Proofs of Propositions 4-6

Proof of Proposition 4: We proceed in two steps, as in Proposition 2.

Step 1: Properties of a hierarchical structure that satisfies conditions (i) and (ii). Such a hierarchical structure must involve $X(N)$ agents, and must satisfy $m_x = K-1$ for $x = 2, .., X(N)$, and $m_1 = m(N)$. Lemma 3 implies that the sum of expected squared errors (which we normalize by $\sigma_\xi^4(1+rN)^2$, and refer to as the cost) is

$$c(N) \equiv \sum_{x=1}^{X(N)-1} \frac{\left(\sum_{x'=x+1}^{X(N)} m_{x'}\right) m_x}{\left[1+r\left(1+\sum_{x'=x+1}^{X(N)} m_{x'}\right)\right]\left[1+r\left(1+\sum_{x'=x}^{X(N)} m_{x'}\right)\right]}. \tag{46}$$

A first property of $c(N)$ is that

$$c(N) - c(N-1) = \frac{[X(N)-1](K-1)}{[1+rN][1+r(N-1)]}. \tag{47}$$

To show equation (47), we assume first that with $N-1$ assets, the top agent works below capacity. Then, by adding one asset, m_1 increases by 1, and equation (46) changes by

$$\frac{D}{1+r(1+D)}\left[\frac{m(N)}{1+rN} - \frac{m(N)-1}{1+r(N-1)}\right],$$

where

$$D = \sum_{x'=2}^{X(N)} m_{x'} = [X(N)-1](K-1) = N-1-m(N).$$

Rearranging this equation, we find equation (47). If, with $N-1$ assets, the top agent works at full capacity, then by adding one asset, we obtain a new top agent with $m_1 = 1$. The term in equation (46) corresponding to this new agent is

$$\frac{D}{[1+rD][1+rN]} = \frac{D}{[1+r(N-1)][1+rN]},$$

and we find again equation (47). A second property of $c(N)$ is that for $n_1, n_2 > 1$,

$$c(n_1) + c(n_2) - c(n_1+n_2-1) + g(n_1,n_2) > 0, \tag{48}$$

where

$$g(n_1,n_2) = \frac{(n_1-1)(n_2-1)[2+r(n_1+n_2)]}{[1+rn_1][1+rn_2][1+r(n_1+n_2-1)]}.$$

To show equation (48), we assume that $n_1 \geq n_2$, and denote the LHS by $f(n_2)$. We first show that $f(n_2) > 0$ for $n_2 \leq K$. Using equation (47), we have

$$\begin{aligned}
c(n_1+n_2-1) - c(n_1) &\geq [X(n_1+n_2-1)-1](K-1)\sum_{n=n_1}^{n_1+n_2-1}\frac{1}{[1+rn][1+r(n-1)]} \\
&= [X(n_1+n_2-1)-1](K-1)\frac{1}{r}\left[\frac{1}{1+rn_1} - \frac{1}{1+r(n_1+n_2-1)}\right] \\
&= \frac{[X(n_1+n_2-1)-1](K-1)(n_2-1)}{[1+rn_1][1+r(n_1+n_2-1)]}
\end{aligned}$$

Using this equation, $c(n_2) = 0$, and $n_1 \geq n_2$, we have

$$f(n_2) \geq \frac{n_2 - 1}{[1 + rn_1][1 + r(n_1 + n_2 - 1)]} \left\{2(n_1 - 1) - [X(n_1 + n_2 - 1) - 1](K - 1)\right\}.$$

Defining ϵ as in Proposition 2, we can write the term in curly brackets as

$$2(n_1 - 1) - [X(n_1) + \epsilon - 1](K - 1) = n_1 - 1 + m(n_1) - \epsilon(K - 1),$$

i.e., as equation (37), which is positive. Therefore, $f(n_2) > 0$ for $n_2 \leq K$, and to show equation (48), it suffices to show that $f(n_2)$ is increasing for $n_2 > K$. Simple algebra shows that

$$g(n_1, n_2) - g(n_1, n_2 - 1) = \frac{(n_1 - 1)(1 + r)[2 + r(n_1 + 2n_2 - 2)]}{[1 + rn_2][1 + r(n_2 - 1)][1 + r(n_1 + n_2 - 1)][1 + r(n_1 + n_2 - 2)]}.$$

