\left(1+\frac{1}{\eta}\right)}\left(c^{\frac{\eta_\mathrm{b}}{1+\eta_\mathrm{b}}}_{\mathrm{b},t}c^{\frac{\eta_\mathrm{s}}{1+\eta_\mathrm{s}}}_{\mathrm{s},t}\right)^{\frac{\alpha}{\frac{\eta_\mathrm{b}}{1+\eta_\mathrm{b}}+\frac{\eta_\mathrm{s}}{1+\eta_\mathrm{s}}}},$$

and by using equation [5.12], A_t can be replaced by a term in the first-best level of output $\hat{Y}^*_{t|t_0}$:

$$\frac{w_t H_t Y^{-\alpha}_t}{1-\varepsilon^{-1}} = \frac{1}{1+\xi}\left(\tilde{c}^{\frac{\eta_\mathrm{b}}{1+\eta_\mathrm{b}}}_{\mathrm{b},t|t_0}\tilde{c}^{\frac{\eta_\mathrm{s}}{1+\eta_\mathrm{s}}}_{\mathrm{s},t|t_0}\right)^{\frac{\alpha}{\frac{\eta_\mathrm{b}}{1+\eta_\mathrm{b}}+\frac{\eta_\mathrm{s}}{1+\eta_\mathrm{s}}}}\frac{\Delta_t^{(1+\xi)\left(1+\frac{1}{\eta}\right)}Y_t^{(1+\xi)\left(1+\frac{1}{\eta}\right)}}{\hat{Y^*}^{\alpha+\xi+\frac{1+\xi}{\eta}}_{t|t_0}},$$

where $\tilde{c}_{\tau,t|t_0} = c_{\tau,t}/c^*_{\tau,t|t_0}$. Noting the definitions of the output gap $\tilde{Y}_{t|t_0} = Y_t/\hat{Y}^*_{t|t_0}$ and the variable $\Psi_{t|t_0}$ in [5.13], the equation above implies:

$$\frac{w_t H_t Y^{-\alpha}_t}{(1-\varepsilon^{-1})\hat{Y^*}^{1-\alpha}_{t|t_0}} = \frac{1}{1+\xi}\Psi_{t|t_0}\Delta_t^{(1+\xi)\left(1+\frac{1}{\eta}\right)}\tilde{Y}^{(1+\xi)\left(1+\frac{1}{\eta}\right)}_{t|t_0}. \quad [\text{A.15.16}]$$

Combining equations [A.15.14], [A.15.15], and [A.15.16] and using the definitions of $\tilde{c}_{\tau,t|t_0}$ and $\tilde{Y}_{\tau,t|t_0}$ again:

$$\frac{1}{\hat{Y^*}^{1-\alpha}_{t|t_0}}\left\{\frac{{c^*}^{\alpha}_{\mathrm{b},t|t_0}}{2}\left(\frac{C^{1-\alpha}_{\mathrm{b},t}}{1-\alpha}-\frac{H^{1+\frac{1}{\eta_\mathrm{b}}}_{\mathrm{b},t}}{1+\frac{1}{\eta_\mathrm{b}}}\right)+\frac{{c^*}^{\alpha}_{\mathrm{s},t|t_0}}{2}\left(\frac{C^{1-\alpha}_{\mathrm{s},t}}{1-\alpha}-\frac{H^{1+\frac{1}{\eta_\mathrm{s}}}_{\mathrm{s},t}}{1+\frac{1}{\eta_\mathrm{s}}}\right)\right\}$$

$$= \frac{\Upsilon^{-(1-\alpha)}_{t|t_0}\tilde{Y}^{1-\alpha}_{t|t_0}}{1-\alpha} - \left(\frac{1}{2}\frac{\eta_\mathrm{b}}{1+\eta_\mathrm{b}}\tilde{c}^{-\alpha}_{\mathrm{b},t|t_0}+\frac{1}{2}\frac{\eta_\mathrm{s}}{1+\eta_\mathrm{s}}\tilde{c}^{-\alpha}_{\mathrm{s},t|t_0}\right)\frac{\Psi_{t|t_0}\Delta_t^{(1+\xi)\left(1+\frac{1}{\eta}\right)}\tilde{Y}^{(1+\xi)\left(1+\frac{1}{\eta}\right)}_{t|t_0}}{1+\xi}. \quad [\text{A.15.17}]$$

Define as follows a variable $\Xi_{t|t_0}$ in terms of the consumption gaps $\tilde{c}_{\tau,t|t_0}$:

$$\Xi_{t|t_0} = \left(\tilde{c}^{\frac{\eta_\mathrm{b}}{1+\eta_\mathrm{b}}}_{\mathrm{b},t|t_0}\tilde{c}^{\frac{\eta_\mathrm{s}}{1+\eta_\mathrm{s}}}_{\mathrm{s},t|t_0}\right)^{\frac{\alpha}{\frac{\eta_\mathrm{b}}{1+\eta_\mathrm{b}}+\frac{\eta_\mathrm{s}}{1+\eta_\mathrm{s}}}}\left(\frac{\frac{\eta_\mathrm{b}}{1+\eta_\mathrm{b}}\tilde{c}^{-\alpha}_{\mathrm{b},t|t_0}+\frac{\eta_\mathrm{s}}{1+\eta_\mathrm{s}}\tilde{c}^{-\alpha}_{\mathrm{s},t|t_0}}{\frac{\eta_\mathrm{b}}{1+\eta_\mathrm{b}}+\frac{\eta_\mathrm{s}}{1+\eta_\mathrm{s}}}\right), \quad [\text{A.15.18}]$$

and with this definition, the definition of $\Psi_{t|t_0}$ in [5.13], and equation [A.15.17], the welfare function [A.15.13] can be written as:

$$\mathscr{W}_{t_0} = \sum_{t=t_0}^{\infty}\beta^{t-t_0}\mathbb{E}_{t_0}\left[\hat{\varphi}^*_{t|t_0}\left\{\frac{\Upsilon^{-(1-\alpha)}_{t|t_0}\tilde{Y}^{1-\alpha}_{t|t_0}}{1-\alpha}-\frac{\Xi_{t|t_0}\Delta_t^{(1+\xi)\left(1+\frac{1}{\eta}\right)}\tilde{Y}^{(1+\xi)\left(1+\frac{1}{\eta}\right)}_{t|t_0}}{(1+\xi)\left(1+\frac{1}{\eta}\right)}\right\}\right]. \quad [\text{A.15.19}]$$

Now consider the variable $\mathit{\Xi}_{t|t_0}$ defined in [A.15.18]. Using Proposition 1, this variable has a well-defined steady-state value given by $\bar{\mathit{\Xi}} = 1$ whatever assumptions are made on η_τ. In the case that $\eta_\tau = \eta$, the formula for $\mathit{\Xi}_{t|t_0}$ simplifies to:

$$\mathit{\Xi}_{t|t_0} = \left(\tilde{c}_{\mathrm{b},t|t_0}^{\frac{1}{2}} \tilde{c}_{\mathrm{s},t|t_0}^{\frac{1}{2}}\right)^{\alpha} \left(\frac{1}{2}\tilde{c}_{\mathrm{b},t|t_0}^{-\alpha} + \frac{1}{2}\tilde{c}_{\mathrm{s},t|t_0}^{-\alpha}\right).$$

A second-order accurate approximation of this equation around the non-stochastic steady state is:

$$\Xi_{t|t_0} + \frac{\Xi_{t|t_0}^2}{2} = \frac{\alpha}{2}(\tilde{\mathsf{c}}_{\mathrm{b},t|t_0} + \tilde{\mathsf{c}}_{\mathrm{s},t|t_0}) + \frac{\alpha^2}{8}(\tilde{\mathsf{c}}_{\mathrm{b},t|t_0} + \tilde{\mathsf{c}}_{\mathrm{s},t|t_0})^2 - \frac{\alpha}{2}(\tilde{\mathsf{c}}_{\mathrm{b},t|t_0} + \tilde{\mathsf{c}}_{\mathrm{s},t|t_0}) + \frac{\alpha^2}{4}(\tilde{\mathsf{c}}_{\mathrm{b},t|t_0}^2 + \tilde{\mathsf{c}}_{\mathrm{s},t|t_0}^2) + \mathscr{O}^3,$$

and since the first-order terms cancel out on the right-hand side, it follows that $\Xi_{t|t_0} = \mathscr{O}^2$, which implies the term in $\Xi_{t|t_0}^2$ can be subsumed into $\mathscr{O}^3$. Expanding the brackets of the remaining terms and simplifying leads to:

$$\Xi_{t|t_0} = \frac{\alpha^2}{2}\left(\frac{\tilde{\mathsf{c}}_{\mathrm{s},t|t_0} - \tilde{\mathsf{c}}_{\mathrm{b},t|t_0}}{2}\right)^2 + \mathscr{O}^3,$$

and then substituting the equations for the consumption gaps from [3.30b] (which hold up to an error of order $\mathscr{O}^2$) and simplifying the resulting expression:

$$\Xi_{t|t_0} = \frac{\alpha^2}{2}\frac{\theta^2}{(1-\theta^2)^2}\left(\frac{1-\beta\lambda}{1-\beta}\right)^2 \tilde{\mathsf{d}}_{t|t_0}^2 + \mathscr{O}^3. \qquad \text{[A.15.20]}$$

In the case where the Frisch elasticities η_τ differ as specified in [A.15.5], the expression for $\mathit{\Xi}_{t|t_0}$ in [A.15.18] becomes:

$$\mathit{\Xi}_{t|t_0} = \left(\tilde{c}_{\mathrm{b},t|t_0}^{\frac{1-\theta}{2}} \tilde{c}_{\mathrm{s},t|t_0}^{\frac{1+\theta}{2}}\right)^{\alpha} \left(\frac{1-\theta}{2}\tilde{c}_{\mathrm{b},t|t_0}^{-\alpha} + \frac{1+\theta}{2}\tilde{c}_{\mathrm{s},t|t_0}^{-\alpha}\right),$$

where the formulas in [A.15.6] have been used. Since [3.30b] implies $(1-\theta)\tilde{\mathsf{c}}_{\mathrm{b},t|t_0} + (1+\theta)\tilde{\mathsf{c}}_{\mathrm{s},t|t_0} = \mathscr{O}^2$, a second-order accurate approximation of $\mathit{\Xi}_{t|t_0}$ is:

$$\mathit{\Xi}_{t|t_0} = \frac{\alpha^2}{2}\left(\frac{1-\theta}{2}\tilde{\mathsf{c}}_{\mathrm{b},t|t_0}^2 + \frac{1+\theta}{2}\tilde{\mathsf{c}}_{\mathrm{s},t|t_0}^2\right) + \mathscr{O}^3.$$

which can be written in terms of the debt gap as follows

$$\Xi_{t|t_0} = \frac{\alpha^2}{2}\frac{\theta^2}{1-\theta^2}\left(\frac{1-\beta\lambda}{1-\beta}\right)^2 \tilde{\mathsf{d}}_{t|t_0}^2 + \mathscr{O}^3, \qquad \text{[A.15.21]}$$

where the formula makes use of [3.30b] again.

Taking a second-order accurate approximation of the terms in brackets in the welfare function [A.15.19]:

$$\frac{\mathit{\Upsilon}_{t|t_0}^{-(1-\alpha)}\tilde{Y}_{t|t_0}^{1-\alpha}}{1-\alpha} - \frac{\mathit{\Xi}_{t|t_0}\mathit{\Delta}_t^{(1+\xi)\left(1+\frac{1}{\eta}\right)}\tilde{Y}_{t|t_0}^{(1+\xi)\left(1+\frac{1}{\eta}\right)}}{(1+\xi)\left(1+\frac{1}{\eta}\right)} = \left(\frac{1}{1-\alpha} - \frac{1}{(1+\xi)\left(1+\frac{1}{\eta}\right)}\right) + (\tilde{\mathsf{Y}}_{t|t_0} - \Upsilon_{t|t_0})$$

$$+\frac{(1-\alpha)}{2}(\tilde{\mathsf{Y}}_{t|t_0}-\Upsilon_{t|t_0})^2 - \frac{\Xi_{t|t_0}+\frac{1}{2}\Xi_{t|t_0}^2}{(1+\xi)\left(1+\frac{1}{\eta}\right)} - (\Delta_t+\tilde{\mathsf{Y}}_{t|t_0}) - \frac{(1+\xi)\left(1+\frac{1}{\eta}\right)}{2}(\Delta_t+\tilde{\mathsf{Y}}_{t|t_0})^2 - \Xi_{t|t_0}(\Delta_t+\tilde{\mathsf{Y}}_{t|t_0})$$

$$+\mathscr{O}^3 = \left(\frac{1}{1-\alpha} - \frac{1}{(1+\xi)\left(1+\frac{1}{\eta}\right)}\right) - \Upsilon_{t|t_0} - \frac{\Xi_{t|t_0}}{(1+\xi)\left(1+\frac{1}{\eta}\right)} - \Delta_t - \frac{\nu}{2}\tilde{\mathsf{Y}}_{t|t_0}^2 + \mathscr{O}^3,$$

where the second equality is obtained by grouping terms, using the definition of the coefficient ν in [5.11], and noting that $\Upsilon_{t|t_0} = \mathscr{O}^2$ ([A.11.21] in the proof of Proposition 11), $\Xi_{t|t_0} = \mathscr{O}^2$ (equations [A.15.20] and [A.15.21]), and $\Delta_t = \mathscr{O}^2$ (Proposition 12). Since the variable $\hat{\varphi}_{t|t_0}^*$ is independent of policy from t_0 onwards (denoted $\mathscr{I}_{t_0}$) and the terms on the right-hand side of the equation above comprise a constant plus second-order terms, it immediately follows that the second-order approximation of the welfare function [A.15.19] is:

$$\mathscr{W}_{t_0} = -\sum_{t=t_0}^{\infty} \beta^{t-t_0}\mathbb{E}_{t_0}\left[\left(\Upsilon_{t|t_0} + \frac{\Xi_{t|t_0}}{(1+\xi)\left(1+\frac{1}{\eta}\right)}\right) + \Delta_t + \frac{\nu}{2}\tilde{Y}^2_{t|t_0}\right] + \mathscr{I}_{t_0} + \mathscr{O}^3. \qquad \text{[A.15.22]}$$

Using the second-order approximation of $\Upsilon_{t|t_0}$ in [A.11.21] and equation [A.15.20] or [A.15.21] as appropriate depending on the assumptions about the Frisch elasticities η_τ:

$$\Upsilon_{t|t_0} + \frac{\Xi_{t|t_0}}{(1+\xi)\left(1+\frac{1}{\eta}\right)} = \frac{\aleph_d}{2}\tilde{d}^2_{t|t_0} + \mathscr{O}^3, \quad \text{where } \aleph_d = \begin{cases} \left(1 + \frac{\alpha\eta}{(1+\xi)(1+\eta)(1-\theta^2)}\right)\aleph_{d,c} & \text{if } \eta_\tau = \eta \\ \left(1 + \frac{\alpha\eta}{(1+\xi)(1+\eta)}\right)\aleph_{d,c} & \text{if [A.15.5]} \end{cases},$$

with $\aleph_{d,c}$ denoting the coefficient $\aleph_d$ in the case of inelastic labour supply from [A.11.21]. Using the equation above and by substituting the expression for the summation of Δ_t from the proof of Proposition 12 in [A.12.8] into [A.15.22], the formulas for the second-order approximation of the welfare function [5.15] and the coefficients in [5.16a] and [5.16b] are obtained. Given that this loss function cannot have a finite value if $\lim_{\ell\to\infty}\beta^\ell\mathbb{E}_t\pi_{t+\ell} \neq 0$ (since $0 < \beta < 1$), the earlier restriction $\lim_{\ell\to\infty}\beta^\ell\mathbb{E}_t\pi_{t+\ell} = 0$ is without loss of generality here. This completes the proof.

