Training and Search during Unemployment*

Johannes Spinnewijn†
London School of Economics

October 27, 2012

Abstract

Displaced workers often experience large losses in earnings even a long time after reemployment. Training programs during unemployment mitigate these losses but also affect the unemployed's willingness to search. This paper analyzes how mandatory training programs affect the optimal design of unemployment insurance and how the training intensity should evolve during the unemployment spell. The introduction of training reverses the optimal consumption dynamics during the unemployment spell and makes it optimal to incentivize the long-term unemployed to find employment despite the depreciation of their human capital. Targeting training programs towards the long-term unemployed, however, is optimal only if the fall in human capital upon displacement is small relative to the depreciation during unemployment.

Keywords: Unemployment, Optimal Insurance, Human Capital, Training

JEL Classification Numbers: H21, J62, J64

*I would like to thank Arthur Campbell, Peter Diamond, Florian Ederer, Jesse Edgerton, Bengt Holmström, Henrik Kleven, Gerard Padró i Miquel, Frans Spinnewyn, Ivan Werning, the editor, three anonymous referees, and seminar participants at MIT, Warwick University and the EEA-ESEM meetings for many helpful comments and suggestions.

†Department of Economics, STICERD R515, LSE, Houghton Street, London WC2A 2AE, United Kingdom (email: j.spinnewijn@lse.ac.uk, web: http://personal.lse.ac.uk/spinnewi/).

1 Introduction

Optimal unemployment insurance trades off the provision of incentives to search for employment and the provision of insurance against the loss of earnings due to unemployment. Displaced workers do not only forego their wages while unemployed, but are also re-employed at substantially lower wages. In the US, one fourth of the re-employed have wages that were at least 25 percent lower than in their previous job (Kling 2006). This wage gap is not only shown to be large, but also long lasting (Jacobson, LaLonde and Sullivan 1993, von Wachter, Song and Manchester 2009). Training programs during unemployment can counter the unemployed's loss in human capital driving the reduction in future wages. Spending on active labor market programs are almost equal to the cost of unemployment benefits in most European countries, of which on average 40 percent is spent on training programs.[1,2] Despite their importance in practice, training programs have received little attention in the optimal unemployment insurance literature, which has analyzed the optimal design of unemployment benefits and taxes mostly in isolation from other unemployment policies.

This paper analyzes how mandatory training programs should be incorporated in the dynamic design of unemployment insurance. The paper aims to make three contributions. First, it shows how the introduction of training programs changes the optimal path of unemployment consumption by affecting when incentives for search are desirable. Second, the paper relates the important issue of when to mandate training programs to different sources of human capital loss and questions the common practice of targeting training programs towards the long-term unemployed. Finally, the paper sheds new light on the complementarity between the insurance value of unemployment benefits and the potential value of training programs as an integral part of unemployment policy.

I consider an unemployed agent in an infinite-horizon model who chooses how much search effort to exert. The agent's human capital depreciates during the unemployment spell, which lowers his productivity upon reemployment. The depreciation can be countered by training efforts during unemployment, which are imposed by the social planner. Both training and search efforts are costly for the agent. If training efforts increase the marginal cost of search (e.g., time spent on training reduces the time available for search), the required participation to training programs implies a negative *lock-in* effect with low exit rates when programs are intensive (Lechner, Miquel and Wunsch 2011). I characterize the optimal unemployment insurance contract, specifying the consumption levels during unemployment and upon reemployment and the training

[1] Spending on labor market programs, active and passive, averages about 3 percent of GDP in the OECD countries.

[2] The impact of training programs has been estimated in the empirical literature. While there is a lot of heterogeneity in impact across different programs (Heckman et al. 1999), a meta-analysis of recent work by Card, Kluve and Weber (2009) suggests that training programs do have a positive long-run effect.

level as a function of the length of the unemployment spell.

Without training technology, the unemployed worker can only be in one of two states. As long as human capital is high, it is worth to provide costly incentives and induce the unemployed to search for employment. The optimal provision of incentives is achieved by decreasing consumption with the length of the unemployment spell, a well-known result established by Shavell and Weiss (1979) and Hopenhayn and Nicolini (1997). When the level of human capital has depreciated too much, the social planner puts the unemployed on *social assistance*, paying constant benefits without encouragement to enter employment again, as shown by Pavoni and Violante (2007) and Pavoni (2009).

The presence of an effective training technology makes that two different states are relevant; the *search-and-training state* and the *training state*. For high levels of human capital, it is optimal to induce search efforts, but training efforts may help to increase the level of human capital or to mitigate the depreciation of human capital. For low levels of human capital, it may be optimal not to induce any search but to impose only training efforts to increase the job seeker's employability first. I focus on CARA preferences with continuous effort expressed in monetary terms and show that the social's planners decision to impose training and to induce search can be analyzed separately from the optimal design of the consumption profile providing these search incentives. I exploit this separability to show the existence of a threshold level of human capital such that the training state is optimal below and the search-and-training state is optimal above this threshold. The training state is always followed by the search-and-training state; the social planner starts inducing the unemployed to search once the level of human capital has increased above the threshold. Moreover, the search-and-training state is absorbent. The social planner thus never stops inducing the long-term unemployed to search, regardless of the length of the unemployment spell.

The dynamics of the optimal unemployment policy and the search incentives in particular thus crucially depend on the integration of the training policy. This translates to the optimal consumption profile as well. Without training, the optimal consumption profile is decreasing for the short-term unemployed, but becomes constant for the long-term unemployed with low human capital. With training, the optimal consumption profile may be constant for the short-term unemployed when they are only asked to train and no incentives for search are needed. However, consumption is always decreasing for the long-term unemployed as they are encouraged to search until they find employment.

Assuming binary search effort, I also show that the level of human capital of the long-term unemployed converges monotonically to a unique stationary level. This implies that the optimal timing and intensity of training is entirely determined by the difference between this stationary level and the level of human capital at the start of the unemployment spell. If the initial level of human capital is lower than the stationary level, training is more intensive towards the start of unemployment. If the initial level of human capital is higher than the stationary level, training is less intensive at the start of unemployment. In both cases, the training intensity converges to the same stationary level, which exactly offsets the depreciation. In practice, the participation to training programs is often required to remain eligible for unemployment benefits like in the New Deal implemented in the United Kingdom in 1998 or extends the eligibility for unemployment benefits like for the "Bildungsgutschein", training vouchers recently introduced in Germany.[3] Training programs are thus mainly targeted to the long-term unemployed. The results of this analysis put this practice into question. The initial and stationary level of human capital can be related to two different sources of human capital loss discussed in the literature; the fall in human capital upon displacement and the depreciation of human capital during unemployment. The former loss depends on the importance of firm and industry-specific skills that become obsolete upon job loss and reduce the employability in other sectors or industries (Neal 1995, Ljungqvist and Sargent 1998). This affects the initial level of human capital in this model. The latter loss depends on the explicit depreciation of basic skills, work habits and confidence during unemployment (Coles and Masters 2000, Falk et al. 2006) and determines the stationary level of human capital in this model. The analysis thus suggests that it is optimal to target mandatory training to the long-term unemployed only if the depreciation in human capital is sufficiently important relative to the fall upon displacement.

I also calibrate the dynamic model to perform some numerical simulations and to calculate the welfare gain from incorporating training as an unemployment policy. The simulations suggest that the welfare gain is substantial when the initial level of human capital is relatively low, but becomes negligible when the initial level of human capital is relatively high, as the optimal training intensity is decreasing in the level of human capital. This is exactly complementary to the insurance value achieved through benefits and taxes, which is low when the initial level of human capital is relatively low. Training programs are thus particularly valuable when the fall in human capital upon displacement is significant. However, if that is the case, participation should be required from the start of the unemployment spell.

The paper is organized as follows. I present the model in Section 2 and set up the social planner's problem in Section 3. In Section 4, I characterize the optimal dynamic unemployment policy analytically, showing first how the recursive problem simplifies for CARA preferences. In Section 5, I present some numerical simulations and calculate the welfare gain from incorporating training programs as an unemployment policy. The last section concludes. The proofs are in appendix.

[3] In the US, similar requirements exist to receive welfare benefits since 1988, when the Family Support Act required welfare recipients to participate in the Job Opportunities and Basic Skill Training program.

Related Literature A large literature has analyzed the dynamics of optimal unemployment insurance with stationary levels of human capital. A central result in this literature is that a job seeker's consumption needs to decline with the length of the unemployment spell to optimally trade-off insurance and the provision of search incentives (e.g., Shavell and Weiss 1979, Hopenhayn and Nicolini 1997). A more recent literature departs from these stationary models and models the depreciation of human capital during the unemployment spell.[4] Shimer and Werning (2006) analyze the optimal timing of consumption in a McCall search model, assuming that human capital depreciation reduces the arrival rate of job offers or deteriorates the distribution of the wages being paid on the job. Pavoni and Violante (2007) and Pavoni (2009) analyze the optimal unemployment insurance contract in a model with endogenous search, assuming that the depreciation of human capital reduces the output upon re-employment and the probability to find employment. Since human capital depreciates, it may reach a level at which the returns to providing costly incentives for search are too low and unemployment consumption becomes constant.[5]

This paper is closest to Pavoni and Violante (2007) and the related papers by Pavoni and Violante (2005) and Wunsch (2008, 2009). Pavoni and Violante (2007) characterize the optimal sequencing of different unemployment policies and the consequences for the optimal consumption path. Their paper introduces social assistance and job monitoring as alternative policies to the standard unemployment policy which induces search through a wedge between employment and unemployment consumption. During job search monitoring, a cost is paid to observe and impose search efforts, but no incentives are needed. During social assistance, no search is induced and thus no incentives are needed either. The exogenous depreciation of human capital determines the optimal timing of these three policies, which naturally affects the optimal consumption profile as well. Like Pavoni and Violante (2007), I characterize the optimal dynamics of the unemployment policy but I consider training as an alternative policy. In contrast to the previous policies, the introduction of training endogenizes the change in human capital which has important consequences for the optimal consumption profile. Another important difference is that I allow for the training policy to be implemented simultaneously with the standard unemployment policy and for the intensity of the

[4] The decrease in human capital, upon displacement and during unemployment, has been central in explaining the persistence of unemployment and the European unemployment dilemma (Pissarides 1992, Ljungqvist and Sargent 1998, Machin and Manning 1999), as well as the negative duration dependence of exit rates (Blanchard and Diamond 1994, Acemoglu 1995). In general, the empirical evidence for the depreciation of human capital during unemployment is mixed. For instance, Frijters and van der Klaauw (2006) find that the re-employment wage distribution deteriorates significantly, in particular during the first six months of unemployment. On the other hand, Card et al. (2007) and Van Ours and Vodopivec (2006) find no significant effect of an increase in unemployment duration on either the wage or the duration of employment in the new job.

[5] In this paper, I assume that human capital only determines the output level, but find in the numerical simulations that results are similar when human capital directly affects the search effectiveness as well. Even without this direct effect, the probability to become employed endogenously decreases during the unemployment spell if no training facility is available and search is continuous.

policies to be changed continuously. This allows me to shed light on the optimal use of the training policy throughout the unemployment policy and how this interacts with the provision of search incentives.

Both Pavoni and Violante (2005) and Wunsch (2008) consider the introduction of a training technology in the setup of Pavoni and Violante (2007), but rely on numerical simulations to shed light on the optimal use of the training technology in the unemployment policy. The training technology differs from the technology considered in this paper. The training policy is binary and cannot be implemented together with search incentives.[6] Training efforts are assumed to be unobservable, just like search effort, and need to be induced by rewarding the trainees with higher unemployment benefits when they successfully increase their observable human capital. Pavoni and Violante (2005) find that the training policy is mostly dominated by other policies in the simulations due to the high fixed cost of the program and the need to provide costly incentives. The effectiveness of training is assumed to be low for low levels of human capital. Hence, the training program does not prevent the long-term unemployed from ending up in social assistance. Similarly, the simulations in Wunsch (2007) suggest that training needs to increase the wage level substantially for the training policy to be optimal. The training policy is more likely to be optimal for intermediate levels of human capital. Wunsch (2009) considers job search assistance, modeled in the same way as the training policy in Pavoni and Violante (2005) and Wunsch (2008), but increasing the search effectiveness of the unemployed rather than the wage level upon re-employment. She provides a comprehensive numerical analysis based on a policy scheme in West Germany and draws several conclusions that correspond to the results established here. In particular, she finds that effective job search assistance delays and may even prevent social assistance when output upon re-employment is unaffected by depreciations. She also finds that job search assistance may be valuable already at the start of the unemployment spell, in particular when the job finding rate is low. In contrast with Pavoni and Violante (2007) and my paper, these papers do not characterize the optimal transition between policies. An important contribution of my paper is to establish analytically the optimal dynamics and the convergence of the unemployment policy in the presence of a training technology. The analysis sheds new light on the determinants of the optimal training policy and how this affects the optimal consumption path during the unemployment spell.

Like in most of the related papers, the characterization of the optimal consumption profile in this paper ignores implementation constraints. A recent literature analyzes how the optimal consumption profile may not be implementable if the unemployed do have savings and savings cannot be restricted (Werning 2002, Shimer and Werning 2008).[7] Unobservable savings affect the set of consumption profiles that are incentive

[6] This is an extreme case of the model considered here when the effort dimensions become extreme substitutes.

[7] If the unemployed have no savings, the consumption profile can naturally be implemented by setting

compatible with a profile of search efforts. The insights regarding the optimal training policy are still expected to generalize as the optimal design of the training policy and search efforts is separable from the design of the consumption profile inducing these search efforts.

2 Model

I consider an agent at the start of an unemployment spell. During each period of unemployment, the agent exerts effort in two dimensions, search and training. Search effort s increases the probability to find employment $\pi(s)$, with $\pi' > 0 \geq \pi''$ and $\pi(0) = 0$.[8] Once the unemployed agent has found a job, he remains employed forever.[9] Training effort t increases the unemployed's human capital θ, which determines his production upon re-employment $y(\theta)$, with $y' > 0 \geq y''$, $lim_{\theta\to\infty} y'(\theta) = 0$ and $lim_{\theta\to 0} y'(\theta) = \infty$.

Training and Search Effort Both searching for employment and participating to training programs are costly for the agent. I assume a convex cost function $\psi(s,t)$ with the cross-derivative $\psi_{s,t}(s,t) \geq 0$. I thus allow the cost of additional search s to increase with the level of training t. This would be implied by the simple fact that time spent on training programs cannot be spent on search. To simplify the analysis, I assume that the search efforts are chosen by the agent, but the training efforts are chosen by the social planner. The unemployment policy thus affects the unemployed's incentives to search for employment both through the insurance the policy provides and the training it imposes.[10] I assume that all costs related to the training program are captured by the cost function $\psi(s,t)$.[11]

Preferences The per-period utility during unemployment and employment are denoted by $u(c, \psi(s,t))$ and $u^e(c^e)$ respectively, where c and c^e are the consumption

unemployment benefits and taxes accordingly. Notice that nearly half of job losers in the United States report zero liquid wealth at the time of job loss (Chetty 2008).

[8] For the convergence analysis in Section 4.3, I assume that search effort is binary, $s \in \{0, \bar{s}\}$. For convenience, I only consider the continuous expressions in the setup of the model.

[9] I only model incentive problems during one unemployment spell. The probability of displacement after re-employment would be captured by a lower (expected) output level upon re-employment $y(\theta)$ and would not affect the analysis otherwise. See Wang and Williamson (1996), Zhao (2000) and Hopenhayn and Nicolini (2009) for a treatment of moral hazard on the job and multiple unemployment spells.

