Appendix to "Intrinsic Inflation Persistence"

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A Proofs of propositions

A.1 Lemmas

Lemma 1 *Let* $\mathscr{D}_\rho \equiv \{z \in \mathbb{C} \mid |z| \leq \rho\}$ *be the closed disc of radius* ρ *and let* $\psi(z) \equiv \sum_{\ell=0}^{\infty} \psi_\ell z^\ell$ *denote the* z*-transform of the sequence of survival probabilities* $\{\psi_\ell\}_{\ell=0}^{\infty}$.

(i) The function $\psi(z)$ *is analytic on* $\mathscr{D}_\rho$ *for some* $\rho > 1$.

(ii) A stationary age distribution $\{\omega_\ell\}_{\ell=0}^{\infty}$ *consistent with the hazard function* $\{\alpha_\ell\}_{\ell=1}^{\infty}$ *is stable if and only if* $\psi(z)$ *has no roots in* $\mathscr{D}_\rho$ *for some* $\rho > 1$.

Proof (i) Let $\alpha_\infty \equiv \lim_{\ell\to\infty} \alpha_\ell$ be the limiting value of the hazard function $\{\alpha_\ell\}_{\ell=1}^{\infty}$. It is assumed that $\alpha_\infty > 0$. Let ϖ be a number lying strictly between $(1-\alpha_\infty)$ and 1, which must satisfy $0 < \varpi < 1$. Given that $\alpha_\ell \to \alpha_\infty$ as $\ell \to \infty$, there must exist a value of $\jmath$ such that $(1-\alpha_\ell) < \varpi$ for all $\ell \geq \jmath$. Since $\psi_\ell = (1-\alpha_\ell)\psi_{\ell-1}$, this implies that $\psi_{\ell+1} \leq \varpi\psi_\ell$ for all $\ell \geq \jmath$ and hence $\psi_\ell \leq \varpi^{\ell-\jmath}\psi_\jmath$.

Now let ρ be any number strictly between 1 and ϖ^{-1}, which implies $0 < \varpi\rho < 1$ and hence that $\sum_{\ell=0}^{\infty} \varpi^\ell |z|^\ell \leq \sum_{\ell=0}^{\infty} (\varpi\rho)^\ell = (1-\varpi\rho)^{-1} < \infty$ for all $z \in \mathscr{D}_\rho$. By applying the triangle inequality to the power series $\psi(z)$, it follows that

$$\left|\sum_{\ell=0}^{\infty} \psi_\ell z^\ell\right| \leq \sum_{\ell=0}^{\infty} \psi_\ell |z|^\ell \leq \sum_{\ell=0}^{\jmath-1} \psi_\ell |z|^\ell + \psi_\jmath \sum_{\ell=\jmath}^{\infty} \varpi^{\ell-\jmath}|z|^\ell = \sum_{\ell=0}^{\jmath-1} \psi_\ell |z|^\ell + \psi_\jmath |z|^\jmath \sum_{\ell=0}^{\infty} \varpi^\ell |z|^\ell,$$

and hence $|\psi(z)| < \infty$ if $z \in \mathscr{D}_\rho$. Therefore, the power series $\psi(z)$ is analytic on $\mathscr{D}_\rho$ for some $\rho > 1$.

(ii) Let $\{\omega_\ell\}_{\ell=0}^{\infty}$ be a stationary age distribution satisfying $\sum_{\ell=0}^{\infty} \omega_\ell = 1$ and consistent with the hazard function $\{\alpha_\ell\}_{\ell=1}^{\infty}$ so that $\omega_\ell = (1-\alpha_\ell)\omega_{\ell-1}$. Now let $\Delta_{\ell,t} \equiv \omega_{\ell,t} - \omega_\ell$ denote the sequence of deviations $\{\Delta_{\ell,t}\}_{\ell=0}^{\infty}$ from this stationary distribution at time t. As $\sum_{\ell=0}^{\infty} \omega_{\ell,t} = 1$, it must be the case that $\sum_{\ell=0}^{\infty} \Delta_{\ell,t} = 0$ and hence $\Delta_{0,t} = -\sum_{\ell=1}^{\infty} \Delta_{\ell,t}$. Thus considering the sequence of deviations $\{\Delta_{\ell,t}\}_{\ell=1}^{\infty}$ is sufficient to know the behaviour of the whole sequence $\{\Delta_{\ell,t}\}_{\ell=0}^{\infty}$.

The laws of motion for the age distribution $\{\omega_{\ell,t}\}_{\ell=0}^{\infty}$ require that $\omega_{\ell,t} = (1-\alpha_\ell)\omega_{\ell-1,t-1}$ for all $\ell \geq 1$. These imply laws of motion for the deviations $\Delta_{\ell,t}$:

$$\Delta_{1,t} = -(1-\alpha_1)\sum_{\ell=1}^{\infty} \Delta_{\ell,t-1}, \quad \text{and } \Delta_{\ell,t} = (1-\alpha_\ell)\Delta_{\ell-1,t-1} \text{ for } \ell = 2, 3, \ldots, \qquad \text{[A.1.1]}$$

using the earlier formula for $\Delta_{0,t-1}$ in terms of the sequence $\{\Delta_{\ell,t-1}\}_{\ell=1}^{\infty}$.

The equations in [A.1.1] define a linear transformation of the sequence $\{\Delta_{\ell,t}\}_{\ell=1}^{\infty}$. Suppose ζ is an eigenvalue of this linear transformation, with the sequence $\{\mathrm{v}_\ell\}_{\ell=1}^{\infty}$ being the corresponding eigenvector. The eigenvalue-eigenvector pair is characterized by

$$\zeta \mathrm{v}_1 = -(1-\alpha_1)\sum_{\ell=1}^{\infty} \mathrm{v}_\ell, \quad \text{and} \quad \zeta \mathrm{v}_\ell = (1-\alpha_\ell)\mathrm{v}_{\ell-1} \quad \text{for} \quad \ell = 2, 3, \ldots. \tag{A.1.2}$$

The stability of the stationary age distribution $\{\omega_\ell\}_{\ell=0}^{\infty}$ is equivalent to all eigenvalues of the linear transformation having modulus less than one.

For a non-zero eigenvalue ζ, note that the equations in [A.1.2] imply $\mathrm{v}_1 \neq 0$, otherwise all elements of the sequence $\{\mathrm{v}_\ell\}_{\ell=1}^{\infty}$ would be zero, which would mean that it could not be an eigenvector (which must be non-zero). Applying [A.1.2] recursively yields

$$(1-\alpha_1)\mathrm{v}_\ell = \zeta^{-(\ell-1)}\left\{\prod_{\jmath=1}^{\ell}(1-\alpha_\jmath)\right\}\mathrm{v}_1 \quad \text{for} \quad \ell = 2, 3, \ldots,$$

and hence $(1-\alpha_1)\mathrm{v}_\ell = \zeta^{-(\ell-1)}\psi_\ell \mathrm{v}_1$ using the definition of the survival probabilities $\{\psi_\ell\}_{\ell=0}^{\infty}$. Substitution into the remaining equation from [A.1.2] implies

$$\left\{\sum_{\ell=0}^{\infty}\psi_\ell\zeta^{-\ell}\right\}\mathrm{v}_1 = 0,$$

which together with $\mathrm{v}_1 \neq 0$ requires that $\psi(\zeta^{-1}) = 0$. Thus, any eigenvalue ζ of the linear transformation from $\{\Delta_{\ell,t}\}_{\ell=1}^{\infty}$ to $\{\Delta_{\ell,t+1}\}_{\ell=1}^{\infty}$ is either zero, or its reciprocal ζ^{-1} is a root of the equation $\psi(z) = 0$. Similarly, the reciprocal of any root of $\psi(z) = 0$ will be an eigenvalue of the linear transformation.

If there is a $\rho > 1$ such that $\psi(z) = 0$ has no roots on $\mathscr{D}_\rho$ then all eigenvalues ζ must have modulus less than one. Conversely, note that $\mathscr{D}_\rho$ is a compact set for any fixed ρ. If this ρ is no more than the threshold found in part (i) then $\psi(z)$ is an analytic function on $\mathscr{D}_\rho$, so it has a finite number of roots in this set. Hence if all eigenvalues ζ have modulus less than one then there exists a minimum value of $|\zeta^{-1}|$, which is greater than one. It follows that there exists a $\rho > 1$ such that $\psi(z) = 0$ has no roots on $\mathscr{D}_\rho$. This completes the proof. ■

Lemma 2 *Suppose $\psi(z) \equiv \sum_{\ell=0}^{\infty}\psi_\ell z^\ell$ is a power series with coefficients satisfying $\psi_0 = 1$ and $0 \leq \psi_{\ell+1} \leq (1-\underline{\alpha})\psi_\ell$ for all $\ell \geq 0$ and for some $0 < \underline{\alpha} \leq 1$. Then there exists a $\rho > 1$ such that the equation $\psi(z) = 0$ has no roots in the set $\mathscr{D}_\rho \equiv \{z \in \mathbb{C} \mid |z| \leq \rho\}$.*

Proof Let ϖ be a number lying strictly between $(1-\underline{\alpha})$ and 1, which must satisfy $0 < \varpi < 1$. Since $\psi_{\ell+1} = (1-\alpha_{\ell+1})\psi_\ell$, the definition of ϖ then implies that $\psi_{\ell+1} \leq \varpi\psi_\ell$ for all $\ell \geq 0$. Now let ρ be any number strictly between 1 and the minimum of ϖ^{-1} and the radius of the disc on which $\psi(z)$ is analytic (greater than one), as established by Lemma 1.

Construct a new function $\mathfrak{F}(z) \equiv (1-\varpi z)\psi(z)$, which inherits the property that it is analytic on $\mathscr{D}_\rho$ from $\psi(z)$ using Lemma 1. Using the definition of $\psi(z)$ and collecting terms in common powers of z:

$$\mathfrak{F}(z) = 1 - \sum_{\ell=1}^{\infty}(\varpi\psi_{\ell-1} - \psi_\ell)z^\ell.$$

The function can be written as $\mathfrak{F}(z) = \mathfrak{F}_0(z) + \mathfrak{F}_1(z)$, where $\mathfrak{F}_0(z) \equiv 1$ and $\mathfrak{F}_1(z) \equiv -\sum_{\ell=1}^{\infty}(\varpi\psi_{\ell-1} -$

$\psi_\ell)z^\ell$ are defined. The modulus of $\mathfrak{F}_1(z)$ satisfies

$$|\mathfrak{F}_1(z)| \leq \sum_{\ell=1}^{\infty} |\varpi\psi_{\ell-1} - \psi_\ell||z|^\ell = \sum_{\ell=1}^{\infty}(\varpi\psi_{\ell-1} - \psi_\ell)|z|^\ell,$$

using the triangle inequality and the positivity of the coefficient of $|z|^\ell$. Now take any $z \in \mathscr{D}_\rho$. Since $|z|^\ell \leq \rho^\ell$, it follows that

$$\sum_{\ell=1}^{\infty}(\varpi\psi_{\ell-1} - \psi_\ell)|z|^\ell \leq \sum_{\ell=1}^{\infty}(\varpi\psi_{\ell-1} - \psi_\ell)\rho^\ell = \varpi\rho - (1 - \varpi\rho)\sum_{\ell=1}^{\infty}\psi_\ell\rho^\ell \leq \varpi\rho,$$

by collecting common terms in ψ_ℓ and using the non-negativity of $\{\psi_\ell\}_{\ell=0}^{\infty}$ together with $\psi_0 = 1$ and $0 < \varpi\rho < 1$. Combining the equations above yields $|\mathfrak{F}_1(z)| \leq \varpi\rho$, and hence $|\mathfrak{F}_1(z)| < |\mathfrak{F}_0(z)|$ for all $z \in \mathscr{D}_\rho$, since $|\mathfrak{F}_0(z)| = 1$.