A.16 Proof of Proposition 16

(i) Optimal monetary policy with commitment starting from date t_0 minimizes the loss function [5.15] subject to the constraints in [5.14a] and [5.14b] for all $t \geq t_0$. The endogenous variables are $\tilde{d}_{t|t_0}$, π_t, $\tilde{Y}_{t|t_0}$, and j_t, and the exogenous variable is $\hat{r}^*_{t|t_0}$, which depends on the growth rate $\hat{g}^*_{t|t_0}$ of the first-best level of real GDP. The Lagrangian for the constrained minimization problem is:

$$\begin{aligned}\mathcal{L}_{t_0} = {}& \frac{1}{2}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\mathbb{E}_{t_0}\left[\aleph_d\tilde{d}^2_{t|t_0} + \aleph_\pi\pi_t^2 + \aleph_Y\tilde{Y}^2_{t|t_0}\right] + \sum_{t=t_0}^{\infty}\beta^{t-t_0}\mathbb{E}_{t_0}\left[\daleth_{t|t_0}\left\{\lambda\tilde{d}_{t|t_0} - \tilde{d}_{t+1|t_0}\right\}\right] \\ &+ \sum_{t=t_0}^{\infty}\beta^{t-t_0}\mathbb{E}_{t_0}\left[\beth_{t|t_0}\left\{\frac{1}{1-\beta\mu}j_{t-1} - \frac{\beta\mu}{1-\beta\mu}j_t - \tilde{d}_{t|t_0} + \lambda\tilde{d}_{t-1|t_0} - \pi_t - \alpha\tilde{Y}_{t|t_0} + \alpha\tilde{Y}_{t-1|t_0}\right.\right. \\ &\left.\left. - \frac{(1-\alpha)(1-\beta)\kappa}{\nu}(\pi_t - \mathbb{E}_{t-1}\pi_t) - \frac{(1-\alpha)(1-\beta)\psi}{(1-\beta\lambda)\nu}(\tilde{d}_{t|t_0} - \mathbb{E}_{t-1}\tilde{d}_{t|t_0}) - \hat{r}^*_{t|t_0}\right\}\right] \\ &+ \sum_{t=t_0}^{\infty}\beta^{t-t_0}\mathbb{E}_{t_0}\left[\gimel_{t|t_0}\left\{\kappa(\pi_t - \beta\pi_{t+1}) - \nu\tilde{Y}_{t|t_0} + \psi\tilde{d}_{t|t_0}\right\}\right] + \Gamma_{t_0}\lim_{t\to\infty}(\beta\mu)^{t-t_0}\mathbb{E}_{t_0}j_t, \end{aligned} \qquad \text{[A.16.1]}$$

where the Lagrangian multipliers $\daleth_{t|t_0}$, $\beth_{t|t_0}$, and $\gimel_{t|t_0}$ are expressed as current values by scaling by β^{t-t_0}. The first-order conditions with respect to $\tilde{d}_{t|t_0}$, π_t, $\tilde{Y}_{t|t_0}$, and j_t are:

$$\aleph_d\tilde{d}_{t|t_0} + \lambda\daleth_{t|t_0} - \beta^{-1}\daleth_{t-1|t_0} - \beth_{t|t_0} + \beta\lambda\mathbb{E}_t\beth_{t+1|t_0} - \frac{(1-\alpha)(1-\beta)\psi}{(1-\beta\lambda)\nu}(\beth_{t|t_0} - \mathbb{E}_{t-1}\beth_{t|t_0}) + \psi\gimel_{t|t_0} = 0; \qquad \text{[A.16.2a]}$$

$$\aleph_\pi\pi_t - \beth_{t|t_0} - \frac{(1-\alpha)(1-\beta)\kappa}{\nu}(\beth_{t|t_0} - \mathbb{E}_{t-1}\beth_{t|t_0}) + \kappa\gimel_{t|t_0} - \kappa\gimel_{t-1|t_0} = 0; \qquad \text{[A.16.2b]}$$

$$\aleph_Y\tilde{Y}_{t|t_0} - \alpha\beth_{t|t_0} + \beta\alpha\mathbb{E}_t\beth_{t+1|t_0} - \nu\gimel_{t|t_0} = 0; \qquad \text{[A.16.2c]}$$

$$\beta\mathbb{E}_t\beth_{t+1|t_0} - \beta\mu\beth_{t|t_0} = 0, \qquad \text{[A.16.2d]}$$

with the notational convention that $\daleth_{t|t_0} = 0$ and $\gimel_{t|t_0} = 0$ for all $t < t_0$. The constrained minimization problem [A.16.1] also has the following transversality conditions:

$$\lim_{t\to\infty}\beta^{t-t_0}\daleth_{t|t_0} = 0, \quad \lim_{t\to\infty}\beta^{t-t_0}\gimel_{t|t_0} = 0, \quad \text{and } \Gamma_{t_0} = \frac{\beta\mu}{1-\beta\mu}\lim_{t\to\infty}\frac{\mathbb{E}_{t_0}\beth_{t|t_0}}{\mu^{t-t_0}}. \qquad \text{[A.16.3]}$$

Since [A.16.2d] implies $\mathbb{E}_t\beth_{t+1|t_0} = \mu\beth_{t|t_0}$ for all $t \geq t_0$, and hence $\mathbb{E}_{t_0}\beth_{t|t_0} = \mu^{t-t_0}\beth_{t_0|t_0}$, it follows that the condition for Γ_{t_0} in [A.16.3] is always satisfied by $\Gamma_{t_0} = (\beta\mu/(1-\beta\mu))\beth_{t_0|t_0}$.

Optimal monetary policy is now characterized, assuming an initial commitment date t_0 arbitrarily far in the past ($t_0 \to -\infty$). The t_0 subscripts are dropped from the debt gap $\tilde{d}_t$, the output gap $\tilde{Y}_t$, the first-best real GDP growth rate $\hat{g}^*_t$ and ex-post real return $\hat{r}^*_t$, and the Lagrangian multipliers $\daleth_t$, $\beth_t$, and $\gimel_t$. The transversality conditions in [A.16.3] reduce to verifying $\beta^\ell \daleth_{t+\ell} \to 0$ and $\beta^\ell \gimel_{t+\ell} \to 0$ as $\ell \to \infty$.

The first-order condition [A.16.2d] and the constraint from [5.14b] on the predictable component $\mathbb{E}_t \tilde{d}_{t+1}$ of the debt gap imply that expected future values of $\tilde{d}_t$ and $\beth_t$ are as follows for all $\ell \geq 0$:

$$\mathbb{E}_t \tilde{d}_{t+\ell} = \lambda^\ell \tilde{d}_t, \quad \text{and} \quad \mathbb{E}_t \beth_{t+\ell} = \mu^\ell \beth_t. \qquad \text{[A.16.4]}$$

The first-order condition [A.16.2a] with respect to the debt gap can be simplified using [A.16.4]:

$$\daleth_{t-1} = \beta\lambda \daleth_t + \beta \aleph_d \tilde{d}_t - \beta(1 - \beta\lambda\mu)\beth_t - \frac{(1-\alpha)(1-\beta)\beta\psi}{(1-\beta\lambda)\nu}(\beth_t - \mu\beth_{t-1}) + \beta\psi\gimel_t. \qquad \text{[A.16.5]}$$

The first-order condition [A.16.2b] with respect to inflation can be simplified using [A.16.4] and the formula for $\aleph_\pi$ from [5.15]:

$$\varepsilon\pi_t - \frac{1}{\kappa}\beth_t - \frac{(1-\alpha)(1-\beta)}{\nu}(\beth_t - \mu\beth_{t-1}) + (\gimel_t - \gimel_{t-1}) = 0. \qquad \text{[A.16.6]}$$

Similarly, the first-order condition [A.16.2c] with respect to the output gap can be simplified using [A.16.4] and the expression for $\aleph_Y$ from [5.15]:

$$\gimel_t = \tilde{Y}_t - \frac{(1-\beta\mu)\alpha}{\nu}\beth_t, \quad \text{and hence} \quad \gimel_t - \gimel_{t-1} = (\tilde{Y}_t - \tilde{Y}_{t-1}) - \frac{(1-\beta\mu)\alpha}{\nu}(\beth_t - \beth_{t-1}). \qquad \text{[A.16.7]}$$

The Phillips curve [5.14a] implies the following expression for the change in the output gap:

$$\tilde{Y}_t - \tilde{Y}_{t-1} = \frac{\kappa}{\nu}\left((\pi_t - \beta\mathbb{E}_t\pi_{t+1}) - (\pi_{t-1} - \beta\mathbb{E}_{t-1}\pi_t)\right) + \frac{\psi}{\nu}(\tilde{d}_t - \tilde{d}_{t-1}),$$

and by putting this together with [A.16.7], rearranging terms, and multiplying both sides by ν/κ:

$$\frac{\nu}{\kappa}(\gimel_t - \gimel_{t-1}) = ((1+\beta)\pi_t - \pi_{t-1} - \beta\pi_{t+1}) + \beta(\pi_{t+1} - \mathbb{E}_t\pi_{t+1}) - \beta(\pi_t - \mathbb{E}_{t-1}\pi_t) \\ + \frac{\psi}{\kappa}(\tilde{d}_t - \tilde{d}_{t-1}) - \frac{(1-\beta\mu)\alpha}{\kappa}(\beth_t - \beth_{t-1}).$$

Multiplying both sides of equation [A.16.2b] by ν/κ and using this together with the equation above to eliminate terms in $\gimel_t$ leads to:

$$\left(1 + \beta + \frac{\varepsilon\nu}{\kappa}\right)\pi_t - \pi_{t-1} - \beta\pi_{t+1} + \beta(\pi_{t+1} - \mathbb{E}_t\pi_{t+1}) - \beta(\pi_t - \mathbb{E}_{t-1}\pi_t) \\ = \frac{\nu}{\kappa^2}\beth_t + \frac{(1-\alpha)(1-\beta)}{\kappa}(\beth_t - \mu\beth_{t-1}) + \frac{(1-\beta\mu)\alpha}{\kappa}(\beth_t - \beth_{t-1}) - \frac{\psi}{\kappa}(\tilde{d}_t - \tilde{d}_{t-1}). \qquad \text{[A.16.8]}$$

This is an expectational difference equation in π_t that can be solved given values of $\beth_t$ and $\tilde{d}_t$. Observe that for any $\varkappa \neq 0$:

$$\varkappa^{-1}\left((\pi_t - \varkappa\pi_{t-1}) - \beta\varkappa(\pi_{t+1} - \varkappa\pi_t)\right) = \left(\varkappa^{-1} + \beta\varkappa\right)\pi_t - \pi_{t-1} - \beta\pi_{t+1}, \qquad \text{[A.16.9]}$$

and define the following quadratic function $\mathcal{Q}(\varkappa)$:

$$\mathcal{Q}(\varkappa) = \beta\varkappa^2 - \left(1 + \beta + \frac{\varepsilon\nu}{\kappa}\right)\varkappa + 1. \qquad \text{[A.16.10]}$$

If $\varkappa \neq 0$ is any number such that $\mathcal{Q}(\varkappa) = 0$ then a comparison of [A.16.8] with [A.16.9] using [A.16.10] shows that inflation must satisfy

$$\varkappa^{-1}(\pi_t - \varkappa\pi_{t-1}) - \beta(\pi_{t+1} - \varkappa\pi_t) - \beta(\epsilon_{t+1} - \epsilon_t) = z_t, \quad \text{where} \quad \epsilon_t = \pi_t - \mathbb{E}_{t-1}\pi_t, \qquad \text{[A.16.11]}$$

and with z_t defined by:

$$z_t = \frac{\nu}{\kappa^2}\beth_t + \frac{(1-\alpha)(1-\beta)}{\kappa}(\beth_t - \mu\beth_{t-1}) + \frac{(1-\beta\mu)\alpha}{\kappa}(\beth_t - \beth_{t-1}) - \frac{\psi}{\kappa}(\tilde{d}_t - \tilde{d}_{t-1}). \qquad \text{[A.16.12]}$$

It can be seen from [A.16.10] that $\mathcal{Q}(0) = 1$, $\mathcal{Q}(1) = -\varepsilon\nu\kappa^{-1}$, and $\mathcal{Q}((1 + \beta + \varepsilon\nu/\kappa)/\beta) = 1$. Since $\mathcal{Q}(\varkappa)$ is a quadratic function of $\varkappa$, this demonstrates the existence of a $\varkappa$ satisfying $0 < \varkappa < 1$ such that $\mathcal{Q}(\varkappa) = 0$, and where this is the smaller of the two roots of the quadratic $\mathcal{Q}(\varkappa)$. The larger root must be $\beta^{-1}\varkappa^{-1}$ using [A.16.10]. Making use of the quadratic root formula for the larger root, the value of $\varkappa$ is as reported in [5.17].