[10] Relaxing this assumption would complicate the analysis as the unemployed's decision to train would interact with his decision to search. Notice, however, that the unemployment policy does not distort the incentives to exert training effort as long as the unemployed capture the resulting increase in productivity when employed. For a fixed search level, the social planner and unemployed agent would then choose the same level of training.

[11] In addition to the private effort cost of participation, training programs may also have financial costs (potentially negative if trainees are temporarily employed in public jobs). In any case, the agent's net consumption $c - \psi(s,t)$ in the optimal policy is the same when either the agent or the social planner pays this cost since the monetary transfer between the agent and social planner will adjust accordingly.

levels during unemployment and upon re-employment respectively. The expected lifetime utility for an agent who starts in unemployment at time $\tau = 0$ equals

$$u(c_0, \psi(s_0, t_0)) + \sum_{\tau=1}^{\infty} \beta^{\tau} [\pi_{\tau}^{e} u^{e}(c_{\tau}^{e}) + (1 - \pi_{\tau}^{e}) u(c_{\tau}, \psi(s_{\tau}, t_{\tau}))],$$

where π_{τ}^{e} denotes the probability to be employed in period τ. The employment probability at time τ equals $\pi_{\tau}^{e} = \pi_{\tau-1}^{e} + \pi(s_{\tau-1})(1 - \pi_{\tau-1}^{e})$. Either the worker was already employed in period $\tau - 1$ and thus still employed in period τ or he found a job with probability $\pi(s_{\tau-1})$ in period $\tau - 1$. I characterize the optimal unemployment policy analytically for CARA preferences with the cost of effort expressed in monetary terms,

$$u(c, \psi(s,t)) = -\exp(-\sigma(c - \psi(s,t)) \text{ and } u^{e}(c^{e}) = -\exp(-\sigma c^{e}).$$

This standard specification is often considered in the optimal unemployment insurance literature (e.g., Werning 2002, Shimer and Werning 2006, 2008), but also in applications of the multi-tasking setting (Holmström and Milgrom 1991). The results on the optimal consumption path apply more generally, making them comparable to the analysis with additive preferences $u(c) - \psi(s)$ in Hopenhayn and Nicolini (1997) and Pavoni and Violante (2007).

Human Capital and Training Human capital θ depreciates during unemployment at an exponential rate δ. I assume a linear training technology for increasing the level of human capital. When an unemployed agent with human capital θ_{τ} exerts training effort t_{τ} in period τ, her human capital in period $\tau + 1$ equals

$$\theta_{\tau+1} = \theta_{\tau}(1 - \delta) + t_{\tau}.$$

Without training, the human capital of long-term unemployed converges to 0 for which there is no added-value of being employed, i.e., $y(0) = 0$. Training efforts counter the depreciation and may even increase the level of human capital during the unemployment spell, $\theta_{\tau+1} > \theta_{\tau}$. The concavity of the output function $y(\theta)$ and the convexity of the cost function $\psi(s,t)$ cause the net return to training to be decreasing in human capital and training respectively. The Inada condition on the output function $lim_{\theta \to \infty} y'(\theta) = 0$ implies that the return to training is bounded. Finally, I assume that no training technology is available during employment such that the level of human capital remains constant once re-employed.[12] Notice that while I only model the depreciation in human capital explicitly, I characterize the optimal contract as a function of the level of human capital at the start of the unemployment spell, which may capture the fall in human

[12] This assumption speaks more to an environment where training is necessary to make or keep the unemployed 'employable'. The opportunity cost of training may well be higher on the job than during unemployment. Moreover, the employer may not be willing to provide training on the job when it cannot capture the rents of training.

capital upon job loss.

3 Social Planner's Problem

The social planner designs an integrated unemployment policy combining three instruments which can vary with the length of the unemployment spell: unemployment consumption c, employment consumption c^e and training t. I assume that the agent's level of human capital at the start of the unemployment spell θ_0 is known to the social planner and allow the unemployment policy to be contingent on this starting level.[13] Following the dual approach as in Spear and Srivastava (1987), I characterize the unemployment policy that provides this agent with a promised level of expected life-time utility V_0 at the lowest expected cost. I consider the optimal policy for any pair (V_0, θ_0). A policy maker's choice of V_0 will depend on her budget constraint, but may depend on the unemployed agent's human capital as well. To provide insurance against the drop in human capital upon displacement, the policy maker may make the promised utility V_0 dependent on the worker's human capital before rather than after displacement.

Rather than writing the optimal contract as a function of the length of the unemployment spell, I determine the optimal contract recursively. The recursive problem has only two state variables: the unemployed agent's current level of human capital θ and the expected life-time utility promised last period to the unemployed agent V. At any point during the unemployment spell, these two state variables summarize all relevant aspects of the agent's unemployment history. I thus characterize the optimal policy variables $\{c(V,\theta), V^e(V,\theta), V^u(V,\theta), s(V,\theta), t(V,\theta)\}$ that minimize the expected cost of the unemployment policy assigning expected life-time utility level V to the unemployed individual with human capital θ. The policy functions $V^e(V,\theta)$ and $V^u(V,\theta)$ determine the promised utilities when respectively employed and unemployed in the next period.[14]

The optimal contract solves

$$C(V,\theta) = \min_{c,V^e,V^u,s,t} c + \beta\left[\pi(s)C^e(V^e,\theta') + (1-\pi(s))C(V^u,\theta')\right]$$

such that

$$u(c,\psi(s,t)) + \beta[\pi(s)V^e + (1-\pi(s))V^u] = V \qquad (PC)$$

$$s \in \arg\max u(c,\psi(s,t)) + \beta\left[\pi(s)V^e + (1-\pi(s))V^u\right] \qquad (IC)$$

[13] I thus ignore unobservable heterogeneity in human capital across job seekers in this analysis. If the social planner were restricted to one plan, he would focus on the starting human capital level of some representative agent.

[14] The optimal contract as a function of the spell length τ can be found by calculating the optimal sequence of state variable pairs $(V_{\tau+1}, \theta_{\tau+1}) = (V^u(V_\tau,\theta_\tau), (1-\delta)\theta_\tau + t(V_\tau,\theta_\tau))$ starting from (V_0,θ_0) and evaluating the optimal consumption levels and training efforts for each of these vectors.

and $\theta' = \theta(1-\delta) + t$. The first order conditions and the two envelope conditions for the Bellman equation are in the appendix. The expected total cost to the social planner consists of the cost this period and the expected cost from the next period on. The cost this period is equal to the unemployment consumption level c. The expected cost from tomorrow on depends on whether the agent finds work today, the respective promised utilities V^e and V^u and the new level of human capital. The social planner is constrained to offer a contract for which the agent's expected utility is equal to V. This is captured by the promise-keeping constraint (PC). The incentive compatibility constraint (IC) captures the fact that agent's search effort is unobserved. He chooses this to maximize his own expected utility taking the unemployment policy as given. The first-order condition for incentive compatibility of the contract implies

$$u_\psi(c, \psi(s,t))\psi_s(s,t) + \beta\pi'(s)\left[V^e - V^u\right] \leq 0, \tag{1}$$

with strict inequality only if $s = 0$. The agent exerts no effort when his marginal return to search is negative for $s = 0$. If the social planner wants to induce the agent to exert costly search efforts $s > 0$, she needs to refrain from providing full insurance and to create a wedge between V^e and V^u.[15]

The expected cost for the social planner to assign V^e to the agent after he has found employment, equals

$$C^e(V^e, \theta') = \min_{c^e, \hat{V}^e} c^e - y(\theta') + \beta C^e(\hat{V}^e, \theta')$$

such that $u^e(c^e)/(1-\beta) = V^e$. Once the agent is re-employed, it is optimal to give the same level of consumption in every future period. Hence,

$$C^e(V^e, \theta') = \frac{(u^e)^{-1}\left(V^e(1-\beta)\right)}{1-\beta} - \frac{y(\theta')}{1-\beta}.$$

The first term of the value function equals the present value of a constant consumption stream $c^e = (u^e)^{-1}\left(V^e(1-\beta)\right)$ that provides the corresponding promised utility level. The second part of the value function equals the present value of the worker's output stream.

4 Optimal Insurance Contract

In this section, I characterize analytically how the optimal policy changes during the unemployment spell. This depends on when during the unemployment spell one of two

[15] In the binary case with $s \in \{0, \bar{s}\}$, the agent only exerts positive effort if

$$u(c, \psi(\bar{s}, t)) - u(c, \psi(0,t)) + \beta\pi(\bar{s})\left[V^e - V^u\right] \geq 0. \tag{2}$$

states is optimal: the *training state*, during which training effort is imposed, but no search effort is induced ($t_\tau > 0, s_\tau = 0$); the *search-and-training state*, during which search effort is induced and training may be imposed ($t_\tau \geq 0, s_\tau > 0$). I characterize the optimal policy profile for a given state and the optimal transition between the two states. I show that the optimal policy starts in the training state when the unemployed worker's human capital is too low initially, but converges to the search-and-training state. The long-term unemployed maintain a positive level of human capital and continue to be encouraged to leave unemployment. Restricting the analysis to binary search effort, I show that the level of human capital of the long-term unemployed converges monotonically to a unique, positive level.

4.1 CARA Preferences with Monetary Costs

I first establish two preliminary results for CARA preferences with monetary costs of effort, which are exploited in the further analysis. First, the cost of the optimal policy is additive in the promised utility V and the level of human capital θ, $C(V,\theta) \equiv h(V) - g(\theta)$. Second, the decision how much training to impose and search to induce can be analyzed separately from the decision how to structure the consumption profile to provide these search incentives optimally.

When an individual's absolute risk aversion is independent of his consumption level, an equal increase in all consumption levels, today and in the future, during employment and unemployment, leaves the margins for search unchanged. Since $u(x+y) = -u(x)u(y)$ and $u(x) = -\frac{u'(x)}{\sigma}$ with CARA preferences, the promised utilities V^e and V^u and marginal utility $u'(c-\psi)$ are all rescaled by $-u(\varepsilon)$ after an ε-increase in all consumption levels. Hence, the marginal return to search,

$$-u'(c-\psi(s,t))\psi_s(s,t) + \beta\pi'(s)[V^e - V^u],$$

and thus the chosen search effort s remains unchanged after an equal increase in all consumption levels. As a consequence, the optimal response to an increase in promised utility V is to increase all consumption levels by the same amount, regardless of the level of human capital. The value function $C(V,\theta)$ determining the minimum cost of the optimal policy for an unemployed worker is therefore additive in the promised utility V and the level of human capital θ, just like for an employed worker.

Proposition 1 *For CARA preferences with monetary cost of efforts, the value function equals*

$$C(V,\theta) = -\frac{\ln(-V(1-\beta))}{\sigma(1-\beta)} - \frac{g(\theta)}{1-\beta}, \tag{3}$$

where

$$g(\theta) = \max_{s,t} -(1-\beta)\left[\psi(s,t)+\kappa(s,t)\right]+\beta\left[\pi(s)\,y(\theta')+(1-\pi(s))g(\theta')\right] \quad (4)$$

$$\kappa(s,t) = \min_{u,V^e,V^u} -\frac{1}{\sigma}\left\{\begin{array}{l}\ln[(\frac{u}{V})/(1-\beta)]+ \\ \quad \frac{\beta}{1-\beta}[\pi(s)\ln(\frac{V^e}{V})+(1-\pi(s))\ln(\frac{V^u}{V})]\end{array}\right\} \quad (5)$$

subject to (PC) and (IC) and $\theta' = \theta(1-\delta)+t$.

The expected cost of the optimal policy for an unemployed worker has similar components as for an employed worker. Like in the case of an employed worker, the first term of $C(V,\theta)$ equals the present value of a constant consumption stream that provides the corresponding promised utility level V. The second term $g(\theta)/(1-\beta)$ represents the expected present value of an unemployed job seeker's human capital, which now equals the present value of his expected output stream net of the expected cost of search and training effort exerted during unemployment and the expected cost of the consumption risk imposed to provide incentives. These different components appear clearly in the Bellman equation (4) characterizing $g(\theta)$. With probability $\pi(s)$, the job seeker finds employment and realizes the output stream $y(\theta')$ from the next period on. With probability $1-\pi(s)$, the job seeker remains unemployed. While unemployed, the job seeker needs to be compensated for his effort cost $\psi(s,t)$ and the consumption risk she faces. The cost of the consumption risk is captured by the risk premium $\kappa(s,t)$. As shown in equation (5), this premium equals the cost of the minimum deviation from perfect consumption smoothing that is required to induce search effort s subject to the promise-keeping constraint. The risk premium depends on the effort levels, but not on the level of human capital θ or the promised utility V. The risk premium is increasing in the induced search effort s and equals 0 for $s=0$. When not inducing search, there is no reason to deviate from perfect consumption smoothing.[16] The risk premium also depends on the training effort t, but only if $\psi_{s,t} \neq 0$. When search and training efforts are substitutes, $\psi_{s,t} > 0$, the premium increases in the training intensity t.

Since the cost of consumption risk is independent of the wealth level for CARA preferences, the risk premium is independent of the promised utility level V. Hence, the promised utility level affects the consumption level, but does not affect the optimal search and training policies. The risk premium does not depend directly on the level of human capital θ either. While the level of human capital is crucial in determining the optimal search and training policies, the optimal consumption profile for those effort levels is independent of the level of human capital. As a consequence, for any search and training profile, the optimal consumption path can be determined independently of θ.

[16] For $s=0$, it is optimal to set $V^u = V^e = u/(1-\beta) = V$ such that $\kappa(0,t)=0$.

Corollary 1 *For CARA preferences with monetary cost of efforts, the optimal search and training policy do not depend on the promised utility level. Conditional on the optimal search and training policy, the optimal consumption policy does not depend on the human capital level.*

The structure of the problem thus allows us to analyze optimal search and training policies in separation of the optimal consumption policy. The first-order conditions for the effort levels imply that

$$\beta\left[\pi(s)\,y'(\theta'))+(1-\pi(s))g'(\theta')\right]-(1-\beta)\left[\psi_t(s,t)+\kappa_t(s,t)\right] \leq 0, \quad (6)$$

$$\beta\pi'(s)\left[y(\theta')-g(\theta')\right]-(1-\beta)\left[\psi_s(s,t)+\kappa_s(s,t)\right] \leq 0, \quad (7)$$

with strictly inequality only if $t=0$ and $s=0$ in the respective equations. The first-order condition for the training intensity clearly shows the trade-off the social planner faces. Training increases the effort cost $\psi(s,t)$ and potentially the cost from providing search incentives $\kappa(s,t)$. However, training also increases the job seeker's level of human capital, which increases the production level as soon as he finds employment, either in the next period or in any future period. The gain thus depends on the weighted sum of marginal products, $\pi(s)\,y'(\theta')+(1-\pi(s))g'(\theta')$.

An increase in the search intensity increases the effort cost and the consumption risk needed to induce the higher search level. However, higher search effort also results in a higher probability of employment. The gain from search thus depends on the difference between the actual production value of human capital and the potential value for an unemployed job seeker, $y(\theta')-g(\theta')$. When the production value of human capital becomes very low, the potential value of the job seeker's human capital could exceed the actual production value, because of the opportunity to increase human capital through training.

4.2 Optimal Transition between States

I now characterize when either the training state $(t_\tau>0, s_\tau=0)$ or the search-and-training state $(t_\tau\geq 0, s_\tau>0)$ is optimal and how the optimal unemployment policy transits between these two states.

