TRIANGLE-FREE MINIMUM DISTANCE GRAPHS IN THE PLANE

KONRAD J. SWANEPOEL

Let P be a set of n points in the plane $\mathbb{R}^2$. Denote the minimum distance occurring between pairs of distinct points of P by $\rho(P)$. Define the *minimum distance graph* $G(P)$ of P to have point set P and edge set the pairs of points in P at distance $\rho(P)$. Note that minimum distance graphs are planar. Let $e(P)$ be the number of edges of $G(P)$. Define $e(n) = \max e(P)$ where the maximum is taken over all sets P of n points in the plane. Harborth [1] showed that $e(n) = \lfloor 3n - \sqrt{12n-3} \rfloor$. See also [3, Theorem 13.12]. The sets P attaining the bound $e(P) = e(n)$ have been characterized by Kupitz [2]; they are all subsets of the triangular lattice.

What happens if we do not allow triangles? Define $f(n) = \max e(P)$ where the maximum is taken over all sets P of n points in the plane such that the minimum distance graph $G(P)$ is triangle-free, i.e., it does not contain any clique of size 3. Euler's formula gives $f(n) \leq 2n-4$ for $n \geq 3$. The square grid gives $f(n) \geq \lfloor 2n - 2\sqrt{n} \rfloor$ for $n \geq 1$ (see Fig. 1 for $n = 11$ and $n = 31$).

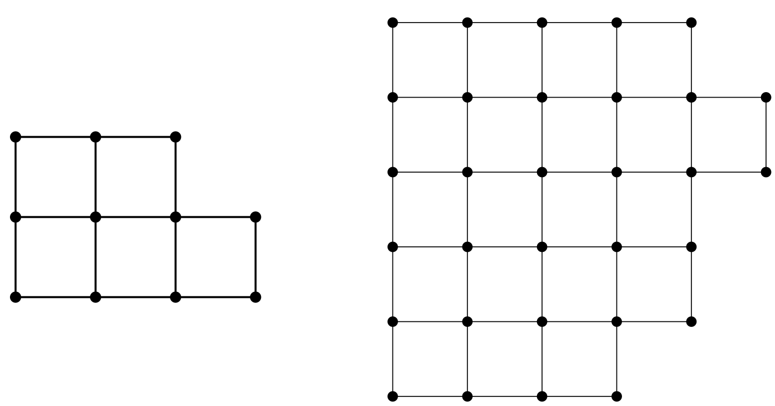

FIGURE 1. Subgraphs of the square lattice on 11 and on 31 points

Problem. *Determine the largest number of edges $f(n)$ in a triangle-free minimum distance graph on $n \geq 1$ points. Is $f(n) = \lfloor 2n - 2\sqrt{n} \rfloor$?*

Harborth's proof does not seem to adapt easily. The square grid is not the only configuration attaining the bound $\lfloor 2n - 2\sqrt{n} \rfloor$, as shown by the vertices of a regular pentagon. The examples in Fig. 2 were discovered by Oloff de Wet (University of South Africa).

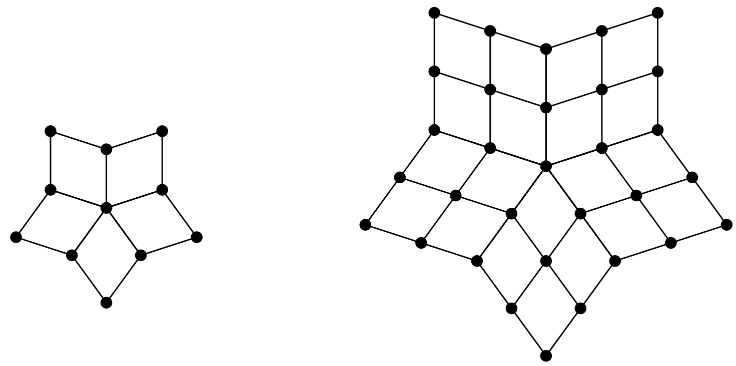

FIGURE 2. Triangle-free minimum distance graphs on 11 and on 31 points

References

[1] H. Harborth, *Lösung zu Problem 664A*, Elemente Math. **29** (1974) 14–15.

[2] Y.S. Kupitz, *On the maximal number of appearances of the minimal distance among n points in the plane*, in: Intuitive Geometry (Szeged, 1991), K. Böröczky et al., eds., Colloq. Math. Soc. János Bolyai **63** (1994) 217–244.

[3] J. Pach and P. K. Agarwal, *Combinatorial Geometry*, Wiley, New York, 1995.

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF SOUTH AFRICA, PO BOX 392, PRETORIA 0003, SOUTH AFRICA

E-mail address: swanekj@unisa.ac.za