Given the innovation ϵ_t to inflation and z_t from [A.16.12], equation [A.16.11] implies that inflation is the solution of the second-order difference equation:

$$(\pi_t - \varkappa\pi_{t-1}) - (\beta\varkappa)^{-1}(\pi_{t-1} - \varkappa\pi_{t-2}) = \epsilon_t - \epsilon_{t-1} - \beta^{-1}z_{t-1}. \qquad \text{[A.16.13]}$$

Now define variables Z_t and u_t as follows in terms of z_t and ϵ_t from [A.16.11] and [A.16.12]:

$$Z_t = \varkappa\sum_{\ell=0}^{\infty}(\beta\varkappa)^\ell \mathbb{E}_t z_{t+\ell}, \quad \text{and} \quad u_t = \epsilon_t - \frac{1}{1-\beta\varkappa}(Z_t - \mathbb{E}_{t-1}Z_t). \qquad \text{[A.16.14]}$$

Noting the results in [A.16.4], observe that:

$$\sum_{\ell=0}^{\infty}(\beta\varkappa)^\ell \mathbb{E}_t\left[\beth_{t+\ell} - \mu\beth_{t+\ell-1}\right] = \beth_t - \mu\beth_{t-1}, \quad \sum_{\ell=0}^{\infty}(\beta\varkappa)^\ell \mathbb{E}_t\beth_{t+\ell} = \left(\sum_{\ell=0}^{\infty}(\beta\mu\varkappa)^\ell\right)\beth_t = \frac{1}{1-\beta\mu\varkappa}\beth_t;$$

$$\sum_{\ell=0}^{\infty}(\beta\varkappa)^\ell \mathbb{E}_t[\beth_{t+\ell} - \beth_{t+\ell-1}] = \left(\frac{1-\beta\varkappa}{1-\beta\mu\varkappa}\right)(\beth_t - \mu\beth_{t-1}) - \left(\frac{1-\mu}{1-\beta\mu\varkappa}\right)\beth_{t-1}; \quad \text{and}$$

$$\sum_{\ell=0}^{\infty}(\beta\varkappa)^\ell \mathbb{E}_t[\tilde{d}_{t+\ell} - \tilde{d}_{t+\ell-1}] = \left(\frac{1-\beta\varkappa}{1-\beta\lambda\varkappa}\right)(\tilde{d}_t - \lambda\tilde{d}_{t-1}) - \left(\frac{1-\lambda}{1-\beta\lambda\varkappa}\right)\tilde{d}_{t-1}.$$

Using [A.16.12] and substituting these results into equation [A.16.14] shows that the infinite sum Z_t is well defined and given by:

$$\begin{aligned} Z_t = {} & \frac{1}{\kappa(1-\beta\mu\varkappa)}\left(\frac{(1-\varkappa)(1-\beta\varkappa)}{\varepsilon}\beth_t - \alpha\varkappa(1-\mu)(1-\beta\mu)\beth_{t-1}\right) + \frac{\varkappa\psi(1-\lambda)}{\kappa(1-\beta\lambda\varkappa)}\tilde{d}_{t-1} \\ & + \frac{\varkappa}{\kappa}\left((1-\alpha)(1-\beta) + \frac{\alpha(1-\beta\mu)(1-\beta\varkappa)}{(1-\beta\mu\varkappa)}\right)(\beth_t - \mu\beth_{t-1}) - \frac{\varkappa\psi(1-\beta\varkappa)}{\kappa(1-\beta\lambda\varkappa)}(\tilde{d}_t - \lambda\tilde{d}_{t-1}), \end{aligned} \qquad \text{[A.16.15]}$$

where equation [A.16.10] is used to deduce $\varkappa\nu/\kappa = (1-\varkappa)(1-\beta\varkappa)/\varepsilon$.

Note that the definition of Z_t in [A.16.14] implies $Z_t = \beta\varkappa\mathbb{E}_t Z_{t+1} + \varkappa z_t$, and hence $\beta^{-1}z_{t-1} = (\beta\varkappa)^{-1}Z_{t-1} - \mathbb{E}_{t-1}Z_t$. It follows that:

$$\frac{Z_t - \mathbb{E}_{t-1}Z_t}{1-\beta\varkappa} - \frac{Z_{t-1} - \mathbb{E}_{t-2}Z_{t-1}}{1-\beta\varkappa} - \beta^{-1}z_{t-1} = \left(\frac{Z_t - \beta\varkappa\mathbb{E}_{t-1}Z_t}{1-\beta\varkappa}\right) - (\beta\varkappa)^{-1}\left(\frac{Z_{t-1} - \beta\varkappa\mathbb{E}_{t-2}Z_{t-1}}{1-\beta\varkappa}\right),$$

and thus by using the definition of u_t in [A.16.14]:

$$\epsilon_t - \epsilon_{t-1} - \beta^{-1}z_{t-1} = \left(\frac{Z_t - \beta\varkappa\mathbb{E}_{t-1}Z_t}{1-\beta\varkappa}\right) - (\beta\varkappa)^{-1}\left(\frac{Z_{t-1} - \beta\varkappa\mathbb{E}_{t-2}Z_{t-1}}{1-\beta\varkappa}\right) + u_t - u_{t-1}.$$

Putting this together with [A.16.13] implies that inflation must satisfy:

$$\begin{aligned} & (\pi_t - \varkappa\pi_{t-1}) - (\beta\varkappa)^{-1}(\pi_{t-1} - \varkappa\pi_{t-2}) \\ & \qquad = \frac{Z_t - \beta\varkappa\mathbb{E}_{t-1}Z_t}{1-\beta\varkappa} - (\beta\varkappa)^{-1}\frac{Z_{t-1} - \beta\varkappa\mathbb{E}_{t-2}Z_{t-1}}{1-\beta\varkappa} + u_t - u_{t-1}. \end{aligned} \qquad \text{[A.16.16]}$$

Now define a variable Π_t in terms of Z_t as follows:

$$\Pi_t = \sum_{\ell=0}^{\infty}\varkappa^\ell\left(\frac{Z_{t-\ell} - \beta\varkappa\mathbb{E}_{t-\ell-1}Z_{t-\ell}}{1-\beta\varkappa}\right), \quad \text{and hence} \quad \Pi_t = \varkappa\Pi_{t-1} + \frac{Z_t - \beta\varkappa\mathbb{E}_{t-1}Z_t}{1-\beta\varkappa}. \qquad \text{[A.16.17]}$$

As $0 < \varkappa < 1$, this definition can be used in conjunction with equation [A.16.16] to deduce inflation must satisfy:

$$(\pi_t - \Pi_t) = (\beta\varkappa)^{-1}(\pi_{t-1} - \Pi_{t-1}) + u_t - \mathfrak{U}_{t-1}, \quad \text{where } \mathfrak{U}_t = (1-\varkappa)\sum_{\ell=0}^{\infty}\varkappa^\ell u_{t-\ell}. \qquad \text{[A.16.18]}$$

Since ϵ_t in [A.16.11] is an innovation ($\mathbb{E}_{t-1}\epsilon_t = 0$), u_t from [A.16.14] must be a martingale difference sequence ($\mathbb{E}_{t-1}u_t = 0$). Note that the definition of $\mathfrak{U}_t$ in [A.16.18] implies $\mathfrak{U}_t = \varkappa\mathfrak{U}_{t-1} + (1-\varkappa)u_t$, and hence $\mathbb{E}_t\mathfrak{U}_{t+\ell} = \varkappa^\ell\mathfrak{U}_t$ using the martingale difference property of u_t. Taking the expectation of equation [A.16.18] at time $t+1$ conditional on period-t information implies $\mathbb{E}_t[\pi_{t+1} - \Pi_{t+1}] = (\beta\varkappa)^{-1}(\pi_t - \Pi_t) - \mathfrak{U}_t$, and by iterating this equation ℓ periods forward:

$$\mathbb{E}_t[\pi_{t+\ell}-\Pi_{t+\ell}] = (\beta\varkappa)^{-\ell}(\pi_t-\Pi_t) - \sum_{\jmath=0}^{\ell-1}(\beta\varkappa)^{-\ell+\jmath+1}\mathbb{E}_t\mathfrak{U}_{t+\ell} = (\beta\varkappa)^{-\ell}(\pi_t-\Pi_t)$$
$$-(\beta\varkappa)^{-\ell+1}\left(\sum_{\jmath=0}^{\ell-1}(\beta\varkappa^2)^{\jmath}\right)\mathfrak{U}_t = \frac{\beta\varkappa}{1-\beta\varkappa^2}\varkappa^{\ell}\mathfrak{U}_t + (\beta\varkappa)^{-\ell}\left(\pi_t-\Pi_t-\frac{\beta\varkappa}{1-\beta\varkappa^2}\mathfrak{U}_t\right). \quad \text{[A.16.19]}$$

Equation [A.16.17] implies that $\mathbb{E}_t\Pi_{t+1}=\varkappa\Pi_t+\mathbb{E}_t\mathsf{Z}_{t+1}$, and hence an expression for $\mathbb{E}_t\Pi_{t+1}$ can be obtained by using [A.16.4] and [A.16.15]:

$$\mathbb{E}_t\Pi_{t+1} = \varkappa\Pi_t + \left(\frac{(1-\varkappa)(1-\beta\varkappa)\mu}{\varepsilon} - \alpha\varkappa(1-\mu)(1-\beta\mu)\right)\frac{\beth_t}{\kappa(1-\beta\mu\varkappa)} + \frac{\varkappa\psi(1-\lambda)}{\kappa(1-\beta\lambda\varkappa)}\tilde{\mathsf{d}}_t. \quad \text{[A.16.20]}$$

The following results are derived from equation [A.16.4]:

$$\sum_{\jmath=0}^{\ell-1}\varkappa^{\ell-1-\jmath}\mathbb{E}_t\tilde{\mathsf{d}}_{t+\jmath} = \left(\sum_{\jmath=0}^{\ell-1}\varkappa^{\ell-1-\jmath}\lambda^{\jmath}\right)\tilde{\mathsf{d}}_t = \left(\frac{\lambda^{\ell}-\varkappa^{\ell}}{\lambda-\varkappa}\right)\tilde{\mathsf{d}}_t, \quad \text{and} \quad \sum_{\jmath=0}^{\ell-1}\varkappa^{\ell-1-\jmath}\mathbb{E}_t\beth_{t+\jmath} = \left(\frac{\mu^{\ell}-\varkappa^{\ell}}{\mu-\varkappa}\right)\beth_t,$$

and with these, equation [A.16.20] can be iterated ℓ periods forward:

$$\mathbb{E}_t\Pi_{t+\ell} = \frac{1}{\kappa(1-\beta\mu\varkappa)}\left(\frac{(1-\varkappa)(1-\beta\varkappa)\mu}{\varepsilon} - \alpha\varkappa(1-\mu)(1-\beta\mu)\right)\left(\frac{\mu^{\ell}-\varkappa^{\ell}}{\mu-\varkappa}\right)\beth_t$$
$$+\frac{\varkappa\psi(1-\lambda)}{\kappa(1-\beta\lambda\varkappa)}\left(\frac{\lambda^{\ell}-\varkappa^{\ell}}{\lambda-\varkappa}\right)\tilde{\mathsf{d}}_t + \varkappa^{\ell}\Pi_t. \quad \text{[A.16.21]}$$

Together with [A.16.19], expectations of inflation ℓ periods ahead must therefore satisfy:

$$\mathbb{E}_t\pi_{t+\ell} = \frac{1}{\kappa(1-\beta\mu\varkappa)}\left(\frac{(1-\varkappa)(1-\beta\varkappa)\mu}{\varepsilon} - \alpha\varkappa(1-\mu)(1-\beta\mu)\right)\left(\frac{\mu^{\ell}-\varkappa^{\ell}}{\mu-\varkappa}\right)\beth_t$$
$$+\frac{\varkappa\psi(1-\lambda)}{\kappa(1-\beta\lambda\varkappa)}\left(\frac{\lambda^{\ell}-\varkappa^{\ell}}{\lambda-\varkappa}\right)\tilde{\mathsf{d}}_t + \varkappa^{\ell}\left(\Pi_t+\frac{\beta\varkappa}{1-\beta\varkappa^2}\mathfrak{U}_t\right) + (\beta\varkappa)^{-\ell}\left(\pi_t-\Pi_t-\frac{\beta\varkappa}{1-\beta\varkappa^2}\mathfrak{U}_t\right). \quad \text{[A.16.22]}$$