The following decomposition proves useful. In a first step, I derive the training policy that maximizes the job seeker's expected net value for a given search profile $\mathbf{s}=\{s_\tau\}_{\tau=0,1,\ldots}$. The search profile specifies for each period τ the job seeker's search effort s_τ when still unemployed. That is,

$$g(\theta|\mathbf{s})=\max_t -(1-\beta)\left[\psi(s_0,t)+\kappa(s_0,t)\right]+\beta\left[\pi(s_0)\,y(\theta')+(1-\pi(s_0))g(\theta'|\mathbf{s}_1)\right]$$

using $\mathbf{s}\equiv\{s_0,\mathbf{s}_1\}$ and $\theta'=(1-\delta)\,\theta+t$. In a second step, I compare the expected net value of all feasible search profiles and select the search profile associated with the highest expected net value,

$$g(\theta) = \max_{\mathbf{s}} g(\theta|\mathbf{s}).$$

By fixing the search profile in the first step, the recursive problem simplifies to a one-dimensional stochastic problem for which standard results apply. I introduce the following assumption.

Assumption 1 *Output $y(\theta)$ is concave in θ, effort cost $\psi(s,t)$ is convex in t and the rivalry between search and training is limited, i.e., $\psi_{s,t}(s,t) \in [0, \bar{\psi}_{s,t}]$ for all (s,t) and some $\bar{\psi}_{s,t} > 0$.*

Both the concavity of the production function $y(\theta)$ and the convexity of the cost function $\psi(s,t)$ in t are natural assumptions. The limited rivalry between search and training serves two purposes. First, the analysis requires the convexity of the sum of the costs $\psi(s,t) + \kappa(s,t)$ in t and $\psi_{s,t}(s,t) + \kappa_{s,t}(s,t) \geq 0$. The curvature of the risk premium is a complicated object depending on both higher order derivatives of the cost and utility function. However, the risk premium $\kappa(s,t)$ depends on the training intensity t only if $\psi_{s,t} \neq 0$. The curvature of the risk premium will therefore be of second order relative to the curvature of the cost function when $\psi_{s,t}$ is small. We can thus always find a positive upper bound $\bar{\psi}_{s,t}$ such that it is sufficient (but not necessary) to assume that $\psi_{s,t} \leq \bar{\psi}_{s,t}$. Second, the limited rivalry simplifies the analysis by excluding cycles of human capital around a particular level with the optimal policy alternating between period(s) of inducing search and period(s) of training to restore the level of human capital.

Lemma 1 *If Assumption 1 holds, the subvalue function $g(\theta|\mathbf{s})$ is concave in θ.*

The concavity of the value function implies that an increase in human capital is more valuable when human capital is low. This is crucial in determining the desirability of different unemployment policies. In particular, I show below how the use of the training state (i.e., $t > 0$, $s = 0$) in the unemployment policy is desirable if and only if human capital is low. The following Lemma establishes the 'only if'-part of the statement.

Lemma 2 *Consider the optimal search profile $\mathbf{s}^*$ for θ_0 given Assumption 1. The introduction of a training period before this policy, $\mathbf{s}^{0,*} = \{0, \mathbf{s}^*\}$, can be an improvement only if $\theta < \theta_0$. If a training period is optimal at θ, human capital increases during that period and never drops below θ afterwards.*

The value of adding a training period at the start of an unemployment policy with search profile $\mathbf{s}^*$ is captured by the difference between the subvalue functions $g(\theta|\{0, \mathbf{s}^*\})$ and $g(\theta|\mathbf{s}^*)$. The Lemma shows that the value-added of such training period is higher for lower levels of human capital. The reason is that the marginal value of human capital is lower when adding a training period due to the fact that the productive use of human capital is deferred by at least one period. Moreover, since a job seeker misses an opportunity to leave unemployment during the training state, such policy can only be optimal if it actually increases the job seeker's human capital. This also implies that if for some level of human capital the optimal policy actually prescribes the training state, the job seeker's human capital will not drop below that level in the remaining unemployment spell.

The gain from inducing search depends on the difference between the actual production value of a job seeker's human capital and the potential net value, $y(\theta) - g(\theta)$. The optimality of the training state for a particular level of human capital θ_0 would imply that this gain is small relative to the cost from inducing search. This follows immediately from the first-order condition with respect to search (see equation (7)). The following Lemma shows that the gain from searching is smaller for lower levels of human capital $\theta < \theta_0$, i.e., $y'(\theta) - g'(\theta) > 0$. This implies that inducing search would not be desirable for this lower range of human capital levels either. Notice that the marginal value of human capital for an unemployed agent $g'(\theta)$ is fully determined by the additional production value if the agent were to find employment. That is, by the envelope theorem, this marginal value is equal to the expected present value of the resulting increase in output upon re-employment,

$$g'(\theta) = \Sigma_\tau \beta^\tau (1-\delta)^\tau \pi_\tau^e y'(\theta_\tau) < \Sigma_\tau \pi_\tau^e y'(\theta_\tau) \leq y'(\theta), \tag{8}$$

where π_τ^e and θ_τ denote respectively the probability of employment and the output level at time τ given the optimal policy. The strict inequality in (8) follows from the positive discounting and depreciation. The weak inequality follows from the concavity of $y(\theta)$ and the optimality of the training state at θ implying that the future level of human capital never drops below this level. Hence, this implies $y'(\theta) > g'(\theta)$.

Lemma 3 *Consider the optimal policy for $\theta_0 > 0$ given Assumption 1. If the optimal policy starts in the training state, then it cannot be optimal to induce positive search at the start of the spell for any $\theta < \theta_0$.*

Lemma 2 and 3 together imply the existence of a cut-off $\bar{\theta}$ such that an unemployed agent with a lower level of human capital starts in the training state and remains in the training state until he increased his level of human capital above this cut-off. When the agent's human capital is above this cut-off, the search-and-training state is optimal and the unemployed agent will be encouraged to search for employment. The agent remains exposed to positive training efforts and his level of human capital will never again drop below the cut-off level $\bar{\theta}$. Once the search-and-training state becomes optimal, it thus remains optimal for the remainder of the unemployment spell. Notice that there may be a lower cut-off $\underline{\theta}$ $\left(< \bar{\theta}\right)$ such that for agents with lower human capital the training efforts required to make the agent sufficiently employable (and thus to exceed the cut-off level $\bar{\theta}$) are too high. The optimal policy would thus never impose training efforts, nor induce search efforts and this from the start of the unemployment spell. Hence, the value of the agent's human capital $g(\theta)$ equals 0 for $\theta < \underline{\theta}$. I disregard this trivial outcome assuming that the training technology is effective such that even for agents with low human capital the potential production value is greater than zero, i.e., $g(\theta) > 0 \; (= y(0))$.

Proposition 2 *Given an effective training technology and Assumption 1, the optimal unemployment policy starts in the training state if and only if human capital is below a cut-off level $\bar{\theta}$. The training state is always followed by the absorbent search-and-training state.*

The optimal sequence of states shows that the optimal design of the dynamic unemployment policy crucially depends on the presence of an effective training technology. The fact that the search-and-training state is absorbent implies that even the long-term unemployed exit unemployment with positive probability. The optimal policy thus never gives up on the unemployed and continues to induce them to find employment. When no training facility is available, the depreciation of human capital causes the production level to be too small compared to the cost of inducing search after a finite time of unemployment. Hence, the unemployed enter *social assistance*, an absorbent policy that pays constant welfare payments and does not encourage the unemployed to find employment (Pavoni and Violante 2007).[17] By combining search and training, social assistance is avoided. The integration of training programs in the unemployment policy thus changes when search incentives are desirable during the unemployment spell; while the unemployed may not be encouraged to search in an initial training phase of the unemployment spell, they are always induced to search in the absorbing search-and-training state. The next section establishes how the optimal policy convergences in the search-and-training state to a stationary state.

4.3 Convergence and Optimal Training Policy

In this Section I analyze the convergence of the optimal unemployment policy. I find that the optimal policy converges to a stationary state which maintains a positive level of human capital. The difference between this stationary level and the level at the start of the unemployment spell fully determines the optimal timing of training during the unemployment spell. To keep the analysis tractable, I assume that search effort can only take two values, $s \in \{0, \bar{s}\}$ and the level $\bar{s}$ is such that the training state is optimal

[17] This requires that without training technology, the output value of human capital converges to a sufficiently low level, relative to the cost of inducing search. Hence, with human capital converging to 0, it is sufficient that $\psi_s(0,0) > 0$ in the case of continuous effort. Notice that with an ineffective training technology, it may be optimal to let human capital depreciate below this level for any job seeker (i.e., not just for those with human capital below $\underline{\theta}$ as discussed above), but still use training to slow down the depreciation while inducing search. However, this would not change the optimal sequencing as characterized by Pavoni and Violante (2007).

below and the absorbing search-and-training state is optimal above a positive cut-off $\bar{\theta}$, like in Proposition 2. The restriction to binary search causes the optimal search policy to be stationary during the search-and-training state, $\bar{\mathbf{s}} = \{\bar{s}\}_{\tau=0,1,...}$. While this simplifies the analysis, the forces driving convergence identified below do not depend on the binary nature of search effort. The numerical simulations in Section 5 suggest that the dynamics generalize for continuous search effort.

The net change in human capital during the search-and-training state depends both on the optimal training policy $t(\theta|\bar{\mathbf{s}})$ and the depreciation rate δ,

$$\theta' - \theta = t(\theta|\bar{\mathbf{s}}) - \delta\theta.$$

By implicit differentiation of the first-order condition with respect to training (see equation (6)), we find that the optimal training intensity is decreasing in the level of human capital,

$$t'(\theta|\bar{\mathbf{s}}) = -(1-\delta)\frac{\beta[\pi(\bar{s})y''(\theta')) + (1-\pi(\bar{s}))g''(\theta'|\bar{\mathbf{s}})]}{\beta[\pi(\bar{s})y''(\theta')) + (1-\pi(\bar{s}))g''(\theta'|\bar{\mathbf{s}})] - (1-\beta)[\psi_{tt}(\bar{s},t) + \kappa_{tt}(\bar{s},t)]} \leq 0. \tag{9}$$

The expression is negative since both the production function $y(\theta)$ and the value function $g(\theta)$ are concave in θ and the sum of the cost functions $\psi(s,t) + \kappa(s,t)$ is convex in t. Since the optimal training efforts are decreasing in θ, while the depreciation is increasing in θ, the net change in human capital is decreasing in θ. As a consequence, the level of human capital converges for the job seekers who remain unemployed to a unique stationary level θ^*. The optimal policy thus converges to a stationary state during which the same level of human capital is maintained through training $t^* = \delta\theta^*$. In this stationary state, the first-order condition for the training intensity simplifies to

$$\beta\pi(\bar{s})\frac{y'(\theta^*)/(1-\beta)}{1-\beta(1-\pi(\bar{s}))(1-\delta)} - [\psi_t(\bar{s},t^*) + \kappa_t(\bar{s},t^*)] = 0. \tag{10}$$

The marginal benefit from training in the stationary state depends on the increase in production upon re-employment $y'(\theta)$, where the discounting reflects that training efforts add to a stock of human capital that depreciates while unemployed.[18] From condition (9), we also find $t'(\theta) \geq -(1-\delta)$. The decrease in the optimal training policy for higher levels of human capital is thus small relative to the depreciation rate. This implies that the net change in human capital becomes smaller when approaching the stationary level. The convergence to the stationary state is therefore monotone.

Proposition 3 *Given a binary search level $s \in \{0, \bar{s}\}$ and Assumption 1, the optimal*

[18] A necessary condition for a positive stationary level is that

$$\beta\pi(\bar{s})\frac{y'(0)/(1-\beta)}{1-\beta(1-\pi(\bar{s}))(1-\delta)} > \psi_t(\bar{s},0) + \kappa_t(\bar{s},0),$$

which is implied by the Inada condition $\lim_{\theta\to 0} y'(\theta) = \infty$.

unemployment policy converges to a unique, stationary policy. The convergence of human capital is monotone.

The monotone convergence implies that the training profile is also monotone during the search-and-training state. The optimal timing of training during unemployment is fully determined by the difference between the level of human capital at the start of the unemployment spell θ_0 and the stationary level of human capital θ^*.

Corollary 2 *If $\theta_0 \geq \theta^*$, training is more intensive for the long-term unemployed. If $\theta_0 < \theta^*$, training is more intensive for the short-term unemployed, except when $\theta_0 < \bar{\theta}$ so that training initially increases during the training state. The total training effort an unemployed worker expects to be exposed to is decreasing in his initial level of human capital θ_0.*

If human capital is relatively high at the start ($\theta_0 \geq \theta^*$), the training intensity is lower and possibly equal to zero at the beginning of the unemployment spell. While the value of training is initially low, it increases as human capital depreciates. If human capital is relatively low at the start ($\underline{\theta} \leq \theta_0 < \theta^*$), the opposite happens. Training is very intensive at the beginning of the unemployment spell and becomes less intensive during the unemployment spell. If human capital is too low at the start ($\theta_0 < \underline{\theta}$), the unemployed agent starts in the training state and only exerts training effort. In the training state, the intensity of training is initially increasing over time. With no immediate employment prospects, deferring training is desirable because the effect of training depreciates over time and the cost of future training is discounted. This follows from the Euler equation characterizing the training profile during the training state,

$$\psi_t(0, t_{\tau-1}) = \beta\,(1-\delta)\,\psi_t(0, t_\tau).$$

Since no job seeker exits unemployment during the training state and a job seeker stays for longer in the training state when his initial level of human capital is lower, the total training effort he expects to exert during the unemployment spell is higher as well.

When search is not binary, the substitutability between search and training efforts also becomes important during the search-and-training state. Since search efforts are more valuable for higher human capital, we expect more search to be induced when human capital is higher. The change in training intensity during the unemployment spell is expected to be more pronounced, the more training increases the marginal cost of search. I explore this further in the numerical simulations in Section 5.

The difference between the level at the start of the spell θ_0 and the stationary level θ^* can be related to the relative importance of two potential sources driving the loss of human capital: the fall in human capital upon displacement reflected in θ_0 (e.g., obsolete firm-specific and industry-specific skills) and the depreciation of human capital during unemployment reflected in θ^* (e.g., detachment from the labor market, loss of work habits,...).[19] The analysis suggests that if the fall in human capital upon displacement becomes more important, relative to the depreciation of human capital during unemployment, training should be targeted more to the short-term unemployed than to the long-term unemployed. The opposite is desirable only if the long-term unemployed are particularly unemployable relative to the short-term unemployed. In practice, training requirements are often imposed on the long-term unemployed. Only after some time in unemployment, one needs to enroll in particular training programs to remain eligible for unemployment benefits.[20] Not all programs have this particular focus on the long-term unemployed. In some countries, training is subsidized from the start of the unemployment spell and the unemployed are allowed to refuse jobs offered by the Public Employment Service if they enroll in these training programs. When offered from the start of the unemployment spell, training is often aimed at young people or at large groups of workers who have been displaced as a result of industrial restructuring (e.g. public training programs in Germany after the unification). This seems consistent with the model's recommendation, as the human capital of these groups may be particularly low or have dropped due to their job loss.[21]

4.4 Optimal Consumption Policy

The optimal path of consumption depends crucially on when it is optimal to provide search incentives. In the training state, the unemployed policy provides no search incentives and perfectly smooths the agent's consumption, as in the first best. In the search-and-training state, the social planner induces the unemployed to search for a job at the expense of consumption smoothing. Since the presence of the training technology changes the convergence and transition between states, the optimal consumption profile is affected as well.

I first characterize the optimal consumption dynamics during the training state and the search-and-training state. As discussed before, the promised utilities are chosen to minimize the cost of the consumption risk to which the job seeker is exposed to induce search. To analyze how this affects the optimal consumption profile when the search level is endogenously chosen as well, it proves useful to study the dynamics of the Lagrange multiplier on the promise-keeping constraint λ. This multiplier equals the shadow cost of the promised utility $C_V(V, \theta)$ and is constant in the first-best.