Using this equation, and equation (47), we find that $f(n_2) - f(n_2 - 1)$ has the same sign as

$$\begin{aligned}
&[X(n_2) - 1](K - 1)[1 + r(n_1 + n_2 - 1)][1 + r(n_1 + n_2 - 2)] \\
&-[X(n_1 + n_2 - 1) - 1](K - 1)[1 + rn_2][1 + r(n_2 - 1)] \\
&+(n_1 - 1)(1 + r)[2 + r(n_1 + 2n_2 - 2)].
\end{aligned}$$

Using the definition of $m(N)$, and some algebra, we can write this as

$$\begin{aligned}
(n_1 - 1)[1 + rn_2][1 + r(n_1 + n_2 - 1)] \quad &- \quad m(n_2)[1 + r(n_1 + n_2 - 1)][1 + r(n_1 + n_2 - 2)] \\
&+ \quad m(n_1 + n_2 - 1)[1 + rn_2][1 + r(n_2 - 1)].
\end{aligned}$$

The third term in this equation is obviously positive. Since

$$\begin{aligned}
(n_1 - 1)[1 + rn_2] - m(n_2)[1 + r(n_1 + n_2 - 2)] \quad &\geq \quad (n_1 - 1)[1 + rn_2] - m(n_2)[1 + 2r(n_1 - 1)] \\
&= \quad n_1 - 1 - m(n_2) + r(n_1 - 1)[n_2 - 2m(n_2)],
\end{aligned}$$

and $n_2 > 2m(n_2)$ for $n_2 > K$, the difference between the first two terms is also positive. Therefore, equation (48) holds.

Step 2: A cost-minimizing hierarchical structure must satisfy conditions (i) and (ii). We use the dynamic programming observation of Proposition 2, and proceed by induction. Consider a cost-minimizing hierarchical structure H, in which the top agent has at least two subordinates, agents 1 and 2. Then, constructing H' as in Proposition 2, we have

$$c(H) = c[N(1)] + c[N(2)] + \frac{[N(1) - 1][N - N(1)]}{[1 + rN(1)][1 + rN]} + \frac{[N(2) - 1][N - N(2)]}{[1 + rN(2)][1 + rN]} + \hat{c},$$

and

$$c(H') = c[N(1)+N(2)-1] + \frac{[N(1)+N(2)-2][N-[N(1)+N(2)-1]]}{[1+r[N(1)+N(2)-1]][1+rN]} + \hat{c}.$$

Simple algrebra shows that $c(H) - c(H')$ is equal to the LHS of equation (48), which is positive, contradicting the optimality of H. Therefore, the top agent in H has only one subordinate, agent 1, and H satisfies condition (i). To show that H satisfies condition (ii), we need to show that the top agent in $H(1)$ works at full capacity, i.e., $m_2 = K-1$. Suppose that $m_2 < K-1$. If $m_2 < m_1$, then we can decrease the cost of H by inverting m_1 and m_2. This is because the sum of the numerators in equation (46) stays constant, while each denominator stays constant or increases. If $m_2 \geq m_1$, then we can decrease the cost of H by adding 1 to m_2 and subtracting 1 from m_1. This is because the sum of the numerators in equation (46) decreases (as shown in the proof of Proposition 2), while each denominator stays constant or increases. Therefore, H satisfies condition (ii). $\|$

Proof of Proposition 5: We will compute the sum of expected squared errors for H_1, H_2, and $\hat{H}_2$. Since all three hierarchies satisfy Condition 1, $E(e_n^2)$ is given by equation (44). Since, in addition, all variables are normal, $E\left[e_n(j)^2\right]$ is equal to the expected squared aggregation loss term, times the expected squared interaction term. Moreover, the latter is given by equation (45).

Consider first H_2. The expected squared aggregation loss term corresponding to $e_1(\cdot)$ is

$$E\left(\lambda_1 - \frac{\lambda_1+\lambda_2}{2}\right)^2 = E\left(\frac{\xi_1-\xi_2}{2}\right)^2 = \frac{\sigma_\xi^2}{2},$$

and the expected squared interaction term is

$$V(\lambda_3+\lambda_4|\lambda_1,\lambda_2) = V(\lambda_3+\lambda_4) = V(2\zeta_{34}+\xi_3+\xi_4) = 4\sigma_\zeta^2 + 2\sigma_\xi^2.$$