By using the Phillips curve in [5.14a] and equation [A.16.7], the expected value ℓ periods ahead of the Lagrangian multiplier $\beth_t$ is:

$$\beta^{\ell}\mathbb{E}_t\beth_{t+\ell} = \frac{\kappa}{\nu}\beta^{\ell}(\mathbb{E}_t\pi_{t+\ell}-\beta\mathbb{E}_t\pi_{t+\ell+1}) + \frac{\psi}{\nu}(\beta\lambda)^{\ell}\tilde{\mathsf{d}}_t - \frac{(1-\beta\mu)\alpha}{\nu}(\beta\mu)^{\ell}\beth_t,$$

and by substituting from equation [A.16.22]:

$$\beta^{\ell}\mathbb{E}_t\beth_{t+\ell} = \frac{\beta^{\ell}}{\nu(1-\beta\mu\varkappa)}\left(\frac{(1-\varkappa)(1-\beta\varkappa)\mu}{\varepsilon} - \alpha\varkappa(1-\mu)(1-\beta\mu)\right)\left(\frac{\mu^{\ell}-\varkappa^{\ell}}{\mu-\varkappa} - \beta\frac{\mu^{\ell+1}-\varkappa^{\ell+1}}{\mu-\varkappa}\right)\beth_t$$
$$+\frac{\varkappa\psi(1-\lambda)}{\nu(1-\beta\lambda\varkappa)}\left(\frac{\lambda^{\ell}-\varkappa^{\ell}}{\lambda-\varkappa} - \beta\frac{\lambda^{\ell+1}-\varkappa^{\ell+1}}{\lambda-\varkappa}\right)\beta^{\ell}\tilde{\mathsf{d}}_t + (1-\beta\varkappa)\varkappa^{\ell}\left(\Pi_t+\frac{\beta\varkappa}{1-\beta\varkappa^2}\mathfrak{U}_t\right) + \frac{\psi}{\nu}(\beta\lambda)^{\ell}\tilde{\mathsf{d}}_t$$
$$-\frac{(1-\beta\mu)\alpha}{\nu}(\beta\mu)^{\ell}\beth_t + (1-\varkappa)\varkappa^{-(\ell+1)}\left(\pi_t-\Pi_t-\frac{\beta\varkappa}{1-\beta\varkappa^2}\mathfrak{U}_t\right).$$

The transversality condition [A.16.3] on the Lagrangian multiplier $\beth_t$ can be satisfied only if $\beta^{\ell}\mathbb{E}_t\beth_{t+\ell}\to 0$ as $\ell\to\infty$. Since $0<\beta<1$, $0<\lambda<1$, $0<\mu<1$, and $0<\varkappa<1$, the equation above shows that this is possible only if:

$$\pi_t-\Pi_t = \frac{\beta\varkappa}{1-\beta\varkappa^2}\mathfrak{U}_t, \quad \text{and} \quad \pi_t-\Pi_t-\mathbb{E}_{t-1}[\pi_t-\Pi_t] = \frac{\beta\varkappa(\mathfrak{U}_t-\mathbb{E}_{t-1}\mathfrak{U}_t)}{1-\beta\varkappa^2} = \frac{\beta\varkappa(1-\varkappa)}{1-\beta\varkappa^2}\mathsf{u}_t, \quad \text{[A.16.23]}$$

where the final equality uses the expression for $\mathfrak{U}_t$ in [A.16.18]. The first equation in [A.16.18] implies:

$$(\pi_t - \Pi_t) - \mathbb{E}_{t-1}[\pi_t - \Pi_t] = \mathsf{u}_t, \quad \text{and therefore} \quad \left(1 - \frac{\beta\varkappa(1-\varkappa)}{1-\beta\varkappa^2}\right)\mathsf{u}_t = 0,$$

in combination with [A.16.23]. Given that $0 < \varkappa < 1$, the coefficient of u_t above is always non-zero, hence $\mathsf{u}_t = 0$ is the unique solution of the equation. With $\mathsf{u}_t = 0$ for all t, it follows from [A.16.18] that $\mathfrak{U}_t = 0$ for all t, and hence inflation must satisfy $\pi_t = \Pi_t$. The predictable component of inflation is therefore given in equation [A.16.20], and the surprise component can be obtained from [A.16.15] and [A.16.17]:

$$\pi_t - \mathbb{E}_{t-1}\pi_t = \left(\frac{(1-\varkappa)}{\varepsilon(1-\beta\mu\varkappa)} + \frac{\varkappa(1-\beta)(1-\alpha)}{(1-\beta\varkappa)} + \frac{\varkappa(1-\beta\mu)\alpha}{(1-\beta\mu\varkappa)}\right)\frac{(\beth_t - \mathbb{E}_{t-1}\beth_t)}{\kappa} - \frac{\varkappa\psi(\tilde{\mathsf{d}}_t - \mathbb{E}_{t-1}\tilde{\mathsf{d}}_t)}{\kappa(1-\beta\lambda\varkappa)}. \quad \text{[A.16.24]}$$

The output gap $\tilde{\mathsf{Y}}_t$ can be obtained by using $\pi_t = \Pi_t$ and [A.16.20] to substitute for the inflation expectations term appearing in the Phillips curve [5.14a]:

$$\tilde{\mathsf{Y}}_t = \frac{\kappa(1-\beta\varkappa)}{\nu}\pi_t + \frac{\psi(1-\beta\varkappa)}{\nu(1-\beta\lambda\varkappa)}\tilde{\mathsf{d}}_t + \frac{\beta}{\nu(1-\beta\mu\varkappa)}\left(\alpha\varkappa(1-\mu)(1-\beta\mu) - \frac{(1-\varkappa)(1-\beta\varkappa)\mu}{\varepsilon}\right)\beth_t. \quad \text{[A.16.25]}$$

Substituting this expression for the output gap into equation [A.16.7], grouping terms, simplifying, and solving for the Lagrangian multiplier $\gimel_t$:

$$\gimel_t = \frac{(1-\beta\varkappa)}{\nu}\left(\kappa\pi_t + \frac{\psi}{(1-\beta\lambda\varkappa)}\tilde{\mathsf{d}}_t - \frac{1}{(1-\beta\mu\varkappa)}\left(\alpha(1-\beta\mu) + \frac{\beta\mu(1-\varkappa)}{\varepsilon}\right)\beth_t\right). \quad \text{[A.16.26]}$$

Now define $\mathcal{F}_t$ to be the following expected discount sum of current and future values of $\gimel_t$:

$$\mathcal{F}_t = \sum_{\ell=0}^{\infty}(\beta\lambda)^{\ell}\mathbb{E}_t\gimel_{t+\ell}. \quad \text{[A.16.27]}$$

Substituting the expression [A.16.26] for $\gimel_t$ into the above, using the formulas for expectations of $\tilde{\mathsf{d}}_t$ and $\beth_t$ from [A.16.4], and summing and simplifying terms leads to:

$$\mathcal{F}_t = \frac{(1-\beta\varkappa)\kappa}{\nu}\sum_{\ell=0}^{\infty}(\beta\lambda)^{\ell}\mathbb{E}_t\pi_{t+\ell} + \frac{(1-\beta\varkappa)\psi}{(1-\beta\lambda\varkappa)(1-\beta\lambda^2)\nu}\tilde{\mathsf{d}}_t$$
$$- \frac{(1-\beta\varkappa)}{(1-\beta\mu\varkappa)(1-\beta\lambda\mu)\nu}\left(\alpha(1-\beta\mu) + \frac{\beta\mu(1-\varkappa)}{\varepsilon}\right)\beth_t. \quad \text{[A.16.28]}$$

With $\pi_t = \Pi_t$ and equation [A.16.21], the terms in inflation can be summed as follows:

$$\sum_{\ell=0}^{\infty}(\beta\lambda)^{\ell}\mathbb{E}_t\pi_{t+\ell} = \frac{\beta\lambda}{\kappa(1-\beta\mu\varkappa)(1-\beta\lambda\mu)(1-\beta\lambda\varkappa)}\left(\frac{(1-\varkappa)(1-\beta\varkappa)\mu}{\varepsilon} - \alpha\varkappa(1-\mu)(1-\beta\mu)\right)\beth_t$$
$$+ \frac{\beta\lambda\varkappa\psi(1-\lambda)}{\kappa(1-\beta\lambda^2)(1-\beta\lambda\varkappa)^2}\tilde{\mathsf{d}}_t + \frac{1}{(1-\beta\lambda\varkappa)}\pi_t,$$

and substituting these into [A.16.28] yields an expression for $\mathcal{F}_t$:

$$\mathcal{F}_t = \frac{(1-\beta\varkappa)\kappa}{(1-\beta\lambda\varkappa)\nu}\pi_t + \frac{(1-\beta\varkappa)(1-\beta\lambda^2\varkappa)\psi}{(1-\beta\lambda^2)(1-\beta\lambda\varkappa)^2\nu}\tilde{\mathsf{d}}_t$$
$$- \frac{(1-\beta\varkappa)}{(1-\beta\lambda\mu)(1-\beta\lambda\varkappa)(1-\beta\mu\varkappa)\nu}\left(\alpha(1-\beta\mu)(1-\beta\lambda\mu\varkappa) + \frac{\beta\mu(1-\varkappa)(1-\lambda)}{\varepsilon}\right)\beth_t. \quad \text{[A.16.29]}$$

Next, taking expectations of equation [A.16.5] at time $t+1$ conditional on period-t information and using [A.16.4] implies:

$$\daleth_t = \beta\lambda\mathbb{E}_t\daleth_{t+1} + \beta\lambda\aleph_{\mathsf{d}}\tilde{\mathsf{d}}_t - \beta\mu(1-\beta\lambda\mu)\beth_t + \beta\psi\mathbb{E}_t\gimel_{t+1},$$

which can be iterated forwards to deduce:

$$\daleth_t = \sum_{\ell=0}^{\infty}(\beta\lambda)^{\ell}\mathbb{E}_t\left[\beta\lambda\aleph_{\mathrm{d}}\tilde{\mathrm{d}}_{t+\ell} - \beta\mu(1-\beta\lambda\mu)\beth_{t+\ell} + \beta\psi\gimel_{t+1+\ell}\right] + \lim_{\ell\to\infty}(\beta\lambda)^{\ell}\mathbb{E}_t\daleth_{t+\ell}.$$

The transversality condition [A.16.3] for the Lagrangian multiplier $\daleth_t$ implies that $(\beta\lambda)^{\ell}\mathbb{E}_t\daleth_{t+\ell}\to 0$ as $\ell\to\infty$ because $0<\lambda<1$. Using this together with the formulas in [A.16.4] for expectations of $\tilde{\mathrm{d}}_t$ and $\beth_t$ and the definition of $\digamma_t$ from [A.16.27], the equation above becomes:

$$\daleth_t = \frac{\beta\lambda\aleph_{\mathrm{d}}}{1-\beta\lambda^2}\tilde{\mathrm{d}}_t - \beta\mu\beth_t + \beta\psi\mathbb{E}_t\digamma_{t+1}.$$

This equation can be substituted back into [A.16.5] to replace terms in $\daleth_t$:

$$\frac{\beta\lambda\aleph_{\mathrm{d}}}{1-\beta\lambda^2}\tilde{\mathrm{d}}_{t-1} - \beta\mu\beth_{t-1} + \beta\psi\mathbb{E}_{t-1}\digamma_t = \beta\lambda\left(\frac{\beta\lambda\aleph_{\mathrm{d}}}{1-\beta\lambda^2}\tilde{\mathrm{d}}_t - \beta\mu\beth_t + \beta\psi\mathbb{E}_t\digamma_{t+1}\right) + \beta\aleph_{\mathrm{d}}\tilde{\mathrm{d}}_t$$
$$- \beta(1-\beta\lambda\mu)\beth_t - \frac{(1-\alpha)(1-\beta)\beta\psi}{(1-\beta\lambda)\nu}(\beth_t - \mu\beth_{t-1}) + \beta\psi\gimel_t,$$

and since the definition [A.16.27] implies $\gimel_t + \beta\lambda\mathbb{E}_t\digamma_{t+1} = \digamma_t$, the equation above can be simplified and written as follows using [A.16.4]:

$$\frac{\aleph_{\mathrm{d}}}{(1-\beta\lambda^2)}(\tilde{\mathrm{d}}_t - \mathbb{E}_{t-1}\tilde{\mathrm{d}}_t) + \psi(\digamma_t - \mathbb{E}_{t-1}\digamma_t) = \left(1 + \frac{(1-\alpha)(1-\beta)\psi}{(1-\beta\lambda)\nu}\right)(\beth_t - \mathbb{E}_{t-1}\beth_t).$$

Using equation [A.16.29] to obtain the surprise component of $\digamma_t$ and substituting into the equation above:

$$\left(\frac{\aleph_{\mathrm{d}}}{1-\beta\lambda^2} + \frac{(1-\beta\varkappa)(1-\beta\lambda^2\varkappa)\psi^2}{(1-\beta\lambda^2)(1-\beta\lambda\varkappa)^2\nu}\right)(\tilde{\mathrm{d}}_t - \mathbb{E}_{t-1}\tilde{\mathrm{d}}_t) + \frac{(1-\beta\varkappa)\kappa\psi}{(1-\beta\lambda\nu)\nu}(\pi_t - \mathbb{E}_{t-1}\pi_t) = \left(1 + \frac{(1-\alpha)(1-\beta)\psi}{(1-\beta\lambda)\nu}\right.$$
$$\left. + \frac{(1-\beta\varkappa)\psi}{(1-\beta\lambda\mu)(1-\beta\lambda\varkappa)(1-\beta\mu\varkappa)\nu}\left(\alpha(1-\beta\mu)(1-\beta\lambda\mu\varkappa) + \frac{\beta\mu(1-\varkappa)(1-\lambda)}{\varepsilon}\right)\right)(\beth_t - \mathbb{E}_{t-1}\beth_t). \quad \text{[A.16.30]}$$