[19] The stationary level is decreasing in the depreciation rate δ by (10). The stationary training level $t^* = \delta\theta^*$ may nevertheless be increasing in δ.

[20] Recent examples are the 'New Deal' in the United Kingdom and the 'Activation of Search Behavior' in Belgium.

[21] The model assumes that training programs can effectively counter the reduction in employability, regardless of the source of human capital loss. In practice, a different type of training may be required. For example, training organized by the Public Employment Service may provide a new set of skills that replace job or sector specific skills that became obsolete and restore the employability of the unemployed in other firms and sectors. The temporary employment in public jobs, however, may be better suited to maintain the employability of the long-term unemployed.

Proposition 4 *In the training state,* $\Delta_{\lambda,\tau} = \lambda_\tau - \lambda_{\tau+1} = 0$*. In the search-and-training state,* $\Delta_{\lambda,\tau} = \lambda_\tau - \lambda_{\tau+1} > 0$.

From the first-order condition FOC_c and the envelope condition EC_V, we find the following Euler equation,

$$\lambda_{\tau+1} = \lambda_\tau - \mu_\tau \frac{\pi'(s_\tau)}{(1-\pi(s_\tau))}. \tag{11}$$

Higher promised utility at $\tau + 1$ relaxes the promise-keeping constraint at τ, but also decreases the incentives to search for a job at τ. Hence, at the optimum, the shadow price of promised utility at $\tau + 1$ equals the shadow price of promised utility at τ minus its impact on incentives for search at τ. In the training state, no search is induced and the Lagrange multiplier on the incentive compatibility constraint μ equals zero. Hence, the shadow price of V remains constant during this training phase of the unemployment spell, like in the first best. In the search-and-training state, the social planner induces the unemployed to search for a job by giving up insurance. The incentive compatibility constraint is thus binding and the Lagrange multiplier on the incentive compatibility constraint is positive, regardless of the presence of training. The shadow price of V thus decreases during unemployment in the search-and-training state.

The Proposition applies for more general preferences. However, in the case of CARA preferences, the shadow cost $C_V(V, \theta)$ is increasing in the promised utility V and independent of the level of human capital, by Proposition 1. The optimal unemployment policy thus prescribes that the expected life-time utility for an unemployed job seeker decreases the longer he is unemployed during the search-and-training state. This reflects the well-known result that search today should not only be induced by immediate rewards, creating a wedge between after-tax wages and unemployment benefits, but the incentives should be spread over all future periods of unemployment (Shavell and Weiss 1979). The introduction of a training program does not change this result. However, the training policy changes when inducing search is desirable. In particular, the training technology reverses when during the unemployment spell it is optimal not to induce any search and thus when consumption can be fully smoothed. This is related to the analysis before. Propositions 2 and 4 together imply that if it is ever optimal to fully smooth consumption, it will be for the short-term unemployed. For the long-term unemployed, it will always be optimal to give up consumption smoothing to induce search efforts. Notice that the training policy also affects how much search is desirable. Since training mitigates the depreciation in human capital, it generally reduces the need to induce search and thus may allow to smooth consumption more, which is further explored in the numerical simulations.

Corollary 3 *For CARA preferences with monetary cost of effort, the net-consumption during unemployment is constant in the training state. Net-consumption during unem-*

ployment and upon re-employment is decreasing with the length of the unemployment spell in the stationary state.

In the training state, the promised utility V remains constant, which results in constant net-consumption $c - \psi$ during the earliest stage of the unemployment spell. During the search-and-training state, the promised utility V decreases, but this does not necessarily result in decreasing net-consumption $c - \psi$. The desired level of search may increase in later periods and higher net-consumption would reduces the marginal utility cost $u'(c-\psi)\psi_s$ of inducing those search levels.[22] However, when the optimal search level is constant - during the stationary state or when search is binary - net consumption will thus be decreasing in the length of the unemployment spell. For CARA preferences, this decrease will be linear, as previously shown in Werning (2002) and Spinnewijn (2009).[23] Notice that the optimal contract makes both consumption during unemployment and upon re-employment dependent on the length of the unemployment spell, to spread incentives optimally over all future states, including employment (Hopenhayn and Nicolini 1997). The consumption levels during unemployment and upon re-employment are, however, disconnected from the actual productivity level as the social planner wants to provide insurance against the loss of human capital. This also affects the implementation of the optimal consumption path. That is, decreasing consumption levels do not necessarily imply that taxes upon re-employment will increase with the length of the unemployment spell.[24]

5 Numerical Simulations

In this section, I calibrate the dynamic model to calculate the optimal unemployment insurance contract numerically. The numerical exercise sheds further light on the dynamics of the different policy variables, in particular on the optimal timing of training and search during the unemployment spell and the consequence for the optimal path of consumption. The analysis illustrates the robust order of policy states and confirms the global convergence to a unique stationary state for continuous search effort. The simulation also suggests that the value-added of the optimal training policy is complementary to the value of an unemployment insurance policy without training. The

[22] For additive preferences in consumption and effort, like in Hopenhayn and Nicolini (1997) and Pavoni and Violante (2007), the shadow cost is inversely related to unemployment consumption, $\lambda_\tau = \frac{1}{u'(c_\tau)}$. Hence, a decreasing shadow cost always results in decreasing unemployment consumption.

[23] The fact that the provision of incentives results in ever decreasing expected utility for the long-term unemployed, and thus ever decreasing consumption for CARA preferences, of course assumes that there is no lower bound on the expected utility (see Pavoni 2007), for instance coming from limited liability or political constraints.

[24] If human capital decreases during the unemployment spell, taxes may initially decrease, but then start increasing when human capital approaches the stationary level. Similarly, the optimal unemployment replacement rate with respect to the potential wage does not necessarily evolve monotonically during unemployment.

welfare gain from the optimal integrated unemployment policy is U-shaped in the level of human capital of the unemployed.

The numerical methodology is based on value function iteration with discretization of the state space. As shown in Lemma 1 for CARA preferences, the effect of V on the value function is known, since it only affects the level of consumption. The relevant state space is thus one-dimensional, with the optimal training policy and consumption differences only depending on the human capital, but not on the promised utility.

5.1 Calibration

The calibration exercise closely follows the previous literature when possible. The one significant difference concerns the explicit modelling of the returns and costs of continuous search and training efforts in this model. No natural empirical moments are readily available to be matched, in particular to get the marginal return and cost of training and the interaction with search. My strategy is as follows. For the calibration of the search parameters, I match the empirical estimates of the exit rate and the elasticity with respect to the unemployment benefit level, evaluated at the current US unemployment insurance scheme. For the calibration of the training parameters, I take the exact same cost of effort parameters as for search. For the marginal return to training and the interaction between the cost of search and training, I show some sensitivity results.

CARA Preferences The unit of time is set to be one month and the monthly discount factor $\beta = 0.996$ to match an annual discount factor of 0.95. The constant coefficient of absolute risk aversion is set equal to $\sigma = 2$.

Human Capital Depreciation Human capital depreciates exponentially at a monthly depreciation rate $\delta = 0.0135$, following Pavoni and Violante (2007). Human capital determines output upon re-employment, $y(\theta) = \theta^{\omega}$ with $\omega < 1$.

Search Costs and Returns The probability to find a job as a function of the continuous search effort s equals $\pi(s) = 1 - \exp(-\rho s)$, following Hopenhayn and Nicolini (1997). I assume a convex monetary cost of search $\psi_0 s^{\psi_1}$. The parameters ρ, ψ_0, ψ_1 and ω are chosen as follows. The calibrated model - evaluated for an insurance contract approximating the current US unemployment insurance system (i.e. a replacement rate of 50 percent for 6 months) - matches two empirical moments; the simulated exit rate equals 0.2, matching the sample exit rate in Spinnewijn (2009), the simulated elasticity in the exit rate with respect to a change in benefits equals 0.5, matching the empirical evidence summarized in Krueger and Meyer (2002). Next to matching these two standard moments, I normalize the stationary level of human capital in the optimal scheme to 1 and assume that the total search cost is such that a job seeker is willing

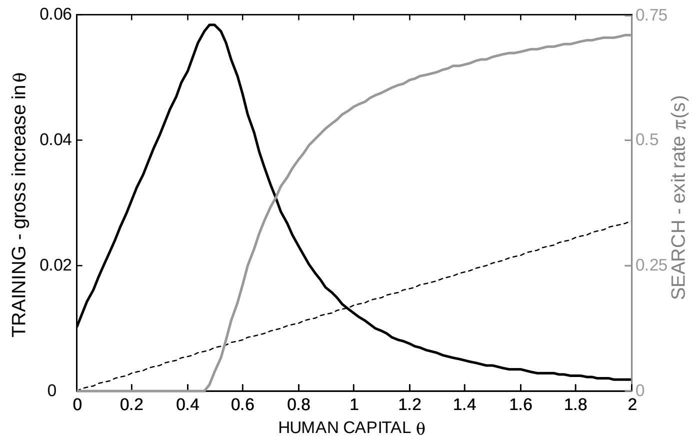

Figure 1: Policy functions: training (black) and search (grey) efforts as a function of human capital θ. Training is expressed in terms of the gross increase in θ (i.e., $z \times t$). Search is expressed in terms of the exit rate (i.e., $\pi(s)$). The depreciation in human capital is presented by the dashed line.

to forego the unemployment benefits if he or she could have the same probability of finding employment without having to search.[25] This calibration exercise implies $\omega = 4/5$, $\rho = 1/8$, $\psi_0 = 1/4$ and $\psi_1 = 6/5$.

Training Costs and Returns I assume the same monetary cost of training $\psi_0 t^{\psi_1}$ and introduce a linear interaction term $\psi_{s,t} st$ in the cost function,

$$\psi(s,t) = \psi_0 s^{\psi_1} + \psi_0 t^{\psi_1} + \psi_{s,t} st.$$

I assume a linear training technology such that the next month's level of human capital equals $\theta' = (1-\delta)\theta + zt$. I show results for different values of z and $\psi_{s,t}$, capturing the effectiveness of training and the complementarity with search respectively. Given the optimal contract and the standard parameter specification ($\psi_{s,t} = 0.02; z = .003$), the long-term unemployed spend half as much effort on training relative to search.

5.2 Optimal Training and Search

I first analyze the interaction between the imposed training and induced search efforts and its determinants. Figure 1 represents the optimal levels of training and search as a function of the level of human capital. In line with Proposition 2, the unemployed agent is in the training state if his human capital is lower than $\bar{\theta} = .46$. For levels above, the agent is induced to search and at the same time required to exert training

[25] Hence, given the current scheme, the cost of the exerted search effort $\psi_0 s^{\psi_1}$, expressed in monetary terms, equals the unemployment benefit level $b = .5$ (for the first six months of unemployment). The evidence by Krueger and Mueller (2010) on the time use by job seekers suggest that this estimate of a job seeker's actual search cost may well be high.

effort. At the stationary level $\theta^* = 1$, the induced exit rate is positive and the increase in human capital due to training $z \times t(\theta)$ equals the depreciation $\delta\theta$.

The simulation illustrates the substitutability between training and search during the search-and-training state itself. The required training effort is decreasing in the level of human capital (and eventually converges to 0), while the induced search effort is increasing in the level of human capital. Search is more valuable when human capital is high, whereas training is more valuable when human capital is low. An increase in the cross-derivative $\psi_{s,t}$ increases the rivalry between training and search efforts. This makes it more likely that the social planner either imposes intensive training programs or induce intensive search and also decreases the level of human capital maintained in the stationary state. This is illustrated in Figure 2, showing the policy functions for a lower and higher value of $\psi_{s,t}$. The substitutability between training and search is not due to the assumption that training does not affect the exit rate directly. Figure 3 shows the respective policy functions when

$$\pi(s, \theta) = 1 - \exp(-\rho\theta s),$$

such that not only the output level, but also the exit rate depends directly on the level of human capital ($\pi_{s,\theta} > 0$). The optimal exit rate is more responsive to human capital, while the optimal level of training is higher overall, particularly for low levels and high levels of human capital in the search-and-training state.

The effectiveness of training policies varies substantially across different programs (Heckman et al. 1999) and this naturally affects the training intensity; if the impact of training z decreases, the desired level of training is lower. The cut-off level $\bar{\theta}$ below which the training state is optimal decreases as well. The optimal level of search is now higher, since leaving unemployment is the only way to avoid the depreciation of human capital. This is illustrated in Figure 4 in Appendix. The search policy function not only shifts up, but also to the left, as the parameter value for z decreases. However, since human capital decreases more rapidly with less effective training, the long-term unemployed search less. Hence, the impact of training effectiveness on the expected unemployment duration is ambiguous. Some empirical evidence (Black et al. 2003) suggest that the anticipation of the participation to training programs decreases the unemployment duration in practice. However, training plays no deterring role here. The social planner can discourage the unemployed from remaining unemployed by lowering future unemployment benefits instead and actually increase its revenues. Training has no impact on output, the social planner will not impose or threaten to impose any training.

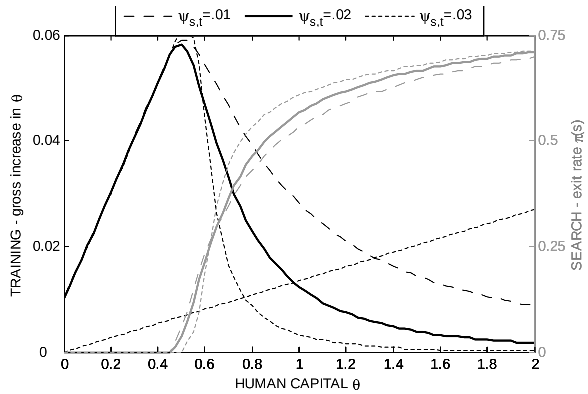

Figure 2: Effort substitutability: training (black) and search (grey) policy functions for different parameter values for the cross-derivative of the effort cost function $\psi_{s,t}$.

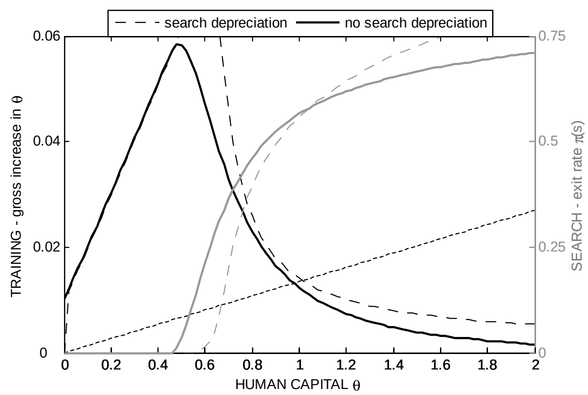

Figure 3: Search skill depreciation: training (black) and search (grey) policy functions with and without search skill depreciation.

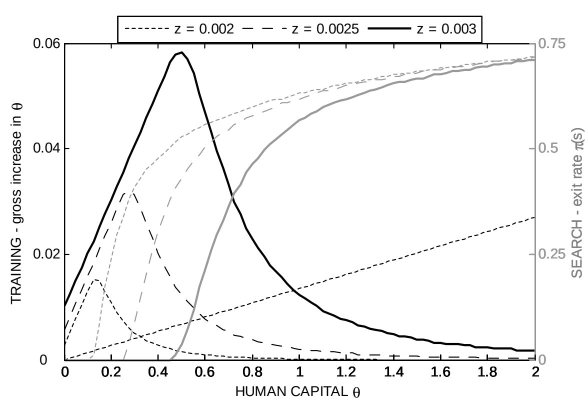

Figure 4: Training effectiveness: training (black) and search (grey) policy functions for different parameter values for the impact of training z.

