Non-planar simplices are not reduced*

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Abstract

A convex body in $\mathbb{R}^n$ which does not properly contain a convex body of the same minimum width is called a reduced body. It is not known whether there exist reduced n-dimensional polytopes for $n \geq 3$. We prove that no n-dimensional simplex is reduced if $n \geq 3$.

Keywords: body of constant width, convex polytope, difference body, projection body, reduced body, simplex.

1 Introduction

Due to E. Heil [4], a convex body $K \subset \mathbb{R}^n$ is called *reduced* if there is no convex body L properly contained in K such that the *minimum width* $\triangle(L)$ (= minimal distance between two different parallel supporting hyperplanes)

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of L is equal to $\triangle(K)$. Reduced bodies are interesting in view of several extremal problems, for example regarding the long-standing question: Which convex body of given minimum width has minimal volume? The extremal body has obviously to be reduced. Every body of constant width in $\mathbb{R}^n$ is reduced, but there are many further examples. For instance, all regular m-gons in $\mathbb{R}^2$ with m odd are reduced, as well as the intersection of the unit ball of $\mathbb{R}^n$ with an orthant of the respective Cartesian coordinate system (for $n = 2$ yielding a quarter of the unit disk). Many geometric properties of reduced bodies were found by M. Lassak [5]. In his paper also the following problem was posed: *Do there exist reduced n-dimensional polytopes for $n \geq 3$?*

Although this question was repeated in [6], the answer is still unknown. Using special geometric properties of tetrahedra (that no longer hold for n-simplices if $n \geq 4$), the authors of [9] proved that there is no reduced 3-simplex. It is our goal to extend this observation to higher dimensions.

2 The result and its proof

For an n-dimensional simplex $S \subset \mathbb{R}^n$, $n \geq 2$, we will use the following notions and abbrevations. The *vertex set* of S is given by $\{x_1, \ldots, x_{n+1}\}$ and, for any $i \in \{1, \ldots, n+1\}$, F_i denotes the unique $(n-1)$-face of S which is opposite to the vertex x_i. We also use some functions defined on the unit sphere S^{n-1} of $\mathbb{R}^n$. Most of their properties considered here hold for arbitrary convex bodies (see [3]), but we introduce them only for simplices. For an arbitrary unit vector $u \in S^{n-1}$ the *width* $w(S, u)$ of S in direction u is the distance of the two different parallel supporting hyperplanes of S which are orthogonal to u. The minimum of the function $w(S, u)$, $u \in S^{n-1}$, is called the *minimum width* or *thickness* of S, and is denoted by $\triangle(S)$. There exists a chord of S parallel to the direction of that minimum and having length $\triangle(S)$ (see [2, §§ 33]). Such a chord is said to be a *thickness chord* of S. Thus, if a segment $[a, b] \subset S$ is a thickness chord of S, then there are different supporting hyperplanes H_1, H_2 of S which are both orthogonal to $[a, b]$ and satisfy $a \in H_1, b \in H_2$. In other words, denoting by $V_1(S, u), u \in S^{n-1}$, the function describing the maximal chord length of S for any direction u, we have

$$\min_{u \in S^{n-1}} V_1(S, u) = \triangle(S). \tag{1}$$

The *brightness function* $V_{n-1}(S, u)$, $u \in S^{n-1}$, of an n-simplex S is the $(n-1)$-volume of the orthogonal projection of S onto the $(n-1)$-subspace orthogonal to u.

In [8] it was shown that for the *volume* $V_n(S)$ of an arbitrary n-simplex

S and *any* direction $u \in S^{n-1}$ the relation

$$V_n(S) = \frac{1}{n} \cdot V_{n-1}(S,u) \cdot V_1(S,u) \tag{2}$$

holds. With (1) this implies in particular

$$V_n(S) = \frac{1}{n} \cdot \max_{u \in S^{n-1}} V_{n-1}(S,u) \cdot \triangle(S), \tag{3}$$

i.e., the maximum brightness and the minimum width of S occur in the same direction.

Now we are ready to prove our

Theorem. *No n-dimensional simplex $S \subset \mathbb{R}^n$, $n \geq 3$, is reduced.*

Proof. We will prove that statement by contradiction. Assuming that S is reduced, it follows firstly that S has to be *equiareal*, i.e., that each $(n-1)$-face F_i must have the same $(n-1)$-volume $V_{n-1}(F_i)$, $i = 1, \ldots, n+1$. Indeed, in the classical formula

$$V_n(S) = \frac{1}{n} \cdot V_{n-1}(F_i) \cdot h_i, \qquad i \in \{1, \ldots, n+1\}, \tag{4}$$

where h_i denotes the length of the i-th altitude of S orthogonal to the affine hull of F_i, h_i is equal to $w(S, u_i)$ with u_i as (outer) normal direction of F_i. If we had $h_i \neq \triangle(S)$ for some $i \in \{1, \ldots, n+1\}$, the corresponding vertex x_i would not belong to a thickness chord of S and could be cut off to get from S a convex body L properly contained in S and satisfying $\triangle(L) = \triangle(S)$, a contradiction to the assumed reducedness of S. Thus we must have $h_i = \triangle(S)$ for all $i \in \{1, \ldots, n+1\}$, implying by (4) that S is equiareal.

