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TWO-OBJECT TWO-BIDDER SIMULTANEOUS AUCTIONS

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Symmetric equilibria are constructed for a class of symmetric auction games. The games all have two identical bidders bidding in two simultaneous sealed-bid auctions for identical objects. Information is complete and the objects are either complements or substitutes. In both cases a continuum of mixed-strategy equilibria are identified. All these equilibria have a surprising structure: The supports of all the mixtures that generate equilibria are two one-dimensional curves, and they surround a two-dimensional set of pure best responses.

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1. Introduction

Offshore oil leases and spectrum licenses are examples of objects sold by the US government through simultaneous auctions of one kind or another. Other examples of simultaneous auctions are abundant around the world. Situations for which a simultaneous design is often recommended are when a bidder's valuation for one object is typically dependent on what other objects she wins. A simultaneous auction allows bidders to express their preferences over sets of objects through their bids, although it does not necessarily result in an assignment that is either efficient or revenue maximizing. Though details of simultaneous design rules differ, they all tend to confront bidders with an "exposure problem": A bidder whose valuation for a set of objects exceeds the sum of his valuations for the separate items in the set may bid above the separate stand-alone valuations of the individual items in the hope of winning the set, risking having overbid on pieces in the event that the grand plan is unsuccessful. Various measures have been used to soften exposure problems: in spectrum auctions the simultaneous designs typically involve ascending prices, which allow bidders time to assess gradually the likelihood of successfully acquiring various combinations of spectrum blocks; and provisions for bid withdrawals are often included. These measures do not completely eliminate the problem however, and the US FCC has recently begun to allow all-or-none bids on subsets of licenses to try to solve the problem directly.[1]

Despite all the attempts to get around these problems, little is known about the structure of equilibria in games generated by simultaneous auctions that present exposure problems.[2] This paper contributes to what is known in a small way: Equilibria are constructed for a class of such auction games. In these auctions two identical bidders are competing for two identical objects in a simultaneous first-price sealed-bid auction. The bidders have complete information about the game. The bidders' valuations for the objects are not additive. Two cases are considered; one in which the objects are complements, and one in which the objects are substitutes. Potentially, inefficiency will arise, either because each bidder wins one object although the objects are complements, or because one of the bidders wins both objects and the objects are substitutes.

In each of these games pure-strategy equilibria exist if the tie-breaking rule is properly defined. If the objects are complements and ties on two objects are broken by awarding both objects to one of the bidders, then bidding half of the value of having two objects on each object constitutes a symmetric equilibrium. If the goods are substitutes, bidding zero (or anything less than the value of having one object) on each of the objects again constitutes a symmetric equilibrium if the tie is broken such that each bidder gets one object. These pure-strategy equilibria are uninteresting ones and sensitive to the tie-breaking rule. Hence, this paper focuses on nonatomic equilibria in the hope, that one can learn something about similar games with incomplete information.

Pure strategies in our games are pairs of real numbers (bids). A mixed strategy is a probability measure on $\mathbb{R}^2$, and we restrict our attention to Borel measures only. Each Borel probability measure can be defined by its cumulative distribution function.

Section 2 deals with the case of complements, that is, the bidder's value-function is superadditive. In this case, the allocation is efficient if one of the bidders wins both of the objects. The bidders can potentially coordinate such that they avoid winning one object only. Such a coordination is possible if the bidders' strategy is supported by a single increasing curve.[3] Once a bidder wins with her first bid she wins with the other one too. Equilibria supported by single increasing curves are shown not to exist. There exist, however, a continuum of symmetric equilibria with supports consisting of two increasing curves. One curve is the inverse image of the other. Surprisingly, the set of pure-strategy best responses in each of these equilibria is all the bid vectors "between" the two curves. In fact, the support is the boundary of the set of pure-strategy best responses.

[1] See Ausubel and Milgrom (2002) for a discussion of some of the issues that the FCC is confronting.
[2] See, however, Krishna and Rosenthal (1996), Rosenthal and Wang (1996), and Szentes and Rosenthal (2003a and 2003b).
[3] That is, the bid vectors are in the form $(x, f(x))$ where f is an increasing function.

Section 3 analyzes the case of substitutes, so the bidder's value function is subadditive. Efficiency requires that each of the bidders wins one object. If the bidders use strategies supported by a single decreasing curve, each bidder wins exactly one object. Such a coordination is shown not to exists in equilibrium. It is shown, however, that there is a continuum of equilibria with supports consisting of two decreasing curves. One curve is strictly above the other. Again, the set of pure best responses in each of these equilibria is the area surrounded by the one-dimensional support.