Note that:

$$\frac{(1-\alpha)(1-\beta)}{(1-\beta\varkappa)} + \frac{\alpha(1-\beta\mu)}{(1-\beta\mu\varkappa)} = \frac{1-\beta}{1-\beta\varkappa} + \frac{\alpha\beta(1-\mu)(1-\varkappa)}{(1-\beta\varkappa)(1-\beta\mu\varkappa)},$$

and hence the coefficient of $\beth_t - \mathbb{E}_{t-1}\beth_t$ in equation [A.16.24] is non-zero, so that equation can be solved for the surprise component of $\beth_t$:

$$\beth_t - \mathbb{E}_{t-1}\beth_t = \frac{\kappa(\pi_t - \mathbb{E}_{t-1}\pi_t) + \frac{\varkappa\psi}{(1-\beta\lambda\varkappa)}(\tilde{\mathrm{d}}_t - \mathbb{E}_{t-1}\tilde{\mathrm{d}}_t)}{\frac{(1-\varkappa)}{\varepsilon(1-\beta\mu\varkappa)} + \frac{\varkappa(1-\beta)(1-\alpha)}{(1-\beta\varkappa)} + \frac{\varkappa(1-\beta\mu)\alpha}{(1-\beta\mu\varkappa)}}. \quad \text{[A.16.31]}$$

Using this equation to eliminate $\beth_t$ from [A.16.30] implies that optimal monetary policy must satisfy the following first-order condition in terms of the debt gap and inflation:

$$\mathfrak{f}_{\mathrm{d}}(\tilde{\mathrm{d}}_t - \mathbb{E}_{t-1}\tilde{\mathrm{d}}_t) = \mathfrak{f}_{\pi}(\pi_t - \mathbb{E}_{t-1}\pi_t), \quad \text{where } \mathfrak{f}_{\mathrm{d}} = \frac{\aleph_{\mathrm{d}}}{1-\beta\lambda^2} + \frac{(1-\beta\varkappa)(1-\beta\lambda^2\varkappa)\psi^2}{(1-\beta\lambda^2)(1-\beta\lambda\varkappa)^2\nu} \quad \text{[A.16.32a]}$$
$$-\frac{\varkappa\psi\left(1 + \frac{(1-\alpha)(1-\beta)\psi}{(1-\beta\lambda)\nu} + \frac{(1-\beta\varkappa)\psi}{(1-\beta\lambda\mu)(1-\beta\lambda\varkappa)(1-\beta\mu\varkappa)\nu}\left(\alpha(1-\beta\mu)(1-\beta\lambda\mu\varkappa) + \frac{\beta\mu(1-\varkappa)(1-\lambda)}{\varepsilon}\right)\right)}{(1-\beta\lambda\varkappa)\left(\frac{(1-\varkappa)}{(1-\beta\mu\varkappa)\varepsilon} + \frac{\varkappa(1-\beta)(1-\alpha)}{(1-\beta\varkappa)} + \frac{\varkappa(1-\beta\mu)\alpha}{(1-\beta\mu\varkappa)}\right)}; \quad \text{[A.16.32b]}$$

$$\text{and } \mathfrak{f}_{\pi} = \frac{\kappa\left(1 + \frac{(1-\alpha)(1-\beta)\psi}{(1-\beta\lambda)\nu} + \frac{(1-\beta\varkappa)\psi}{(1-\beta\lambda\mu)(1-\beta\lambda\varkappa)(1-\beta\mu\varkappa)\nu}\left(\alpha(1-\beta\mu)(1-\beta\lambda\mu\varkappa) + \frac{\beta\mu(1-\varkappa)(1-\lambda)}{\varepsilon}\right)\right)}{\frac{(1-\varkappa)}{(1-\beta\mu\varkappa)\varepsilon} + \frac{\varkappa(1-\beta)(1-\alpha)}{(1-\beta\varkappa)} + \frac{\varkappa(1-\beta\mu)\alpha}{(1-\beta\mu\varkappa)}}$$
$$-\frac{(1-\beta\varkappa)\kappa\psi}{(1-\beta\lambda\varkappa)\nu}. \quad \text{[A.16.32c]}$$

Now consider the second constraint in [5.14b] at time $t+1$ and take expectations conditional on period-t information. Using the law of iterated expectations and the first constraint in [5.14b], this equation reduces to:

$$\mathrm{j}_t = \beta\mu\mathbb{E}_t\mathrm{j}_{t+1} + (1-\beta\mu)\mathbb{E}_t\left[\pi_{t+1} + \alpha(\tilde{\mathrm{Y}}_{t+1} - \tilde{\mathrm{Y}}_t) + \hat{\mathrm{r}}^*_{t+1}\right].$$

Iterating this equation forwards and using the third constraint from [5.14b] (the transversality condition) implies that the bond yield j_t must satisfy:

$$\mathrm{j}_t = (1-\beta\mu)\sum_{\ell=1}^{\infty}(\beta\mu)^{\ell-1}\mathbb{E}_t\left[\pi_{t+\ell} + \alpha(\tilde{\mathrm{Y}}_{t+\ell} - \tilde{\mathrm{Y}}_{t+\ell-1}) + \hat{\mathrm{r}}^*_{t+\ell}\right],$$

which can be used to substitute for j_t in the second constraint from [5.14b]:

$$\frac{1}{1-\beta\mu}\mathrm{j}_{t-1} - \tilde{\mathrm{d}}_t + \lambda\tilde{\mathrm{d}}_{t-1} - \frac{(1-\alpha)(1-\beta)\kappa}{\nu}(\pi_t - \mathbb{E}_{t-1}\pi_t) - \frac{(1-\alpha)(1-\beta)\psi}{(1-\beta\lambda)\nu}(\tilde{\mathrm{d}}_t - \mathbb{E}_{t-1}\tilde{\mathrm{d}}_t)$$
$$- \sum_{\ell=0}^{\infty}(\beta\mu)^{\ell}\mathbb{E}_t\pi_{t+\ell} - \alpha\sum_{\ell=0}^{\infty}(\beta\mu)^{\ell}\mathbb{E}_t[\tilde{\mathrm{Y}}_{t+\ell} - \tilde{\mathrm{Y}}_{t+\ell-1}] = \sum_{\ell=0}^{\infty}(\beta\mu)^{\ell}\mathbb{E}_t\hat{\mathrm{r}}^*_{t+\ell}. \quad \text{[A.16.33]}$$

Note that by collecting terms with the same date, the expected discounted sum of changes in the output gap $\tilde{\mathrm{Y}}_t$ can be written as follows:

$$\sum_{\ell=0}^{\infty}(\beta\mu)^{\ell}\mathbb{E}_t[\tilde{\mathrm{Y}}_{t+\ell} - \tilde{\mathrm{Y}}_{t+\ell-1}] = (1-\beta\mu)\sum_{\ell=0}^{\infty}(\beta\mu)^{\ell}\mathbb{E}_t\tilde{\mathrm{Y}}_{t+\ell} - \tilde{\mathrm{Y}}_{t-1}. \quad \text{[A.16.34]}$$

Next, by substituting the expression for the output gap $\tilde{\mathrm{Y}}_t$ from [A.16.25] and using the conditional expectations from [A.16.4], the following sum of terms in inflation and the level of the output gap can be derived:

$$\sum_{\ell=0}^{\infty}(\beta\mu)^{\ell}\mathbb{E}_t\left[\pi_{t+\ell} + \alpha(1-\beta\mu)\tilde{\mathrm{Y}}_{t+\ell}\right] = \left(1 + \frac{(1-\beta\varkappa)(1-\beta\mu)\alpha\kappa}{\nu}\right)\sum_{\ell=0}^{\infty}(\beta\mu)^{\ell}\mathbb{E}_t\pi_{t+\ell}$$
$$+ \frac{(1-\beta\varkappa)(1-\beta\mu)\alpha\psi}{(1-\beta\lambda\varkappa)(1-\beta\lambda\mu)\nu}\tilde{\mathrm{d}}_t + \frac{(1-\beta\mu)\beta\alpha}{(1-\beta\mu^2)(1-\beta\mu\varkappa)\nu}\left(\alpha\varkappa(1-\mu)(1-\beta\mu) - \frac{(1-\varkappa)(1-\beta\varkappa)\mu}{\varepsilon}\right)\daleth_t. \quad \text{[A.16.35]}$$

An expression for the sum of the terms in inflation can be obtained using $\pi_t = \Pi_t$ and the formula from [A.16.21]:

$$\sum_{\ell=0}^{\infty}(\beta\mu)^{\ell}\mathbb{E}_t\pi_{t+\ell} = \frac{\beta\mu}{\kappa(1-\beta\mu^2)(1-\beta\mu\varkappa)^2}\left(\frac{(1-\varkappa)(1-\beta\varkappa)\mu}{\varepsilon} - \alpha\varkappa(1-\mu)(1-\beta\mu)\right)\daleth_t$$
$$+ \frac{\beta\mu\varkappa\psi(1-\lambda)}{\kappa(1-\beta\lambda\mu)(1-\beta\lambda\varkappa)(1-\beta\mu\varkappa)}\tilde{\mathrm{d}}_t + \frac{1}{(1-\beta\mu\varkappa)}\pi_t. \quad \text{[A.16.36]}$$

Equations [A.16.34], [A.16.35], and [A.16.36] can be combined to deduce:

$$\sum_{\ell=0}^{\infty}(\beta\mu)^{\ell}\mathbb{E}_t\left[\pi_{t+\ell} + \alpha(1-\beta\mu)\tilde{\mathrm{Y}}_{t+\ell}\right] = \frac{1}{(1-\beta\mu\varkappa)}\left(1 + \frac{(1-\beta\varkappa)(1-\beta\mu)\alpha\kappa}{\nu}\right)\pi_t$$
$$+ \frac{\beta}{(1-\beta\mu^2)(1-\beta\mu\varkappa)^2}\left(\frac{\mu}{\kappa} - \frac{(1-\mu)(1-\beta\mu)\alpha}{\nu}\right)\left(\frac{(1-\varkappa)(1-\beta\varkappa)\mu}{\varepsilon} - \alpha\varkappa(1-\mu)(1-\beta\mu)\right)\daleth_t$$
$$+ \frac{\psi}{(1-\beta\lambda\mu)(1-\beta\lambda\varkappa)(1-\beta\mu\varkappa)}\left(\frac{\beta\mu\varkappa(1-\lambda)}{\kappa} + \frac{\alpha(1-\beta\varkappa)(1-\beta\mu)(1-\beta\lambda\mu\varkappa)}{\nu}\right)\tilde{\mathrm{d}}_t,$$

and by substituting this into [A.16.33] and subtracting the expectation of the same equation conditional on period $t-1$ information:

$$\left(\frac{1}{1-\beta\mu\varkappa}+\frac{(1-\alpha)(1-\beta)\kappa}{\nu}+\frac{(1-\beta\varkappa)(1-\beta\mu)\alpha\kappa}{(1-\beta\mu\varkappa)\nu}\right)(\pi_t-\mathbb{E}_{t-1}\pi_t)+\left(1+\frac{(1-\alpha)(1-\beta)\psi}{(1-\beta\lambda)\nu}\right.$$
$$\left.+\frac{\psi}{(1-\beta\lambda\mu)(1-\beta\lambda\varkappa)(1-\beta\mu\varkappa)}\left(\frac{\beta\mu\varkappa(1-\lambda)}{\kappa}+\frac{\alpha(1-\beta\varkappa)(1-\beta\mu)(1-\beta\lambda\mu\varkappa)}{\nu}\right)\right)(\tilde{\mathsf{d}}_t-\mathbb{E}_{t-1}\tilde{\mathsf{d}}_t)$$
$$+\frac{\beta}{(1-\beta\mu^2)(1-\beta\mu\varkappa)^2}\left(\frac{\mu}{\kappa}-\frac{(1-\mu)(1-\beta\mu)\alpha}{\nu}\right)\left(\frac{(1-\varkappa)(1-\beta\varkappa)\mu}{\varepsilon}\right.$$
$$\left.-\alpha\varkappa(1-\mu)(1-\beta\mu)\right)(\beth_t-\mathbb{E}_{t-1}\beth_t)=-\wp_t,\quad\text{where }\ \wp_t=\sum_{\ell=0}^{\infty}(\beta\mu)^{\ell}\left(\mathbb{E}_t\hat{\mathsf{r}}^*_{t+\ell}-\mathbb{E}_{t-1}\hat{\mathsf{r}}^*_{t+\ell}\right).\tag{A.16.37}$$

By following the same steps as in the proof of Proposition 13, the definition of $\wp_t$ is equivalent to that given in the proposition. Substituting equation [A.16.31] into [A.16.37] shows that optimal monetary policy must satisfy:

$$\mathfrak{c}_{\mathsf{d}}(\tilde{\mathsf{d}}_t-\mathbb{E}_{t-1}\tilde{\mathsf{d}}_t)+\mathfrak{c}_\pi(\pi_t-\mathbb{E}_{t-1}\pi_t)=-\wp_t,\quad\text{where}\tag{A.16.38a}$$

$$\mathfrak{c}_{\mathsf{d}}=1+\frac{\psi}{(1-\beta\lambda\mu)(1-\beta\lambda\varkappa)(1-\beta\mu\varkappa)}\left(\frac{\beta\mu\varkappa(1-\lambda)}{\kappa}+\frac{\alpha(1-\beta\varkappa)(1-\beta\mu)(1-\beta\lambda\mu\varkappa)}{\nu}\right)$$
$$+\frac{(1-\alpha)(1-\beta)\psi}{(1-\beta\lambda)\nu}+\frac{\beta\varkappa\psi\left(\frac{\mu}{\kappa}-\frac{(1-\mu)(1-\beta\mu)\alpha}{\nu}\right)\left(\frac{(1-\varkappa)(1-\beta\varkappa)\mu}{\varepsilon}-\alpha\varkappa(1-\mu)(1-\beta\mu)\right)}{(1-\beta\mu^2)(1-\beta\mu\varkappa)^2(1-\beta\lambda\varkappa)\left(\frac{(1-\varkappa)}{(1-\beta\mu\varkappa)\varepsilon}+\frac{\varkappa(1-\beta)(1-\alpha)}{(1-\beta\varkappa)}+\frac{\varkappa(1-\beta\mu)\alpha}{(1-\beta\mu\varkappa)}\right)};\tag{A.16.38b}$$

$$\text{and }\ \mathfrak{c}_\pi=\frac{1}{1-\beta\mu\varkappa}+\frac{(1-\alpha)(1-\beta)\kappa}{\nu}+\frac{(1-\beta\varkappa)(1-\beta\mu)\alpha\kappa}{(1-\beta\mu\varkappa)\nu}$$
$$+\frac{\beta\kappa\left(\frac{\mu}{\kappa}-\frac{(1-\mu)(1-\beta\mu)\alpha}{\nu}\right)\left(\frac{(1-\varkappa)(1-\beta\varkappa)\mu}{\varepsilon}-\alpha\varkappa(1-\mu)(1-\beta\mu)\right)}{(1-\beta\mu^2)(1-\beta\mu\varkappa)^2\left(\frac{(1-\varkappa)}{(1-\beta\mu\varkappa)\varepsilon}+\frac{\varkappa(1-\beta)(1-\alpha)}{(1-\beta\varkappa)}+\frac{\varkappa(1-\beta\mu)\alpha}{(1-\beta\mu\varkappa)}\right)}.\tag{A.16.38c}$$