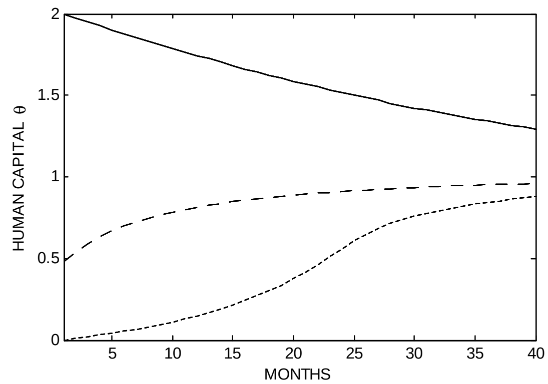

Figure 5: Human capital θ during the unemployment spell, starting with high human capital ($\theta_0 > \theta^*$ - solid), low human capital ($\theta_0 < \theta^*$ - dashed) and no human capital ($\theta_0 = 0$ - dotted).

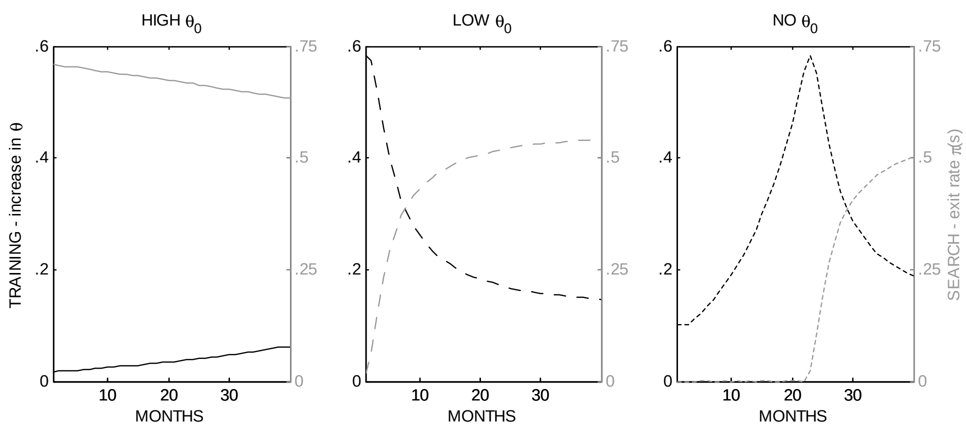

Figure 6: Training (black) and search (grey) during the unemployment spell, shown for a high starting level of human capital ($\theta_0 > \theta^*$) in the left panel, for a low level ($\theta_0 < \theta^*$) in the center panel and for a zero level ($\theta_0 = 0$) in the right panel.

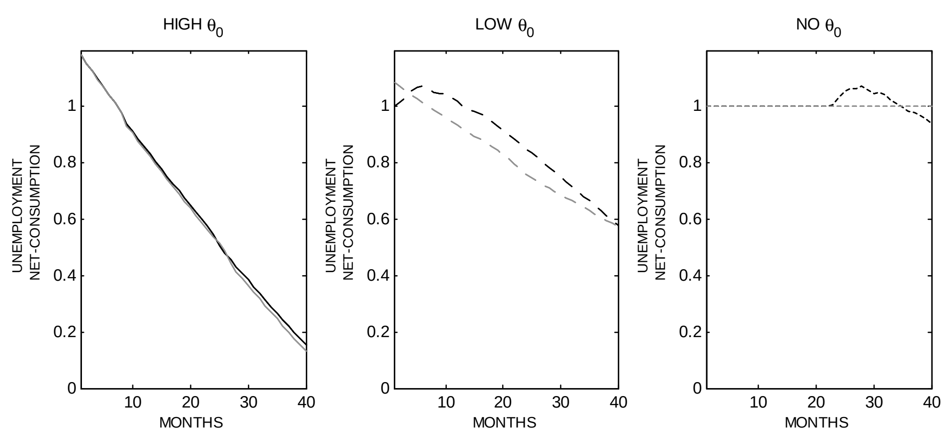

Figure 7: Net-consumption during the unemployment spell with training technology (black) and without training technology (grey), shown for a high starting level of human capital ($\theta_0 > \theta^*$) in the left panel, for a low level ($\theta_0 < \theta^*$) in the center panel and for a zero level ($\theta_0 = 0$) in the right panel.

5.3 Optimal Dynamics

The dynamics of the optimal unemployment policy crucially depend on whether the initial human capital level θ_0 is higher or lower than the stationary level θ^* and, if lower, whether the initial level is above or below the cut-off $\bar{\theta}$. I show the optimal dynamics for different starting values θ_0 in the three respective regions; $[0, \bar{\theta})$, $[\bar{\theta}, \theta^*)$ and $[\theta^*, \infty)$.

The change in human capital during the unemployment spell depends on difference between the gain in human capital due to the training efforts and the loss in human capital due to depreciation. From Figure 1, it follows that human capital will increase when it is below θ^* and decrease when it is above θ^*. Figure 5 shows how the unemployed's human capital evolves with the length of the unemployment spell for the different values of θ_0. In line with Propositions 3, the convergence is global and monotone. Regardless of the initial level, human capital converges to the positive, stationary level θ^* for the long-term unemployed. Figure 6 shows the associated optimal training path. If human capital is high at the start ($\theta_0 \geq \theta^*$), the training intensity starts low, but increases during the unemployment spell. If human capital is low at the start ($\bar{\theta} \leq \theta_0 < \theta^*$), the training intensity starts high, but decreases during the unemployment spell. When human capital is too low at the start ($\theta_0 < \bar{\theta}$), the training intensity initially increases during the training state, but starts decreasing once θ passes $\bar{\theta}$. When the unemployed worker starts without any valued human capital, this training stage takes more than twenty months, as shown in the right panel of Figure 6.

The panels in Figure 6 also show the duration-dependence of the exit rates, for which the trend is opposite to the training intensity. The desirability to induce search determines the optimal path of consumption during unemployment and upon re-employment. Figure 7 shows net consumption $c - \psi$ during the unemployment spell. If human capital is sufficiently low, no incentives for search are needed and net consumption is perfectly smooth initially, as shown in the right panel. Immediately after the training state, net-consumption is initially increasing, but when human capital approaches the stationary level, the optimal consumption path starts decreasing. The presence of the training technology changes the optimal consumption profile. Without training, net consumption is decreasing as long as search induced, but eventually will become constant as the level of human capital converges to 0, as shown by the dashed line in the panels of Figure 7. In the absence of a training technology, the consumption scheme is the social planner's only instrument to both insure the unemployed and provide incentives for search. Since training mitigates the effect of human capital depreciation, it reduces the need for search for a given level of human capital. This generally allows the social planner to focus more on insurance and smooth the marginal utility of consumption. Net-consumption will not be as rapidly decreasing in the beginning of the unemployment spell, particularly when it is optimal to impose intensive training as shown in the center panel of Figure 7. This suggests that training as an active labor market policy is more complementary to a continental European unemployment insurance scheme with low incentives for search (high and slowly decreasing benefits) than to the US unemployment insurance scheme with high incentives for search (low and rapidly decreasing benefits).

5.4 Welfare Gains

To evaluate the advantages of training programs for unemployment policies, I compare the welfare gains from the optimal schemes with and without training programs. I calculate the welfare gain as a function of the level of human capital at the start of the unemployment spell. I set the expected cost for the two schemes equal to the calibrated expected cost of the current US unemployment insurance scheme, which depends on the starting level of human capital. Under normal economic conditions, the scheme provides a monthly unemployment benefit for a maximum duration of six months of about 50 percent of the pre-unemployment earnings. I set the monthly pre-unemployment earnings equal to 1, which is the output level to which the long-term unemployed converge given the optimal contract. The welfare gains are expressed in terms of the per-period consumption an unemployed job seeker without insurance is willing to pay for the unemployment insurance scheme.

Figure 8 shows the welfare gain as a function of the level of human capital at the start of unemployment for three different schemes: the optimal scheme with training, the optimal scheme without training and the current scheme. By construction, the optimal scheme which allows for the use of training dominates the optimal scheme without training, and both schemes dominate the current scheme. However, the relative welfare gains are very different depending on the initial level of human capital. Although the effectiveness of training remains unchanged, the additional welfare gain from introducing training is negligible for high starting levels of human capital, while it is very large for low starting levels of human capital. The gain increases exponentially up to twenty percent of the stationary output level as θ_0 decreases. The pattern is opposite for the welfare gain from financial insurance. Both the value of the current scheme and the additional value of the optimal insurance scheme without training are increasing in the starting level of human capital. As human capital is higher, the output upon re-employment $y(\theta)$ is higher, which increases both the willingness to pay for unemployment benefits and the scope for consumption smoothing.

The welfare analysis suggests that training is complementary as an unemployment policy, creating high value when financial insurance falls short. This results in the U-shape for the optimal policy's welfare gain as a function of human capital. This also implies that if the conditions apply under which it is optimal to increase the training intensity during the unemployment spell (i.e., $\theta_0 > \theta^*$), the introduction of training programs may add very little value to an unemployment insurance scheme that is optimally designed. This questions the focus on the long-term unemployed by many training policies implemented in practice.

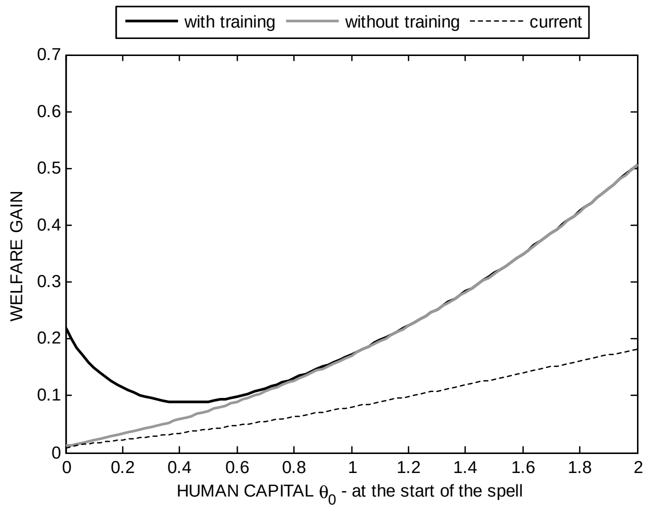

Figure 8: Welfare comparison: welfare gain in terms of per-period consumption for the optimal policy with training, the optimal policy without training and the current policy in comparison with no policy, as a function of the level of human capital at the start of the unemployment spell.

6 Conclusion

The secular trend of increasing production mobility, technological innovations and shifts in consumer demand forces workers to switch jobs and industries frequently. Job mobility does not only involve the risk of unemployment, but also the risk of wage loss (Low, Meghir and Pistaferri 2010). Displaced workers are often reemployed at lower wages and the persistent nature of this shock makes insurance against wage risk imperative. I approach training programs as a complementary unemployment policy to deal with the loss in wages. If the skill set of a displaced worker becomes redundant, incorporating training programs in unemployment policies is particularly valuable and these programs should be targeted towards the recently displaced. Only if the depreciation of human capital during the unemployment spell is sufficiently important, training should be targeted towards the long-term unemployed, as we often observe in practice.

The model has focused on the provision of insurance under moral hazard, ignoring political constraints and unobservable heterogeneity, two potential explanations for the focus on long-term unemployed. Requiring training programs only for the long-term unemployed helps *deterring* job seekers from remaining unemployed (Besley and Coate 1992, Black, Smith, Berger and Noel 2003) and allows the job seekers with high human capital to leave unemployment before the start of the costly programs. Notice that without political constraints, the social planner would always prefer to deter the unemployed by threatening to lower the monetary transfers rather than by imposing ineffective training programs. Moreover, if a menu of schedules could be offered, requiring intensive training from the start in the schedule designed for the job seekers with low human capital is not likely to encourage job seekers with high human capital to pretend they have low human capital.

The introduction of training programs also changes the optimal design of the consumption scheme. The integrated approach of the different aspects of unemployment policies is thus crucial. Firms may play an important role as well. The loss of skills increases the importance of inducing firms to internalize the costs of displacing workers. Firms could also be subsidized by governments to hire and train low-skilled workers who are not employable otherwise. The analysis here has shed light on the subsidy governments should be willing to pay for firms to play this role.

References

[1] Acemoglu, D., 1995. Public Policy in a Model of Long-term Unemployment. Economica 62, 161-78.

[2] Acemoglu, D., 2009. Introduction to Modern Economic Growth. Princeton university Press. 990p.

[3] Akerlof, G., 1978. The Economics of "Tagging" as Applied to the Optimal Income Tax, Welfare Programs, and Manpower Planning. American Economic Review 68(1), 8-19.

[4] Besley, T. and Coate, S., 1992. Workfare versus Welfare: Incentive Arguments for Work Requirements in Poverty-Alleviation Programs. American Economic Review 82(1), 249-261.

[5] Black, D., Smith, J., Berger, M., Noel, B., 2003. Is the Threat of Reemployment Services More Effective Than the Services Themselves? Evidence from Random Assignment in the UI System. American Economic Review 93(4), 1313-1327.

[6] Blanchard, O., Diamond, P., 1994. Ranking, Unemployment Duration and Wage Determination. Review of Economic Studies 61(3), 417-434.

[7] Card, D., Chetty, R., Weber, A., 2007. Cash-on-Hand and Competing Models of Intertemporal Behavior: New Evidence from the Labor Market. Quarterly Journal of Economics 122(4), 1511-1560.

[8] Card, D., Kluve, J., Weber, A., 2009. Active Labor Market Policy Evaluations. CESifo working paper 2570.

[9] Chetty, R., 2008. Moral Hazard vs. Liquidity and Optimal Unemployment Insurance. Journal of Political Economy 116(2), 173-234.

[10] Coles, M., Masters, A. 2000. Retraining and long-term unemployment in a model of unlearning by not doing. European Economic Review 44(9), 1801-1822.

[11] Falk, A., Huffman, D., Sunde, U., 2006. Do I have What It Takes? Equilibrium Search with Type Uncertainty and Non-Participation. IZA Discussion Paper 2531.

[12] Frijters, P., van der Klaauw, B., 2006. Job Search with Nonparticipation. The Economic Journal 508, 45-83.

[13] Heckman, J., LaLonde, R., Smith, J. 1999. The Economics and Econometrics of Active Labor Market Programs. In: Ashenfelter, O., Card, D. (Eds.), Handbook of Labour Economics, Vol. 3. Amsterdam: North-Holland, 1865-2097.

[14] Holmström, B., Milgrom, P., 1991. Multi-Tasking Principal-Agent Analyses: Linear Contracts, Asset Ownership and Job Design. The Journal of Law, Economics & Organization 7, 24-52.

[15] Hopenhayn, H., Nicolini, J., 1997. Optimal Unemployment Insurance. Journal of Political Economy 105(2), 412-438.

[16] Hopenhayn, H., Nicolini, J., 2009. Optimal Unemployment Insurance and Employment History. Review of Economic Studies 76, 1049-1070.

[17] Jacobson, L., LaLonde, R., Sullivan, D., 1993. Earnings Losses of Displaced Workers. American Economic Review 83(4), 685-709.

[18] Kling, J., 2006. Fundamentally Restructuring Unemployment Insurance: Wage-loss Insurance and Temporary Earnings Replacement Accounts. Hamilton Project Discussion Paper 2006-05.

[19] Krueger, A., Meyer, B., 2002. Labor Supply Effects of Social Insurance. In: Auerbach, A., and Feldstein, M. (Eds.), Handbook of Public Economics, Vol. 4. Amsterdam: North-Holland, pp. 2327-2392.