As a constant function, $\mathfrak{F}_0(z)$ must be analytic, and consequently $\mathfrak{F}_1(z)$ inherits this property from $\mathfrak{F}(z)$. Since $\mathfrak{F}(z) = \mathfrak{F}_0(z) + \mathfrak{F}_1(z)$, Rouché's Theorem[1] implies that $\mathfrak{F}(z)$ and $\mathfrak{F}_0(z)$ have the same number of zeros on $\mathscr{D}_\rho$. Since $\mathfrak{F}_0(z)$ clearly has no zeros on this set, neither has $\mathfrak{F}(z)$. Because its definition ensures that $\mathfrak{F}(z)$ inherits any roots of $\psi(z) = 0$, this precludes $\psi(z)$ having a zero in $\mathscr{D}_\rho$ as well. This completes the proof. ■

Lemma 3 *The sequence of recursive parameters $\{\varphi_i\}_{i=1}^{\infty}$ generating the hazard function $\{\alpha_\ell\}_{\ell=1}^{\infty}$ using* [3.1] *can be written as*

$$\varphi_i = (-1)^i \sum_{(\jmath_1,\ldots,\jmath_{i+1})\in\mathscr{C}_{i+1}} \prod_{\ell=1}^{i+1} \sigma_{\jmath_\ell}, \tag{A.1.3}$$

where the sequence $\{\sigma_\ell\}_{\ell=1}^{\infty}$ is defined by $\sigma_1 \equiv -(1-\alpha_1)$ and $\sigma_\ell \equiv \alpha_\ell - \alpha_{\ell-1}$, and where $\{\mathscr{C}_i\}_{i=2}^{\infty}$ is a sequence of sets $\mathscr{C}_i$, with each $\mathscr{C}_i$ being a subset of the set of sequences

$$\mathscr{P}_i \equiv \left\{ (\jmath_1,\ldots,\jmath_i) \in \mathbb{N}^i \;\middle|\; 1 \leq \jmath_\ell \leq \ell \right\}. \tag{A.1.4}$$

Proof Define a sequence $\{\phi_i\}_{i=1}^{\infty}$ with $\phi_1 \equiv 1-\alpha$ and $\phi_i \equiv -\varphi_{i-1}$ for $i \geq 2$. With these definitions, the recursion [3.2] for the survival function $\{\psi_\ell\}_{\ell=0}^{\infty}$ reduces to:

$$\psi_\ell = \sum_{i=1}^{\ell} \phi_i\psi_{\ell-i},$$

with initial condition $\psi_0 = 1$. Using the initial condition, the order of the recursion can be reversed to yield

$$\phi_i = \psi_i - \sum_{j=1}^{i-1} \psi_{i-j}\phi_j. \tag{A.1.5}$$

The definition of the survival probabilities means that $\psi_i = \prod_{\ell=1}^{i}(1-\alpha_\ell)$, and the definition of the sequence $\{\sigma_\ell\}_{\ell=1}^{\infty}$ in the statement of the Lemma implies $1-\alpha_\ell = \sum_{j=1}^{\ell}(-\sigma_j)$. It follows that

$$\psi_i = \prod_{\ell=1}^{i}\sum_{j=1}^{\ell}(-\sigma_j) = (-1)^i \sum_{\jmath_1=1}^{1} \cdots \sum_{\jmath_i=1}^{i} \prod_{\ell=1}^{i} \sigma_{\jmath_\ell},$$

[1]See any text on complex analysis, such as Gamelin (2001), for further details about the theorem.

where the order of summation and multiplication is reversed in the final expression for ψ_i. Note that the definition of the set $\mathscr{P}_i$ of sequences $(\jmath_1,\ldots,\jmath_i)$ in [A.1.4] implies that ψ_i can be written as a sum of products $\prod_{\ell=1}^{i}\sigma_{\jmath_\ell}$ over all sequences in the set $\mathscr{P}_i$:

$$\psi_i = (-1)^i \sum_{(\jmath_1,\ldots,\jmath_i)\in\mathscr{P}_i} \prod_{\ell=1}^{i} \sigma_{\jmath_\ell}. \tag{A.1.6}$$

Now let $\mathscr{C}_1 \equiv \mathscr{P}_1 \equiv \{\ (1)\ \}$, where the expression for $\mathscr{P}_1$ comes from [A.1.4], and define the sets $\mathscr{C}_i$ in the sequence $\{\mathscr{C}_i\}_{i=2}^{\infty}$ with the recursion

$$\mathscr{C}_i \equiv \mathscr{P}_i \backslash \left(\bigcup_{j=1}^{i-1} (\mathscr{C}_j \times \mathscr{P}_{i-j}) \right), \tag{A.1.7}$$

in terms of the sequence $\{\mathscr{P}_i\}_{i=1}^{\infty}$ specified in [A.1.4]. Observe that $\mathscr{C}_i \subseteq \mathscr{P}_i$ is well defined if $\mathscr{C}_j \subseteq \mathscr{P}_j$ for all $j = 1,\ldots,i-1$ because $(\jmath_1,\ldots,\jmath_{i-j}) \in \mathscr{P}_{i-j}$ implies $\jmath_\ell \leq \ell + j$. Since $\mathscr{C}_1 \subseteq \mathscr{P}_1$ by definition, the claim that $\mathscr{C}_i \subseteq \mathscr{P}_i$ for all i follows by induction.

Now consider the following claim about the sequence of sets $\{\mathscr{C}_i\}_{i=1}^{\infty}$ defined by [A.1.7]:

$$(\mathscr{C}_j \times \mathscr{P}_{i-j}) \cap (\mathscr{C}_k \times \mathscr{P}_{i-k}) = \emptyset, \quad \text{for all } \ i,j,k \in \mathbb{N} \ \text{ with } \ j,k < i \ \ , \ \ j \neq k. \tag{A.1.8}$$

Suppose for contradiction that $(\mathscr{C}_j \times \mathscr{P}_{i-j}) \cap (\mathscr{C}_k \times \mathscr{P}_{i-k}) \neq \emptyset$, and without loss of generality take $j > k$. Hence there is a sequence $(\jmath_1,\ldots,\jmath_i) \in \mathbb{N}^i$ such that $(\jmath_1,\ldots,\jmath_j) \in \mathscr{C}_j$, $(\jmath_1,\ldots,\jmath_k) \in \mathscr{C}_k$, and $(\jmath_{k+1},\ldots,\jmath_i) \in \mathscr{P}_{i-k}$. This implies that $(\jmath_{k+1},\ldots,\jmath_j) \in \mathscr{P}_{j-k}$ because the first $j-k$ terms of a sequence of length $i-k > j-k$ in $\mathscr{P}_{i-k}$ must necessarily belong to $\mathscr{P}_{j-k}$ given the definition in [A.1.4]. Thus it follows that there exists a $(\jmath_1,\ldots,\jmath_j) \in \mathscr{C}_j \cap (\mathscr{C}_k \times \mathscr{P}_{j-k})$ for some $k < j$. However, this directly contradicts the definition of $\mathscr{C}_j$ in [A.1.7]. Therefore, [A.1.8] must be true.

Given the recursion for $\{\phi_i\}_{i=1}^{\infty}$ in [A.1.5] and the expression for ψ_i in [A.1.6], the following provides a formula for ϕ_i:

$$\phi_i = \left\{ (-1)^i \sum_{(\jmath_1,\ldots,\jmath_i)\in\mathscr{P}_i} \prod_{\ell=1}^{i} \sigma_{\jmath_\ell} \right\} - \sum_{j=1}^{i-1} \phi_j \left\{ (-1)^{i-j} \sum_{(\jmath_1,\ldots,\jmath_{i-j})\in\mathscr{P}_{i-j}} \prod_{\ell=1}^{i-j} \sigma_{\jmath_\ell} \right\}. \tag{A.1.9}$$

It is claimed that the following equation holds for all $i = 1,2,\ldots$:

$$\phi_i = (-1)^i \sum_{(\jmath_1,\ldots,\jmath_i)\in\mathscr{C}_i} \prod_{\ell=1}^{i} \sigma_{\jmath_\ell}, \tag{A.1.10}$$

Suppose this statement has already been proved for $j = 1,\ldots,i-1$ and substitute it into [A.1.9] to obtain:

$$\phi_i = \left((-1)^i \sum_{(\jmath_1,\ldots,\jmath_i)\in\mathscr{P}_i} \prod_{\ell=1}^{i} \sigma_{\jmath_\ell} \right) - \sum_{j=1}^{i-1} \left((-1)^i \sum_{(\jmath_1,\ldots,\jmath_i)\in(\mathscr{C}_j\times\mathscr{P}_{i-j})} \prod_{\ell=1}^{i} \sigma_{\jmath_\ell} \right), \tag{A.1.11}$$

where the following has been used:

$$\left\{ (-1)^j \sum_{(\jmath_1,\ldots,\jmath_j)\in\mathscr{C}_j} \prod_{\ell=1}^{j} \sigma_{\jmath_\ell} \right\} \left\{ (-1)^{i-j} \sum_{(\jmath_1,\ldots,\jmath_{i-j})\in\mathscr{P}_{i-j}} \prod_{\ell=1}^{i-j} \sigma_{\jmath_\ell} \right\} = (-1)^i \sum_{(\jmath_1,\ldots,\jmath_i)\in(\mathscr{C}_j\times\mathscr{P}_{i-j})} \prod_{\ell=1}^{i} \sigma_{\jmath_\ell}.$$

It follows from [A.1.11] that [A.1.10] holds for i if the sets $\mathscr{C}_j \times \mathscr{P}_{i-j}$ and $\mathscr{C}_k \times \mathscr{P}_{i-k}$ are disjoint for all $j \neq k$, which is the claim [A.1.8] established earlier. Now note that the definitions of $\mathscr{C}_1$, σ_1 and ϕ_1 imply that [A.1.10] holds for $i = 1$. Therefore, the expression for ϕ_i in [A.1.10] is verified for all i by induction. Since $\varphi_i = (-1)\phi_{i+1}$ by definition, equation [A.1.3] is demonstrated for the particular sets $\{\mathscr{C}_i\}_{i=2}^{\infty}$ characterized in [A.1.7]. This completes the proof. ■

A.2 Proof of Proposition 1

Let $\psi(z) \equiv \sum_{\ell=0}^{\infty} \psi_\ell z^\ell$ denote the z-transform of the survival probabilities $\{\psi_\ell\}_{\ell=0}^{\infty}$. Lemma 1 demonstrates that $\psi(z)$ is analytic on $\mathscr{D}_\rho \equiv \left\{z \in \mathbb{C} \;\middle|\; |z| \leq \rho\right\}$ for some $\rho > 1$. Since $1 \in \mathscr{D}_\rho$, it follows that $\psi(1) = \sum_{\ell=0}^{\infty} \psi_\ell$ is finite (and positive given that $\psi_0 = 1$ and $\psi_\ell \geq 0$). Define $\omega_0 = \psi(1)^{-1}$ and $\omega_\ell = \omega_0 \psi_\ell$ for $\ell \geq 1$. By construction, the sequence $\{\omega_\ell\}_{\ell=0}^{\infty}$ satisfies $\sum_{\ell=0}^{\infty} \omega_\ell = 1$, and $\omega_\ell = (1 - \alpha_\ell)\omega_{\ell-1}$ since $\psi_\ell = (1 - \alpha_\ell)\psi_{\ell-1}$. Note also that

$$\sum_{\ell=1}^{\infty} \alpha_\ell \omega_{\ell-1} = \omega_0 \sum_{\ell=1}^{\infty} \alpha_\ell \psi_{\ell-1} = \omega_0 \sum_{\ell=1}^{\infty} (\psi_{\ell-1} - \psi_\ell) = \omega_0,$$

as $\psi_\ell = (1 - \alpha_\ell)\psi_{\ell-1}$ and $\psi_0 = 1$. This confirms that $\{\omega_\ell\}_{\ell=0}^{\infty}$ is a stationary age distribution. There can be only one such distribution because $\{\omega_\ell\}_{\ell=0}^{\infty}$ must satisfy $\omega_\ell = (1 - \alpha_\ell)\omega_{\ell-1}$ for all $\ell \geq 1$. This leaves only ω_0 to be determined, but this is pinned down by the requirement $\sum_{\ell=0}^{\infty} \omega_\ell = 1$.

Now suppose that $\alpha_\ell \geq \underline{\alpha}$ for all ℓ for some $\underline{\alpha}$ satisfying $0 < \underline{\alpha} < 1$. Since $\psi_{\ell+1} = (1 - \alpha_{\ell+1})\psi_\ell$, this implies $0 \leq \psi_{\ell+1} \leq (1 - \underline{\alpha})\psi_\ell$ for all ℓ. Hence Lemma 2 implies that there exists a $\rho > 1$ such that $\psi(z) = 0$ has no roots on $\mathscr{D}_\rho$. Lemma 1 shows that this condition implies that the stationary age distribution is stable, completing the proof.

A.3 Proof of Proposition 2

The first step is to derive the standard representation of the Phillips curve [2.6] from equations [2.3], [2.4] and [2.5]. Let $\psi(z) \equiv \sum_{\ell=0}^{\infty} \psi_\ell z^\ell$ and $\omega(z) \equiv \sum_{\ell=0}^{\infty} \omega_\ell z^\ell$ be the z-transforms of the sequences of survival probabilities $\{\psi_\ell\}_{\ell=0}^{\infty}$ and the age distribution $\{\omega_\ell\}_{\ell=0}^{\infty}$. Written in terms of the lag and forward operators $\mathbb{L}$ and $\mathbb{F}$, equations [2.4] and [2.5] become:

$$\mathsf{r}_t = \psi(\beta)^{-1} \mathbb{E}_t \left[\psi(\beta\mathbb{F})\mathsf{p}_t^*\right], \quad \text{and} \quad \mathsf{p}_t = \omega(\mathbb{L})\mathsf{r}_t. \tag{A.3.1}$$

Note that $\omega_\ell = \psi_\ell \omega_0$, so $\omega_0 = \psi(1)^{-1}$ since $\omega(1) = 1$. This justifies the relationship $\omega(z) = \psi(1)^{-1}\psi(z)$ between $\omega(z)$ and $\psi(z)$. By using this result, eliminating the reset price r_t from [A.3.1], and substituting the expression for p_t^* from [2.3]:

$$\left\{\mathbb{I} - \psi(1)^{-1}\psi(\beta)^{-1}\psi(\mathbb{L})\mathbb{E}_t\psi(\beta\mathbb{F})\right\} \mathsf{p}_t = \nu \left\{\psi(1)^{-1}\psi(\beta)^{-1}\psi(\mathbb{L})\mathbb{E}_t\psi(\beta\mathbb{F})\right\} \mathsf{x}_t, \tag{A.3.2}$$

where $\mathbb{I}$ denotes the identity operator.