2. The Case of Complements

Assume that a bidder's utility is one if she has two objects and zero otherwise. We will construct a continuum of symmetric mixed-strategy equilibria in this game. The support of these equilibria consists of two increasing curves connecting the points $(0, 0)$ and $(1/2, 1/2)$. One curve is the inverse image of the other. An interesting feature of these equilibria is that the set of pure-strategy best responses is a much larger set than the support. Before constructing these equilibria we analyze the best responses against an arbitrary atomless strategy.

In the proposition below we show that the set of pure best responses is a lattice no matter what atomless strategy the opponent plays. ($X \subset \mathbb{R}^2$ is said to be a lattice if $a, b \in X$ then $a \vee b, a \wedge b \in X$.)

Proposition 1. *Let F be a cdf on $\mathbb{R}^2_+$ such that the marginal distributions of F are atomless. Then the set of pure-strategy best responses to F is a lattice.*

Proof. Suppose that (x_1, y_1) and (x_2, y_2) are both best responses, where $x_1 > x_2$ and $y_1 < y_2$. Then randomizing with probability half on (x_1, y_1) and (x_2, y_2) is also a best response. The expected probability of winning two objects with this mixture is

$$\frac{1}{2}(F(x_1, y_1) + F(x_2, y_2)),$$

and the expected cost is

$$\frac{1}{2}(F(x_1, \infty)x_1 + F(x_2, \infty)x_2 + F(\infty, y_1)y_1 + F(\infty, y_2)y_2).$$

We have to show that $(x_1, y_1) \vee (x_2, y_2) = (x_1, y_2)$ and $(x_1, y_1) \wedge (x_2, y_2) = (x_2, y_1)$ are also best responses. Consider the following mixed strategy. Play both (x_1, y_2) and (x_2, y_1) with probability one-half. The expected cost of this mixed strategy is the same as that of the previous mixed strategy. However, the probability of winning two objects is

$$\frac{1}{2}(F(x_1, y_2) + F(x_2, y_1)).$$

This expression is larger than the expected revenue of the previous mixed strategy by the F-measure of the rectangle spanned by the points (x_1, y_1), (x_1, y_2), (x_2, y_1), and (x_2, y_2). Since the previous mixed strategy was a best response, the F-measure of this rectangle must be zero and the new randomization must also be a best response. If a mixed strategy is a best response, all of those pure-strategies which are played with positive probability are also best responses. Therefore (x_1, y_2) and (x_2, y_1) are also best responses. □

It is also clear from the proof of the previous proposition, that if a, b are in the set of best responses, then either $a \leq (\geq) b$ or the F-measure of the open rectangle spanned by $a, b, a \vee b$ and $a \wedge b$ is zero. In particular, this must be true for the points in the support of any equilibrium. This implies that the support of any equilibrium can contain neither a decreasing curve nor a two dimensional object. This observation suggests that in a symmetric equilibria the support of the strategy consists of at most two increasing curves. Does there exist a symmetric equilibrium strategy supported on a single increasing curve? Suppose there exists such an equilibrium. If there was an atom at the bottom of the support, then an infinitesimal increase from this atom is obviously a profitable deviation, a contradiction. If there is no atom at the bottom, then it must generate zero profit, since it does not have a chance to win. If the top of the support makes zero profit, then every other bid vector between the bottom and the top must make a strictly positive profit. This means that the bidder cannot be indifferent on the support, a contradiction. Is it possible to construct an atomless equilibrium with a support consisting of more than two increasing curves? Notice that none of the curves can be located between the other two. That is, if a and b belong to different curves, then no curve can intersect with the rectangle spanned by $a, b, a \vee b$ and $a \wedge b$. Therefore the support can always be seen to be the union of two curves that are not necessarily connected. In what follows a continuum of equilibrium strategies is constructed. The supports of each of these strategies consist of two increasing curves.