[20] Krueger, A., Mueller, A., 2010. Job Search and Unemployment Insurance: New Evidence from Time Use Data. Journal of Public Economics 94(3-4), 298-307.

[21] Lechner, M., Miquel, R., Wunsch, C., 2011. Long-run Effects of Public Sector Sponsored Training in West-Germany. Journal of European Economic Association 9(4), 742-784.

[22] Ljungqvist, L., Sargent, T., 1998. The European Unemployment Dilemma. Journal of Political Economy 106(3), 514-550.

[23] Low, H., Meghir, C., Pistaferri, L. 2010. Wage risk and employment risk over the life-cycle. American Economic Review 100(4), 1432-1467.

[24] Machin, S., Manning, A., 1999. The Causes and Consequences of Long-Term Unemployment in Europe. In: Ashenfelter, O., Card, D. (Eds.), Handbook of Labour Economics, Vol. 3. Amsterdam: North-Holland, pp. 3085-3139.

[25] Neal, D. 1995. Industry-Specific Human Capital: Evidence from Displaced Workers. Journal of Labor Economics 13(4), 653-677.

[26] Pavoni, N., Violante, G., 2005. Optimal Welfare-to-Works Programs. Discussion Paper 143. Federal Reserve Bank of Minneapolis.

[27] Pavoni, N., Violante, G., 2007. Optimal Welfare-to-Works Programs. Review of Economic Studies 74(1), 283-318.

[28] Pavoni, N. 2007. On Optimal Unemployment Compensation. Journal of Monetary Economics. 54, 1612-1630.

[29] Pavoni, N. 2009. Optimal Unemployment Insurance, with Human Capital Depreciation and Duration Dependence. International Economic Review 50(2), 323-362.

[30] Pissarides, C. 1992. Loss of Skill during Unemployment and the Persistence of Employment Shocks. Quarterly Journal of Economics 107, 1371-1391.

[31] Shavell, S., Weiss, L., 1979. The Optimal Payment of Unemployment Insurance Benefits Over Time. Journal of Political Economy 87, 1347-1362.

[32] Shimer, R., Werning, I., 2006. On the Optimal Timing of Benefits with Heterogeneous Workers and Human Capital Depreciation. NBER Working Paper 12230.

[33] Shimer, R., Werning, I., 2008. Liquidity and Insurance for the Unemployed. American Economic Review 98(5), 1922-1942.

[34] Spear, S., Srivastava, S., 1987. On repeated Moral Hazard with Discounting. Review of Economic Studies 53, 599-617.

[35] Spinnewijn, J. 2009. Unemployed but Optimistic: Optimal Insurance Design with Biased Beliefs. MIT mimeo

[36] Van Ours, J., Vodopivec, M., 2006. How Shortening the Potential Duration of Unemployment Benefits Affects the Duration of Unemployment: Evidence from a Natural Experiment. Journal of Labor Economics 24(2), 351-378.

[37] von Wachter, T., Song, J., Manchester, J., 2009. Long-Term Earnings Losses due to Mass Layoffs During the 1982 Recession: An Analysis Using U.S. Administrative Data from 1974 to 2004. Working Paper

[38] Wang, C., Williamson, S., 1996. Unemployment Insurance with Moral Hazard in a Dynamic Economy. Carnegie-Rochester Conference Series on Public Policy 44, 1-41.

[39] Werning, I., 2002. Optimal Unemployment Insurance with Unobservable Savings. MIT Working Paper.

[40] Wunsch, C., 2008. Optimal Use of Labour Market Policies. University of St. Gallen Working Paper.

[41] Wunsch, C., 2009. Optimal Use of Labor Market Policies: The Role of Job Search Assistance. forthcoming in the Review of Economics and Statistics.

[42] Zhao, R., 2000. The Optimal Unemployment Insurance Contract: Why a Replacement Ratio? University of Chicago Working Paper.

Appendix: Proofs and Derivations

First Order and Envelope Conditions.
The optimal contract solves

$$C(V,\theta) = \min_{c,V^u,V^e,s,t} c + \beta\left[\pi(s)\tfrac{(u^e)^{-1}((1-\beta)V^e)-y(\theta')}{1-\beta} + (1-\pi(s))C(V^u,\theta')\right]$$

such that

$$\begin{aligned} V - u(c,\psi(s,t)) - \beta[\pi(s)V^e + (1-\pi(s))V^u] &= 0 && (PC)\\ u_\psi(c,\psi(s,t))\psi_s(s,t) + \beta\pi'(s)\left[V^e - V^u\right] &\leq 0, && (IC) \end{aligned}$$

where λ and μ denote the Lagrange multipliers on the individual rationality constraint and incentive compatibility constraint respectively.
For an interior solution, the first order conditions are

$$\begin{aligned} 0 &= 1 - \lambda u_c - \mu u_{c,\psi}\psi_s && (FOC_c)\\ 0 &= C_V^e(V^e,\theta') - \lambda - \mu\frac{\pi'(s)}{\pi(s)} && (FOC_{V^e})\\ 0 &= C_V(V^u,\theta') - \lambda + \mu\frac{\pi'(s)}{(1-\pi(s))} && (FOC_{V^u})\\ 0 &= \beta\pi'(s)\left[C^e(V^e,\theta') - C(V^u,\theta')\right] - \mu(u_{\psi,\psi}\left(\psi_s\right)^2 && (FOC_s)\\ & \quad + u_\psi\psi_{s,s} + \beta\pi''(s)\left[V^e - V^u\right])\\ 0 &= \beta\left[\pi(s)C_\theta^e(V^e,\theta') + (1-\pi(s))C_\theta(V^u,\theta')\right] && (FOC_t)\\ & \quad -\lambda u_\psi\psi_t - \mu\left[u_{\psi,\psi}\psi_s\psi_t + u_\psi\psi_{s,t}\right] \end{aligned}$$

The envelope conditions are

$$\begin{aligned} C_V(V,\theta) &= \lambda && (EC_V)\\ C_\theta(V,\theta) &= \beta\left[\pi(s)C_\theta^e(V^e,\theta') + (1-\pi(s))C_\theta(V^u,\theta')\right](1-\delta).\square && (EC_\theta) \end{aligned}$$

Proof of Proposition 1.
The optimal consumption level for a promised value V^e when employed equals $c^e = -\frac{1}{\sigma}\ln(-V^e(1-\beta))$. Hence,

$$C^e(V^e,\theta') = \frac{-\ln(-V^e(1-\beta))}{\sigma(1-\beta)} - \frac{y(\theta')}{1-\beta}.$$

For the value function during unemployment, I rewrite the problem in terms of $\alpha_{V^e} = V^e/V$, $\alpha_{V^u} = V^u/V$ and $\alpha_{V^u} = u/V$. This implies the unemployment consumption level, $c = -\frac{\ln(-\alpha_u V)}{\sigma} + \psi(s,t)$, which is additive in V. The PC and IC constraints are independent of the state variable V,

$$\alpha_u + \beta[\pi(s)\,\alpha_{V^e} + (1-\pi(s))\alpha_{V^u}] = 1 \qquad (PC)$$

$$\alpha_{V^e} - \alpha_{V^u} \leq -\frac{\sigma\alpha_u\psi_s(s,t)}{\beta\pi'(s)}. \qquad (IC)$$

Following the expression for the value function upon re-employment, I now make the following guess for the value function during unemployment,

$$C(V,\theta) = -\frac{\ln(-V(1-\beta))}{\sigma(1-\beta)} - \frac{g(\theta)}{1-\beta}$$

for some function $g(\theta)$. For this guess, the Bellman equation equals

$$\begin{aligned} C(V,\theta) = \min_{\alpha,s,t}\{&-\frac{\ln(-\alpha_u V)}{\sigma} + \psi(s,t) \\ &+\beta\left[\pi(s)\frac{-\ln(-\alpha_{V^e}V(1-\beta))}{\sigma(1-\beta)} + (1-\pi(s))\frac{-\ln(-\alpha_{V^u}V(1-\beta))}{\sigma(1-\beta)}\right] \\ &-\beta\left[\pi(s)\frac{y(\theta')}{1-\beta} + (1-\pi(s))\frac{g(\theta')}{1-\beta}\right]\} \end{aligned}$$

subject to PC and IC, with $\theta' = \theta(1-\delta) + t$. Isolating the promised utility V, this can be rewritten as

$$\begin{aligned} C(V,\theta) = &-\frac{\ln(-V(1-\beta))}{\sigma(1-\beta)} + \\ &\min_{\alpha,s,t}\left\{\begin{array}{c} \psi(s,t) - \frac{1}{\sigma}\ln(\frac{\alpha_u}{1-\beta}) - \frac{1}{\sigma}\frac{\beta}{1-\beta}[\pi(s)\ln(\alpha_{V^e}) + (1-\pi(s))\ln(\alpha_{V^u})] \\ -\beta\left[\pi(s)\frac{y(\theta')}{1-\beta} + (1-\pi(s))\frac{g(\theta')}{1-\beta}\right] \end{array}\right\}, \end{aligned}$$

subject to PC and IC. The value function is thus additive in V and θ. Since the PC and IC constraints are also independent of V, this confirms our guess with

$$\begin{aligned} g(\theta) = \max_{\alpha,s,t}\{&\beta\left[\pi(s)\,y(\theta') + (1-\pi(s))g(\theta')\right] - (1-\beta)\psi(s,t) \\ &+\frac{(1-\beta)}{\sigma}\left[\ln(\frac{\alpha_u}{1-\beta}) + \frac{\beta}{1-\beta}[\pi(s)\ln(\alpha_{V^e}) + (1-\pi(s))\ln(\alpha_{V^u})]\right]\}. \end{aligned}$$

Since the optimal controls $\{\alpha, s, t\}$ do not depend on V, an increase in V results in an equal increase in all consumption levels. We also see that the terms containing α in the objective function are separable from θ, while the PC and IC constraint are independent of θ. Hence, we can write

$$\begin{aligned} g(\theta) = \max_{s,t}\{&\beta\left[\pi(s)\,y(\theta') + (1-\pi(s))g(\theta')\right] - (1-\beta)\psi(s,t) \\ &-(1-\beta)\min_{\alpha}\left\{-\frac{1}{\sigma}\left[\ln(\frac{\alpha_u}{1-\beta}) + \beta\pi(s)\ln(\alpha_{V^e}) + \beta(1-\pi(s))\ln(\alpha_{V^u})\right]\right\}\}. \end{aligned}$$

This corresponds to expression (4) in the proposition with

$$\kappa(s,t) \equiv \min_{\alpha} -\frac{1}{\sigma}\left[\ln(\frac{\alpha_u}{1-\beta}) + \frac{\beta}{1-\beta}\left[\pi(s)\ln(\alpha_{V^e}) + (1-\pi(s))\ln(\alpha_{V^u})\right]\right].$$

Rewriting this in terms of promised utility levels, using $\alpha_{V^e} = V^e/V$, $\alpha_{V^u} = V^u/V$ and $\alpha_{V^u} = u/V$, we find expression (5) in the Proposition. This minimization is subject to the PC and IC constraints. Notice that for $s = 0$, it is optimal to set $\alpha_u = (1-\beta)$ and $\alpha_{V^e} = \alpha_{V^u} = 1$ such that $\kappa(0,t) = 0$. For an interior solution, we can substitute the constraints into the objective function,

$$\begin{aligned} \frac{\alpha_{V^e}}{\alpha_u} &= -\tfrac{\sigma\psi_s(s,t)}{\beta\pi'(s)} + \tfrac{\alpha_{V^u}}{\alpha_u} && (IC) \\ \frac{1}{\alpha_u} &= 1 + \beta[-\pi(s)\tfrac{\sigma\psi_s(s,t)}{\beta\pi'(s)} + \tfrac{\alpha_{V^u}}{\alpha_u}], && (PC) \end{aligned}$$

and find the unconstrained optimization,

$$\begin{aligned} \kappa(s,t) = \min_{\alpha_{V^u}/\alpha_u} -\frac{1}{\sigma(1-\beta)} \{ &\ln(1+\beta[-\pi(s)\tfrac{\sigma\psi_s(s,t)}{\beta\pi'(s)} + \tfrac{\alpha_{V^u}}{\alpha_u}]) - (1-\beta)\ln(1-\beta) \\ &+ \beta\left[\pi(s)\ln(-\tfrac{\sigma\psi_s(s,t)}{\beta\pi'(s)} + \tfrac{\alpha_{V^u}}{\alpha_u}) + (1-\pi(s))\ln(\tfrac{\alpha_{V^u}}{\alpha_u})\right]\}, \end{aligned}$$

which depends on only one control variable, α_{V^u}/α_u. From the above expression, we can infer how $\kappa(s,t)$ depends on t. From the envelope condition with respect to t, we find

$$\kappa_t(s,t) = \frac{\pi(s)}{\pi'(s)}\psi_{s,t}(s,t)\frac{\alpha_u}{1-\beta}\left[1 + \frac{1}{\alpha_{V^e}}\right]. \tag{12}$$

Hence, $\kappa_t(s,t) \geq 0$ if and only if $\psi_{s,t}(s,t) \geq 0$. While the first-order derivative $\kappa_t(s,t)$ has a simple form due to the envelope condition, the second-order derivative $\kappa_{tt}(s,t)$ and the cross-derivative $\kappa_{s,t}(s,t)$ are a complicated function depending on the relative size of higher-order derivatives of the cost and utility function. However, it is clear that $\kappa_{tt}(s,t) = \kappa_{s,t}(s,t) = 0$ if $\psi_{s,t}(s,t) = 0$ for all (s,t).□

Proof of Corollary 1.

The corollary follows immediately from Proposition 1. In the minimization problem in (5) for a given (s,t), the objective function and the constraints IC and PC are independent of θ. This implies that the optimal promised utilities and thus the consumption levels are independent of θ, given (s,t). Moreover, when rewriting the minimization subject to PC and IC in terms of $\alpha_{V^e} = V^e/V$, $\alpha_{V^u} = V^u/V$ and $\alpha_{V^u} = u/V$, the objective function and the constraints become independent of V as well. Hence, the optimal choice of $u/V, V^e/V$ and V^u/V is independent of θ and V, conditional on s and t. This also implies that the minimized risk premium $\kappa(s,t)$ is independent of V. Hence, from the optimization problem in (4), it follows that the optimal choice of s and t are independent of V.□

Proof of Lemma 1.

Consider a search profile $\mathbf{s} = \{s_\tau\}_{\tau=0,1,\dots}$. Let $z = (z_1, z_2)$ be a state vector with $z_1 \in \{\mathcal{E}, \mathcal{U}\}$ and $z_2 \in \mathbb{N}_+$, where z_1 determines whether the agent is employed or unemployed and z_2 determines the search intensity for the unemployed agent by the mapping $\mathcal{S} : \mathbb{N}_+ \to \mathbb{R}$. The initial state equals $z = \{\mathcal{U}, 0\}$. The transition rules from z to z' are as follows. For the second variable of the state vector, we have $z_2' = z_2 + 1$. For the first variable, if $z_1 = \mathcal{E}$, then $z_1' = \mathcal{E}$. If $z_1 = \mathcal{U}$, then $z_1' = \mathcal{E}$ with probability $\pi(\mathcal{S}(z_1))$ and $z_1' = \mathcal{U}$ with probability $1 - \pi(\mathcal{S}(z_1))$.