The left-hand side of [A.3.2] is

$$\left\{\mathbb{I} - \psi(1)^{-1}\psi(\beta)^{-1}\psi(\mathbb{L})\mathbb{E}_t\psi(\beta\mathbb{F})\right\} \mathsf{p}_t = \mathsf{p}_t - \frac{\sum_{\jmath=0}^{\infty} \psi_\jmath \sum_{\ell=0}^{\infty} \beta^\ell \psi_\ell \mathbb{E}_{t-\jmath}\mathsf{p}_{t-\jmath+\ell}}{\sum_{\jmath=0}^{\infty} \psi_\jmath \sum_{\ell=0}^{\infty} \beta^\ell \psi_\ell}. \tag{A.3.3}$$

The definition of inflation $\pi_t = \mathsf{p}_t - \mathsf{p}_{t-1}$ implies $\mathsf{p}_{t-\jmath+\ell} = \mathsf{p}_{t-\jmath} + \pi_{t-\jmath+1} + \cdots + \pi_{t-\jmath+\ell}$, so

$$\left\{\mathbb{I} - \psi(1)^{-1}\psi(\beta)^{-1}\psi(\mathbb{L})\mathbb{E}_t\psi(\beta\mathbb{F})\right\} \mathsf{p}_t = \frac{\sum_{\ell=0}^{\infty} \psi_\ell(\mathsf{p}_t - \mathsf{p}_{t-\ell})}{\sum_{\ell=0}^{\infty} \psi_\ell} - \frac{\sum_{\jmath=0}^{\infty} \sum_{\ell=0}^{\infty} \psi_\jmath \left(\sum_{i=\ell}^{\infty} \beta^i \psi_i\right) \mathbb{E}_{t-\jmath}\pi_{t-\jmath+\ell}}{\sum_{\jmath=0}^{\infty} \sum_{\ell=0}^{\infty} \beta^\ell \psi_\jmath \psi_\ell}.$$

The definition of inflation also implies $\mathsf{p}_t - \mathsf{p}_{t-\ell} = \pi_{t-\ell+1} + \cdots + \pi_t$, thus

$$\left\{\mathbb{I} - \psi(1)^{-1}\psi(\beta)^{-1}\psi(\mathbb{L})\mathbb{E}_t\psi(\beta\mathbb{F})\right\}\mathsf{p}_t = \frac{\sum_{\ell=1}^{\infty}\psi_\ell}{\sum_{\ell=0}^{\infty}\psi_\ell}\pi_t + \frac{\sum_{\ell=1}^{\infty}\left(\sum_{i=\ell+1}^{\infty}\psi_i\right)\pi_{t-\ell}}{\sum_{\ell=0}^{\infty}\psi_\ell} - \frac{\sum_{\jmath=0}^{\infty}\sum_{\ell=0}^{\infty}\psi_\jmath\left(\sum_{i=\ell}^{\infty}\beta^i\psi_i\right)\mathbb{E}_{t-\jmath}\pi_{t-\jmath+\ell}}{\sum_{\jmath=0}^{\infty}\sum_{\ell=0}^{\infty}\beta^\ell\psi_\jmath\psi_\ell}. \qquad \text{[A.3.4]}$$

The right-hand side of [A.3.2] is

$$\nu\left\{\psi(1)^{-1}\psi(\beta)^{-1}\psi(\mathbb{L})\mathbb{E}_t\psi(\beta\mathbb{F})\right\}\mathsf{x}_t = \nu\frac{\sum_{\jmath=0}^{\infty}\sum_{\ell=0}^{\infty}\beta^\ell\psi_\jmath\psi_\ell\mathbb{E}_{t-\jmath}\mathsf{x}_{t-\jmath+\ell}}{\sum_{\jmath=0}^{\infty}\sum_{\ell=0}^{\infty}\beta^\ell\psi_\jmath\psi_\ell}. \qquad \text{[A.3.5]}$$

Using the expressions in [A.3.4] and [A.3.5], it is seen that [A.3.2] is equivalent to the standard Phillips curve equation [2.6] with the coefficients:

$$\mathrm{a}_\ell = -\frac{\sum_{i=\ell+1}\psi_i}{\sum_{i=1}^{\infty}\psi_i}, \quad \mathrm{b}_{\jmath\ell} = \frac{\psi_\jmath\sum_{i=\ell}^{\infty}\psi_i}{\sum_{i=1}^{\infty}\sum_{h=0}^{\infty}\beta^h\psi_i\psi_h}, \quad \text{and } \mathrm{c}_{\jmath\ell} = \frac{\beta^\ell\psi_\jmath\psi_\ell}{\sum_{i=1}^{\infty}\sum_{h=0}^{\infty}\beta^h\psi_i\psi_h}.$$

Now suppose the hazard function implies that the stationary age distribution of prices is stable. As Lemma 1 shows, this is equivalent to there being a $\rho > 1$ such that $\psi(z)$ has no roots in the set $\mathscr{D}_\rho \equiv \left\{z \in \mathbb{C} \,\middle|\, |z| \leq \rho\right\}$. Under this condition, the function $\phi(z) \equiv \psi(z)^{-1}$ is analytic on $\mathscr{D}_\rho$, which is equivalent to $\phi(z)$ being equal to its Taylor expansion around $z = 0$ for all $z \in \mathscr{D}_\rho$. Thus, $\phi(z) \equiv 1 - \sum_{\ell=1}^{\infty}\phi_\ell z^\ell$ for some sequence of numbers $\{\phi_\ell\}_{\ell=1}^{\infty}$, with $\sum_{\ell=1}^{\infty}|\phi_\ell| < \infty$ since $\mathscr{D}_\rho$ encloses the unit circle. The first term in the Taylor series of $\phi(z)$ is 1 because $\psi(0) = \psi_0 = 1$.

Since $\phi(z)\psi(z) = 1$ for all $|z| \leq 1$, it follows that $\mathbb{I} = \psi(\mathbb{L})\phi(\mathbb{L})$, which allows the left-hand side of [A.3.2] to be expressed equivalently as follows:

$$\left\{\mathbb{I} - \psi(1)^{-1}\psi(\beta)^{-1}\psi(\mathbb{L})\mathbb{E}_t\psi(\beta\mathbb{F})\right\}\mathsf{p}_t = \psi(1)^{-1}\psi(\beta)^{-1}\psi(\mathbb{L})\left\{\psi(1)\psi(\beta)\phi(\mathbb{L}) - \mathbb{E}_t\psi(\beta\mathbb{F})\right\}\mathsf{p}_t. \qquad \text{[A.3.6]}$$

It also follows from $\phi(z)\psi(z) = 1$ that $\mathbb{I} = \psi(\beta\mathbb{F})\phi(\beta\mathbb{F})$, and thus $\phi(\mathbb{L}) = \mathbb{I}\phi(\mathbb{L}) = \psi(\beta\mathbb{F})\phi(\beta\mathbb{F})\phi(\mathbb{L})$. Furthermore, note that the power series $\phi(\mathbb{L}) \equiv \sum_{\ell=0}^{\infty}\phi_\ell\mathbb{L}^\ell$ contains only non-negative powers of the lag operator $\mathbb{L}$, so $\phi(\mathbb{L})\mathsf{p}_t = \mathbb{E}_t\phi(\mathbb{L})\mathsf{p}_t$. Putting these two results together implies $\phi(\mathbb{L})\mathsf{p}_t = \mathbb{E}_t\psi(\beta\mathbb{F})\phi(\beta\mathbb{F})\phi(\mathbb{L})\mathsf{p}_t$. Then observe that because the power series $\psi(\beta\mathbb{F}) \equiv \sum_{\ell=0}^{\infty}\beta^\ell\psi_\ell\mathbb{F}^\ell$ contains only non-negative powers of $\mathbb{F}$, the law of iterated expectations (from which it follows that the conditional expectation operator $\mathbb{E}_t$ commutes with all non-negative powers of the forward operator $\mathbb{F}$) implies

$$\phi(\mathbb{L})\mathsf{p}_t = \mathbb{E}_t\left[\psi(\beta\mathbb{F})\left\{\mathbb{E}_t\phi(\beta\mathbb{F})\phi(\mathbb{L})\right\}\mathsf{p}_t\right].$$

This result, together with [A.3.6], and noting $\phi(\beta\mathbb{F})\phi(\mathbb{L}) = \phi(\mathbb{L})\phi(\beta\mathbb{F})$, $\psi(1) = \phi(1)^{-1}$ and $\psi(\beta) = \phi(\beta)^{-1}$, yields

$$\left\{\mathbb{I} - \psi(1)^{-1}\psi(\beta)^{-1}\psi(\mathbb{L})\mathbb{E}_t\psi(\beta\mathbb{F})\right\}\mathsf{p}_t = \left\{\psi(1)^{-1}\psi(\beta)^{-1}\psi(\mathbb{L})\mathbb{E}_t\psi(\beta\mathbb{F})\right\}\mathbb{E}_t\left\{\phi(1)^{-1}\phi(\beta)^{-1}\phi(\mathbb{L})\phi(\beta\mathbb{F}) - \mathbb{I}\right\}\mathsf{p}_t. \qquad \text{[A.3.7]}$$

Equating this expression to the right-hand side of [A.3.2] leads to the following equation that is exactly equivalent to the Phillips curve [2.6]:

$$\left\{\psi(1)^{-1}\psi(\beta)^{-1}\psi(\mathbb{L})\mathbb{E}_t\psi(\beta\mathbb{F})\right\}\left(\mathbb{E}_t\left[\left\{\phi(1)^{-1}\phi(\beta)^{-1}\phi(\mathbb{L})\phi(\beta\mathbb{F}) - \mathbb{I}\right\}\mathsf{p}_t\right] - \nu\mathsf{x}_t\right) = 0. \qquad \text{[A.3.8]}$$

Now define the function $\chi(z) \equiv \phi(1)^{-1}\phi(\beta)^{-1}\phi(z)\phi(\beta z^{-1}) - 1$, which is analytic on $\mathscr{A}_\rho \equiv \left\{z \in \mathbb{C} \,\middle|\, \beta\rho^{-1} \leq |z| \leq \rho\right\}$ given that $\phi(z)$ is analytic and has no roots on $\mathscr{D}_\rho$. Notice that $\chi(1) = 0$, so it follows that there is another function $\theta(z)$ analytic on $\mathscr{A}_\rho$ such that $\chi(z) = (1-z)\theta(z)$. The function $\theta(z)$ is equal to its Laurent series expansion $\theta(z) = \sum_{\ell\to-\infty}^{\infty} \theta_\ell z^\ell$ for all $z \in \mathscr{A}_\rho$. Since $\mathscr{A}_\rho$ includes the unit circle, it follows that $\sum_{\ell\to-\infty}^{\infty} |\theta_\ell| < \infty$. Make the following definitions of sequences $\{\lambda_\ell\}_{\ell=1}^{\infty}$ and $\{\xi_\ell\}_{\ell=1}^{\infty}$, and coefficient κ appearing in the new Phillips curve [2.7]:

$$\lambda_\ell \equiv -\frac{\theta_\ell}{\theta_0}, \quad \xi_\ell \equiv -\frac{\theta_{-\ell}}{\theta_0}, \quad \text{and} \quad \kappa \equiv \frac{1}{\theta_0}.$$

With these definitions, the sequences clearly satisfy $\sum_{\ell=1}^{\infty} |\lambda_\ell| < \infty$ and $\sum_{\ell=1}^{\infty} |\xi_\ell| < \infty$ (it can be shown that $\theta_0 \neq 0$ using the argument presented in the proof of Proposition 6).

Now define

$$\mathsf{d}_t \equiv \pi_t - \sum_{\ell=1}^{\infty} \lambda_\ell \pi_{t-\ell} - \sum_{\ell=1}^{\infty} \xi_\ell \mathbb{E}_t \pi_{t+\ell} - \nu\kappa \mathsf{x}_t, \tag{A.3.9}$$

and note that the definitions above imply $\mathsf{d}_t = \kappa\{\mathbb{E}_t[\theta(\mathbb{L})\pi_t] - \nu \mathsf{x}_t\}$. Since $\pi_t = (\mathbb{I} - \mathbb{L})\mathsf{p}_t$ and $\chi(\mathbb{L}) = (\mathbb{I}-\mathbb{L})\theta(\mathbb{L})$, it follows that $\theta(\mathbb{L})\pi_t = \chi(\mathbb{L})\mathsf{p}_t$ and hence $\mathsf{d}_t = \kappa\{\mathbb{E}_t[\chi(\mathbb{L})\mathsf{p}_t] - \nu\mathsf{x}_t\}$. Therefore, comparing this expression for d_t to equation [A.3.8], the Phillips curve [2.6] is equivalent to $\left\{\psi(1)^{-1}\psi(\beta)^{-1}\psi(\mathbb{L})\mathbb{E}_t\psi(\beta\mathbb{F})\right\}\mathsf{d}_t = 0$, and thus to

$$\psi(\mathbb{L})\mathbb{E}_t[\psi(\beta\mathbb{F})\mathsf{d}_t] = 0, \tag{A.3.10}$$

holding in all time periods t. Let $\mathsf{e}_t \equiv \mathbb{E}_t[\psi(\beta\mathbb{F})\mathsf{d}_t]$, with equation [A.3.10] being equivalent to $\psi(\mathbb{L})\mathsf{e}_t = 0$ for all t.