Let h be an atomless cdf on $[0, 1/2]$ such that $h(x)(1-x)/x$ is also a cdf on $[0, 1/2]$. Let g be implicitly defined on $[0, 1/2]$ by the following equation:

$$h(g(x)) = \frac{1-x}{x} h(x).$$

Since $h(x)(1-x)/x$ is a cdf g is well-defined. The support of the strategy is the curve g and its inverse image, that is,

$$\{(x, g(x)) : x \in [0, 1/2]\} \cup \{(g(x), x) : x \in [0, 1/2]\}.$$

The measure on the curves are defined as follows:

$$\mu\{(x, g(x)) : x \in [0, a]\} = \mu\{(g(x), x) : x \in [0, a]\} = \frac{h(x) + h(g(x))}{2}.$$

This strategy is obviously symmetric with respect to the objects. To see that such an h (and hence g) really exists notice the following. If f is a cdf on $[0, 1/2]$ and $h(x) = (x/(1-x))^n f(x)$ $(n \in \mathbb{N}\backslash\{0\})$ then $h(x)(1-x)/x$ is also a cdf on $[0, 1/2]$. Therefore, there is indeed a continuum of such strategies.

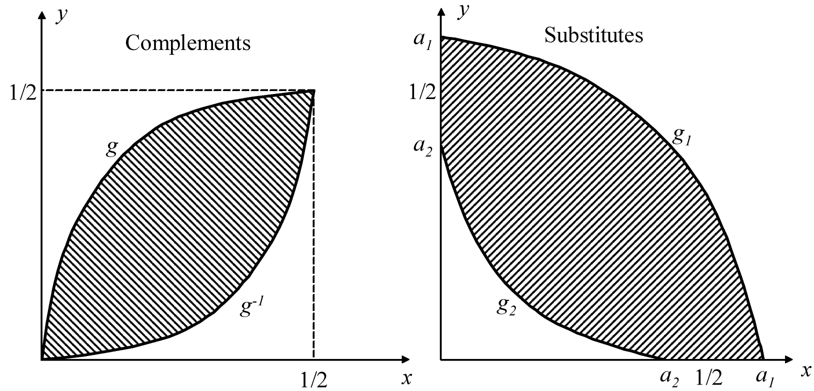

Fig. 1.

Theorem 1. *The strategy defined above constitutes a symmetric equilibrium. The bidders break even and the set of pure-strategy best responses is*

$$\{(x, y) : x \in [0, 1/2], y \in g^{-1}(x), g(x)\}.$$

The support of the equilibrium strategy depicted on the left picture of Fig. 1. The shaded area is the set of pure best responses.

Proof. First, we show that the expected payoff of a bid vector (x, y) where $x \in [0, 1/2], y \in [g^{-1}(x), g(x)]$ is zero. The probability of winning two objects by bidding (x, y) is clearly

$$\frac{h(x) + h(y)}{2}.$$

The expected cost of bidding (x, y) is

$$\frac{h(x) + h(g(x))}{2}x + \frac{h(y) + h(g(y))}{2}y.$$

Observe that the payoff is additively separable with respect to the bids. Collecting the terms containing x together we get

$$h(x) - \frac{h(x) + h(g(x))}{2}x.$$

Recall that $h(g(x)) = ((1 - x)/x)h(x)$. Therefore the previous expression is zero. Hence, the claim about the set of pure-strategy best responses is established.

It still remains to show that there does not exist any profitable deviation. It is obviously enough to restrict attention to deviations where both of the bids are between zero and one-half. Since the strategy is symmetric with respect to the bids, one can consider the following arbitrary deviation: (x, y) where $x \in [0, 1/2)$ and $y \in (g(x), 1/2]$. Observe that the probability that the bid vector (x, y) wins two objects is exactly the same as the probability that the bid vector $(x, g(x))$ wins two objects. However the expected cost of $(x, g(x))$ is strictly smaller than that of $(x \, . \, y)$. □

Notice that the support is one dimensional, and the set of pure best responses is two dimensional. In fact, the support is the boundary of the set of best responses.

In all of the equilibria constructed above the bidders are breaking even. However, the allocation is not always efficient and therefore the surplus is not fully extracted.

Intuition. In what follows we provide an intuition for the increasing nature of the supports of these equilibria. Suppose that (x_1, y_1) is in the support and the bidder decides to bid $x_2 > x_1$ on the first object. That is she bids more aggressively on the first object. Then the probability of winning the first object goes up compared to the one generated by the bid vector (x_1, y_1). Since the objects are complements bidding x_2 instead of x_1 on the first object makes the second object more valuable. Therefore the bidder should bid more aggressively on the second object too. That is why, one would expect the bid x_2 be matched up with a bid y_2, such that $y_2 > y_1$.