The problem determining the optimal training intensity for a given search profile $\mathbf{s}$ can be re-written as a stochastic dynamic program with state vector $z \in \{\mathcal{E}, \mathcal{U}\} \times \mathbb{N}_+$. That is,

$$g(\theta, z) = \max_{\theta' \in G(\theta, z)} u(\theta, \theta', z) + \beta E g(\theta', z')$$

with

$$\begin{aligned} u(\theta, \theta', \mathcal{U}, z_2) &= -(1-\beta)\left[\psi\left(\mathcal{S}(z_2), \theta' - (1-\delta)\theta\right) + \kappa\left(\mathcal{S}(z_2), \theta' - (1-\delta)\theta\right)\right] \\ u(\theta, \theta', \mathcal{E}, z_2) &= (1-\beta) y(\theta), \end{aligned}$$

and with $G(\theta; \mathcal{U}, z_2) = [(1-\delta)\theta, \infty)$ and $G(\theta; \mathcal{E}, z_2) = \{\theta\}$, two convex sets. Denote $\psi(s,t) + \kappa(s,t)$ by $\tilde{\psi}(s,t)$. Since $\tilde{\psi}(s,t)$ is convex in t,

$$\begin{aligned} &\tilde{\psi}\left(\mathcal{S}(z_2), \alpha\left[\theta_1' - (1-\delta)\theta_1\right] + (1-\alpha)\left[\theta_2' - (1-\delta)\theta_2\right]\right) \\ &\qquad \leq \alpha\tilde{\psi}\left(\mathcal{S}(z_2), \theta_1' - (1-\delta)\theta_1\right) + (1-\alpha)\tilde{\psi}\left(\mathcal{S}(z_2), \theta_2' - (1-\delta)\theta_2\right) \end{aligned}$$

$$\begin{aligned} &\Leftrightarrow u\left(\alpha\theta_1 + (1-\alpha)\theta_2, \alpha\theta_1' + (1-\alpha)\theta_2', \mathcal{U}, z_2\right) \\ &\qquad \geq \alpha u\left(\theta_1, \theta_1', u\right) + (1-\alpha) u\left(\theta_2, \theta_2', \mathcal{U}, z_2\right) \end{aligned}$$

Hence, $u(\theta, \theta', \mathcal{U}, z_2)$ is a concave function given the state vector $(\mathcal{U}, z_2)$. The concavity of $u(\theta, \theta', \mathcal{E}, z_2)$ given the state vector $(\mathcal{E}, z_2)$ simply follows from the concavity of $y(\theta)$. Hence, by Proposition 16.4 in Acemoglu (2009), the function $g(\theta, z)$ is concave in θ for all z and for any choice of $\mathcal{S}$ defining $\mathbf{s}$ and $g(\theta|\mathbf{s}) = g(\theta, \mathcal{U}, 0)$.□

Proof of Lemma 2.

Consider a positive level of human capital θ_0 and denote the optimal search profile at θ_0 by $\mathbf{s}^* = \{s_\tau\}_{\tau=0,1,\dots}$,

$$\mathbf{s}^* = \arg\max_{\mathbf{s}} g(\theta_0|\mathbf{s}).$$

I analyze whether the policy with search profile $\mathbf{s}^*$ which is optimal for θ_0 can be dominated for some other θ by an alternative policy with search profile $\mathbf{s}^{0,*} = \{0, \mathbf{s}^*\}$ which introduces a training period $(t_0 > 0, s_0 = 0)$ before starting the search profile $\mathbf{s}^*$.

The subvalue function for this alternative policy equals

$$g\left(\theta|\mathbf{s}^{0,*}\right) = \max_{t} -\psi\left(0,t\right) + \beta g\left(\theta\left(1-\delta\right)+t|\mathbf{s}^{*}\right). \tag{13}$$

When positive, the optimal training intensity $t\left(\theta|\mathbf{s}^{0,*}\right)$, given the search profile $\mathbf{s}^{0,*}$, satisfies

$$\psi_t\left(0, t\left(\theta|\mathbf{s}^{0,*}\right)\right) = \beta g'\left(\theta\left(1-\delta\right)+t\left(\theta|\mathbf{s}^{0,*}\right)|\mathbf{s}^{*}\right).$$

By the concavity of $g\left(\theta|\mathbf{s}^{*}\right)$ and the convexity of $\psi\left(0,t\right)$ in t, the optimal training intensity $t\left(\theta|\mathbf{s}^{0,*}\right)$ is decreasing in θ. Denote by $\tilde{\theta}$ the cut-off level for which $t\left(\theta|\mathbf{s}^{0,*}\right) = \delta\theta$. This cut-off is positive, but finite by the Inada conditions on the output function, $lim_{\theta\to 0} y'\left(\theta\right) = \infty$ and $lim_{\theta\to\infty} y'\left(\theta\right) = 0$.

For any $\theta \geq \tilde{\theta}$, the level of human capital decreases during the training period, i.e., $\theta\left(1-\delta\right)+t\left(\theta|\mathbf{s}^{0,*}\right) \leq \theta$. However, in this case, it can never be optimal to start with a training period, since

$$\begin{aligned} g\left(\theta|\mathbf{s}^{*}\right) &\geq g\left(\theta\left(1-\delta\right)+t\left(\theta|\mathbf{s}^{0,*}\right)|\mathbf{s}^{*}\right) \\ &> -\psi\left(0,t\left(\theta|\mathbf{s}^{0,*}\right)\right) + \beta g\left(\theta\left(1-\delta\right)+t\left(\theta|\mathbf{s}^{0,*}\right)|\mathbf{s}^{*}\right) \\ &= g\left(\theta|\mathbf{s}^{0,*}\right). \end{aligned}$$

The weak inequality follows because $\theta\left(1-\delta\right)+t\left(\theta|\mathbf{s}^{0,*}\right) \leq \theta$ for $\theta \geq \tilde{\theta}$. The strict inequality follows trivially because we subtract the positive cost of training and discount the positive net value.

We thus only consider levels of human capital $\theta < \tilde{\theta}$. From the envelope condition of (13), we find

$$g'\left(\theta|\mathbf{s}^{0,*}\right) = \beta\left(1-\delta\right)g'\left(\theta\left(1-\delta\right)+t\left(\theta|\mathbf{s}^{0,*}\right)|\mathbf{s}^{*}\right) < g'\left(\theta\left(1-\delta\right)+t\left(\theta|\mathbf{s}^{0,*}\right)|\mathbf{s}^{*}\right).$$

The inequality follows by $\beta\left(1-\delta\right) < 1$. Moreover, since $\theta < \tilde{\theta}$, the level of human capital increases during the initial training period, i.e., $\theta\left(1-\delta\right)+t\left(\theta|\mathbf{s}^{0,*}\right) > \theta$. Hence, $g'\left(\theta\left(1-\delta\right)+t\left(\theta|\mathbf{s}^{0,*}\right)|\mathbf{s}^{*}\right) < g'\left(\theta|\mathbf{s}^{*}\right)$, which implies

$$g'\left(\theta|\mathbf{s}^{0,*}\right) < g'\left(\theta|\mathbf{s}^{*}\right).$$

That is, the marginal value of human capital is lower when a training period is added to the policy. This inequality helps signing the difference in value for the two policies when $\theta < \tilde{\theta}$. That is,

$$g\left(\theta|\mathbf{s}^{0,*}\right) - g\left(\theta|\mathbf{s}^{*}\right) = g\left(\theta_0|\mathbf{s}^{0,*}\right) - g\left(\theta_0|\mathbf{s}^{*}\right) + \int_{\theta_0}^{\theta}\left[g'\left(x|\mathbf{s}^{0,*}\right) - g'\left(x|\mathbf{s}^{*}\right)\right]dx.$$

Since $g\left(\theta_0|\mathbf{s}^{0,*}\right) \leq g\left(\theta_0|\mathbf{s}^{*}\right)$ by the optimality of s^{*} at θ_0, $g\left(\theta|\mathbf{s}^{0,*}\right) \leq g\left(\theta|\mathbf{s}^{*}\right)$ for

$\theta \in \left[\theta_0, \tilde{\theta}\right]$. Hence, if $g\left(\theta|\mathbf{s}^{0,*}\right) > g\left(\theta|\mathbf{s}^*\right)$, then $\theta < \theta_0$, which proves the first part of the Lemma.

For the second part of the Lemma, consider now some different starting level of human capital, again denoted by θ_0, for which the policy with search profile $\mathbf{s}^{0,*} = (0, \mathbf{s}^*)$ is optimal. From the argument above, we know that since the policy starts in the training state, the level of human capital will be higher in the next period, i.e., $\theta_0(1-\delta) + t\left(\theta_0|\mathbf{s}^{0,*}\right) > \theta_0$. Moreover, as long as the training state is optimal, the level of human capital increases. That is, if $\mathbf{s}^{0,*}$ specifies zero search effort for τ periods, the level of human capital increases up to period τ. I now analyze what happens to human capital when the optimal unemployment policy leaves the training state. Consider the case where $\mathbf{s}^{0,*}$ specifies zero search effort at $\tau = 0$ and some positive search effort ε at $\tau = 1$ (i.e., the profile $\mathbf{s}^*$ specifies search effort ε at $\tau = 0$). Now I argue that the optimal level of training at $\tau = 1$ is higher than the optimal training level at $\tau = 0$ if $\psi_{s,t}$ is bounded from above by some level $\bar{\psi}_{s,t}$. This is sufficient for the level of human capital to further increase during this first period in the search-and-training state.

The optimal training levels $t\left(\theta|\mathbf{s}^{0,*}\right)$ and $t\left(\theta|\mathbf{s}^*\right)$ solve respectively

$$(1-\beta)\left[\psi_t(0,t) + \kappa_t(0,t)\right] = \beta g'(\theta(1-\delta) + t|\mathbf{s}^*), \tag{14}$$

$$(1-\beta)\left[\psi_t(\varepsilon,t) + \kappa_t(\varepsilon,t)\right] = \beta\left[\pi(\varepsilon)\, y'(\theta(1-\delta)+t) + (1-\pi(\varepsilon))\, g'(\theta(1-\delta)+t|\mathbf{s}_1^*)\right] \tag{15}$$

I introduce the notation $\hat{\theta}\left(\theta|\mathbf{s}^{0,*}\right) = \theta(1-\delta) + t\left(\theta|\mathbf{s}^{0,*}\right)$ for the next period's level of human capital given this period's optimal training level for a search profile $\mathbf{s}^{0,*}$. Notice that the search profile $\mathbf{s}^*$ is optimal at $\hat{\theta}\left(\theta_0|\mathbf{s}^{0,*}\right)$ by the optimality of $\mathbf{s}^{0,*}$ at θ_0. Now using the respective envelope conditions,

$$g'\left(\theta_0|\mathbf{s}^{0,*}\right) = \beta(1-\delta)\, g'(\hat{\theta}\left(\theta_0|\mathbf{s}^{0,*}\right)|\mathbf{s}^*),$$

$$g'\left(\hat{\theta}\left(\theta_0|\mathbf{s}^{0,*}\right)|\mathbf{s}^*\right) = \beta(1-\delta)\left[\begin{array}{c} \pi(\varepsilon)\, y'(\hat{\theta}\left(\theta_0|\mathbf{s}^{0,*}\right)(1-\delta) + t\left(\hat{\theta}\left(\theta_0|\mathbf{s}^{0,*}\right)|\mathbf{s}^*\right)) \\ +(1-\pi(\varepsilon))\, g'(\hat{\theta}\left(\theta_0|\mathbf{s}^{0,*}\right)(1-\delta) + t\left(\hat{\theta}\left(\theta_0|\mathbf{s}^{0,*}\right)|\mathbf{s}^*\right)|\mathbf{s}_1^*) \end{array}\right],$$

we can rewrite the conditions (14) and (15) as

$$g'\left(\theta_0|\mathbf{s}^{0,*}\right) = (1-\delta)(1-\beta)\left[\psi_t(0, t\left(\theta_0|\mathbf{s}^{0,*}\right)) + \kappa_t\left(0, t\left(\theta_0|\mathbf{s}^{0,*}\right)\right)\right],$$

$$g'\left(\hat{\theta}\left(\theta_0|\mathbf{s}^{0,*}\right)|\mathbf{s}^*\right) = (1-\delta)(1-\beta)\left[\psi_t(\varepsilon, t\left(\hat{\theta}\left(\theta_0|\mathbf{s}^{0,*}\right)|\mathbf{s}^*\right)) + \kappa_t\left(\varepsilon, t\left(\hat{\theta}\left(\theta_0|\mathbf{s}^{0,*}\right)|\mathbf{s}^*\right)\right)\right].$$

Importantly, the value of additional human capital is higher after the training state. That is, $g'\left(\hat{\theta}\left(\theta_0|\mathbf{s}^{0,*}\right)|\mathbf{s}^*\right) > \beta(1-\delta)\, g'\left(\hat{\theta}\left(\theta_0|\mathbf{s}^{0,*}\right)|\mathbf{s}^*\right) = g'\left(\theta_0|\mathbf{s}^{0,*}\right)$, using again $\beta(1-\delta) < 1$ and the envelope condition. Hence, there exists an upper bound $\bar{\psi}_{s,t} > 0$ such that $\psi_{s,t}(s,t) \in \left[0, \bar{\psi}_{s,t}\right]$ is sufficient (but not necessary) for

$$t\left(\theta_0|\mathbf{s}^{0,*}\right) \leq t\left(\hat{\theta}\left(\theta_0|\mathbf{s}^{0,*}\right)|\mathbf{s}^*\right).$$

That is, the upper bound on $\psi_{s,t}(s,t)$ is sufficient for $\psi(s,t)+\kappa(s,t)$ to be convex in t, but also for $[\psi_t(\varepsilon,t)+\kappa_t(\varepsilon,t)]-[\psi_t(0,t)+\kappa_t(0,t)]$ to be small. Since both $\hat{\theta}\left(\theta_0|\mathbf{s}^{0,*}\right) > \theta_0$ and $t\left(\hat{\theta}\left(\theta_0|\mathbf{s}^{0,*}\right)|\mathbf{s}^*\right) \geq t\left(\theta_0|\mathbf{s}^{0,*}\right)$, we find

$$\hat{\theta}\left(\theta_0|\mathbf{s}^{0,*}\right)(1-\delta)+t\left(\hat{\theta}\left(\theta_0|\mathbf{s}^{0,*}\right)|\mathbf{s}^*\right) > \theta_0(1-\delta)+t\left(\theta_0|\mathbf{s}^{0,*}\right) > \hat{\theta}\left(\theta_0|\mathbf{s}^{0,*}\right).$$

Hence, human capital further increases when a level is reached for which the training state is no longer optimal. This proves the second part of the Lemma.□

Proof of Lemma 3.

Consider a positive level of human capital θ_0 for which the optimal policy starts with a training period. Like for Lemma 2, I denote the corresponding search profile by $\mathbf{s}^{0,*} = \{0, \mathbf{s}^*\}$ with subvalue function $g\left(\theta|\mathbf{s}^{0,*}\right)$. The optimality of the training state at θ_0 (i.e., $s_0 = 0$) implies that an increase in search in the initial period s_0 cannot increase the net value $g\left(\theta|\mathbf{s}^{0,*}\right)$. I denote this change by $\Delta(\theta)$. Hence,

$$\begin{aligned}\Delta(\theta_0) = \beta\pi'(0)\left[y(\hat{\theta}\left(\theta_0|\mathbf{s}^{0,*}\right)) - g(\hat{\theta}\left(\theta_0|\mathbf{s}^{0,*}\right)|\mathbf{s}^*)\right] \\ -(1-\beta)\left[\psi_s(0,t\left(\theta_0|\mathbf{s}^{0,*}\right)) + \kappa_s\left(0,t\left(\theta_0|\mathbf{s}^{0,*}\right)\right)\right] \leq 0\end{aligned}$$

This corresponds to condition (7) in the main text, using the notation $\hat{\theta}\left(\theta|\mathbf{s}^{0,*}\right) \equiv \theta(1-\delta)+t\left(\theta|\mathbf{s}^{0,*}\right)$ as in the proof of Lemma 2. Notice that in response to a change in s_0 it may be optimal to change the entire search and training profile. However, since these profiles have been chosen optimally, the effect on the net value of these changes will be only of second order by the envelope theorem.