Since assets are symmetric, the sum of expected squared errors is

$$\sum_{n=1}^{4} E(e_n^2) = 4E(e_1^2) = 4E\left[e_1(\cdot)^2\right] = 4\frac{\sigma_\xi^2}{2}(4\sigma_\zeta^2 + 2\sigma_\xi^2) = 8\sigma_\xi^2\sigma_\zeta^2 + 4\sigma_\xi^4.$$

Consider next $\hat{H}_2$. The expected squared aggregation loss term corresponding to $e_1(\cdot)$ is

$$E\left(\lambda_1 - \frac{\lambda_1+\lambda_3}{2}\right)^2 = E\left(\frac{\zeta_{12}+\xi_1-\zeta_{34}-\xi_3}{2}\right)^2 = \frac{\sigma_\zeta^2+\sigma_\xi^2}{2},$$

and the expected squared interaction term is

$$V(\lambda_2+\lambda_4|\lambda_1,\lambda_3) = V(\lambda_2|\lambda_1) + V(\lambda_4|\lambda_3) = 2V(\lambda_2|\lambda_1) = 2\frac{\sigma_\xi^2(2\sigma_\zeta^2+\sigma_\xi^2)}{\sigma_\zeta^2+\sigma_\xi^2}.$$

Since assets are symmetric, the sum of expected squared errors is

$$\sum_{n=1}^{4} E(e_n^2) = 4E(e_1^2) = 4E\left[e_1(\cdot)^2\right] = 4\frac{\sigma_\zeta^2 + \sigma_\xi^2}{2}\left[2\frac{\sigma_\xi^2(2\sigma_\zeta^2 + \sigma_\xi^2)}{\sigma_\zeta^2 + \sigma_\xi^2}\right] = 8\sigma_\xi^2\sigma_\zeta^2 + 4\sigma_\xi^4,$$

the same as for H_2.

Consider finally H_1. The expected squared aggregation loss term corresponding to $e_1(1)$ is

$$E\left(\lambda_1 - \frac{\lambda_1 + \lambda_2}{2}\right)^2 = \frac{\sigma_\xi^2}{2},$$

and the expected squared interaction term is

$$V[\lambda_3 + E(\lambda_4|\lambda_1,\lambda_2,\lambda_3)|\lambda_1,\lambda_2] = V[\lambda_3 + E(\lambda_4|\lambda_3)] = V\left[\frac{2\sigma_\zeta^2 + \sigma_\xi^2}{\sigma_\zeta^2 + \sigma_\xi^2}\lambda_3\right] = \frac{(2\sigma_\zeta^2 + \sigma_\xi^2)^2}{\sigma_\zeta^2 + \sigma_\xi^2}.$$

The expected squared aggregation loss term corresponding to $e_1(\cdot)$ is

$$E\left(\lambda_1 - \frac{\lambda_1 + \lambda_2 + \lambda_3}{3}\right)^2 = E\left(\frac{\zeta_{12} + 2\xi_1 - \xi_2 - \zeta_{34} - \xi_3}{3}\right)^2 = \frac{2\sigma_\zeta^2 + 6\sigma_\xi^2}{9},$$

and the expected squared interaction term is

$$V(\lambda_4|\lambda_1,\lambda_2,\lambda_3) = V(\lambda_4|\lambda_3) = \frac{\sigma_\xi^2(2\sigma_\zeta^2 + \sigma_\xi^2)}{\sigma_\zeta^2 + \sigma_\xi^2}.$$

The expected squared aggregation loss term corresponding to $e_3(\cdot)$ is

$$E\left(\lambda_3 - \frac{\lambda_1 + \lambda_2 + \lambda_3}{3}\right)^2 = E\left(\frac{2\zeta_{34} + 2\xi_3 - 2\zeta_{12} - \xi_1 - \xi_2}{3}\right)^2 = \frac{8\sigma_\zeta^2 + 6\sigma_\xi^2}{9},$$

and the expected squared interaction term is as for $e_1(\cdot)$. Since assets 1 and 2 are symmetric, and there is no error for asset 4, the sum of expected squared errors is