We do not claim a full characterization of atomless equilibria. In fact, we exhibit an equilibrium different from those described before. An interesting symmetric equilibrium can be constructed by the following strategy: Select an object randomly, and bid on this object randomly according to the cdf $x/(1-x)$ on $[0, 1/2]$, and bid zero on the other object. That is, although the bidders wish to win two objects they bid only on one of them. We show that this strategy constitutes a symmetric equilibrium with the following tie-breaking rule. If there is a tie on one of the objects, whoever bids higher on the other object wins both.

Claim 1. *The strategy described above constitutes a symmetric equilibrium with the tie-breaking rule defined above. The set of pure-strategy best responses is*

$$\{(x, y) : x, y \in [0, 1/2]\}.$$

Proof. We show that the bid vector (x, y) makes zero profit whenever $x \in [0, 1/2]$ and $y \in [0, 1/2]$. The probability of winning two objects is clearly

$$\frac{1}{2}\frac{x}{1-x} + \frac{1}{2}\frac{y}{1-y}.$$

The expected cost is

$$\left(\frac{1}{2} + \frac{1}{2}\frac{x}{1-x}\right)x + \left(\frac{1}{2} + \frac{1}{2}\frac{y}{1-y}\right)y.$$

Collecting the terms containing x together in the payoff, one can get

$$\frac{1}{2}\frac{x}{1-x} - \left(\frac{1}{2} + \frac{1}{2}\frac{x}{1-x}\right)x,$$

which is zero. The same is true for the terms containing y. Therefore the payoff of (x, y) is zero. Hence the claim about the set of best responses is established. Profitable deviation obviously does not exist. □

Although the bidders are desperate to win both of the objects, in the previous equilibrium they always place positive bid on one only in the hope that they win the other one when the tie is broken.

3. The Case of Substitutes

Assume that each of the bidders would like to win one object only. That is, the utility of a bidder is one if she wins at least one object and zero otherwise. We will exhibit a continuum of symmetric atomless equilibria. (Symmetry is understood not only with respect to the players but also with respect to the objects.) The support consists of two decreasing curves. The inverse image of either of these curves is the curve itself.

Before constructing these equilibria we again characterize the set of best responses against an arbitrary atomless strategy. Recall that in Proposition 1 we have shown that when the objects are complements, the set of pure-strategy best responses is always a lattice. That is, if a, b are best responses then so are $a \vee b$ and $a \wedge b$. We will show that if the objects are substitutes, and $a \vee b$ and $a \wedge b$ are best responses, then so are a and b. Let us call the set $X \subset \mathbb{R}^2$ an *anti-lattice* if $a \vee b \in X$, $a \wedge b \in X$ implies $a \in X$, $b \in X$.

Proposition 2. *Let F be a cdf on $\mathbb{R}^2_+$ such that the marginal distributions of F are atomless. Then the set of pure-strategy best responses for F is an anti-lattice.*

Proof. Suppose that (x_1, y_1) and (x_2, y_2) are both best responses, where $x_1 > x_2$ and $y_1 > y_2$. Then randomizing with probability one-half on these vectors is also a best response. The probability of winning at least one object with this mixed strategy is

$$\frac{1}{2}(F(x_1,\infty) + F(x_2,\infty) + F(\infty,y_1) + F(\infty,y_2) - F(x_1,y_1) - F(x_2,y_2)), \tag{1}$$

and the expected payment is

$$\frac{1}{2}(F(x_1,\infty)x_1 + F(x_2,\infty)x_2 + F(\infty,y_1)y_1 + F(\infty,y_2)y_2).$$

Now consider the following mixed strategy. Play both (x_1, y_2) and (x_2, y_1) with probability one-half. The expected payment of this mixed strategy is the same as that of the previous mixed strategy. However, the probability of winning at least one object is clearly

$$\frac{1}{2}(F(x_1,\infty) + F(x_2,\infty) + F(\infty,y_1) + F(\infty,y_2) - F(x_1,y_2) - F(x_2,y_1)).$$

This expression is larger than (1) by the F-measure of the rectangle spanned by the points (x_1, y_1), (x_1, y_2), (x_2, y_1) and (x_2, y_1). Since the previous mixed strategy was a best response, the F-measure of this rectangle must be zero and the new randomization must also be a best response. If a mixed strategy is a best response, all those pure strategies which are played with positive probability are also best responses. Therefore (x_1, y_2) and (x_2, y_1) are best responses. □