Solving the simultaneous equations [A.16.32a] and [A.16.38a] yields expressions for the surprise components of the debt gap and inflation:

$$\tilde{\mathsf{d}}_t-\mathbb{E}_{t-1}\tilde{\mathsf{d}}_t=-\mathfrak{b}_{\mathsf{d}}\wp_t,\quad\text{and}\quad\pi_t-\mathbb{E}_{t-1}\pi_t=-\mathfrak{b}_\pi\wp_t,\quad\text{where }\ \mathfrak{b}_{\mathsf{d}}=\frac{\mathfrak{f}_\pi}{\mathfrak{c}_{\mathsf{d}}\mathfrak{f}_\pi+\mathfrak{c}_\pi\mathfrak{f}_{\mathsf{d}}}\ \text{ and }\ \mathfrak{b}_\pi=\frac{\mathfrak{f}_{\mathsf{d}}}{\mathfrak{c}_{\mathsf{d}}\mathfrak{f}_\pi+\mathfrak{c}_\pi\mathfrak{f}_{\mathsf{d}}},\tag{A.16.39}$$

where the solution is given in terms of the coefficients defined in [A.16.32b]–[A.16.32c] and [A.16.38b]–[A.16.38c]. Note that since the equations are linear, and a solution is known to exist for general $\wp_t\neq 0$, it must be the case that $\mathfrak{c}_{\mathsf{d}}\mathfrak{f}_\pi+\mathfrak{c}_\pi\mathfrak{f}_{\mathsf{d}}\neq 0$. Given the first constraint in [5.14b], it follows that the solution for the debt gap is:

$$\tilde{\mathsf{d}}_t=\lambda\tilde{\mathsf{d}}_{t-1}-\mathfrak{b}_{\mathsf{d}}\wp_t.\tag{A.16.40}$$

To write the solution for inflation, combine equations [A.16.20] (with $\pi_t=\Pi_t$) and [A.16.39] to obtain:

$$\pi_t=\varkappa\pi_{t-1}+\left(\frac{(1-\varkappa)(1-\beta\varkappa)\mu}{\varepsilon}-\alpha\varkappa(1-\mu)(1-\beta\mu)\right)\frac{\beth_{t-1}}{\kappa(1-\beta\mu\varkappa)}+\frac{\varkappa\psi(1-\lambda)}{\kappa(1-\beta\lambda\varkappa)}\tilde{\mathsf{d}}_{t-1}-\mathfrak{b}_\pi\wp_t.\tag{A.16.41}$$

Next, combine [A.16.4] and [A.16.39] to derive the solution for the Lagrangian multiplier $\beth_t$:

$$\beth_t=\mu\beth_{t-1}-\mathfrak{b}_\beth\wp_t,\quad\text{where }\ \mathfrak{b}_\beth=\frac{\frac{\varkappa\psi}{1-\beta\lambda\varkappa}\mathfrak{b}_{\mathsf{d}}+\kappa\mathfrak{b}_\pi}{\frac{(1-\varkappa)}{(1-\beta\mu\varkappa)\varepsilon}+\frac{\varkappa(1-\beta)(1-\alpha)}{(1-\beta\varkappa)}+\frac{\varkappa(1-\beta\mu)\alpha}{(1-\beta\mu\varkappa)}}.\tag{A.16.42}$$

Using equations [A.16.40], [A.16.41], and [A.16.42], an explicit solution for inflation is:

$$\pi_t = \mathfrak{a}_{\pi,1}\pi_{t-1} + \mathfrak{a}_{\pi,2}\pi_{t-2} + \mathfrak{a}_{\pi,3}\pi_{t-3} - \mathfrak{b}_{\pi,0}\wp_t - \mathfrak{b}_{\pi,1}\wp_{t-1} - \mathfrak{b}_{\pi,2}\wp_{t-2}, \quad \text{where} \tag{A.16.43a}$$

$$\mathfrak{a}_{\pi,1} = \lambda + \mu + \varkappa, \quad \mathfrak{a}_{\pi,2} = -(\lambda\mu + \lambda\varkappa + \mu\varkappa), \quad \mathfrak{a}_{\pi,3} = \lambda\mu\varkappa, \quad \mathfrak{b}_{\pi,0} = \mathfrak{b}_\pi, \tag{A.16.43b}$$

$$\mathfrak{b}_{\pi,1} = \frac{\varkappa\psi(1-\lambda)}{\kappa(1-\beta\lambda\varkappa)}\mathfrak{b}_{\mathrm{d}} + \left(\frac{(1-\varkappa)(1-\beta\varkappa)\mu}{\varepsilon} - \alpha\varkappa(1-\mu)(1-\beta\mu)\right)\frac{\mathfrak{b}_\beth}{\kappa(1-\beta\lambda\varkappa)} - (\lambda+\mu)\mathfrak{b}_\pi, \quad \text{and}$$

$$\mathfrak{b}_{\pi,2} = \lambda\mu\mathfrak{b}_\pi - \frac{\varkappa\psi(1-\lambda)\mu}{\kappa(1-\beta\lambda\varkappa)}\mathfrak{b}_{\mathrm{d}} - \left(\frac{(1-\varkappa)(1-\beta\varkappa)\mu}{\varepsilon} - \alpha\varkappa(1-\mu)(1-\beta\mu)\right)\frac{\lambda\mathfrak{b}_\beth}{\kappa(1-\beta\lambda\varkappa)} - (\lambda+\mu)\mathfrak{b}_\pi.$$

The solution for the output gap $\tilde{\mathsf{Y}}_t$ can be obtained by first solving for $\tilde{\mathsf{d}}_t$, $\beth_t$, and π_t using [A.16.40]–[A.16.42], and substituting these into equation [A.16.25].

(ii) With strict inflation targeting ($\pi_t = 0$ for all t), the Phillip curve [5.14a] implies the output gap is given by:

$$\tilde{\mathsf{Y}}_t = \frac{\psi}{\nu}\tilde{\mathsf{d}}_t, \tag{A.16.44}$$

where the t_0 subscript is dropped (assuming $t_0 \to -\infty$). The bond yield j_t is obtained by taking expectations of the second equation in [5.14b] at date $t+1$ conditional on date-t information, and using the law of iterated expectations, $\pi_t = 0$, and the first equation from [5.14b]:

$$\mathsf{j}_t = \beta\mu\mathbb{E}_t\mathsf{j}_{t+1} + (1-\beta\mu)\mathbb{E}_t\left[\alpha(\tilde{\mathsf{Y}}_{t+1} - \tilde{\mathsf{Y}}_t) + \hat{\mathsf{r}}^*_{t+1}\right].$$

Substituting for $\tilde{\mathsf{Y}}_t$ from [A.16.44] and using the first equation in [5.14b] to write expectations of the future debt gap in terms of the current gap:

$$\mathsf{j}_t = \beta\mu\mathbb{E}_t\mathsf{j}_{t+1} - \frac{(1-\beta\mu)(1-\lambda)\alpha\psi}{\nu}\tilde{\mathsf{d}}_t + (1-\beta\mu)\mathbb{E}_t\hat{\mathsf{r}}^*_{t+1}.$$

Iterating this equation forwards and using the transversality condition from [5.14b] and $\mathbb{E}_t\tilde{\mathsf{d}}_{t+\ell} = \lambda^\ell\tilde{\mathsf{d}}_t$ from [A.16.4]:

$$\mathsf{j}_t = -\frac{(1-\beta\mu)(1-\lambda)\alpha\psi}{\nu}\sum_{\ell=0}^{\infty}(\beta\mu)^\ell\mathbb{E}_t\tilde{\mathsf{d}}_{t+\ell} + (1-\beta\mu)\sum_{\ell=0}^{\infty}(\beta\mu)^\ell\mathbb{E}_t\hat{\mathsf{r}}^*_{t+\ell+1} + \lim_{\ell\to\infty}(\beta\mu)^\ell\mathbb{E}_t\mathsf{j}_{t+\ell}$$

$$= -\frac{(1-\beta\mu)(1-\lambda)\alpha\psi}{(1-\beta\lambda\mu)\nu}\tilde{\mathsf{d}}_t + (1-\beta\mu)\sum_{\ell=0}^{\infty}(\beta\mu)^\ell\mathbb{E}_t\hat{\mathsf{r}}^*_{t+\ell+1}. \tag{A.16.45}$$

Subtracting the expectation of the second equation in [5.14b] conditional on date $t-1$ information from that equation and using $\pi_t = 0$ implies:

$$\left(1 + \frac{(1-\alpha)(1-\beta)\psi}{(1-\beta\lambda)\nu}\right)(\tilde{\mathsf{d}}_t - \mathbb{E}_{t-1}\tilde{\mathsf{d}}_t) + \alpha(\tilde{\mathsf{Y}}_t - \mathbb{E}_{t-1}\tilde{\mathsf{Y}}_t) + \frac{\beta\mu}{1-\beta\mu}(\mathsf{j}_t - \mathbb{E}_{t-1}\mathsf{j}_t) = -(\hat{\mathsf{r}}^*_t - \mathbb{E}_{t-1}\hat{\mathsf{r}}^*_t).$$

Substituting the expressions for $\tilde{\mathsf{Y}}_t$ and j_t from [A.16.44] and [A.16.45] leads to:

$$\left(1 + \frac{(1-\alpha)(1-\beta)\psi}{(1-\beta\lambda)\nu} + \frac{\alpha(1-\beta\mu)\psi}{(1-\beta\lambda\mu)\nu}\right)(\tilde{\mathsf{d}}_t - \mathbb{E}_{t-1}\tilde{\mathsf{d}}_t) = -\sum_{\ell=0}^{\infty}(\beta\mu)^\ell(\mathbb{E}_t\hat{\mathsf{r}}^*_{t+\ell} - \mathbb{E}_{t-1}\hat{\mathsf{r}}^*_{t+\ell}).$$

Rearranging the (non-zero) coefficient of $\tilde{\mathsf{d}}_t - \mathbb{E}_{t-1}\tilde{\mathsf{d}}_t$, using equation [A.16.4] and the definition of $\wp_t$ from [A.16.37], the solution for the debt gap under strict inflation targeting is:

$$\tilde{\mathsf{d}}_t = \lambda\tilde{\mathsf{d}}_{t-1} - \frac{1}{1 + \left(\frac{(1-\beta)}{(1-\beta\lambda)} + \frac{\alpha\beta(1-\lambda)(1-\mu)}{(1-\beta\mu)(1-\beta\lambda\mu)}\right)\frac{\psi}{\nu}}\wp_t. \tag{A.16.46}$$

The solution for the debt gap under the optimal policy in [A.16.40] is proportional to the above, so the definition of the optimal policy weight χ is such that:

$$\frac{1-\chi}{1 + \left(\frac{(1-\beta)}{(1-\beta\lambda)} + \frac{\alpha\beta(1-\lambda)(1-\mu)}{(1-\beta\mu)(1-\beta\lambda\mu)}\right)\frac{\psi}{\nu}} = \mathfrak{b}_{\mathrm{d}},$$

and hence χ is equal to the following (with $\mathfrak{b}_{\mathrm{d}}$ from [A.16.39]):

$$\chi = \left(1 + \frac{\mathfrak{b}_{\mathrm{d}}}{(1-\mathfrak{b}_{\mathrm{d}}) + \left(\frac{(1-\beta)}{(1-\beta\lambda)} + \frac{\alpha\beta(1-\lambda)(1-\mu)}{(1-\beta\mu)(1-\beta\lambda\mu)}\right)\frac{\psi}{\nu}}\right)^{-1}. \qquad \text{[A.16.47]}$$

(iii) This special case assumes $\mu = 0$ and $\psi = 0$. With $\mu = 0$, equation [A.16.4] implies that the Lagrangian multiplier $\beth_t$ is a martingale difference sequence ($\mathbb{E}_{t-1}\beth_t = 0$). Equation [A.16.24] reduces to:

$$\pi_t - \mathbb{E}_{t-1}\pi_t = \frac{1}{\kappa}\left(\frac{(1-\varkappa)}{\varepsilon} + \frac{\varkappa(1-\beta)(1-\alpha)}{(1-\beta\varkappa)} + \alpha\varkappa\right)\beth_t. \qquad \text{[A.16.48]}$$

Observe that

$$\frac{(1-\varkappa)}{\varepsilon} + \frac{\varkappa(1-\beta)(1-\alpha)}{(1-\beta\varkappa)} + \alpha\varkappa = \frac{(1-\varkappa)}{\varepsilon} + \frac{\varkappa}{(1-\beta\varkappa)}\left((1-\beta) + \alpha\beta(1-\varkappa)\right),$$

and by using $(1-\varkappa)/\varepsilon = \varkappa\nu/(\kappa(1-\beta\varkappa))$ from [A.16.10], equation [A.16.48] can be written as:

$$\pi_t - \mathbb{E}_{t-1}\pi_t = \frac{\varkappa}{\kappa(1-\beta\varkappa)}\left(\frac{\nu}{\kappa} + (1-\beta) + \alpha\beta(1-\varkappa)\right)\beth_t. \qquad \text{[A.16.49]}$$