Note that by comparing [A.3.9] to [2.7], the new Phillips curve equation is equivalent to $\mathsf{d}_t = 0$ for all t. Suppose the new Phillips curve equation [2.7] holds. Thus $\mathsf{d}_t = 0$ for all t and hence $\mathsf{e}_t = 0$ for all t as well. It follows that $\psi(\mathbb{L})\mathsf{e}_t = 0$, so the original Phillips curve [2.6] must hold.

Conversely, suppose the original Phillips curve [2.6] holds, which implies $\psi(\mathbb{L})\mathsf{e}_t = 0$ using [A.3.10]. Given the stability of the stationary age distribution, it has been shown that $\psi(z) = 0$ has no roots on or inside the unit circle. Thus if $\mathsf{e}_{t_0} \neq 0$ for some t_0, it follows from $\psi(\mathbb{L})\mathsf{e}_t = 0$ that e_t is unbounded for time periods before t_0. Now given the location of the roots of $\psi(z) = 0$, it follows from $0 < \beta < 1$ that $\psi(\beta z) = 0$ has no roots on or inside the unit circle. Hence if $\mathsf{e}_t = 0$ for all t, the only bounded solution of $\mathsf{e}_t \equiv \mathbb{E}_t[\psi(\beta\mathbb{F})\mathsf{d}_t]$ is $\mathsf{d}_t = 0$ for all t. On the other hand, if e_t is unbounded over all time periods t, then d_t must also be unbounded. If d_t is unbounded then equation [A.3.9] shows that either inflation π_t or real marginal cost x_t must be unbounded. Consequently, if attention is restricted to bounded rational expectations solutions (as is conventional), the original Phillips curve [2.6] implies $\mathsf{e}_t = 0$ for all t, and hence $\mathsf{d}_t = 0$ for all t. This then demonstrates that the new Phillips curve [2.7] must hold, completing the proof.

A.4 Proof of Proposition 3

Let $\psi(z) \equiv \sum_{\ell=0}^{\infty} \psi_\ell z^\ell$ denote the z-transform of the sequence of survival probabilities $\{\psi_\ell\}_{\ell=0}^{\infty}$ generated by some hazard function $\{\alpha_\ell\}_{\ell=1}^{\infty}$ from parameters α and $\{\varphi_i\}_{i=1}^{n}$ using the recursion [3.1]. Define the polynomial

$$\phi(z) = 1 - \left(1 - \alpha + \sum_{i=1}^{n} \varphi_i\right) z + \sum_{j=1}^{n} \varphi_j z^{j+1} \tag{A.4.1}$$

using these parameters. Since the recursion in [3.1] is equivalent to [3.2], by multiplying the power series $\phi(z)$ and $\psi(z)$ and noting that $\psi_0 = 1$, it follows that $\phi(z)\psi(z) = 1$ for all z for which $\psi(z)$ is analytic.

The hazard function implies a unique stationary age distribution $\{\omega_\ell\}_{\ell=0}^{\infty}$, with its z-transform denoted by $\omega(z) \equiv \sum_{\ell=0}^{\infty} \omega_\ell z^\ell$. Since $\omega_\ell = (1-\alpha_\ell)\omega_{\ell-1}$ and $\psi_\ell = (1-\alpha_\ell)\psi_{\ell-1}$, it follows that $\omega(z)$

is a multiple of $\psi(z)$. In particular, as $\psi_0 = 1$, it must be the case that $\omega(z) = \omega_0 \psi(z)$. As $\{\omega_\ell\}_{\ell=0}^{\infty}$ is a probability distribution, it follows that $\omega(1) = 1$, and thus $\omega_0 = \psi(1)^{-1}$ and $\omega(z) = \psi(1)^{-1}\psi(z)$. Together with $\phi(z)\psi(z) = 1$, it is established that $\phi(z)\omega(z) = \psi(1)^{-1}$. Since $\psi(1)^{-1} = \phi(1)$:

$$\omega(z) = \phi(1)\phi(z)^{-1}. \qquad \text{[A.4.2]}$$

(i) Let $\bar{\alpha}$ denote the average probability of price adjustment, calculated with respect to the stationary age distribution of prices at the beginning of any period. This distribution is given by $\{\omega_{\ell-1}\}_{\ell=1}^{\infty}$, so $\bar{\alpha} = \sum_{\ell=1}^{\infty} \omega_{\ell-1}\alpha_\ell$. Using the fact that $\omega_\ell = (1-\alpha_\ell)\omega_{\ell-1}$, it follows that $\omega_{\ell-1}\alpha_\ell = \omega_{\ell-1} - \omega_\ell$ and thus

$$\bar{\alpha} = \sum_{\ell=1}^{\infty}(\omega_{\ell-1} - \omega_\ell) = \sum_{\ell=0}^{\infty}\omega_\ell - \sum_{\ell=1}^{\infty}\omega_\ell = \omega_0. \qquad \text{[A.4.3]}$$

The fraction of newly set prices is ω_0. Since $\omega_0 = \omega(0)$ and $\phi(0) = 1$, it follows from [A.4.1] and [A.4.2] that

$$\bar{\alpha} = \omega_0 = \phi(1) = \alpha, \qquad \text{[A.4.4]}$$

for all values of $\{\varphi_i\}_{i=1}^{n}$.

(ii) Now consider the expected duration of a newly set price. If $\varsigma_\ell \equiv 1 - \alpha_\ell$ denotes the probability of price stickiness in the current period if ℓ periods have elapsed since the last change then $\alpha_\ell \prod_{\jmath=1}^{\ell-1} \varsigma_\jmath$ is the probability that a price will survive for exactly ℓ periods after first being set before being changed. The expected duration is denoted by $\hbar$:

$$\hbar \equiv \sum_{\ell=1}^{\infty} \ell\alpha_\ell \prod_{\jmath=1}^{\ell-1} \varsigma_\jmath.$$

The definition of the survival probabilities $\{\psi_\ell\}_{\ell=0}^{\infty}$ implies $\psi_{\ell-1} = \prod_{\jmath=1}^{\ell-1}\varsigma_\jmath$. Together with $\alpha_\ell\psi_{\ell-1} = \psi_{\ell-1} - \psi_\ell$, the expected duration is given by

$$\hbar = \sum_{\ell=1}^{\infty} \ell\alpha_\ell\psi_{\ell-1} = \sum_{\ell=1}^{\infty}\ell(\psi_{\ell-1} - \psi_\ell) = \sum_{\ell=0}^{\infty}(\ell+1)\psi_\ell - \sum_{\ell=0}^{\infty}\ell\psi_\ell = \sum_{\ell=0}^{\infty}\psi_\ell = \psi(1). \qquad \text{[A.4.5]}$$

As $\phi(z)\psi(z) = 1$, it follows that $\psi(1) = \phi(1)^{-1}$. The result in [A.4.4] then implies that $\hbar = \alpha^{-1}$.

(iii) Let $\hbar_\alpha$ denote the average age of the prices that are changed. Using Bayes' law, the probability that a price has age ℓ conditional on being changed is the product of α_ℓ and $\omega_{\ell-1}$ divided by $\alpha = \omega_0$. Since $\omega_{\ell-1}/\omega_0 = \psi_{\ell-1}$, it follows that $\hbar_\alpha$ is given by

$$\hbar_\alpha = \sum_{\ell=1}^{\infty}\ell\alpha_\ell\psi_{\ell-1}. \qquad \text{[A.4.6]}$$

The result in [A.4.5] then implies $\hbar_\alpha = \hbar = \alpha^{-1}$.

Let $\hbar_\varsigma$ denote the average age of the prices that are not changed. Again, using Bayes' law, the probability that a price has age ℓ conditional on not being changed is the product of $\varsigma_\ell = 1 - \alpha_\ell$ and $\omega_{\ell-1}$ divided by $1 - \alpha$. Thus $\hbar_\varsigma$ is given by

$$\hbar_\varsigma = \sum_{\ell=1}^{\infty}\ell\frac{\varsigma_\ell\omega_{\ell-1}}{1-\alpha}.$$

Using $\varsigma_\ell = 1 - \alpha_\ell$ and $\omega_{\ell-1} = \alpha\psi_{\ell-1}$ since $\omega_0 = \alpha$:

$$\hbar_\varsigma = \frac{1}{1-\alpha}\left(\sum_{\ell=1}^{\infty} \ell\omega_{\ell-1} - \alpha\sum_{\ell=1}^{\infty} \ell\alpha_\ell\psi_{\ell-1}\right) = \frac{1}{1-\alpha}\left(\sum_{\ell=0}^{\infty} \omega_\ell + \sum_{\ell=0}^{\infty} \ell\omega_\ell - \alpha\sum_{\ell=1}^{\infty} \ell\alpha_\ell\psi_{\ell-1}\right).$$

Note that $\sum_{\ell=0}^{\infty} \omega_\ell = 1$. From the definition of $\omega(z)$ it follows that $\omega'(z) = \sum_{\ell=0}^{\infty} \ell\omega_\ell z^{\ell-1}$ and thus $\omega'(1) = \sum_{\ell=0}^{\infty} \ell\omega_\ell$. Substituting these results and using the expression for $\hbar_\alpha$ from [A.4.6] together with $\hbar_\alpha = \alpha^{-1}$ to deduce:

$$\hbar_\varsigma = \frac{1}{1-\alpha}\left(1 + \omega'(1) - \alpha\alpha^{-1}\right) = \frac{\omega'(1)}{1-\alpha}. \qquad \text{[A.4.7]}$$

Differentiation of both sides of [A.4.2] yields:

$$\omega'(z) = -\frac{\phi(1)\phi'(z)}{\phi(z)^2},$$

and hence $\omega'(1) = -\phi'(1)\phi(1)^{-1}$. Differentiation of the polynomial $\phi(z)$ in [A.4.1] implies $\phi'(z) = -\left(1 - \alpha + \sum_{i=1}^{n} \varphi_i\right) + \sum_{\jmath=1}^{n} (\jmath+1)\varphi_\jmath z^\jmath$, from which it follows that $\phi'(1) = -\left(1 - \alpha - \sum_{i=1}^{n} i\varphi_i\right)$. And since $\phi(1) = \alpha$:

$$\omega'(1) = \left(1 - \alpha - \sum_{i=1}^{n} i\varphi_i\right)\alpha^{-1}. \qquad \text{[A.4.8]}$$

Therefore, using [A.4.7], the difference between the average ages of prices conditional on adjustment and non-adjustment is

$$\hbar_\alpha - \hbar_\varsigma = \alpha^{-1} - \left(1 - \alpha - \sum_{i=1}^{n} i\varphi_i\right)\alpha^{-1}(1-\alpha)^{-1} = \left(\sum_{i=1}^{n} i\varphi_i\right)\alpha^{-1}(1-\alpha)^{-1}.$$

(iv) Let $\hbar \equiv \sum_{\ell=0}^{\infty} \ell\omega_\ell$ denote the average age of prices actually in use according to the stationary distribution $\{\omega_\ell\}_{\ell=0}^{\infty}$. Using the definition of $\omega(z)$ it follows that $\hbar = \omega'(1)$. Hence, [A.4.8] implies

$$\hbar = \left(1 - \alpha - \sum_{i=1}^{n} i\varphi_i\right)\alpha^{-1} = \left(1 - \sum_{i=1}^{n} i\varphi_i\right)\alpha^{-1} - 1. \qquad \text{[A.4.9]}$$

(v) The hazard function recursion [3.1] implies that the probability of adjusting the most recently set price is

$$\alpha_1 = \alpha - \sum_{i=1}^{n} \varphi_i.$$

So α_1 is clearly strictly decreasing in each φ_i.

Let $\alpha_\infty \equiv \lim_{\ell\to\infty} \alpha_\ell$ be the limiting value of the hazard function for price spells of arbitrarily long duration. The recursion for the hazard function is equivalent to the linear recursion for the survival probabilities $\{\psi_\ell\}_{\ell=0}^{\infty}$ in [3.2]. The recursion [3.2] is a linear difference equation with $\phi(z^{-1}) = 0$ in [A.4.1] being the characteristic polynomial (since $\phi(z)\psi(z) = 1$).

Now consider parameter values α and $\{\varphi_i\}_{i=1}^{n}$ such that $\phi(z) = 0$ has no repeated roots. This will be without loss of generality because there is always a set of parameters implying no repeated roots arbitrarily close to parameters for which there are repeated roots. With no repeated roots, the solution for the sequence of survival probabilities $\{\psi_\ell\}_{\ell=0}^{\infty}$ takes the following general form

$$\psi_\ell = \sum_{j=1}^{n+1} \varkappa_j \zeta_j^\ell, \qquad \text{[A.4.10]}$$

for some sequence of coefficients $\{\varkappa_j\}_{j=1}^{n+1}$, and a sequence $\{\zeta_j\}_{j=1}^{n+1}$ where each ζ_j is a reciprocal of one of the $n+1$ distinct roots of $\phi(z)=0$, that is, $\phi(\zeta_j^{-1})=0$.