Moreover, combining (4) and (3), we obtain

$$V_{n-1}(F_i) = \max_{u \in S^{n-1}} V_{n-1}(S,u), \qquad i = 1, \ldots, n+1. \tag{5}$$

From [3, § 4.1] we read off that the brightness function of S has the representation

$$V_{n-1}(S,u) = \frac{1}{2} \sum_{i=1}^{n+1} |\langle v_i, u \rangle|, \qquad u \in S^{n-1}, \tag{6}$$

where $v_i := V_{n-1}(F_i) \cdot u_i$. Due to $\sum_{i=1}^{n+1} v_i = o$ (Minkowski's existence theorem, cf. [3, Appendix A]) this can also be written in the form

$$V_{n-1}(S,u) = \sum_{i \in I(u)} \langle v_i, u \rangle, \qquad u \in S^{n-1}, \tag{7}$$

where $I(u) := \{j \in \{1, \ldots, n+1\} : \langle v_j, u\rangle \geq 0\}$. From (7) it follows that

$$\max_{u \in S^{n-1}} V_{n-1}(S, u) = ||\sum_{i \in I^*} v_i||, \tag{8}$$

where the nonempty index set $I^* \subseteq \{1, \ldots, n+1\}$ is determined by

$$||\sum_{i \in I^*} v_i|| = \max_{I \subseteq \{1, \ldots, n+1\}} ||\sum_{i \in I} v_i||.$$

Without loss of generality, we may consider $\{v_1, \ldots, v_{n+1}\}$ as a system of *unit vectors* since S is assumed to be equiareal. Therefore we can continue with the following

Lemma. *Given $m > 3$ unit vectors $v_1, \ldots, v_m$ in $\mathbb{R}^n$. Then there exist distinct indices i, j such that $||v_i + v_j|| > 1$.*

Proof. Suppose that $||v_i + v_j|| \leq 1$ for all $1 \leq i < j \leq m$. Squaring we obtain $||v_i||^2 + 2\langle v_i, v_j\rangle + ||v_j||^2 \leq 1$, implying $2\langle v_i, v_j\rangle \leq -1$. Hence,

$$||\sum_{i=1}^{m} v_i||^2 = \sum_{i=1}^{m} ||v_i||^2 + 2\sum_{i<j} \langle v_i, v_j\rangle \leq m - \binom{m}{2},$$

yielding $m - \binom{m}{2} \geq 0$. Thus $m \leq 3$, contradicting the hypothesis. □

In view of (8), this lemma says that for an equiareal n-simplex S, $n \geq 3$, the quantity $\max_{u \in S^{n-1}} V_{n-1}(S, u)$ cannot be equal to the $(n-1)$-volume of an $(n-1)$-face, i.e., (5) is not satisfied. By (3) it follows that no such simplex has its mimimum width in the normal direction of an $(n-1)$-face, i.e., its vertices are not contained in thickness chords and can be cut off without decreasing $\triangle(S)$. Thus, there is no reduced n-simplex for $n \geq 3$. □

3 Concluding remarks

1. Our theorem might be considered as a starting point to solve M. Lassak's problem for all convex n-polytopes (e.g. by some inductional approach based on the cardinality of the vertex set). However, the method presented here can no longer be used. Namely, the function $V_{n-1}(S, u), u \in S^{n-1}$, considered above is known to be the support function of the so-called *projection body* ΠS of the simplex S, and $V_1(S, u), u \in S^{n-1}$, is the radius function of the *difference body* $DS = S + (-S)$ of S. In these terms, relation (2) says that ΠS and

DS are polar reciprocal with respect to the sphere of radius $\sqrt{n \cdot V_n(S)}$ which is centred at the origin. (For definitions and many properties of the bodies ΠS and DS, associated with S, the reader should consult [2, §§ 30 and §§ 33] and [3, § 4.1 and § 3.2].) It was proved in [7] that for all convex n-polytopes which are not simplices such a polarity (even with respect to spheres of arbitrary radii) does no longer hold. Thus our conclusion from (2) to (3) is, in general, no longer true.

2. To get a dualization of the famous Jung theorem (cf. [2], §§ 44), W. Blaschke erroneously assumed that the minimum width of a *regular* n-simplex in $\mathbb{R}^n$ is attained at the normal directions of its $(n-1)$-faces, see [1]. (Blaschke's assumption is true only for $n = 2$, and his statement for higher dimensions was corrected by P. Steinhagen [10].) From our considerations it follows that *no equiareal* n*-simplex*, $n \geq 3$, has the property assumed by Blaschke.

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