HOMOGENEOUS UTILITY FUNCTIONS AND EQUALITY IN "THE OPTIMUM TOWN". A NOTE[1]

Nicholas Stern

St. Catherine's College, Oxford, England

In his recent paper on "The Optimum Town" [2], Mirrlees showed that it would be optimum to have equality of utility in his model only when the utility function of the individual (where all individuals are identical) took a special form. Mirrlees gave three examples of classes of utility functions that would give equality at the optimum. The most important of these classes consisted of utility functions homogeneous in the consumption good (c) and land occupied (a). It is rather common to use homogeneous utility functions in both empirical estimation and numerical examples—see for example, Solow [5] and Dixit [1] for their use in discussions of equilibrium and optimality (respectively) in towns. It might be thought, therefore, that this case of equality was of some importance. The purpose of this note is to point out that most simple elaborations of, or similar models to, the model Mirrlees used in the initial part of the paper for his discussion of equality would not give the homogeneity-implies-equality (HIE) result. The consequences for the HIE result of two of the modifications considered in this note are implicit in Mirrlees' results but our understanding of the distribution of utility in these models is helped by making these consequences explicit. The reasons we shall usually want utility to be unequal are discussed by Mirrlees [2] and Riley [4].

The two other utility functions that give equality which were noted by Mirrlees are of less interest—for one of these functions, $u = e^{kc}w(a, r)$, the equality result is robust to simple generalisations, for the other, $u = v(c, az(r))$, it is not (r is the distance from the centre of the town and k is a constant).

Mirrlees' problem in the initial part of the paper was the maximisation of $\int u(c, a, r) r f(r) dr$ where $f(r)$ is the population density (so that where all land is occupied $af = 1$). The constraints were a fixed population $2\pi N$ and a fixed amount of the consumption good $2\pi Y$, i.e.,[2]

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[2] Mirrlees' [2] numbers for equations (1)–(4) are (5), (6), (8), (9).

$$\int rf(r)\,dr = N \tag{1}$$

$$\int crf(r)\,dr = Y \tag{2}$$

Using Lagrange multipliers μ, λ for these constraints we have the first order conditions w.r.t. to c and f

$$u_c = \lambda \tag{3}$$

and

$$u - cu_c - au_a = \mu \tag{4}$$

Mirrlees expresses the first order condition (4), with respect to f, neatly as follows: "The effect of adding an extra man to the town should be the same—in terms of welfare—regardless of the place in which space is found for him. Suppose we put him at distance r. Then we give him consumption $c(r)$ at someone's expense and we reduce the area given to people at distance r by $a(r)$ in aggregate." The HIE result is immediate from (4) since u homogenous degree α in c and a implies $cu_c + au_a$ is equal to αu thus

$$(1-\alpha)u = \mu \tag{5}$$

and u is independent of r provided $\alpha \neq 1$.[1]

There will be an analogue of condition (4) for most optimum town models and it is the modifications to this condition for most generalisations of the simple Mirrlees model that cause the HIE result to disappear. The effect of adding an extra man to the town will always include $-(cu_c + au_a)$, but if there is any other cost to the remaining population of adding a man at r *that depends on* r (call this cost $\varphi(r)$) then the HIE result disappears. Equation (5) becomes

$$(1-\alpha)u - \varphi(r) = \mu$$

Then u is a monotonic increasing (decreasing) function of $\varphi(r)$ for $\alpha < 1 (\alpha > 1)$. It will often be easy to see how φ, and thus utility, depends on r. $\alpha = 1$ implies φ is constant. $\alpha > 1$ is excluded if u is concave in (c, a), as assumed by Mirrlees (see preceding footnote).

We consider three modifications to the Mirrlees model—involving relaxation

[1] The case $\alpha = 1$ is rather peculiar, since it implies $\mu = 0$. Thus, wherever we put the extra man we have zero net addition to the utility integral. This suggests the solution for this case may not be unique. Equal utility may well be one of the solutions of course.

Note that if we write (1) as the pair of constraints (1a) $\int rfdr \leq N$ and (1b) $\int rfdr \geq N$ then (1a) "bites" for $\alpha < 1$ and (1b) "bites" for $\alpha > 1$. In other words maximizing subject to (1a) alone with $\alpha > 1$ would lead us to solutions where (1a) did not bind. This is to be expected since $\alpha > 1$ corresponds to "increasing returns" in the utility function (as a function of (c, a)) and should make us watchful for solutions with concentrations of utility.

of each of the main assumptions—to see that in each case a φ function enters. First we modify the utility function for environmental externalities; second, we suppose that land is needed for transportation (or that there are congestion costs), and third we relax the assumption that work hours are fixed. The first two are considered by Mirrlees although their effect on the HIE result is not.

1. *Environmental externalities.* The effect of adding an extra man now includes the diseconomies he imposes on others through increased population density. Thus, $\varphi = -fu_f$ (note $u_f < 0$ usually) which will in general depend on r. See Mirrlees [2] equation (48).

2. *Land required for roads/congestion costs.* The distance r at which we place the extra man will affect road requirements/congestion costs at all points nearer to the centre than r. Thus φ will be an increasing function of r. In the Mirrlees case where road width k is required per traveller at any point then $\varphi(r) = \int_0^r k u_a dr$ since u_a is the marginal utility of the displaced land. With more general congestion cost models (see e.g., Dixit [1]) φ will be the integral of the extra congestion cost between 0 and r.

3. *Variable work hours.* Suppose that the utility function depends only on consumption and land (so that if homogeneous it satisfies two of Mirrlees' sufficiency conditions for equality at the optimum) but that leisure hours are fixed so that the further a man lives out the less work he does. Again we shall have φ increasing with r since the further out we place the extra man the more output is lost. (Dixit [1] incorporates this assumption together with this utility function in his model, and Riley [4] also considers variable work hours.)

Mirrlees shows that $u = v(c, az(r))$ will also give equal utility since (3) and (4) become two equations in c and az which are independent of r and thus c and az are equal for everyone. This argument disappears if a φ function enters equation (4). This case is of little interest when combined with the fixed work hours assumption since it removes all aspects of required travel to the centre. The problem reduces to sharing out the goods consumption and land (thus we shall want $z(r)$ to fall to zero at some point to achieve a finite town, if land area is not constrained *a priori*).

The equality result with the utility function $u = e^{kc}w(a, r)$ is robust to changes in the assumptions of the model since equality is a consequence of equation (3) only ($\lambda = u_c = ku$).[1]

We should note that the Rawlsian social welfare function[2] (where we maximize the welfare of the worst off) will usually give equality. We can write the Rawlsian problem

$$\text{maximize} \int \bar{u} r f \, dr$$

$$\text{subject to } u \geq \bar{u} \tag{6}$$

[1] To keep utility concave in (c, a) it is preferable to use $u = -e^{-kc}w(a, r)$.
[2] See Rawls [3].

and any other constraints, for example $\int crf\,dr = Y$ (2)

Taking a Lagrange multiplier $\nu(r)$ for constraint (6) and maximizing w.r.t. c (which we suppose for the moment does not appear in constraints other than (2) and (6)) we have $\nu u_c - \lambda rf = 0$. Thus ν will in general be positive and $u = \bar{u}$. It is clear this result is robust to modifications of the model since $\nu > 0$ will usually result from considering variation of any variable entering the utility function. The Rawlsian problem has been considered and the equality resulted noted by Dixit [1] and Riley [4].

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