In the special case, equation [A.16.37] reduces to:

$$(\tilde{\mathsf{d}}_t - \mathbb{E}_{t-1}\tilde{\mathsf{d}}_t) + \left(1 + \frac{(1-\alpha)(1-\beta)\kappa}{\nu} + \frac{\alpha(1-\beta\varkappa)\kappa}{\nu}\right)(\pi_t - \mathbb{E}_{t-1}\pi_t) + \frac{\alpha^2\beta\varkappa}{\nu}\beth_t = -\wp_t.$$

Rearranging the terms of the coefficient of $\pi_t - \mathbb{E}_{t-1}\pi_t$ and substituting for $\pi_t - \mathbb{E}_{t-1}\pi_t$ from [A.16.49] yields:

$$(\tilde{\mathsf{d}}_t - \mathbb{E}_{t-1}\tilde{\mathsf{d}}_t) + \frac{\varkappa}{(1-\beta\varkappa)\nu}\left(\left(\frac{\nu}{\kappa} + (1-\beta) + \alpha\beta(1-\varkappa)\right)^2 + \alpha^2\beta(1-\beta\varkappa)\right)\beth_t = -\wp_t. \qquad \text{[A.16.50]}$$

Next, in the special case, equation [A.16.30] reduces to the following, with the loss function coefficient $\aleph_{\mathrm{d}}$ taken from [5.16b]:

$$\frac{\aleph_{\mathrm{d}}}{1-\beta\lambda^2}(\tilde{\mathsf{d}}_t - \mathbb{E}_{t-1}\tilde{\mathsf{d}}_t) = \beth_t, \quad \text{where } \aleph_{\mathrm{d}} = \frac{\alpha\theta^2(1-\beta\lambda)^2\left(1 + \frac{\alpha\eta}{(1+\xi)(1+\eta)}\right)}{(1-\theta^2)(1-\beta)^2}.$$

Substituting into equation [A.16.50] and solving for $\tilde{\mathsf{d}}_t - \mathbb{E}_{t-1}\tilde{\mathsf{d}}_t$ leads to:

$$\tilde{\mathsf{d}}_t - \mathbb{E}_{t-1}\tilde{\mathsf{d}}_t = -\frac{1}{1 + \frac{\alpha\theta^2(1-\beta\lambda)^2\left(1+\frac{\alpha\eta}{(1+\xi)(1+\eta)}\right)}{(1-\theta^2)(1-\beta)^2(1-\beta\lambda^2)(1-\beta\varkappa)}\frac{\varkappa}{\nu}\left(\left(\frac{\nu}{\kappa} + (1-\beta) + \alpha\beta(1-\varkappa)\right)^2 + \alpha^2\beta(1-\beta\varkappa)\right)}\wp_t.$$

Using equation [A.16.4], the solution can be written in the form $\tilde{\mathsf{d}}_t = \lambda\tilde{\mathsf{d}}_{t-1} - (1-\chi)\wp_t$, where comparison with the above equation confirms the expression for χ in [5.18a]. The expression for χ is consistent with [A.16.47] given that $\mathfrak{b}_{\mathrm{d}} = 1-\chi$ and $\psi = 0$.

With $\tilde{\mathsf{d}}_t - \mathbb{E}_{t-1}\tilde{\mathsf{d}}_t = -(1-\chi)\wp_t$, equation [A.16.50] implies:

$$\beth_t = -\frac{(1-\beta\varkappa)\frac{\nu}{\varkappa}\chi}{\left(\frac{\nu}{\kappa} + (1-\beta) + \alpha\beta(1-\varkappa)\right)^2 + \alpha^2\beta(1-\beta\varkappa)}\wp_t. \qquad \text{[A.16.51]}$$

In the special case, equation [A.16.20] (with $\pi_t = \Pi_t$) reduces to:

$$\mathbb{E}_t\pi_{t+1} = \varkappa\pi_t - \frac{\alpha\varkappa}{\kappa}\beth_t,$$

and with [A.16.49], this implies that inflation is given by:

$$\pi_t = \varkappa\pi_{t-1} + \frac{\varkappa}{\kappa(1-\beta\varkappa)}\left(\frac{\nu}{\kappa} + (1-\beta) + \alpha\beta(1-\varkappa)\right)\beth_t - \frac{\alpha\varkappa}{\kappa}\beth_{t-1}. \qquad \text{[A.16.52]}$$

Substituting equation [A.16.51] into the above confirms the solution for inflation in [5.18b]. Equation [A.16.25] for the output gap $\tilde{\mathsf{Y}}_t$ reduces to the following in the special case:

$$\tilde{\mathsf{Y}}_t = \frac{(1-\beta\varkappa)\kappa}{\nu}\pi_t + \frac{\alpha\beta\varkappa}{\nu}\beth_t,$$

and hence:

$$\tilde{\mathsf{Y}}_t = \varkappa \tilde{\mathsf{Y}}_{t-1} + \frac{(1-\beta\varkappa)\kappa}{\nu}(\pi_t - \varkappa\pi_{t-1}) + \frac{\alpha\beta\varkappa}{\nu}(\beth_t - \varkappa\beth_{t-1}).$$

Substituting from equation [A.16.52], collecting terms, and simplifying leads to:

$$\tilde{\mathsf{Y}}_t = \varkappa \tilde{\mathsf{Y}}_{t-1} + \frac{\varkappa}{\nu}\left(\left(\frac{\nu}{\kappa} + (1-\beta) + \alpha\beta(2-\varkappa)\right)\beth_t - \alpha\beth_{t-1}\right),$$

and using [A.16.51] to replace terms in $\beth_t$ confirms the solution for the output gap given in equation [5.18c]. The real interest rate gap is $\tilde{\rho}_t = \rho_t - \hat{\rho}^*_t$, which is related to the output gap according to $\tilde{\rho}_t = \alpha(\mathbb{E}_t\tilde{\mathsf{Y}}_{t+1} - \tilde{\mathsf{Y}}_t)$ given equation [3.9a] and $\hat{\rho}^*_t = \alpha\mathbb{E}_t\hat{\mathsf{g}}^*_{t+1}$. Using the solution for the output gap in [5.18c]:

$$\mathbb{E}_t\tilde{\mathsf{Y}}_{t+1} = \varkappa\tilde{\mathsf{Y}}_t + \frac{\alpha(1-\beta\varkappa)\chi}{\left(\frac{\nu}{\kappa} + (1-\beta) + \alpha\beta(1-\varkappa)\right)^2 + \alpha^2\beta(1-\beta\varkappa)}\wp_t.$$

It follows that the real interest rate gap is given by:

$$\tilde{\rho}_t = -\alpha(1-\varkappa)\tilde{\mathsf{Y}}_t + \frac{\alpha^2(1-\beta\varkappa)\chi}{\left(\frac{\nu}{\kappa} + (1-\beta) + \alpha\beta(1-\varkappa)\right)^2 + \alpha^2\beta(1-\beta\varkappa)}\wp_t,$$

and hence:

$$\tilde{\rho}_t = \varkappa\tilde{\rho}_{t-1} - \alpha(1-\varkappa)(\tilde{\mathsf{Y}}_t - \varkappa\tilde{\mathsf{Y}}_{t-1}) + \frac{\alpha^2(1-\beta\varkappa)\chi}{\left(\frac{\nu}{\kappa} + (1-\beta) + \alpha\beta(1-\varkappa)\right)^2 + \alpha^2\beta(1-\beta\varkappa)}(\wp_t - \varkappa\wp_{t-1}).$$

Substituting from the solution for $\tilde{\mathsf{Y}}_t$ in [5.18c] then collecting terms and simplifying confirms the expression for $\tilde{\rho}_t$ in [5.18d].

(iv) With the assumptions on the type-specific Frisch elasticities from Proposition 15 that lead to $\psi = 0$, the log-linearized equation for the growth rate of first-best output $\hat{Y}^*_{t|t_0}$ from [5.12] is

$$\hat{\mathsf{g}}^*_{t|t_0} = \left(\frac{1 + \xi + \frac{1+\xi}{\eta}}{\alpha + \xi + \frac{1+\xi}{\eta}}\right)(\mathsf{A}_t - \mathsf{A}_{t-1}),$$

where the terms in the consumption ratios cancel out because of equation [A.15.6]. It follows that $\hat{\mathsf{g}}^*_{t|t_0} = \hat{\mathsf{g}}_t$, which is independent of t_0. Similarly, the output gap $\tilde{\mathsf{Y}}_{t|t_0}$ is such that $\tilde{\mathsf{Y}}_{t|t_0} = \tilde{\mathsf{Y}}_t$, independent of t_0 up to a first-order approximation. Since expectations of real GDP growth $\hat{\mathsf{g}}_t$ determine $\hat{\mathsf{r}}^*_{t|t_0}$, it is also the case that $\hat{\mathsf{r}}^*_{t|t_0} = \hat{\mathsf{r}}^*_t$. With $\psi = 0$, the Phillips curve constraint in [5.14a] reduces to:

$$\tilde{\mathsf{Y}}_t = \frac{\kappa}{\nu}(\pi_t - \beta\mathbb{E}_t\pi_{t+1}). \qquad \text{[A.16.53]}$$

The transversality condition in [5.14b] is satisfied automatically when $\mu = 0$. With $\mu = 0$ and $\psi = 0$, the second constraint in [5.14b] at $t = t_0$ becomes:

$$\mathsf{j}_{t-1} - \tilde{\mathsf{d}}_{t|t} - \pi_t - \frac{(1-\alpha)(1-\beta)\kappa}{\nu}(\pi_t - \mathbb{E}_{t-1}\pi_t) - \alpha\tilde{\mathsf{Y}}_t + \alpha\tilde{\mathsf{Y}}_{t-1} = \hat{\mathsf{r}}^*_t, \qquad \text{[A.16.54]}$$

using the definition $\tilde{\mathsf{d}}_{t-1|t} = 0$. The first constraint in [5.14b] then requires $\mathbb{E}_{t-1}\tilde{\mathsf{d}}_{t|t} = 0$, and using this in conjunction with [A.16.54] implies $\mathsf{j}_{t-1} = \mathbb{E}_{t-1}\hat{\mathsf{r}}^*_t + \alpha\mathbb{E}_{t-1}\tilde{\mathsf{Y}}_t - \alpha\tilde{\mathsf{Y}}_{t-1} + \mathbb{E}_{t-1}\pi_t$. Substituting this into equation [A.16.54] to replace j_{t-1} leads to:

$$\tilde{\mathsf{d}}_{t|t} = -\left(1 + \frac{(1-\alpha)(1-\beta)\kappa}{\nu}\right)(\pi_t - \mathbb{E}_{t-1}\pi_t) - \alpha(\tilde{\mathsf{Y}}_t - \mathbb{E}_{t-1}\tilde{\mathsf{Y}}_t) - \wp_t, \qquad \text{[A.16.55]}$$

where $\wp_t = \hat{\mathsf{r}}^*_t - \mathbb{E}_{t-1}\hat{\mathsf{r}}^*_t$ coincides with the definition given in [A.16.37] when $\mu = 0$ and $\hat{\mathsf{r}}^*_{t|t_0} = \hat{\mathsf{r}}^*_t$. The variable $\wp_t$ is a martingale difference sequence ($\mathbb{E}_{t-1}\wp_t = 0$).

Following the same steps as in the proof of Proposition 14, the loss function [5.15] at $t = t_0$ can be expressed in terms of $\tilde{\mathsf{d}}_{t|t}$, π_t, and $\tilde{\mathsf{Y}}_t$ (with $\tilde{\mathsf{Y}}_{t|t_0} = \tilde{\mathsf{Y}}_t$) as follows:

$$\mathscr{L}_t = \frac{1}{2}\sum_{\ell=0}^{\infty}\beta^\ell\mathbb{E}_t\left[\frac{\aleph_{\mathsf{d}}}{1-\beta\lambda^2}\tilde{\mathsf{d}}^2_{t+\ell|t+\ell} + \aleph_\pi\pi^2_{t+\ell} + \aleph_{\mathsf{Y}}\tilde{\mathsf{Y}}^2_{t+\ell}\right]. \qquad \text{[A.16.56]}$$

The constraints on monetary policy in [5.14a] and [5.14b] are equivalent to [A.16.53] and [A.16.55]. In minimizing the loss function [A.16.56] subject to [A.16.53] and [A.16.55], observe that starting from any time t, the constraints from that date onwards and the continuation value of the loss function are independent of all variables realized prior to t. Thus, in a Markovian discretionary policy equilibrium, expectations of future values of $\tilde{\mathsf{d}}_{t|t}$, π_t, and $\tilde{\mathsf{Y}}_t$ are taken as independent of current policy actions.