Without loss of generality, order the sequence $\{\zeta_j\}_{j=1}^{n+1}$ so that $|\zeta_1| \geq |\zeta_2| \geq \cdots \geq |\zeta_{n+1}|$. As $\psi_\ell = (1-\alpha_\ell)\psi_{\ell-1}$, it follows that $\alpha_\ell = 1-(\psi_\ell/\psi_{\ell-1})$ and hence:

$$\alpha_\ell = 1 - \frac{\sum_{j=1}^{n+1}\varkappa_j\zeta_j^\ell}{\sum_{j=1}^{n+1}\varkappa_j\zeta_j^{\ell-1}} = 1 - \frac{\zeta_1 + \sum_{j=2}^{n+1}\zeta_j\frac{\varkappa_j}{\varkappa_1}\left(\frac{\zeta_j}{\zeta_1}\right)^{\ell-1}}{1+\sum_{j=2}^{n+1}\frac{\varkappa_j}{\varkappa_1}\left(\frac{\zeta_j}{\zeta_1}\right)^{\ell-1}}.$$

With no repeated roots, $\zeta_1 \neq \zeta_2$, so a necessary condition for the limit $\lim_{\ell\to\infty}\alpha_\ell$ to exist is that $|\zeta_1| > |\zeta_2|$ (using the ordering of the roots), which also requires ζ_1 to be a real number. Under this condition, $\alpha_\infty \equiv \lim_{\ell\to\infty}\alpha_\ell = 1-\zeta_1$. For this limit to be economically meaningful and ensure $\alpha_\infty > 0$, it is necessary that $0 \leq \zeta_1 < 1$.

It is known that $\phi(z)\psi(z) = 1$, so $\phi(1) = \psi(1)^{-1}$, which is necessarily positive since $\psi_0 = 1$ and $\psi_\ell \geq 0$ for all ℓ. As ζ_1 is the largest of the reciprocals of the roots of $\phi(z)=0$, there must be no value of ζ between ζ_1 and 1 such that $\phi(\zeta^{-1})=0$. Since $\phi(z)$ in [A.4.1] is a polynomial, it is a continuous function. Together with $\phi(1)>0$ and the absence of any value of ζ between ζ_1 and 1 such that $\phi(\zeta^{-1})=0$, it must be the case that $\phi'(\zeta_1^{-1})<0$.

The value of ζ_1 is characterized by $\phi(\zeta_1^{-1})=0$, so the change in ζ_1 resulting from a change in a parameter φ_i is implicitly determined by the condition $\phi(\zeta_1^{-1})=0$. Differentiating this condition yields

$$\left.\frac{\partial\zeta_1^{-1}}{\partial\varphi_i}\right|_{\phi(\zeta_1^{-1})=0} = -\frac{1}{\zeta_1^{i+1}\phi'(\zeta_1^{-1})}. \qquad \text{[A.4.11]}$$

As $\alpha_\infty = 1-(\zeta_1^{-1})^{-1}$, it follows that $\partial\alpha_\infty/\partial\zeta_1^{-1} = \zeta_1^2$, and thus using the chain rule with [A.4.11]:

$$\frac{\partial\alpha_\infty}{\partial\varphi_i} = -\frac{1}{\zeta_1^{i-1}\phi'(\zeta_1^{-1})} > 0,$$

since $\phi'(\zeta_1^{-1})<0$ as demonstrated above.

(vi) In what follows, suppose that $n=\infty$ in the hazard function recursion [3.1]. This is without loss of generality because any superfluous φ_i parameters can be set to zero. Equation [3.1] implies

$$\alpha_{\ell+1} - \alpha_\ell = \sum_{i=1}^{\ell}\varphi_i\left(\prod_{j=\ell+1-i}^{\ell}(1-\alpha_j)\right)^{-1} - \sum_{i=1}^{\ell-1}\varphi_i\left(\prod_{j=\ell-i}^{\ell-1}(1-\alpha_j)\right)^{-1},$$

and by combining overlapping terms and extracting common factors:

$$\alpha_{\ell+1} - \alpha_\ell = \varphi_\ell\left(\prod_{j=1}^{\ell}(1-\alpha_j)\right)^{-1} + \sum_{i=1}^{\ell-1}\varphi_i\left(\prod_{j=\ell-i}^{\ell}(1-\alpha_j)\right)^{-1}\left\{(1-\alpha_{\ell-i})-(1-\alpha_\ell)\right\}.$$

Therefore, the change in the hazard function is given by:

$$\alpha_{\ell+1} - \alpha_\ell = \sum_{i=1}^{\ell-1} \varphi_i(\alpha_\ell - \alpha_{\ell-i}) \left(\prod_{j=\ell-i}^{\ell} (1-\alpha_j) \right)^{-1} + \varphi_\ell \left(\prod_{j=1}^{\ell} (1-\alpha_j) \right)^{-1}. \quad \text{[A.4.12]}$$

It follows that $\varphi_i = 0$ for all i implies $\alpha_\ell = \alpha$ for all ℓ. Similarly, suppose $\alpha_\ell = \alpha_1$ for all ℓ. It follows from [A.4.12] that $\varphi_i = 0$ for all i.

(vii) Suppose that $\varphi_i \geq 0$ for all i. It follows immediately from [A.4.12] that $\alpha_2 \geq \alpha_1$. Now suppose that $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_{i-1} \leq \alpha_\ell$ has already been established for some ℓ. Given this supposition, it follows that $\alpha_\ell - \alpha_{\ell-i} \geq 0$ for all $i = 1, \ldots, \ell-1$. Equation [A.4.12] then implies that $\alpha_{\ell+1} \geq \alpha_\ell$. This proves $\alpha_{\ell+1} \geq \alpha_\ell$ for all ℓ by induction.

(viii) Define the sequence $\{\sigma_\ell\}_{\ell=1}^{\infty}$ using $\sigma_1 = -(1-\alpha_1)$ and $\sigma_\ell = \alpha_\ell - \alpha_{\ell-1}$ for $\ell \geq 2$. If $\alpha_{\ell+1} \leq \alpha_\ell$ for all ℓ then $\sigma_\ell \leq 0$ for all ℓ. It follows from the expression for φ_i in equation [A.1.3] justified by Lemma 3 that φ_i is the product of $(-1)^i$ and $i+1$ non-positive terms. Hence, $\varphi_i \leq 0$ for all i is established. This completes the proof.

A.5 Proof of Proposition 4

Let $\psi(z) \equiv \sum_{\ell=0}^{\infty} \psi_\ell z^\ell$ denote the z-transform of the sequence of survival probabilities $\{\psi_\ell\}_{\ell=0}^{\infty}$ generated by a hazard function $\{\alpha_\ell\}_{\ell=1}^{\infty}$. If the hazard function implies the stationary age distribution is stable then Lemma 1 shows there exists a $\rho > 1$ such that $\psi(z)$ has no roots in $\mathscr{D}_\rho \equiv \left\{ z \in \mathbb{C} \;\middle|\; |z| \leq \rho \right\}$. Define the function $\phi(z) \equiv \psi(z)^{-1}$ on $\mathscr{D}_\rho$, which is analytic because $\psi(z) \neq 0$ for all $z \in \mathscr{D}_\rho$.

Since $\phi(z)$ is an analytic function, it is equal to its Taylor series expansion around $z = 0$ (contained in $\mathscr{D}_\rho$). Thus $\phi(z) \equiv 1 - \sum_{i=1}^{\infty} \phi_i z^\ell$ for some sequence $\{\phi_i\}_{i=1}^{\infty}$ (the leading term of the Taylor series is 1 because $\psi(0) = \psi_0 = 1$). As $z = 1$ belongs to $\mathscr{D}_\rho$, it follows that $\sum_{i=1}^{\infty} |\phi_i| < \infty$.

The definition of $\phi(z)$ requires $\phi(z)\psi(z) = 1$ for all $z \in \mathscr{D}_\rho$. Multiplying the power series for $\phi(z)$ and $\psi(z)$ yields

$$\phi(z)\psi(z) = \psi_0 + \sum_{\ell=1}^{\infty} \left(\psi_\ell - \sum_{i=1}^{\ell} \phi_i \psi_{\ell-i} \right) z^\ell.$$

Since $\psi_0 = 1$ always, $\phi(z)\psi(z) = 1$ holds for all $z \in \mathscr{D}_\rho$ if and only if $\psi_\ell = \sum_{i=1}^{\ell} \phi_i \psi_{\ell-i}$ is true for all ℓ. Define α and $\{\varphi_i\}_{i=1}^{\infty}$ according to $\alpha \equiv 1 - \sum_{i=1}^{\infty} \phi_i$ and $\varphi_i \equiv -\phi_{i+1}$. With these definitions, the recursion for $\{\psi_\ell\}_{\ell=0}^{\infty}$ in [3.2] holds with $n = \infty$, which is equivalent to the original recursion for the hazard function in [3.1]. Given the definitions, it has also been shown that $\sum_{i=1}^{\infty} |\varphi_i| < \infty$. This completes the proof.

A.6 Proof of Proposition 5

(i) Define the sequence of probabilities of price stickiness $\{\varsigma_\ell\}_{\ell=1}^{\infty}$ as $\varsigma_\ell \equiv 1 - \alpha_\ell$ using the hazard function $\{\alpha_\ell\}_{\ell=1}^{\infty}$. If the parameters α and $\{\varphi_i\}_{i=1}^{n}$ generate a well-defined hazard function then it follows that $0 \leq \varsigma_\ell \leq 1$ for all ℓ.

Using the hazard function recursion [3.1], the sequence $\{\varsigma_\ell\}_{\ell=1}^{\infty}$ satisfies

$$\varsigma_\ell = \left(1 - \alpha + \sum_{i=1}^{n} \varphi_i \right) - \sum_{i=1}^{\min\{\ell-1,n\}} \frac{\varphi_i}{\prod_{j=\ell-i}^{\ell-1} \varsigma_j}, \quad \text{[A.6.1]}$$

for all ℓ.

Consider the claim

$$\sum_{j=i}^{n} \varphi_j \leq \alpha. \qquad \text{[A.6.2]}$$

Since [A.6.1] implies $\varsigma_1 = 1 - \alpha + \sum_{i=1}^{n} \varphi_i$, the requirement $\varsigma_1 \leq 1$ implies that [A.6.2] is true for $i = 1$.

Now suppose that the claim [A.6.2] has been proved for all $i = 1, \ldots, k$ for some k. If $\varphi_k \geq 0$ then the result $\sum_{j=k+1}^{n} \varphi_j \leq \alpha$ follows automatically from $\sum_{j=k}^{n} \varphi_j \leq \alpha$, proving the statement [A.6.2] for the case $i = k + 1$ as well.

Consider the case $\varphi_k < 0$. Using [A.6.1], the requirement $\varsigma_{k+1} \leq 1$ is equivalent to

$$-\sum_{i=1}^{k-1} \frac{\varphi_i}{\prod_{j=k+1-i}^{k} \varsigma_j} - \frac{\varphi_k}{\prod_{j=1}^{k} \varsigma_j} \leq \alpha - \sum_{i=1}^{n} \varphi_i. \qquad \text{[A.6.3]}$$

Since $0 \leq \varsigma_1 \leq 1$ and $\varphi_k < 0$ in the case under consideration, it follows from [A.6.3] that

$$-\sum_{i=1}^{k-2} \frac{\varphi_i}{\prod_{j=k+1-i}^{k} \varsigma_j} - \frac{(\varphi_{k-1} + \varphi_k)}{\prod_{j=2}^{k} \varsigma_j} \leq \alpha - \sum_{i=1}^{n} \varphi_i. \qquad \text{[A.6.4]}$$

Now if $\varphi_{k-1} + \varphi_k \geq 0$ then $\sum_{j=k+1}^{n} \varphi_j \leq \alpha$ would follow from $\sum_{j=k-1}^{n} \varphi_j \leq \alpha$, proving the statement [A.6.2] for $i = k + 1$. If not, then since $0 \leq \varsigma_2 \leq 1$, inequality [A.6.4] together with $\varphi_{k-1} + \varphi_k < 0$ implies that

$$-\sum_{i=1}^{k-3} \frac{\varphi_i}{\prod_{j=k+1-i}^{k} \varsigma_j} - \frac{(\varphi_{k-2} + \varphi_{k-1} + \varphi_k)}{\prod_{j=3}^{k} \varsigma_j} \leq \alpha - \sum_{i=1}^{n} \varphi_i. \qquad \text{[A.6.5]}$$

By again considering the two cases for the sign of $\varphi_{k-2} + \varphi_{k-1} + \varphi_k$ the claim [A.6.2] for $i = k + 1$ either follows, or a new inequality is deduced alone the pattern of [A.6.3]–[A.6.5] above. This process terminates either with [A.6.2] proved for $i = k + 1$ or the inequality

$$-\frac{\sum_{i=1}^{k} \varphi_i}{\varsigma_k} \leq \alpha - \sum_{i=1}^{n} \varphi_i.$$

Since $0 \leq \varsigma_k \leq 1$ and the claim [A.6.2] is known to be true for $i = 1$, it follows that

$$-\sum_{i=1}^{k} \varphi_i \leq \alpha - \sum_{i=1}^{n} \varphi_i,$$

which proves that [A.6.2] holds for $i = k + 1$. Thus, [A.6.2] is true for $i = k + 1$ in all cases, so it follows for all $i = 1, \ldots, n$ by induction.

Next, consider the claim

$$-(1 - \alpha) \leq \sum_{j=i}^{n} \varphi_i. \qquad \text{[A.6.6]}$$

Noting that [A.6.1] implies $\varsigma_1 = 1 - \alpha + \sum_{i=1}^{n} \varphi_i$, the requirement $\varsigma_1 \geq 0$ means that [A.6.6] must hold for $i = 1$.