It follows from the proof of the previous proposition that if $a \vee b$ and $a \wedge b$ are in the set of best responses, then the F-measure of the rectangle spanned by $a, b, a \vee b$ and $a \wedge b$ must be zero. In particular, for any mixed-strategy equilibrium the following statement must be true: If $a \vee b$ and $a \wedge b$ belong to the support of the equilibrium then the intersection of the support and the open rectangle spanned by $a, b, a \vee b$ and $a \wedge b$ has F-measure zero. Hence, one can conclude that the support of any atomless symmetric equilibrium strategy cannot contain any increasing curves and cannot be two dimensional. This observation suggests equilibria supported on two decreasing curves. One can exclude the existence of an equilibrium supported on one or more than two decreasing curves.

The support of the strategy is two continuous strictly decreasing curves g_1 and g_2. g_1 is defined on $[0, a_1]$ and g_2 on $[0, a_2]$ where $a_1 \in (1/2, 1)$ and $a_2 \in (0, 1/2)$. They are defined implicitly by the following equations:

$$G^1(g_1(x)) = a_1 - G^1(x),$$
$$G^2(g_2(x)) = a_2 - G^2(x)$$

where $G^2(x)$ is a strictly increasing function on $[0, a_2]$, $G^2(0) = 0$, and $G^2(a_2) = 1 - a_1$. Furthermore

$$G^1(x) = \begin{cases} \dfrac{x}{1-x} G^2(x) & \text{if } x \in [0, a_2] \\[2ex] \dfrac{x}{1-x}(1 - a_1) & \text{if } x \in [a_2, a_1]. \end{cases}$$

Notice that G^1 is continuous since $G^2(a_2) = 1 - a_1$. The measures on the curves are defined as follows:

$$\mu_1(\{(y, g_1(y)) | y \in [0, x)\}) = G^1(x)$$
$$\mu_2(\{(y, g_2(y)) | y \in [0, x)\}) = G^2(x).$$

Observe that μ_1 and μ_2 indeed define a probability measure on the union of g_1 and g_2. This is because the μ_1-measure of g_1 is a_1 and the μ_2-measure of g_2 is $1 - a_1$. Clearly there are a continuum of strategies described above.

For convenience let us define G^2 as $1 - a_1$ and g_2 as zero on $[a_2, a_1]$.

Theorem 2. *The strategy described above generates a symmetric mixed-strategy equilibrium in the first-price auction. The set of pure-strategy best responses is*

$$\{(x, y) : x \in [0, a_1],\ y \in [g_2(x), g_1(x)]\}.$$

The support of the equilibrium strategy depicted on the right picture of Fig. 1. The shaded area is the set of pure best responses.

Before we prove this theorem we show that one curve is always above the other, and the strategy described above is indeed symmetric with respect to the objects.

Lemma 1. *$g_1 > g_2$ and both of them are decreasing. The strategy described above is symmetric with respect to the objects.*

Proof. First we show that g_i $(i = 1, 2)$ is symmetric on its domain, that is, $g_i(g_i(x)) = x$. Observe that by the definition of g_i,

$$G^i(g_i(g_i(x))) = a_i - G^i(g_i(x)) = a_i - (a_i - G^i(x)) = G^i(x).$$

Since G^i is strictly increasing $g_i(g_i(x)) = x$. Notice, the measures on the two curves are also symmetric.

Next, we show that $g_1 > g_2$ on $[a_2, a_1]$.

$$g_1(x) = a_1 - G^1(g_1(g_1(x))) = a_1 - G^1(x) > a_2 - G^2(x) = g_2(x).$$

The second equality follows from the symmetry of g_1 and the inequality follows from $a_1 > a_2$ and $G^1(x) < G^2(x)$.

We show that g_i is strictly decreasing. If $x < y$ then $G^i(x) < G^i(y)$ and $a_i - G^i(x) > a_i - G^i(y)$. But then, by the definition of g_i, $g_i(x) > g_i(y)$. □

We are ready to prove Theorem 2.