The discretionary policy can be found by minimizing the period loss function $\mathfrak{L}_t$ below subject to the constraints [A.16.53] and [A.16.55], taking expectations as given:

$$\mathfrak{L}_t = \frac{1}{2}\left(\frac{\aleph_{\mathrm{d}}}{1-\beta\lambda^2}\tilde{\mathsf{d}}^2_{t+\ell|t+\ell} + \aleph_\pi \pi^2_{t+\ell} + \aleph_{\mathrm{Y}}\tilde{\mathsf{Y}}^2_{t+\ell}\right). \qquad \text{[A.16.57]}$$

By substituting the Phillips curve [A.16.53] into the constraint [A.16.55]:

$$\tilde{\mathsf{d}}_{t|t} = -\left(1 + \frac{(1-\alpha)(1-\beta)\kappa}{\nu}\right)(\pi_t - \mathbb{E}_{t-1}\pi_t) - \frac{\alpha\kappa}{\nu}\left((\pi_t - \mathbb{E}_{t-1}\pi_t) - \beta(\mathbb{E}_t\pi_{t+1} - \mathbb{E}_{t-1}\pi_{t+1})\right) - \wp_t, \qquad \text{[A.16.58]}$$

which is then substituted into the period loss function [A.16.57] along with the Phillips curve [A.16.53] to write the debt gap $\tilde{\mathsf{d}}_{t|t}$ and output gap $\tilde{\mathsf{Y}}_t$ in terms of inflation π_t:

$$\mathfrak{L}_t = \frac{\aleph_{\mathrm{d}}}{2(1-\beta\lambda^2)}\left(-\left(1 + \frac{(1-\alpha)(1-\beta)\kappa}{\nu} + \frac{\alpha\kappa}{\nu}\right)(\pi_t - \mathbb{E}_{t-1}\pi_t) + \frac{\alpha\beta\kappa}{\nu}(\mathbb{E}_t\pi_{t+1} - \mathbb{E}_{t-1}\pi_{t+1}) - \wp_t\right)^2 + \frac{\aleph_\pi}{2}\pi_t^2 + \frac{\aleph_{\mathrm{Y}}}{2}\left(\frac{\kappa}{\nu}(\pi_t - \beta\mathbb{E}_t\pi_{t+1})\right)^2. \qquad \text{[A.16.59]}$$

The first-order condition for minimizing $\mathfrak{L}_t$ with respect to inflation π_t is:

$$\left(1 + \frac{(1-\alpha)(1-\beta)\kappa}{\nu} + \frac{\alpha\kappa}{\nu}\right)\frac{\aleph_{\mathrm{d}}}{1-\beta\lambda^2}\tilde{\mathsf{d}}_{t|t} = \aleph_\pi\pi_t + \frac{\kappa}{\nu}\aleph_{\mathrm{Y}}\tilde{\mathsf{Y}}_t,$$

and by using the expressions for $\aleph_\pi$ and $\aleph_{\mathrm{Y}}$ in [5.15] and simplifying:

$$\left(1 + (1-\beta+\alpha\beta)\frac{\kappa}{\nu}\right)\frac{\aleph_{\mathrm{d}}}{1-\beta\lambda^2}\tilde{\mathsf{d}}_{t|t} = \kappa(\varepsilon\pi_t + \tilde{\mathsf{Y}}_t). \qquad \text{[A.16.60]}$$

The discretionary policy equilibrium is found by substituting the first-order condition [A.16.60] into the constraints [A.16.53] and [A.16.55]. Using [A.16.60] to substitute for $\tilde{\mathsf{Y}}_t$ in the Phillips curve [A.16.53]:

$$\left(1 + \frac{\varepsilon\nu}{\kappa}\right)\pi_t = \beta\mathbb{E}_t\pi_{t+1} + \frac{1}{\kappa}\left(1-\beta+\alpha\beta+\frac{\nu}{\kappa}\right)\frac{\aleph_{\mathrm{d}}}{1-\beta\lambda^2}\tilde{\mathsf{d}}_{t|t}.$$

Since $0<\beta<1$ and ε, ν, and κ are all positive, the unique stationary equilibrium for inflation (requiring $\lim_{\ell\to\infty}\beta^\ell\mathbb{E}_t\pi_{t+\ell}=0$) is found by iterating this equation forwards:

$$\pi_t = \frac{\aleph_{\mathrm{d}}\left(1-\beta+\alpha\beta+\frac{\nu}{\kappa}\right)}{(1-\beta\lambda^2)(\kappa+\varepsilon\nu)}\sum_{\ell=0}^{\infty}\left(\beta\left(1+\frac{\varepsilon\nu}{\kappa}\right)^{-1}\right)^\ell\mathbb{E}_t\tilde{\mathsf{d}}_{t+\ell|t+\ell} = \frac{\aleph_{\mathrm{d}}\left(1-\beta+\alpha\beta+\frac{\nu}{\kappa}\right)}{(1-\beta\lambda^2)(\kappa+\varepsilon\nu)}\tilde{\mathsf{d}}_{t|t}, \qquad \text{[A.16.61]}$$

using $\mathbb{E}_t\tilde{\mathsf{d}}_{t+\ell|t+\ell}=0$ for all $\ell\geq 1$, which follows from $\mathbb{E}_t\tilde{\mathsf{d}}_{t+1|t+1}=0$ by the law of iterated expectations. With $\mathbb{E}_{t-1}\tilde{\mathsf{d}}_{t|t}=0$ it follows that $\mathbb{E}_{t-1}\pi_t=0$, in which case the constraint [A.16.58] becomes:

$$\tilde{\mathsf{d}}_{t|t} = -\frac{\kappa}{\nu}\left(1-\beta+\alpha\beta+\frac{\nu}{\kappa}\right)\pi_t - \wp_t. \qquad \text{[A.16.62]}$$

Combining this equation with [A.16.61] leads to the solution for $\tilde{\mathsf{d}}_{t|t}$:

$$\tilde{\mathsf{d}}_{t|t} = -\left(1 + \frac{\aleph_{\mathrm{d}}\left(1-\beta+\alpha\beta+\frac{\nu}{\kappa}\right)^2}{\nu(1-\beta\lambda^2)\left(1+\frac{\varepsilon\nu}{\kappa}\right)}\right)^{-1}\wp_t,$$

which can be written as follows using the expression for $\aleph_{\mathrm{d}}$ in [5.16b]:

$$\tilde{\mathsf{d}}_{t|t} = -(1-\chi')\wp_t, \quad \text{where } \chi' = \left(1 + \frac{\nu\frac{(1-\theta^2)(1-\beta)^2(1-\beta\lambda^2)}{\alpha\theta^2(1-\beta\lambda)^2\left(1+\frac{\alpha\eta}{(1+\xi)(1+\eta)}\right)}\left(1+\frac{\varepsilon\nu}{\kappa}\right)}{\left(1-\beta+\alpha\beta+\frac{\nu}{\kappa}\right)^2}\right)^{-1}. \qquad \text{[A.16.63]}$$

For any arbitrary t_0, by following the proof of Proposition 14 (equation [A.14.3]), $\tilde{\mathsf{d}}_{t|t_0} = \tilde{\mathsf{d}}_{t|t} + \lambda\tilde{\mathsf{d}}_{t-1|t_0}$, and hence $\tilde{\mathsf{d}}_{t|t_0} = \lambda\tilde{\mathsf{d}}_{t-1|t_0} - (1-\chi')\wp_t$. Taking the case of $t_0 \to -\infty$ for comparison with the optimal policy under commitment leads to the (Markovian) discretionary policy equilibrium $\tilde{\mathsf{d}}_t = \lambda\tilde{\mathsf{d}}_{t-1} - (1-\chi')\wp_t$. The solution for inflation is obtained by substituting [A.16.63] into [A.16.62]. With $\mathbb{E}_{t-1}\pi_t = 0$, the Phillips curve [A.16.53] reduces to $\tilde{\mathsf{Y}}_t = (\kappa/\nu)\pi_t$, so the solution for the output gap $\tilde{\mathsf{Y}}_t$ is obtained immediately from the inflation solution. Finally, the real interest rate gap $\tilde{\rho}_t = \alpha\mathbb{E}_t[\tilde{\mathsf{Y}}_{t+1} - \tilde{\mathsf{Y}}_t]$ solution is obtained from the output gap solution noting $\mathbb{E}_{t-1}\tilde{\mathsf{Y}}_t = 0$. In summary, the (Markovian) discretionary policy equilibrium is:

$$\pi_t = -\frac{\frac{\nu}{\kappa}\chi'}{1-\beta+\alpha\beta+\frac{\nu}{\kappa}}\wp_t, \quad \tilde{\mathsf{Y}}_t = -\frac{\chi'}{1-\beta+\alpha\beta+\frac{\nu}{\kappa}}\wp_t, \quad \text{and} \quad \tilde{\rho}_t = \frac{\alpha\chi'}{1-\beta+\alpha\beta+\frac{\nu}{\kappa}}\wp_t. \qquad [\text{A.16.64}]$$

Since $\mathbb{E}_{t-1}\wp_t = 0$, inflation, the output gap, and the real interest rate gap are all serially uncorrelated.

Comparing the expressions for χ and χ' in [5.18a] and [A.16.63], the inequality $\chi \geq \chi'$ is equivalent to:

$$\frac{\varkappa\left(\left(1-\beta+\alpha\beta+\frac{\nu}{\kappa}-\alpha\beta\varkappa\right)^2+\alpha^2\beta(1-\beta\varkappa)\right)}{\nu(1-\beta\varkappa)} \geq \frac{\left(1-\beta+\alpha\beta+\frac{\nu}{\kappa}\right)^2}{\nu\left(1+\frac{\varepsilon\nu}{\kappa}\right)}.$$

Using [A.16.10], it follows that $\varepsilon\nu/\kappa = (1-\varkappa)(1-\beta\varkappa)/\varkappa$, so the inequality above is equivalent to:

$$(1-\beta\varkappa+\beta\varkappa^2)\left(\left(1-\beta+\alpha\beta+\frac{\nu}{\kappa}-\alpha\beta\varkappa\right)^2+\alpha^2\beta(1-\beta\varkappa)\right)-(1-\beta\varkappa)\left(1-\beta+\alpha\beta+\frac{\nu}{\kappa}\right)^2 \geq 0. \qquad [\text{A.16.65}]$$

By expanding the bracket of the first term below, observe that:

$$\left(1-\beta+\alpha\beta+\frac{\nu}{\kappa}-\alpha\beta\varkappa\right)^2+\alpha^2\beta(1-\beta\varkappa) = \left(1-\beta+\alpha\beta+\frac{\nu}{\kappa}\right)^2 - 2\alpha\beta\varkappa\left(1-\beta+\alpha\beta+\frac{\nu}{\kappa}\right)+\alpha^2\beta(1-\beta\varkappa+\beta\varkappa^2),$$

and substituting this into [A.16.65] shows that inequality is equivalent to:

$$\beta\varkappa^2\left(1-\beta+\alpha\beta+\frac{\nu}{\kappa}\right)^2 - 2\alpha\beta\varkappa\left(1-\beta+\alpha\beta+\frac{\nu}{\kappa}\right)(1-\beta\varkappa+\beta\varkappa^2)+\alpha^2\beta(1-\beta\varkappa+\beta\varkappa^2)^2 \geq 0.$$

Cancelling β from the expression and grouping terms:

$$\left(\alpha(1-\beta\varkappa+\beta\varkappa^2)\right)^2 - 2\left(\alpha(1-\beta\varkappa+\beta\varkappa^2)\right)\left(\varkappa\left(1-\beta+\alpha\beta+\frac{\nu}{\kappa}\right)\right)+\left(\varkappa\left(1-\beta+\alpha\beta+\frac{\nu}{\kappa}\right)\right)^2 \geq 0,$$

which is exactly equal to a squared term, thus $\chi \geq \chi'$ is equivalent to:

$$\left(\alpha(1-\beta\varkappa+\beta\varkappa^2)-\varkappa\left(1-\beta+\alpha\beta+\frac{\nu}{\kappa}\right)\right)^2 \geq 0. \qquad [\text{A.16.66}]$$

Noting that $1-\beta\varkappa+\beta\varkappa^2 = \varkappa+(1-\varkappa)(1-\beta\varkappa)$ and $(1-\varkappa)(1-\beta\varkappa) = \varepsilon\nu\varkappa/\kappa$ from [A.16.10], the inequality [A.16.66] implies that $\chi = \chi'$ holds only when:

$$\alpha\varkappa\left(1+\frac{\varepsilon\nu}{\kappa}\right)-\varkappa\left(1-\beta+\alpha\beta+\frac{\nu}{\kappa}\right) = 0, \quad \text{or equivalently} \quad (\alpha-1)(1-\beta)+(\varepsilon-1)\frac{\nu}{\kappa} = 0,$$

since $\varkappa > 0$. For any other parameters, [A.16.66] implies $\chi > \chi'$, so this holds generically, completing the proof.

A.17 Results for the model where discount factors are internalized

In this version of the model, both individual households and the policymaker internalize the effects of their actions on the discount factors in [2.2]. To ensure that both types of households continue to have the same marginal propensities to consume in the neighbourhood of the non-stochastic steady state, a type-specific constant is subtracted from the period utility functions so that the steady-state value of utility is the same for both household types. The changes to the results are summarized below.

The term λ appearing in the coefficients of the log-linearized equations from Proposition 4 is replaced by λ', the solution of the quadratic equation $\beta\lambda(\lambda')^2-(1-\beta+2\beta\lambda)\lambda'+\lambda = 0$ in the unit interval $(0 < \lambda' < 1)$.

The formula for the root is:

$$\lambda' = \frac{2\lambda}{1 - \beta + 2\beta\lambda + \sqrt{(1 - \beta + 2\beta\lambda)^2 - 4\beta}},$$

and it can be seen that this root satisfies $\lambda < \lambda' < 1$. The coefficient $\aleph_d$ of the debt gap in the loss function from Proposition 12 is replaced by:

$$\aleph_d' = \frac{\alpha\theta^2(1 - \beta\lambda')(1 + \beta\lambda' - 2\beta\lambda\lambda')}{(1 - \theta^2)(1 - \beta)^2}.$$

In the tradeoff between fluctuations in the debt gap and fluctuations in inflation, the optimal policy weight χ on the debt gap from Proposition 13 is replaced by:

$$\chi' = \left(1 + \frac{\varepsilon(1 + \varepsilon\xi)\sigma(1 - \theta^2)(1 - \beta)^2(1 - \beta(\lambda')^2)(1 - \beta\mu^2)}{\alpha\theta^2(1 - \sigma)(1 - \beta\sigma)(1 - \beta\lambda')(1 + \beta\lambda' - 2\beta\lambda\lambda')}\right)^{-1}.$$

It can be shown that $\chi' \geq \chi$ for all parameter values.