Now suppose that the statement [A.6.6] has been proved for $i = 1, \ldots, k$ for some k. Given equation [A.6.1], the inequality $\varsigma_{k+1} \geq 0$ holds if and only if

$$\sum_{i=1}^{k} \frac{\varphi_i}{\prod_{j=k+1-i}^{k} \varsigma_j} \leq \left(1 - \alpha + \sum_{i=1}^{n} \varphi_i\right). \qquad \text{[A.6.7]}$$

Multiplying both sides by the non-negative term $\prod_{j=1}^{k} \varsigma_j$ leads to an equivalent inequality:

$$\sum_{i=1}^{k-1} \left(\prod_{j=1}^{k-i} \varsigma_j\right) \varphi_i + \varphi_k \leq \left(1 - \alpha + \sum_{i=1}^{n} \varphi_i\right) \prod_{j=1}^{k} \varsigma_j. \qquad \text{[A.6.8]}$$

If $\varphi_k < 0$ then the inequality $-(1-\alpha) \leq \sum_{j=k+1}^{n} \varphi_j$ follows automatically from $-(1-\alpha) \leq \sum_{j=k}^{n} \varphi_j$, proving the statement [A.6.6] for $i = k+1$. On the other hand, if $\varphi_k \leq 0$ then inequality [A.6.8] together with the requirement $0 \leq \varsigma_1 \leq 1$ implies

$$\sum_{i=1}^{k-2} \left(\prod_{j=1}^{k-i} \varsigma_j\right) \varphi_i + \varsigma_1(\varphi_{k-1} + \varphi_k) \leq \left(1 - \alpha + \sum_{i=1}^{n} \varphi_i\right) \prod_{j=1}^{k} \varsigma_j. \qquad \text{[A.6.9]}$$

If $\varphi_{k-1} + \varphi_k < 0$ then $-(1-\alpha) \leq \sum_{j=k+1}^{n} \varphi_j$ follows from knowing $-(1-\alpha) \leq \sum_{j=k-1}^{n} \varphi_j$, proving [A.6.6] for $i = k+1$. But if $\varphi_{k-1} + \varphi_k \geq 0$ then [A.6.9] and $0 \leq \varsigma_2 \leq 1$ imply:

$$\sum_{i=1}^{k-3} \left(\prod_{j=1}^{k-i} \varsigma_j\right) \varphi_i + \varsigma_1\varsigma_2(\varphi_{k-2} + \varphi_{k-1} + \varphi_k) \leq \left(1 - \alpha + \sum_{i=1}^{n} \varphi_i\right) \prod_{j=1}^{k} \varsigma_j. \qquad \text{[A.6.10]}$$

Proceeding this way, the claim [A.6.6] either follows, or the following inequality is eventually deduced:

$$\left(\prod_{j=1}^{k-1} \varsigma_j\right) \left(\sum_{i=1}^{k} \varphi_i\right) \leq \left(1 - \alpha + \sum_{i=1}^{n} \varphi_i\right) \prod_{j=1}^{k} \varsigma_j.$$

Since [A.6.6] is known to be true for $i = 1$ and as $0 \leq \varsigma_k \leq 1$, it follows that:

$$\left(\prod_{j=1}^{k-1} \varsigma_j\right) \left(\sum_{i=1}^{k} \varphi_i\right) \leq \left(1 - \alpha + \sum_{i=1}^{n} \varphi_i\right) \left(\prod_{j=1}^{k-1} \varsigma_j\right),$$

from which the statement [A.6.6] is proved for $i = k+1$. Thus [A.6.6] is demonstrated for all $i = 1, \ldots, n$ by induction. Therefore $-(1-\alpha) \leq \sum_{j=i}^{n} \varphi_j \leq \alpha$ for all $i = 1, \ldots, n$.

(ii) Suppose $n = 1$ and $\varphi \equiv \varphi_1$. In the case $\varphi = 0$, the restriction $0 \leq \alpha \leq 1$ is clearly all that is required for the hazard function to be well defined. Thus assume $\varphi \neq 0$ in what follows.

The hazard function recursion [3.1] in the case $n = 1$ reduces to

$$\alpha_\ell = (\alpha - \varphi) + \frac{\varphi}{1 - \alpha_{\ell-1}}, \qquad \text{[A.6.11]}$$

and the linear recursion for the survival probabilities [3.2] becomes:

$$\psi_\ell = (1 - \alpha + \varphi)\psi_{\ell-1} - \varphi\psi_{\ell-2}. \qquad \text{[A.6.12]}$$

Define the quadratic equation $\phi(z) = 1-(1-\alpha+\varphi)z-\varphi z^2$. Note that $\phi(z^{-1}) = 0$ is the characteristic equation for the sequence of survival probabilities $\{\psi_\ell\}_{\ell=0}^{\infty}$. Let ζ_1 and ζ_2 denote the reciprocals of the two roots of $\phi(z) = 0$. The quadratic can thus be written as $\phi(z) = (1 - \zeta_1 z)(1 - \zeta_2 z)$. By equating coefficients of powers of z, it follows that $1 - \alpha + \varphi = \zeta_1 + \zeta_2$ and $\varphi = \zeta_1\zeta_2$. Note that $\alpha_1 = \alpha - \varphi$, which must be a well-defined probability, so $\varphi \leq \alpha$ is always required.

The roots ζ_1 and ζ_2 are real numbers when the following condition is satisfied:

$$(1 - \alpha + \varphi)^2 - 4\varphi = \varphi^2 - 2(1 + \alpha)\varphi + (1 - \alpha)^2 \geq 0. \qquad \text{[A.6.13]}$$

Interpreted as a quadratic in φ, it is straightforward to see that it has two positive real roots. The condition above is satisfied when φ is below the smaller of the two roots:

$$\varphi \leq (1 + \alpha) - \sqrt{(1 + \alpha)^2 - (1 - \alpha)^2} = \left(1 - \sqrt{\alpha}\right)^2. \qquad \text{[A.6.14]}$$

The sum of the roots of the quadratic in [A.6.13] is $2(1 + \alpha)$, so the larger root is greater than α, which is in the range where $\varphi \leq \alpha$ is violated.

Consider first the case where $\varphi > 0$. Suppose it is claimed that there is an upper bound $\bar{\alpha}$ for the hazard function $\{\alpha_\ell\}_{\ell=1}^{\infty}$. If $\alpha_{\ell-1} \leq \bar{\alpha}$ then [A.6.11] implies

$$\alpha_\ell \leq (\alpha - \varphi) + \frac{\varphi}{1 - \bar{\alpha}}.$$

Hence $\bar{\alpha}$ is valid upper bound for $\{\alpha_\ell\}_{\ell=1}^{\infty}$ (satisfying $0 < \bar{\alpha} < 1$) if the following inequality holds:

$$(\alpha - \varphi) + \frac{\varphi}{1 - \bar{\alpha}} \leq \bar{\alpha},$$

which is equivalent to:

$$1 - (1 - \alpha + \varphi)(1 - \bar{\alpha})^{-1} + \varphi(1 - \bar{\alpha})^{-2} \leq 0.$$

Since in the case $\varphi > 0$, [A.6.11] implies the hazard function is strictly increasing as long as it remains well defined. Thus the hazard function is well defined if and only if $\varphi \leq \alpha$ and there is some bound $\bar{\alpha}$ satisfying $0 < \bar{\alpha} < 1$ such that $\phi((1 - \bar{\alpha})^{-1}) \leq 0$. This requires $\phi(z) = 0$ to have real roots, which in turn requires the inequality in [A.6.14] to be satisfied. Furthermore, one of the real roots must be strictly greater than one to ensure $0 < \bar{\alpha} < 1$. Note that $\phi(0) = 1$ and $\phi(1) = \alpha > 0$, and that the product of the roots of $\phi(z) = 0$ is φ^{-1}. Under the condition [A.6.14], $\varphi < 1$, so the product of the roots is greater than one. The sum of the roots is positive, so both must be positive. Thus, $\varphi \leq \alpha$ and [A.6.14] are necessary and sufficient for the hazard function to be well defined in the case $\varphi > 0$.

Now consider the case where $\varphi < 0$. Since $\alpha_1 = \alpha - \varphi$, it is necessary to assume $\varphi \geq -(1 - \alpha)$ to ensure α_1 is a well-defined probability. Note that any negative value of φ satisfies [A.6.13], so both ζ_1 and ζ_2 are real numbers. As $\zeta_1\zeta_2 = \varphi$, one of these numbers must be positive and the other negative. Without loss of generality, assume $\zeta_1 > 0$ and $\zeta_2 < 0$. Since $\zeta_1 + \zeta_2 = 1 - \alpha + \varphi$ and as $\alpha_1 = \alpha - \varphi$ is well defined, it follows that $\zeta_1 > -\zeta_2$. Noting that $\phi(0) = 1$ and $\phi(1) = \alpha$, so as $\phi(\zeta_1^{-1}) = 0$ and $\phi(\zeta_2^{-1}) = 0$ it must be the case that $\zeta_1 < 1$ (otherwise $\phi(z)$ would have to change sign twice between 0 and 1, implying that both ζ_1 and ζ_2 would be positive).

Since ζ_1 and ζ_2 are distinct numbers in the case $\varphi < 0$, the survival probabilities $\{\psi_\ell\}_{\ell=0}^{\infty}$ can be expressed as $\psi_\ell = \varkappa_1\zeta_1^\ell + \varkappa_2\zeta_2^\ell$, where $\varkappa_1$ and $\varkappa_2$ are real numbers. Consequently:

$$\psi_\ell = \varkappa_1\zeta_1^\ell\left\{1 + \frac{\varkappa_2}{\varkappa_1}\left(\frac{\zeta_2}{\zeta_1}\right)^\ell\right\}, \quad \text{and} \quad \psi_\ell - \psi_{\ell+1} = \varkappa_1(1 - \zeta_1)\zeta_1^\ell\left\{1 + \frac{\varkappa_2(1 - \zeta_2)}{\varkappa_1(1 - \zeta_1)}\left(\frac{\zeta_2}{\zeta_1}\right)^\ell\right\}. \qquad \text{[A.6.15]}$$

The hazard function recursion [A.6.11] implies $\alpha_1 = \alpha - \varphi$ and $\alpha_2 = (\alpha - \varphi) + \varphi/(1 - \alpha + \varphi)$. Given the restriction $\varphi \geq -(1 - \alpha)$ that ensures α_1 is well defined in the case $\varphi < 0$, the probability α_2 is well defined if and only if $\varphi \geq -(\alpha - \varphi)(1 - \alpha + \varphi)$. Rearranging this inequality shows that it is equivalent to

$$\varphi^2 - 2\alpha\varphi - \alpha(1 - \alpha) \leq 0.$$

Interpreted as a quadratic in φ, the above inequality has one positive and one negative root. Given that $\varphi < 0$ in the case under consideration, the relevant restriction is that

$$\varphi \geq \alpha - \sqrt{\alpha^2 + \alpha(1 - \alpha)} = -\sqrt{\alpha}(1 - \sqrt{\alpha}). \qquad \text{[A.6.16]}$$

Notice that $\sqrt{\alpha}(1 - \sqrt{\alpha}) \leq 1 - \alpha$, so the requirement $\varphi \geq -(1 - \alpha)$ is automatically satisfied when [A.6.16] holds.

The condition [A.6.16] is thus seen to be equivalent to α_1 and α_2 being well defined in the case $\varphi < 0$. This is itself equivalent to $0 \leq \psi_2 \leq \psi_1 \leq \psi_0 = 1$ because $\psi_\ell = (1 - \alpha_\ell)\psi_{\ell-1}$. By using [A.6.15], $\psi_0 - \psi_1 = \varkappa_1(1 - \zeta_1) + \varkappa_2(1 - \zeta_2) \geq 0$ and $\psi_1 - \psi_2 = \varkappa_1(1 - \zeta_1)\zeta_1 + \varkappa_2(1 - \zeta_2)\zeta_2 \geq 0$. Since $0 < \zeta_1 < 1$ and $\zeta_2 < 0$, it follows from the first inequality that at least one of $\varkappa_1$ and $\varkappa_2$ must be non-negative, and thus from the second inequality that $\varkappa_1 \geq 0$.

Since $\zeta_1 > -\zeta_2$, the terms $(\varkappa_2/\varkappa_1)(\zeta_2/\zeta_1)^i$ and $(\varkappa_2(1 - \zeta_2)/\varkappa_1(1 - \zeta_1))(\zeta_2/\zeta_1)^i$ in [A.6.15] must alternate in sign and decline in absolute value as ℓ increases. Because $\varkappa_1$, ζ_1 and $(1 - \zeta_1)$ are non-negative, the inequalities $0 \leq \psi_2 \leq \psi_1 \leq \psi_0$ imply $0 \leq \psi_\ell \leq \psi_{\ell-1}$ for all ℓ, which ensure the hazard function is well defined everywhere. Since this condition is equivalent to [A.6.16], the proof is complete.