Proof of Theorem 2. Let (x, y) be a bid vector such that $x \in [0, a_1]$, $y \in [g_2(x), g_1(x)]$. The probability of winning at least one object is

$$1 - G_1(a_1) + G_1(x) + G_1(y).$$

($1 - G_1(a_1)$ is the probability that the opponent is bidding on g_2.) The bidder's expected payment is clearly

$$(G_1(x) + G_2(x))x + (G_1(y) + G_2(y))y.$$

Observe that $G_1(z) = (G_1(z)+G_2(z))z$ for all $z \in [0, a_1]$, and therefore the expected payoff is $1 - G_1(a_1)$. That is, if there does not exist a profitable deviation, the set of pure-strategy best responses is

$$\{(x, y) : x \in [0, a_1],\ y \in [g_2(x), g_1(x)]\}.$$

It remains to show that there does not exist any profitable deviation. One can clearly restrict attention to deviations where the bids are smaller than a_1. Consider first a deviation (x, y) where $y > g_1(x)$. By decreasing the second bid to $g_1(x)$ the bidder still wins with probability one but pays less. Therefore this deviation cannot be profitable. Suppose now that $y < g_2(x)$. Then, the probability of winning at least one object is

$$G_1(x) + G_2(x) + G_1(y) + G_2(y). \tag{2}$$

The expected payment is

$$(G_1(x) + G_2(x))x + (G_1(y) + G_2(y))y = G_1(x) + G_1(y) \tag{3}$$

because $G_1(z) = (G_1(z) + G_2(z))z$ for all $z \in [0, a_1]$. Since $G_1(a_1) + G_2(a_2) = 1$, (2) can be rewritten as

$$1 - G_1(a_1) + G_1(x) + G_1(y) - (G_2(a_2) - G_2(x) - G_2(y)).$$

Using (3) and the previous expression the payoff of the deviation is

$$1 - G_1(a_1) - (G_2(a_2) - G_2(x) - G_2(y)).$$

Since $y < g_2(x)$, $G_2(a_2) > G_2(x) + G_2(y)$ and therefore the payoff of the deviation is less than $1 - G_1(a_1)$ which is the payoff on the support. □

Observe that the ex-post allocation is sometimes inefficient, because a bidder may win both of the objects. Again, the one dimensional support is the boundary of the two-dimensional set of pure best responses.

Intuition. Again, we provide an intuition for the decreasing nature of the supports of these equilibria. Suppose that (x_1, y_1) is in the support and the bidder decides to bid $x_2 > x_1$ on the first object. That is, she bids more aggressively on the first object. Then the probability of winning the first object goes up compared to the one generated by the bid vector (x_1, y_1). Since the objects are substitutes, bidding x_2 instead of x_1 on the first object makes the second object less valuable. Therefore, the bidder should bid less aggressively on the second object. This explains that the bid x_2 be matched up with a bid y_2, such that $y_2 < y_1$.

4. Discussion

Propositions 1 and 2 express unusually strong statements. Specific attributes of the sets of best responses are characterized against an arbitrary strategy.[4] The attribute of the equilibria structure that the one-dimensional support is the boundary of the two-dimensional set of best responses seems to be particularly interesting. Szentes and Rosenthal (2003a) also exhibits an equilibrium in a complete information auction game with similar attributes, where two identical bidders are competing for three identical objects in simultaneous first-price auctions. The bidders' marginal valuation is one for the second object and zero for the first and the third ones. They show an equilibrium which is supported on the surface of a regular tetrahedron. The set of best responses is shown to be the tetrahedron itself.

In the case of complements, assuming a positive marginal value of the first object which is less than the marginal value of the second object does not change the main results (but makes the computations slightly more tedious). The driving force of the results is the increasing marginal valuation. In the case of substitutes, a positive marginal value of the second object could have been assumed which is less than that of the first one.

[4]Milgrom and Roberts (1990) identify a class of games, called *supermodular games*, where the set of best responses is always a lattice. Although, our games are not supermodular, the case of complements has similar features.

In this paper we have only analyzed first-price auctions. However, the structure of equilibria and in particular the topological attributes of the supports are the same in a second-price auction too. As a matter of fact, for each equilibrium support of the first-price auction there is a corresponding equilibrium in the second-price auction, such that the support of the second-price equilibrium is a monotonic transformation of the first-price one. This is formalized in Szentes (2003).

What happens if there are more than two objects? If the marginal valuations of the bidders are increasing in the number of objects they win, one can generalize our results. One can construct a continuum of equilibria in the following forms. The support of these equilibria consists of curves in the following form: $(x, \ldots, x, g(x))$, where g is an increasing function. The number of curves constituting the support is equal to the number of objects. We do not know what happens if the bidder's marginal valuation is decreasing.

What happens if there are more than two bidders? We conjecture but could not prove that there do not exist nonatomic mixed-strategy equilibria in those games.

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