I now show that the gain from searching in the initial period decreases when decreasing θ below θ_0, i.e., $\Delta'(\theta) > 0$. This implies that introducing search in the initial period remains dominated for human capital $\theta < \theta_0$. Notice that a change in θ changes the optimal search profile $\mathbf{s}^*$ from the next period on when search is continuous, but the impact of a change in $\mathbf{s}^*$ on the subvalue function $g(\theta|\mathbf{s}^*)$ is unknown. Hence, to make this argument, I consider a setting where the set of potential search levels is discrete. That is, $s_\tau \in \{0, \varepsilon, 2\varepsilon, ...\}$ with ε small. This allows me to consider a cut-off $\tilde{\theta} < \theta_0$ for which a different search profile $\widetilde{\mathbf{s}}^* \neq \mathbf{s}^{0,*}$ becomes optimal, while the search profile $\mathbf{s}^{0,*}$ is optimal for $\theta \in \left[\tilde{\theta}, \theta_0\right]$. I show below that the gain of an increase in search s_0 keeping $\mathbf{s}^*$ fixed (i.e., the introduction of positive search in the initial period, keeping the search profile from the next period on fixed) strictly decreases when lowering human capital for $\theta \in \left[\tilde{\theta}, \theta_0\right]$. By consequence, it cannot be that the new optimal search profile $\widetilde{\mathbf{s}}^*$ at $\tilde{\theta}$ involves positive search in the initial period, i.e., $s_0 = 0$ for the new optimal profile. The above argument can then be repeated starting from the profile $\widetilde{\mathbf{s}}^*$ which is optimal at $\tilde{\theta}$ and so on. The argument continues to hold for $\varepsilon \to 0$.

The change in the gain from inducing search in the initial period when changing human capital for $\theta \in \left[\tilde{\theta}, \theta_0\right]$ equals

$$\begin{aligned} \Delta'(\theta) &= \beta\pi'(0)\left[y'(\hat{\theta}(\theta|\mathbf{s}^{0,*})) - g'(\hat{\theta}(\theta|\mathbf{s}^{0,*})|\mathbf{s}^*)\right]\left[(1-\delta) + t'(\theta|\mathbf{s}^{0,*})\right] \qquad (16) \\ & - (1-\beta)\left[\psi_{s,t}(0, t(\theta|\mathbf{s}^{0,*})) + \kappa_{s,t}\left(0, t(\theta|\mathbf{s}^{0,*})\right)\right] t'\left(\theta|\mathbf{s}^{0,*}\right). \end{aligned}$$

This accounts for the change in this period's optimal training level $t'\left(\theta|\mathbf{s}^{0,*}\right)$ and the next period's level of human capital $\hat{\theta}'\left(\theta|\mathbf{s}^{0,*}\right)$ when changing θ, which may have a first-order effect on the impact of an increase in search s_0. By construction, the search profile $\mathbf{s}^*$ is still optimal from the next period on when changing $\theta \in \left[\tilde{\theta}, \theta_0\right]$, so the subvalue function $g(\theta|\mathbf{s}^*)$ remains the relevant one. Notice that also the future training levels may change, but the impact of this is captured by $g'(\hat{\theta}\left(\theta_0|\mathbf{s}^{0,*}\right)|\mathbf{s}^*)$.

I now determine the sign of $\Delta'(\theta)$ considering the different terms in (16). First, the optimal training intensity in the initial period decreases in the initial level of human capital. By implicit differentiation,

$$t'\left(\theta|\mathbf{s}^{0,*}\right) = -(1-\delta)\frac{\beta g''\left(\hat{\theta}\left(\theta|\mathbf{s}^{0,*}\right)|\mathbf{s}^*\right)}{\beta g''\left(\hat{\theta}\left(\theta|\mathbf{s}^{0,*}\right)|\mathbf{s}^*\right) - (1-\beta)\psi_{tt}\left(0, t\left(\theta|\mathbf{s}^{0,*}\right)\right)}.$$

By the concavity of $g(\theta|s)$ and the convexity in t of $\psi(s,t)$, $-(1-\delta) < t'\left(\theta|\mathbf{s}^{0,*}\right) < 0$. Hence, $(1-\delta) + t'\left(\theta|\mathbf{s}^{0,*}\right) > 0$. Second, if $\psi_{s,t}(s,t) + \kappa_{s,t}(s,t) \geq 0$, then it is sufficient that the difference $y'(\hat{\theta}\left(\theta|\mathbf{s}^{0,*}\right)) - g'(\hat{\theta}\left(\theta|\mathbf{s}^{0,*}\right)|\mathbf{s}^*) > 0$ for $\Delta'(\theta) > 0$. Since the curvature of $\kappa(s,t)$ with respect to t is of second order when $\psi_{s,t}(s,t)$ is small, there exists again an upper bound $\bar{\psi}_{s,t} > 0$ such that $\psi_{s,t}(s,t) \in \left[0, \bar{\psi}_{s,t}\right]$ is sufficient (but not necessary) for $\psi_{s,t}(s,t) + \kappa_{s,t}(s,t) \geq 0$.

I therefore complete this proof by showing that $y'(\hat{\theta}\left(\theta|\mathbf{s}^{0,*}\right)) - g'(\hat{\theta}\left(\theta|\mathbf{s}^{0,*}\right)|\mathbf{s}^*) > 0$. The value of additional human capital for an unemployed job seeker only depends on the expected increase in output when employed. Applying the envelope condition, we find

$$g'(\hat{\theta}\left(\theta|\mathbf{s}^{0,*}\right)|\mathbf{s}^*) = \Sigma_{\tau=0,1,\ldots}\beta^\tau (1-\delta)^\tau \pi_\tau y'(\hat{\theta}_\tau)$$

where π^e_τ and $\hat{\theta}_\tau$ denote respectively the probability of employment and the human capital level at time τ given the optimal policy at $\hat{\theta}\left(\theta|\mathbf{s}^{0,*}\right)$. The impact of the change in the policy variables in response to the change in θ is again of second order by the envelope theorem. Since the optimal policy starts in the training state for $\theta \in \left[\tilde{\theta}, \theta_0\right]$, we know that the future levels of human capital $\hat{\theta}_\tau$ are bounded from below by $\hat{\theta}\left(\theta|\mathbf{s}^{0,*}\right) = \theta(1-\delta) + t\left(\theta|\mathbf{s}^{0,*}\right)$ by Lemma 2. Hence, it immediately follows that

$$g'(\hat{\theta}\left(\theta|\mathbf{s}^{0,*}\right)|\mathbf{s}^*) < \Sigma_{\tau=0,1,\ldots}\beta^\tau (1-\delta)^\tau \pi_\tau y'(\hat{\theta}\left(\theta|\mathbf{s}^{0,*}\right)) < y'(\hat{\theta}\left(\theta|\mathbf{s}^{0,*}\right)).$$

The first inequality follows from the concavity of $y(\theta)$. The second inequality follows since $\beta(1-\delta) < 1$. This proves the Lemma.□

Proof of Proposition 2.

The Proposition simply puts Lemma 2 and 3 together. In the presence of an effective training technology, the two Lemma's imply the existence of a cut-off $\bar{\theta}$ such that the training state is optimal for any $\theta \in [0, \bar{\theta}]$, while the search-and-training state is optimal for any $\theta \in [\bar{\theta}, \infty)$. The level of human capital strictly increases during the training state until it exceeds $\bar{\theta}$. The human capital level remains above $\bar{\theta}$ by Lemma 2. Hence, the search-and-training state is absorbent. Notice that although $y(0) - g(0) < 0$ by the effective training technology, the return to search for $\theta = 0$ depends on $y(t(0|\mathbf{s}^*)) - g(t(0|\mathbf{s}^*)$. Hence, the cut-off level $\bar{\theta}$ will only be strictly positive (and thus the training state will be optimal for $\theta \in [0, \bar{\theta}]$) if

$$\beta\pi'(0)\{y(t(0|\mathbf{s}^*)) - g(t(0|\mathbf{s}^*))\} - (1-\beta)[\psi_s(0, t(0|\mathbf{s}^*)) + \kappa_s(0, t(0|\mathbf{s}^*))] < 0.$$

□

Proof of Proposition 3.

Denote the stationary search profile in the search-and-training state by $\bar{\mathbf{s}} = \{\bar{s}, \bar{s}, ...\}$ and the net change in human capital by $n(\theta) = t(\theta|\bar{\mathbf{s}}) - \delta\theta$. Notice that $n(\theta) > 0$ for θ sufficiently low, while $n(\theta) < 0$ for θ sufficiently high by the Inada conditions $lim_{\theta\to\infty} y'(\theta) = 0$ and $lim_{\theta\to 0} y'(\theta) = \infty$. By continuity, there exists a positive θ^* such that $t(\theta^*|\bar{\mathbf{s}}) = \delta\theta^*$. By implicit differentiation of the first-order condition with respect to training (6), we find that the optimal training intensity is decreasing in the level of human capital,

$$t'(\theta|\bar{\mathbf{s}}) = -(1-\delta)\frac{\beta[\pi(\bar{s})y''(\theta')) + (1-\pi(\bar{s}))g''(\theta'|\bar{\mathbf{s}})]}{\beta[\pi(\bar{s})y''(\theta')) + (1-\pi(\bar{s}))g''(\theta'|\bar{\mathbf{s}})] - (1-\beta)[\psi_{tt}(\bar{s},t) + \kappa_{tt}(\bar{s},t)]} \leq 0.$$

The numerator of the fraction is negative, since $y''(\theta') < 0$ and $g''(\theta'|\bar{\mathbf{s}}) < 0$. The denominator is also negative, but smaller than the numerator, since $\psi_{tt}(\bar{s}, t) + \kappa_{tt}(\bar{s}, t) > 0$. Hence, $-(1-\delta) \leq t'(\theta|\bar{\mathbf{s}}) \leq 0$. As a consequence, $-1 < n'(\theta) < 0$ for all θ. This implies that human capital converges to a unique stationary level θ^* and the convergence is monotone. Re-expressing the value function in this stationary state and using the envelope condition, we find

$$\begin{aligned} g(\theta^*) &= \frac{\beta\pi(\bar{s})y(\theta^*) - (1-\beta)[\psi(\bar{s}, \delta\theta^*) + \kappa(\bar{s}, \delta\theta^*)]}{1 - \beta(1-\pi(\bar{s}))} \\ g'(\theta^*) &= \frac{\beta\pi(\bar{s})y'(\theta^*)(1-\delta)}{1-\beta(1-\pi(\bar{s}))(1-\delta)}. \end{aligned}$$

Substituting this into the first-order condition for training (6), we find

$$\beta\pi(\bar{s})\frac{y'(\theta^*)/(1-\beta)}{1-\beta(1-\pi(\bar{s}))(1-\delta)} = \psi_t(\bar{s}, t^*) + \kappa_t(\bar{s}, t^*).$$

Notice that the stationary level θ^* is positive, since $lim_{\theta\to 0}y'(\theta) = \infty$ and thus

$$\beta\pi(\bar{s})\frac{y'(0)/(1-\beta)}{1-\beta(1-\pi(\bar{s}))(1-\delta)} > \psi_t(\bar{s},0)+\kappa_t(\bar{s},0).$$

□

Proof of Corollary 2.
For $\theta_0 < \bar{\theta}$, it is optimal to start in the training state by Proposition 2 and the level of training is increasing by Lemma 2. Using the envelope condition and the first-order condition for training in period τ during the training state,

$$\begin{aligned} g'(\theta_\tau|\mathbf{s}^*) &= \beta(1-\delta)g'(\theta_\tau(1-\delta)+t_\tau|\mathbf{s}_1^*) \\ \psi_t(0,t_\tau) &= \beta g'(\theta_\tau(1-\delta)+t_\tau|\mathbf{s}_1^*), \end{aligned}$$

with $\mathbf{s}^* = \{s_0, \mathbf{s}_1^*\}$, we find the Euler equation describing the change in training,

$$\psi_t(0,t_\tau) = \beta(1-\delta)\psi_t(0,t_{\tau+1}).$$

With $\beta(1-\delta) < 1$, the convexity of t indeed implies that t is increasing during the training state.

For $\theta_0 \geq \bar{\theta}$, it is optimal to start in the search-and-training state by Proposition 2. Since θ converges monotonically to θ^* by Proposition 3 and $t'(\theta|\bar{\mathbf{s}}) \leq 0$ during the search-and-training state, training increases during the unemployment spell when $\bar{\theta} \leq \theta_0 \leq \theta^*$, while it decreases when $\theta_0 \geq \theta^*$.

The total training effort an unemployed agents expects to be exposed to equals $Et = t_0 + \sum_{\tau=1}^{\infty}(1-\pi_\tau^e)t_\tau$, where π_τ^e and t_τ denote respectively the probability of employment and the training level at time τ given the optimal policy. Since no one exits the training state before θ exceeds $\bar{\theta}$, the total training effort exerted before reaching $\bar{\theta}$ is clearly lower for agents who started with a higher level of human capital. Moreover, since the convergence of human capital is monotone and $t'(\theta|\bar{\mathbf{s}}) \leq 0$ during the search-and-training state, while the probability to leave unemployment $\pi(\bar{s})$ is constant, the total expected training effort is decreasing in θ during this state as well.
□

Proof of Proposition 4.
Using FOC_{V^u} at time τ and EC_V at time $\tau+1$, we find

$$\Delta_{\lambda,\tau} \equiv \lambda_\tau - \lambda_{\tau+1} = \mu_\tau\frac{\pi'(s_\tau)}{(1-\pi(s_\tau))}.$$

If the training state is optimal in period τ (i.e., $s_\tau = 0$), $\mu_\tau = 0$ and thus $\lambda_\tau = \lambda_{\tau+1}$. If the search-and-training state is optimal in period τ (i.e., $s_\tau > 0$), $\mu_\tau > 0$ by the Kuhn-Tucker conditions. Since $\pi'(s_\tau) > 0$, this implies $\Delta_{\lambda,\tau} > 0$ in the search-and-training state. The same argument applies for binary search with $\Delta_{\lambda,\tau} = \mu_\tau \frac{\pi(\bar{s}_\tau)}{(1-\pi(\bar{s}_\tau))}$.□

Proof of Corollary 3.

From FOC_c, $\frac{1}{u'(c-\psi)} = \lambda$ when $\mu = 0$. Since λ remains constant during the training state by Proposition 4, net-consumption thus remains constant as well. From EC_V, $C_V(V,\theta) = \lambda$. From Proposition 1, $C_V(V,\theta) = \frac{-1}{\sigma(1-\beta)V}$. Since $s_\tau > 0$ in a stationary state with $\theta^* > 0$ and $t^* = \delta\theta^*$, λ and thus V decreases during the stationary state. For CARA preferences, we have that

$$\begin{aligned} c^e(V,\theta) &= (u^e)^{-1}((1-\beta)\alpha_{V^e}(\theta)V) \\ c(V,\theta) - \psi(s(\theta), t(\theta)) &= u^{-1}(\alpha_u(\theta)V). \end{aligned}$$

Notice that the policy functions $\alpha_{V^e}(\theta)$ and $\alpha_u(\theta)$ are positive and only depend on θ by Proposition 1. The inverse functions $(u^e)^{-1}$ and u^{-1} are strictly increasing in their respective arguments. Since θ is constant and V does decrease during the stationary state, the net-consumption levels during unemployment and upon re-employment are decreasing with the length of the unemployment spell in the stationary state.□