A.7 Proof of Proposition 6

(i) Let $\psi(z) \equiv \sum_{\ell=0}^{\infty} \psi_\ell z^\ell$ and $\omega(z) \equiv \sum_{\ell=0}^{\infty} \omega_\ell z^\ell$ denote the z-transforms of the survival probabilities $\{\psi_\ell\}_{\ell=0}^{\infty}$ and the stationary age distribution $\{\omega_\ell\}_{\ell=0}^{\infty}$. Equations [2.4] and [2.5] for the reset price r_t and price level p_t can be written in terms of the lag and forward operators $\mathbb{L}$ and $\mathbb{F}$ and the power series $\psi(z)$ and $\omega(z)$:

$$\mathsf{r}_t = \psi(\beta)^{-1}\mathbb{E}_t\left[\psi(\beta\mathbb{F})\mathsf{p}_t^*\right], \quad \text{and} \;\; \mathsf{p}_t = \omega(\mathbb{L})\mathsf{r}_t. \qquad \text{[A.7.1]}$$

Suppose that the hazard function $\{\alpha_\ell\}_{\ell=1}^{\infty}$ is generated by the recursion [3.1] using parameters α and $\{\varphi_i\}_{i=1}^{n}$. Define the polynomial $\phi(z) \equiv 1 - (1 - \alpha + \sum_{i=1}^{n} \varphi_i)\, z + \sum_{j=1}^{n} \varphi_j z^{j+1}$. Lemma 1 shows that $\psi(z)$ is analytic on the set $\mathscr{D}_\rho \equiv \left\{z \in \mathbb{C} \;\middle|\; |z| \leq \rho\right\}$ for some $\rho > 1$. Note that the equivalent recursion [3.2] for the survival probabilities and $\psi_0 = 1$ imply $\phi(z)\psi(z) = 1$ for all $z \in \mathscr{D}_\rho$.

Now multiply both sides of the equation in [A.7.1] for the reset price r_t by $\phi(\beta\mathbb{F})$ and take conditional expectations at time t:

$$\mathbb{E}_t\left[\phi(\beta\mathbb{F})\mathsf{r}_t\right] = \mathbb{E}_t\left[\phi(\beta\mathbb{F})\left\{\psi(\beta)^{-1}\mathbb{E}_t\left[\psi(\beta\mathbb{F})\mathsf{p}_t^*\right]\right\}\right] = \mathbb{E}_t\left[\psi(\beta)^{-1}\mathbb{E}_t\left[\phi(\beta\mathbb{F})\psi(\beta\mathbb{L})\mathsf{p}_t^*\right]\right] = \psi(\beta)^{-1}\mathsf{p}_t^*. \qquad \text{[A.7.2]}$$

This result follows first because $\phi(\beta\mathbb{F})$ contains only non-negative powers of $\mathbb{F}$, so it commutes with the conditional expectation $\mathbb{E}_t[\cdot]$ operator inside another conditional expectation. Second, $\phi(z)\psi(z) = 1$, hence $\phi(\beta\mathbb{F})\psi(\beta\mathbb{F}) = \mathbb{I}$, where $\mathbb{I}$ is the identity operator. Next, note that because $\omega_\ell = (1 - \alpha_\ell)\omega_{\ell-1}$ and $\psi_\ell = (1 - \alpha_\ell)\psi_{\ell-1}$, the functions $\omega(z)$ and $\psi(z)$ are proportional. Thus $\omega(z) = (\omega(1)/\psi(1))\psi(z)$, and $\omega(z) = \phi(1)\psi(z)$, since $\psi(1)^{-1} = \phi(1)$, and $\omega(1) = 1$ because $\{\omega_\ell\}_{\ell=0}^{\infty}$ is a probability distribution. It follows that $\phi(z)\omega(z) = \phi(1)$ for all $z \in \mathscr{D}_\rho$. Multiplying both sides of the equation for p_t in [A.7.1] by $\phi(\mathbb{L})$ yields

$$\phi(\mathbb{L})\mathsf{p}_t = \phi(\mathbb{L})\omega(\mathbb{L})\mathsf{r}_t = \phi(1)\mathbb{I}\mathsf{r}_t = \phi(1)\mathsf{r}_t. \qquad \text{[A.7.3]}$$

Now multiply both sides of equation [A.7.2] by $\phi(1)$ and note that $\psi(\beta)^{-1} = \phi(\beta)$, and then substitute the expression for p_t^* from [2.3]:

$$\mathbb{E}_t\left[\phi(\beta\mathbb{F})\phi(1)\mathsf{r}_t\right] = \phi(1)\phi(\beta)(\mathsf{p}_t + \nu\mathsf{x}_t).$$

Substitute the formula for $\phi(1)\mathsf{r}_t$ from [A.7.3] into the above and divide both sides by $\phi(1)\phi(\beta)$:

$$\mathbb{E}_t\left[\left\{\frac{\phi(\mathbb{L})}{\phi(1)}\frac{\phi(\beta\mathbb{F})}{\phi(\beta)} - 1\right\}\mathsf{p}_t\right] = \nu\mathsf{x}_t. \quad \text{[A.7.4]}$$

Define the Laurent polynomial $\chi(z)$ as follows:

$$\chi(z) \equiv \frac{\phi(z)}{\phi(1)}\frac{\phi(\beta z^{-1})}{\phi(\beta)} - 1, \quad \text{[A.7.5]}$$

so that equation [A.7.4] is equivalent to $\mathbb{E}_t\left[\chi(\mathbb{L})\mathsf{p}_t\right] = \nu\mathsf{x}_t$, noting that $\mathbb{F} \equiv \mathbb{L}^{-1}$. For algebraic convenience, define the sequence of coefficients $\{\phi_\jmath\}_{\jmath=1}^{n+1}$ by $\phi_1 \equiv (1 - \alpha + \sum_{i=1}^{n}\varphi_i)$ and $\phi_\jmath \equiv -\varphi_{\jmath-1}$ for $\jmath = 2, \ldots, n+1$ in terms of the parameters of the recursion [3.1]. With these definitions the polynomial $\phi(z)$ can be written as $\phi(z) \equiv 1 - \sum_{\jmath=1}^{n+1}\phi_\jmath z^\jmath$. The Laurent polynomial $\chi(z)$ can be written explicitly using this expression:

$$\chi(z) = \vartheta\left\{\left(1 - \sum_{\jmath=1}^{n+1}\phi_\jmath z^\jmath\right)\left(1 - \sum_{\jmath=1}^{n+1}\beta^\jmath\phi_\jmath z^{-\jmath}\right) - \left(1 - \sum_{\jmath=1}^{n+1}\phi_\jmath\right)\left(1 - \sum_{\jmath=1}^{n+1}\beta^\jmath\phi_\jmath\right)\right\},$$

where $\vartheta \equiv \phi(1)^{-1}\phi(\beta)^{-1}$ is defined. Expanding the brackets to obtain an expression of the form $\chi(z) = \sum_{\ell=-(n+1)}^{n+1}\chi_\ell z^\ell$ and equating powers of z implies that $\chi(z)$ can be written as

$$\chi(z) = \chi_0 + \sum_{\ell=1}^{n+1}\chi_\ell\left\{z^\ell + \beta^\ell z^{-\ell}\right\}, \quad \text{where } \chi_\ell = -\vartheta\left\{\phi_\ell - \sum_{\jmath=1}^{n+1-\ell}\beta^\jmath\phi_\jmath\phi_{\jmath+\ell}\right\} \text{ for } \ell \geq 1, \quad \text{[A.7.6]}$$

since $\chi_{-\ell} = \beta^\ell\chi_\ell$ for all ℓ. As the definition in [A.7.5] implies $\chi(1) = 0$, it follows that $\chi_0 = -\sum_{\ell=1}^{n+1}(1+\beta^\ell)\chi_\ell$. Furthermore, $\chi(1) = 0$ implies that there exists a Laurent polynomial $\theta(z)$ such that $\chi(z) = (1-z)\theta(z)$. Given the degree of $\chi(z)$, this Laurent polynomial must have the form $\theta(z) = \sum_{\ell=-(n+1)}^{n}\theta_\ell z^\ell$. Multiplying $\theta(z)$ by $1-z$ and equating powers of z yields an expression for $\chi(z)$:

$$\chi(z) = \theta_{-(n+1)}z^{-(n+1)} + \sum_{\ell=-n}^{n}(\theta_\ell - \theta_{\ell-1})z^\ell - \theta_n z^{n+1}.$$

Equating coefficients of powers of z with those in [A.7.6] implies $\chi_{n+1} = -\theta_n$, $\beta^{n+1}\chi_{n+1} = \theta_{-(n+1)}$, and $\chi_\ell = \theta_\ell - \theta_{\ell-1}$ for all $\ell = -n, \ldots, n$. Iterating these relationships then implies

$$\theta_\ell = -\sum_{\jmath=\ell+1}^{n+1}\chi_\jmath, \quad \text{and } \theta_{-\ell} = \sum_{\jmath=\ell}^{n+1}\beta^\jmath\chi_\jmath, \quad \text{[A.7.7]}$$

for all $\ell = 1, \ldots, n+1$. Combining these expressions with those for χ_i in [A.7.6] yields

$$\theta_\ell = \vartheta\sum_{i=\ell+1}^{n+1}\left\{\phi_i - \sum_{j=1}^{n+1-i}\beta^j\phi_j\phi_{i+j}\right\} = \vartheta\sum_{i=\ell+1}^{n+1}\phi_i\left\{1 - \sum_{j=1}^{i-\ell-1}\beta^j\phi_j\right\}, \quad \text{[A.7.8a]}$$

where a change in the order of summation has been made in the final term. Similarly,

$$\theta_{-\ell} = -\vartheta \sum_{i=\ell}^{n+1} \beta^i \left\{ \phi_i - \sum_{j=1}^{n+1-i} \beta^j \phi_j \phi_{i+j} \right\} = -\vartheta \sum_{i=\ell}^{n+1} \beta^i \phi_i \left\{ 1 - \sum_{j=1}^{i-\ell} \phi_j \right\}. \qquad \text{[A.7.8b]}$$

The original definitions of the terms of the sequence $\{\phi_j\}_{j=1}^{n+1}$ are $\phi_1 = 1-\alpha+\sum_{i=1}^{n} \varphi_i$ and $\phi_j = -\varphi_{j-1}$ for $j = 2, \ldots, n+1$. Substituting the original parameters α and $\{\varphi_j\}_{j=1}^{n}$ back into [A.7.8a] and [A.7.8b] yields

$$\theta_\ell = -\vartheta \left\{ \varphi_\ell + \sum_{i=\ell+1}^{n} \varphi_i \left(1 - \beta(1-\alpha_1) + \sum_{j=1}^{i-\ell-1} \beta^{j+1} \varphi_j \right) \right\}, \quad \text{for } \ell = 1, \ldots, n; \qquad \text{[A.7.9a]}$$

$$\theta_{-(\ell+1)} = \vartheta \beta^{\ell+1} \left\{ \varphi_\ell + \sum_{i=\ell+1}^{n} \beta^{i-\ell} \varphi_i \left(\alpha_1 + \sum_{j=1}^{i-\ell-1} \varphi_j \right) \right\}, \quad \text{for } \ell = 1, \ldots, n; \qquad \text{[A.7.9b]}$$

$$\theta_0 = \vartheta \left\{ (1-\alpha_1) - \sum_{i=1}^{n} \varphi_i \left(1 - \beta(1-\alpha_1) + \sum_{j=1}^{i-1} \beta^{j+1} \varphi_j \right) \right\}; \text{ and} \qquad \text{[A.7.9c]}$$

$$\theta_{-1} = -\vartheta\beta \left\{ (1-\alpha_1) - \sum_{i=1}^{n} \beta^i \varphi_i \left(\alpha_1 + \sum_{j=1}^{i-1} \varphi_j \right) \right\}. \qquad \text{[A.7.9d]}$$

Since the definition of $\theta(z)$ requires $\chi(z) = (1-z)\theta(z)$, and as inflation is defined by $\pi_t = (\mathbb{I}-\mathbb{L})\mathsf{p}_t$, it follows from equations [A.7.4] and [A.7.5] that $\mathbb{E}_t [\theta(\mathbb{L})\pi_t] = \nu x_t$. Make the following definitions of the coefficient κ and the sequences $\{\lambda_\ell\}_{\ell=1}^{n}$ and $\{\xi_\ell\}_{\ell=1}^{n+1}$ in terms of the elements of the sequence $\{\theta_\ell\}_{\ell=-(n+1)}^{n}$ from [A.7.9]:

$$\lambda_\ell \equiv -\frac{\theta_\ell}{\theta_0}, \quad \xi_\ell \equiv -\frac{\theta_{-\ell}}{\theta_0}, \quad \text{and} \quad \kappa \equiv \frac{1}{\theta_0}, \qquad \text{[A.7.10]}$$

noting that $\theta_0 > 0$ ensures these definitions are valid. With these definition, the Laurent polynomial $\theta(z)$ is given by $\theta(z) = \kappa^{-1} \left\{ 1 - \sum_{\ell=1}^{n} \lambda_\ell z^\ell - \sum_{\ell=1}^{n+1} \xi_\ell z^{-\ell} \right\}$, and so $\mathbb{E}_t [\theta(\mathbb{L})\pi_t] = \nu x_t$ is equivalent to the Phillips curve in [4.3].