$$\begin{aligned}
\sum_{n=1}^{4} E(e_n^2) &= 2E(e_1^2) + E(e_3^2) \\
&= 2\left[\frac{\sigma_\xi^2}{2}\frac{(2\sigma_\zeta^2 + \sigma_\xi^2)^2}{\sigma_\zeta^2 + \sigma_\xi^2} + \frac{2\sigma_\zeta^2 + 6\sigma_\xi^2}{9}\frac{\sigma_\xi^2(2\sigma_\zeta^2 + \sigma_\xi^2)}{\sigma_\zeta^2 + \sigma_\xi^2}\right] + \frac{8\sigma_\zeta^2 + 6\sigma_\xi^2}{9}\frac{\sigma_\xi^2(2\sigma_\zeta^2 + \sigma_\xi^2)}{\sigma_\zeta^2 + \sigma_\xi^2} \\
&= \sigma_\xi^2\frac{2\sigma_\zeta^2 + \sigma_\xi^2}{\sigma_\zeta^2 + \sigma_\xi^2}\left(\frac{10}{3}\sigma_\zeta^2 + 3\sigma_\xi^2\right).
\end{aligned}$$

Simple algebra shows that this is smaller than for H_2 and $\hat{H}_2$, for all σ_ζ^2 and σ_ξ^2. ||

Proof of Proposition 6: For $\sigma_\zeta^2/\sigma_\xi^2$ sufficiently large, it is optimal that agents know the group components for the assets they examine. Indeed, suppose that this is not the case for one agent. If the agent examines two assets in the same group, then the group component will not be reflected in the weighting of the agent's group relative to the other group. If the agent examines two assets in different groups, then the group component will not be reflected in the relative weighting of the two assets. Finally, if the agent examines one asset and a subordinate's portfolio (which contains at least one asset in a different group), then the group component will not be correctly reflected in the relative weighting of the asset and the portfolio.

Given that agents know the group components, we can assume without loss of generality, that their information for the assets they examine is as follows:

	Agent 1,1	Agent 1	Agent ·
H_1	ζ_{12}, ξ_2	ζ_{34}	ζ_{34}
$\hat{H}_1$	ζ_{12}, ζ_{34}	ζ_{12}	ζ_{34}

and

	Agent 1	Agent 2
H_2	ζ_{12}, ξ_2	ζ_{34}, ξ_4
$\hat{H}_2$	ζ_{12}, ζ_{34}	ζ_{12}, ζ_{34}

To determine agents' optimal decision rules, we equations (18) and (19). The only difference with the case where factor loadings are observed perfectly, is that when defining $\Gamma(j)$ and $I(j)$, we replace λ_n by its expectation conditional on the information of the agent who examines asset n.

The lowest order term associated to the first-best investment in asset 1 is

$$F_1^* = -(\zeta_{12} + \xi_1)(2\zeta_{12} + \xi_1 + \xi_2 + 2\zeta_{34} + \xi_3 + \xi_4),$$

and those for assets 2, 3, and 4, are symmetric. For H_2, the lowest order term associated to the investment in asset 1 is

$$F_1 = -\zeta_{12}(2\zeta_{12} + \xi_2) - \frac{2\zeta_{12} + \xi_2}{2}(2\zeta_{34} + \xi_4) - \sigma_\xi^2,$$

that for asset 2 is

$$F_2 = -(\zeta_{12} + \xi_2)(2\zeta_{12} + \xi_2) - \frac{2\zeta_{12} + \xi_2}{2}(2\zeta_{34} + \xi_4),$$

and those for assets 3 and 4 are symmetric. For $\hat{H}_2$, we have

$$F_1 = -\zeta_{12}(2\zeta_{12} + 2\zeta_{34}) - \sigma_\xi^2,$$

and symmetrically for assets 2, 3, and 4. For $\hat{H}_1$, the lowest order terms are the same as for $\hat{H}_2$, since in both hierarchies an agent examining an asset has the same information on all assets as his superiors. Finally, for H_1, we have

$$F_1 = -\zeta_{12}(2\zeta_{12} + \xi_2) - \frac{2\zeta_{12} + \xi_2}{2} 2\zeta_{34} - \sigma_\xi^2,$$

$$F_2 = -(\zeta_{12} + \xi_2)(2\zeta_{12} + \xi_2) - \frac{2\zeta_{12} + \xi_2}{2} 2\zeta_{34},$$

$$F_3 = F_4 = -\zeta_{34}(2\zeta_{12} + \xi_2 + 2\zeta_{34}) - \sigma_\xi^2.$$

It is easy to check that for $\sigma_\zeta^2/\sigma_\xi^2$ sufficiently large, and for each n, $E[(F_n - F_n^*)^2]$ is smallest for H_2. ||