(ii) First consider the expression for θ_0 in [A.7.9c]. By expanding the bracket and changing the order of summation:

$$\theta_0 = \vartheta \left\{ (1-\alpha_1) - (1-\beta(1-\alpha_1)) \left(\sum_{i=1}^{n} \varphi_i \right) - \sum_{i=1}^{n-1} \beta^{i+1} \varphi_i \left(\sum_{j=i+1}^{n} \varphi_j \right) \right\}.$$

Adding and subtracting terms in the final summation to obtain an equivalent expression:

$$\theta_0 = \vartheta \left\{ (1-\alpha_1) - (1-\beta(1-\alpha_1)) \left(\sum_{i=1}^{n} \varphi_i \right) - \left(\sum_{i=1}^{n} \beta^{i+1} \varphi_i \right) \left(\sum_{j=1}^{n} \varphi_j \right) + \sum_{i=1}^{n} \beta^{i+1} \varphi_i \left(\sum_{j=1}^{i} \varphi_j \right) \right\}.$$

The definition of the polynomial $\phi(z)$ implies $\phi(1) = \alpha_1 + \sum_{i=1}^{n} \varphi_i$ and $\phi(\beta) = 1 - \beta(1-\alpha_1) + \sum_{i=1}^{n} \beta^{i+1} \varphi_i$. By defining the sums $s_i \equiv \sum_{j=1}^{i} \varphi_j$ for $i = 0, \ldots, n$ (with $s_0 = 0$) and noting that

$\phi(1) - \alpha_1 = \sum_{i=1}^{n} \varphi_i$:

$$\theta_0 = \vartheta \left\{ (1-\alpha_1) - (\phi(1)-\alpha_1)\,\phi(\beta) + \sum_{i=1}^{n} \beta^{i+1}(s_i - s_{i-1})s_i \right\}.$$

Rearranging the first two terms leads to

$$\theta_0 = \vartheta \left\{ (1-\alpha_1)(1-\phi(\beta)) + (1-\phi(1))\phi(\beta) + \sum_{i=1}^{n} \beta^{i+1}(s_i - s_{i-1})s_i \right\}. \qquad \text{[A.7.11]}$$

Note that $(s_i - s_{i-1})s_i = (1/2)\left\{(s_i^2 - s_{i-1}^2) + (s_i - s_{i-1})^2\right\}$, and thus

$$\sum_{i=1}^{n} \beta^{i+1}(s_i - s_{i-1})s_i = \frac{1}{2}\left\{ \sum_{i=1}^{n} \beta^{i+1}\varphi_i^2 + (1-\beta)\sum_{i=1}^{n-1} \beta^{i+1}s_i^2 + \beta^{n+1}s_n^2 - \beta^2 s_0^2 \right\},$$

since $s_i - s_{i-1} = \varphi_i$. Using $s_0 = 0$ and substituting this result into [A.7.11]:

$$\theta_0 = \vartheta \left\{ (1-\alpha_1)(1-\phi(\beta)) + (1-\phi(1))\phi(\beta) + \frac{1}{2}\left\{ \sum_{i=1}^{n} \beta^{i+1}\varphi_i^2 + (1-\beta)\sum_{i=1}^{n-1} \beta^{i+1}s_i^2 + \beta^{n+1}s_n^2 \right\} \right\}. \qquad \text{[A.7.12]}$$

Now observe that $\phi(1) = \psi(1)^{-1}$, and $\psi(1) = \sum_{\ell=0}^{\infty} \psi_\ell$, so $0 < \phi(1) < 1$ because $\psi_0 \equiv 1$, $\psi_\ell \geq 0$, $\sum_{\ell=0}^{\infty} \psi_\ell < \infty$, and $\psi_1 > 0$ under the assumption $\alpha_1 < 1$. Similarly, $\phi(\beta) = \psi(\beta)^{-1}$ and $\psi(\beta) = \sum_{\ell=0}^{\infty} \beta^\ell \psi_\ell$. Since $0 < \beta < 1$, it follows that $0 < \phi(\beta) < 1$. Together these results establish that $\vartheta > 0$ since $\vartheta \equiv \phi(1)^{-1}\phi(\beta)^{-1}$. Given $\alpha_1 < 1$, the parameter α_1 must satisfy $0 \leq \alpha_1 < 1$. Consequently, the first two terms in the brackets in [A.7.12] are strictly positive and all other terms are non-negative. Thus, it is shown that $\theta_0 > 0$. The proof of $\theta_0 > 0$ then automatically shows $\kappa > 0$.

(iii) Now consider the value of ξ_1, which requires examining θ_{-1}. Let $s_i \equiv \alpha_1 + \sum_{j=1}^{i} \varphi_j$, and so $s_i - s_{i-1} = \varphi_i$ for all $i = 1, \ldots, n$, and $s_0 = \alpha_1$. The expression for θ_{-1} in [A.7.9d] can be written as

$$-\theta_{-1} = \vartheta \left\{ \beta(1-\alpha_1) - \sum_{i=1}^{n} \beta^{i+1}(s_i - s_{i-1})s_{i-1} \right\}. \qquad \text{[A.7.13]}$$

Note that $(s_i - s_{i-1})s_{i-1} = (1/2)\left\{(s_i^2 - s_{i-1}^2) - (s_i - s_{i-1})^2\right\}$, and hence

$$\sum_{i=1}^{n} \beta^{i+1}(s_i - s_{i-1})s_{i-1} = \frac{1}{2}\left\{ (1-\beta)\sum_{i=1}^{n-1} \beta^{i+1}s_i^2 + \beta^{n+1}s_n^2 - \beta^2 s_0^2 - \sum_{i=1}^{n} \beta^{i+1}\varphi_i^2 \right\}.$$

Also note that $\sum_{i=1}^{n} \beta^{i+1}\varphi_i = \sum_{i=1}^{n} \beta^{i+1}(s_i - s_{i-1}) = (1-\beta)\sum_{i=1}^{n-1} \beta^{i+1}s_i + \beta^{n+1}s_n - \beta^2 s_0$. By adding and subtracting a multiple of these equal terms to the equation above:

$$\begin{aligned}\sum_{i=1}^{n} \beta^{i+1}(s_i - s_{i-1})s_{i-1} = \frac{1}{2}\sum_{i=1}^{n} \beta^{i+1}\varphi_i - \frac{1}{2}\left\{ (1-\beta)\sum_{i=1}^{n-1} \beta^{i+1}s_i + \beta^{n+1}s_n - \frac{1}{2}\beta^2\alpha_1 \right\} \\ + \frac{1}{2}\left\{ (1-\beta)\sum_{i=1}^{n-1} \beta^{i+1}s_i^2 + \beta^{n+1}s_n^2 - \beta^2\alpha_1^2 - \sum_{i=1}^{n} \beta^{i+1}\varphi_i^2 \right\},\end{aligned}$$

recalling that $s_0 = \alpha_1$. Since $\sum_{i=1}^{\infty} \beta^{i+1}\varphi_i = \phi(\beta) - 1 + \beta(1-\alpha_1)$, the above equation can be rearranged as follows:

$$\sum_{i=1}^{n} \beta^{i+1}(s_i - s_{i-1})s_{i-1} = -\frac{\beta}{2}\left\{\alpha_1^2\beta - \alpha_1\beta + \alpha_1 - 1\right\} + \frac{1}{2}(1-\phi(\beta))$$
$$+\frac{1}{2}\left\{(1-\beta)\sum_{i=1}^{n-1}\beta^{i+1}s_i(1-s_i) + \beta^{n+1}s_n(1-s_n) + \sum_{i=1}^{n}\beta^{i+1}\varphi_i^2\right\}.$$

Substituting this result into equation [A.7.13] yields:

$$-\theta_{-1} = \frac{\vartheta}{2}\left\{\beta(1-\alpha_1)(1-\alpha_1\beta) + (1-\phi(\beta))\right\}$$
$$+\frac{\vartheta}{2}\left\{(1-\beta)\sum_{i=1}^{n-1}\beta^{i+1}s_i(1-s_i) + \beta^{n+1}s_n(1-s_n) + \sum_{i=1}^{n}\beta^{i+1}\varphi_i^2\right\}.$$

Proposition 5 demonstrates that $0 \leq s_i \leq 1$ for all $i = 1, \ldots, n$ is necessary for the hazard function $\{\alpha_\ell\}_{\ell=1}^{\infty}$ to be well defined. Since $0 < \alpha_1 < 1$, $0 < \beta < 1$, and $0 < \phi(\beta) < 1$, $\vartheta > 0$ and $\theta_0 > 0$ as shown earlier, it follows that $\xi_1 = -\theta_{-1}/\theta_0$ is strictly positive.

(iv) Next, note that

$$1 - \beta(1-\alpha_1) + \sum_{j=1}^{i}\beta^{j+1}\varphi_j = (1-\beta)\left\{1 + \sum_{h=0}^{i-1}\beta^{h+1}s_h\right\} + \beta^{i+1}s_i, \quad \text{where } s_j \equiv \alpha_1 + \sum_{h=1}^{j}\varphi_j.$$

Using equations [A.7.9a] and [A.7.10], the coefficients of lagged inflation $\{\lambda_\ell\}_{\ell=1}^{n}$ can be expressed as

$$\lambda_\ell = \left(\frac{\vartheta}{\theta_0}\right)\varphi_\ell + \sum_{i=\ell+1}^{n}\left\{\left(\frac{\vartheta}{\theta_0}\right)\left((1-\beta)\left(1 + \sum_{j=0}^{i-\ell-2}\beta^{j+1}s_j\right) + \beta^{i-\ell}s_{j-i-1}\right)\right\}\varphi_i.$$

Proposition 5 shows that $0 \leq s_i \leq 1$ for all $i = 0, 1, \ldots, n$. Since $\vartheta > 0$ and $\theta_0 > 0$, it follows that λ_ℓ is a weighted sum of $\varphi_\ell, \ldots, \varphi_n$.

(v) Similarly, equations [A.7.9b] and [A.7.10] show that the coefficients on future inflation $\{\xi_\ell\}_{\ell=2}^{n+1}$ are:

$$\xi_\ell = -\left\{\left(\frac{\vartheta\beta^\ell}{\theta_0}\right)\varphi_{\ell-1} + \sum_{i=\ell}^{n}\left(\frac{\vartheta\beta^{i+1}s_{i-\ell}}{\theta_0}\right)\varphi_i\right\}, \quad \text{for } \ell = 2, \ldots, n+1,$$

where s_i is as defined above. Thus ξ_ℓ for $\ell \geq 2$ is the negative of a weighted sum of the parameters $\varphi_{\ell-1}, \ldots, \varphi_n$.

(vi) Note that [A.7.5] implies $\chi(\beta) = 0$. Since $\chi(z) = (1-z)\theta(z)$, it must be the case that $\theta(\beta) = 0$ also. The definition of $\theta(z)$ then implies that $\sum_{\ell\to-(n+1)}^{n}\beta^\ell\theta_\ell = 0$. The result follows by using [A.7.10].

(vii) Finally, to derive the restrictions across the sequences of coefficients $\{\lambda_\ell\}_{\ell=1}^{n}$ and $\{\xi_\ell\}_{\ell=1}^{n+1}$, use the definition in [A.7.10] and equation [A.7.7] to deduce:

$$(1-\beta)\sum_{i=\ell}^{n}\beta^i\lambda_i = -\frac{(1-\beta)}{\theta_0}\sum_{i=\ell}^{n}\beta^i\theta_i = \frac{(1-\beta)}{\theta_0}\sum_{i=\ell}^{n}\sum_{j=i+1}^{n+1}\beta^i\chi_j = \frac{(1-\beta)}{\theta_0}\sum_{i=\ell+1}^{n+1}\left\{\sum_{j=\ell}^{i-1}\beta^j\right\}\chi_i,$$

using a change in the order of summation to derive the final equality. Using the formula for the geometric sum yields

$$(1-\beta)\sum_{i=\ell}^{n}\beta^{i}\lambda_i = \frac{1}{\theta_0}\sum_{i=\ell+1}^{n+1}(\beta^{\ell}-\beta^{i})\chi_i. \qquad \text{[A.7.14]}$$

Thus, adding β to the equation above in the case of $\ell = 1$ and substituting for θ_0 using [A.7.7]

$$\beta + (1-\beta)\sum_{i=1}^{n}\beta^{i}\lambda_i = \frac{1}{\theta_0}\left\{\sum_{i=2}^{n+1}(\beta-\beta^{i})\chi_i - \beta\sum_{i=1}^{n+1}\chi_i\right\} = -\frac{1}{\theta_0}\sum_{i=1}^{n+1}\beta^{i}\chi_i = -\frac{\theta_{-1}}{\theta_0},$$

with the expression for θ_{-1} taken from [A.7.7]. Given the definition in [A.7.10], the equation for ξ_1 is confirmed. Now substract the expression in [A.7.14] for $\ell \geq 2$ from $\beta^{\ell}\lambda_{\ell-1}$:

$$\beta^{\ell}\lambda_{\ell-1} - (1-\beta)\sum_{i=\ell}^{n}\beta^{i}\lambda_i = -\frac{\beta^{\ell}\theta_{\ell-1}}{\theta_0} - \frac{1}{\theta_0}\sum_{i=\ell+1}^{n+1}(\beta^{\ell}-\beta^{i})\chi_i = \frac{1}{\theta_0}\left\{\beta^{\ell}\sum_{i=\ell}^{n+1}\chi_i - \sum_{i=\ell+1}^{n+1}(\beta^{\ell}-\beta^{i})\chi_i\right\},$$

making use of equations [A.7.7] and [A.7.10]. It follows that

$$\beta^{\ell}\lambda_{\ell-1} - (1-\beta)\sum_{i=\ell}^{n}\beta^{i}\lambda_i = \frac{1}{\theta_0}\sum_{i=\ell}^{n+1}\beta^{i}\chi_i = \frac{\theta_{-\ell}}{\theta_0},$$

using [A.7.7] again. Therefore the equation for ξ_ℓ is verified for $\ell \geq 2$. This completes the proof.